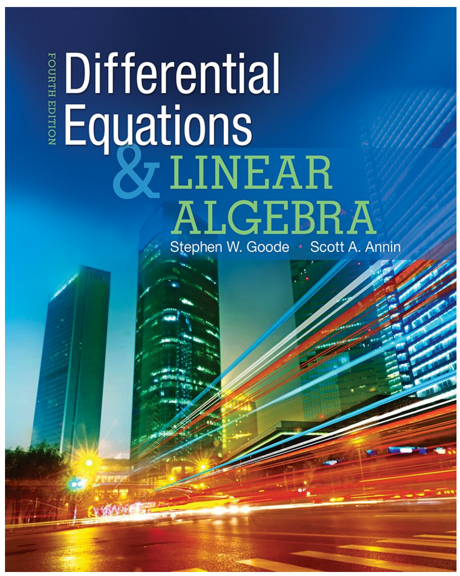
A Solution Manual For

Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015



Nasser M. Abbasi

May 15, 2024

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1.1 problem Problem 7

Internal problem ID [2587]
Internal file name [`OUTPUT/2079_Sunday_June_05_2022_02_47_48_AM_5325985/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.2, Basic Ideas and Terminology. page 21
Problem number: Problem 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

`[[_2nd_order, _missing_x]]`

$$\boxed{y'' - 25y = 0}$$

1.1.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -25$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 25\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 25 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -25$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-25)} \\ &= \pm 5\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +5 \\ \lambda_2 &= -5\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 5 \\ \lambda_2 &= -5\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(5)x} + c_2 e^{(-5)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{5x} + c_2 \mathrm{e}^{-5x}$$

<u>Summary</u>

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{5x} + c_2 \mathrm{e}^{-5x} \tag{1}$$

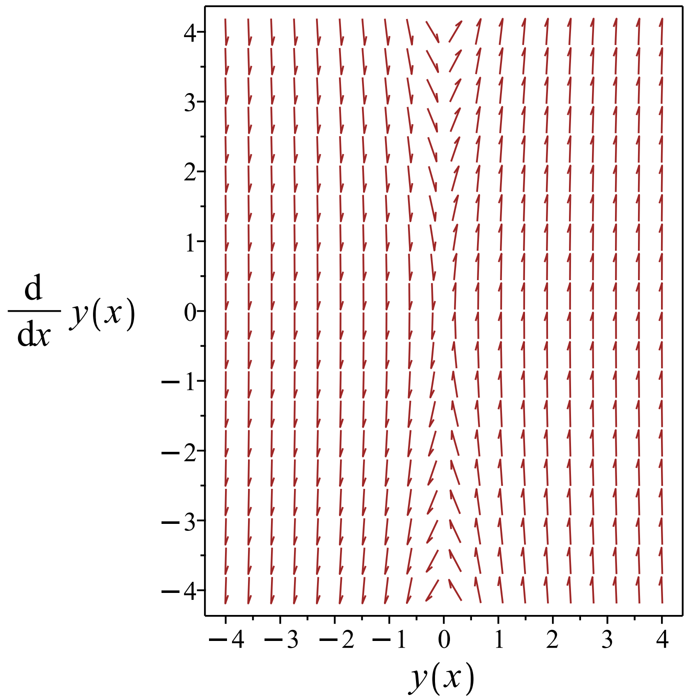


Figure 1: Slope field plot

Verification of solutions

$$y = c_1 e^{5x} + c_2 e^{-5x}$$

Verified OK.

1.1.2 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$y'y'' - 25yy' = 0$$

Integrating the above w.r.t x gives

$$\int (y'y'' - 25yy')\, dx = 0$$
$$\frac{y'^2}{2} - \frac{25y^2}{2} = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{25y^2 + 2c_1} \tag{1}$$
$$y' = -\sqrt{25y^2 + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$
\int \frac{1}{\sqrt{25y^2+2c_1}}dy = \int dx
$$
$$
\frac{\ln\left(y\sqrt{25}+\sqrt{25y^2+2c_1}\right)\sqrt{25}}{25} = x + c_2
$$

Raising both side to exponential gives

$$
\mathrm{e}^{\frac{\ln\left(y\sqrt{25}+\sqrt{25y^2+2c_1}\right)\sqrt{25}}{25}} = \mathrm{e}^{x+c_2}
$$

Which simplifies to

$$
\left(5y+\sqrt{25y^2+2c_1}\right)^{\frac{1}{5}} = c_3\mathrm{e}^{x}
$$

Solving equation (2)

Integrating both sides gives

$$
\int -\frac{1}{\sqrt{25y^2+2c_1}}dy = \int dx
$$
$$
-\frac{\ln\left(y\sqrt{25}+\sqrt{25y^2+2c_1}\right)\sqrt{25}}{25} = x + c_4
$$

Raising both side to exponential gives

$$
\mathrm{e}^{-\frac{\ln\left(y\sqrt{25}+\sqrt{25y^2+2c_1}\right)\sqrt{25}}{25}} = \mathrm{e}^{x+c_4}
$$

Which simplifies to

$$
\frac{1}{\left(5y+\sqrt{25y^2+2c_1}\right)^{\frac{1}{5}}} = c_5\mathrm{e}^{x}
$$

Summary

The solution(s) found are the following

$$
y = \frac{\left(\mathrm{e}^{10x}c_3^{10}-2c_1\right)\mathrm{e}^{-5x}}{10c_3^5} \tag{1}
$$
$$
y = -\frac{\left(2c_1c_5^{10}\mathrm{e}^{10x}-1\right)\mathrm{e}^{-5x}}{10c_5^5} \tag{2}
$$

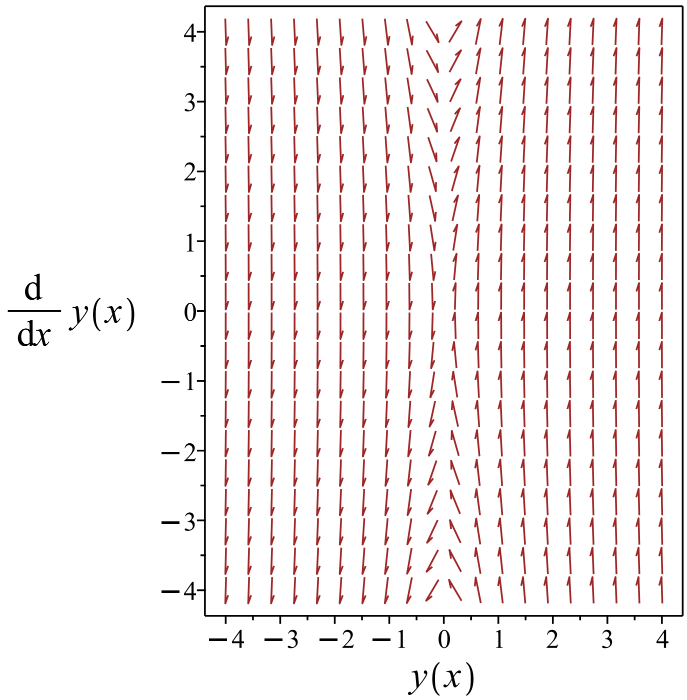


Figure 2: Slope field plot

Verification of solutions

$$y = \frac{\left(e^{10x} c_3^{10} - 2c_1\right) e^{-5x}}{10 c_3^5}$$

Verified OK.

$$y = -\frac{\left(2 c_1 c_5^{10} e^{10x} - 1\right) e^{-5x}}{10 c_5^5}$$

Verified OK.

1.1.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 25y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= 0 \\
C &= -25
\end{aligned}
\tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{25}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 25 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = 25z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 25$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-5x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \mathrm{e}^{-5x} \end{aligned}$$

Which simplifies to

$$y_1 = e^{-5x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= e^{-5x} \int \frac{1}{e^{-10x}}\, dx \\ &= e^{-5x} \left(\frac{e^{10x}}{10}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(e^{-5x}\right) + c_2 \left(e^{-5x} \left(\frac{e^{10x}}{10}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-5x} + \frac{c_2 e^{5x}}{10} \tag{1}$$

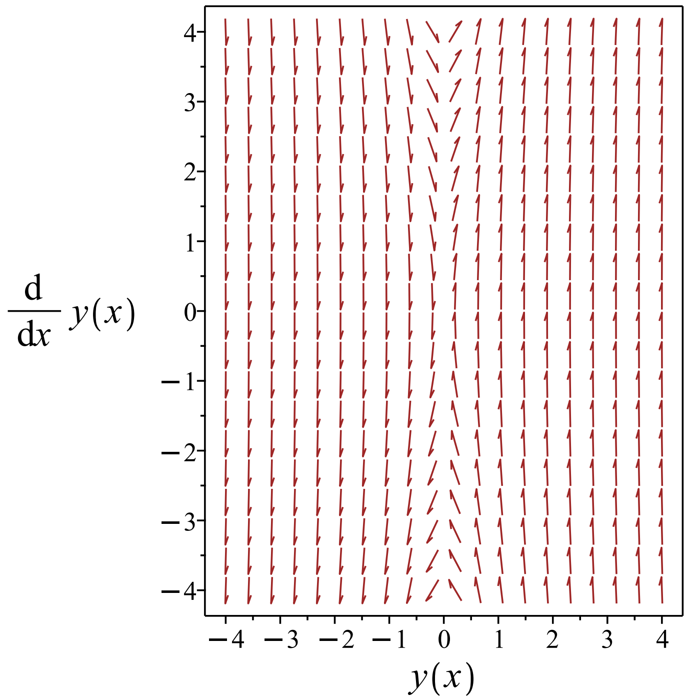


Figure 3: Slope field plot

<u>Verification of solutions</u>

$$y = c_1 e^{-5x} + \frac{c_2 e^{5x}}{10}$$

Verified OK.

1.1.4 Maple step by step solution

Let's solve

$y'' - 25y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 25 = 0$

- Factor the characteristic polynomial

  $(r - 5)(r + 5) = 0$

- Roots of the characteristic polynomial

$$r = (-5, 5)$$

- 1st solution of the ODE

$$y_1(x) = e^{-5x}$$

- 2nd solution of the ODE

$$y_2(x) = e^{5x}$$

- General solution of the ODE

$$y = c_1 y_1(x) + c_2 y_2(x)$$

- Substitute in solutions

$$y = c_1 e^{-5x} + c_2 e^{5x}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(y(x),x$2)-25*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{5x} + c_2 e^{-5x}$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 22

```
DSolve[y''[x]-25*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{5x} + c_2 e^{-5x}$$

1.2 problem Problem 8

Internal problem ID [2588]
Internal file name [`OUTPUT/2080_Sunday_June_05_2022_02_47_50_AM_77417296/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.2, Basic Ideas and Terminology. page 21
Problem number: Problem 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 4y = 0}$$

1.2.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 4\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(4)} \\ &= \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +2i \\ \lambda_2 &= -2i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2i \\ \lambda_2 &= -2i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(2x) + c_2 \sin(2x))$$

Or

$$y = c_1 \cos(2x) + c_2 \sin(2x)$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(2x) + c_2 \sin(2x) \tag{1}$$

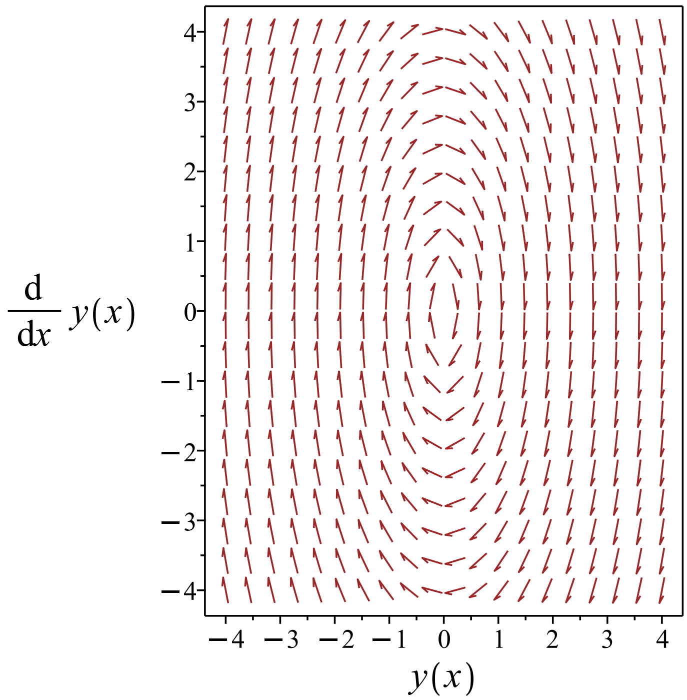


Figure 4: Slope field plot

Verification of solutions

$$y = c_1 \cos(2x) + c_2 \sin(2x)$$

Verified OK.

1.2.2 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$y'y'' + 4yy' = 0$$

Integrating the above w.r.t x gives

$$\int (y'y'' + 4yy')\, dx = 0$$
$$\frac{y'^2}{2} + 2y^2 = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{-4y^2 + 2c_1} \tag{1}$$
$$y' = -\sqrt{-4y^2 + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$
\int \frac{1}{\sqrt{-4y^2+2c_1}}dy = \int dx
$$

$$
\frac{\arctan\left(\frac{2y}{\sqrt{-4y^2+2c_1}}\right)}{2} = x + c_2
$$

Solving equation (2)

Integrating both sides gives

$$
\int -\frac{1}{\sqrt{-4y^2+2c_1}}dy = \int dx
$$

$$
-\frac{\arctan\left(\frac{2y}{\sqrt{-4y^2+2c_1}}\right)}{2} = x + c_3
$$

Summary

The solution(s) found are the following

$$
\frac{\arctan\left(\frac{2y}{\sqrt{-4y^2+2c_1}}\right)}{2} = x + c_2 \tag{1}
$$

$$
-\frac{\arctan\left(\frac{2y}{\sqrt{-4y^2+2c_1}}\right)}{2} = x + c_3 \tag{2}
$$

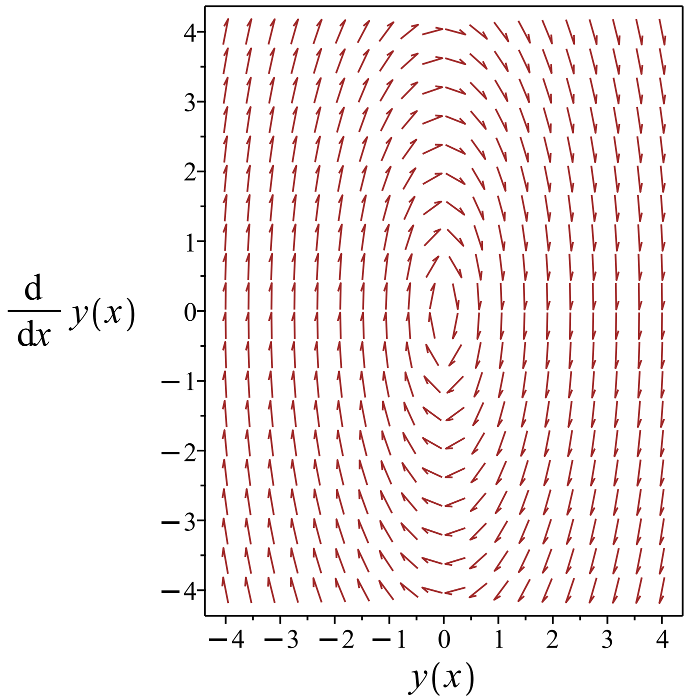


Figure 5: Slope field plot

Verification of solutions

$$\frac{\arctan\left(\frac{2y}{\sqrt{-4y^2+2c_1}}\right)}{2} = x + c_2$$

Verified OK.

$$-\frac{\arctan\left(\frac{2y}{\sqrt{-4y^2+2c_1}}\right)}{2} = x + c_3$$

Verified OK.

1.2.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= 0 \\
C &= 4
\end{aligned}
\tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = ye^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$
z''(x) = rz(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-4}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -4 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = -4z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 3: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(2x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(2x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(2x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= \cos(2x) \int \frac{1}{\cos(2x)^2}\, dx \\
&= \cos(2x) \left(\frac{\tan(2x)}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\cos(2x)) + c_2\left(\cos(2x)\left(\frac{\tan(2x)}{2}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2} \tag{1}$$

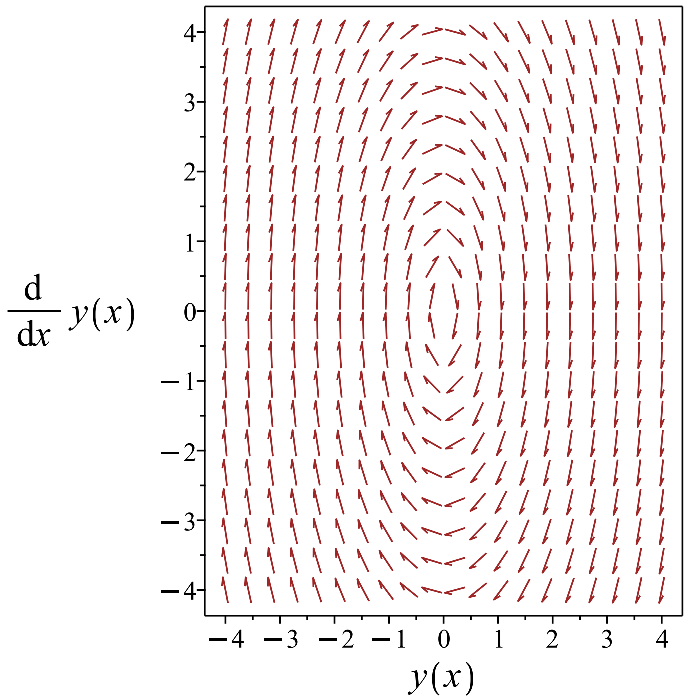


Figure 6: Slope field plot

Verification of solutions

$$y = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2}$$

Verified OK.

1.2.4 Maple step by step solution

Let's solve

$y'' + 4y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + 4 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

$$r = (-2\,\mathrm{I}, 2\,\mathrm{I})$$

- 1st solution of the ODE

$$y_1(x) = \cos(2x)$$

- 2nd solution of the ODE

$$y_2(x) = \sin(2x)$$

- General solution of the ODE

$$y = c_1 y_1(x) + c_2 y_2(x)$$

- Substitute in solutions

$$y = c_1 \cos(2x) + c_2 \sin(2x)$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(y(x),x$2)+4*y(x)=0,y(x), singsol=all)
```

$$y(x) = \sin(2x)\, c_1 + c_2 \cos(2x)$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 20

```
DSolve[y''[x]+4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \cos(2x) + c_2 \sin(2x)$$

1.3 problem Problem 9

Internal problem ID [2589]
Internal file name [`OUTPUT/2081_Sunday_June_05_2022_02_47_52_AM_20594809/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.2, Basic Ideas and Terminology. page 21
Problem number: Problem 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + y' - 2y = 0}$$

1.3.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = -2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + \lambda\,\mathrm{e}^{\lambda x} - 2\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda - 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = -2$ into the above gives

$$
\begin{aligned}
\lambda_{1,2} &= \frac{-1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{1^2 - (4)(1)(-2)} \\
&= -\frac{1}{2} \pm \frac{3}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
\lambda_1 &= -\frac{1}{2} + \frac{3}{2} \\
\lambda_2 &= -\frac{1}{2} - \frac{3}{2}
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
\lambda_1 &= 1 \\
\lambda_2 &= -2
\end{aligned}
$$

Since roots are real and distinct, then the solution is

$$
\begin{aligned}
y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\
y &= c_1 e^{(1)x} + c_2 e^{(-2)x}
\end{aligned}
$$

Or

$$
y = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-2x}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-2x} \tag{1}
$$

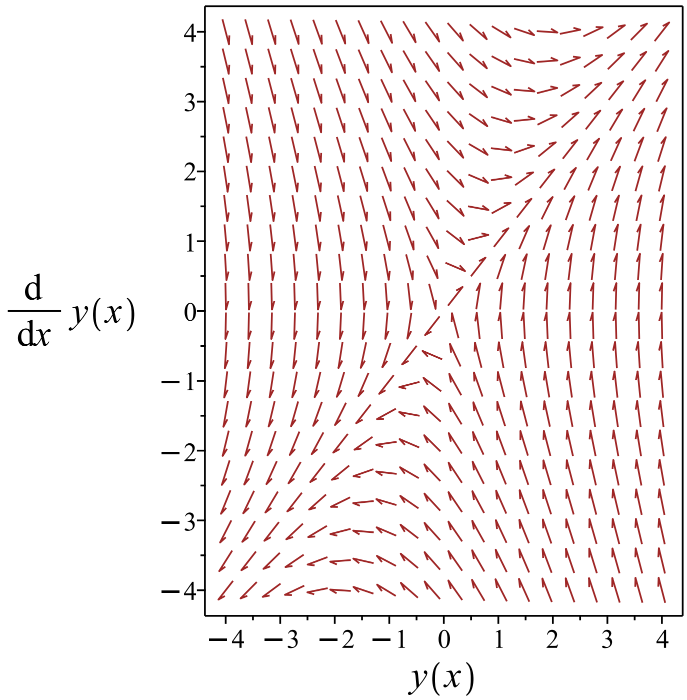


Figure 7: Slope field plot

Verification of solutions

$$y = c_1 e^{x} + c_2 e^{-2x}$$

Verified OK.

1.3.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 9 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{9z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 5: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{9}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{3x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\, dx} \\ &= z_1 e^{-\frac{x}{2}} \\ &= z_1\left(\mathrm{e}^{-\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-x}}{(y_1)^2}\, dx \\ &= y_1\left(\frac{\mathrm{e}^{3x}}{3}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-2x}\right) + c_2 \left(\mathrm{e}^{-2x} \left(\frac{\mathrm{e}^{3x}}{3}\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-2x} + \frac{c_2 \mathrm{e}^{x}}{3} \tag{1}
$$

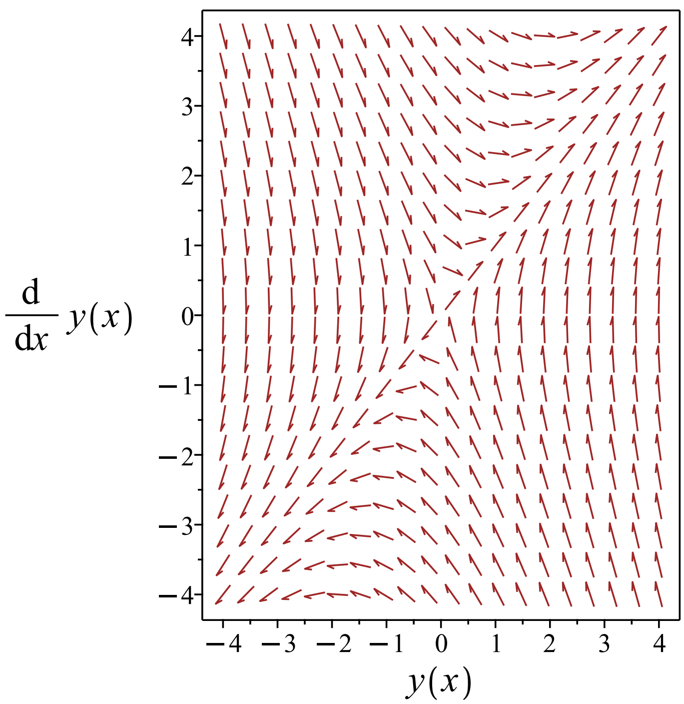


Figure 8: Slope field plot

Verification of solutions

$$
y = c_1 \mathrm{e}^{-2x} + \frac{c_2 \mathrm{e}^{x}}{3}
$$

Verified OK.

1.3.3 Maple step by step solution

Let's solve

$y'' + y' - 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + r - 2 = 0$

- Factor the characteristic polynomial

  $(r+2)(r-1) = 0$

- Roots of the characteristic polynomial

  $r = (-2, 1)$

- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^{-2x}$

- 2nd solution of the ODE

  $y_2(x) = \mathrm{e}^{x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 17

```
dsolve(diff(y(x),x$2)+diff(y(x),x)-2*y(x)=0,y(x), singsol=all)
```

$$y(x) = \left(c_2 \mathrm{e}^{3x} + c_1\right) \mathrm{e}^{-2x}$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 20

```
DSolve[y''[x]+y'[x]-2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{-2x} + c_2 e^x$$

1.4 problem Problem 10

Internal problem ID [2590]
Internal file name [`OUTPUT/2082_Sunday_June_05_2022_02_47_54_AM_75794518/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.2, Basic Ideas and Terminology. page 21
Problem number: Problem 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' + y^2 = 0}$$

1.4.1 Solving as quadrature ode

Integrating both sides gives

$$\int -\frac{1}{y^2} dy = x + c_1$$
$$\frac{1}{y} = x + c_1$$

Solving for y gives these solutions

$$y_1 = \frac{1}{x + c_1}$$

Summary
The solution(s) found are the following

$$y = \frac{1}{x + c_1} \tag{1}$$

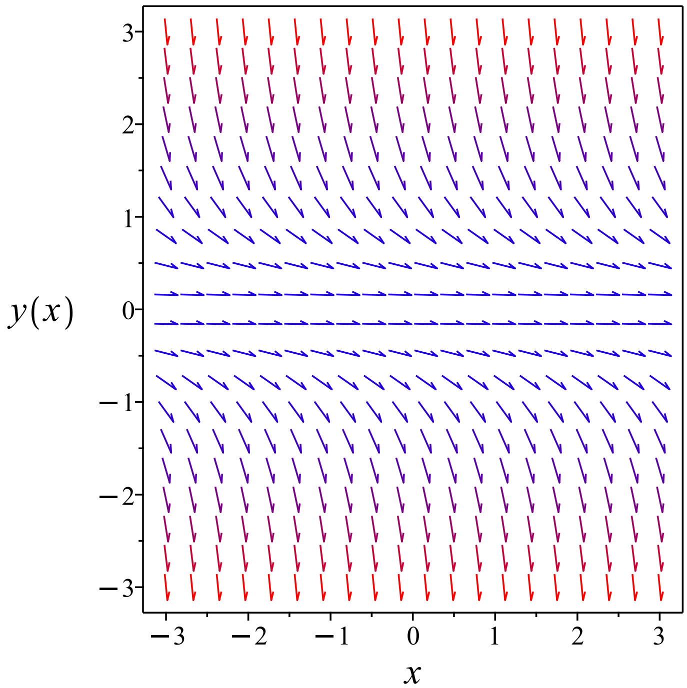


Figure 9: Slope field plot

Verification of solutions

$$
y = \frac{1}{x + c_1}
$$

Verified OK.

1.4.2 Maple step by step solution

Let's solve

$y' + y^2 = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^2} = -1$

- Integrate both sides with respect to x

  $\int \frac{y'}{y^2} dx = \int (-1)\, dx + c_1$

- Evaluate integral

$$-\frac{1}{y} = -x + c_1$$

- Solve for y

$$y = -\frac{1}{-x + c_1}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 9

```
dsolve(diff(y(x),x)=-y(x)^2,y(x), singsol=all)
```

$$y(x) = \frac{1}{c_1 + x}$$

✓ Solution by Mathematica

Time used: 0.092 (sec). Leaf size: 18

```
DSolve[y'[x]==-y[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{x - c_1}$$
$$y(x) \to 0$$

1.5 problem Problem 11

Internal problem ID [2591]
Internal file name [`OUTPUT/2083_Sunday_June_05_2022_02_47_55_AM_11922254/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.2, Basic Ideas and Terminology. page 21
Problem number: Problem 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y' - \frac{y}{2x} = 0}$$

1.5.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{y}{2x} \end{aligned}$$

Where $f(x) = \frac{1}{2x}$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{y}\,dy &= \frac{1}{2x}\,dx \\
\int \frac{1}{y}\,dy &= \int \frac{1}{2x}\,dx \\
\ln(y) &= \frac{\ln(x)}{2} + c_1 \\
y &= \mathrm{e}^{\frac{\ln(x)}{2} + c_1} \\
&= c_1\sqrt{x}
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\sqrt{x} \tag{1}$$

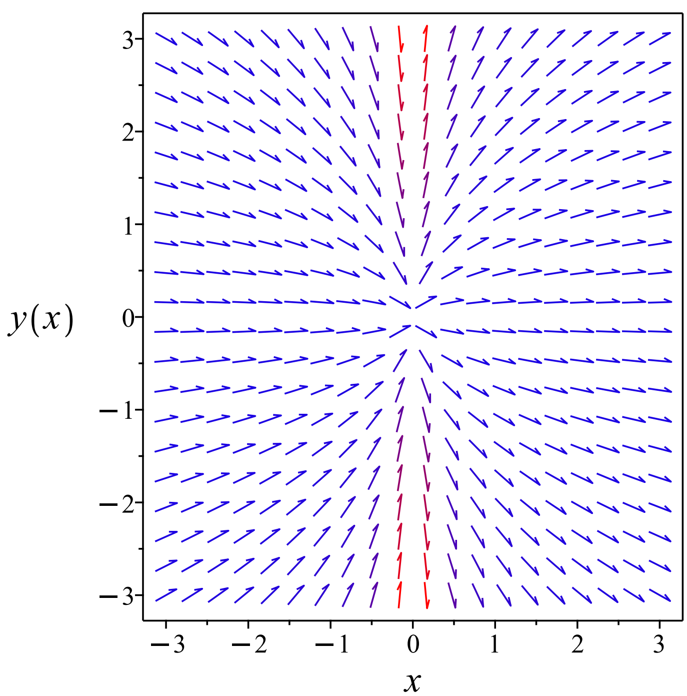


Figure 10: Slope field plot

Verification of solutions

$$y = c_1\sqrt{x}$$

Verified OK.

1.5.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{1}{2x} \\ q(x) &= 0 \end{aligned}$$

Hence the ode is

$$y' - \frac{y}{2x} = 0$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int -\frac{1}{2x}dx} \\ &= \frac{1}{\sqrt{x}} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}\mu y &= 0 \\ \frac{d}{dx}\left(\frac{y}{\sqrt{x}}\right) &= 0 \end{aligned}$$

Integrating gives

$$\frac{y}{\sqrt{x}} = c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{\sqrt{x}}$ results in

$$y = c_1\sqrt{x}$$

Summary
The solution(s) found are the following

$$y = c_1\sqrt{x} \tag{1}$$

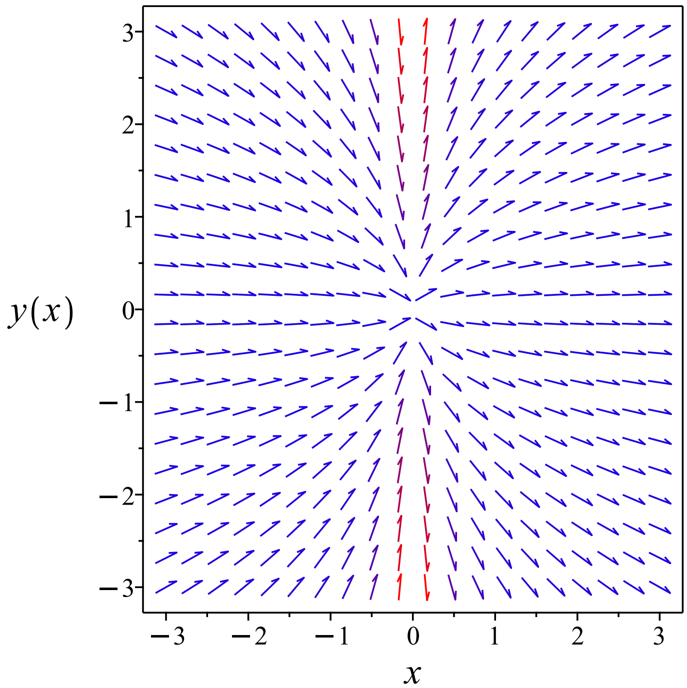


Figure 11: Slope field plot

Verification of solutions

$$y = c_1\sqrt{x}$$

Verified OK.

1.5.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x + \frac{u(x)}{2} = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u}{2x} \end{aligned}$$

Where $f(x) = -\frac{1}{2x}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= -\frac{1}{2x}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{1}{2x}\,dx \\
\ln(u) &= -\frac{\ln(x)}{2} + c_2 \\
u &= e^{-\frac{\ln(x)}{2}+c_2} \\
&= \frac{c_2}{\sqrt{x}}
\end{aligned}
$$

Therefore the solution y is

$$
\begin{aligned}
y &= xu \\
&= \sqrt{x}\,c_2
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = \sqrt{x}\,c_2 \tag{1}
$$

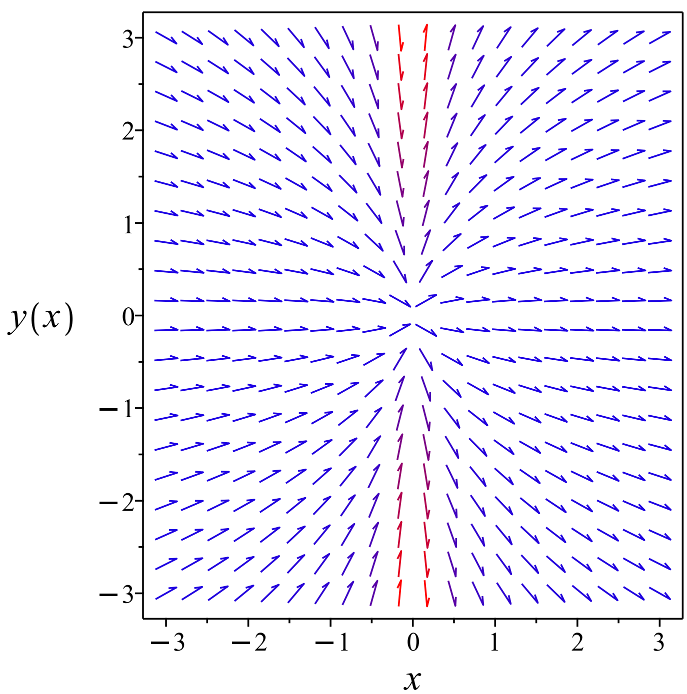


Figure 12: Slope field plot

Verification of solutions

$$y = \sqrt{x}\, c_2$$

Verified OK.

1.5.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y}{2x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 8: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \sqrt{x}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\sqrt{x}} dy \end{aligned}$$

Which results in

$$S = \frac{y}{\sqrt{x}}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y}{2x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y}{2x^{\frac{3}{2}}} \\ S_y &= \frac{1}{\sqrt{x}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{\sqrt{x}} = c_1$$

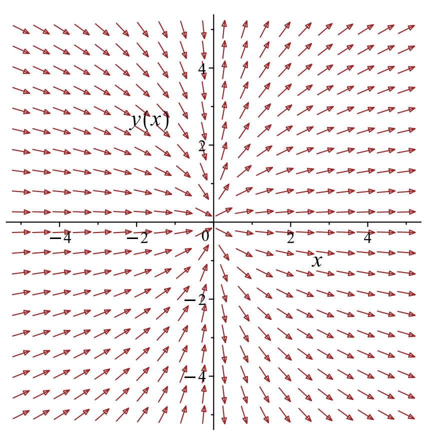
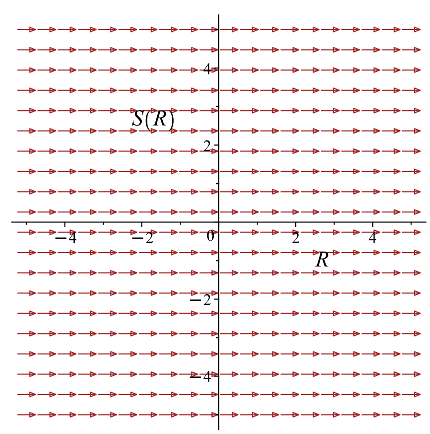
Which simplifies to

$$\frac{y}{\sqrt{x}} = c_1$$

Which gives

$$y = c_1\sqrt{x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y}{2x}$ 	$R = x$ $S = \frac{y}{\sqrt{x}}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$y = c_1\sqrt{x} \tag{1}$$

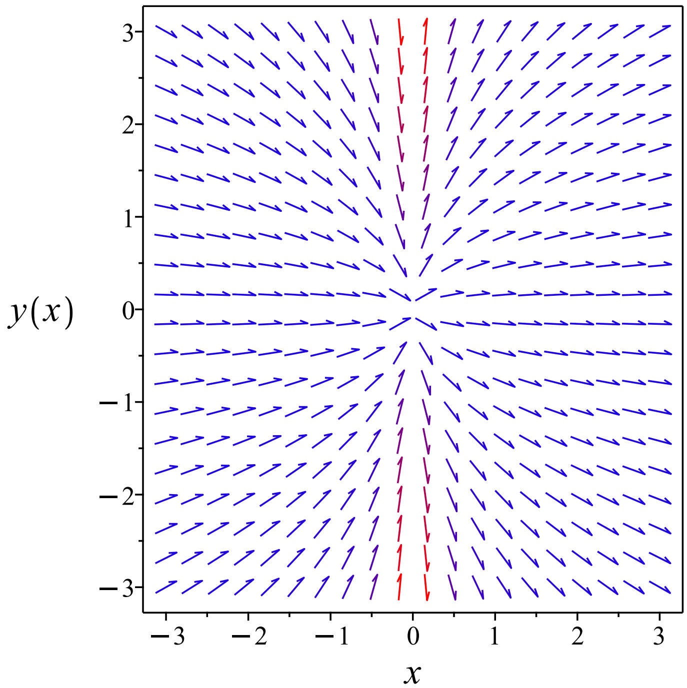


Figure 13: Slope field plot

Verification of solutions

$$y = c_1\sqrt{x}$$

Verified OK.

1.5.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(\frac{2}{y}\right) \mathrm{d}y &= \left(\frac{1}{x}\right) \mathrm{d}x \\
\left(-\frac{1}{x}\right) \mathrm{d}x + \left(\frac{2}{y}\right) \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -\frac{1}{x} \\
N(x, y) &= \frac{2}{y}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x}\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{2}{y}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int M\, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int -\frac{1}{x}\, \mathrm{d}x \\
\phi &= -\ln(x) + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{2}{y}$. Therefore equation (4) becomes

$$
\frac{2}{y} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{2}{y}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y)\, \mathrm{d}y &= \int \left(\frac{2}{y}\right) \mathrm{d}y \\
f(y) &= 2\ln(y) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) + 2\ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) + 2\ln(y)$$

The solution becomes

$$y = e^{\frac{\ln(x)}{2}+\frac{c_1}{2}}$$

<u>Summary</u>
The solution(s) found are the following

$$y = e^{\frac{\ln(x)}{2}+\frac{c_1}{2}} \tag{1}$$

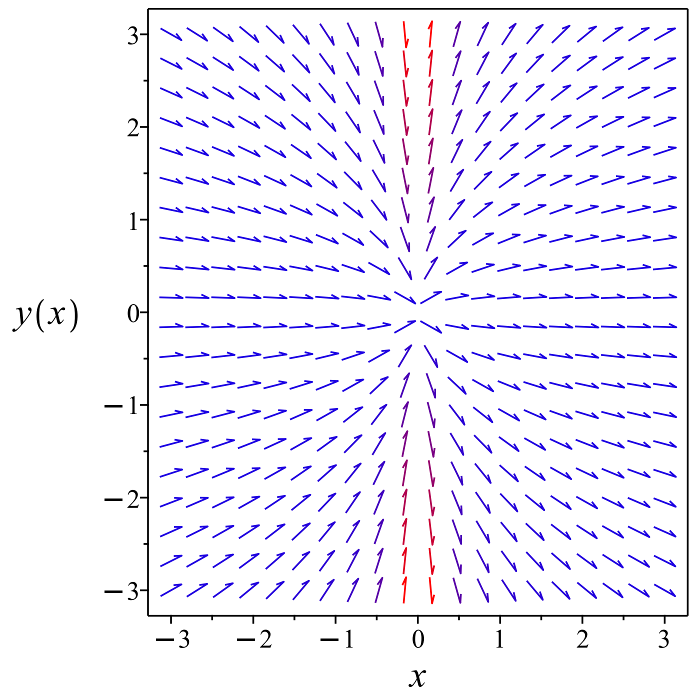


Figure 14: Slope field plot

<u>Verification of solutions</u>

$$y = e^{\frac{\ln(x)}{2}+\frac{c_1}{2}}$$

Verified OK.

1.5.6 Maple step by step solution

Let's solve

$y' - \frac{y}{2x} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = \frac{1}{2x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y}dx = \int \frac{1}{2x}dx + c_1$

- Evaluate integral

  $\ln(y) = \frac{\ln(x)}{2} + c_1$

- Solve for y

  $\left\{y = \frac{\sqrt{e^{-2c_1}x}}{e^{-2c_1}}, y = -\frac{\sqrt{e^{-2c_1}x}}{e^{-2c_1}}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 9

```
dsolve(diff(y(x),x)=y(x)/(2*x),y(x), singsol=all)
```

$$y(x) = c_1\sqrt{x}$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 18

```
DSolve[y'[x]==y[x]/(2*x),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1\sqrt{x}$$
$$y(x) \to 0$$

1.6 problem Problem 12

Internal problem ID [2592]
Internal file name [`OUTPUT/2084_Sunday_June_05_2022_02_47_57_AM_18256709/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.2, Basic Ideas and Terminology. page 21
Problem number: Problem 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 2y' + 5y = 0}$$

1.6.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 2, C = 5$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 2\lambda e^{\lambda x} + 5 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2\lambda + 5 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = 5$ into the above gives

$$
\begin{aligned}
\lambda_{1,2} &= \frac{-2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{2^2 - (4)(1)(5)} \\
&= -1 \pm 2i
\end{aligned}
$$

Hence

$$
\begin{aligned}
\lambda_1 &= -1 + 2i \\
\lambda_2 &= -1 - 2i
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
\lambda_1 &= -1 + 2i \\
\lambda_2 &= -1 - 2i
\end{aligned}
$$

Since roots are complex conjugate of each others, then let the roots be

$$
\lambda_{1,2} = \alpha \pm i\beta
$$

Where $\alpha = -1$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$
y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))
$$

Which becomes

$$
y = e^{-x}(c_1 \cos(2x) + c_2 \sin(2x))
$$

Summary

The solution(s) found are the following

$$
y = \mathrm{e}^{-x}(c_1 \cos(2x) + c_2 \sin(2x)) \tag{1}
$$

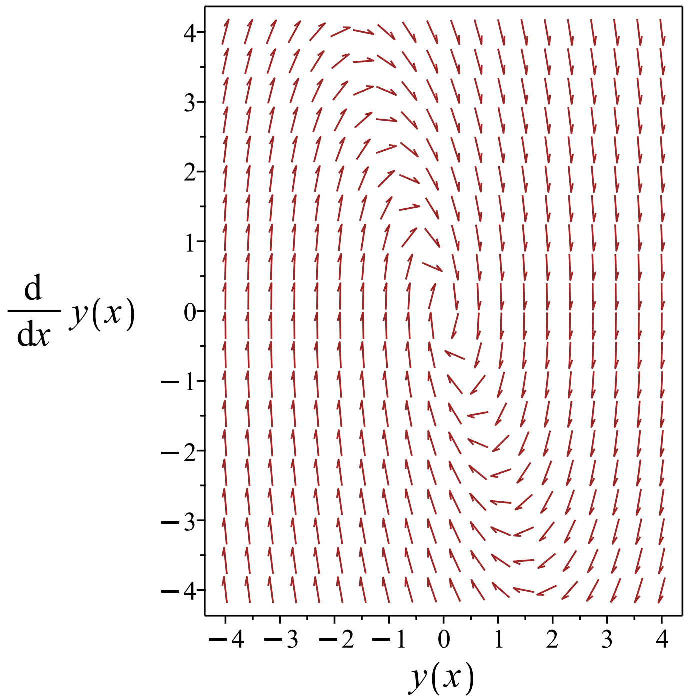


Figure 15: Slope field plot

Verification of solutions

$$y = e^{-x}(c_1 \cos(2x) + c_2 \sin(2x))$$

Verified OK.

1.6.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y' + 5y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= 5 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 11: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(2x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dx} \\ &= z_1 e^{-x} \\ &= z_1 \left(\mathrm{e}^{-x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}\cos(2x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-2x}}{(y_1)^2}\,dx \\ &= y_1 \left(\frac{\tan(2x)}{2}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-x} \cos(2x)\right) + c_2 \left(\mathrm{e}^{-x} \cos(2x) \left(\frac{\tan(2x)}{2}\right)\right)
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-x} \cos(2x) + \frac{c_2 \mathrm{e}^{-x} \sin(2x)}{2} \tag{1}
$$

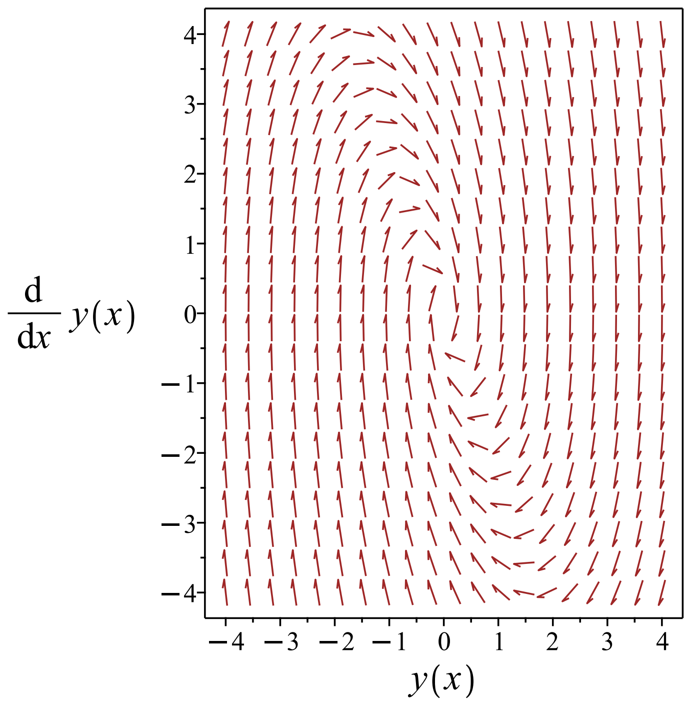


Figure 16: Slope field plot

<u>Verification of solutions</u>

$$
y = c_1 \mathrm{e}^{-x} \cos(2x) + \frac{c_2 \mathrm{e}^{-x} \sin(2x)}{2}
$$

Verified OK.

1.6.3 Maple step by step solution

Let's solve

$y'' + 2y' + 5y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + 2r + 5 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-2)\pm(\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (-1 - 2\,\mathrm{I}, -1 + 2\,\mathrm{I})$

- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^{-x}\cos(2x)$

- 2nd solution of the ODE

  $y_2(x) = \mathrm{e}^{-x}\sin(2x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^{-x}\cos(2x) + c_2 \mathrm{e}^{-x}\sin(2x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
dsolve(diff(y(x),x$2)+2*diff(y(x),x)+5*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{-x}(\sin(2x)\,c_1 + c_2\cos(2x))$$

✓ Solution by Mathematica

Time used: 0.018 (sec). Leaf size: 26

```
DSolve[y''[x]+2*y'[x]+5*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x}(c_2\cos(2x) + c_1\sin(2x))$$

1.7 problem Problem 13

Internal problem ID [2593]
Internal file name [`OUTPUT/2085_Sunday_June_05_2022_02_47_59_AM_37621237/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.2, Basic Ideas and Terminology. page 21
Problem number: Problem 13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

`[[_2nd_order, _missing_x]]`

$$\boxed{y'' - 9y = 0}$$

1.7.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -9$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 9\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -9$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-9)} \\ &= \pm 3\end{aligned}$$

Hence

$$\lambda_1 = +3$$
$$\lambda_2 = -3$$

Which simplifies to

$$\lambda_1 = 3$$
$$\lambda_2 = -3$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(3)x} + c_2 e^{(-3)x}$$

Or

$$y = c_1 \mathrm{e}^{3x} + c_2 \mathrm{e}^{-3x}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{3x} + c_2 \mathrm{e}^{-3x} \tag{1}$$

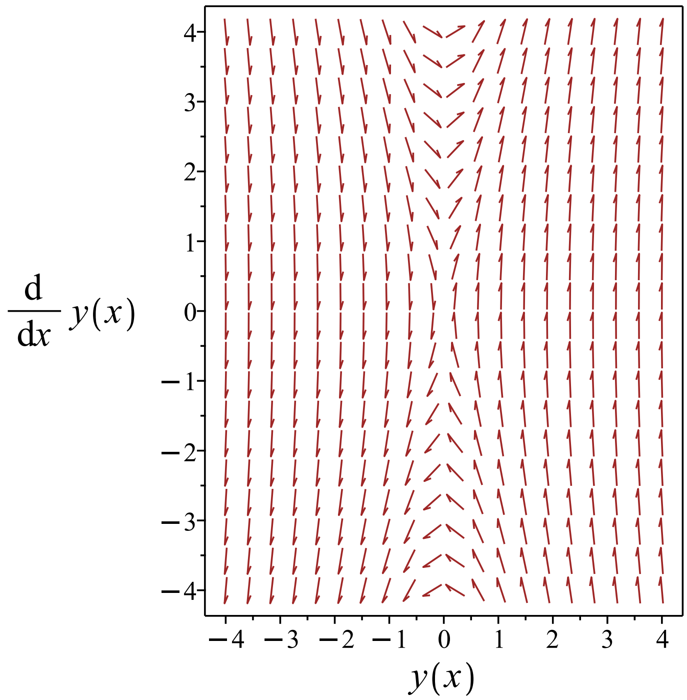


Figure 17: Slope field plot

Verification of solutions

$$y = c_1 e^{3x} + c_2 e^{-3x}$$

Verified OK.

1.7.2 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$y'y'' - 9yy' = 0$$

Integrating the above w.r.t x gives

$$\int (y'y'' - 9yy')\, dx = 0$$
$$\frac{y'^2}{2} - \frac{9y^2}{2} = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{9y^2 + 2c_1} \tag{1}$$
$$y' = -\sqrt{9y^2 + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$
\begin{aligned}
\int \frac{1}{\sqrt{9y^2+2c_1}}dy &= \int dx \\
\frac{\ln\left(y\sqrt{9}+\sqrt{9y^2+2c_1}\right)\sqrt{9}}{9} &= x+c_2
\end{aligned}
$$

Raising both side to exponential gives

$$
\mathrm{e}^{\frac{\ln\left(y\sqrt{9}+\sqrt{9y^2+2c_1}\right)\sqrt{9}}{9}} = \mathrm{e}^{x+c_2}
$$

Which simplifies to

$$
\left(3y+\sqrt{9y^2+2c_1}\right)^{\frac{1}{3}} = c_3\mathrm{e}^{x}
$$

Solving equation (2)

Integrating both sides gives

$$
\begin{aligned}
\int -\frac{1}{\sqrt{9y^2+2c_1}}dy &= \int dx \\
-\frac{\ln\left(y\sqrt{9}+\sqrt{9y^2+2c_1}\right)\sqrt{9}}{9} &= x+c_4
\end{aligned}
$$

Raising both side to exponential gives

$$
\mathrm{e}^{-\frac{\ln\left(y\sqrt{9}+\sqrt{9y^2+2c_1}\right)\sqrt{9}}{9}} = \mathrm{e}^{x+c_4}
$$

Which simplifies to

$$
\frac{1}{\left(3y+\sqrt{9y^2+2c_1}\right)^{\frac{1}{3}}} = c_5\mathrm{e}^{x}
$$

Summary

The solution(s) found are the following

$$
y = \frac{\left(\mathrm{e}^{6x}c_3^6-2c_1\right)\mathrm{e}^{-3x}}{6c_3^3} \tag{1}
$$

$$
y = -\frac{\left(2c_1c_5^6\mathrm{e}^{6x}-1\right)\mathrm{e}^{-3x}}{6c_5^3} \tag{2}
$$

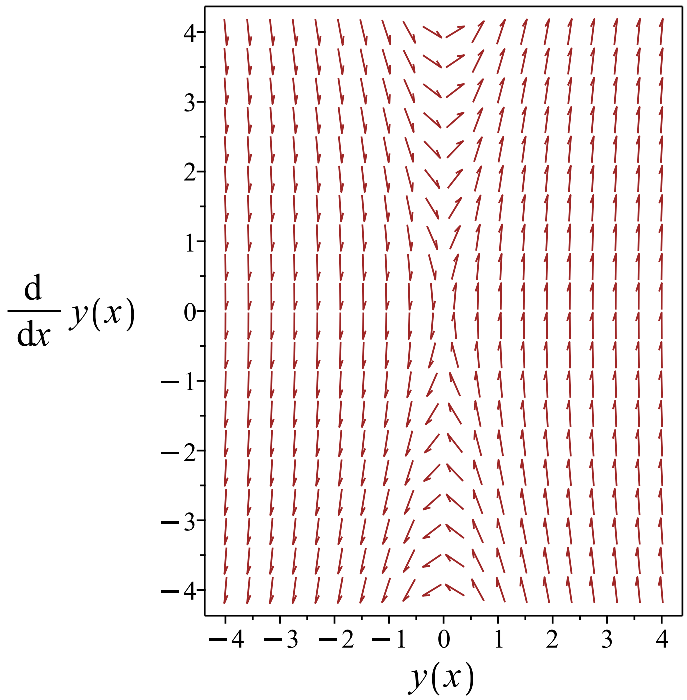


Figure 18: Slope field plot

Verification of solutions

$$y = \frac{(e^{6x}c_3^6 - 2c_1)\, e^{-3x}}{6c_3^3}$$

Verified OK.

$$y = -\frac{(2c_1c_5^6e^{6x} - 1)\, e^{-3x}}{6c_5^3}$$

Verified OK.

1.7.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 9y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= 0 \\
C &= -9
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A} \, dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{9}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 9 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = 9z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x) \, e^{-\int \frac{B}{2A} \, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 13: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 9$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-3x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \mathrm{e}^{-3x} \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-3x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \mathrm{e}^{-3x} \int \frac{1}{\mathrm{e}^{-6x}}\, dx \\ &= \mathrm{e}^{-3x} \left(\frac{\mathrm{e}^{6x}}{6}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-3x}\right) + c_2 \left(\mathrm{e}^{-3x} \left(\frac{\mathrm{e}^{6x}}{6}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-3x} + \frac{c_2 \mathrm{e}^{3x}}{6} \tag{1}$$

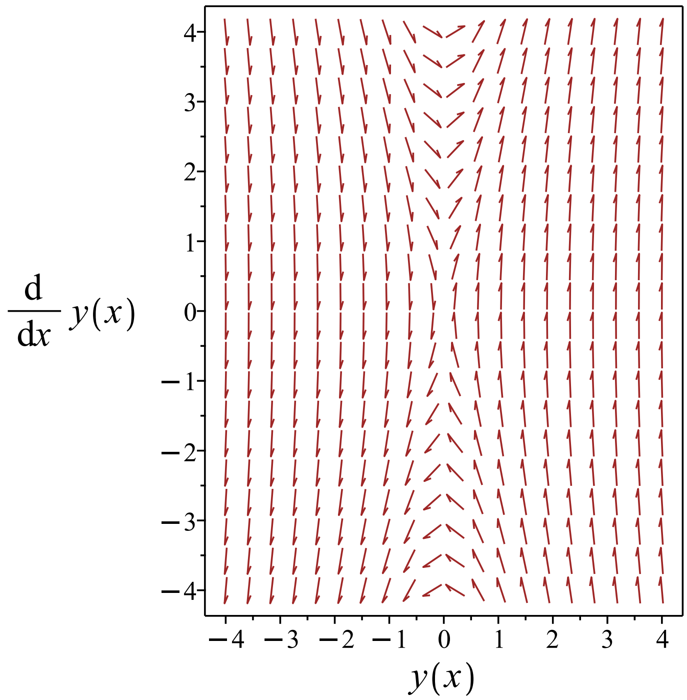


Figure 19: Slope field plot

<u>Verification of solutions</u>

$$
y = c_1 e^{-3x} + \frac{c_2 e^{3x}}{6}
$$

Verified OK.

1.7.4 Maple step by step solution

Let's solve

$y'' - 9y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 9 = 0$

- Factor the characteristic polynomial

  $(r-3)(r+3) = 0$

- Roots of the characteristic polynomial

$$r = (-3, 3)$$

- 1st solution of the ODE

$$y_1(x) = e^{-3x}$$

- 2nd solution of the ODE

$$y_2(x) = e^{3x}$$

- General solution of the ODE

$$y = c_1 y_1(x) + c_2 y_2(x)$$

- Substitute in solutions

$$y = c_1 e^{-3x} + c_2 e^{3x}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(y(x),x$2)-9*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{3x} + e^{-3x} c_2$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 22

```
DSolve[y''[x]-9*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-3x}\left(c_1 e^{6x} + c_2\right)$$

1.8 problem Problem 14

Internal problem ID [2594]
Internal file name [`OUTPUT/2086_Sunday_June_05_2022_02_48_00_AM_34534614/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.2, Basic Ideas and Terminology. page 21
Problem number: Problem 14.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "exact linear second order ode", "second_order_integrable_as_is", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{x^2y'' + 5xy' + 3y = 0}$$

1.8.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} + 5xrx^{r-1} + 3x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r + 5r\,x^r + 3x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) + 5r + 3 = 0$$

Or

$$r^2 + 4r + 3 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= -3 \\ r_2 &= -1 \end{aligned}$$

Since the roots are real and distinct, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = \frac{c_1}{x^3} + \frac{c_2}{x}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1}{x^3} + \frac{c_2}{x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x^3} + \frac{c_2}{x}$$

Verified OK.

1.8.2 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$x^2y'' + 5xy' + 3y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned}
p(x) &= \frac{5}{x} \\
q(x) &= \frac{3}{x^2}
\end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)} dx \\
&= \int \mathrm{e}^{-\left(\int \frac{5}{x}dx\right)} dx \\
&= \int e^{-5\ln(x)}\, dx \\
&= \int \frac{1}{x^5} dx \\
&= -\frac{1}{4x^4}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{3}{x^2}}{\frac{1}{x^{10}}} \\
&= 3x^8
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + 3x^8 y(\tau) &= 0
\end{aligned}$$

But in terms of τ

$$3x^8 = \frac{3}{16\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{3y(\tau)}{16\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$16\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + 3y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$16\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 3\tau^r = 0$$

Simplifying gives

$$16r(r-1)\,\tau^r + 0\,\tau^r + 3\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$16r(r-1) + 0 + 3 = 0$$

Or

$$16r^2 - 16r + 3 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$
\begin{aligned}
r_1 &= \frac{1}{4} \\
r_2 &= \frac{3}{4}
\end{aligned}
$$

Since the roots are real and distinct, then the general solution is

$$
y(\tau) = c_1 y_1 + c_2 y_2
$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$
y(\tau) = c_1 \tau^{\frac{1}{4}} + c_2 \tau^{\frac{3}{4}}
$$

The above solution is now transformed back to y using (6) which results in

$$
y = \frac{\sqrt{2}\left(-\frac{1}{x^4}\right)^{\frac{1}{4}}\left(c_2\sqrt{-\frac{1}{x^4}} + 2c_1\right)}{4}
$$

Summary
The solution(s) found are the following

$$
y = \frac{\sqrt{2}\left(-\frac{1}{x^4}\right)^{\frac{1}{4}}\left(c_2\sqrt{-\frac{1}{x^4}} + 2c_1\right)}{4} \tag{1}
$$

Verification of solutions

$$
y = \frac{\sqrt{2}\left(-\frac{1}{x^4}\right)^{\frac{1}{4}}\left(c_2\sqrt{-\frac{1}{x^4}} + 2c_1\right)}{4}
$$

Verified OK.

1.8.3 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$
x^2 y'' + 5xy' + 3y = 0 \tag{1}
$$

Becomes

$$
y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}
$$

Where

$$
\begin{aligned}
p(x) &= \frac{5}{x} \\
q(x) &= \frac{3}{x^2}
\end{aligned}
$$

Applying change of variables $\tau = g(x)$ to (2) results

$$
\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}
$$

Where τ is the new independent variable, and

$$
\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} & (4) \\
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} & (5)
\end{aligned}
$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$
\begin{aligned}
\tau' &= \frac{1}{c}\sqrt{q} \\
&= \frac{\sqrt{3}\,\sqrt{\frac{1}{x^2}}}{c} \\
\tau'' &= -\frac{\sqrt{3}}{c\sqrt{\frac{1}{x^2}}\,x^3}
\end{aligned} \tag{6}
$$

Substituting the above into (4) results in

$$
\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{-\frac{\sqrt{3}}{c\sqrt{\frac{1}{x^2}}\,x^3} + \frac{5}{x}\frac{\sqrt{3}\,\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{3}\,\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\
&= \frac{4c\sqrt{3}}{3}
\end{aligned}
$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + \frac{4c\sqrt{3}\left(\frac{d}{d\tau}y(\tau)\right)}{3} + c^2 y(\tau) &= 0
\end{aligned} \tag{7}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = \mathrm{e}^{-\frac{2\sqrt{3}\,c\tau}{3}} \left(c_1 \cosh\left(\frac{\sqrt{3}\,c\tau}{3}\right) + i c_2 \sinh\left(\frac{\sqrt{3}\,c\tau}{3}\right)\right)$$

Now from (6)

$$\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int \sqrt{3}\,\sqrt{\frac{1}{x^2}}dx}{c} \\
&= \frac{\sqrt{3}\,\sqrt{\frac{1}{x^2}}\,x \ln(x)}{c}
\end{aligned}$$

Substituting the above into the solution obtained gives

$$y = \frac{(i c_2 + c_1)\,x^2 - i c_2 + c_1}{2x^3}$$

Summary

The solution(s) found are the following

$$y = \frac{(i c_2 + c_1)\,x^2 - i c_2 + c_1}{2x^3} \tag{1}$$

Verification of solutions

$$y = \frac{(i c_2 + c_1)\,x^2 - i c_2 + c_1}{2x^3}$$

Verified OK.

1.8.4 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$x^2 y'' + 5 x y' + 3 y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$
\begin{aligned}
p(x) &= \frac{5}{x} \\
q(x) &= \frac{3}{x^2}
\end{aligned}
$$

Applying change of variables on the depndent variable $y = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$
v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}
$$

Let the coefficient of $v(x)$ above be zero. Hence

$$
\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}
$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$
\frac{n(n-1)}{x^2} + \frac{5n}{x^2} + \frac{3}{x^2} = 0 \tag{5}
$$

Solving (5) for n gives

$$
n = -1 \tag{6}
$$

Substituting this value in (3) gives

$$
\begin{aligned}
v''(x) + \frac{3v'(x)}{x} &= 0 \\
v''(x) + \frac{3v'(x)}{x} &= 0 \qquad (7)
\end{aligned}
$$

Using the substitution

$$
u(x) = v'(x)
$$

Then (7) becomes

$$
u'(x) + \frac{3u(x)}{x} = 0 \tag{8}
$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{3u}{x} \end{aligned}$$

Where $f(x) = -\frac{3}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= -\frac{3}{x}\,dx \\ \int \frac{1}{u}\,du &= \int -\frac{3}{x}\,dx \\ \ln(u) &= -3\ln(x) + c_1 \\ u &= \mathrm{e}^{-3\ln(x)+c_1} \\ &= \frac{c_1}{x^3} \end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\,dx + c_2 \\ &= -\frac{c_1}{2x^2} + c_2 \end{aligned}$$

Hence

$$\begin{aligned} y &= v(x)\,x^n \\ &= \frac{-\frac{c_1}{2x^2} + c_2}{x} \\ &= \frac{2c_2x^2 - c_1}{2x^3} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{-\frac{c_1}{2x^2} + c_2}{x} \tag{1}$$

Verification of solutions

$$y = \frac{-\frac{c_1}{2x^2} + c_2}{x}$$

Verified OK.

1.8.5 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int \left(x^2 y'' + 5xy' + 3y\right) dx = 0$$
$$y' x^2 + 3yx = c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{3}{x}$$
$$q(x) = \frac{c_1}{x^2}$$

Hence the ode is

$$y' + \frac{3y}{x} = \frac{c_1}{x^2}$$

The integrating factor μ is

$$\mu = e^{\int \frac{3}{x} dx}$$
$$= x^3$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{c_1}{x^2}\right)$$
$$\frac{d}{dx}\left(x^3 y\right) = \left(x^3\right)\left(\frac{c_1}{x^2}\right)$$
$$d\left(x^3 y\right) = (c_1 x)\ dx$$

Integrating gives

$$x^3 y = \int c_1 x \, dx$$
$$x^3 y = \frac{c_1 x^2}{2} + c_2$$

Dividing both sides by the integrating factor $\mu = x^3$ results in

$$y = \frac{c_1}{2x} + \frac{c_2}{x^3}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1}{2x} + \frac{c_2}{x^3} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{2x} + \frac{c_2}{x^3}$$

Verified OK.

1.8.6 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$x^2y'' + 5xy' + 3y = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\int \left(x^2y'' + 5xy' + 3y\right) dx = 0$$
$$y'x^2 + 3yx = c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{3}{x}$$
$$q(x) = \frac{c_1}{x^2}$$

Hence the ode is

$$y' + \frac{3y}{x} = \frac{c_1}{x^2}$$

The integrating factor μ is

$$\mu = e^{\int \frac{3}{x}dx}$$
$$= x^3$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{c_1}{x^2}\right)\\
\frac{\mathrm{d}}{\mathrm{d}x}\left(x^3 y\right) &= \left(x^3\right)\left(\frac{c_1}{x^2}\right)\\
\mathrm{d}\left(x^3 y\right) &= (c_1 x)\,\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
x^3 y &= \int c_1 x \,\mathrm{d}x\\
x^3 y &= \frac{c_1 x^2}{2} + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = x^3$ results in

$$
y = \frac{c_1}{2x} + \frac{c_2}{x^3}
$$

Summary
The solution(s) found are the following

$$
y = \frac{c_1}{2x} + \frac{c_2}{x^3} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1}{2x} + \frac{c_2}{x^3}
$$

Verified OK.

1.8.7 Solving using Kovacic algorithm

Writing the ode as

$$
\begin{aligned}
x^2 y'' + 5xy' + 3y &= 0 \qquad (1)\\
Ay'' + By' + Cy &= 0 \qquad (2)
\end{aligned}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= x^2\\
B &= 5x \qquad (3)\\
C &= 3
\end{aligned}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A}\,dx}
$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 15: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2x} + (-)(0) \\ &= -\frac{1}{2x} \\ &= -\frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(0) + 2\left(-\frac{1}{2x}\right)(0) + \left(\left(\frac{1}{2x^2}\right) + \left(-\frac{1}{2x}\right)^2 - \left(\frac{3}{4x^2}\right)\right) &= 0 \\ 0 &= 0\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int -\frac{1}{2x} dx} \\ &= \frac{1}{\sqrt{x}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{5x}{x^2}\, dx} \\ &= z_1 e^{-\frac{5\ln(x)}{2}} \\ &= z_1 \left(\frac{1}{x^{\frac{5}{2}}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x^3}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{5x}{x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-5\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{x^2}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{x^3}\right) + c_2 \left(\frac{1}{x^3} \left(\frac{x^2}{2}\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = \frac{c_1}{x^3} + \frac{c_2}{2x} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1}{x^3} + \frac{c_2}{2x}
$$

Verified OK.

1.8.8 Solving as exact linear second order ode ode

An ode of the form

$$
p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)
$$

is exact if

$$
p''(x) - q'(x) + r(x) = 0 \tag{1}
$$

For the given ode we have

$$
\begin{aligned}
p(x) &= x^2 \\
q(x) &= 5x \\
r(x) &= 3 \\
s(x) &= 0
\end{aligned}
$$

Hence

$$
\begin{aligned}
p''(x) &= 2 \\
q'(x) &= 5
\end{aligned}
$$

Therefore (1) becomes

$$
2 - (5) + (3) = 0
$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$
(p(x)\, y' + (q(x) - p'(x))\, y)' = s(x)
$$

Integrating gives

$$
p(x)\, y' + (q(x) - p'(x))\, y = \int s(x)\, dx
$$

Substituting the above values for p, q, r, s gives

$$
y'x^2 + 3yx = c_1
$$

We now have a first order ode to solve which is

$$
y'x^2 + 3yx = c_1
$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
y' + p(x)y = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= \frac{3}{x} \\
q(x) &= \frac{c_1}{x^2}
\end{aligned}
$$

Hence the ode is

$$
y' + \frac{3y}{x} = \frac{c_1}{x^2}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int \frac{3}{x} dx} \\
&= x^3
\end{aligned}
$$

The ode becomes

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{c_1}{x^2}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(x^3 y\right) &= \left(x^3\right)\left(\frac{c_1}{x^2}\right) \\
\mathrm{d}\left(x^3 y\right) &= (c_1 x)\ \mathrm{d}x
\end{aligned}$$

Integrating gives

$$\begin{aligned}
x^3 y &= \int c_1 x \,\mathrm{d}x \\
x^3 y &= \frac{c_1 x^2}{2} + c_2
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x^3$ results in

$$y = \frac{c_1}{2x} + \frac{c_2}{x^3}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{2x} + \frac{c_2}{x^3} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{2x} + \frac{c_2}{x^3}$$

Verified OK.

1.8.9 Maple step by step solution

Let's solve

$x^2 y'' + 5xy' + 3y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{5y'}{x} - \frac{3y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{5y'}{x} + \frac{3y}{x^2} = 0$

- Multiply by denominators of the ODE

  $x^2y'' + 5xy' + 3y = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right)t'(x)$

  - Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2}y(t)\right)t'(x)^2 + t''(x)\left(\frac{d}{dt}y(t)\right)$

  - Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) + 5\frac{d}{dt}y(t) + 3y(t) = 0$

- Simplify

  $\frac{d^2}{dt^2}y(t) + 4\frac{d}{dt}y(t) + 3y(t) = 0$

- Characteristic polynomial of ODE

  $r^2 + 4r + 3 = 0$

- Factor the characteristic polynomial

  $(r+3)(r+1) = 0$

- Roots of the characteristic polynomial

  $r = (-3, -1)$

- 1st solution of the ODE

  $y_1(t) = \mathrm{e}^{-3t}$

- 2nd solution of the ODE

  $y_2(t) = \mathrm{e}^{-t}$

- General solution of the ODE

$y(t) = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y(t) = c_1 \mathrm{e}^{-3t} + c_2 \mathrm{e}^{-t}$

- Change variables back using $t = \ln(x)$

  $y = \frac{c_1}{x^3} + \frac{c_2}{x}$

- Simplify

  $y = \frac{c_1}{x^3} + \frac{c_2}{x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 15

```
dsolve(x^2*diff(y(x),x$2)+5*x*diff(y(x),x)+3*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_2 x^2 + c_1}{x^3}$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 18

```
DSolve[x^2*y''[x]+5*x*y'[x]+3*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_2 x^2 + c_1}{x^3}$$

1.9 problem Problem 15

Internal problem ID [2595]
Internal file name [`OUTPUT/2087_Sunday_June_05_2022_02_48_02_AM_91072846/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.2, Basic Ideas and Terminology. page 21
Problem number: Problem 15.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler], [_2nd_order, _linear, `_with_symmetry_[0,F(
   x)]`]]
```

$$\boxed{x^2y'' - 3xy' + 4y = 0}$$

1.9.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} - 3xrx^{r-1} + 4x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r - 3r\,x^r + 4x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) - 3r + 4 = 0$$

Or

$$r^2 - 4r + 4 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= 2 \\ r_2 &= 2 \end{aligned}$$

Since the roots are equal, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = x^r$ and $y_2 = x^r \ln(x)$. Hence

$$y = c_1 x^2 + c_2 x^2 \ln(x)$$

Summary
The solution(s) found are the following

$$y = c_1 x^2 + c_2 x^2 \ln(x) \tag{1}$$

Verification of solutions

$$y = c_1 x^2 + c_2 x^2 \ln(x)$$

Verified OK.

1.9.2 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$x^2 y'' - 3xy' + 4y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{3}{x} \\ q(x) &= \frac{4}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\
&= \int \mathrm{e}^{-\left(\int -\frac{3}{x}dx\right)}dx \\
&= \int e^{3\ln(x)}\,dx \\
&= \int x^3 dx \\
&= \frac{x^4}{4}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{4}{x^2}}{x^6} \\
&= \frac{4}{x^8}
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0$$
$$\frac{d^2}{d\tau^2}y(\tau) + \frac{4y(\tau)}{x^8} = 0$$

But in terms of τ

$$\frac{4}{x^8} = \frac{1}{4\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{y(\tau)}{4\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$4\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$4\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + \tau^r = 0$$

Simplifying gives

$$4r(r-1)\,\tau^r + 0\,\tau^r + \tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$4r(r-1) + 0 + 1 = 0$$

Or

$$4r^2 - 4r + 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{2}$$
$$r_2 = \frac{1}{2}$$

Since the roots are equal, then the general solution is

$$y(\tau) = c_1y_1 + c_2y_2$$

Where $y_1 = \tau^r$ and $y_2 = \tau^r \ln(\tau)$. Hence

$$y(\tau) = c_1\sqrt{\tau} + c_2\sqrt{\tau}\,\ln(\tau)$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{(c_2\ln(x^4) - 2c_2\ln(2) + c_1)\sqrt{x^4}}{2}$$

Summary

The solution(s) found are the following

$$
y = \frac{\left(c_2 \ln \left(x^4\right) - 2c_2 \ln \left(2\right) + c_1\right) \sqrt{x^4}}{2} \tag{1}
$$

Verification of solutions

$$
y = \frac{\left(c_2 \ln \left(x^4\right) - 2c_2 \ln \left(2\right) + c_1\right) \sqrt{x^4}}{2}
$$

Verified OK.

1.9.3 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$
x^2 y'' - 3xy' + 4y = 0 \tag{1}
$$

Becomes

$$
y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}
$$

Where

$$
\begin{aligned}
p(x) &= -\frac{3}{x} \\
q(x) &= \frac{4}{x^2}
\end{aligned}
$$

Applying change of variables $\tau = g(x)$ to (2) results

$$
\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}
$$

Where τ is the new independent variable, and

$$
p_1(\tau) = \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'\left(x\right)^2} \tag{4}
$$

$$
q_1(\tau) = \frac{q(x)}{\tau'\left(x\right)^2} \tag{5}
$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$
\begin{aligned}
\tau' &= \frac{1}{c}\sqrt{q} \\
&= \frac{2\sqrt{\frac{1}{x^2}}}{c} \qquad (6) \\
\tau'' &= -\frac{2}{c\sqrt{\frac{1}{x^2}}\, x^3}
\end{aligned}
$$

Substituting the above into (4) results in

$$
\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{-\frac{2}{c\sqrt{\frac{1}{x^2}}\,x^3} - \frac{3}{x}\frac{2\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{2\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\
&= -2c
\end{aligned}
$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) - 2c\left(\frac{d}{d\tau}y(\tau)\right) + c^2 y(\tau) &= 0 \qquad (7)
\end{aligned}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
y(\tau) = \mathrm{e}^{c\tau} c_1
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int 2\sqrt{\frac{1}{x^2}}dx}{c} \\
&= \frac{2\sqrt{\frac{1}{x^2}}\,x\ln(x)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = c_1 x^2
$$

Summary

The solution(s) found are the following

$$
y = c_1 x^2 \tag{1}
$$

Verification of solutions

$$
y = c_1 x^2
$$

Verified OK.

1.9.4 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$x^2y'' - 3xy' + 4y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{3}{x} \\ q(x) &= \frac{4}{x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{3n}{x^2} + \frac{4}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 2 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{v'(x)}{x} &= 0 \\ v''(x) + \frac{v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{u}{x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= -\frac{1}{x}\, dx \\ \int \frac{1}{u}\, du &= \int -\frac{1}{x}\, dx \\ \ln(u) &= -\ln(x) + c_1 \\ u &= \mathrm{e}^{-\ln(x) + c_1} \\ &= \frac{c_1}{x} \end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\, dx + c_2 \\ &= c_1 \ln(x) + c_2 \end{aligned}$$

Hence

$$\begin{aligned} y &= v(x)\, x^n \\ &= (c_1 \ln(x) + c_2)\, x^2 \\ &= (c_1 \ln(x) + c_2)\, x^2 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = (c_1 \ln(x) + c_2)\, x^2 \tag{1}$$

Verification of solutions

$$y = (c_1 \ln(x) + c_2)\, x^2$$

Verified OK.

1.9.5 Solving using Kovacic algorithm

Writing the ode as

$$
\begin{align}
x^2y'' - 3xy' + 4y &= 0 \tag{1}\\
Ay'' + By' + Cy &= 0 \tag{2}
\end{align}
$$

Comparing (1) and (2) shows that

$$
\begin{align}
A &= x^2 \\
B &= -3x \tag{3}\\
C &= 4
\end{align}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = ye^{\int \frac{B}{2A}\,dx}
$$

Then (2) becomes

$$
z''(x) = rz(x) \tag{4}
$$

Where r is given by

$$
\begin{align}
r &= \frac{s}{t} \tag{5}\\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{align}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-1}{4x^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{align}
s &= -1 \\
t &= 4x^2
\end{align}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\,dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 17: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = -\frac{1}{4x^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = -\frac{1}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_{\infty}^{-} = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_{\infty}^{-} - (\alpha_{c_1}^{-}) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_{\infty}$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (-)[\sqrt{r}]_{\infty} \\ &= \frac{1}{2x} + (-)(0) \\ &= \frac{1}{2x} \\ &= \frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-3x}{x^2}\,dx} \\
&= z_1 e^{\frac{3\ln(x)}{2}} \\
&= z_1\left(x^{\frac{3}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x^2$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-3x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{3\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(\ln(x))
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^2\right) + c_2\left(x^2(\ln(x))\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x^2 + c_2 x^2 \ln(x) \tag{1}$$

Verification of solutions

$$y = c_1 x^2 + c_2 x^2 \ln(x)$$

Verified OK.

1.9.6 Maple step by step solution

Let's solve

$x^2y'' - 3xy' + 4y = 0$

- Highest derivative means the order of the ODE is 2

y''

- Isolate 2nd derivative

$y'' = \frac{3y'}{x} - \frac{4y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' - \frac{3y'}{x} + \frac{4y}{x^2} = 0$

- Multiply by denominators of the ODE

$x^2y'' - 3xy' + 4y = 0$

- Make a change of variables

$t = \ln(x)$

□ Substitute the change of variables back into the ODE

  ◦ Calculate the 1st derivative of y with respect to x , using the chain rule

  $y' = \left(\frac{d}{dt}y(t)\right)t'(x)$

  ◦ Compute derivative

  $y' = \frac{\frac{d}{dt}y(t)}{x}$

  ◦ Calculate the 2nd derivative of y with respect to x , using the chain rule

  $y'' = \left(\frac{d^2}{dt^2}y(t)\right)t'(x)^2 + t''(x)\left(\frac{d}{dt}y(t)\right)$

  ◦ Compute derivative

  $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

Substitute the change of variables back into the ODE

$x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) - 3\frac{d}{dt}y(t) + 4y(t) = 0$

- Simplify

$\frac{d^2}{dt^2}y(t) - 4\frac{d}{dt}y(t) + 4y(t) = 0$

- Characteristic polynomial of ODE

$r^2 - 4r + 4 = 0$

- Factor the characteristic polynomial

  $(r-2)^2 = 0$
- Root of the characteristic polynomial

  $r = 2$
- 1st solution of the ODE

  $y_1(t) = e^{2t}$
- Repeated root, multiply $y_1(t)$ by t to ensure linear independence

  $y_2(t) = t\,e^{2t}$
- General solution of the ODE

  $y(t) = c_1 y_1(t) + c_2 y_2(t)$
- Substitute in solutions

  $y(t) = c_1 e^{2t} + c_2 t\,e^{2t}$
- Change variables back using $t = \ln(x)$

  $y = c_1 x^2 + c_2 x^2 \ln(x)$
- Simplify

  $y = x^2(c_1 + c_2 \ln(x))$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 14

```
dsolve(x^2*diff(y(x),x$2)-3*x*diff(y(x),x)+4*y(x)=0,y(x), singsol=all)
```

$$y(x) = x^2(c_2 \ln(x) + c_1)$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 18

```
DSolve[x^2*y''[x]-3*x*y'[x]+4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^2(2c_2 \log(x) + c_1)$$

1.10 problem Problem 16

Internal problem ID [2596]
Internal file name [`OUTPUT/2088_Sunday_June_05_2022_02_48_04_AM_36533908/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.2, Basic Ideas and Terminology. page 21
Problem number: Problem 16.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{x^2y'' - 3xy' + 13y = 0}$$

1.10.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} - 3xrx^{r-1} + 13x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r - 3r\,x^r + 13x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) - 3r + 13 = 0$$

Or

$$r^2 - 4r + 13 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= 2 - 3i \\ r_2 &= 2 + 3i \end{aligned}$$

The roots are complex conjugate of each others. Let the roots be

$$\begin{aligned} r_1 &= \alpha + i\beta \\ r_2 &= \alpha - i\beta \end{aligned}$$

Where in this case $\alpha = 2$ and $\beta = -3$. Hence the solution becomes

$$\begin{aligned} y &= c_1 x^{r_1} + c_2 x^{r_2} \\ &= c_1 x^{\alpha + i\beta} + c_2 x^{\alpha - i\beta} \\ &= x^\alpha \left(c_1 x^{i\beta} + c_2 x^{-i\beta}\right) \\ &= x^\alpha \left(c_1 e^{\ln\left(x^{i\beta}\right)} + c_2 e^{\ln\left(x^{-i\beta}\right)}\right) \\ &= x^\alpha \left(c_1 e^{i(\beta \ln x)} + c_2 e^{-i(\beta \ln x)}\right) \end{aligned}$$

Using the values for $\alpha = 2, \beta = -3$, the above becomes

$$y = x^2 \left(c_1 e^{-3i \ln(x)} + c_2 e^{3i \ln(x)}\right)$$

Using Euler relation, the expression $c_1 e^{iA} + c_2 e^{-iA}$ is transformed to $c_1 \cos A + c_1 \sin A$ where the constants are free to change. Applying this to the above result gives

$$y = x^2 (c_1 \cos(3 \ln(x)) + c_2 \sin(3 \ln(x)))$$

Summary

The solution(s) found are the following

$$y = x^2 (c_1 \cos(3 \ln(x)) + c_2 \sin(3 \ln(x))) \tag{1}$$

Verification of solutions

$$y = x^2 (c_1 \cos(3 \ln(x)) + c_2 \sin(3 \ln(x)))$$

Verified OK.

1.10.2 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$x^2y'' - 3xy' + 13y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{3}{x} \\ q(x) &= \frac{13}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int e^{-\left(\int p(x)dx\right)}dx \\ &= \int e^{-\left(\int -\frac{3}{x}dx\right)}dx \\ &= \int e^{3\ln(x)}\,dx \\ &= \int x^3 dx \\ &= \frac{x^4}{4} \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned} q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\ &= \frac{\frac{13}{x^2}}{x^6} \\ &= \frac{13}{x^8} \end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned} \frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + \frac{13y(\tau)}{x^8} &= 0 \end{aligned}$$

But in terms of τ

$$\frac{13}{x^8} = \frac{13}{16\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{13y(\tau)}{16\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$16\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + 13y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$16\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 13\tau^r = 0$$

Simplifying gives

$$16r(r-1)\,\tau^r + 0\,\tau^r + 13\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$16r(r-1) + 0 + 13 = 0$$

Or

$$16r^2 - 16r + 13 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{2} - \frac{3i}{4}$$
$$r_2 = \frac{1}{2} + \frac{3i}{4}$$

The roots are complex conjugate of each others. Let the roots be

$$r_1 = \alpha + i\beta$$
$$r_2 = \alpha - i\beta$$

Where in this case $\alpha = \frac{1}{2}$ and $\beta = -\frac{3}{4}$. Hence the solution becomes

$$\begin{aligned} y(\tau) &= c_1 \tau^{r_1} + c_2 \tau^{r_2} \\ &= c_1 \tau^{\alpha+i\beta} + c_2 \tau^{\alpha-i\beta} \\ &= \tau^{\alpha}\left(c_1 \tau^{i\beta} + c_2 \tau^{-i\beta}\right) \\ &= \tau^{\alpha}\left(c_1 e^{\ln(\tau^{i\beta})} + c_2 e^{\ln(\tau^{-i\beta})}\right) \\ &= \tau^{\alpha}\left(c_1 e^{i(\beta \ln \tau)} + c_2 e^{-i(\beta \ln \tau)}\right) \end{aligned}$$

Using the values for $\alpha = \frac{1}{2}, \beta = -\frac{3}{4}$, the above becomes

$$y(\tau) = \tau^{\frac{1}{2}}\left(c_1 e^{-\frac{3i\ln(\tau)}{4}} + c_2 e^{\frac{3i\ln(\tau)}{4}}\right)$$

Using Euler relation, the expression $c_1 e^{iA} + c_2 e^{-iA}$ is transformed to $c_1 \cos A + c_1 \sin A$ where the constants are free to change. Applying this to the above result gives

$$y(\tau) = \sqrt{\tau}\left(c_1 \cos\left(\frac{3\ln(\tau)}{4}\right) + c_2 \sin\left(\frac{3\ln(\tau)}{4}\right)\right)$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{\left(c_1 \cos\left(-\frac{3\ln(2)}{2} + 3\ln(x)\right) + c_2 \sin\left(-\frac{3\ln(2)}{2} + 3\ln(x)\right)\right) x^2}{2}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{\left(c_1 \cos\left(-\frac{3\ln(2)}{2} + 3\ln(x)\right) + c_2 \sin\left(-\frac{3\ln(2)}{2} + 3\ln(x)\right)\right) x^2}{2} \tag{1}$$

Verification of solutions

$$y = \frac{\left(c_1 \cos\left(-\frac{3\ln(2)}{2}+3\ln(x)\right)+c_2 \sin\left(-\frac{3\ln(2)}{2}+3\ln(x)\right)\right)x^2}{2}$$

Verified OK.

1.10.3 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$x^2y''-3xy'+13y=0 \tag{1}$$

Becomes

$$y''+p(x)\,y'+q(x)\,y=0 \tag{2}$$

Where

$$\begin{aligned}p(x)&=-\frac{3}{x}\\ q(x)&=\frac{13}{x^2}\end{aligned}$$

Applying change of variables $\tau=g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau)+p_1\left(\frac{d}{d\tau}y(\tau)\right)+q_1y(\tau)=0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau)=\frac{\tau''(x)+p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau)=\frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1=c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}\tau'&=\frac{1}{c}\sqrt{q}\\ &=\frac{\sqrt{13}\,\sqrt{\frac{1}{x^2}}}{c} \qquad (6)\\ \tau''&=-\frac{\sqrt{13}}{c\sqrt{\frac{1}{x^2}}\,x^3}\end{aligned}$$

Substituting the above into (4) results in

$$
\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{-\frac{\sqrt{13}}{c\sqrt{\frac{1}{x^2}}\,x^3} - \frac{3}{x}\frac{\sqrt{13}\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{13}\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\
&= -\frac{4c\sqrt{13}}{13}
\end{aligned}
$$

Therefore ode (3) now becomes

$$
\begin{gathered}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) = 0 \\
\frac{d^2}{d\tau^2}y(\tau) - \frac{4c\sqrt{13}\left(\frac{d}{d\tau}y(\tau)\right)}{13} + c^2 y(\tau) = 0
\end{gathered} \tag{7}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
y(\tau) = \mathrm{e}^{\frac{2\sqrt{13}\,c\tau}{13}}\left(c_1 \cos\left(\frac{3\sqrt{13}\,c\tau}{13}\right) + c_2 \sin\left(\frac{3\sqrt{13}\,c\tau}{13}\right)\right)
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int \sqrt{13}\sqrt{\frac{1}{x^2}}dx}{c} \\
&= \frac{\sqrt{13}\sqrt{\frac{1}{x^2}}\,x\ln(x)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = x^2(c_1 \cos(3\ln(x)) + c_2 \sin(3\ln(x)))
$$

Summary

The solution(s) found are the following

$$
y = x^2(c_1 \cos(3\ln(x)) + c_2 \sin(3\ln(x))) \tag{1}
$$

Verification of solutions

$$
y = x^2(c_1 \cos(3\ln(x)) + c_2 \sin(3\ln(x)))
$$

Verified OK.

1.10.4 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$x^2y'' - 3xy' + 13y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{3}{x} \\ q(x) &= \frac{13}{x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{3n}{x^2} + \frac{13}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 2 + 3i \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \left(\frac{4+6i}{x} - \frac{3}{x}\right) v'(x) &= 0 \\ v''(x) + \frac{(1+6i)\, v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{(1+6i)\,u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{(-1-6i)\,u}{x} \end{aligned}$$

Where $f(x) = \frac{-1-6i}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= \frac{-1-6i}{x}\,dx \\ \int \frac{1}{u}\,du &= \int \frac{-1-6i}{x}\,dx \\ \ln(u) &= (-1-6i)\ln(x) + c_1 \\ u &= \mathrm{e}^{(-1-6i)\ln(x)+c_1} \\ &= c_1 \mathrm{e}^{(-1-6i)\ln(x)} \end{aligned}$$

Which simplifies to

$$u(x) = \frac{c_1 x^{-6i}}{x}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\,dx + c_2 \\ &= \frac{i c_1 x^{-6i}}{6} + c_2 \end{aligned}$$

Hence

$$\begin{aligned} y &= v(x)\,x^n \\ &= \left(\frac{i c_1 x^{-6i}}{6} + c_2\right) x^{2+3i} \\ &= c_2 x^{2+3i} + \frac{i c_1 x^{2-3i}}{6} \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \left(\frac{i c_1 x^{-6i}}{6} + c_2\right) x^{2+3i} \tag{1}$$

Verification of solutions

$$y = \left(\frac{i c_1 x^{-6i}}{6} + c_2\right) x^{2+3i}$$

Verified OK.

1.10.5 Solving using Kovacic algorithm

Writing the ode as

$$x^2 y'' - 3xy' + 13y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -3x \\ C &= 13 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-37}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -37 \\
t &= 4x^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(-\frac{37}{4x^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 19: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{37}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{37}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + 3i \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - 3i
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{37}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{37}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + 3i \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - 3i
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{37}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2} + 3i$	$\frac{1}{2} - 3i$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}+3i$	$\frac{1}{2}-3i$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2} - 3i$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= \frac{1}{2} - 3i - \left(\frac{1}{2} - 3i\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + (-)[\sqrt{r}]_\infty \\
&= \frac{\frac{1}{2} - 3i}{x} + (-)(0) \\
&= \frac{\frac{1}{2} - 3i}{x} \\
&= \frac{\frac{1}{2} - 3i}{x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(\frac{\frac{1}{2}-3i}{x}\right)(0)+\left(\left(\frac{-\frac{1}{2}+3i}{x^2}\right)+\left(\frac{\frac{1}{2}-3i}{x}\right)^2-\left(-\frac{37}{4x^2}\right)\right)=0$$
$$0=0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}z_1(x)&=pe^{\int \omega\,dx}\\&=\mathrm{e}^{\int \frac{\frac{1}{2}-3i}{x}dx}\\&=x^{\frac{1}{2}-3i}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1&=z_1e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\&=z_1e^{-\int \frac{1}{2}\frac{-3x}{x^2}\,dx}\\&=z_1e^{\frac{3\ln(x)}{2}}\\&=z_1\left(x^{\frac{3}{2}}\right)\end{aligned}$$

Which simplifies to

$$y_1=x^{2-3i}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2&=y_1\int \frac{e^{\int -\frac{-3x}{x^2}\,dx}}{(y_1)^2}\,dx\\&=y_1\int \frac{e^{3\ln(x)}}{(y_1)^2}\,dx\\&=y_1\left(-\frac{ix^{6i}}{6}\right)\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^{2-3i}\right) + c_2 \left(x^{2-3i} \left(-\frac{i x^{6i}}{6}\right)\right)
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = c_1 x^{2-3i} - \frac{i c_2 x^{2+3i}}{6} \tag{1}
$$

<u>Verification of solutions</u>

$$
y = c_1 x^{2-3i} - \frac{i c_2 x^{2+3i}}{6}
$$

Verified OK.

1.10.6 Maple step by step solution

Let's solve

$x^2 y'' - 3 x y' + 13 y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{3 y'}{x} - \frac{13 y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{3 y'}{x} + \frac{13 y}{x^2} = 0$

- Multiply by denominators of the ODE

  $x^2 y'' - 3 x y' + 13 y = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

- ○ Calculate the 1st derivative of y with respect to x , using the chain rule

  $y' = \left(\frac{d}{dt} y(t)\right) t'(x)$

- Compute derivative

  $y' = \frac{\frac{d}{dt}y(t)}{x}$
- Calculate the 2nd derivative of y with respect to x , using the chain rule

  $y'' = \left(\frac{d^2}{dt^2}y(t)\right)t'(x)^2 + t''(x)\left(\frac{d}{dt}y(t)\right)$
- Compute derivative

  $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) - 3\frac{d}{dt}y(t) + 13y(t) = 0$

- Simplify

  $\frac{d^2}{dt^2}y(t) - 4\frac{d}{dt}y(t) + 13y(t) = 0$
- Characteristic polynomial of ODE

  $r^2 - 4r + 13 = 0$
- Use quadratic formula to solve for r

  $r = \frac{4\pm(\sqrt{-36})}{2}$
- Roots of the characteristic polynomial

  $r = (2 - 3\,\mathrm{I}, 2 + 3\,\mathrm{I})$
- 1st solution of the ODE

  $y_1(t) = \mathrm{e}^{2t}\cos(3t)$
- 2nd solution of the ODE

  $y_2(t) = \mathrm{e}^{2t}\sin(3t)$
- General solution of the ODE

  $y(t) = c_1y_1(t) + c_2y_2(t)$
- Substitute in solutions

  $y(t) = c_1\mathrm{e}^{2t}\cos(3t) + c_2\mathrm{e}^{2t}\sin(3t)$
- Change variables back using $t = \ln(x)$

  $y = c_1x^2\cos(3\ln(x)) + c_2x^2\sin(3\ln(x))$
- Simplify

  $y = x^2(c_1\cos(3\ln(x)) + c_2\sin(3\ln(x)))$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 23

```
dsolve(x^2*diff(y(x),x$2)-3*x*diff(y(x),x)+13*y(x)=0,y(x), singsol=all)
```

$$y(x) = x^2(c_1 \sin(3\ln(x)) + c_2 \cos(3\ln(x)))$$

✓ Solution by Mathematica
Time used: 0.029 (sec). Leaf size: 26

```
DSolve[x^2*y''[x]-3*x*y'[x]+13*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^2(c_2 \cos(3\log(x)) + c_1 \sin(3\log(x)))$$

1.11 problem Problem 17

Internal problem ID [2597]
Internal file name [`OUTPUT/2089_Sunday_June_05_2022_02_48_05_AM_11285012/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.2, Basic Ideas and Terminology. page 21
Problem number: Problem 17.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2x^2y'' - xy' + y = 9x^2}$$

1.11.1 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 2x^2, B = -x, C = 1, f(x) = 9x^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$2x^2y'' - xy' + y = 0$$

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$2x^2(r(r-1))x^{r-2} - xrx^{r-1} + x^r = 0$$

Simplifying gives

$$2r(r-1)\,x^r - r\,x^r + x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$2r(r-1) - r + 1 = 0$$

Or

$$2r^2 - 3r + 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= 1 \\ r_2 &= \frac{1}{2} \end{aligned}$$

Since the roots are real and distinct, then the general solution is

$$y = c_1y_1 + c_2y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = c_1x + \sqrt{x}\,c_2$$

Next, we find the particular solution to the ODE

$$2x^2y'' - xy' + y = 9x^2$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x$$
$$y_2 = \sqrt{x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & \sqrt{x} \\ \frac{d}{dx}(x) & \frac{d}{dx}(\sqrt{x}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & \sqrt{x} \\ 1 & \frac{1}{2\sqrt{x}} \end{vmatrix}$$

Therefore

$$W = (x)\left(\frac{1}{2\sqrt{x}}\right) - (\sqrt{x})(1)$$

Which simplifies to

$$W = -\frac{\sqrt{x}}{2}$$

Which simplifies to

$$W = -\frac{\sqrt{x}}{2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{9x^{\frac{5}{2}}}{-x^{\frac{5}{2}}}\, dx$$

Which simplifies to

$$u_1 = -\int (-9)\, dx$$

Hence

$$u_1 = 9x$$

And Eq. (3) becomes

$$u_2 = \int \frac{9x^3}{-x^{\frac{5}{2}}}\, dx$$

Which simplifies to

$$u_2 = \int -9\sqrt{x} dx$$

Hence

$$u_2 = -6x^{\frac{3}{2}}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = 3x^2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= 3x^2 + c_1 x + \sqrt{x}\, c_2 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = 3x^2 + c_1 x + \sqrt{x}\, c_2 \tag{1}$$

Verification of solutions

$$y = 3x^2 + c_1 x + \sqrt{x}\, c_2$$

Verified OK.

1.11.2 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$2x^2y'' - xy' + y = 0$$

In normal form the ode

$$2x^2y'' - xy' + y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{1}{2x} \\ q(x) &= \frac{1}{2x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)} dx \\
&= \int \mathrm{e}^{-\left(\int -\frac{1}{2x} dx\right)} dx \\
&= \int e^{\frac{\ln(x)}{2}} \, dx \\
&= \int \sqrt{x} dx \\
&= \frac{2x^{\frac{3}{2}}}{3}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{1}{2x^2}}{x} \\
&= \frac{1}{2x^3}
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}
\frac{d^2}{d\tau^2} y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2} y(\tau) + \frac{y(\tau)}{2x^3} &= 0
\end{aligned}$$

But in terms of τ

$$\frac{1}{2x^3} = \frac{2}{9\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2} y(\tau) + \frac{2y(\tau)}{9\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$9\left(\frac{d^2}{d\tau^2} y(\tau)\right)\tau^2 + 2y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$9\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 2\tau^r = 0$$

Simplifying gives

$$9r(r-1)\,\tau^r + 0\,\tau^r + 2\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$9r(r-1) + 0 + 2 = 0$$

Or

$$9r^2 - 9r + 2 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{3}$$
$$r_2 = \frac{2}{3}$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1 y_1 + c_2 y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = c_1\tau^{\frac{1}{3}} + c_2\tau^{\frac{2}{3}}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{c_1 2^{\frac{1}{3}} 3^{\frac{2}{3}} \left(x^{\frac{3}{2}}\right)^{\frac{1}{3}}}{3} + \frac{c_2 2^{\frac{2}{3}} 3^{\frac{1}{3}} \left(x^{\frac{3}{2}}\right)^{\frac{2}{3}}}{3}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{c_1 2^{\frac{1}{3}} 3^{\frac{2}{3}} \left(x^{\frac{3}{2}}\right)^{\frac{1}{3}}}{3} + \frac{c_2 2^{\frac{2}{3}} 3^{\frac{1}{3}} \left(x^{\frac{3}{2}}\right)^{\frac{2}{3}}}{3}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \left(x^{\frac{3}{2}}\right)^{\frac{1}{3}}$$

$$y_2 = \left(x^{\frac{3}{2}}\right)^{\frac{2}{3}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \left(x^{\frac{3}{2}}\right)^{\frac{1}{3}} & \left(x^{\frac{3}{2}}\right)^{\frac{2}{3}} \\ \frac{d}{dx}\left(\left(x^{\frac{3}{2}}\right)^{\frac{1}{3}}\right) & \frac{d}{dx}\left(\left(x^{\frac{3}{2}}\right)^{\frac{2}{3}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \left(x^{\frac{3}{2}}\right)^{\frac{1}{3}} & \left(x^{\frac{3}{2}}\right)^{\frac{2}{3}} \\ \frac{\sqrt{x}}{2\left(x^{\frac{3}{2}}\right)^{\frac{2}{3}}} & \frac{\sqrt{x}}{\left(x^{\frac{3}{2}}\right)^{\frac{1}{3}}} \end{vmatrix}$$

Therefore

$$W = \left(\left(x^{\frac{3}{2}}\right)^{\frac{1}{3}}\right)\left(\frac{\sqrt{x}}{\left(x^{\frac{3}{2}}\right)^{\frac{1}{3}}}\right) - \left(\left(x^{\frac{3}{2}}\right)^{\frac{2}{3}}\right)\left(\frac{\sqrt{x}}{2\left(x^{\frac{3}{2}}\right)^{\frac{2}{3}}}\right)$$

Which simplifies to

$$W = \frac{\sqrt{x}}{2}$$

Which simplifies to

$$W = \frac{\sqrt{x}}{2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{9\left(x^{\frac{3}{2}}\right)^{\frac{2}{3}} x^2}{x^{\frac{5}{2}}} dx$$

Which simplifies to

$$u_1 = -\int \frac{9\left(x^{\frac{3}{2}}\right)^{\frac{2}{3}}}{\sqrt{x}} dx$$

Hence

$$u_1 = -6\sqrt{x}\left(x^{\frac{3}{2}}\right)^{\frac{2}{3}}$$

And Eq. (3) becomes

$$u_2 = \int \frac{9\left(x^{\frac{3}{2}}\right)^{\frac{1}{3}} x^2}{x^{\frac{5}{2}}} dx$$

Which simplifies to

$$u_2 = \int \frac{9\left(x^{\frac{3}{2}}\right)^{\frac{1}{3}}}{\sqrt{x}} dx$$

Hence

$$u_2 = 9\sqrt{x}\left(x^{\frac{3}{2}}\right)^{\frac{1}{3}}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = 3x^2$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\frac{c_1 2^{\frac{1}{3}} 3^{\frac{2}{3}} \left(x^{\frac{3}{2}}\right)^{\frac{1}{3}}}{3} + \frac{c_2 2^{\frac{2}{3}} 3^{\frac{1}{3}} \left(x^{\frac{3}{2}}\right)^{\frac{2}{3}}}{3}\right) + (3x^2)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = \frac{c_1 2^{\frac{1}{3}} 3^{\frac{2}{3}} \left(x^{\frac{3}{2}}\right)^{\frac{1}{3}}}{3} + \frac{c_2 2^{\frac{2}{3}} 3^{\frac{1}{3}} \left(x^{\frac{3}{2}}\right)^{\frac{2}{3}}}{3} + 3x^2 \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 2^{\frac{1}{3}} 3^{\frac{2}{3}} \left(x^{\frac{3}{2}}\right)^{\frac{1}{3}}}{3} + \frac{c_2 2^{\frac{2}{3}} 3^{\frac{1}{3}} \left(x^{\frac{3}{2}}\right)^{\frac{2}{3}}}{3} + 3x^2
$$

Verified OK.

1.11.3 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$
Ay''(x) + By'(x) + Cy(x) = f(x)
$$

Where $A = 2x^2, B = -x, C = 1, f(x) = 9x^2$. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$
2x^2y'' - xy' + y = 0
$$

In normal form the ode

$$
2x^2y'' - xy' + y = 0 \tag{1}
$$

Becomes

$$
y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}
$$

Where

$$p(x) = -\frac{1}{2x}$$
$$q(x) = \frac{1}{2x^2}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$
$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}
\tau' &= \frac{1}{c}\sqrt{q} \\
&= \frac{\sqrt{2}\,\sqrt{\frac{1}{x^2}}}{2c} \\
\tau'' &= -\frac{\sqrt{2}}{2c\sqrt{\frac{1}{x^2}}\,x^3}
\end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{-\frac{\sqrt{2}}{2c\sqrt{\frac{1}{x^2}}\,x^3} - \frac{1}{2x}\frac{\sqrt{2}\,\sqrt{\frac{1}{x^2}}}{2c}}{\left(\frac{\sqrt{2}\,\sqrt{\frac{1}{x^2}}}{2c}\right)^2} \\
&= -\frac{3c\sqrt{2}}{2}
\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) - \frac{3c\sqrt{2}\left(\frac{d}{d\tau}y(\tau)\right)}{2} + c^2 y(\tau) &= 0
\end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = \mathrm{e}^{\frac{3\sqrt{2}\,c\tau}{4}} \left(c_1 \cosh\left(\frac{\sqrt{2}\,c\tau}{4}\right) + ic_2 \sinh\left(\frac{\sqrt{2}\,c\tau}{4}\right)\right)$$

Now from (6)

$$\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int \frac{\sqrt{2}\sqrt{\frac{1}{x^2}}}{2} dx}{c} \\
&= \frac{\sqrt{2}\sqrt{\frac{1}{x^2}}\,x\ln(x)}{2c}
\end{aligned}$$

Substituting the above into the solution obtained gives

$$y = x^{\frac{3}{4}} \left(c_1 \cosh\left(\frac{\ln(x)}{4}\right) + ic_2 \sinh\left(\frac{\ln(x)}{4}\right)\right)$$

Now the particular solution to this ODE is found

$$2x^2y'' - xy' + y = 9x^2$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}
y_1 &= \left(x^{\frac{3}{2}}\right)^{\frac{1}{3}} \\
y_2 &= \left(x^{\frac{3}{2}}\right)^{\frac{2}{3}}
\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \left(x^{\frac{3}{2}}\right)^{\frac{1}{3}} & \left(x^{\frac{3}{2}}\right)^{\frac{2}{3}} \\ \frac{d}{dx}\left(\left(x^{\frac{3}{2}}\right)^{\frac{1}{3}}\right) & \frac{d}{dx}\left(\left(x^{\frac{3}{2}}\right)^{\frac{2}{3}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \left(x^{\frac{3}{2}}\right)^{\frac{1}{3}} & \left(x^{\frac{3}{2}}\right)^{\frac{2}{3}} \\ \frac{\sqrt{x}}{2\left(x^{\frac{3}{2}}\right)^{\frac{2}{3}}} & \frac{\sqrt{x}}{\left(x^{\frac{3}{2}}\right)^{\frac{1}{3}}} \end{vmatrix}$$

Therefore

$$W = \left(\left(x^{\frac{3}{2}}\right)^{\frac{1}{3}}\right)\left(\frac{\sqrt{x}}{\left(x^{\frac{3}{2}}\right)^{\frac{1}{3}}}\right) - \left(\left(x^{\frac{3}{2}}\right)^{\frac{2}{3}}\right)\left(\frac{\sqrt{x}}{2\left(x^{\frac{3}{2}}\right)^{\frac{2}{3}}}\right)$$

Which simplifies to

$$W = \frac{\sqrt{x}}{2}$$

Which simplifies to

$$W = \frac{\sqrt{x}}{2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{9\left(x^{\frac{3}{2}}\right)^{\frac{2}{3}} x^2}{x^{\frac{5}{2}}}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{9\left(x^{\frac{3}{2}}\right)^{\frac{2}{3}}}{\sqrt{x}} dx$$

Hence

$$u_1 = -6\sqrt{x}\left(x^{\frac{3}{2}}\right)^{\frac{2}{3}}$$

And Eq. (3) becomes

$$u_2 = \int \frac{9\left(x^{\frac{3}{2}}\right)^{\frac{1}{3}} x^2}{x^{\frac{5}{2}}} dx$$

Which simplifies to

$$u_2 = \int \frac{9\left(x^{\frac{3}{2}}\right)^{\frac{1}{3}}}{\sqrt{x}} dx$$

Hence

$$u_2 = 9\sqrt{x}\left(x^{\frac{3}{2}}\right)^{\frac{1}{3}}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = 3x^2$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p \\
&= \left(x^{\frac{3}{4}}\left(c_1 \cosh\left(\frac{\ln(x)}{4}\right) + ic_2 \sinh\left(\frac{\ln(x)}{4}\right)\right)\right) + \left(3x^2\right) \\
&= 3x^2 + x^{\frac{3}{4}}\left(c_1 \cosh\left(\frac{\ln(x)}{4}\right) + ic_2 \sinh\left(\frac{\ln(x)}{4}\right)\right)
\end{aligned}$$

Which simplifies to

$$y = ix^{\frac{3}{4}} \sinh\left(\frac{\ln(x)}{4}\right) c_2 + x^{\frac{3}{4}} \cosh\left(\frac{\ln(x)}{4}\right) c_1 + 3x^2$$

Summary
The solution(s) found are the following

$$y = i x^{\frac{3}{4}} \sinh\left(\frac{\ln(x)}{4}\right) c_2 + x^{\frac{3}{4}} \cosh\left(\frac{\ln(x)}{4}\right) c_1 + 3x^2 \tag{1}$$

Verification of solutions

$$y = i x^{\frac{3}{4}} \sinh\left(\frac{\ln(x)}{4}\right) c_2 + x^{\frac{3}{4}} \cosh\left(\frac{\ln(x)}{4}\right) c_1 + 3x^2$$

Verified OK.

1.11.4 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 2x^2, B = -x, C = 1, f(x) = 9x^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$2x^2y'' - xy' + y = 0$$

In normal form the ode

$$2x^2y'' - xy' + y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = -\frac{1}{2x}$$
$$q(x) = \frac{1}{2x^2}$$

Applying change of variables on the depndent variable $y = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{n}{2x^2} + \frac{1}{2x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 1 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{3v'(x)}{2x} &= 0 \\ v''(x) + \frac{3v'(x)}{2x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{3u(x)}{2x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{3u}{2x} \end{aligned}$$

Where $f(x) = -\frac{3}{2x}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= -\frac{3}{2x}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{3}{2x}\,dx \\
\ln(u) &= -\frac{3\ln(x)}{2} + c_1 \\
u &= \mathrm{e}^{-\frac{3\ln(x)}{2}+c_1} \\
&= \frac{c_1}{x^{\frac{3}{2}}}
\end{aligned}
$$

Now that $u(x)$ is known, then

$$
\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\,dx + c_2 \\
&= -\frac{2c_1}{\sqrt{x}} + c_2
\end{aligned}
$$

Hence

$$
\begin{aligned}
y &= v(x)\,x^n \\
&= \left(-\frac{2c_1}{\sqrt{x}} + c_2\right)x \\
&= c_2x - 2c_1\sqrt{x}
\end{aligned}
$$

Now the particular solution to this ODE is found

$$
2x^2y'' - xy' + y = 9x^2
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1y_1 + u_2y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= x \\
y_2 &= \sqrt{x}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$
W = \begin{vmatrix} x & \sqrt{x} \\ \frac{d}{dx}(x) & \frac{d}{dx}\left(\sqrt{x}\right) \end{vmatrix}
$$

Which gives

$$
W = \begin{vmatrix} x & \sqrt{x} \\ 1 & \frac{1}{2\sqrt{x}} \end{vmatrix}
$$

Therefore

$$
W = (x)\left(\frac{1}{2\sqrt{x}}\right) - \left(\sqrt{x}\right)(1)
$$

Which simplifies to

$$
W = -\frac{\sqrt{x}}{2}
$$

Which simplifies to

$$
W = -\frac{\sqrt{x}}{2}
$$

Therefore Eq. (2) becomes

$$
u_1 = -\int \frac{9x^{\frac{5}{2}}}{-x^{\frac{5}{2}}}\, dx
$$

Which simplifies to

$$u_1 = -\int (-9)\,dx$$

Hence

$$u_1 = 9x$$

And Eq. (3) becomes

$$u_2 = \int \frac{9x^3}{-x^{\frac{5}{2}}}\,dx$$

Which simplifies to

$$u_2 = \int -9\sqrt{x}dx$$

Hence

$$u_2 = -6x^{\frac{3}{2}}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = 3x^2$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p \\
&= \left(\left(-\frac{2c_1}{\sqrt{x}} + c_2\right)x\right) + (3x^2) \\
&= 3x^2 + \left(-\frac{2c_1}{\sqrt{x}} + c_2\right)x
\end{aligned}$$

Which simplifies to

$$y = c_2x - 2c_1\sqrt{x} + 3x^2$$

Summary

The solution(s) found are the following

$$y = c_2x - 2c_1\sqrt{x} + 3x^2 \tag{1}$$

Verification of solutions

$$y = c_2x - 2c_1\sqrt{x} + 3x^2$$

Verified OK.

1.11.5 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned} y' &= B'v + v'B \\ y'' &= B''v + B'v' + v''B + v'B' \\ &= v''B + 2v' + B' + B''v \end{aligned}$$

And now the original ode becomes

$$\begin{aligned} A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\ ABv'' + \left(2AB' + B^2\right) v' + (AB'' + BB' + CB)\, v &= 0 \end{aligned} \tag{1}$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + \left(2AB' + B^2\right) v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + \left(2AB' + B^2\right) u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned} A &= 2x^2 \\ B &= -x \\ C &= 1 \\ F &= 9x^2 \end{aligned}$$

The above shows that for this ode

$$\begin{aligned} AB'' + BB' + CB &= \left(2x^2\right)(0) + (-x)(-1) + (1)(-x) \\ &= 0 \end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$-2x^3v'' + (-3x^2)\,v' = 0$$

Now by applying $v' = u$ the above becomes

$$-2x^3u'(x) - 3x^2u(x) = 0$$

Which is now solved for u. In canonical form the ODE is

$$\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= -\frac{3u}{2x}
\end{aligned}$$

Where $f(x) = -\frac{3}{2x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= -\frac{3}{2x}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{3}{2x}\,dx \\
\ln(u) &= -\frac{3\ln(x)}{2} + c_1 \\
u &= \mathrm{e}^{-\frac{3\ln(x)}{2}+c_1} \\
&= \frac{c_1}{x^{\frac{3}{2}}}
\end{aligned}$$

The ode for v now becomes

$$\begin{aligned}
v' &= u \\
&= \frac{c_1}{x^{\frac{3}{2}}}
\end{aligned}$$

Which is now solved for v. Integrating both sides gives

$$\begin{aligned}
v(x) &= \int \frac{c_1}{x^{\frac{3}{2}}}\,\mathrm{d}x \\
&= -\frac{2c_1}{\sqrt{x}} + c_2
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= Bv \\
&= (-x)\left(-\frac{2c_1}{\sqrt{x}} + c_2\right) \\
&= -c_2x + 2c_1\sqrt{x}
\end{aligned}$$

And now the particular solution $y_p(x)$ will be found. The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x$$
$$y_2 = \sqrt{x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & \sqrt{x} \\ \frac{d}{dx}(x) & \frac{d}{dx}\left(\sqrt{x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & \sqrt{x} \\ 1 & \frac{1}{2\sqrt{x}} \end{vmatrix}$$

Therefore

$$W = (x)\left(\frac{1}{2\sqrt{x}}\right) - \left(\sqrt{x}\right)(1)$$

Which simplifies to

$$W = -\frac{\sqrt{x}}{2}$$

Which simplifies to

$$W = -\frac{\sqrt{x}}{2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{9x^{\frac{5}{2}}}{-x^{\frac{5}{2}}}\, dx$$

Which simplifies to

$$u_1 = -\int (-9)\, dx$$

Hence

$$u_1 = 9x$$

And Eq. (3) becomes

$$u_2 = \int \frac{9x^3}{-x^{\frac{5}{2}}}\, dx$$

Which simplifies to

$$u_2 = \int -9\sqrt{x}dx$$

Hence

$$u_2 = -6x^{\frac{3}{2}}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = 3x^2$$

Hence the complete solution is

$$\begin{aligned} y(x) &= y_h + y_p \\ &= \left(-c_2 x + 2c_1\sqrt{x}\right) + \left(3x^2\right) \\ &= -c_2 x + 2c_1\sqrt{x} + 3x^2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -c_2 x + 2c_1\sqrt{x} + 3x^2 \tag{1}$$

Verification of solutions

$$y = -c_2x + 2c_1\sqrt{x} + 3x^2$$

Verified OK.

1.11.6 Solving using Kovacic algorithm

Writing the ode as

$$2x^2y'' - xy' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2 \\ B &= -x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{3}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 21: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{3}{16x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{3}{16x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{4}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\
&= \frac{1}{4} - \left(\frac{1}{4}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{4x} + (-)(0) \\
&= \frac{1}{4x} \\
&= \frac{1}{4x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{1}{4x}\right)(0) + \left(\left(-\frac{1}{4x^2}\right) + \left(\frac{1}{4x}\right)^2 - \left(-\frac{3}{16x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \frac{1}{4x} dx} \\
&= x^{\frac{1}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-x}{2x^2} \, dx} \\
&= z_1 e^{\frac{\ln(x)}{4}} \\
&= z_1 \left(x^{\frac{1}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \sqrt{x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{2x^2} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{\frac{\ln(x)}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(2\sqrt{x}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\sqrt{x}\right) + c_2 \left(\sqrt{x}\left(2\sqrt{x}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
2x^2y'' - xy' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1\sqrt{x} + 2c_2 x$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= \sqrt{x} \\ y_2 &= 2x \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \sqrt{x} & 2x \\ \frac{d}{dx}\left(\sqrt{x}\right) & \frac{d}{dx}(2x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \sqrt{x} & 2x \\ \frac{1}{2\sqrt{x}} & 2 \end{vmatrix}$$

Therefore

$$W = \left(\sqrt{x}\right)(2) - (2x)\left(\frac{1}{2\sqrt{x}}\right)$$

Which simplifies to

$$W = \sqrt{x}$$

Which simplifies to

$$W = \sqrt{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{18x^3}{2x^{\frac{5}{2}}}\, dx$$

Which simplifies to

$$u_1 = -\int 9\sqrt{x} dx$$

Hence

$$u_1 = -6x^{\frac{3}{2}}$$

And Eq. (3) becomes

$$u_2 = \int \frac{9x^{\frac{5}{2}}}{2x^{\frac{5}{2}}}\, dx$$

Which simplifies to

$$u_2 = \int \frac{9}{2} dx$$

Hence

$$u_2 = \frac{9x}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = 3x^2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\sqrt{x} + 2c_2x\right) + \left(3x^2\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\sqrt{x} + 2c_2x + 3x^2 \tag{1}$$

Verification of solutions

$$y = c_1\sqrt{x} + 2c_2x + 3x^2$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   <- LODE of Euler type successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve(2*x^2*diff(y(x),x$2)-x*diff(y(x),x)+y(x)=9*x^2,y(x), singsol=all)
```

$$y(x) = c_2x + c_1\sqrt{x} + 3x^2$$

✓ Solution by Mathematica

Time used: 0.016 (sec). Leaf size: 23

```
DSolve[2*x^2*y''[x]-x*y'[x]+y[x]==9*x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 3x^2 + c_2x + c_1\sqrt{x}$$

1.12 problem Problem 18

Internal problem ID [2598]
Internal file name [`OUTPUT/2090_Sunday_June_05_2022_02_48_08_AM_7065649/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.2, Basic Ideas and Terminology. page 21
Problem number: Problem 18.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_1", "second_order_change_of_variable_on_y_method_2", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{x^2 y'' - 4xy' + 6y = x^4 \sin(x)}$$

1.12.1 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -4x, C = 6, f(x) = x^4 \sin(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - 4xy' + 6y = 0$$

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} - 4xrx^{r-1} + 6x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r - 4r\,x^r + 6x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) - 4r + 6 = 0$$

Or

$$r^2 - 5r + 6 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = 2$$
$$r_2 = 3$$

Since the roots are real and distinct, then the general solution is

$$y = c_1y_1 + c_2y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = c_2x^3 + c_1x^2$$

Next, we find the particular solution to the ODE

$$x^2y'' - 4xy' + 6y = x^4 \sin(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= x^2 \\ y_2 &= x^3 \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^2 & x^3 \\ \frac{d}{dx}(x^2) & \frac{d}{dx}(x^3) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix}$$

Therefore

$$W = (x^2)(3x^2) - (x^3)(2x)$$

Which simplifies to

$$W = x^4$$

Which simplifies to

$$W = x^4$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^7 \sin(x)}{x^6}\, dx$$

Which simplifies to

$$u_1 = -\int x \sin(x)\, dx$$

Hence

$$u_1 = -\sin(x) + \cos(x)\, x$$

And Eq. (3) becomes

$$u_2 = \int \frac{\sin(x)\, x^6}{x^6}\, dx$$

Which simplifies to

$$u_2 = \int \sin(x)\, dx$$

Hence

$$u_2 = -\cos(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = (-\sin(x) + \cos(x)\, x)\, x^2 - \cos(x)\, x^3$$

Which simplifies to

$$y_p(x) = -x^2 \sin(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= x^2(c_2 x - \sin(x) + c_1) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = x^2(c_2x - \sin(x) + c_1) \tag{1}$$

Verification of solutions

$$y = x^2(c_2x - \sin(x) + c_1)$$

Verified OK.

1.12.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = -\frac{4}{x}$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int -\frac{4}{x}\,dx} \\ &= \frac{1}{x^2} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)y)'' &= \sin(x) \\ \left(\frac{y}{x^2}\right)'' &= \sin(x) \end{aligned}$$

Integrating once gives

$$\left(\frac{y}{x^2}\right)' = -\cos(x) + c_1$$

Integrating again gives

$$\left(\frac{y}{x^2}\right) = c_1x - \sin(x) + c_2$$

Hence the solution is

$$y = \frac{c_1x - \sin(x) + c_2}{\frac{1}{x^2}}$$

Or

$$y = c_1x^3 + c_2x^2 - x^2\sin(x)$$

Summary
The solution(s) found are the following

$$y = c_1 x^3 + c_2 x^2 - x^2 \sin(x) \tag{1}$$

Verification of solutions

$$y = c_1 x^3 + c_2 x^2 - x^2 \sin(x)$$

Verified OK.

1.12.3 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' - 4xy' + 6y = 0$$

In normal form the ode

$$x^2y'' - 4xy' + 6y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{4}{x} \\ q(x) &= \frac{6}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)} dx \\
&= \int \mathrm{e}^{-\left(\int -\frac{4}{x}dx\right)} dx \\
&= \int e^{4\ln(x)}\, dx \\
&= \int x^4 dx \\
&= \frac{x^5}{5} \qquad (6)
\end{aligned}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{6}{x^2}}{x^8} \\
&= \frac{6}{x^{10}} \qquad (7)
\end{aligned}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + \frac{6y(\tau)}{x^{10}} &= 0
\end{aligned}$$

But in terms of τ

$$\frac{6}{x^{10}} = \frac{6}{25\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{6y(\tau)}{25\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$25\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + 6y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$25\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 6\tau^r = 0$$

Simplifying gives

$$25r(r-1)\,\tau^r + 0\,\tau^r + 6\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$25r(r-1) + 0 + 6 = 0$$

Or

$$25r^2 - 25r + 6 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= \frac{2}{5} \\ r_2 &= \frac{3}{5} \end{aligned}$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1 y_1 + c_2 y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = c_1 \tau^{\frac{2}{5}} + c_2 \tau^{\frac{3}{5}}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{c_1 5^{\frac{3}{5}} (x^5)^{\frac{2}{5}}}{5} + \frac{c_2 5^{\frac{2}{5}} (x^5)^{\frac{3}{5}}}{5}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{c_1 5^{\frac{3}{5}} (x^5)^{\frac{2}{5}}}{5} + \frac{c_2 5^{\frac{2}{5}} (x^5)^{\frac{3}{5}}}{5}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = (x^5)^{\frac{2}{5}}$$

$$y_2 = (x^5)^{\frac{3}{5}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} (x^5)^{\frac{2}{5}} & (x^5)^{\frac{3}{5}} \\ \frac{d}{dx}\left((x^5)^{\frac{2}{5}}\right) & \frac{d}{dx}\left((x^5)^{\frac{3}{5}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} (x^5)^{\frac{2}{5}} & (x^5)^{\frac{3}{5}} \\ \frac{2x^4}{(x^5)^{\frac{3}{5}}} & \frac{3x^4}{(x^5)^{\frac{2}{5}}} \end{vmatrix}$$

Therefore

$$W = \left((x^5)^{\frac{2}{5}}\right)\left(\frac{3x^4}{(x^5)^{\frac{2}{5}}}\right) - \left((x^5)^{\frac{3}{5}}\right)\left(\frac{2x^4}{(x^5)^{\frac{3}{5}}}\right)$$

Which simplifies to

$$W = x^4$$

Which simplifies to

$$W = x^4$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{(x^5)^{\frac{3}{5}} x^4 \sin(x)}{x^6} \, dx$$

Which simplifies to

$$u_1 = -\int \frac{(x^5)^{\frac{3}{5}} \sin(x)}{x^2} dx$$

Hence

$$u_1 = \frac{(x^5)^{\frac{3}{5}} \cos(x)}{x^2} - \frac{(x^5)^{\frac{3}{5}} \sin(x)}{x^3}$$

And Eq. (3) becomes

$$u_2 = \int \frac{(x^5)^{\frac{2}{5}} x^4 \sin(x)}{x^6} \, dx$$

Which simplifies to

$$u_2 = \int \frac{(x^5)^{\frac{2}{5}} \sin(x)}{x^2} dx$$

Hence

$$u_2 = -\frac{(x^5)^{\frac{2}{5}} \cos(x)}{x^2}$$

Which simplifies to

$$u_1 = \frac{(-\sin(x) + \cos(x) x) (x^5)^{\frac{3}{5}}}{x^3}$$

$$u_2 = -\frac{(x^5)^{\frac{2}{5}} \cos(x)}{x^2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = (-\sin(x) + \cos(x) x) x^2 - \cos(x) x^3$$

Which simplifies to

$$y_p(x) = -x^2 \sin(x)$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\frac{c_1 5^{\frac{3}{5}} (x^5)^{\frac{2}{5}}}{5} + \frac{c_2 5^{\frac{2}{5}} (x^5)^{\frac{3}{5}}}{5}\right) + \left(-x^2 \sin(x)\right)
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = \frac{c_1 5^{\frac{3}{5}} (x^5)^{\frac{2}{5}}}{5} + \frac{c_2 5^{\frac{2}{5}} (x^5)^{\frac{3}{5}}}{5} - x^2 \sin(x) \tag{1}
$$

<u>Verification of solutions</u>

$$
y = \frac{c_1 5^{\frac{3}{5}} (x^5)^{\frac{2}{5}}}{5} + \frac{c_2 5^{\frac{2}{5}} (x^5)^{\frac{3}{5}}}{5} - x^2 \sin(x)
$$

Verified OK.

1.12.4 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$
Ay''(x) + By'(x) + Cy(x) = f(x)
$$

Where $A = x^2, B = -4x, C = 6, f(x) = x^4 \sin(x)$. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$
x^2 y'' - 4xy' + 6y = 0
$$

In normal form the ode

$$
x^2 y'' - 4xy' + 6y = 0 \tag{1}
$$

Becomes

$$
y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}
$$

Where

$$
\begin{aligned}
p(x) &= -\frac{4}{x} \\
q(x) &= \frac{6}{x^2}
\end{aligned}
$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}\tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{6}\,\sqrt{\frac{1}{x^2}}}{c} \\ \tau'' &= -\frac{\sqrt{6}}{c\sqrt{\frac{1}{x^2}}\,x^3}\end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned}p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\ &= \frac{-\frac{\sqrt{6}}{c\sqrt{\frac{1}{x^2}}\,x^3} - \frac{4}{x}\frac{\sqrt{6}\,\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{6}\,\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\ &= -\frac{5c\sqrt{6}}{6}\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) - \frac{5c\sqrt{6}\,\left(\frac{d}{d\tau}y(\tau)\right)}{6} + c^2 y(\tau) &= 0\end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = \mathrm{e}^{\frac{5\sqrt{6}\,c\tau}{12}}\left(c_1\cosh\left(\frac{\sqrt{6}\,c\tau}{12}\right) + ic_2\sinh\left(\frac{\sqrt{6}\,c\tau}{12}\right)\right)$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int \sqrt{6}\,\sqrt{\frac{1}{x^2}}dx}{c} \\
&= \frac{\sqrt{6}\,\sqrt{\frac{1}{x^2}}\,x\ln(x)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = x^{\frac{5}{2}}\left(c_1\cosh\left(\frac{\ln(x)}{2}\right) + ic_2\sinh\left(\frac{\ln(x)}{2}\right)\right)
$$

Now the particular solution to this ODE is found

$$
x^2y'' - 4xy' + 6y = x^4\sin(x)
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1y_1 + u_2y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \left(x^5\right)^{\frac{2}{5}} \\
y_2 &= \left(x^5\right)^{\frac{3}{5}}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \left(x^5\right)^{\frac{2}{5}} & \left(x^5\right)^{\frac{3}{5}} \\ \frac{d}{dx}\left(\left(x^5\right)^{\frac{2}{5}}\right) & \frac{d}{dx}\left(\left(x^5\right)^{\frac{3}{5}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \left(x^5\right)^{\frac{2}{5}} & \left(x^5\right)^{\frac{3}{5}} \\ \frac{2x^4}{\left(x^5\right)^{\frac{3}{5}}} & \frac{3x^4}{\left(x^5\right)^{\frac{2}{5}}} \end{vmatrix}$$

Therefore

$$W = \left(\left(x^5\right)^{\frac{2}{5}}\right)\left(\frac{3x^4}{\left(x^5\right)^{\frac{2}{5}}}\right) - \left(\left(x^5\right)^{\frac{3}{5}}\right)\left(\frac{2x^4}{\left(x^5\right)^{\frac{3}{5}}}\right)$$

Which simplifies to

$$W = x^4$$

Which simplifies to

$$W = x^4$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\left(x^5\right)^{\frac{3}{5}} x^4 \sin(x)}{x^6}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{\left(x^5\right)^{\frac{3}{5}} \sin(x)}{x^2} dx$$

Hence

$$u_1 = \frac{\left(x^5\right)^{\frac{3}{5}} \cos(x)}{x^2} - \frac{\left(x^5\right)^{\frac{3}{5}} \sin(x)}{x^3}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\left(x^5\right)^{\frac{2}{5}} x^4 \sin(x)}{x^6}\, dx$$

Which simplifies to

$$u_2 = \int \frac{(x^5)^{\frac{2}{5}} \sin(x)}{x^2} dx$$

Hence

$$u_2 = -\frac{(x^5)^{\frac{2}{5}} \cos(x)}{x^2}$$

Which simplifies to

$$u_1 = \frac{(-\sin(x) + \cos(x)\, x)\,(x^5)^{\frac{3}{5}}}{x^3}$$

$$u_2 = -\frac{(x^5)^{\frac{2}{5}} \cos(x)}{x^2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = (-\sin(x) + \cos(x)\, x)\, x^2 - \cos(x)\, x^3$$

Which simplifies to

$$y_p(x) = -x^2 \sin(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(x^{\frac{5}{2}}\left(c_1 \cosh\left(\frac{\ln(x)}{2}\right) + i c_2 \sinh\left(\frac{\ln(x)}{2}\right)\right)\right) + (-x^2 \sin(x)) \\ &= -x^2 \sin(x) + x^{\frac{5}{2}}\left(c_1 \cosh\left(\frac{\ln(x)}{2}\right) + i c_2 \sinh\left(\frac{\ln(x)}{2}\right)\right) \end{aligned}$$

Which simplifies to

$$y = i \sinh\left(\frac{\ln(x)}{2}\right) x^{\frac{5}{2}} c_2 + \cosh\left(\frac{\ln(x)}{2}\right) x^{\frac{5}{2}} c_1 - x^2 \sin(x)$$

Summary

The solution(s) found are the following

$$y = i \sinh\left(\frac{\ln(x)}{2}\right) x^{\frac{5}{2}} c_2 + \cosh\left(\frac{\ln(x)}{2}\right) x^{\frac{5}{2}} c_1 - x^2 \sin(x) \tag{1}$$

Verification of solutions

$$
y = i \sinh\left(\frac{\ln (x)}{2}\right) x^{\frac{5}{2}} c_2 + \cosh\left(\frac{\ln (x)}{2}\right) x^{\frac{5}{2}} c_1 - x^2 \sin (x)
$$

Verified OK.

1.12.5 Solving as second order change of variable on y method 1 ode

This is second order non-homogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$
x^2y'' - 4xy' + 6y = 0
$$

In normal form the given ode is written as

$$
y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}
$$

Where

$$
\begin{aligned}
p(x) &= -\frac{4}{x} \\
q(x) &= \frac{6}{x^2}
\end{aligned}
$$

Calculating the Liouville ode invariant Q given by

$$
\begin{aligned}
Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\
&= \frac{6}{x^2} - \frac{\left(-\frac{4}{x}\right)'}{2} - \frac{\left(-\frac{4}{x}\right)^2}{4} \\
&= \frac{6}{x^2} - \frac{\left(\frac{4}{x^2}\right)}{2} - \frac{\left(\frac{16}{x^2}\right)}{4} \\
&= \frac{6}{x^2} - \left(\frac{2}{x^2}\right) - \frac{4}{x^2} \\
&= 0
\end{aligned}
$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$
y = v(x)\, z(x) \tag{3}
$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$\begin{aligned} z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)} \\ &= e^{-\int \frac{-\frac{4}{x}}{2}} \\ &= x^2 \end{aligned} \tag{5}$$

Hence (3) becomes

$$y = v(x)\, x^2 \tag{4}$$

Applying this change of variable to the original ode results in

$$v''(x) = \sin(x)$$

Which is now solved for $v(x)$ Integrating once gives

$$v'(x) = -\cos(x) + c_1$$

Integrating again gives

$$v(x) = -\sin(x) + c_1 x + c_2$$

Now that $v(x)$ is known, then

$$\begin{aligned} y &= v(x)\, z(x) \\ &= (c_1 x - \sin(x) + c_2)\,(z(x)) \end{aligned} \tag{7}$$

But from (5)

$$z(x) = x^2$$

Hence (7) becomes

$$y = (c_1 x - \sin(x) + c_2)\, x^2$$

Therefore the homogeneous solution y_h is

$$y_h = (c_1 x - \sin(x) + c_2)\, x^2$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \left(x^5\right)^{\frac{2}{5}}$$
$$y_2 = \left(x^5\right)^{\frac{3}{5}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \left(x^5\right)^{\frac{2}{5}} & \left(x^5\right)^{\frac{3}{5}} \\ \frac{d}{dx}\left(\left(x^5\right)^{\frac{2}{5}}\right) & \frac{d}{dx}\left(\left(x^5\right)^{\frac{3}{5}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \left(x^5\right)^{\frac{2}{5}} & \left(x^5\right)^{\frac{3}{5}} \\ \frac{2x^4}{\left(x^5\right)^{\frac{3}{5}}} & \frac{3x^4}{\left(x^5\right)^{\frac{2}{5}}} \end{vmatrix}$$

Therefore

$$W = \left(\left(x^5\right)^{\frac{2}{5}}\right)\left(\frac{3x^4}{\left(x^5\right)^{\frac{2}{5}}}\right) - \left(\left(x^5\right)^{\frac{3}{5}}\right)\left(\frac{2x^4}{\left(x^5\right)^{\frac{3}{5}}}\right)$$

Which simplifies to

$$W = x^4$$

Which simplifies to

$$W = x^4$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\left(x^5\right)^{\frac{3}{5}} x^4 \sin\left(x\right)}{x^6} dx$$

Which simplifies to

$$u_1 = -\int \frac{\left(x^5\right)^{\frac{3}{5}} \sin\left(x\right)}{x^2} dx$$

Hence

$$u_1 = \frac{\left(x^5\right)^{\frac{3}{5}} \cos\left(x\right)}{x^2} - \frac{\left(x^5\right)^{\frac{3}{5}} \sin\left(x\right)}{x^3}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\left(x^5\right)^{\frac{2}{5}} x^4 \sin\left(x\right)}{x^6} dx$$

Which simplifies to

$$u_2 = \int \frac{\left(x^5\right)^{\frac{2}{5}} \sin\left(x\right)}{x^2} dx$$

Hence

$$u_2 = -\frac{\left(x^5\right)^{\frac{2}{5}} \cos\left(x\right)}{x^2}$$

Which simplifies to

$$u_1 = \frac{\left(-\sin\left(x\right) + \cos\left(x\right) x\right) \left(x^5\right)^{\frac{3}{5}}}{x^3}$$

$$u_2 = -\frac{\left(x^5\right)^{\frac{2}{5}} \cos\left(x\right)}{x^2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-\sin\left(x\right) + \cos\left(x\right) x\right) x^2 - \cos\left(x\right) x^3$$

Which simplifies to

$$y_p(x) = -x^2 \sin(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left((c_1 x - \sin(x) + c_2)\, x^2\right) + \left(-x^2 \sin(x)\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = (c_1 x - \sin(x) + c_2)\, x^2 - x^2 \sin(x) \tag{1}$$

Verification of solutions

$$y = (c_1 x - \sin(x) + c_2)\, x^2 - x^2 \sin(x)$$

Verified OK.

1.12.6 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -4x, C = 6, f(x) = x^4 \sin(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. Solving for y_h from

$$x^2 y'' - 4xy' + 6y = 0$$

In normal form the ode

$$x^2 y'' - 4xy' + 6y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = -\frac{4}{x}$$
$$q(x) = \frac{6}{x^2}$$

Applying change of variables on the depndent variable $y = v(x) \, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{4n}{x^2} + \frac{6}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 3 \tag{6}$$

Substituting this value in (3) gives

$$v''(x) + \frac{2v'(x)}{x} = 0$$
$$v''(x) + \frac{2v'(x)}{x} = 0 \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{2u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= -\frac{2u}{x}
\end{aligned}$$

Where $f(x) = -\frac{2}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= -\frac{2}{x}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{2}{x}\,dx \\
\ln(u) &= -2\ln(x) + c_1 \\
u &= \mathrm{e}^{-2\ln(x)+c_1} \\
&= \frac{c_1}{x^2}
\end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\,dx + c_2 \\
&= -\frac{c_1}{x} + c_2
\end{aligned}$$

Hence

$$\begin{aligned}
y &= v(x)\,x^n \\
&= \left(-\frac{c_1}{x} + c_2\right)x^3 \\
&= (c_2 x - c_1)\,x^2
\end{aligned}$$

Now the particular solution to this ODE is found

$$x^2y'' - 4xy' + 6y = x^4\sin(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= x^2 \\
y_2 &= x^3
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$
W = \begin{vmatrix} x^2 & x^3 \\ \frac{d}{dx}(x^2) & \frac{d}{dx}(x^3) \end{vmatrix}
$$

Which gives

$$
W = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix}
$$

Therefore

$$
W = (x^2)(3x^2) - (x^3)(2x)
$$

Which simplifies to

$$
W = x^4
$$

Which simplifies to

$$
W = x^4
$$

Therefore Eq. (2) becomes

$$
u_1 = -\int \frac{x^7 \sin(x)}{x^6} \, dx
$$

Which simplifies to

$$u_1 = -\int x \sin(x)\, dx$$

Hence

$$u_1 = -\sin(x) + \cos(x)\, x$$

And Eq. (3) becomes

$$u_2 = \int \frac{\sin(x)\, x^6}{x^6}\, dx$$

Which simplifies to

$$u_2 = \int \sin(x)\, dx$$

Hence

$$u_2 = -\cos(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = (-\sin(x) + \cos(x)\, x)\, x^2 - \cos(x)\, x^3$$

Which simplifies to

$$y_p(x) = -x^2 \sin(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\left(-\frac{c_1}{x} + c_2\right) x^3\right) + \left(-x^2 \sin(x)\right) \\ &= -x^2 \sin(x) + \left(-\frac{c_1}{x} + c_2\right) x^3 \end{aligned}$$

Which simplifies to

$$y = x^2(-\sin(x) + c_2 x - c_1)$$

Summary

The solution(s) found are the following

$$y = x^2(-\sin(x) + c_2 x - c_1) \tag{1}$$

Verification of solutions

$$y = x^2(-\sin(x) + c_2x - c_1)$$

Verified OK.

1.12.7 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' - 4xy' + 6y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -4x \\ C &= 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 22: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4x}{x^2}\,dx} \\
&= z_1 e^{2\ln(x)} \\
&= z_1\left(x^2\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^2
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{4\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^2\right) + c_2\left(x^2(x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$
x^2y'' - 4xy' + 6y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_2x^3 + c_1x^2$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= x^2 \\ y_2 &= x^3 \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^2 & x^3 \\ \frac{d}{dx}(x^2) & \frac{d}{dx}(x^3) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix}$$

Therefore

$$W = (x^2)(3x^2) - (x^3)(2x)$$

Which simplifies to

$$W = x^4$$

Which simplifies to

$$W = x^4$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^7 \sin(x)}{x^6} dx$$

Which simplifies to

$$u_1 = -\int x \sin(x)\, dx$$

Hence

$$u_1 = -\sin(x) + \cos(x)\, x$$

And Eq. (3) becomes

$$u_2 = \int \frac{\sin(x)\, x^6}{x^6} dx$$

Which simplifies to

$$u_2 = \int \sin(x)\, dx$$

Hence

$$u_2 = -\cos(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = (-\sin(x) + \cos(x)\, x)\, x^2 - \cos(x)\, x^3$$

Which simplifies to

$$y_p(x) = -x^2 \sin(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_2 x^3 + c_1 x^2\right) + \left(-x^2 \sin(x)\right) \end{aligned}$$

Which simplifies to

$$y = x^2(c_2x + c_1) - x^2 \sin(x)$$

Summary

The solution(s) found are the following

$$y = x^2(c_2x + c_1) - x^2 \sin(x) \tag{1}$$

Verification of solutions

$$y = x^2(c_2x + c_1) - x^2 \sin(x)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   <- LODE of Euler type successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(x^2*diff(y(x),x$2)-4*x*diff(y(x),x)+6*y(x)=x^4*sin(x),y(x), singsol=all)
```

$$y(x) = x^2(c_2x - \sin(x) + c_1)$$

✓ Solution by Mathematica

Time used: 0.028 (sec). Leaf size: 20

```
DSolve[x^2*y''[x]-4*x*y'[x]+6*y[x]==x^4*Sin[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^2(-\sin(x) + c_2x + c_1)$$

1.13 problem Problem 19

Internal problem ID [2599]
Internal file name [`OUTPUT/2091_Sunday_June_05_2022_02_48_10_AM_51967793/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.2, Basic Ideas and Terminology. page 21
Problem number: Problem 19.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - (a + b)\, y' + aby = 0}$$

1.13.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -a-b, C = ab$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + (-a - b)\, \lambda\, e^{\lambda x} + ab\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + (-a - b)\, \lambda + ab = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -a - b, C = ab$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{a+b}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-a - b^2 - (4)\,(1)\,(ab)} \\
&= \frac{a}{2} + \frac{b}{2} \pm \frac{\sqrt{(b-a)^2}}{2}
\end{aligned}$$

Hence

$$\lambda_1 = \frac{a}{2} + \frac{b}{2} + \frac{\sqrt{(b-a)^2}}{2}$$
$$\lambda_2 = \frac{a}{2} + \frac{b}{2} - \frac{\sqrt{(b-a)^2}}{2}$$

Which simplifies to

$$\lambda_1 = \frac{a}{2} + \frac{b}{2} + \frac{\sqrt{(b-a)^2}}{2}$$
$$\lambda_2 = \frac{a}{2} + \frac{b}{2} - \frac{\sqrt{(b-a)^2}}{2}$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{\left(\frac{a}{2}+\frac{b}{2}+\frac{\sqrt{(b-a)^2}}{2}\right)x} + c_2 e^{\left(\frac{a}{2}+\frac{b}{2}-\frac{\sqrt{(b-a)^2}}{2}\right)x}$$

Or

$$y = c_1 \mathrm{e}^{\left(\frac{a}{2}+\frac{b}{2}+\frac{\sqrt{(b-a)^2}}{2}\right)x} + c_2 \mathrm{e}^{\left(\frac{a}{2}+\frac{b}{2}-\frac{\sqrt{(b-a)^2}}{2}\right)x}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{\left(\frac{a}{2}+\frac{b}{2}+\frac{\sqrt{(b-a)^2}}{2}\right)x} + c_2 \mathrm{e}^{\left(\frac{a}{2}+\frac{b}{2}-\frac{\sqrt{(b-a)^2}}{2}\right)x} \tag{1}$$

Verification of solutions

$$
y = c_1 \mathrm{e}^{\left(\frac{a}{2}+\frac{b}{2}+\frac{\sqrt{(b-a)^2}}{2}\right)x} + c_2 \mathrm{e}^{\left(\frac{a}{2}+\frac{b}{2}-\frac{\sqrt{(b-a)^2}}{2}\right)x}
$$

Verified OK.

1.13.2 Solving using Kovacic algorithm

Writing the ode as

$$
y'' + (-a-b)\, y' + aby = 0 \tag{1}
$$

$$
Ay'' + By' + Cy = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= -a-b \\
C &= ab
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = ye^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$
z''(x) = rz(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{a^2 - 2ab + b^2}{4} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= a^2 - 2ab + b^2 \\
t &= 4
\end{aligned}
$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}a^2 - \frac{1}{2}ab + \frac{1}{4}b^2\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 23: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}a^2 - \frac{1}{2}ab + \frac{1}{4}b^2$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{\frac{\sqrt{(b-a)^2}\, x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-a-b}{1}\, dx} \\
&= z_1 e^{\left(\frac{a}{2}+\frac{b}{2}\right)x} \\
&= z_1\left(\mathrm{e}^{\frac{(a+b)x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{\frac{(\operatorname{csgn}(-b+a)(-b+a)+a+b)x}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-a-b}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{(a+b)x}}{(y_1)^2}\, dx \\
&= y_1\left(-\frac{\mathrm{e}^{-\operatorname{csgn}(-b+a)(-b+a)x}}{\sqrt{(-b+a)^2}}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{\frac{(\operatorname{csgn}(-b+a)(-b+a)+a+b)x}{2}}\right) + c_2\left(\mathrm{e}^{\frac{(\operatorname{csgn}(-b+a)(-b+a)+a+b)x}{2}}\left(-\frac{\mathrm{e}^{-\operatorname{csgn}(-b+a)(-b+a)x}}{\sqrt{(-b+a)^2}}\right)\right)
\end{aligned}$$

Simplifying the solution $y = c_1 \mathrm{e}^{\frac{(\mathrm{csgn}(-b+a)(-b+a)+a+b)x}{2}} - \frac{c_2 \,\mathrm{csgn}(-b+a) \mathrm{e}^{-\frac{x(\mathrm{csgn}(-b+a)(-b+a)-a-b)}{2}}}{-b+a}$ to $y = c_1 \mathrm{e}^{ax} - \frac{c_2 \mathrm{e}^{bx}}{-b+a}$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{ax} - \frac{c_2 \mathrm{e}^{bx}}{-b+a} \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{ax} - \frac{c_2 \mathrm{e}^{bx}}{-b+a}$$

Verified OK.

1.13.3 Maple step by step solution

Let's solve

$y'' + (-a-b)\, y' + aby = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + (-a-b)\, r + ab = 0$

- Factor the characteristic polynomial

  $(b-r)\,(a-r) = 0$

- Roots of the characteristic polynomial

  $r = (a, b)$

- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^{ax}$

- 2nd solution of the ODE

  $y_2(x) = \mathrm{e}^{bx}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^{ax} + c_2 \mathrm{e}^{bx}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 17

```
dsolve(diff(y(x),x$2)-(a+b)*diff(y(x),x)+a*b*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{ax} + c_2 e^{bx}$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 22

```
DSolve[y''[x]-(a+b)*y'[x]+a*b*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 e^{ax} + c_1 e^{bx}$$

1.14 problem Problem 20

Internal problem ID [2600]
Internal file name [`OUTPUT/2092_Sunday_June_05_2022_02_48_12_AM_84334121/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.2, Basic Ideas and Terminology. page 21
Problem number: Problem 20.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 2y'a + ya^2 = 0}$$

1.14.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2a, C = a^2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 2a\lambda\,\mathrm{e}^{\lambda x} + a^2\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$a^2 - 2a\lambda + \lambda^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2a, C = a^2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2a}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(-2a)^2 - (4)(1)(a^2)} \\ &= a\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -a$. Therefore the solution is

$$y = c_1 e^{ax} + c_2 x\, e^{ax} \tag{1}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{ax} + c_2 x\, e^{ax} \tag{1}$$

Verification of solutions

$$y = c_1 e^{ax} + c_2 x\, e^{ax}$$

Verified OK.

1.14.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = -2a$. Therefore, there is an integrating factor given by

$$\begin{aligned}M(x) &= e^{\frac{1}{2}\int p\, dx} \\ &= e^{\int -2a\, dx} \\ &= e^{-ax}\end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned}(M(x)y)'' &= 0 \\ \left(e^{-ax} y\right)'' &= 0\end{aligned}$$

Integrating once gives

$$\left(\mathrm{e}^{-ax}y\right)' = c_1$$

Integrating again gives

$$\left(\mathrm{e}^{-ax}y\right) = c_1 x + c_2$$

Hence the solution is

$$y = \frac{c_1 x + c_2}{\mathrm{e}^{-ax}}$$

Or

$$y = c_1 x\,\mathrm{e}^{ax} + c_2\mathrm{e}^{ax}$$

Summary
The solution(s) found are the following

$$y = c_1 x\,\mathrm{e}^{ax} + c_2\mathrm{e}^{ax} \tag{1}$$

Verification of solutions

$$y = c_1 x\,\mathrm{e}^{ax} + c_2\mathrm{e}^{ax}$$

Verified OK.

1.14.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y'a + ya^2 = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2a \\ C &= a^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{0}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 0 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = 0 \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 25: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - -\infty\\ &= \infty\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{-2a}{1}\,dx}\\ &= z_1 e^{ax}\\ &= z_1(\mathrm{e}^{ax})\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{ax}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{-2a}{1}\,dx}}{(y_1)^2}\,dx\\ &= y_1 \int \frac{e^{2ax}}{(y_1)^2}\,dx\\ &= y_1(x)\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^{ax}) + c_2(\mathrm{e}^{ax}(x))
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{ax} + c_2 x\,\mathrm{e}^{ax} \tag{1}
$$

Verification of solutions

$$
y = c_1 \mathrm{e}^{ax} + c_2 x\,\mathrm{e}^{ax}
$$

Verified OK.

1.14.4 Maple step by step solution

Let's solve

$y'' - 2y'a + ya^2 = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $a^2 - 2ra + r^2 = 0$

- Factor the characteristic polynomial

  $(a - r)^2 = 0$

- Root of the characteristic polynomial

  $r = a$

- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^{ax}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,\mathrm{e}^{ax}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

$$y = c_1 e^{ax} + c_2 x \, e^{ax}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(diff(y(x),x$2)-2*a*diff(y(x),x)+a^2*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{ax}(c_2 x + c_1)$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 18

```
DSolve[y''[x]-2*a*y'[x]+a^2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{ax}(c_2 x + c_1)$$

1.15 problem Problem 21

Internal problem ID [2601]
Internal file name [`OUTPUT/2093_Sunday_June_05_2022_02_48_14_AM_32397012/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.2, Basic Ideas and Terminology. page 21
Problem number: Problem 21.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 2y'a + (a^2 + b^2)\, y = 0}$$

1.15.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2a, C = a^2 + b^2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 2a\lambda\, e^{\lambda x} + (a^2 + b^2)\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$a^2 - 2a\lambda + b^2 + \lambda^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2a, C = a^2 + b^2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2a}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-2a^2 - (4)(1)(a^2 + b^2)} \\ &= a \pm \sqrt{-b^2}\end{aligned}$$

Hence

$$\lambda_1 = a + \sqrt{-b^2}$$
$$\lambda_2 = a - \sqrt{-b^2}$$

Which simplifies to

$$\lambda_1 = a + \sqrt{-b^2}$$
$$\lambda_2 = a - \sqrt{-b^2}$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{\left(a+\sqrt{-b^2}\right)x} + c_2 e^{\left(a-\sqrt{-b^2}\right)x}$$

Or

$$y = c_1 \mathrm{e}^{\left(a+\sqrt{-b^2}\right)x} + c_2 \mathrm{e}^{\left(a-\sqrt{-b^2}\right)x}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{\left(a+\sqrt{-b^2}\right)x} + c_2 \mathrm{e}^{\left(a-\sqrt{-b^2}\right)x} \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{\left(a+\sqrt{-b^2}\right)x} + c_2 \mathrm{e}^{\left(a-\sqrt{-b^2}\right)x}$$

Verified OK.

1.15.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y'a + (a^2 + b^2)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2a \\ C &= a^2 + b^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-b^2}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -b^2 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = (-b^2)\, z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 27: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -b^2$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{x\sqrt{-b^2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2a}{1}\,dx} \end{aligned}$$

$$
\begin{aligned}
&= z_1 e^{ax} \\
&= z_1(\mathrm{e}^{ax})
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{\left(a+\sqrt{-b^2}\right)x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2a}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{2ax}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{\sqrt{-b^2}\,\mathrm{e}^{-2x\sqrt{-b^2}}}{2b^2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{\left(a+\sqrt{-b^2}\right)x}\right) + c_2 \left(\mathrm{e}^{\left(a+\sqrt{-b^2}\right)x} \left(\frac{\sqrt{-b^2}\,\mathrm{e}^{-2x\sqrt{-b^2}}}{2b^2}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{\left(a+\sqrt{-b^2}\right)x} + \frac{c_2\sqrt{-b^2}\,\mathrm{e}^{\left(a-\sqrt{-b^2}\right)x}}{2b^2} \tag{1}
$$

Verification of solutions

$$
y = c_1 \mathrm{e}^{\left(a+\sqrt{-b^2}\right)x} + \frac{c_2\sqrt{-b^2}\,\mathrm{e}^{\left(a-\sqrt{-b^2}\right)x}}{2b^2}
$$

Verified OK.

1.15.3 Maple step by step solution

Let's solve

$y'' - 2y'a + (a^2 + b^2)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $a^2 - 2ra + b^2 + r^2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(2a)\pm\left(\sqrt{-4b^2}\right)}{2}$

- Roots of the characteristic polynomial

  $r = \left(a - \sqrt{-b^2}, a + \sqrt{-b^2}\right)$

- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^{\left(a-\sqrt{-b^2}\right)x}$

- 2nd solution of the ODE

  $y_2(x) = \mathrm{e}^{\left(a+\sqrt{-b^2}\right)x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^{\left(a-\sqrt{-b^2}\right)x} + c_2 \mathrm{e}^{\left(a+\sqrt{-b^2}\right)x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
dsolve(diff(y(x),x$2)-2*a*diff(y(x),x)+(a^2+b^2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{ax}(c_1 \sin(bx) + c_2 \cos(bx))$$

✓ Solution by Mathematica

Time used: 0.018 (sec). Leaf size: 31

```
DSolve[y''[x]-2*a*y'[x]+(a^2+b^2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{x(a-ib)}\left(c_2 e^{2ibx} + c_1\right)$$

1.16 problem Problem 22

Internal problem ID [2602]
Internal file name [`OUTPUT/2094_Sunday_June_05_2022_02_48_15_AM_92038629/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.2, Basic Ideas and Terminology. page 21
Problem number: Problem 22.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - y' - 6y = 0}$$

1.16.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -1, C = -6$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - \lambda \,\mathrm{e}^{\lambda x} - 6 \,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - \lambda - 6 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -1, C = -6$ into the above gives

$$
\begin{aligned}
\lambda_{1,2} &= \frac{1}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{-1^2 - (4)(1)(-6)} \\
&= \frac{1}{2} \pm \frac{5}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
\lambda_1 &= \frac{1}{2} + \frac{5}{2} \\
\lambda_2 &= \frac{1}{2} - \frac{5}{2}
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
\lambda_1 &= 3 \\
\lambda_2 &= -2
\end{aligned}
$$

Since roots are real and distinct, then the solution is

$$
\begin{aligned}
y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\
y &= c_1 e^{(3)x} + c_2 e^{(-2)x}
\end{aligned}
$$

Or

$$
y = c_1 \mathrm{e}^{3x} + c_2 \mathrm{e}^{-2x}
$$

Summary
The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{3x} + c_2 \mathrm{e}^{-2x} \tag{1}
$$

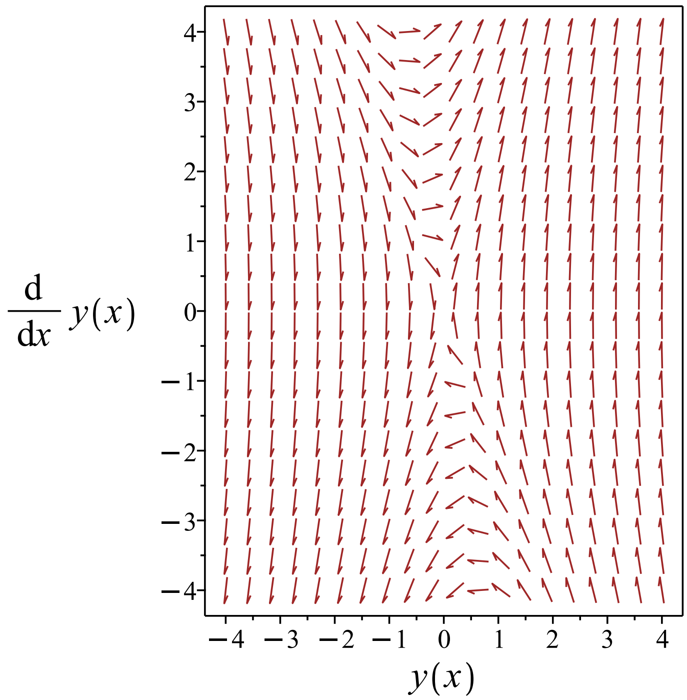


Figure 20: Slope field plot

Verification of solutions

$$y = c_1 e^{3x} + c_2 e^{-2x}$$

Verified OK.

1.16.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y' - 6y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -1 \\ C &= -6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{25}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 25 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{25z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 29: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{25}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{5x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-1}{1}\,dx} \\
&= z_1 e^{\frac{x}{2}} \\
&= z_1 \left(\mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{x}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{\mathrm{e}^{5x}}{5}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-2x}\right) + c_2\left(\mathrm{e}^{-2x}\left(\frac{\mathrm{e}^{5x}}{5}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-2x} + \frac{c_2 \mathrm{e}^{3x}}{5} \tag{1}
$$

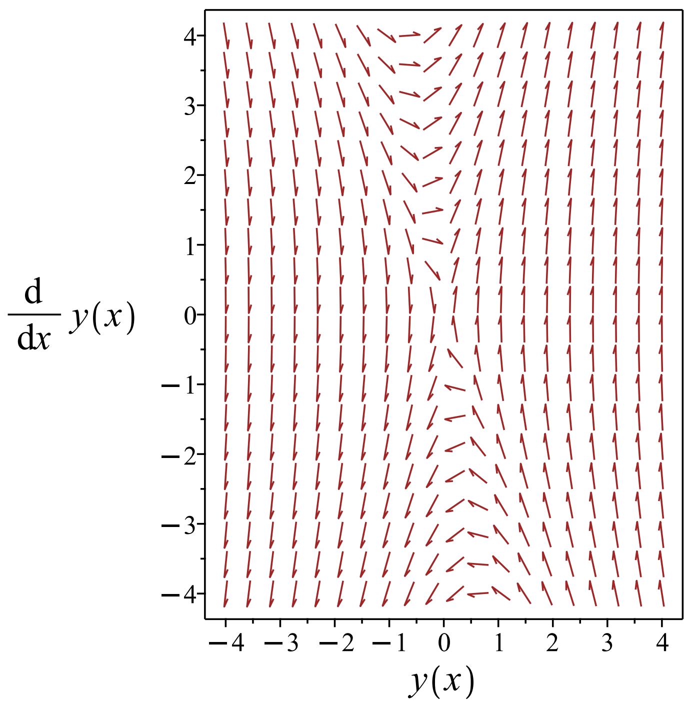


Figure 21: Slope field plot

Verification of solutions

$$
y = c_1 \mathrm{e}^{-2x} + \frac{c_2 \mathrm{e}^{3x}}{5}
$$

Verified OK.

1.16.3 Maple step by step solution

Let's solve

$y'' - y' - 6y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - r - 6 = 0$

- Factor the characteristic polynomial

  $(r + 2)(r - 3) = 0$

- Roots of the characteristic polynomial

  $r = (-2, 3)$

- 1st solution of the ODE

  $y_1(x) = e^{-2x}$

- 2nd solution of the ODE

  $y_2(x) = e^{3x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{-2x} + c_2 e^{3x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(y(x),x$2)-diff(y(x),x)-6*y(x)=0,y(x), singsol=all)
```

$$y(x) = \left(c_1 e^{5x} + c_2\right) e^{-2x}$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 22

```
DSolve[y''[x]-y'[x]-6*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-2x}\left(c_2 e^{5x} + c_1\right)$$

1.17 problem Problem 23

Internal problem ID [2603]
Internal file name [`OUTPUT/2095_Sunday_June_05_2022_02_48_17_AM_78263782/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.2, Basic Ideas and Terminology. page 21
Problem number: Problem 23.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 6y' + 9y = 0}$$

1.17.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 6, C = 9$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 6\lambda\, \mathrm{e}^{\lambda x} + 9\, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 6\lambda + 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 6, C = 9$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-6}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(6)^2 - (4)(1)(9)} \\ &= -3\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 3$. Therefore the solution is

$$y = c_1 e^{-3x} + c_2 x e^{-3x} \tag{1}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-3x} + c_2 x e^{-3x} \tag{1}$$

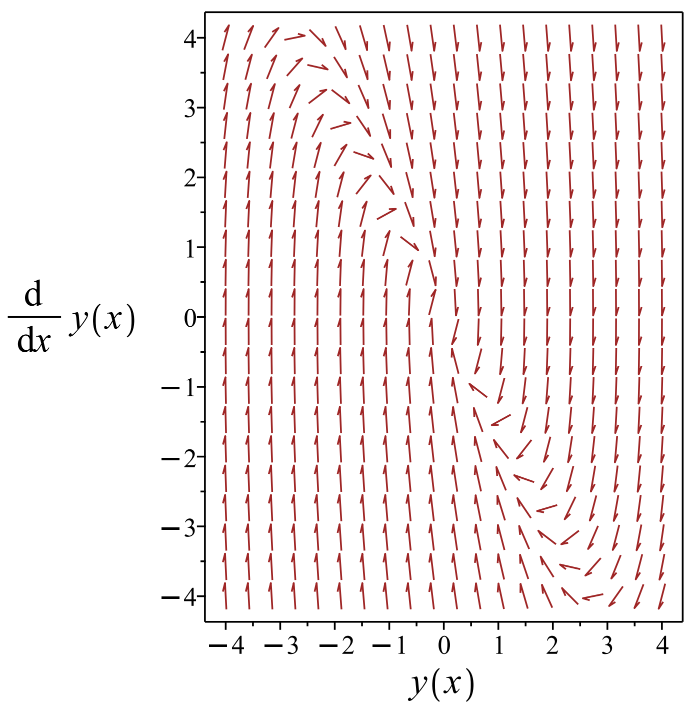


Figure 22: Slope field plot

Verification of solutions

$$y = c_1 e^{-3x} + c_2 x e^{-3x}$$

Verified OK.

1.17.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$
y'' + p(x)\,y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)
$$

Where $p(x) = 6$. Therefore, there is an integrating factor given by

$$
\begin{aligned}
M(x) &= e^{\frac{1}{2}\int p\,dx} \\
&= e^{\int 6\,dx} \\
&= \mathrm{e}^{3x}
\end{aligned}
$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$
\begin{aligned}
(M(x)y)'' &= 0 \\
\left(\mathrm{e}^{3x} y\right)'' &= 0
\end{aligned}
$$

Integrating once gives

$$
\left(\mathrm{e}^{3x} y\right)' = c_1
$$

Integrating again gives

$$
\left(\mathrm{e}^{3x} y\right) = c_1 x + c_2
$$

Hence the solution is

$$
y = \frac{c_1 x + c_2}{\mathrm{e}^{3x}}
$$

Or

$$
y = c_1 x\,\mathrm{e}^{-3x} + c_2 \mathrm{e}^{-3x}
$$

Summary
The solution(s) found are the following

$$
y = c_1 x\,\mathrm{e}^{-3x} + c_2 \mathrm{e}^{-3x} \tag{1}
$$

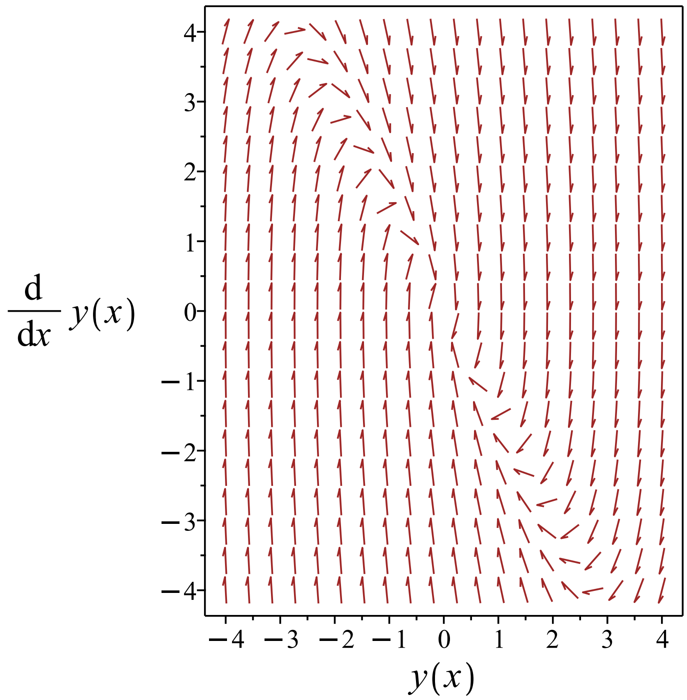


Figure 23: Slope field plot

Verification of solutions

$$y = c_1 x \, \mathrm{e}^{-3x} + c_2 \mathrm{e}^{-3x}$$

Verified OK.

1.17.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 6y' + 9y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 6 \\ C &= 9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 31: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - -\infty\\ &= \infty\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{6}{1}\,dx}\\ &= z_1 e^{-3x}\\ &= z_1\left(\mathrm{e}^{-3x}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-3x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{6}{1}\,dx}}{(y_1)^2}\,dx\\ &= y_1 \int \frac{e^{-6x}}{(y_1)^2}\,dx\\ &= y_1(x)\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-3x}\right) + c_2\left(\mathrm{e}^{-3x}(x)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-3x} + c_2 x\,\mathrm{e}^{-3x} \tag{1}
$$

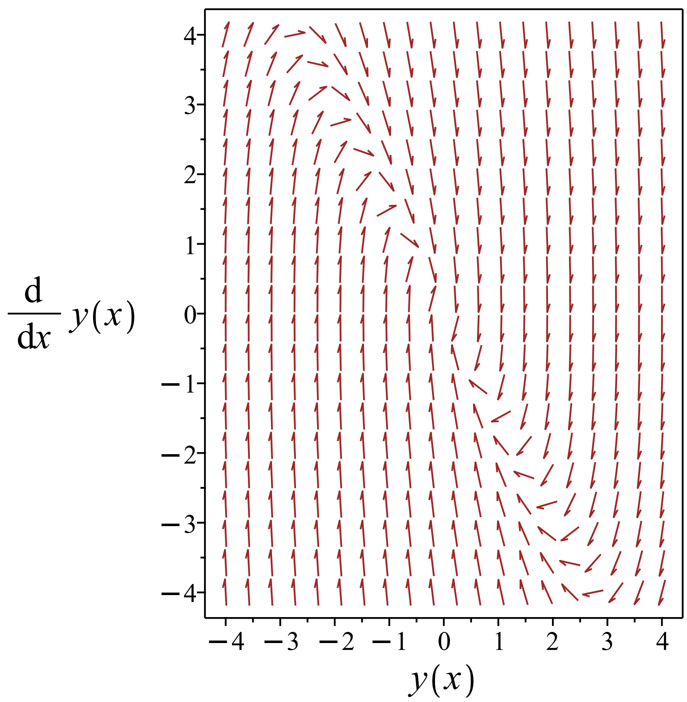


Figure 24: Slope field plot

Verification of solutions

$$
y = c_1 \mathrm{e}^{-3x} + c_2 x\,\mathrm{e}^{-3x}
$$

Verified OK.

1.17.4 Maple step by step solution

Let's solve

$y'' + 6y' + 9y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + 6r + 9 = 0$

- Factor the characteristic polynomial

  $(r+3)^2 = 0$

- Root of the characteristic polynomial

  $r = -3$

- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^{-3x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,\mathrm{e}^{-3x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^{-3x} + c_2 x\,\mathrm{e}^{-3x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(diff(y(x),x$2)+6*diff(y(x),x)+9*y(x)=0,y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{-3x}(c_2 x + c_1)$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 18

```
DSolve[y''[x]+6*y'[x]+9*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-3x}(c_2 x + c_1)$$

1.18 problem Problem 24

Internal problem ID [2604]
Internal file name [`OUTPUT/2096_Sunday_June_05_2022_02_48_18_AM_5147740/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.2, Basic Ideas and Terminology. page 21
Problem number: Problem 24.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "exact linear second order ode", "second_order_integrable_as_is", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{x^2y'' + xy' - y = 0}$$

1.18.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} + xrx^{r-1} - x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r + r\,x^r - x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) + r - 1 = 0$$

Or

$$r^2 - 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= -1 \\ r_2 &= 1 \end{aligned}$$

Since the roots are real and distinct, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = \frac{c_1}{x} + c_2 x$$

Summary
The solution(s) found are the following

$$y = \frac{c_1}{x} + c_2 x \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x} + c_2 x$$

Verified OK.

1.18.2 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$x^2y'' + xy' - y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= -\frac{1}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\ &= \int \mathrm{e}^{-\left(\int \frac{1}{x}dx\right)}dx \\ &= \int e^{-\ln(x)}\,dx \\ &= \int \frac{1}{x}dx \\ &= \ln(x) \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}q_1(\tau) &= \frac{q(x)}{\tau'(x)^2}\\&= \frac{-\frac{1}{x^2}}{\frac{1}{x^2}}\\&= -1\end{aligned}\tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0\\\frac{d^2}{d\tau^2}y(\tau) - y(\tau) &= 0\end{aligned}$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda\tau} - e^{\lambda\tau} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-1)}\\&= \pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +1\\\lambda_2 &= -1\end{aligned}$$

Which simplifies to

$$\lambda_1 = 1$$
$$\lambda_2 = -1$$

Since roots are real and distinct, then the solution is

$$y(\tau) = c_1 e^{\lambda_1 \tau} + c_2 e^{\lambda_2 \tau}$$
$$y(\tau) = c_1 e^{(1)\tau} + c_2 e^{(-1)\tau}$$

Or

$$y(\tau) = c_1 \mathrm{e}^{\tau} + c_2 \mathrm{e}^{-\tau}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{c_1 x^2 + c_2}{x}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1 x^2 + c_2}{x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x^2 + c_2}{x}$$

Verified OK.

1.18.3 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$x^2 y'' + x y' - y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = -\frac{1}{x^2}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}\tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{-\frac{1}{x^2}}}{c} \\ \tau'' &= \frac{1}{c\sqrt{-\frac{1}{x^2}}\,x^3}\end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned}p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\ &= \frac{\frac{1}{c\sqrt{-\frac{1}{x^2}}\,x^3} + \frac{1}{x}\frac{\sqrt{-\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{-\frac{1}{x^2}}}{c}\right)^2} \\ &= 0\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + c^2 y(\tau) &= 0\end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int \sqrt{-\frac{1}{x^2}}dx}{c} \\
&= \frac{\sqrt{-\frac{1}{x^2}}\,x\ln(x)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = \frac{(ic_2 + c_1)\,x^2 - ic_2 + c_1}{2x}
$$

Summary

The solution(s) found are the following

$$
y = \frac{(ic_2 + c_1)\,x^2 - ic_2 + c_1}{2x} \tag{1}
$$

Verification of solutions

$$
y = \frac{(ic_2 + c_1)\,x^2 - ic_2 + c_1}{2x}
$$

Verified OK.

1.18.4 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$
x^2y'' + xy' - y = 0 \tag{1}
$$

Becomes

$$
y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}
$$

Where

$$
\begin{aligned}
p(x) &= \frac{1}{x} \\
q(x) &= -\frac{1}{x^2}
\end{aligned}
$$

Applying change of variables on the depndent variable $y = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$
v''(x) + \left(\frac{2n}{x} + p\right)v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right)v(x) = 0 \tag{3}
$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{n}{x^2} - \frac{1}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 1 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{3v'(x)}{x} &= 0 \\ v''(x) + \frac{3v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{3u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{3u}{x} \end{aligned}$$

Where $f(x) = -\frac{3}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= -\frac{3}{x}\, dx \\ \int \frac{1}{u}\, du &= \int -\frac{3}{x}\, dx \\ \ln(u) &= -3\ln(x) + c_1 \\ u &= e^{-3\ln(x)+c_1} \\ &= \frac{c_1}{x^3} \end{aligned}$$

Now that $u(x)$ is known, then

$$
\begin{aligned}
v'(x) &= u(x)\\
v(x) &= \int u(x)\,dx + c_2\\
&= -\frac{c_1}{2x^2} + c_2
\end{aligned}
$$

Hence

$$
\begin{aligned}
y &= v(x)\,x^n\\
&= \left(-\frac{c_1}{2x^2} + c_2\right) x\\
&= \left(-\frac{c_1}{2x^2} + c_2\right) x
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \left(-\frac{c_1}{2x^2} + c_2\right) x \tag{1}
$$

Verification of solutions

$$
y = \left(-\frac{c_1}{2x^2} + c_2\right) x
$$

Verified OK.

1.18.5 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$
\begin{aligned}
\int \left(x^2y'' + xy' - y\right) dx &= 0\\
y'x^2 - yx &= c_1
\end{aligned}
$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
y' + p(x)y = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{1}{x}\\
q(x) &= \frac{c_1}{x^2}
\end{aligned}
$$

Hence the ode is

$$y' - \frac{y}{x} = \frac{c_1}{x^2}$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int -\frac{1}{x}dx} \\ &= \frac{1}{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{c_1}{x^2}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x}\right) &= \left(\frac{1}{x}\right)\left(\frac{c_1}{x^2}\right) \\ \mathrm{d}\left(\frac{y}{x}\right) &= \left(\frac{c_1}{x^3}\right)\,\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{y}{x} &= \int \frac{c_1}{x^3}\,\mathrm{d}x \\ \frac{y}{x} &= -\frac{c_1}{2x^2} + c_2\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$y = -\frac{c_1}{2x} + c_2 x$$

Summary
The solution(s) found are the following

$$y = -\frac{c_1}{2x} + c_2 x \tag{1}$$

Verification of solutions

$$y = -\frac{c_1}{2x} + c_2 x$$

Verified OK.

1.18.6 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned} y' &= B'v + v'B \\ y'' &= B''v + B'v' + v''B + v'B' \\ &= v''B + 2v' + B' + B''v \end{aligned}$$

And now the original ode becomes

$$\begin{aligned} A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\ ABv'' + \left(2AB' + B^2\right) v' + \left(AB'' + BB' + CB\right) v &= 0 \end{aligned} \tag{1}$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + \left(2AB' + B^2\right) v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + \left(2AB' + B^2\right) u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= -1 \\ F &= 0 \end{aligned}$$

The above shows that for this ode

$$\begin{aligned} AB'' + BB' + CB &= \left(x^2\right)(0) + (x)(1) + (-1)(x) \\ &= 0 \end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$x^3v'' + (3x^2)\, v' = 0$$

Now by applying $v' = u$ the above becomes

$$x^2(u'(x)\, x + 3u(x)) = 0$$

Which is now solved for u. In canonical form the ODE is

$$\begin{aligned}
u' &= F(x, u) \\
&= f(x)g(u) \\
&= -\frac{3u}{x}
\end{aligned}$$

Where $f(x) = -\frac{3}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\, du &= -\frac{3}{x}\, dx \\
\int \frac{1}{u}\, du &= \int -\frac{3}{x}\, dx \\
\ln(u) &= -3\ln(x) + c_1 \\
u &= \mathrm{e}^{-3\ln(x)+c_1} \\
&= \frac{c_1}{x^3}
\end{aligned}$$

The ode for v now becomes

$$\begin{aligned}
v' &= u \\
&= \frac{c_1}{x^3}
\end{aligned}$$

Which is now solved for v. Integrating both sides gives

$$\begin{aligned}
v(x) &= \int \frac{c_1}{x^3}\, \mathrm{d}x \\
&= -\frac{c_1}{2x^2} + c_2
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y(x) &= Bv \\
&= (x)\left(-\frac{c_1}{2x^2} + c_2\right) \\
&= \left(-\frac{c_1}{2x^2} + c_2\right) x
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = \left(-\frac{c_1}{2x^2} + c_2\right) x \tag{1}$$

Verification of solutions

$$y = \left(-\frac{c_1}{2x^2} + c_2\right) x$$

Verified OK.

1.18.7 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$x^2 y'' + x y' - y = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\int (x^2 y'' + x y' - y)\, dx = 0$$
$$y' x^2 - y x = c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{1}{x}$$
$$q(x) = \frac{c_1}{x^2}$$

Hence the ode is

$$y' - \frac{y}{x} = \frac{c_1}{x^2}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{1}{x} dx}$$
$$= \frac{1}{x}$$

The ode becomes

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{c_1}{x^2}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x}\right) &= \left(\frac{1}{x}\right)\left(\frac{c_1}{x^2}\right) \\
\mathrm{d}\left(\frac{y}{x}\right) &= \left(\frac{c_1}{x^3}\right)\,\mathrm{d}x
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\frac{y}{x} &= \int \frac{c_1}{x^3}\,\mathrm{d}x \\
\frac{y}{x} &= -\frac{c_1}{2x^2} + c_2
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$y = -\frac{c_1}{2x} + c_2 x$$

<u>Summary</u>
The solution(s) found are the following

$$y = -\frac{c_1}{2x} + c_2 x \tag{1}$$

<u>Verification of solutions</u>

$$y = -\frac{c_1}{2x} + c_2 x$$

Verified OK.

1.18.8 Solving using Kovacic algorithm

Writing the ode as

$$x^2 y'' + x y' - y = 0 \tag{1}$$

$$A y'' + B y' + C y = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}
A &= x^2 \\
B &= x \\
C &= -1
\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 33: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2x} + (-)(0) \\ &= -\frac{1}{2x} \\ &= -\frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2x}\right)(0) + \left(\left(\frac{1}{2x^2}\right) + \left(-\frac{1}{2x}\right)^2 - \left(\frac{3}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int -\frac{1}{2x} dx} \\ &= \frac{1}{\sqrt{x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{x}{x^2} \, dx} \\ &= z_1 e^{-\frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{1}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{x^2}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{x}\right) + c_2 \left(\frac{1}{x}\left(\frac{x^2}{2}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1}{x} + \frac{c_2 x}{2} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1}{x} + \frac{c_2 x}{2}
$$

Verified OK.

1.18.9 Solving as exact linear second order ode ode

An ode of the form

$$
p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)
$$

is exact if

$$
p''(x) - q'(x) + r(x) = 0 \tag{1}
$$

For the given ode we have

$$
\begin{aligned}
p(x) &= x^2 \\
q(x) &= x \\
r(x) &= -1 \\
s(x) &= 0
\end{aligned}
$$

Hence

$$
\begin{aligned}
p''(x) &= 2 \\
q'(x) &= 1
\end{aligned}
$$

Therefore (1) becomes

$$
2-(1)+(-1)=0
$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$
(p(x)\,y'+(q(x)-p'(x))\,y)'=s(x)
$$

Integrating gives

$$
p(x)\,y'+(q(x)-p'(x))\,y=\int s(x)\;dx
$$

Substituting the above values for p, q, r, s gives

$$
y'x^2-yx=c_1
$$

We now have a first order ode to solve which is

$$
y'x^2-yx=c_1
$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
y'+p(x)y=q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{1}{x} \\
q(x) &= \frac{c_1}{x^2}
\end{aligned}
$$

Hence the ode is

$$
y'-\frac{y}{x}=\frac{c_1}{x^2}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{1}{x}dx} \\
&= \frac{1}{x}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{c_1}{x^2}\right) \\
\frac{d}{dx}\left(\frac{y}{x}\right) &= \left(\frac{1}{x}\right)\left(\frac{c_1}{x^2}\right) \\
d\left(\frac{y}{x}\right) &= \left(\frac{c_1}{x^3}\right) dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{y}{x} &= \int \frac{c_1}{x^3}\, dx \\
\frac{y}{x} &= -\frac{c_1}{2x^2} + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$
y = -\frac{c_1}{2x} + c_2 x
$$

Summary
The solution(s) found are the following

$$
y = -\frac{c_1}{2x} + c_2 x \tag{1}
$$

Verification of solutions

$$
y = -\frac{c_1}{2x} + c_2 x
$$

Verified OK.

1.18.10 Maple step by step solution

Let's solve

$x^2 y'' + x y' - y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y'}{x} + \frac{y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} - \frac{y}{x^2} = 0$

- Multiply by denominators of the ODE

  $x^2y'' + xy' - y = 0$

- Make a change of variables

  $t = \ln(x)$

- ☐ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right)t'(x)$

  - Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2}y(t)\right)t'(x)^2 + t''(x)\left(\frac{d}{dt}y(t)\right)$

  - Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) + \frac{d}{dt}y(t) - y(t) = 0$

- Simplify

  $\frac{d^2}{dt^2}y(t) - y(t) = 0$

- Characteristic polynomial of ODE

  $r^2 - 1 = 0$

- Factor the characteristic polynomial

  $(r-1)(r+1) = 0$

- Roots of the characteristic polynomial

  $r = (-1, 1)$

- 1st solution of the ODE

  $y_1(t) = e^{-t}$

- 2nd solution of the ODE

  $y_2(t) = e^{t}$

- General solution of the ODE

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

- Substitute in solutions

$$y(t) = c_1 e^{-t} + c_2 e^{t}$$

- Change variables back using $t = \ln(x)$

$$y = \frac{c_1}{x} + c_2 x$$

- Simplify

$$y = \frac{c_1}{x} + c_2 x$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)-y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_2 x^2 + c_1}{x}$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 16

```
DSolve[x^2*y''[x]+x*y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_1}{x} + c_2 x$$

1.19 problem Problem 25

Internal problem ID [2605]
Internal file name [`OUTPUT/2097_Sunday_June_05_2022_02_48_20_AM_48551106/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.2, Basic Ideas and Terminology. page 21
Problem number: Problem 25.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{x^2y'' + 5xy' + 4y = 0}$$

1.19.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} + 5xrx^{r-1} + 4x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r + 5r\,x^r + 4x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) + 5r + 4 = 0$$

Or

$$r^2 + 4r + 4 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= -2 \\ r_2 &= -2 \end{aligned}$$

Since the roots are equal, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = x^r$ and $y_2 = x^r \ln(x)$. Hence

$$y = \frac{c_1}{x^2} + \frac{c_2 \ln(x)}{x^2}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{x^2} + \frac{c_2 \ln(x)}{x^2} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x^2} + \frac{c_2 \ln(x)}{x^2}$$

Verified OK.

1.19.2 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$x^2 y'' + 5xy' + 4y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{5}{x} \\ q(x) &= \frac{4}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\
&= \int \mathrm{e}^{-\left(\int \frac{5}{x}dx\right)}dx \\
&= \int e^{-5\ln(x)}\,dx \\
&= \int \frac{1}{x^5}dx \\
&= -\frac{1}{4x^4}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{4}{x^2}}{\frac{1}{x^{10}}} \\
&= 4x^8
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0$$

$$\frac{d^2}{d\tau^2}y(\tau) + 4x^8 y(\tau) = 0$$

But in terms of τ

$$4x^8 = \frac{1}{4\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{y(\tau)}{4\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$4\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$4\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + \tau^r = 0$$

Simplifying gives

$$4r(r-1)\,\tau^r + 0\,\tau^r + \tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$4r(r-1) + 0 + 1 = 0$$

Or

$$4r^2 - 4r + 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{2}$$
$$r_2 = \frac{1}{2}$$

Since the roots are equal, then the general solution is

$$y(\tau) = c_1y_1 + c_2y_2$$

Where $y_1 = \tau^r$ and $y_2 = \tau^r \ln(\tau)$. Hence

$$y(\tau) = c_1\sqrt{\tau} + c_2\sqrt{\tau}\,\ln(\tau)$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{\left(-2c_2\ln(2) + c_2\ln\left(-\frac{1}{x^4}\right) + c_1\right)\sqrt{-\frac{1}{x^4}}}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(-2c_2 \ln(2) + c_2 \ln\left(-\frac{1}{x^4}\right) + c_1\right)\sqrt{-\frac{1}{x^4}}}{2} \tag{1}$$

Verification of solutions

$$y = \frac{\left(-2c_2 \ln(2) + c_2 \ln\left(-\frac{1}{x^4}\right) + c_1\right)\sqrt{-\frac{1}{x^4}}}{2}$$

Verified OK.

1.19.3 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$x^2y'' + 5xy' + 4y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{5}{x} \\ q(x) &= \frac{4}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{2\sqrt{\frac{1}{x^2}}}{c} \\ \tau'' &= -\frac{2}{c\sqrt{\frac{1}{x^2}}\, x^3} \end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$
\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{-\frac{2}{c\sqrt{\frac{1}{x^2}}\,x^3} + \frac{5}{x}\frac{2\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{2\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\
&= 2c
\end{aligned}
$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + 2c\left(\frac{d}{d\tau}y(\tau)\right) + c^2 y(\tau) &= 0 \qquad (7)
\end{aligned}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
y(\tau) = \mathrm{e}^{-c\tau}c_1
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int 2\sqrt{\frac{1}{x^2}}dx}{c} \\
&= \frac{2\sqrt{\frac{1}{x^2}}\,x\ln(x)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = \frac{c_1}{x^2}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1}{x^2} \qquad (1)
$$

Verification of solutions

$$
y = \frac{c_1}{x^2}
$$

Verified OK.

1.19.4 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$x^2y'' + 5xy' + 4y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{5}{x} \\ q(x) &= \frac{4}{x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{5n}{x^2} + \frac{4}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = -2 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{v'(x)}{x} &= 0 \\ v''(x) + \frac{v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= -\frac{u}{x}
\end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= -\frac{1}{x}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{1}{x}\,dx \\
\ln(u) &= -\ln(x) + c_1 \\
u &= \mathrm{e}^{-\ln(x)+c_1} \\
&= \frac{c_1}{x}
\end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\,dx + c_2 \\
&= c_1 \ln(x) + c_2
\end{aligned}$$

Hence

$$\begin{aligned}
y &= v(x)\,x^n \\
&= \frac{c_1 \ln(x) + c_2}{x^2} \\
&= \frac{c_1 \ln(x) + c_2}{x^2}
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 \ln(x) + c_2}{x^2} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \ln(x) + c_2}{x^2}$$

Verified OK.

1.19.5 Solving using Kovacic algorithm

Writing the ode as

$$x^2 y'' + 5xy' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 5x \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 35: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_{\infty}^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_{\infty}^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_{\infty}^{-} = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_{\infty}^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_{\infty}$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x} + (-)\,(0) \\
&= \frac{1}{2x} \\
&= \frac{1}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \frac{1}{2x} dx} \\
&= \sqrt{x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{5x}{x^2} \, dx} \\
&= z_1 e^{-\frac{5 \ln(x)}{2}} \\
&= z_1 \left(\frac{1}{x^{\frac{5}{2}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{1}{x^2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{5x}{x^2} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-5 \ln(x)}}{(y_1)^2} \, dx \\
&= y_1(\ln(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{x^2}\right) + c_2 \left(\frac{1}{x^2}(\ln(x))\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{x^2} + \frac{c_2 \ln(x)}{x^2} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x^2} + \frac{c_2 \ln(x)}{x^2}$$

Verified OK.

1.19.6 Maple step by step solution

Let's solve

$x^2y'' + 5xy' + 4y = 0$

- Highest derivative means the order of the ODE is 2

  y''
- Isolate 2nd derivative

  $y'' = -\frac{5y'}{x} - \frac{4y}{x^2}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{5y'}{x} + \frac{4y}{x^2} = 0$
- Multiply by denominators of the ODE

  $x^2y'' + 5xy' + 4y = 0$
- Make a change of variables

  $t = \ln(x)$

☐ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right) t'(x)$
  - Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$
  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2}y(t)\right) t'(x)^2 + t''(x)\left(\frac{d}{dt}y(t)\right)$
  - Compute derivative

$y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

Substitute the change of variables back into the ODE

$x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) + 5\frac{d}{dt}y(t) + 4y(t) = 0$

- Simplify

  $\frac{d^2}{dt^2}y(t) + 4\frac{d}{dt}y(t) + 4y(t) = 0$
- Characteristic polynomial of ODE

  $r^2 + 4r + 4 = 0$
- Factor the characteristic polynomial

  $(r+2)^2 = 0$
- Root of the characteristic polynomial

  $r = -2$
- 1st solution of the ODE

  $y_1(t) = \mathrm{e}^{-2t}$
- Repeated root, multiply $y_1(t)$ by t to ensure linear independence

  $y_2(t) = t\,\mathrm{e}^{-2t}$
- General solution of the ODE

  $y(t) = c_1 y_1(t) + c_2 y_2(t)$
- Substitute in solutions

  $y(t) = c_1\mathrm{e}^{-2t} + c_2 t\,\mathrm{e}^{-2t}$
- Change variables back using $t = \ln(x)$

  $y = \frac{c_1}{x^2} + \frac{c_2 \ln(x)}{x^2}$
- Simplify

  $y = \frac{c_1}{x^2} + \frac{c_2 \ln(x)}{x^2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(x^2*diff(y(x),x$2)+5*x*diff(y(x),x)+4*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_2 \ln(x) + c_1}{x^2}$$

✓ Solution by Mathematica

Time used: 0.016 (sec). Leaf size: 18

```
DSolve[x^2*y''[x]+5*x*y'[x]+4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2c_2 \log(x) + c_1}{x^2}$$

1.20 problem Problem 28

Internal problem ID [2606]
Internal file name [`OUTPUT/2098_Sunday_June_05_2022_02_48_21_AM_8392845/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.2, Basic Ideas and Terminology. page 21
Problem number: Problem 28.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[`y=_G(x,y')`]
```

$$\boxed{y' - \frac{e^x - \sin(y)}{x \cos(y)} = 0}$$

1.20.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
(x \cos(y))\, \mathrm{d}y &= (\mathrm{e}^{x} - \sin(y))\, \mathrm{d}x \\
(-\mathrm{e}^{x} + \sin(y))\, \mathrm{d}x + (x \cos(y))\, \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -\mathrm{e}^{x} + \sin(y) \\
N(x, y) &= x \cos(y)
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-\mathrm{e}^{x} + \sin(y)) \\
&= \cos(y)
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x \cos(y)) \\
&= \cos(y)
\end{aligned}
$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x}=M \tag{1}$$
$$\frac{\partial \phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int M\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int -\mathrm{e}^{x}+\sin(y)\,\mathrm{d}x$$
$$\phi=-\mathrm{e}^{x}+x\sin(y)+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y}=x\cos(y)+f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y}=x\cos(y)$. Therefore equation (4) becomes

$$x\cos(y)=x\cos(y)+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y)=0$$

Therefore

$$f(y)=c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi=-\mathrm{e}^{x}+x\sin(y)+c_1$$

But since ϕ itself is a constant function, then let $\phi=c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1=-\mathrm{e}^{x}+x\sin(y)$$

Summary
The solution(s) found are the following

$$-e^{x}+x \sin \left(y\right) = c_{1} \tag{1}$$

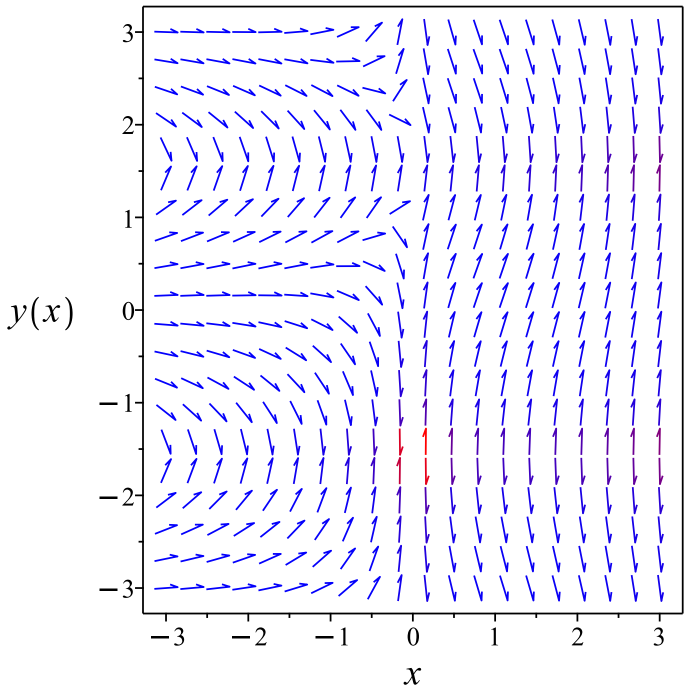


Figure 25: Slope field plot

Verification of solutions

$$-e^{x}+x \sin \left(y\right) = c_{1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(diff(y(x),x)=(exp(x)-sin(y(x)))/(x*cos(y(x))),y(x), singsol=all)
```

$$y(x) = \arcsin\left(\frac{-c_1 + e^x}{x}\right)$$

✓ Solution by Mathematica

Time used: 11.572 (sec). Leaf size: 16

```
DSolve[y'[x]==(Exp[x]-Sin[y[x]])/(x*Cos[y[x]]),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \arcsin\left(\frac{e^x + c_1}{x}\right)$$

1.21 problem Problem 29

Internal problem ID [2607]
Internal file name [`OUTPUT/2099_Sunday_June_05_2022_02_48_27_AM_75787558/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.2, Basic Ideas and Terminology. page 21
Problem number: Problem 29.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[_rational, [_1st_order, `_with_symmetry_[F(x)*G(y),0]`], [
   _Abel, `2nd type`, `class B`]]
```

$$\boxed{y' - \frac{1-y^2}{2+2yx} = 0}$$

1.21.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(2xy + 2)\,\mathrm{d}y &= \left(-y^2 + 1\right)\mathrm{d}x \\ \left(y^2 - 1\right)\mathrm{d}x + (2xy + 2)\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= y^2 - 1 \\ N(x, y) &= 2xy + 2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(y^2 - 1\right) \\ &= 2y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(2xy + 2) \\ &= 2y\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x}=M \tag{1}$$
$$\frac{\partial \phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int M\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int y^2-1\,\mathrm{d}x$$
$$\phi=\left(y^2-1\right)x+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y}=2xy+f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y}=2xy+2$. Therefore equation (4) becomes

$$2xy+2=2xy+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y)=2$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y=\int (2)\,\mathrm{d}y$$
$$f(y)=2y+c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi=\left(y^2-1\right)x+2y+c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = (y^2 - 1)\,x + 2y$$

Summary
The solution(s) found are the following

$$x(y^2 - 1) + 2y = c_1 \tag{1}$$

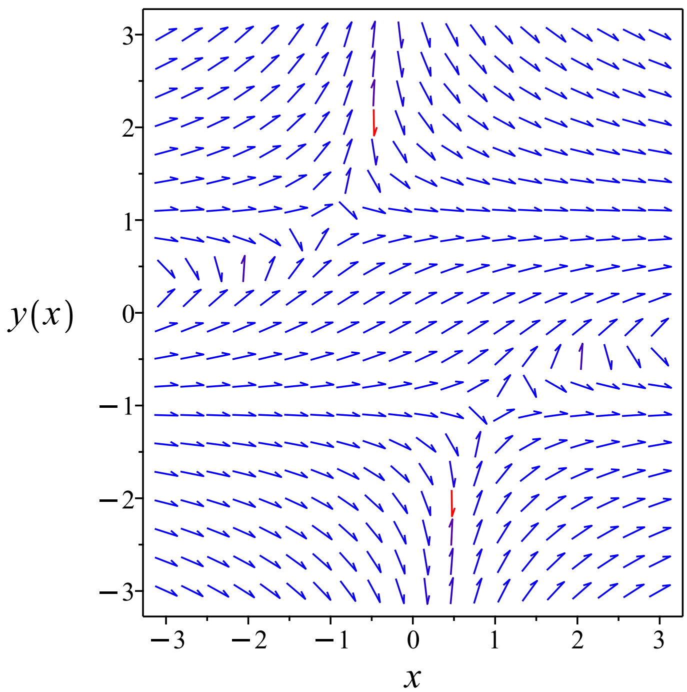


Figure 26: Slope field plot

Verification of solutions

$$x(y^2 - 1) + 2y = c_1$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
      -> Trying a Liouvillian solution using Kovacics algorithm
         A Liouvillian solution exists
         Reducible group (found an exponential solution)
      <- Kovacics algorithm successful
   <- Abel AIR successful: ODE belongs to the 2F1 3-parameter class
<- inverse linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(diff(y(x),x)=(1-y(x)^2)/(2*(1+x*y(x))),y(x), singsol=all)
```

$$c_1 + \frac{1}{(y(x)-1)(xy(x)+x+2)} = 0$$

✓ Solution by Mathematica

Time used: 0.463 (sec). Leaf size: 58

```
DSolve[y'[x]==(1-y[x]^2)/(2*(1+x*y[x])),y[x],x,IncludeSingularSolutions -> True]
```

$$\begin{aligned}
y(x) &\to -\frac{1+\sqrt{x^2+c_1x+1}}{x} \\
y(x) &\to \frac{-1+\sqrt{x^2+c_1x+1}}{x} \\
y(x) &\to -1 \\
y(x) &\to 1
\end{aligned}$$

1.22 problem Problem 30

Internal problem ID [2608]
Internal file name [`OUTPUT/2100_Sunday_June_05_2022_02_48_29_AM_15930513/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.2, Basic Ideas and Terminology. page 21
Problem number: Problem 30.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[[_1st_order, `_with_symmetry_[F(x),G(y)]`]]
```

$$\boxed{y' - \frac{(1 - y\,\mathrm{e}^{yx})\,\mathrm{e}^{-yx}}{x} = 0}$$

With initial conditions

$$[y(1) = 0]$$

1.22.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= -\frac{(y\,\mathrm{e}^{xy} - 1)\,\mathrm{e}^{-xy}}{x} \end{aligned}$$

The x domain of $f(x,y)$ when $y = 0$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $f(x,y)$ when $x = 1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Now we will look at the continuity of

$$
\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{(y\,\mathrm{e}^{xy}-1)\,\mathrm{e}^{-xy}}{x}\right) \\
&= -\frac{(\mathrm{e}^{xy}+y\,\mathrm{e}^{xy}x)\,\mathrm{e}^{-xy}}{x} + (y\,\mathrm{e}^{xy}-1)\,\mathrm{e}^{-xy}
\end{aligned}
$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 0$ is

$$
\{x < 0 \lor 0 < x\}
$$

And the point $x_0 = 1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 1$ is

$$
\{-\infty < y < \infty\}
$$

And the point $y_0 = 0$ is inside this domain. Therefore solution exists and is unique.

1.22.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$
M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}
$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$
\frac{d}{dx}\phi(x,y) = 0
$$

Hence

$$
\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}
$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2\phi}{\partial x \partial y} = \frac{\partial^2\phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(\mathrm{e}^{xy}x)\,\mathrm{d}y &= (1 - y\,\mathrm{e}^{xy})\,\mathrm{d}x\\ (y\,\mathrm{e}^{xy} - 1)\,\mathrm{d}x + (\mathrm{e}^{xy}x)\,\mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= y\,\mathrm{e}^{xy} - 1\\ N(x,y) &= \mathrm{e}^{xy}x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y\,\mathrm{e}^{xy} - 1)\\ &= \mathrm{e}^{xy}(xy+1)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(\mathrm{e}^{xy}x)\\ &= \mathrm{e}^{xy}(xy+1)\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = M \tag{1}$$

$$\frac{\partial\phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int M \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int y\, \mathrm{e}^{xy} - 1 \mathrm{d}x \\
\phi &= -x + \mathrm{e}^{xy} + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \mathrm{e}^{xy} x + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{xy} x$. Therefore equation (4) becomes

$$
\mathrm{e}^{xy} x = \mathrm{e}^{xy} x + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$
f(y) = c_1
$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$
\phi = -x + \mathrm{e}^{xy} + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
c_1 = -x + \mathrm{e}^{xy}
$$

The solution becomes

$$
y = \frac{\ln(x + c_1)}{x}
$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \ln(c_1 + 1)$$

$$c_1 = 0$$

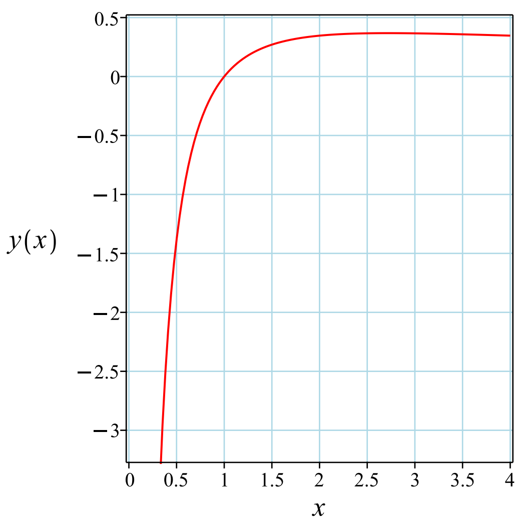
Substituting c_1 found above in the general solution gives

$$y = \frac{\ln(x)}{x}$$

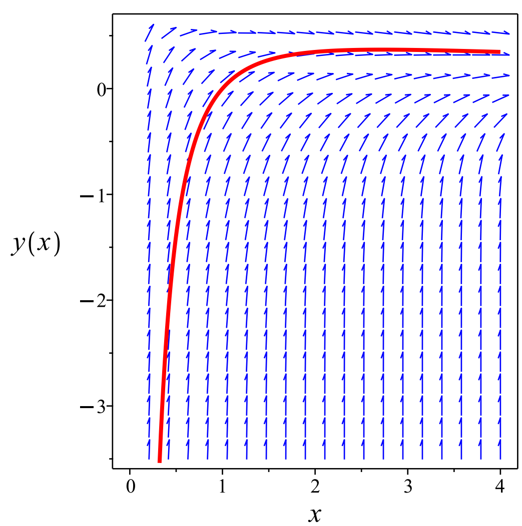
Summary

The solution(s) found are the following

$$y = \frac{\ln(x)}{x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{\ln(x)}{x}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4`[1, -y/x], [-x, (x*y-1)/x]
```

✓ Solution by Maple
Time used: 0.062 (sec). Leaf size: 10

```
dsolve([diff(y(x),x)=(1-y(x)*exp(x*y(x)))/(x*exp(x*y(x))),y(1) = 0],y(x), singsol=all)
```

$$y(x) = \frac{\ln(x)}{x}$$

✓ Solution by Mathematica
Time used: 0.403 (sec). Leaf size: 11

```
DSolve[{y'[x]==(1-y[x]*Exp[x*y[x]])/(x*Exp[x*y[x]]),{y[1]==0}},y[x],x,IncludeSingularSolutio
```

$$y(x) \to \frac{\log(x)}{x}$$

1.23 problem Problem 31

Internal problem ID [2609]
Internal file name [`OUTPUT/2101_Sunday_June_05_2022_02_48_34_AM_38515979/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.2, Basic Ideas and Terminology. page 21
Problem number: Problem 31.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[`y=_G(x,y')`]
```

$$\boxed{y' - \frac{x^2(1-y^2) + y\,e^{\frac{y}{x}}}{x\left(e^{\frac{y}{x}} + 2x^2 y\right)} = 0}$$

1.23.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(x\left(\mathrm{e}^{\frac{y}{x}} + 2x^2y\right)\right)\mathrm{d}y &= \left(-y^2x^2 + y\,\mathrm{e}^{\frac{y}{x}} + x^2\right)\mathrm{d}x \\
\left(y^2x^2 - y\,\mathrm{e}^{\frac{y}{x}} - x^2\right)\mathrm{d}x + \left(x\left(\mathrm{e}^{\frac{y}{x}} + 2x^2y\right)\right)\mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= y^2x^2 - y\,\mathrm{e}^{\frac{y}{x}} - x^2 \\
N(x,y) &= x\left(\mathrm{e}^{\frac{y}{x}} + 2x^2y\right)
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(y^2x^2 - y\,\mathrm{e}^{\frac{y}{x}} - x^2\right) \\
&= \frac{(-y-x)\,\mathrm{e}^{\frac{y}{x}} + 2x^3y}{x}
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(x\left(\mathrm{e}^{\frac{y}{x}} + 2x^2 y\right)\right) \\
&= \frac{(-y+x)\,\mathrm{e}^{\frac{y}{x}} + 6x^3 y}{x}
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{x\,\mathrm{e}^{\frac{y}{x}} + 2x^3 y}\left(\left(2x^2 y - \mathrm{e}^{\frac{y}{x}} - \frac{y\,\mathrm{e}^{\frac{y}{x}}}{x}\right) - \left(\mathrm{e}^{\frac{y}{x}} + 2x^2 y + x\left(-\frac{y\,\mathrm{e}^{\frac{y}{x}}}{x^2} + 4xy\right)\right)\right) \\
&= -\frac{2}{x}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \,\mathrm{d}x} \\
&= e^{\int -\frac{2}{x} \,\mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-2\ln(x)} \\
&= \frac{1}{x^2}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{x^2}\left(y^2 x^2 - y\,\mathrm{e}^{\frac{y}{x}} - x^2\right) \\
&= \frac{x^2(y^2-1) - y\,\mathrm{e}^{\frac{y}{x}}}{x^2}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^2}\left(x\left(\mathrm{e}^{\frac{y}{x}} + 2x^2 y\right)\right) \\
&= \frac{\mathrm{e}^{\frac{y}{x}} + 2x^2 y}{x}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\left(\frac{x^2(y^2-1) - y\,\mathrm{e}^{\frac{y}{x}}}{x^2}\right) + \left(\frac{\mathrm{e}^{\frac{y}{x}} + 2x^2y}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \frac{x^2(y^2-1) - y\,\mathrm{e}^{\frac{y}{x}}}{x^2}\,\mathrm{d}x$$
$$\phi = x\,y^2 + \mathrm{e}^{\frac{y}{x}} - x + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 2xy + \frac{\mathrm{e}^{\frac{y}{x}}}{x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{\mathrm{e}^{\frac{y}{x}}+2x^2y}{x}$. Therefore equation (4) becomes

$$\frac{\mathrm{e}^{\frac{y}{x}} + 2x^2y}{x} = \frac{\mathrm{e}^{\frac{y}{x}} + 2x^2y}{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = x\,y^2 + \mathrm{e}^{\frac{y}{x}} - x + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x\,y^2 + \mathrm{e}^{\frac{y}{x}} - x$$

Summary

The solution(s) found are the following

$$y^2 x + \mathrm{e}^{\frac{y}{x}} - x = c_1 \tag{1}$$

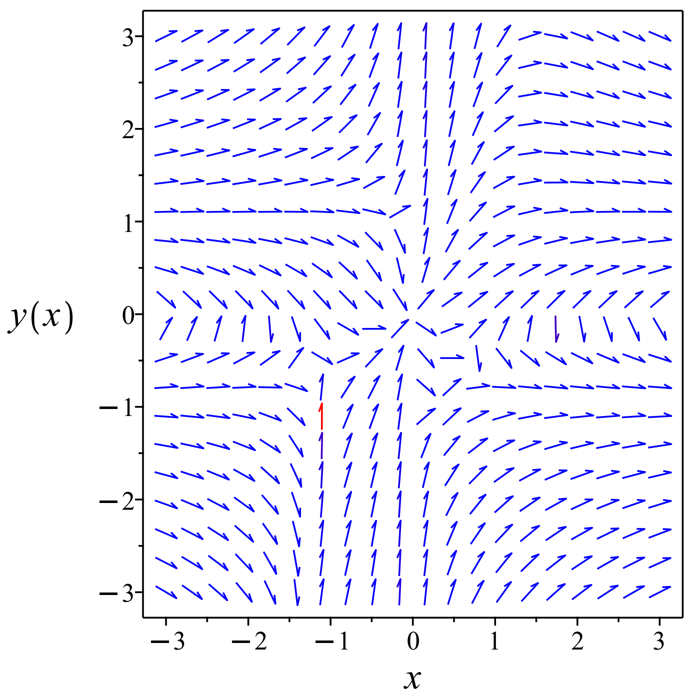


Figure 28: Slope field plot

Verification of solutions

$$y^2 x + \mathrm{e}^{\frac{y}{x}} - x = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 21

```
dsolve(diff(y(x),x)=(x^2*(1-y(x)^2)+y(x)*exp(y(x)/x))/(x*(exp(y(x)/x)+2*x^2*y(x))),y(x), sin
```

$$y(x) = \mathrm{RootOf}\left(\mathrm{e}^{\_Z} + x^3\_Z^2 + c_1 - x\right) x$$

✓ Solution by Mathematica

Time used: 0.293 (sec). Leaf size: 24

```
DSolve[y'[x]==(x^2*(1-y[x]^2)+y[x]*Exp[y[x]/x])/(x*(Exp[y[x]/x]+2*x^2*y[x])),y[x],x,IncludeS
```

$$\text{Solve}\left[xy(x)^2 + e^{\frac{y(x)}{x}} - x = c_1, y(x)\right]$$

1.24 problem Problem 32

Internal problem ID [2610]
Internal file name [`OUTPUT/2102_Sunday_June_05_2022_02_48_38_AM_97526757/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.2, Basic Ideas and Terminology. page 21
Problem number: Problem 32.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "bernoulli", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_Bernoulli]
```

$$\boxed{y' - \frac{\cos(x) - 2y^2 x}{2x^2 y} = 0}$$

With initial conditions

$$\left[y(\pi) = \frac{1}{\pi}\right]$$

1.24.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= \frac{\cos(x) - 2x\,y^2}{2x^2 y} \end{aligned}$$

The x domain of $f(x, y)$ when $y = \frac{1}{\pi}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = \pi$ is inside this domain. The y domain of $f(x, y)$ when $x = \pi$ is

$$\{y < 0 \lor 0 < y\}$$

And the point $y_0 = \frac{1}{\pi}$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{\cos(x) - 2x\,y^2}{2x^2 y}\right) \\
&= -\frac{2}{x} - \frac{\cos(x) - 2x\,y^2}{2x^2 y^2}
\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = \frac{1}{\pi}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = \pi$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = \pi$ is

$$\{y < 0 \lor 0 < y\}$$

And the point $y_0 = \frac{1}{\pi}$ is inside this domain. Therefore solution exists and is unique.

1.24.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned}
y' &= \frac{\cos(x) - 2x\,y^2}{2x^2 y} \\
y' &= \omega(x, y)
\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 37: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \frac{1}{x^2 y}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{x^2 y}} dy \end{aligned}$$

Which results in

$$S = \frac{y^2 x^2}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{\cos(x) - 2x\, y^2}{2x^2 y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= x\, y^2 \\ S_y &= x^2 y \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{\cos(x)}{2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{\cos(R)}{2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{\sin(R)}{2} + c_1 \tag{4}$$

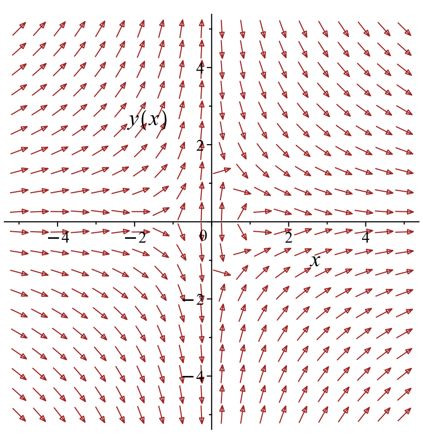
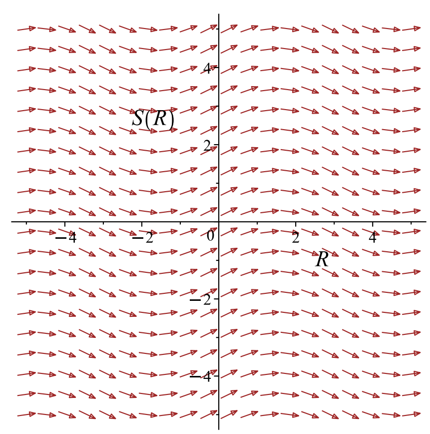
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y^2 x^2}{2} = \frac{\sin(x)}{2} + c_1$$

Which simplifies to

$$\frac{y^2 x^2}{2} = \frac{\sin(x)}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{\cos(x)-2x\,y^2}{2x^2 y}$ 	$R = x$ $S = \frac{y^2 x^2}{2}$	$\frac{dS}{dR} = \frac{\cos(R)}{2}$ 

Initial conditions are used to solve for c_1. Substituting $x = \pi$ and $y = \frac{1}{\pi}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{2} = c_1$$

$$c_1 = \frac{1}{2}$$

Substituting c_1 found above in the general solution gives

$$\frac{y^2 x^2}{2} = \frac{\sin(x)}{2} + \frac{1}{2}$$

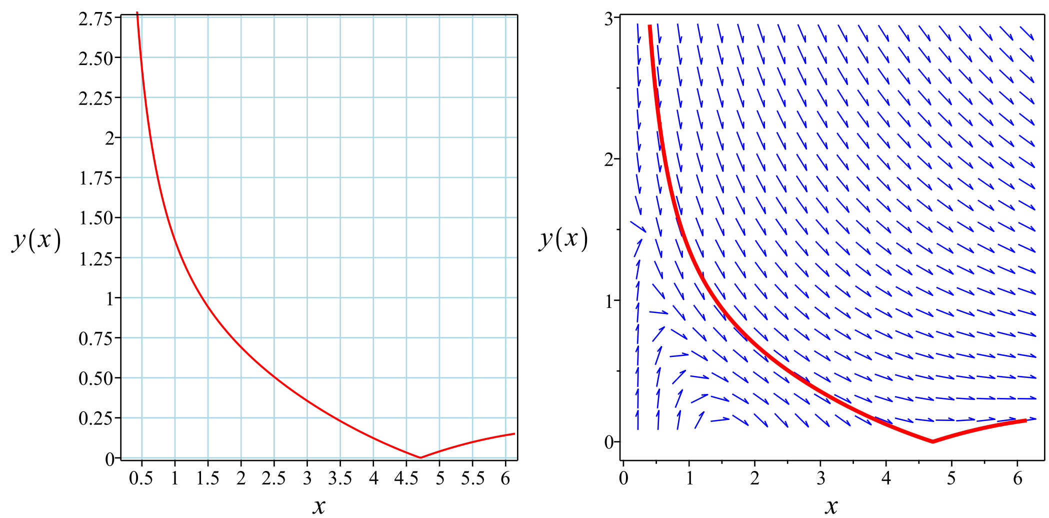
Solving for y from the above gives

$$y = \frac{\sqrt{1+\sin(x)}}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{\sqrt{1+\sin(x)}}{x} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{\sqrt{1+\sin(x)}}{x}$$

Verified OK.

1.24.3 Solving as bernoulli ode

In canonical form, the ODE is

$$
\begin{aligned}
y' &= F(x,y) \\
&= \frac{\cos(x) - 2x\,y^2}{2x^2 y}
\end{aligned}
$$

This is a Bernoulli ODE.

$$
y' = -\frac{1}{x}y + \frac{\cos(x)}{2x^2}\frac{1}{y} \tag{1}
$$

The standard Bernoulli ODE has the form

$$
y' = f_0(x)y + f_1(x)y^n \tag{2}
$$

The first step is to divide the above equation by y^n which gives

$$
\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}
$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= -\frac{1}{x} \\
f_1(x) &= \frac{\cos(x)}{2x^2} \\
n &= -1
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$
y'y = -\frac{y^2}{x} + \frac{\cos(x)}{2x^2} \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= y^2
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = 2yy' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
\frac{w'(x)}{2} &= -\frac{w(x)}{x} + \frac{\cos(x)}{2x^2} \\
w' &= -\frac{2w}{x} + \frac{\cos(x)}{x^2}
\end{aligned}
\tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= \frac{2}{x} \\
q(x) &= \frac{\cos(x)}{x^2}
\end{aligned}
$$

Hence the ode is

$$
w'(x) + \frac{2w(x)}{x} = \frac{\cos(x)}{x^2}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int \frac{2}{x} dx} \\
&= x^2
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)\left(\frac{\cos(x)}{x^2}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}(x^2 w) &= (x^2)\left(\frac{\cos(x)}{x^2}\right) \\
\mathrm{d}(x^2 w) &= \cos(x)\,\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
x^2 w &= \int \cos(x)\,\mathrm{d}x \\
x^2 w &= \sin(x) + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = x^2$ results in

$$
w(x) = \frac{\sin(x)}{x^2} + \frac{c_1}{x^2}
$$

which simplifies to

$$w(x) = \frac{\sin(x) + c_1}{x^2}$$

Replacing w in the above by y^2 using equation (5) gives the final solution.

$$y^2 = \frac{\sin(x) + c_1}{x^2}$$

Initial conditions are used to solve for c_1. Substituting $x = \pi$ and $y = \frac{1}{\pi}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{\pi^2} = \frac{c_1}{\pi^2}$$

$$c_1 = 1$$

Substituting c_1 found above in the general solution gives

$$y^2 = \frac{1 + \sin(x)}{x^2}$$

The above simplifies to

$$y^2 x^2 - \sin(x) - 1 = 0$$

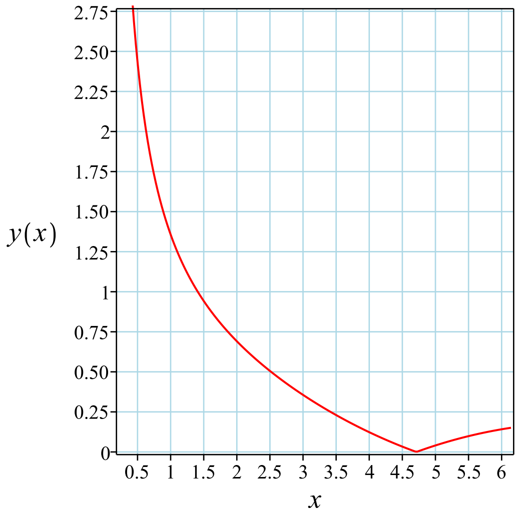
Solving for y from the above gives

$$y = \frac{\sqrt{1 + \sin(x)}}{x}$$

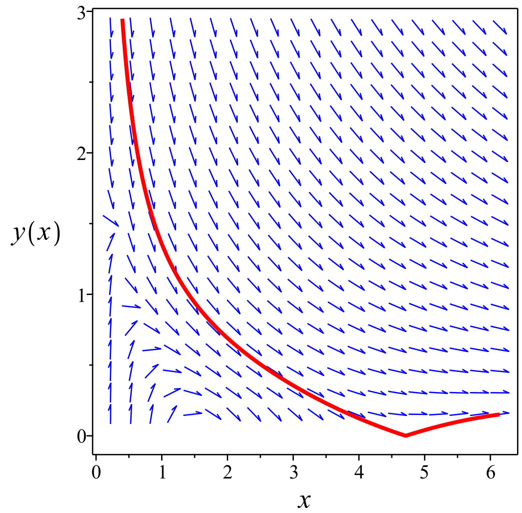
Summary

The solution(s) found are the following

$$y = \frac{\sqrt{1 + \sin(x)}}{x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{\sqrt{1+\sin(x)}}{x}$$

Verified OK.

1.24.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(2x^2 y)\,\mathrm{d}y &= (\cos(x) - 2x\,y^2)\,\mathrm{d}x \\ (-\cos(x) + 2x\,y^2)\,\mathrm{d}x + (2x^2 y)\,\mathrm{d}y &= 0 \qquad (2\mathrm{A})\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -\cos(x) + 2x\,y^2 \\ N(x, y) &= 2x^2 y\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\cos(x) + 2x\,y^2\right) \\ &= 4xy\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(2x^2 y\right) \\ &= 4xy\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int M \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int -\cos(x) + 2x\,y^2 \mathrm{d}x$$
$$\phi = y^2x^2 - \sin(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 2x^2y + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 2x^2y$. Therefore equation (4) becomes

$$2x^2y = 2x^2y + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = y^2x^2 - \sin(x) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = y^2x^2 - \sin(x)$$

Initial conditions are used to solve for c_1. Substituting $x = \pi$ and $y = \frac{1}{\pi}$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1$$

$$c_1 = 1$$

Substituting c_1 found above in the general solution gives

$$y^2x^2 - \sin(x) = 1$$

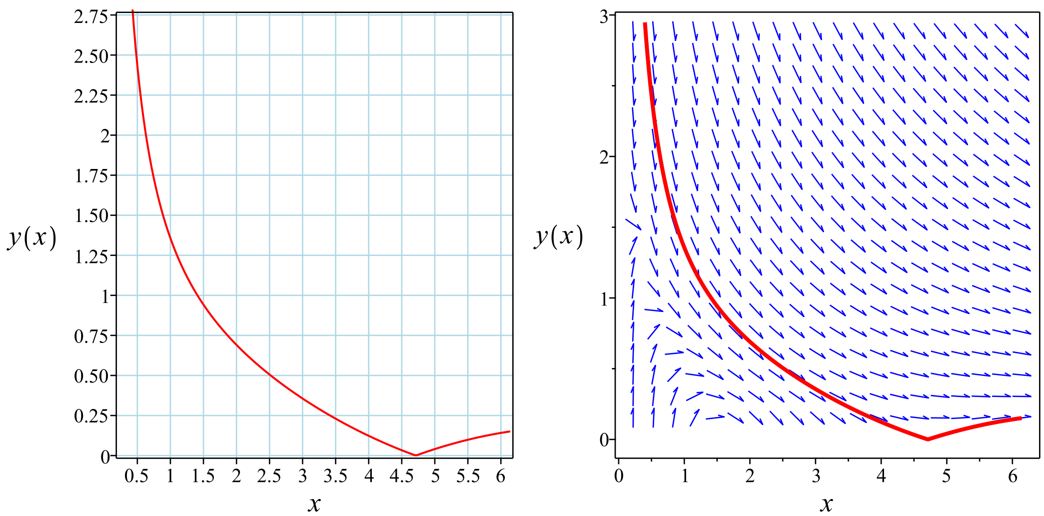
Solving for y from the above gives

$$y = \frac{\sqrt{1+\sin(x)}}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{\sqrt{1+\sin(x)}}{x} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{\sqrt{1+\sin(x)}}{x}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.172 (sec). Leaf size: 14

```
dsolve([diff(y(x),x)=(cos(x)-2*x*y(x)^2)/(2*x^2*y(x)),y(Pi) = 1/Pi],y(x), singsol=all)
```

$$y(x) = \frac{\sqrt{\sin(x) + 1}}{x}$$

✓ Solution by Mathematica

Time used: 0.342 (sec). Leaf size: 17

```
DSolve[{y'[x]==(Cos[x]-2*x*y[x]^2)/(2*x^2*y[x]),{y[Pi]==1/Pi}},y[x],x,IncludeSingularSolutio
```

$$y(x) \to \frac{\sqrt{\sin(x) + 1}}{x}$$

1.25 problem Problem 33

Internal problem ID [2611]
Internal file name [`OUTPUT/2103_Sunday_June_05_2022_02_48_42_AM_58942872/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.2, Basic Ideas and Terminology. page 21
Problem number: Problem 33.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' = \sin(x)}$$

1.25.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int \sin(x)\, \mathrm{d}x \\ &= -\cos(x) + c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = -\cos(x) + c_1 \tag{1}$$

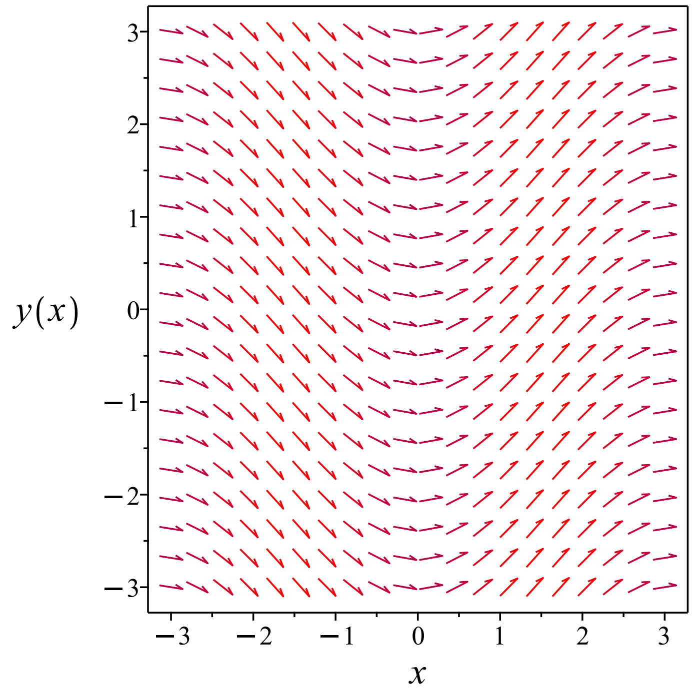


Figure 32: Slope field plot

Verification of solutions

$$y = -\cos(x) + c_1$$

Verified OK.

1.25.2 Maple step by step solution

Let's solve

$y' = \sin(x)$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int y' dx = \int \sin(x)\, dx + c_1$

- Evaluate integral

  $y = -\cos(x) + c_1$

- Solve for y

  $y = -\cos(x) + c_1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 10

```
dsolve(diff(y(x),x)=sin(x),y(x), singsol=all)
```

$$y(x) = -\cos(x) + c_1$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 12

```
DSolve[y'[x]==Sin[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\cos(x) + c_1$$

1.26 problem Problem 34

Internal problem ID [2612]
Internal file name [`OUTPUT/2104_Sunday_June_05_2022_02_48_44_AM_47541874/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.2, Basic Ideas and Terminology. page 21
Problem number: Problem 34.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' = \frac{1}{x^{\frac{2}{3}}}}$$

1.26.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{1}{x^{\frac{2}{3}}}\, \mathrm{d}x \\ &= 3x^{\frac{1}{3}} + c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = 3x^{\frac{1}{3}} + c_1 \tag{1}$$

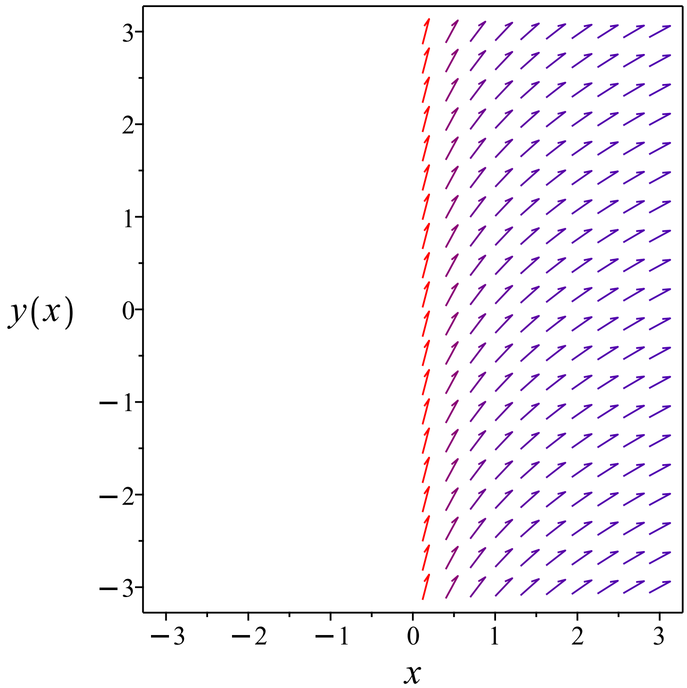


Figure 33: Slope field plot

Verification of solutions

$$y = 3x^{\frac{1}{3}} + c_1$$

Verified OK.

1.26.2 Maple step by step solution

Let's solve

$y' = \frac{1}{x^{\frac{2}{3}}}$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int y' dx = \int \frac{1}{x^{\frac{2}{3}}} dx + c_1$

- Evaluate integral

  $y = 3x^{\frac{1}{3}} + c_1$

- Solve for y

$$y = 3x^{\frac{1}{3}} + c_1$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 11

```
dsolve(diff(y(x),x)=x^(-2/3),y(x), singsol=all)
```

$$y(x) = 3x^{\frac{1}{3}} + c_1$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 15

```
DSolve[y'[x]==x^(-2/3),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 3\sqrt[3]{x} + c_1$$

1.27 problem Problem 35

Internal problem ID [2613]
Internal file name [`OUTPUT/2105_Sunday_June_05_2022_02_48_45_AM_32814056/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.2, Basic Ideas and Terminology. page 21
Problem number: Problem 35.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_ode_quadrature", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _quadrature]]
```

$$\boxed{y'' = x\,\mathrm{e}^{x}}$$

1.27.1 Solving as second order ode quadrature ode

Integrating once gives

$$y' = (x-1)\,\mathrm{e}^{x} + c_1$$

Integrating again gives

$$y = (x-2)\,\mathrm{e}^{x} + c_1 x + c_2$$

Summary
The solution(s) found are the following

$$y = (x-2)\,\mathrm{e}^{x} + c_1 x + c_2 \tag{1}$$

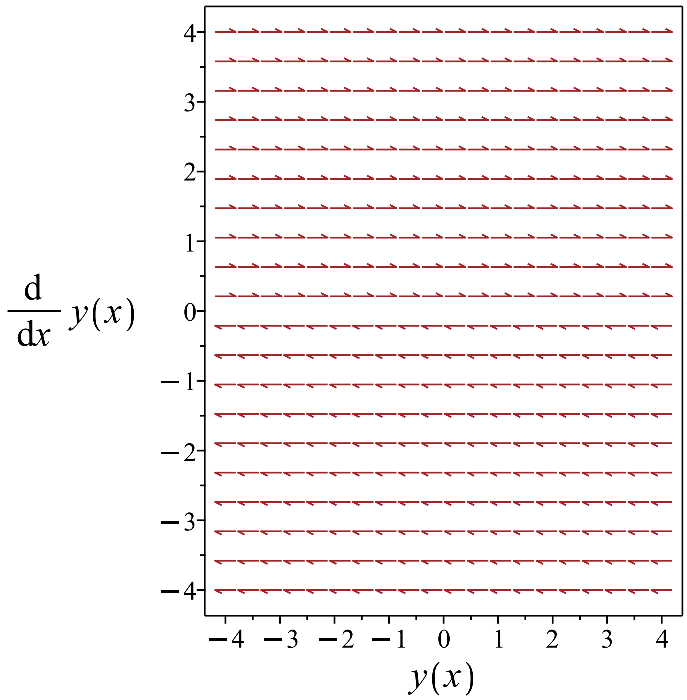


Figure 34: Slope field plot

Verification of solutions

$$y = (x - 2)\,e^{x} + c_1 x + c_2$$

Verified OK.

1.27.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 0, f(x) = x\,e^{x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 0$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 0$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(0)^2 - (4)(1)(0)}\\ &= 0\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 0$. Therefore the solution is

$$y = c_1 1 + c_2 x \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_2 x + c_1$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x\,\mathrm{e}^x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\,\mathrm{e}^x, \mathrm{e}^x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, x\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 x\,\mathrm{e}^x + A_2 \mathrm{e}^x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1\mathrm{e}^{x} + A_1x\,\mathrm{e}^{x} + A_2\mathrm{e}^{x} = x\,\mathrm{e}^{x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1, A_2 = -2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x\,\mathrm{e}^{x} - 2\,\mathrm{e}^{x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_2x + c_1) + (x\,\mathrm{e}^{x} - 2\,\mathrm{e}^{x}) \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_2x + c_1 + x\,\mathrm{e}^{x} - 2\,\mathrm{e}^{x} \tag{1}$$

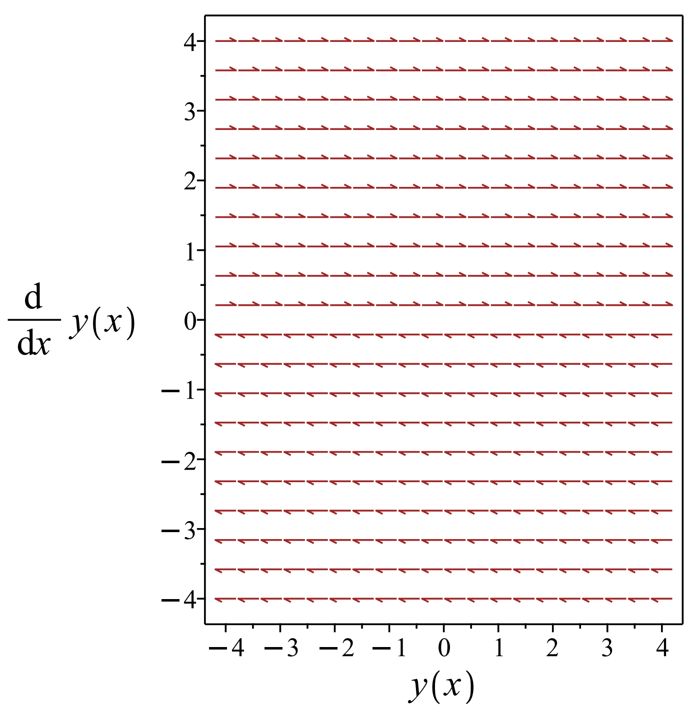


Figure 35: Slope field plot

Verification of solutions

$$y = c_2 x + c_1 + x\,\mathrm{e}^{x} - 2\,\mathrm{e}^{x}$$

Verified OK.

1.27.3 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int y'' dx = \int x\,\mathrm{e}^{x} dx$$
$$y' = (x-1)\,\mathrm{e}^{x} + c_1$$

Which is now solved for y. Integrating both sides gives

$$\begin{aligned} y &= \int x\,\mathrm{e}^{x} - \mathrm{e}^{x} + c_1 \, \mathrm{d}x \\ &= x\,\mathrm{e}^{x} + c_1 x - 2\,\mathrm{e}^{x} + c_2 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = x\,\mathrm{e}^{x} + c_1 x - 2\,\mathrm{e}^{x} + c_2 \tag{1}$$

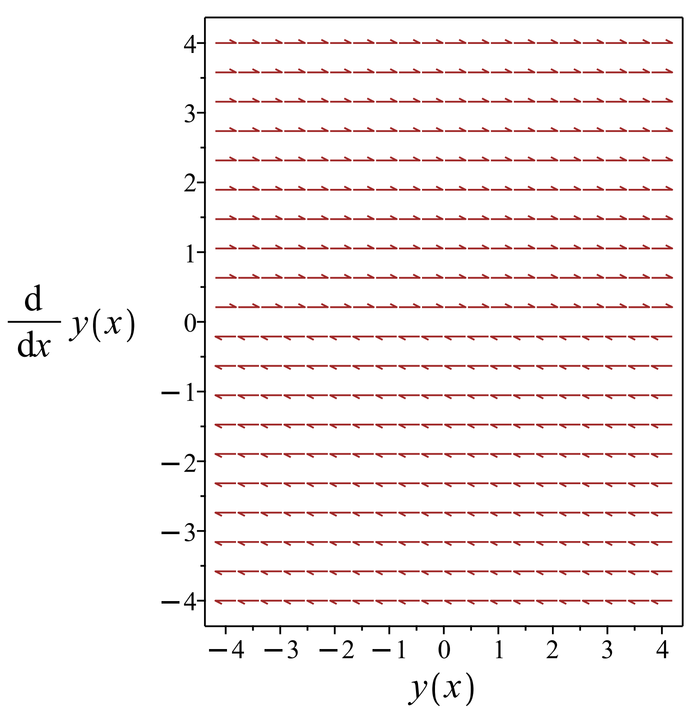


Figure 36: Slope field plot

Verification of solutions

$$y = x\,\mathrm{e}^{x} + c_1 x - 2\,\mathrm{e}^{x} + c_2$$

Verified OK.

1.27.4 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) - x\,\mathrm{e}^{x} = 0$$

Which is now solve for $p(x)$ as first order ode. Integrating both sides gives

$$\begin{aligned} p(x) &= \int x\,\mathrm{e}^{x}\,\mathrm{d}x \\ &= (x-1)\,\mathrm{e}^{x} + c_1 \end{aligned}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = (x-1)\,\mathrm{e}^{x} + c_1$$

Integrating both sides gives

$$\begin{aligned} y &= \int x\,\mathrm{e}^{x} - \mathrm{e}^{x} + c_1\,\mathrm{d}x \\ &= x\,\mathrm{e}^{x} + c_1 x - 2\,\mathrm{e}^{x} + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = x\,\mathrm{e}^{x} + c_1 x - 2\,\mathrm{e}^{x} + c_2 \tag{1}$$

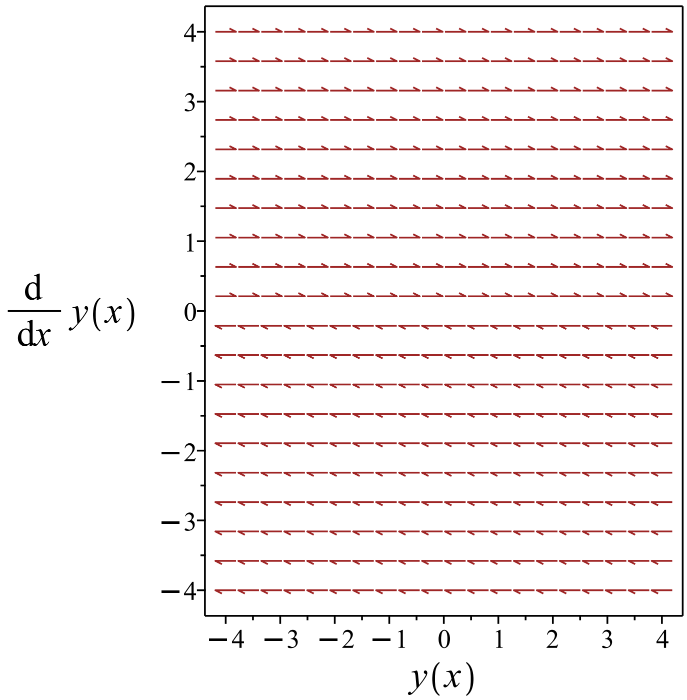


Figure 37: Slope field plot

Verification of solutions

$$
y = x\,\mathrm{e}^{x} + c_1 x - 2\,\mathrm{e}^{x} + c_2
$$

Verified OK.

1.27.5 Solving using Kovacic algorithm

Writing the ode as

$$
y'' = 0 \tag{1}
$$

$$
Ay'' + By' + Cy = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= 0 \\
C &= 0
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A}\,dx}
$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 41: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}
y_1 &= z_1 \\
&= 1
\end{aligned}$$

Which simplifies to

$$y_1 = 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\
&= 1 \int \frac{1}{1}\,dx \\
&= 1(x)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(1) + c_2(1(x))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_2 x + c_1
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
x\,\mathrm{e}^{x}
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{x\,\mathrm{e}^{x}, \mathrm{e}^{x}\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{1, x\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 x\,\mathrm{e}^{x} + A_2 \mathrm{e}^{x}
$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
2A_1 \mathrm{e}^{x} + A_1 x\,\mathrm{e}^{x} + A_2 \mathrm{e}^{x} = x\,\mathrm{e}^{x}
$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1, A_2 = -2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x\,e^x - 2\,e^x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_2 x + c_1) + (x\,e^x - 2\,e^x) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_2 x + c_1 + x\,e^x - 2\,e^x \tag{1}$$

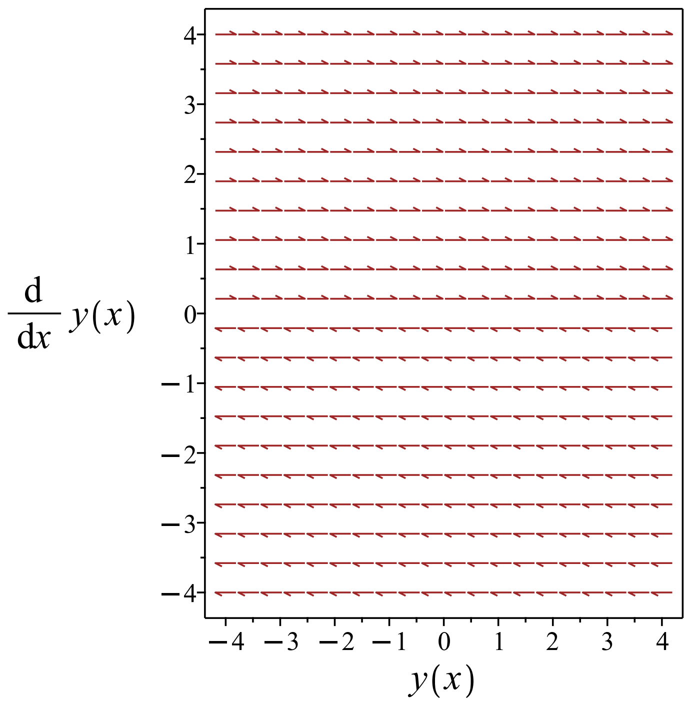


Figure 38: Slope field plot

Verification of solutions

$$y = c_2 x + c_1 + x\,e^x - 2\,e^x$$

Verified OK.

1.27.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y''+q(x)\,y'+r(x)\,y=s(x)$$

is exact if

$$p''(x)-q'(x)+r(x)=0 \tag{1}$$

For the given ode we have

$$\begin{aligned}p(x)&=1\\q(x)&=0\\r(x)&=0\\s(x)&=x\,\mathrm{e}^{x}\end{aligned}$$

Hence

$$\begin{aligned}p''(x)&=0\\q'(x)&=0\end{aligned}$$

Therefore (1) becomes

$$0-(0)+(0)=0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$\left(p(x)\,y'+\left(q(x)-p'(x)\right)y\right)'=s(x)$$

Integrating gives

$$p(x)\,y'+\left(q(x)-p'(x)\right)y=\int s(x)\,dx$$

Substituting the above values for p,q,r,s gives

$$y'=\int x\,\mathrm{e}^{x}\,dx$$

We now have a first order ode to solve which is

$$y'=(x-1)\,\mathrm{e}^{x}+c_1$$

Integrating both sides gives

$$
\begin{aligned}
y &= \int x\,\mathrm{e}^{x} - \mathrm{e}^{x} + c_1 \,\mathrm{d}x \\
&= x\,\mathrm{e}^{x} + c_1 x - 2\,\mathrm{e}^{x} + c_2
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = x\,\mathrm{e}^{x} + c_1 x - 2\,\mathrm{e}^{x} + c_2 \tag{1}
$$

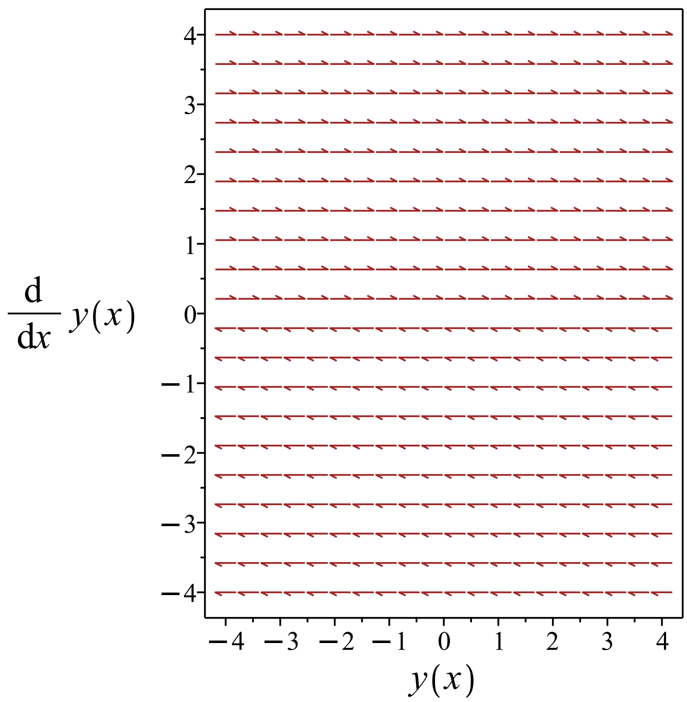


Figure 39: Slope field plot

Verification of solutions

$$
y = x\,\mathrm{e}^{x} + c_1 x - 2\,\mathrm{e}^{x} + c_2
$$

Verified OK.

1.27.7 Maple step by step solution

Let's solve

$y'' = x\,\mathrm{e}^{x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm \left(\sqrt{0}\right)}{2}$

- Roots of the characteristic polynomial

  $r = 0$

- 1st solution of the homogeneous ODE

  $y_1(x) = 1$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 + c_2 x + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = x\,\mathrm{e}^{x}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 1$

  - Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\left(\int x^2 \mathrm{e}^x dx\right) + x\left(\int x\,\mathrm{e}^x dx\right)$$

- Compute integrals

$$y_p(x) = (x-2)\,\mathrm{e}^x$$

- Substitute particular solution into general solution to ODE

$$y = (x-2)\,\mathrm{e}^x + c_2 x + c_1$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(diff(y(x),x$2)=x*exp(x),y(x), singsol=all)
```

$$y(x) = (-2+x)\,\mathrm{e}^x + c_1 x + c_2$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 19

```
DSolve[y''[x]==x*Exp[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^x(x-2) + c_2 x + c_1$$

1.28 problem Problem 36

Internal problem ID [2614]
Internal file name [`OUTPUT/2106_Sunday_June_05_2022_02_48_47_AM_58618917/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.2, Basic Ideas and Terminology. page 21
Problem number: Problem 36.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_ode_quadrature", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _quadrature]]
```

$$\boxed{y'' = x^n}$$

1.28.1 Solving as second order ode quadrature ode

Integrating once gives

$$y' = \frac{x^{n+1}}{n+1} + c_1$$

Integrating again gives

$$y = \frac{x^{2+n}}{(2+n)\,(n+1)} + c_1 x + c_2$$

Summary

The solution(s) found are the following

$$y = \frac{x^{2+n}}{(2+n)(n+1)} + c_1 x + c_2 \tag{1}$$

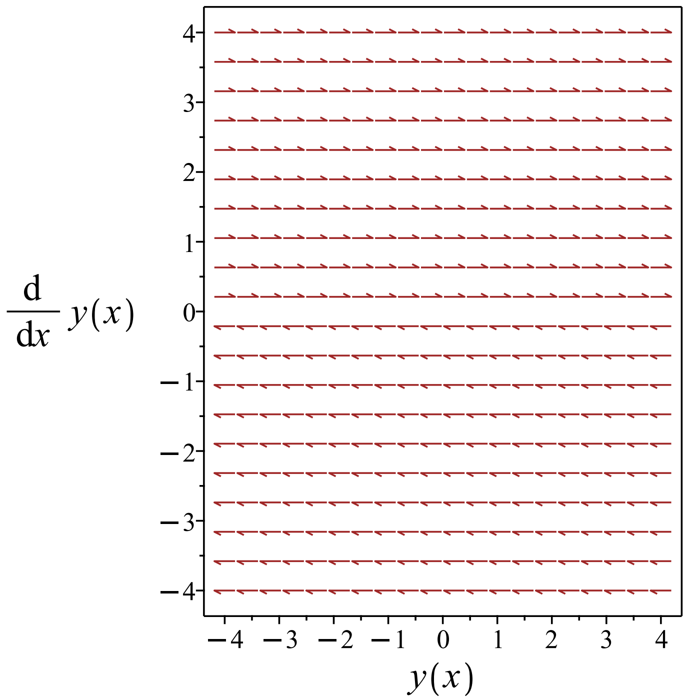


Figure 40: Slope field plot

Verification of solutions

$$y = \frac{x^{2+n}}{(2+n)(n+1)} + c_1 x + c_2$$

Verified OK.

1.28.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 0, f(x) = x^n$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$y''=0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x)+By'(x)+Cy(x)=0$$

Where in the above $A=1,B=0,C=0$. Let the solution be $y=e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x}=0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2=0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2}=\frac{-B}{2A}\pm\frac{1}{2A}\sqrt{B^2-4AC}$$

Substituting $A=1,B=0,C=0$ into the above gives

$$\begin{aligned}\lambda_{1,2}&=\frac{0}{(2)(1)}\pm\frac{1}{(2)(1)}\sqrt{(0)^2-(4)(1)(0)}\\&=0\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2}=0$. Therefore the solution is

$$y=c_1 1+c_2 x \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h=c_2 x+c_1$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x)=u_1 y_1+u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= 1 \\
y_2 &= x
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$
W = \begin{vmatrix} 1 & x \\ \frac{d}{dx}(1) & \frac{d}{dx}(x) \end{vmatrix}
$$

Which gives

$$
W = \begin{vmatrix} 1 & x \\ 0 & 1 \end{vmatrix}
$$

Therefore

$$
W = (1)(1) - (x)(0)
$$

Which simplifies to

$$
W = 1
$$

Which simplifies to

$$
W = 1
$$

Therefore Eq. (2) becomes

$$
u_1 = -\int \frac{x\, x^n}{1}\, dx
$$

Which simplifies to

$$u_1 = -\int x^{n+1} dx$$

Hence

$$u_1 = -\frac{x^{2+n}}{2+n}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^n}{1} dx$$

Which simplifies to

$$u_2 = \int x^n dx$$

Hence

$$u_2 = \frac{x^{n+1}}{n+1}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{x^{2+n}}{2+n} + \frac{x^{n+1}x}{n+1}$$

Which simplifies to

$$y_p(x) = \frac{x^{2+n}}{(2+n)(n+1)}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_2 x + c_1) + \left(\frac{x^{2+n}}{(2+n)(n+1)}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_2 x + c_1 + \frac{x^{2+n}}{(2+n)(n+1)} \tag{1}$$

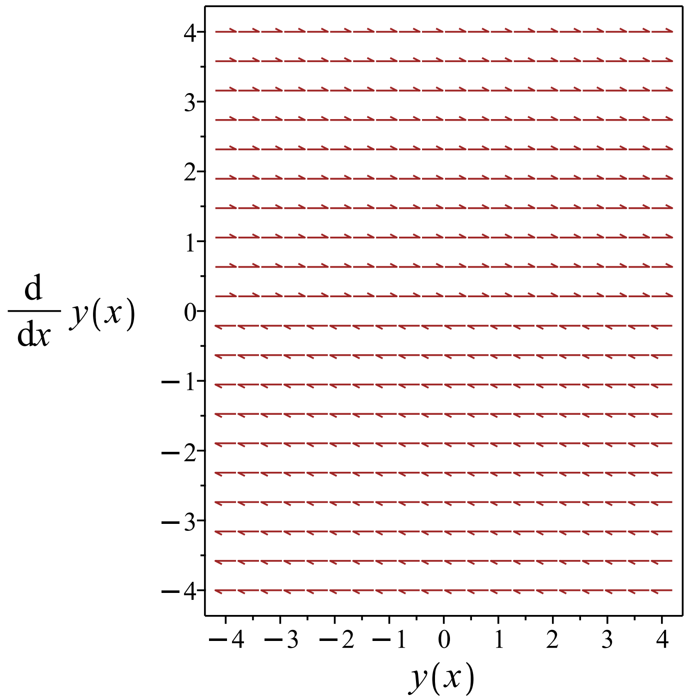


Figure 41: Slope field plot

Verification of solutions

$$y = c_2 x + c_1 + \frac{x^{2+n}}{(2+n)(n+1)}$$

Verified OK.

1.28.3 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int y'' dx = \int x^n dx$$
$$y' = \frac{x^{n+1}}{n+1} + c_1$$

Which is now solved for y. Integrating both sides gives

$$\begin{aligned} y &= \int \frac{nc_1 + x^{n+1} + c_1}{n+1} \, dx \\ &= \frac{c_1 x + \frac{x^{2+n}}{2+n} + c_1 n x}{n+1} + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 x + \frac{x^{2+n}}{2+n} + c_1 n x}{n+1} + c_2 \tag{1}$$

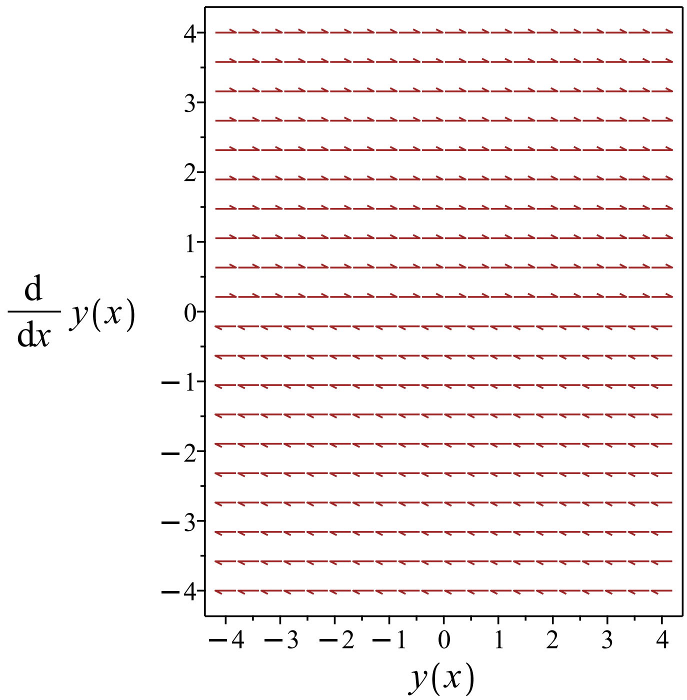


Figure 42: Slope field plot

Verification of solutions

$$y = \frac{c_1 x + \frac{x^{2+n}}{2+n} + c_1 n x}{n+1} + c_2$$

Verified OK.

1.28.4 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) - x^n = 0$$

Which is now solve for $p(x)$ as first order ode. Integrating both sides gives

$$\begin{aligned} p(x) &= \int x^n \,\mathrm{d}x \\ &= \frac{x^{n+1}}{n+1} + c_1 \end{aligned}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = \frac{x^{n+1}}{n+1} + c_1$$

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{nc_1 + x^{n+1} + c_1}{n+1} \,\mathrm{d}x \\ &= \frac{c_1 x + \frac{x^{2+n}}{2+n} + c_1 n x}{n+1} + c_2 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1 x + \frac{x^{2+n}}{2+n} + c_1 n x}{n+1} + c_2 \tag{1}$$

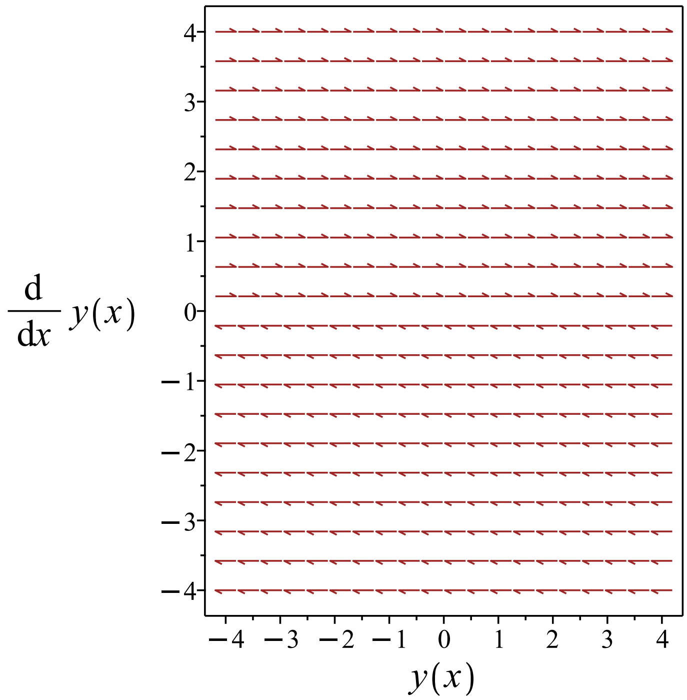


Figure 43: Slope field plot

Verification of solutions

$$y = \frac{c_1 x + \frac{x^{2+n}}{2+n} + c_1 n x}{n+1} + c_2$$

Verified OK.

1.28.5 Solving using Kovacic algorithm

Writing the ode as

$$y'' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 43: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= 1 \end{aligned}$$

Which simplifies to

$$y_1 = 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= 1 \int \frac{1}{1}\,dx \\ &= 1(x) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(1) + c_2(1(x))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_2 x + c_1
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= 1 \\
y_2 &= x
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} 1 & x \\ \frac{d}{dx}(1) & \frac{d}{dx}(x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} 1 & x \\ 0 & 1 \end{vmatrix}$$

Therefore

$$W = (1)(1) - (x)(0)$$

Which simplifies to

$$W = 1$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x\, x^n}{1}\, dx$$

Which simplifies to

$$u_1 = -\int x^{n+1} dx$$

Hence

$$u_1 = -\frac{x^{2+n}}{2+n}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^n}{1}\, dx$$

Which simplifies to

$$u_2 = \int x^n dx$$

Hence

$$u_2 = \frac{x^{n+1}}{n+1}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{x^{2+n}}{2+n} + \frac{x^{n+1}x}{n+1}$$

Which simplifies to

$$y_p(x) = \frac{x^{2+n}}{(2+n)(n+1)}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_2x + c_1) + \left(\frac{x^{2+n}}{(2+n)(n+1)}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_2x + c_1 + \frac{x^{2+n}}{(2+n)(n+1)} \tag{1}$$

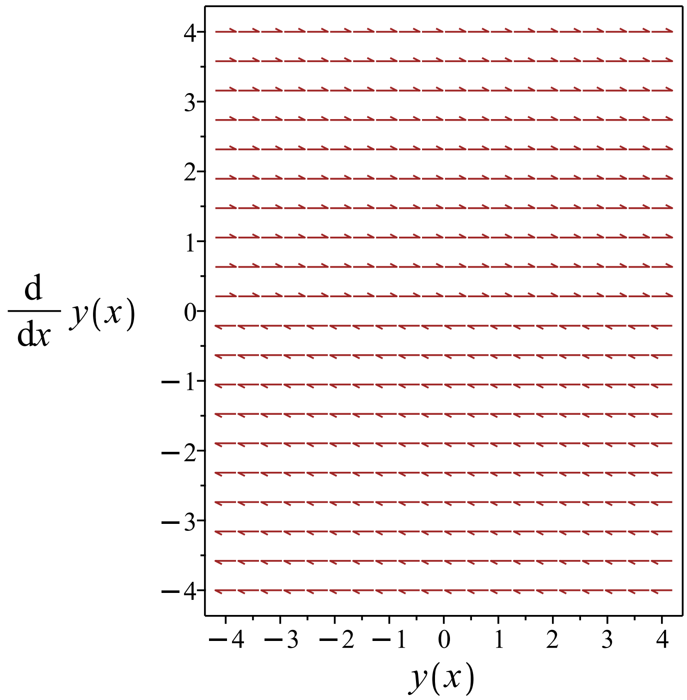


Figure 44: Slope field plot

Verification of solutions

$$y = c_2 x + c_1 + \frac{x^{2+n}}{(2+n)(n+1)}$$

Verified OK.

1.28.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x) y'' + q(x) y' + r(x) y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 0 \\ r(x) &= 0 \\ s(x) &= x^n \end{aligned}$$

Hence

$$
\begin{aligned}
p''(x) &= 0 \\
q'(x) &= 0
\end{aligned}
$$

Therefore (1) becomes

$$
0 - (0) + (0) = 0
$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$
(p(x)\,y' + (q(x) - p'(x))\,y)' = s(x)
$$

Integrating gives

$$
p(x)\,y' + (q(x) - p'(x))\,y = \int s(x)\;dx
$$

Substituting the above values for p, q, r, s gives

$$
y' = \int x^n\,dx
$$

We now have a first order ode to solve which is

$$
y' = \frac{x^{n+1}}{n+1} + c_1
$$

Integrating both sides gives

$$
\begin{aligned}
y &= \int \frac{nc_1 + x^{n+1} + c_1}{n+1}\,\mathrm{d}x \\
&= \frac{c_1 x + \frac{x^{2+n}}{2+n} + c_1 n x}{n+1} + c_2
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = \frac{c_1 x + \frac{x^{2+n}}{2+n} + c_1 n x}{n+1} + c_2 \tag{1}
$$

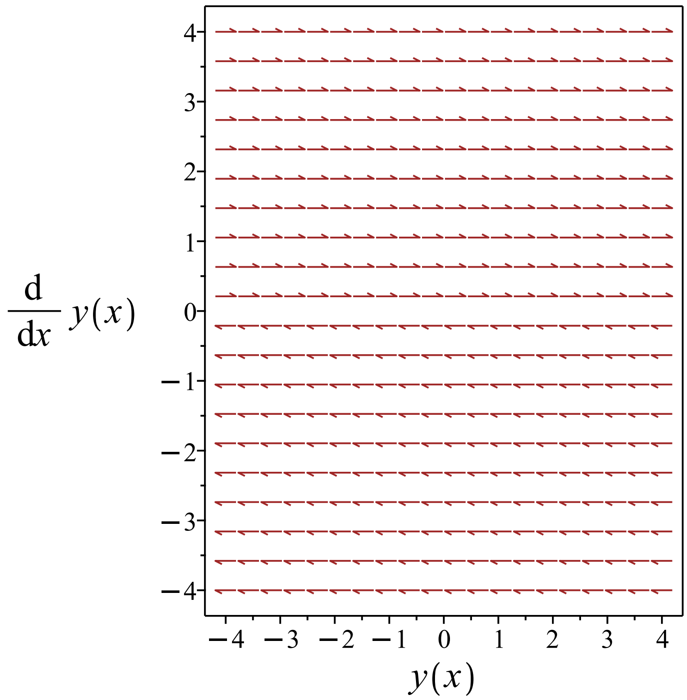


Figure 45: Slope field plot

Verification of solutions

$$y = \frac{c_1 x + \frac{x^{2+n}}{2+n} + c_1 n x}{n+1} + c_2$$

Verified OK.

1.28.7 Maple step by step solution

Let's solve

$y'' = x^n$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{0})}{2}$

- Roots of the characteristic polynomial

  $r = 0$
- 1st solution of the homogeneous ODE

  $y_1(x) = 1$
- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE

  $y = c_1 + c_2 x + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = x^n\right]$
  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$
  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 1$
  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\left(\int x^{n+1} dx\right) + x\left(\int x^n dx\right)$
  - Compute integrals

    $y_p(x) = \frac{x^{2+n}}{(2+n)(n+1)}$

- Substitute particular solution into general solution to ODE

  $y = c_2 x + c_1 + \frac{x^{2+n}}{(2+n)(n+1)}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 25

```
dsolve(diff(y(x),x$2)=x^n,y(x), singsol=all)
```

$$y(x) = \frac{x^{2+n}}{(2+n)(n+1)} + c_1 x + c_2$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 28

```
DSolve[y''[x]==x^n,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^{n+2}}{n^2 + 3n + 2} + c_2 x + c_1$$

1.29 problem Problem 37

Internal problem ID [2615]
Internal file name [`OUTPUT/2107_Sunday_June_05_2022_02_48_48_AM_91022340/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.2, Basic Ideas and Terminology. page 21
Problem number: Problem 37.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' = \ln(x)\, x^2}$$

With initial conditions

$$[y(1) = 2]$$

1.29.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 0$$
$$q(x) = \ln(x)\, x^2$$

Hence the ode is

$$y' = \ln(x)\, x^2$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = \ln(x)\, x^2$ is

$$\{0 < x\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

1.29.2 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int \ln(x)\, x^2 \, \mathrm{d}x \\ &= \frac{x^3 \ln(x)}{3} - \frac{x^3}{9} + c_1 \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = -\frac{1}{9} + c_1$$

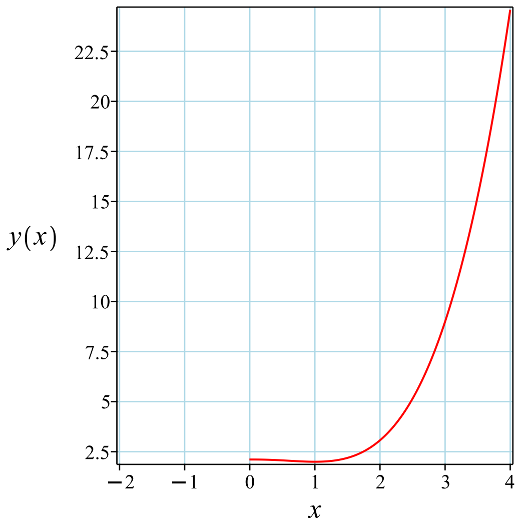
$$c_1 = \frac{19}{9}$$

Substituting c_1 found above in the general solution gives

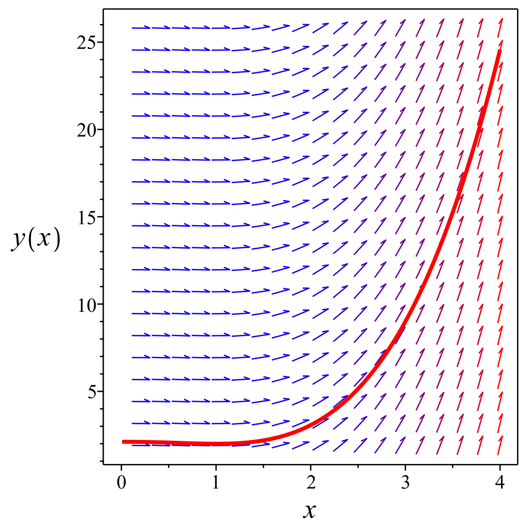
$$y = \frac{x^3 \ln(x)}{3} - \frac{x^3}{9} + \frac{19}{9}$$

Summary
The solution(s) found are the following

$$y = \frac{x^3 \ln(x)}{3} - \frac{x^3}{9} + \frac{19}{9} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{x^3 \ln(x)}{3} - \frac{x^3}{9} + \frac{19}{9}$$

Verified OK.

1.29.3 Maple step by step solution

Let's solve

$[y' = \ln(x)\, x^2, y(1) = 2]$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int y' dx = \int \ln(x)\, x^2 dx + c_1$

- Evaluate integral

  $y = \frac{x^3 \ln(x)}{3} - \frac{x^3}{9} + c_1$

- Solve for y

  $y = \frac{x^3 \ln(x)}{3} - \frac{x^3}{9} + c_1$

- Use initial condition $y(1) = 2$

  $2 = -\frac{1}{9} + c_1$

- Solve for c_1

  $c_1 = \frac{19}{9}$

- Substitute $c_1 = \frac{19}{9}$ into general solution and simplify

  $y = \frac{x^3 \ln(x)}{3} - \frac{x^3}{9} + \frac{19}{9}$

- Solution to the IVP

  $y = \frac{x^3 \ln(x)}{3} - \frac{x^3}{9} + \frac{19}{9}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve([diff(y(x),x)=x^2*ln(x),y(1) = 2],y(x), singsol=all)
```

$$y(x) = \frac{x^3 \ln(x)}{3} - \frac{x^3}{9} + \frac{19}{9}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 23

```
DSolve[{y'[x]==x^2*Log[x],{y[1]==2}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{9}\left(-x^3 + 3x^3 \log(x) + 19\right)$$

1.30 problem Problem 38

Internal problem ID [2616]
Internal file name [`OUTPUT/2108_Sunday_June_05_2022_02_48_51_AM_90750636/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.2, Basic Ideas and Terminology. page 21
Problem number: Problem 38.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_ode_quadrature", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _quadrature]]
```

$$\boxed{y'' = \cos(x)}$$

With initial conditions

$$[y(0) = 2, y'(0) = 1]$$

1.30.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= 0 \\ F &= \cos(x) \end{aligned}$$

Hence the ode is

$$y'' = \cos(x)$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $F = \cos(x)$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

1.30.2 Solving as second order ode quadrature ode

Integrating once gives

$$y' = \sin(x) + c_1$$

Integrating again gives

$$y = -\cos(x) + c_1x + c_2$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = -\cos(x) + c_1x + c_2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 0$ in the above gives

$$2 = c_2 - 1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \sin(x) + c_1$$

substituting $y' = 1$ and $x = 0$ in the above gives

$$1 = c_1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

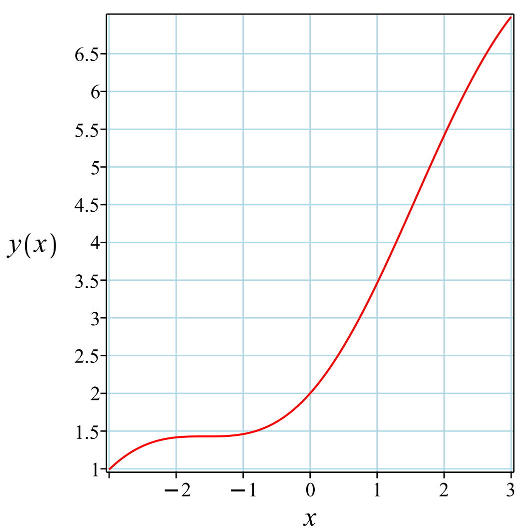
$$c_1 = 1$$
$$c_2 = 3$$

Substituting these values back in above solution results in

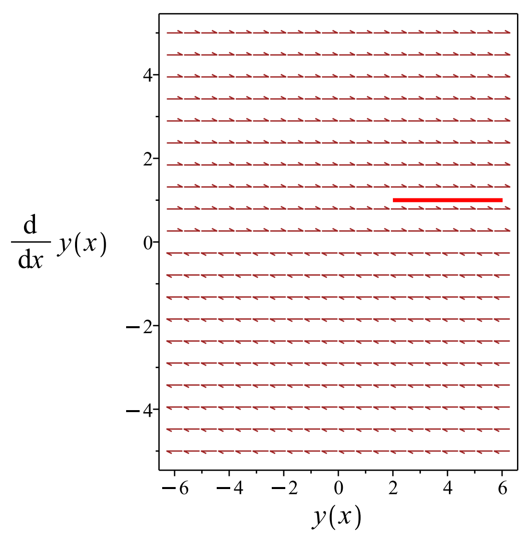
$$y = -\cos(x) + 3 + x$$

Summary
The solution(s) found are the following

$$y = -\cos(x) + 3 + x \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\cos(x) + 3 + x$$

Verified OK.

1.30.3 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 0, f(x) = \cos(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 0$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 0$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(0)^2 - (4)(1)(0)}\\ &= 0\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 0$. Therefore the solution is

$$y = c_1 1 + c_2 x \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_2 x + c_1$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, x\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x) + A_2 \sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1 \cos(x) - A_2 \sin(x) = \cos(x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -1, A_2 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\cos(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_2 x + c_1) + (-\cos(x)) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_2 x + c_1 - \cos(x) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 0$ in the above gives

$$2 = c_1 - 1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_2 + \sin(x)$$

substituting $y' = 1$ and $x = 0$ in the above gives

$$1 = c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 3$$
$$c_2 = 1$$

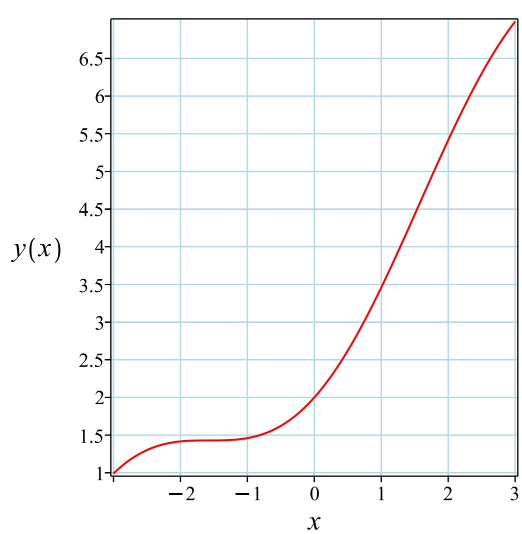
Substituting these values back in above solution results in

$$y = -\cos(x) + 3 + x$$

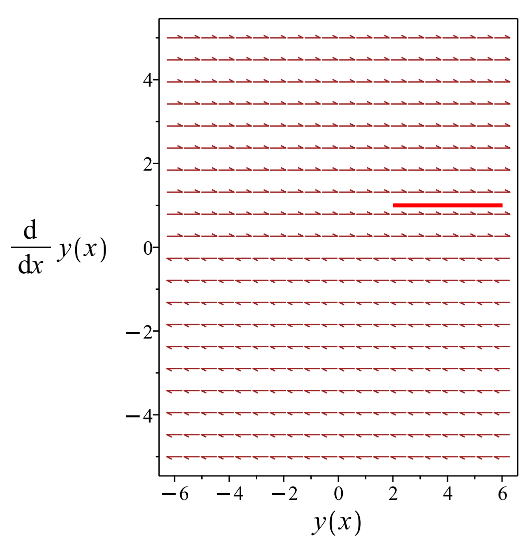
Summary

The solution(s) found are the following

$$y = -\cos(x) + 3 + x \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\cos(x) + 3 + x$$

Verified OK.

1.30.4 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int y'' dx = \int \cos(x)\, dx$$
$$y' = \sin(x) + c_1$$

Which is now solved for y. Integrating both sides gives

$$\begin{aligned} y &= \int \sin(x) + c_1 \, \mathrm{d}x \\ &= -\cos(x) + c_1 x + c_2 \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = -\cos(x) + c_1 x + c_2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 0$ in the above gives

$$2 = c_2 - 1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \sin(x) + c_1$$

substituting $y' = 1$ and $x = 0$ in the above gives

$$1 = c_1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

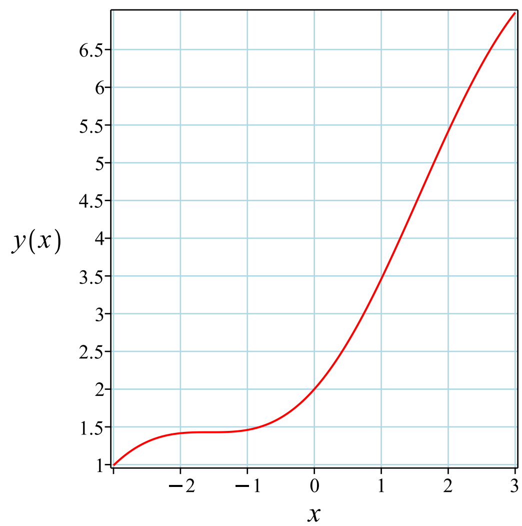
$$\begin{aligned} c_1 &= 1 \\ c_2 &= 3 \end{aligned}$$

Substituting these values back in above solution results in

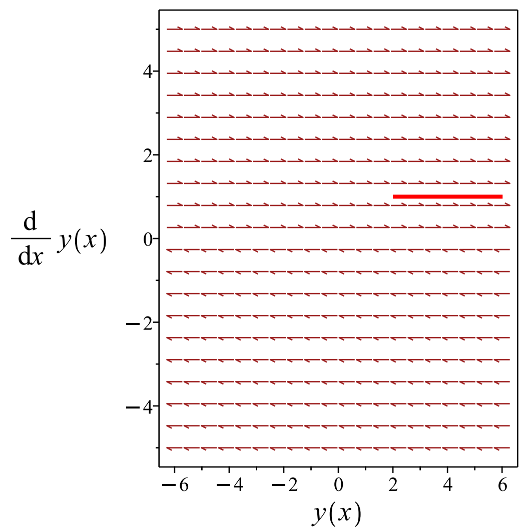
$$y = -\cos(x) + 3 + x$$

Summary
The solution(s) found are the following

$$y = -\cos(x) + 3 + x \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\cos(x) + 3 + x$$

Verified OK.

1.30.5 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) - \cos(x) = 0$$

Which is now solve for $p(x)$ as first order ode. Integrating both sides gives

$$\begin{aligned} p(x) &= \int \cos(x)\, \mathrm{d}x \\ &= \sin(x) + c_1 \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $p = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1$$

$$c_1 = 1$$

Substituting c_1 found above in the general solution gives

$$p(x) = 1 + \sin(x)$$

Since $p = y'$ then the new first order ode to solve is

$$y' = 1 + \sin(x)$$

Integrating both sides gives

$$\begin{aligned} y &= \int 1 + \sin(x)\, \mathrm{d}x \\ &= -\cos(x) + c_2 + x \end{aligned}$$

Initial conditions are used to solve for c_2. Substituting $x = 0$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = c_2 - 1$$

$$c_2 = 3$$

Substituting c_2 found above in the general solution gives

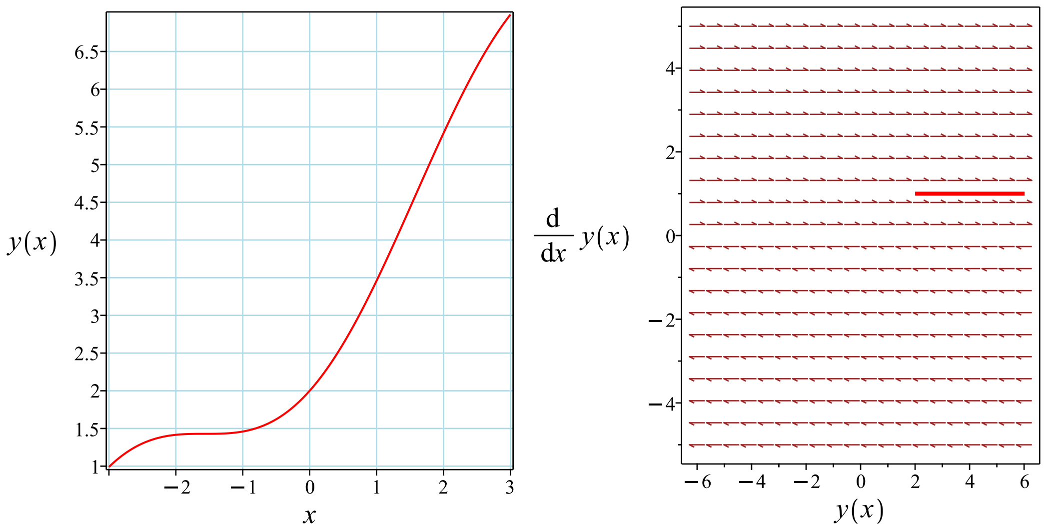
$$y = -\cos(x) + 3 + x$$

Initial conditions are used to solve for the constants of integration.

Summary

The solution(s) found are the following

$$y = -\cos(x) + 3 + x \tag{1}$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$y = -\cos(x) + 3 + x$$

Verified OK.

1.30.6 Solving using Kovacic algorithm

Writing the ode as

$$y'' = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 46: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= 1 \end{aligned}$$

Which simplifies to

$$y_1 = 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= 1 \int \frac{1}{1}\,dx \\ &= 1(x) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(1) + c_2(1(x))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_2 x + c_1
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
\cos(x)
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\cos(x), \sin(x)\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{1, x\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 \cos(x) + A_2 \sin(x)
$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
-A_1 \cos(x) - A_2 \sin(x) = \cos(x)
$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -1, A_2 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\cos(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_2 x + c_1) + (-\cos(x)) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_2 x + c_1 - \cos(x) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 0$ in the above gives

$$2 = c_1 - 1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_2 + \sin(x)$$

substituting $y' = 1$ and $x = 0$ in the above gives

$$1 = c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 3 \\ c_2 &= 1 \end{aligned}$$

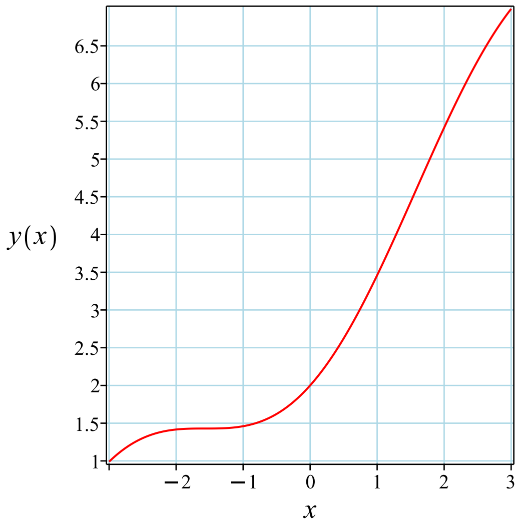
Substituting these values back in above solution results in

$$y = -\cos(x) + 3 + x$$

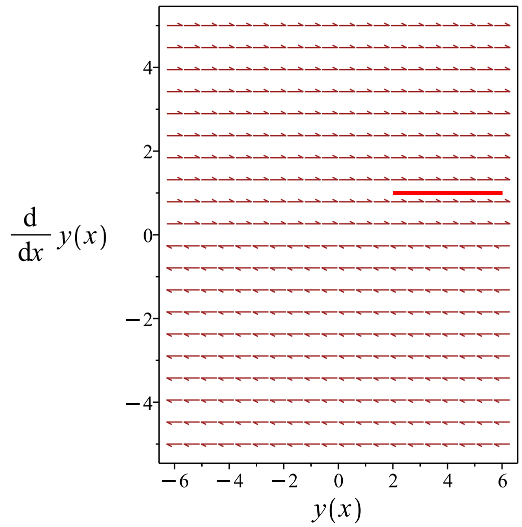
Summary

The solution(s) found are the following

$$y = -\cos(x) + 3 + x \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\cos(x) + 3 + x$$

Verified OK.

1.30.7 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 0 \\ r(x) &= 0 \\ s(x) &= \cos(x) \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 0 \\ q'(x) &= 0 \end{aligned}$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\, y' + (q(x) - p'(x))\, y)' = s(x)$$

Integrating gives

$$p(x)\, y' + (q(x) - p'(x))\, y = \int s(x)\; dx$$

Substituting the above values for p, q, r, s gives

$$y' = \int \cos(x)\; dx$$

We now have a first order ode to solve which is

$$y' = \sin(x) + c_1$$

Integrating both sides gives

$$\begin{aligned} y &= \int \sin(x) + c_1 \,\mathrm{d}x \\ &= -\cos(x) + c_1 x + c_2 \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = -\cos(x) + c_1 x + c_2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 0$ in the above gives

$$2 = c_2 - 1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \sin(x) + c_1$$

substituting $y' = 1$ and $x = 0$ in the above gives

$$1 = c_1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 1$$
$$c_2 = 3$$

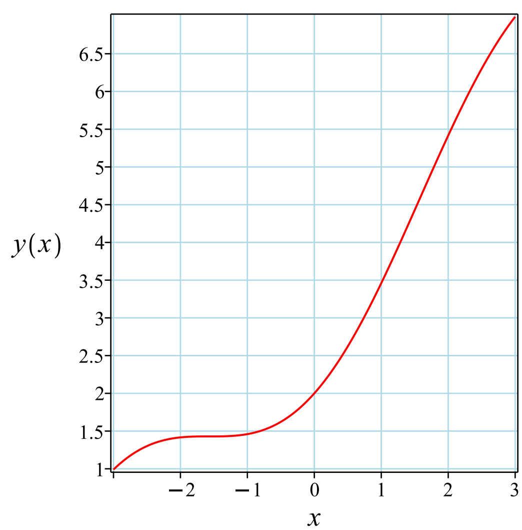
Substituting these values back in above solution results in

$$y = -\cos(x) + 3 + x$$

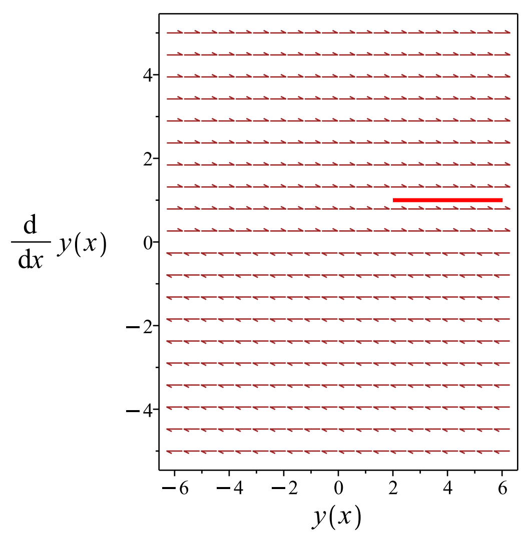
Summary

The solution(s) found are the following

$$y = -\cos(x) + 3 + x \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\cos(x) + 3 + x$$

Verified OK.

1.30.8 Maple step by step solution

Let's solve

$$\left[y'' = \cos(x), y(0) = 2, y'\Big|_{\{x=0\}} = 1\right]$$

- Highest derivative means the order of the ODE is 2

  y''
- Characteristic polynomial of homogeneous ODE

  $r^2 = 0$
- Use quadratic formula to solve for r

  $r = \frac{0 \pm \left(\sqrt{0}\right)}{2}$
- Roots of the characteristic polynomial

  $r = 0$
- 1st solution of the homogeneous ODE

  $y_1(x) = 1$
- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE

  $y = c_1 + c_2 x + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = \cos(x)\right]$
- Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$
- Compute Wronskian

  $W(y_1(x), y_2(x)) = 1$
- Substitute functions into equation for $y_p(x)$

$y_p(x) = -\left(\int \cos(x)\, x dx\right) + x\left(\int \cos(x)\, dx\right)$

- Compute integrals

  $y_p(x) = -\cos(x)$

- Substitute particular solution into general solution to ODE

  $y = c_2 x + c_1 - \cos(x)$

- Check validity of solution $y = c_2 x + c_1 - \cos(x)$
  - Use initial condition $y(0) = 2$

    $2 = c_1 - 1$
  - Compute derivative of the solution

    $y' = c_2 + \sin(x)$
  - Use the initial condition $y'\Big|_{\{x=0\}} = 1$

    $1 = c_2$
  - Solve for c_1 and c_2

    $\{c_1 = 3, c_2 = 1\}$
  - Substitute constant values into general solution and simplify

    $y = -\cos(x) + 3 + x$

- Solution to the IVP

  $y = -\cos(x) + 3 + x$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 11

```
dsolve([diff(y(x),x$2)=cos(x),y(0) = 2, D(y)(0) = 1],y(x), singsol=all)
```

$$y(x) = -\cos(x) + x + 3$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 12

```
DSolve[{y''[x]==Cos[x],{y[0]==2,y'[0]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x - \cos(x) + 3$$

1.31 problem Problem 39

Internal problem ID [2617]
Internal file name [`OUTPUT/2109_Sunday_June_05_2022_02_48_53_AM_70023368/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.2, Basic Ideas and Terminology. page 21
Problem number: Problem 39.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _quadrature]]
```

$$\boxed{y''' = 6x}$$

With initial conditions

$$[y(0) = 1, y'(0) = -1, y''(0) = -4]$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' = 0$$

The characteristic equation is

$$\lambda^3 = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 0$$
$$\lambda_3 = 0$$

Therefore the homogeneous solution is

$$y_h(x) = c_3x^2 + c_2x + c_1$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = 1$$
$$y_2 = x$$
$$y_3 = x^2$$

Now the particular solution to the given ODE is found

$$y''' = 6x$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, x, x^2\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x, x^2\}]$$

Since x is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2, x^3\}]$$

Since x^2 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^3, x^4\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_2x^4 + A_1x^3$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$24xA_2 + 6A_1 = 6x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{1}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x^4}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_3x^2 + c_2x + c_1\right) + \left(\frac{x^4}{4}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_3x^2 + c_2x + c_1 + \frac{1}{4}x^4 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$1 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = x^3 + 2c_3x + c_2$$

substituting $y' = -1$ and $x = 0$ in the above gives

$$-1 = c_2 \tag{2A}$$

Taking two derivatives of the solution gives

$$y'' = 3x^2 + 2c_3$$

substituting $y'' = -4$ and $x = 0$ in the above gives

$$-4 = 2c_3 \tag{3A}$$

Equations {1A,2A,3A} are now solved for $\{c_1, c_2, c_3\}$. Solving for the constants gives

$$c_1 = 1$$
$$c_2 = -1$$
$$c_3 = -2$$

Substituting these values back in above solution results in

$$y = -2x^2 - x + 1 + \frac{1}{4}x^4$$

Summary

The solution(s) found are the following

$$y = -2x^2 - x + 1 + \frac{1}{4}x^4 \tag{1}$$

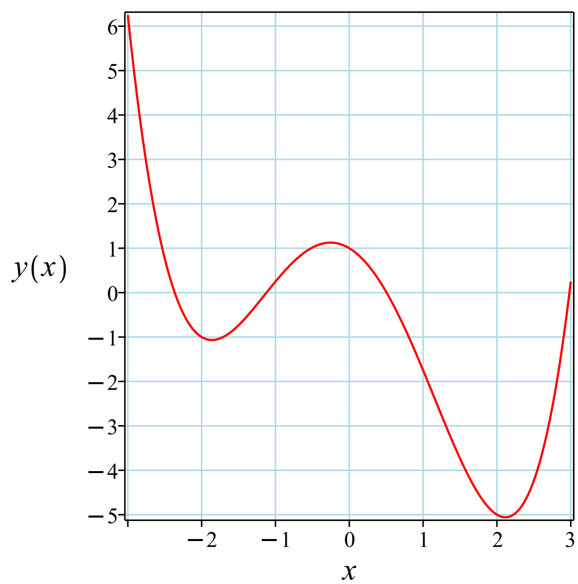


Figure 53: Solution plot

Verification of solutions

$$y = -2x^2 - x + 1 + \frac{1}{4}x^4$$

Verified OK.

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 18

```
dsolve([diff(y(x),x$3)=6*x,y(0) = 1, D(y)(0) = -1, (D@@2)(y)(0) = -4],y(x), singsol=all)
```

$$y(x) = \frac{1}{4}x^4 - 2x^2 + 1 - x$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 22

```
DSolve[{y'''[x]==6*x,{y[0]==2,y'[0]==-1,y''[0]==-4}},y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to \frac{1}{4}\left(x^4 - 8x^2 - 4x + 8\right)$$

1.32 problem Problem 40

Internal problem ID [2618]
Internal file name [`OUTPUT/2110_Sunday_June_05_2022_02_48_55_AM_14258921/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.2, Basic Ideas and Terminology. page 21
Problem number: Problem 40.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_ode_quadrature", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _quadrature]]
```

$$\boxed{y'' = x\,\mathrm{e}^{x}}$$

With initial conditions

$$[y(0) = 3, y'(0) = 4]$$

1.32.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= 0 \\ F &= x\,\mathrm{e}^{x} \end{aligned}$$

Hence the ode is

$$y'' = x\,\mathrm{e}^{x}$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $F = x\,\mathrm{e}^{x}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

1.32.2 Solving as second order ode quadrature ode

Integrating once gives

$$y' = (x-1)\,\mathrm{e}^{x} + c_1$$

Integrating again gives

$$y = (x-2)\,\mathrm{e}^{x} + c_1 x + c_2$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = (x-2)\,\mathrm{e}^{x} + c_1 x + c_2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 0$ in the above gives

$$3 = -2 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = e^x + (x - 2)e^x + c_1$$

substituting $y' = 4$ and $x = 0$ in the above gives

$$4 = c_1 - 1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 5$$
$$c_2 = 5$$

Substituting these values back in above solution results in

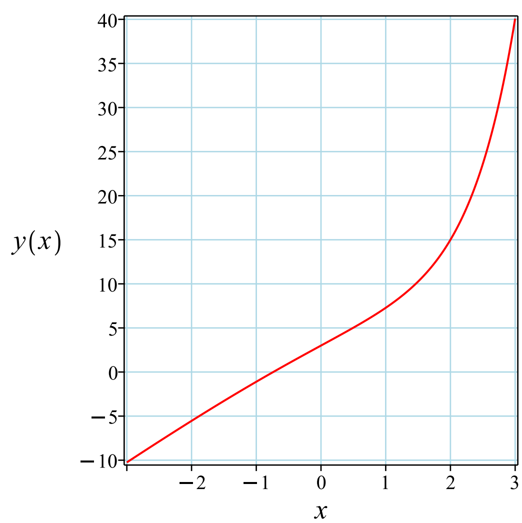
$$y = x e^x - 2e^x + 5 + 5x$$

Which simplifies to

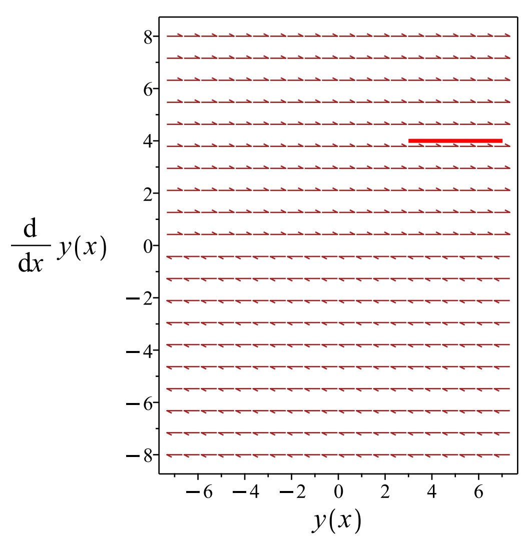
$$y = (x - 2)e^x + 5x + 5$$

Summary
The solution(s) found are the following

$$y = (x - 2)e^x + 5x + 5 \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = (x - 2)e^x + 5x + 5$$

Verified OK.

1.32.3 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 0, f(x) = x\,\mathrm{e}^{x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 0$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 0$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{(0)^2 - (4)\,(1)\,(0)}\\ &= 0\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 0$. Therefore the solution is

$$y = c_1 1 + c_2 x \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_2 x + c_1$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x\,e^{x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\,e^{x}, e^{x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, x\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 x\,e^{x} + A_2 e^{x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 e^{x} + A_1 x\,e^{x} + A_2 e^{x} = x\,e^{x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1, A_2 = -2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x\,e^{x} - 2\,e^{x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_2 x + c_1) + (x\,e^{x} - 2\,e^{x}) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_2 x + c_1 + x\,e^{x} - 2\,e^{x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 0$ in the above gives

$$3 = -2 + c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_2 + x\,e^x - e^x$$

substituting $y' = 4$ and $x = 0$ in the above gives

$$4 = c_2 - 1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 5$$
$$c_2 = 5$$

Substituting these values back in above solution results in

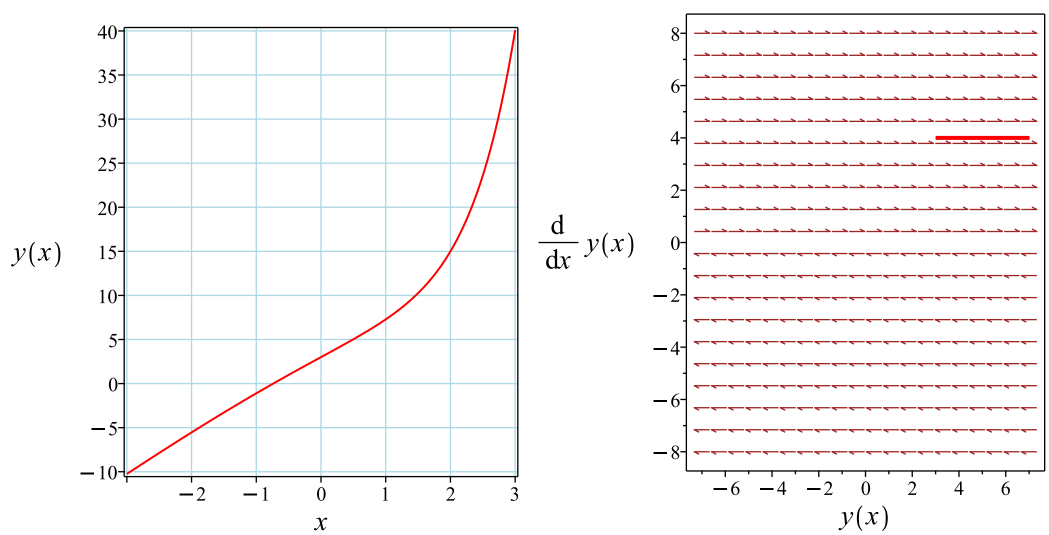
$$y = x\,e^x - 2\,e^x + 5 + 5x$$

Which simplifies to

$$y = (x - 2)\,e^x + 5x + 5$$

Summary
The solution(s) found are the following

$$y = (x - 2)\,e^x + 5x + 5 \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = (x - 2)\,\mathrm{e}^{x} + 5x + 5$$

Verified OK.

1.32.4 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\begin{aligned}\int y''dx &= \int x\,\mathrm{e}^{x}dx\\ y' &= (x - 1)\,\mathrm{e}^{x} + c_1\end{aligned}$$

Which is now solved for y. Integrating both sides gives

$$\begin{aligned}y &= \int x\,\mathrm{e}^{x} - \mathrm{e}^{x} + c_1 \,\mathrm{d}x\\ &= x\,\mathrm{e}^{x} + c_1 x - 2\,\mathrm{e}^{x} + c_2\end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = x\,\mathrm{e}^{x} + c_1 x - 2\,\mathrm{e}^{x} + c_2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 0$ in the above gives

$$3 = -2 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = x\,\mathrm{e}^{x} - \mathrm{e}^{x} + c_1$$

substituting $y' = 4$ and $x = 0$ in the above gives

$$4 = c_1 - 1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned}c_1 &= 5\\ c_2 &= 5\end{aligned}$$

Substituting these values back in above solution results in

$$y = x\,e^{x} - 2\,e^{x} + 5 + 5x$$

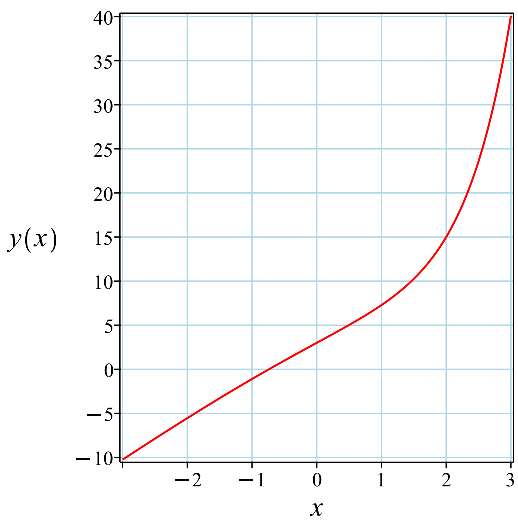
Which simplifies to

$$y = (x-2)\,e^{x} + 5x + 5$$

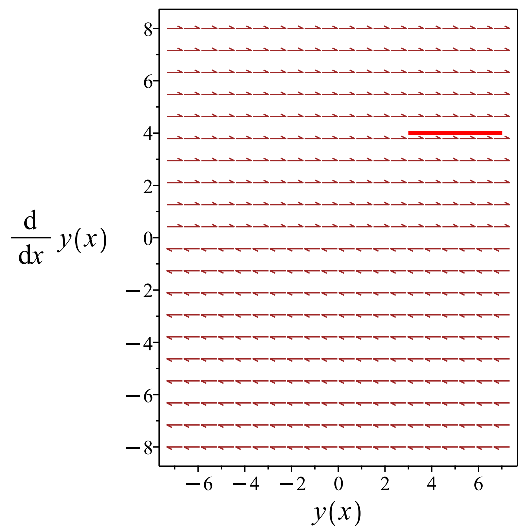
Summary

The solution(s) found are the following

$$y = (x-2)\,e^{x} + 5x + 5 \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = (x-2)\,e^{x} + 5x + 5$$

Verified OK.

1.32.5 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) - x\,\mathrm{e}^{x} = 0$$

Which is now solve for $p(x)$ as first order ode. Integrating both sides gives

$$\begin{aligned} p(x) &= \int x\,\mathrm{e}^{x}\,\mathrm{d}x \\ &= (x-1)\,\mathrm{e}^{x} + c_1 \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $p = 4$ in the above solution gives an equation to solve for the constant of integration.

$$4 = c_1 - 1$$

$$c_1 = 5$$

Substituting c_1 found above in the general solution gives

$$p(x) = x\,\mathrm{e}^{x} - \mathrm{e}^{x} + 5$$

Since $p = y'$ then the new first order ode to solve is

$$y' = x\,\mathrm{e}^{x} - \mathrm{e}^{x} + 5$$

Integrating both sides gives

$$\begin{aligned} y &= \int x\,\mathrm{e}^{x} - \mathrm{e}^{x} + 5\,\mathrm{d}x \\ &= x\,\mathrm{e}^{x} - 2\,\mathrm{e}^{x} + c_2 + 5x \end{aligned}$$

Initial conditions are used to solve for c_2. Substituting $x = 0$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = -2 + c_2$$

$$c_2 = 5$$

Substituting c_2 found above in the general solution gives

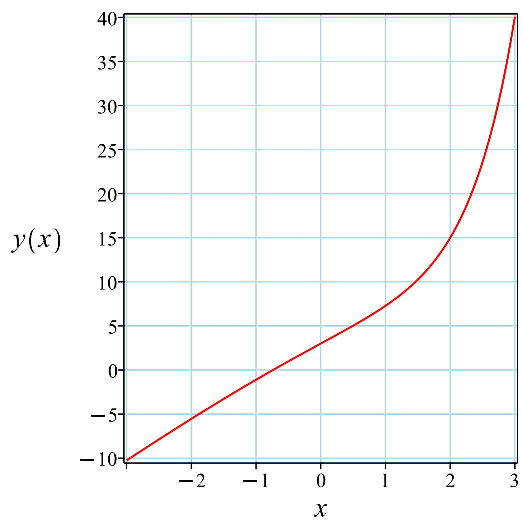
$$y = x\,e^{x} - 2\,e^{x} + 5 + 5x$$

Initial conditions are used to solve for the constants of integration.

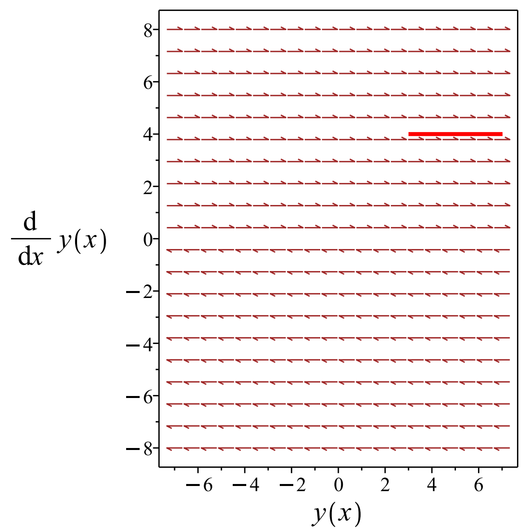
Summary

The solution(s) found are the following

$$y = x\,e^{x} - 2\,e^{x} + 5 + 5x \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = x\,e^{x} - 2\,e^{x} + 5 + 5x$$

Verified OK.

1.32.6 Solving using Kovacic algorithm

Writing the ode as

$$y'' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= 0 \\
C &= 0
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{0}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 0 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = 0 \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 48: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= 1
\end{aligned}
$$

Which simplifies to

$$y_1 = 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= 1 \int \frac{1}{1}\,dx \\ &= 1(x) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(1) + c_2(1(x)) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_2 x + c_1$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x\,\mathrm{e}^{x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\,e^{x}, e^{x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, x\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 x\,e^{x} + A_2 e^{x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 e^{x} + A_1 x\,e^{x} + A_2 e^{x} = x\,e^{x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1, A_2 = -2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x\,e^{x} - 2\,e^{x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_2 x + c_1) + (x\,e^{x} - 2\,e^{x}) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_2 x + c_1 + x\,e^{x} - 2\,e^{x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 0$ in the above gives

$$3 = -2 + c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_2 + x\,\mathrm{e}^x - \mathrm{e}^x$$

substituting $y' = 4$ and $x = 0$ in the above gives

$$4 = c_2 - 1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 5$$
$$c_2 = 5$$

Substituting these values back in above solution results in

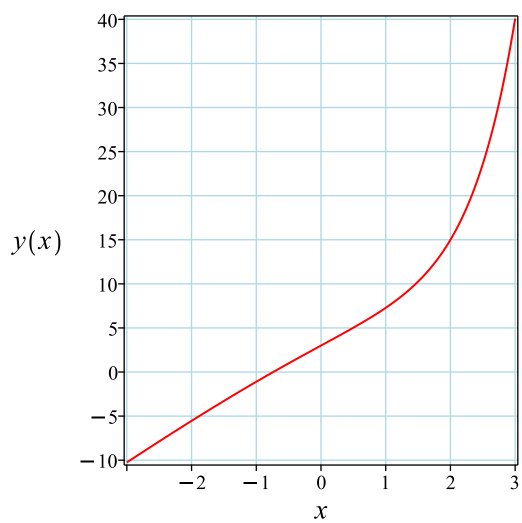
$$y = x\,\mathrm{e}^x - 2\,\mathrm{e}^x + 5 + 5x$$

Which simplifies to

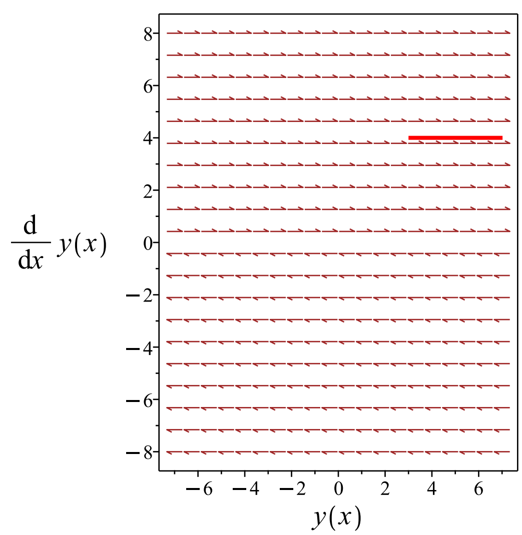
$$y = (x - 2)\,\mathrm{e}^x + 5x + 5$$

Summary
The solution(s) found are the following

$$y = (x - 2)\,\mathrm{e}^x + 5x + 5 \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = (x - 2)\,\mathrm{e}^x + 5x + 5$$

Verified OK.

1.32.7 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 0 \\ r(x) &= 0 \\ s(x) &= x\,\mathrm{e}^{x} \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 0 \\ q'(x) &= 0 \end{aligned}$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\,y' + (q(x) - p'(x))\,y)' = s(x)$$

Integrating gives

$$p(x)\,y' + (q(x) - p'(x))\,y = \int s(x)\,dx$$

Substituting the above values for p, q, r, s gives

$$y' = \int x\,\mathrm{e}^{x}\,dx$$

We now have a first order ode to solve which is

$$y' = (x - 1)\,\mathrm{e}^{x} + c_1$$

Integrating both sides gives

$$
\begin{aligned}
y &= \int x\,\mathrm{e}^{x} - \mathrm{e}^{x} + c_1 \,\mathrm{d}x \\
&= x\,\mathrm{e}^{x} + c_1 x - 2\,\mathrm{e}^{x} + c_2
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = x\,\mathrm{e}^{x} + c_1 x - 2\,\mathrm{e}^{x} + c_2 \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 0$ in the above gives

$$
3 = -2 + c_2 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = x\,\mathrm{e}^{x} - \mathrm{e}^{x} + c_1
$$

substituting $y' = 4$ and $x = 0$ in the above gives

$$
4 = c_1 - 1 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 5 \\
c_2 &= 5
\end{aligned}
$$

Substituting these values back in above solution results in

$$
y = x\,\mathrm{e}^{x} - 2\,\mathrm{e}^{x} + 5 + 5x
$$

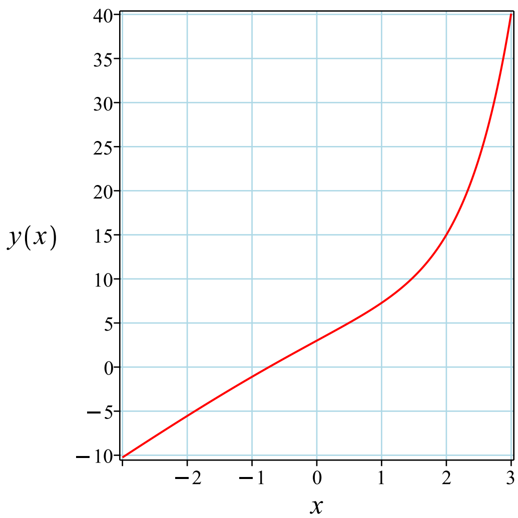
Which simplifies to

$$
y = (x - 2)\,\mathrm{e}^{x} + 5x + 5
$$

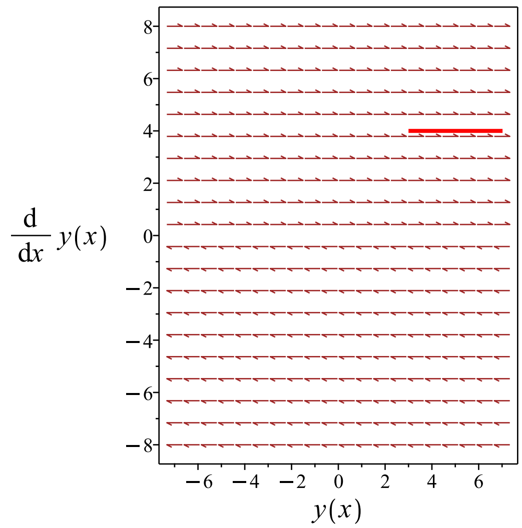
<u>Summary</u>

The solution(s) found are the following

$$
y = (x - 2)\,\mathrm{e}^{x} + 5x + 5 \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = (x-2)\,e^{x} + 5x + 5$$

Verified OK.

1.32.8 Maple step by step solution

Let's solve

$$\left[y'' = x\,e^{x}, y(0) = 3, y'\Big|_{\{x=0\}} = 4\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{0})}{2}$

- Roots of the characteristic polynomial

  $r = 0$

- 1st solution of the homogeneous ODE

$y_1(x) = 1$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

$y_2(x) = x$

- General solution of the ODE

$y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

$y = c_1 + c_2 x + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

$\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = x\,\mathrm{e}^x\right]$

- ◦ Wronskian of solutions of the homogeneous equation

$W(y_1(x), y_2(x)) = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$

- ◦ Compute Wronskian

$W(y_1(x), y_2(x)) = 1$

- ◦ Substitute functions into equation for $y_p(x)$

$y_p(x) = -\left(\int x^2 \mathrm{e}^x dx\right) + x\left(\int x\,\mathrm{e}^x dx\right)$

- ◦ Compute integrals

$y_p(x) = (x-2)\,\mathrm{e}^x$

- Substitute particular solution into general solution to ODE

$y = (x-2)\,\mathrm{e}^x + c_2 x + c_1$

□ Check validity of solution $y = (x-2)\,\mathrm{e}^x + c_2 x + c_1$

- ◦ Use initial condition $y(0) = 3$

$3 = -2 + c_1$

- ◦ Compute derivative of the solution

$y' = \mathrm{e}^x + (x-2)\,\mathrm{e}^x + c_2$

- ◦ Use the initial condition $y'\Big|_{\{x=0\}} = 4$

$4 = c_2 - 1$

- Solve for c_1 and c_2

  $\{c_1 = 5, c_2 = 5\}$

- Substitute constant values into general solution and simplify

  $y = (x - 2) e^x + 5x + 5$

- Solution to the IVP

  $y = (x - 2) e^x + 5x + 5$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 15

```
dsolve([diff(y(x),x$2)=x*exp(x),y(0) = 3, D(y)(0) = 4],y(x), singsol=all)
```

$$y(x) = (-2 + x) e^x + 5x + 5$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 18

```
DSolve[{y''[x]==x*Exp[x],{y[0]==3,y'[0]==4}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^x(x - 2) + 5(x + 1)$$

1.33 problem Problem 45

Internal problem ID [2619]
Internal file name [`OUTPUT/2111_Sunday_June_05_2022_02_48_57_AM_59657886/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.2, Basic Ideas and Terminology. page 21
Problem number: Problem 45.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + y' - 6y = 0}$$

1.33.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = -6$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + \lambda\,\mathrm{e}^{\lambda x} - 6\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda - 6 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = -6$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{-1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{1^2 - (4)(1)(-6)} \\
&= -\frac{1}{2} \pm \frac{5}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= -\frac{1}{2} + \frac{5}{2} \\
\lambda_2 &= -\frac{1}{2} - \frac{5}{2}
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= 2 \\
\lambda_2 &= -3
\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}
y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\
y &= c_1 e^{(2)x} + c_2 e^{(-3)x}
\end{aligned}$$

Or

$$y = \mathrm{e}^{2x} c_1 + c_2 \mathrm{e}^{-3x}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{2x} c_1 + c_2 \mathrm{e}^{-3x} \tag{1}$$

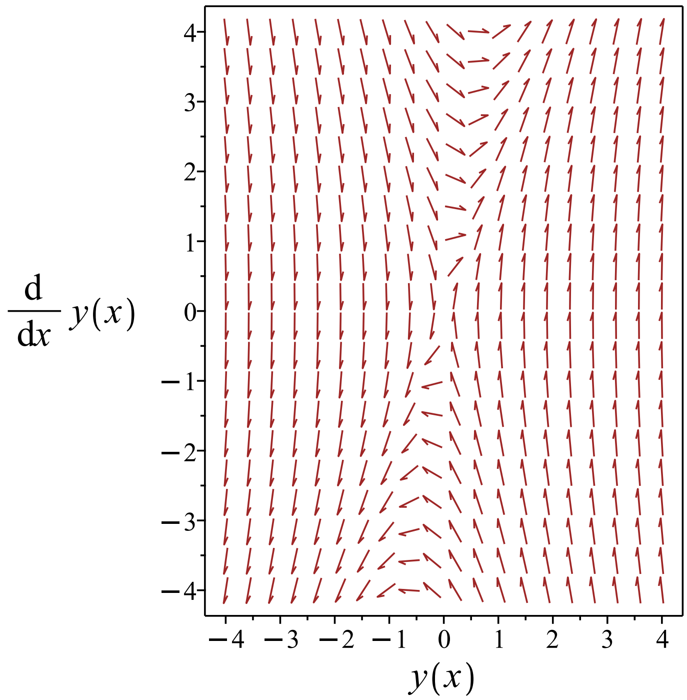


Figure 60: Slope field plot

Verification of solutions

$$
y = e^{2x}c_1 + c_2e^{-3x}
$$

Verified OK.

1.33.2 Solving using Kovacic algorithm

Writing the ode as

$$
y'' + y' - 6y = 0 \tag{1}
$$

$$
Ay'' + By' + Cy = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= 1 \\
C &= -6
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = ye^{\int \frac{B}{2A}\,dx}
$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{25}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 25 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{25z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 50: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{25}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{5x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dx} \\ &= z_1 e^{-\frac{x}{2}} \\ &= z_1\left(\mathrm{e}^{-\frac{x}{2}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-3x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-x}}{(y_1)^2}\,dx \\ &= y_1\left(\frac{\mathrm{e}^{5x}}{5}\right)\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-3x}\right) + c_2\left(\mathrm{e}^{-3x}\left(\frac{\mathrm{e}^{5x}}{5}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-3x} + \frac{c_2 \mathrm{e}^{2x}}{5} \tag{1}
$$

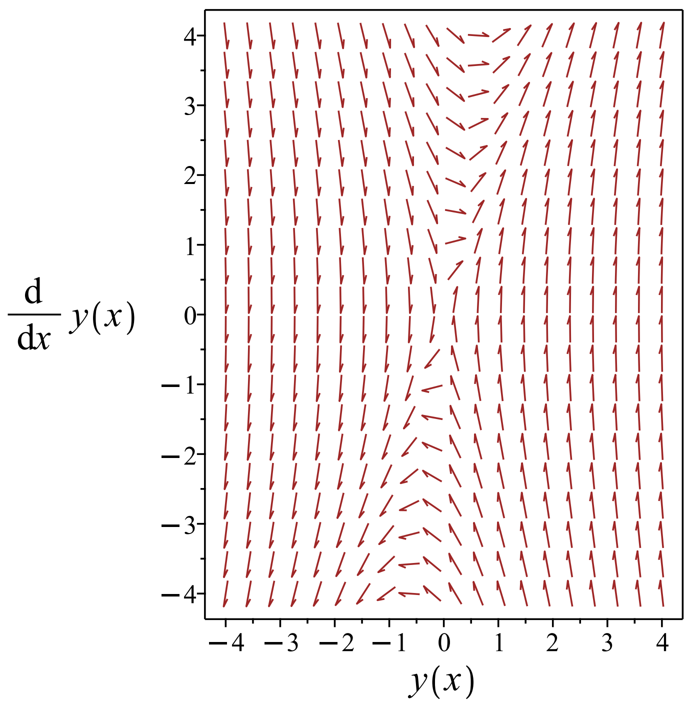


Figure 61: Slope field plot

Verification of solutions

$$
y = c_1 \mathrm{e}^{-3x} + \frac{c_2 \mathrm{e}^{2x}}{5}
$$

Verified OK.

1.33.3 Maple step by step solution

Let's solve

$y'' + y' - 6y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + r - 6 = 0$

- Factor the characteristic polynomial

  $(r+3)(r-2) = 0$

- Roots of the characteristic polynomial

  $r = (-3, 2)$

- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^{-3x}$

- 2nd solution of the ODE

  $y_2(x) = \mathrm{e}^{2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^{-3x} + c_2 \mathrm{e}^{2x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(y(x),x$2)+diff(y(x),x)-6*y(x)=0,y(x), singsol=all)
```

$$y(x) = \left(c_2 \mathrm{e}^{5x} + c_1\right) \mathrm{e}^{-3x}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 19

```
DSolve[y''[x]==x*Exp[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^x(x-2) + c_2 x + c_1$$

1.34 problem Problem 46

Internal problem ID [2620]
Internal file name [`OUTPUT/2112_Sunday_June_05_2022_02_48_59_AM_63405669/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.2, Basic Ideas and Terminology. page 21
Problem number: Problem 46.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

`[[_Emden, _Fowler]]`

$$\boxed{x^2y'' - xy' - 8y = 0}$$

1.34.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} - xrx^{r-1} - 8x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r - r\,x^r - 8x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) - r - 8 = 0$$

Or

$$r^2 - 2r - 8 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= -2 \\ r_2 &= 4 \end{aligned}$$

Since the roots are real and distinct, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = \frac{c_1}{x^2} + c_2 x^4$$

Summary
The solution(s) found are the following

$$y = \frac{c_1}{x^2} + c_2 x^4 \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x^2} + c_2 x^4$$

Verified OK.

1.34.2 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$x^2 y'' - x y' - 8y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= -\frac{8}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\
&= \int \mathrm{e}^{-\left(\int -\frac{1}{x}dx\right)}dx \\
&= \int e^{\ln(x)}\,dx \\
&= \int x dx \\
&= \frac{x^2}{2}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{-\frac{8}{x^2}}{x^2} \\
&= -\frac{8}{x^4}
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) - \frac{8y(\tau)}{x^4} &= 0
\end{aligned}$$

But in terms of τ

$$-\frac{8}{x^4} = -\frac{2}{\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) - \frac{2y(\tau)}{\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 - 2y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} - 2\tau^r = 0$$

Simplifying gives

$$r(r-1)\,\tau^r + 0\,\tau^r - 2\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$r(r-1) + 0 - 2 = 0$$

Or

$$r^2 - r - 2 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= -1 \\ r_2 &= 2 \end{aligned}$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1y_1 + c_2y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = \frac{c_1}{\tau} + c_2\tau^2$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{2c_1}{x^2} + \frac{c_2x^4}{4}$$

Summary
The solution(s) found are the following

$$y = \frac{2c_1}{x^2} + \frac{c_2 x^4}{4} \tag{1}$$

Verification of solutions

$$y = \frac{2c_1}{x^2} + \frac{c_2 x^4}{4}$$

Verified OK.

1.34.3 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$x^2 y'' - x y' - 8y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= -\frac{8}{x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{n}{x^2} - \frac{8}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 4 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{7v'(x)}{x} &= 0 \\ v''(x) + \frac{7v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{7u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{7u}{x} \end{aligned}$$

Where $f(x) = -\frac{7}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= -\frac{7}{x}\, dx \\ \int \frac{1}{u}\, du &= \int -\frac{7}{x}\, dx \\ \ln(u) &= -7\ln(x) + c_1 \\ u &= \mathrm{e}^{-7\ln(x)+c_1} \\ &= \frac{c_1}{x^7} \end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\, dx + c_2 \\ &= -\frac{c_1}{6x^6} + c_2 \end{aligned}$$

Hence

$$
\begin{aligned}
y &= v(x)\, x^n \\
&= \left(-\frac{c_1}{6x^6} + c_2\right) x^4 \\
&= \frac{6c_2x^6 - c_1}{6x^2}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \left(-\frac{c_1}{6x^6} + c_2\right) x^4 \tag{1}
$$

Verification of solutions

$$
y = \left(-\frac{c_1}{6x^6} + c_2\right) x^4
$$

Verified OK.

1.34.4 Solving using Kovacic algorithm

Writing the ode as

$$
x^2y'' - xy' - 8y = 0 \tag{1}
$$

$$
Ay'' + By' + Cy = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= x^2 \\
B &= -x \\
C &= -8
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = ye^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$
z''(x) = rz(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{35}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 35 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{35}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 52: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{35}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{5}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{35}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{35}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{5}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{35}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{5}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -\frac{5}{2} - \left(-\frac{5}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{5}{2x} + (-)\,(0) \\
&= -\frac{5}{2x} \\
&= -\frac{5}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{5}{2x}\right)(0) + \left(\left(\frac{5}{2x^2}\right) + \left(-\frac{5}{2x}\right)^2 - \left(\frac{35}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega\, dx} \\ &= \mathrm{e}^{\int -\frac{5}{2x} dx} \\ &= \frac{1}{x^{\frac{5}{2}}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x}{x^2}\, dx} \\ &= z_1 e^{\frac{\ln(x)}{2}} \\ &= z_1\left(\sqrt{x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{\ln(x)}}{(y_1)^2}\, dx \\ &= y_1\left(\frac{x^6}{6}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{x^2}\right) + c_2 \left(\frac{1}{x^2} \left(\frac{x^6}{6}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1}{x^2} + \frac{c_2 x^4}{6} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1}{x^2} + \frac{c_2 x^4}{6}
$$

Verified OK.

1.34.5 Maple step by step solution

Let's solve

$x^2 y'' - x y' - 8y = 0$

- Highest derivative means the order of the ODE is 2

  y''
- Isolate 2nd derivative

  $y'' = \frac{y'}{x} + \frac{8y}{x^2}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{y'}{x} - \frac{8y}{x^2} = 0$
- Multiply by denominators of the ODE

  $x^2 y'' - x y' - 8y = 0$
- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt} y(t)\right) t'(x)$

- Compute derivative

  $y' = \frac{\frac{d}{dt}y(t)}{x}$

- Calculate the 2nd derivative of y with respect to x , using the chain rule

  $y'' = \left(\frac{d^2}{dt^2}y(t)\right) t'(x)^2 + t''(x) \left(\frac{d}{dt}y(t)\right)$

- Compute derivative

  $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $x^2 \left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) - \frac{d}{dt}y(t) - 8y(t) = 0$

- Simplify

  $\frac{d^2}{dt^2}y(t) - 2\frac{d}{dt}y(t) - 8y(t) = 0$

- Characteristic polynomial of ODE

  $r^2 - 2r - 8 = 0$

- Factor the characteristic polynomial

  $(r+2)(r-4) = 0$

- Roots of the characteristic polynomial

  $r = (-2, 4)$

- 1st solution of the ODE

  $y_1(t) = \mathrm{e}^{-2t}$

- 2nd solution of the ODE

  $y_2(t) = \mathrm{e}^{4t}$

- General solution of the ODE

  $y(t) = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y(t) = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{4t}$

- Change variables back using $t = \ln(x)$

  $y = \frac{c_1}{x^2} + c_2 x^4$

- Simplify

  $y = \frac{c_1}{x^2} + c_2 x^4$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 15

```
dsolve(x^2*diff(y(x),x$2)-x*diff(y(x),x)-8*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_2x^6 + c_1}{x^2}$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 18

```
DSolve[x^2*y''[x]-x*y'[x]-8*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_2x^6 + c_1}{x^2}$$

1.35 problem Problem 47

Internal problem ID [2621]
Internal file name [`OUTPUT/2113_Sunday_June_05_2022_02_49_00_AM_99743443/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.2, Basic Ideas and Terminology. page 21
Problem number: Problem 47.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{x^2y'' - 3xy' + 4y = \ln(x)\, x^2}$$

1.35.1 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -3x, C = 4, f(x) = \ln(x)\, x^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$.

Solving for y_h from

$$x^2y'' - 3xy' + 4y = 0$$

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} - 3xrx^{r-1} + 4x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r - 3r\,x^r + 4x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) - 3r + 4 = 0$$

Or

$$r^2 - 4r + 4 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = 2$$
$$r_2 = 2$$

Since the roots are equal, then the general solution is

$$y = c_1y_1 + c_2y_2$$

Where $y_1 = x^r$ and $y_2 = x^r \ln(x)$. Hence

$$y = c_1x^2 + c_2x^2 \ln(x)$$

Next, we find the particular solution to the ODE

$$x^2y'' - 3xy' + 4y = \ln(x)\,x^2$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= x^2 \\
y_2 &= \ln(x)\, x^2
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$
W = \begin{vmatrix} x^2 & \ln(x)\, x^2 \\ \frac{d}{dx}(x^2) & \frac{d}{dx}(\ln(x)\, x^2) \end{vmatrix}
$$

Which gives

$$
W = \begin{vmatrix} x^2 & \ln(x)\, x^2 \\ 2x & x + 2\ln(x)\, x \end{vmatrix}
$$

Therefore

$$
W = \left(x^2\right)\left(x + 2\ln(x)\, x\right) - \left(\ln(x)\, x^2\right)(2x)
$$

Which simplifies to

$$
W = x^3
$$

Which simplifies to

$$
W = x^3
$$

Therefore Eq. (2) becomes

$$
u_1 = -\int \frac{\ln(x)^2\, x^4}{x^5}\, dx
$$

Which simplifies to

$$u_1 = -\int \frac{\ln(x)^2}{x} dx$$

Hence

$$u_1 = -\frac{\ln(x)^3}{3}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^4 \ln(x)}{x^5} dx$$

Which simplifies to

$$u_2 = \int \frac{\ln(x)}{x} dx$$

Hence

$$u_2 = \frac{\ln(x)^2}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\ln(x)^3 x^2}{6}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= x^2 \left(\frac{\ln(x)^3}{6} + c_1 + c_2 \ln(x) \right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = x^2 \left(\frac{\ln(x)^3}{6} + c_1 + c_2 \ln(x) \right) \tag{1}$$

Verification of solutions

$$y = x^2 \left(\frac{\ln(x)^3}{6} + c_1 + c_2 \ln(x) \right)$$

Verified OK.

1.35.2 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' - 3xy' + 4y = 0$$

In normal form the ode

$$x^2y'' - 3xy' + 4y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{3}{x} \\ q(x) &= \frac{4}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$
\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)} dx \\
&= \int \mathrm{e}^{-\left(\int -\frac{3}{x}dx\right)} dx \\
&= \int e^{3\ln(x)}\, dx \\
&= \int x^3 dx \\
&= \frac{x^4}{4} \qquad (6)
\end{aligned}
$$

Using (6) to evaluate q_1 from (5) gives

$$
\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{4}{x^2}}{x^6} \\
&= \frac{4}{x^8} \qquad (7)
\end{aligned}
$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$
\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + \frac{4y(\tau)}{x^8} &= 0
\end{aligned}
$$

But in terms of τ

$$
\frac{4}{x^8} = \frac{1}{4\tau^2}
$$

Hence the above ode becomes

$$
\frac{d^2}{d\tau^2}y(\tau) + \frac{y(\tau)}{4\tau^2} = 0
$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$
4\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + y(\tau) = 0
$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$
4\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + \tau^r = 0
$$

Simplifying gives

$$4r(r-1)\,\tau^r + 0\,\tau^r + \tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$4r(r-1) + 0 + 1 = 0$$

Or

$$4r^2 - 4r + 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{2}$$
$$r_2 = \frac{1}{2}$$

Since the roots are equal, then the general solution is

$$y(\tau) = c_1 y_1 + c_2 y_2$$

Where $y_1 = \tau^r$ and $y_2 = \tau^r \ln(\tau)$. Hence

$$y(\tau) = c_1\sqrt{\tau} + c_2\sqrt{\tau}\,\ln(\tau)$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{(c_2 \ln(x^4) - 2c_2 \ln(2) + c_1)\sqrt{x^4}}{2}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{(c_2 \ln(x^4) - 2c_2 \ln(2) + c_1)\sqrt{x^4}}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \sqrt{x^4} \\
y_2 &= \frac{\ln\left(x^4\right)\sqrt{x^4}}{2} - \ln\left(2\right)\sqrt{x^4}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$
W = \begin{vmatrix} \sqrt{x^4} & \frac{\ln\left(x^4\right)\sqrt{x^4}}{2} - \ln\left(2\right)\sqrt{x^4} \\ \frac{d}{dx}\left(\sqrt{x^4}\right) & \frac{d}{dx}\left(\frac{\ln\left(x^4\right)\sqrt{x^4}}{2} - \ln\left(2\right)\sqrt{x^4}\right) \end{vmatrix}
$$

Which gives

$$
W = \begin{vmatrix} \sqrt{x^4} & \frac{\ln\left(x^4\right)\sqrt{x^4}}{2} - \ln\left(2\right)\sqrt{x^4} \\ \frac{2x^3}{\sqrt{x^4}} & \frac{2\sqrt{x^4}}{x} + \frac{\ln\left(x^4\right)x^3}{\sqrt{x^4}} - \frac{2\ln\left(2\right)x^3}{\sqrt{x^4}} \end{vmatrix}
$$

Therefore

$$
W = \left(\sqrt{x^4}\right)\left(\frac{2\sqrt{x^4}}{x} + \frac{\ln\left(x^4\right)x^3}{\sqrt{x^4}} - \frac{2\ln\left(2\right)x^3}{\sqrt{x^4}}\right) - \left(\frac{\ln\left(x^4\right)\sqrt{x^4}}{2} - \ln\left(2\right)\sqrt{x^4}\right)\left(\frac{2x^3}{\sqrt{x^4}}\right)
$$

Which simplifies to

$$
W = 2x^3
$$

Which simplifies to

$$
W = 2x^3
$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\left(\frac{\ln(x^4)\sqrt{x^4}}{2} - \ln(2)\sqrt{x^4}\right)\ln(x)\,x^2}{2x^5}dx$$

Which simplifies for $0 < x$ to

$$u_1 = -\int -\frac{(-2\ln(x) + \ln(2))\ln(x)}{2x}dx$$

Hence

$$u_1 = -\frac{\ln(x)^3}{3} + \frac{\ln(2)\ln(x)^2}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\sqrt{x^4}\,\ln(x)\,x^2}{2x^5}dx$$

Which simplifies for $0 < x$ to

$$u_2 = \int \frac{\ln(x)}{2x}dx$$

Hence

$$u_2 = \frac{\ln(x)^2}{4}$$

Which simplifies to

$$u_1 = -\frac{\ln(x)^2(4\ln(x) - 3\ln(2))}{12}$$

$$u_2 = \frac{\ln(x)^2}{4}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{\ln(x)^2(4\ln(x) - 3\ln(2))\sqrt{x^4}}{12} + \frac{\ln(x)^2\left(\frac{\ln(x^4)\sqrt{x^4}}{2} - \ln(2)\sqrt{x^4}\right)}{4}$$

Which simplifies to

$$y_p(x) = \frac{\ln(x)^3 x^2}{6}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{(c_2 \ln(x^4) - 2c_2 \ln(2) + c_1)\sqrt{x^4}}{2}\right) + \left(\frac{\ln(x)^3 x^2}{6}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{(c_2 \ln(x^4) - 2c_2 \ln(2) + c_1)\sqrt{x^4}}{2} + \frac{\ln(x)^3 x^2}{6} \tag{1}$$

Verification of solutions

$$y = \frac{(c_2 \ln(x^4) - 2c_2 \ln(2) + c_1)\sqrt{x^4}}{2} + \frac{\ln(x)^3 x^2}{6}$$

Verified OK. `{0 < x}`

1.35.3 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -3x, C = 4, f(x) = \ln(x)\, x^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - 3xy' + 4y = 0$$

In normal form the ode

$$x^2y'' - 3xy' + 4y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = -\frac{3}{x}$$
$$q(x) = \frac{4}{x^2}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$
$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{2\sqrt{\frac{1}{x^2}}}{c} \\ \tau'' &= -\frac{2}{c\sqrt{\frac{1}{x^2}}\, x^3} \end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned} p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\ &= \frac{-\frac{2}{c\sqrt{\frac{1}{x^2}}\, x^3} - \frac{3}{x}\frac{2\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{2\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\ &= -2c \end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned} y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2} y(\tau) - 2c\left(\frac{d}{d\tau} y(\tau)\right) + c^2 y(\tau) &= 0 \end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = e^{c\tau} c_1$$

Now from (6)

$$\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int 2\sqrt{\frac{1}{x^2}}dx}{c} \\
&= \frac{2\sqrt{\frac{1}{x^2}}\,x\ln(x)}{c}
\end{aligned}$$

Substituting the above into the solution obtained gives

$$y = c_1 x^2$$

Now the particular solution to this ODE is found

$$x^2y'' - 3xy' + 4y = \ln(x)\,x^2$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}
y_1 &= \sqrt{x^4} \\
y_2 &= \frac{\ln(x^4)\sqrt{x^4}}{2} - \ln(2)\sqrt{x^4}
\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \sqrt{x^4} & \frac{\ln(x^4)\sqrt{x^4}}{2} - \ln(2)\sqrt{x^4} \\ \frac{d}{dx}\left(\sqrt{x^4}\right) & \frac{d}{dx}\left(\frac{\ln(x^4)\sqrt{x^4}}{2} - \ln(2)\sqrt{x^4}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \sqrt{x^4} & \frac{\ln(x^4)\sqrt{x^4}}{2} - \ln(2)\sqrt{x^4} \\ \frac{2x^3}{\sqrt{x^4}} & \frac{2\sqrt{x^4}}{x} + \frac{\ln(x^4)x^3}{\sqrt{x^4}} - \frac{2\ln(2)x^3}{\sqrt{x^4}} \end{vmatrix}$$

Therefore

$$W = \left(\sqrt{x^4}\right)\left(\frac{2\sqrt{x^4}}{x} + \frac{\ln(x^4)\,x^3}{\sqrt{x^4}} - \frac{2\ln(2)\,x^3}{\sqrt{x^4}}\right) - \left(\frac{\ln(x^4)\sqrt{x^4}}{2} - \ln(2)\sqrt{x^4}\right)\left(\frac{2x^3}{\sqrt{x^4}}\right)$$

Which simplifies to

$$W = 2x^3$$

Which simplifies to

$$W = 2x^3$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\left(\frac{\ln(x^4)\sqrt{x^4}}{2} - \ln(2)\sqrt{x^4}\right)\ln(x)\,x^2}{2x^5}\,dx$$

Which simplifies for $0 < x$ to

$$u_1 = -\int -\frac{(-2\ln(x) + \ln(2))\ln(x)}{2x}dx$$

Hence

$$u_1 = -\frac{\ln(x)^3}{3} + \frac{\ln(2)\ln(x)^2}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\sqrt{x^4}\, \ln(x)\, x^2}{2x^5}\, dx$$

Which simplifies for $0 < x$ to

$$u_2 = \int \frac{\ln(x)}{2x} dx$$

Hence

$$u_2 = \frac{\ln(x)^2}{4}$$

Which simplifies to

$$u_1 = -\frac{\ln(x)^2\, (4\ln(x) - 3\ln(2))}{12}$$

$$u_2 = \frac{\ln(x)^2}{4}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{\ln(x)^2\, (4\ln(x) - 3\ln(2))\, \sqrt{x^4}}{12} + \frac{\ln(x)^2 \left(\frac{\ln(x^4)\sqrt{x^4}}{2} - \ln(2)\sqrt{x^4}\right)}{4}$$

Which simplifies to

$$y_p(x) = \frac{\ln(x)^3\, x^2}{6}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 x^2) + \left(\frac{\ln(x)^3\, x^2}{6}\right) \\ &= \frac{\ln(x)^3\, x^2}{6} + c_1 x^2 \end{aligned}$$

Which simplifies to

$$y = x^2 \left(\frac{\ln(x)^3}{6} + c_1\right)$$

Summary
The solution(s) found are the following

$$y = x^2 \left(\frac{\ln(x)^3}{6} + c_1\right) \tag{1}$$

Verification of solutions

$$y = x^2 \left(\frac{\ln(x)^3}{6} + c_1\right)$$

Verified OK. `{0 < x}`

1.35.4 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -3x, C = 4, f(x) = \ln(x)\, x^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - 3xy' + 4y = 0$$

In normal form the ode

$$x^2y'' - 3xy' + 4y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = -\frac{3}{x}$$
$$q(x) = \frac{4}{x^2}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{3n}{x^2} + \frac{4}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 2 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{v'(x)}{x} &= 0 \\ v''(x) + \frac{v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{u}{x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= -\frac{1}{x}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{1}{x}\,dx \\
\ln(u) &= -\ln(x) + c_1 \\
u &= \mathrm{e}^{-\ln(x)+c_1} \\
&= \frac{c_1}{x}
\end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\,dx + c_2 \\
&= c_1 \ln(x) + c_2
\end{aligned}$$

Hence

$$\begin{aligned}
y &= v(x)\,x^n \\
&= (c_1 \ln(x) + c_2)\,x^2 \\
&= (c_1 \ln(x) + c_2)\,x^2
\end{aligned}$$

Now the particular solution to this ODE is found

$$x^2y'' - 3xy' + 4y = \ln(x)\,x^2$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}
y_1 &= x^2 \\
y_2 &= \ln(x)\,x^2
\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^2 & \ln(x)\, x^2 \\ \frac{d}{dx}(x^2) & \frac{d}{dx}(\ln(x)\, x^2) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^2 & \ln(x)\, x^2 \\ 2x & x + 2\ln(x)\, x \end{vmatrix}$$

Therefore

$$W = (x^2)(x + 2\ln(x)\, x) - (\ln(x)\, x^2)(2x)$$

Which simplifies to

$$W = x^3$$

Which simplifies to

$$W = x^3$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\ln(x)^2 x^4}{x^5}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{\ln(x)^2}{x}\, dx$$

Hence

$$u_1 = -\frac{\ln(x)^3}{3}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^4 \ln(x)}{x^5} dx$$

Which simplifies to

$$u_2 = \int \frac{\ln(x)}{x} dx$$

Hence

$$u_2 = \frac{\ln(x)^2}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\ln(x)^3 x^2}{6}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left((c_1 \ln(x) + c_2) x^2\right) + \left(\frac{\ln(x)^3 x^2}{6}\right) \\ &= \frac{\ln(x)^3 x^2}{6} + (c_1 \ln(x) + c_2) x^2 \end{aligned}$$

Which simplifies to

$$y = x^2 \left(\frac{\ln(x)^3}{6} + c_1 \ln(x) + c_2\right)$$

Summary

The solution(s) found are the following

$$y = x^2 \left(\frac{\ln(x)^3}{6} + c_1 \ln(x) + c_2\right) \tag{1}$$

Verification of solutions

$$y = x^2 \left(\frac{\ln(x)^3}{6} + c_1 \ln(x) + c_2\right)$$

Verified OK. `{0 < x}`

1.35.5 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' - 3xy' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -3x \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 54: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = -\frac{1}{4x^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = -\frac{1}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + (-)(0) \\
&= \frac{1}{2x} \\
&= \frac{1}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right) \right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \frac{1}{2x} dx} \\
&= \sqrt{x}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-3x}{x^2}\,dx} \\
&= z_1 e^{\frac{3\ln(x)}{2}} \\
&= z_1\left(x^{\frac{3}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^2
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-3x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{3\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(\ln(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^2\right) + c_2\left(x^2(\ln(x))\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$
x^2y'' - 3xy' + 4y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 x^2 + c_2 x^2 \ln(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= x^2 \\ y_2 &= \ln(x)\, x^2 \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^2 & \ln(x)\, x^2 \\ \frac{d}{dx}(x^2) & \frac{d}{dx}(\ln(x)\, x^2) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^2 & \ln(x)\, x^2 \\ 2x & x + 2\ln(x)\, x \end{vmatrix}$$

Therefore

$$W = (x^2)(x + 2\ln(x)\, x) - (\ln(x)\, x^2)(2x)$$

Which simplifies to

$$W = x^3$$

Which simplifies to

$$W = x^3$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\ln(x)^2 x^4}{x^5} dx$$

Which simplifies to

$$u_1 = -\int \frac{\ln(x)^2}{x} dx$$

Hence

$$u_1 = -\frac{\ln(x)^3}{3}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^4 \ln(x)}{x^5} dx$$

Which simplifies to

$$u_2 = \int \frac{\ln(x)}{x} dx$$

Hence

$$u_2 = \frac{\ln(x)^2}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\ln(x)^3 x^2}{6}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 x^2 + c_2 x^2 \ln(x)\right) + \left(\frac{\ln(x)^3 x^2}{6}\right) \end{aligned}$$

Which simplifies to

$$y = x^2(c_1 + c_2 \ln(x)) + \frac{\ln(x)^3 x^2}{6}$$

Summary

The solution(s) found are the following

$$y = x^2(c_1 + c_2 \ln(x)) + \frac{\ln(x)^3 x^2}{6} \tag{1}$$

Verification of solutions

$$y = x^2(c_1 + c_2 \ln(x)) + \frac{\ln(x)^3 x^2}{6}$$

Verified OK. {0 < x}

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   <- LODE of Euler type successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 20

```
dsolve(x^2*diff(y(x),x$2)-3*x*diff(y(x),x)+4*y(x)=x^2*ln(x),y(x), singsol=all)
```

$$y(x) = x^2\left(c_2 + \ln(x) c_1 + \frac{\ln(x)^3}{6}\right)$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 27

```
DSolve[x^2*y''[x]-3*x*y'[x]+4*y[x]==x^2*Log[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{6}x^2\left(\log^3(x) + 12c_2\log(x) + 6c_1\right)$$

2 Chapter 1, First-Order Differential Equations. Section 1.4, Separable Differential Equations. page 43

2.1 problem Problem 1

Internal problem ID [2622]
Internal file name [`OUTPUT/2114_Sunday_June_05_2022_02_49_02_AM_83416222/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.4, Separable Differential Equations. page 43
Problem number: Problem 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - 2yx = 0}$$

2.1.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= 2xy \end{aligned}$$

Where $f(x) = 2x$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned} \frac{1}{y}\,dy &= 2x\,dx \\ \int \frac{1}{y}\,dy &= \int 2x\,dx \\ \ln(y) &= x^2 + c_1 \\ y &= \mathrm{e}^{x^2+c_1} \\ &= c_1 \mathrm{e}^{x^2} \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{x^2} \tag{1}$$

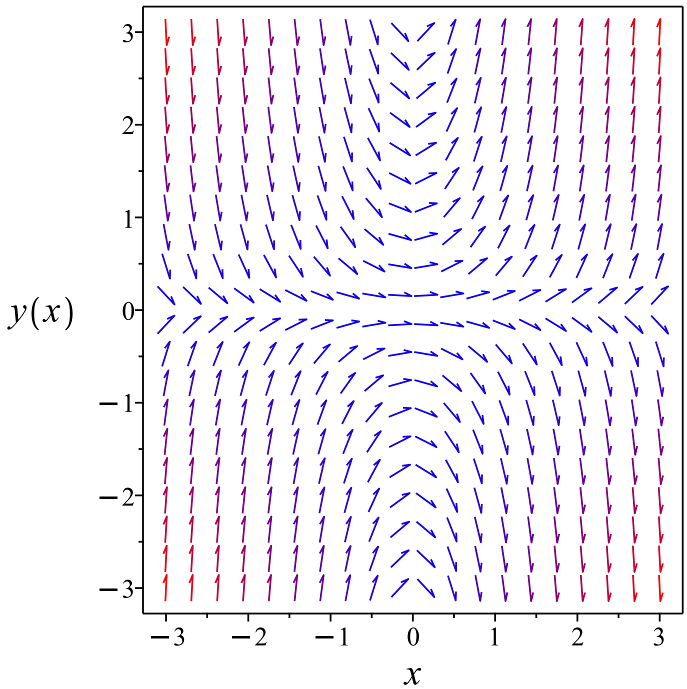


Figure 62: Slope field plot

Verification of solutions

$$y = c_1 e^{x^2}$$

Verified OK.

2.1.2 Maple step by step solution

Let's solve

$y' - 2yx = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = 2x$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int 2x dx + c_1$

- Evaluate integral

  $\ln(y) = x^2 + c_1$

- Solve for y

  $y = e^{x^2 + c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 10

```
dsolve(diff(y(x),x)=2*x*y(x),y(x), singsol=all)
```

$$y(x) = e^{x^2} c_1$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 18

```
DSolve[y'[x]==2*x*y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{x^2}$$
$$y(x) \to 0$$

2.2 problem Problem 2

Internal problem ID [2623]
Internal file name [`OUTPUT/2115_Sunday_June_05_2022_02_49_05_AM_4677793/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.4, Separable Differential Equations. page 43
Problem number: Problem 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{y^2}{x^2+1} = 0}$$

2.2.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{y^2}{x^2+1} \end{aligned}$$

Where $f(x) = \frac{1}{x^2+1}$ and $g(y) = y^2$. Integrating both sides gives

$$\begin{aligned} \frac{1}{y^2}\,dy &= \frac{1}{x^2+1}\,dx \\ \int \frac{1}{y^2}\,dy &= \int \frac{1}{x^2+1}\,dx \\ -\frac{1}{y} &= \arctan(x) + c_1 \end{aligned}$$

Which results in

$$y = -\frac{1}{\arctan(x) + c_1}$$

Summary
The solution(s) found are the following

$$y = -\frac{1}{\arctan(x) + c_1} \tag{1}$$

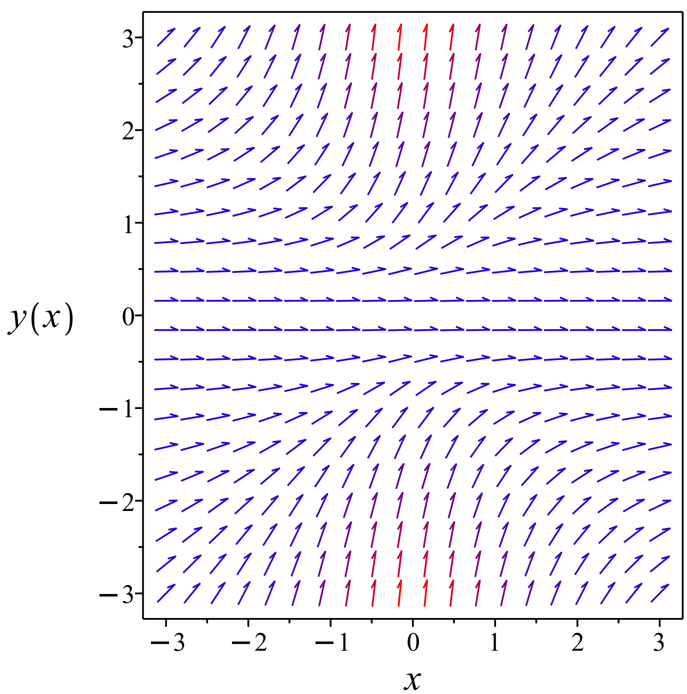


Figure 63: Slope field plot

Verification of solutions

$$y = -\frac{1}{\arctan(x) + c_1}$$

Verified OK.

2.2.2 Maple step by step solution

Let's solve

$y' - \frac{y^2}{x^2+1} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^2} = \frac{1}{x^2+1}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y^2}dx = \int \frac{1}{x^2+1}dx + c_1$

- Evaluate integral

  $-\frac{1}{y} = \arctan(x) + c_1$

- Solve for y

  $y = -\frac{1}{\arctan(x)+c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(diff(y(x),x)=y(x)^2/(x^2+1),y(x), singsol=all)
```

$$y(x) = \frac{1}{-\arctan(x) + c_1}$$

✓ Solution by Mathematica

Time used: 0.16 (sec). Leaf size: 19

```
DSolve[y'[x]==y[x]^2/(x^2+1),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{\arctan(x) + c_1}$$
$$y(x) \to 0$$

2.3 problem Problem 3

Internal problem ID [2624]
Internal file name [`OUTPUT/2116_Sunday_June_05_2022_02_49_06_AM_88882434/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.4, Separable Differential Equations. page 43
Problem number: Problem 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_separable]`

$$\boxed{e^{y+x}y' = 1}$$

2.3.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= e^{-y}e^{-x} \end{aligned}$$

Where $f(x) = e^{-x}$ and $g(y) = e^{-y}$. Integrating both sides gives

$$\begin{aligned} \frac{1}{e^{-y}}\, dy &= e^{-x}\, dx \\ \int \frac{1}{e^{-y}}\, dy &= \int e^{-x}\, dx \\ e^{y} &= -e^{-x} + c_1 \end{aligned}$$

Which results in

$$y = \ln\left(-1 + c_1 \mathrm{e}^{x}\right) - x$$

Summary

The solution(s) found are the following

$$y = \ln\left(-1 + c_1 \mathrm{e}^{x}\right) - x \tag{1}$$

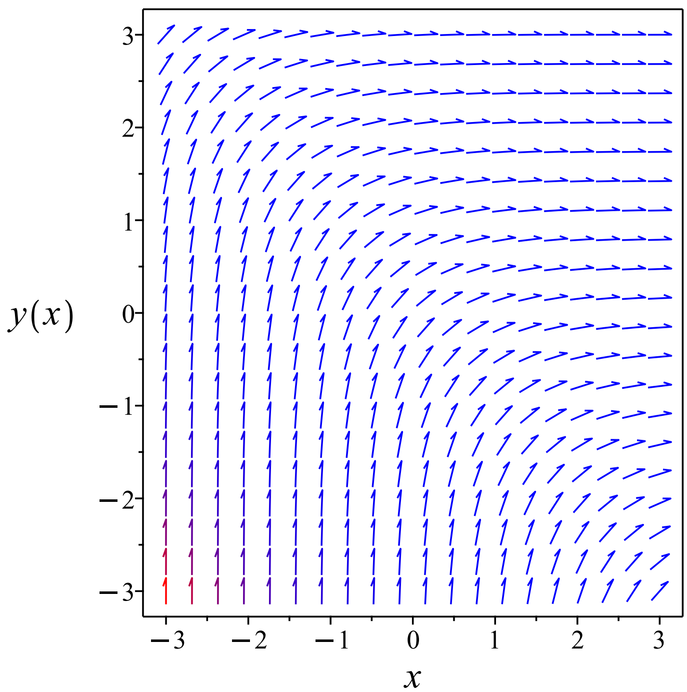


Figure 64: Slope field plot

Verification of solutions

$$y = \ln\left(-1 + c_1 \mathrm{e}^{x}\right) - x$$

Verified OK.

2.3.2 Maple step by step solution

Let's solve

$e^{y+x}y' = 1$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $y'e^{y} = \frac{1}{e^{x}}$

- Integrate both sides with respect to x

  $\int y'e^{y}dx = \int \frac{1}{e^{x}}dx + c_1$

- Evaluate integral

  $e^{y} = -\frac{1}{e^{x}} + c_1$

- Solve for y

  $y = \ln(-1 + c_1 e^{x}) - x$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(exp(x+y(x))*diff(y(x),x)-1=0,y(x), singsol=all)
```

$$y(x) = \ln(e^{x}c_1 - 1) - x$$

✓ Solution by Mathematica

Time used: 0.097 (sec). Leaf size: 16

```
DSolve[Exp[x+y[x]]*y'[x]-1==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \log \left(-e^{-x} + c_1\right)$$

2.4 problem Problem 4

Internal problem ID [2625]
Internal file name [`OUTPUT/2117_Sunday_June_05_2022_02_49_09_AM_36847014/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.4, Separable Differential Equations. page 43
Problem number: Problem 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{y}{\ln(x)\, x} = 0}$$

2.4.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{y}{\ln(x)\, x} \end{aligned}$$

Where $f(x) = \frac{1}{\ln(x)x}$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned} \frac{1}{y}\, dy &= \frac{1}{\ln(x)\, x}\, dx \\ \int \frac{1}{y}\, dy &= \int \frac{1}{\ln(x)\, x}\, dx \\ \ln(y) &= \ln(\ln(x)) + c_1 \\ y &= \mathrm{e}^{\ln(\ln(x))+c_1} \\ &= c_1 \ln(x) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \ln(x) \tag{1}$$

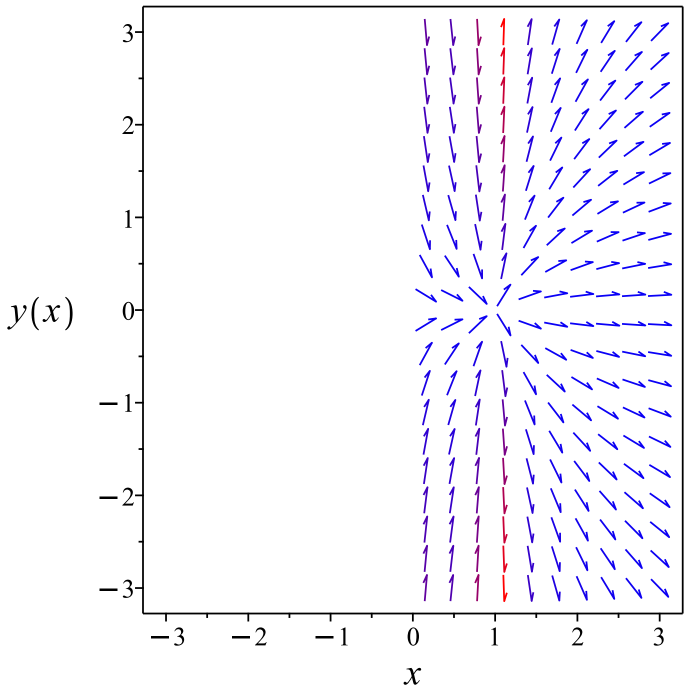


Figure 65: Slope field plot

Verification of solutions

$$y = c_1 \ln(x)$$

Verified OK.

2.4.2 Maple step by step solution

Let's solve

$y' - \frac{y}{\ln(x)x} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = \frac{1}{\ln(x)x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int \frac{1}{\ln(x) x} dx + c_1$
- Evaluate integral

  $\ln(y) = \ln(\ln(x)) + c_1$
- Solve for y

  $y = e^{c_1} \ln(x)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 8

```
dsolve(diff(y(x),x)=y(x)/(x*ln(x)),y(x), singsol=all)
```

$$y(x) = \ln(x) c_1$$

✓ Solution by Mathematica

Time used: 0.026 (sec). Leaf size: 15

```
DSolve[y'[x]==y[x]/(x*Log[x]),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \log(x)$$
$$y(x) \to 0$$

2.5 problem Problem 5

Internal problem ID [2626]
Internal file name [`OUTPUT/2118_Sunday_June_05_2022_02_49_11_AM_50037576/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.4, Separable Differential Equations. page 43
Problem number: Problem 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y-(x-1)\,y'=0}$$

2.5.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{y}{x-1} \end{aligned}$$

Where $f(x)=\frac{1}{x-1}$ and $g(y)=y$. Integrating both sides gives

$$\begin{aligned} \frac{1}{y}\,dy &= \frac{1}{x-1}\,dx \\ \int \frac{1}{y}\,dy &= \int \frac{1}{x-1}\,dx \\ \ln(y) &= \ln(x-1)+c_1 \\ y &= e^{\ln(x-1)+c_1} \\ &= c_1(x-1) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1(x - 1) \tag{1}$$

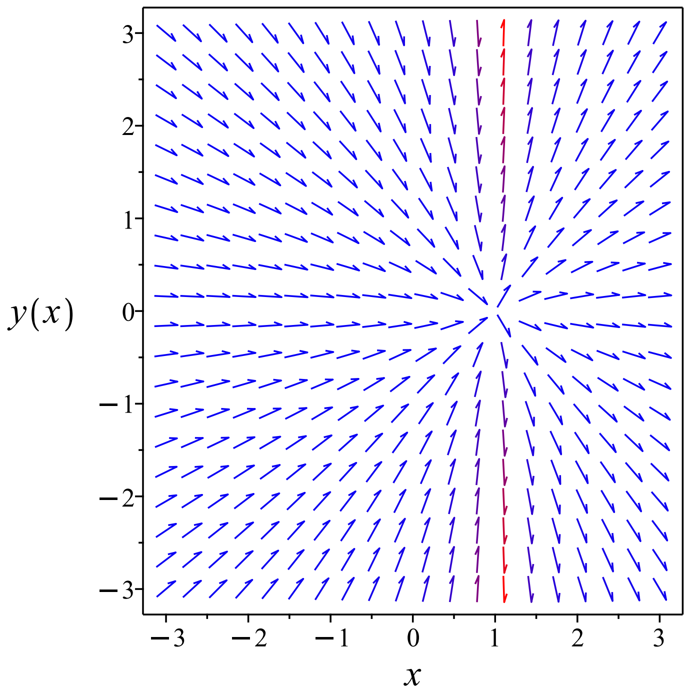


Figure 66: Slope field plot

Verification of solutions

$$y = c_1(x - 1)$$

Verified OK.

2.5.2 Maple step by step solution

Let's solve

$y - (x - 1)\, y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = \frac{1}{x-1}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int \frac{1}{x-1} dx + c_1$

- Evaluate integral

  $\ln (y) = \ln (x-1) + c_1$

- Solve for y

  $y = e^{c_1}(x-1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 9

```
dsolve(y(x)-(x-1)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = c_1(x-1)$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 16

```
DSolve[y[x]-(x-1)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1(x-1)$$
$$y(x) \to 0$$

2.6 problem Problem 6

Internal problem ID [2627]
Internal file name [`OUTPUT/2119_Sunday_June_05_2022_02_49_12_AM_27167669/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.4, Separable Differential Equations. page 43
Problem number: Problem 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{2x(y-1)}{x^2+3} = 0}$$

2.6.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{x(2y-2)}{x^2+3} \end{aligned}$$

Where $f(x) = \frac{x}{x^2+3}$ and $g(y) = 2y - 2$. Integrating both sides gives

$$\begin{aligned} \frac{1}{2y-2}\,dy &= \frac{x}{x^2+3}\,dx \\ \int \frac{1}{2y-2}\,dy &= \int \frac{x}{x^2+3}\,dx \\ \frac{\ln(y-1)}{2} &= \frac{\ln(x^2+3)}{2} + c_1 \end{aligned}$$

Raising both side to exponential gives

$$\sqrt{y-1} = \mathrm{e}^{\frac{\ln(x^2+3)}{2}+c_1}$$

Which simplifies to

$$\sqrt{y-1} = c_2\sqrt{x^2+3}$$

Which simplifies to

$$y = c_2^2\left(x^2+3\right)\mathrm{e}^{2c_1}+1$$

Summary

The solution(s) found are the following

$$y = c_2^2\left(x^2+3\right)\mathrm{e}^{2c_1}+1 \tag{1}$$

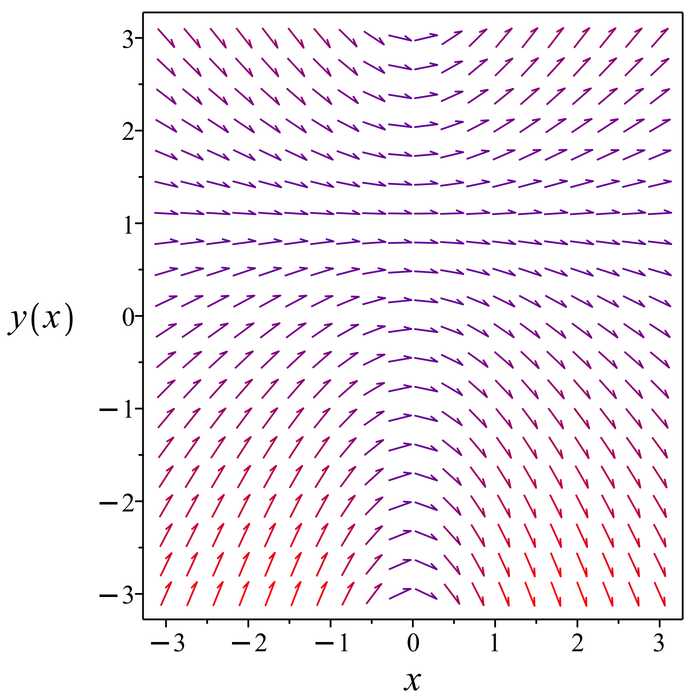


Figure 67: Slope field plot

Verification of solutions

$$y = c_2^2\left(x^2+3\right)\mathrm{e}^{2c_1}+1$$

Verified OK.

2.6.2 Maple step by step solution

Let's solve

$y' - \frac{2x(y-1)}{x^2+3} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y-1} = \frac{2x}{x^2+3}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y-1}dx = \int \frac{2x}{x^2+3}dx + c_1$

- Evaluate integral

  $\ln(y-1) = \ln(x^2+3) + c_1$

- Solve for y

  $y = e^{c_1}x^2 + 3\,e^{c_1} + 1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(diff(y(x),x)=(2*x*(y(x)-1))/(x^2+3),y(x), singsol=all)
```

$$y(x) = c_1x^2 + 3c_1 + 1$$

✓ Solution by Mathematica

Time used: 0.031 (sec). Leaf size: 20

```
DSolve[y'[x]==(2*x*(y[x]-1))/(x^2+3),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 1 + c_1(x^2 + 3)$$
$$y(x) \to 1$$

2.7 problem Problem 7

Internal problem ID [2628]
Internal file name [`OUTPUT/2120_Sunday_June_05_2022_02_49_15_AM_13431656/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.4, Separable Differential Equations. page 43
Problem number: Problem 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_separable]`

$$\boxed{-xy' + y + 2y'x^2 = 3}$$

2.7.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{-y+3}{x\,(2x-1)} \end{aligned}$$

Where $f(x) = \frac{1}{x(2x-1)}$ and $g(y) = -y+3$. Integrating both sides gives

$$\begin{aligned} \frac{1}{-y+3}\,dy &= \frac{1}{x\,(2x-1)}\,dx \\ \int \frac{1}{-y+3}\,dy &= \int \frac{1}{x\,(2x-1)}\,dx \\ -\ln(y-3) &= \ln(2x-1) - \ln(x) + c_1 \end{aligned}$$

Raising both side to exponential gives

$$\frac{1}{y-3} = e^{\ln(2x-1)-\ln(x)+c_1}$$

Which simplifies to

$$\frac{1}{y-3} = c_2 e^{\ln(2x-1)-\ln(x)}$$

Which simplifies to

$$y = \frac{3c_2\left(2\,e^{c_1} - \frac{e^{c_1}}{x}\right) + 1}{c_2\left(2\,e^{c_1} - \frac{e^{c_1}}{x}\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{3c_2\left(2\,e^{c_1} - \frac{e^{c_1}}{x}\right) + 1}{c_2\left(2\,e^{c_1} - \frac{e^{c_1}}{x}\right)} \tag{1}$$

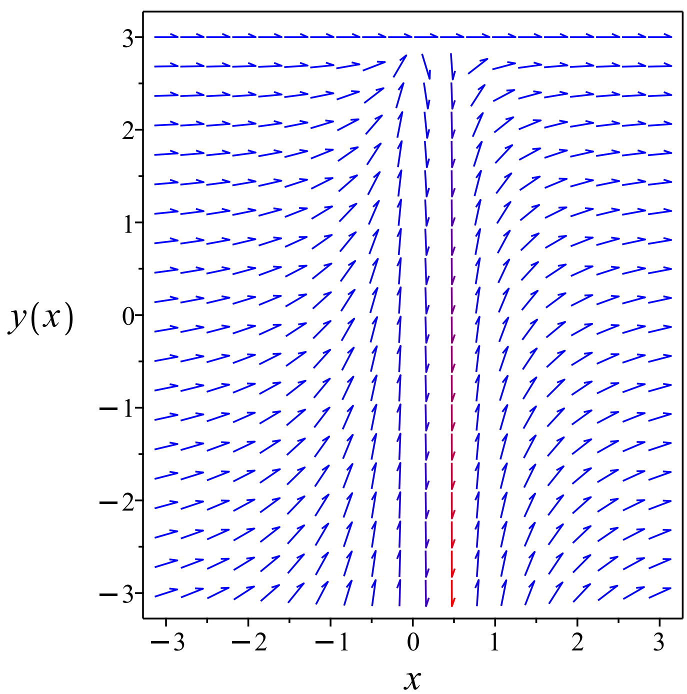


Figure 68: Slope field plot

Verification of solutions

$$
y=\frac{3c_2\left(2\,\mathrm{e}^{c_1}-\frac{\mathrm{e}^{c_1}}{x}\right)+1}{c_2\left(2\,\mathrm{e}^{c_1}-\frac{\mathrm{e}^{c_1}}{x}\right)}
$$

Verified OK.

2.7.2 Maple step by step solution

Let's solve

$-xy'+y+2y'x^2=3$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{-y+3}=\frac{1}{2x^2-x}$

- Integrate both sides with respect to x

  $\int\frac{y'}{-y+3}dx=\int\frac{1}{2x^2-x}dx+c_1$

- Evaluate integral

  $-\ln\left(-y+3\right)=\ln\left(2x-1\right)-\ln\left(x\right)+c_1$

- Solve for y

  $y=\frac{6\,\mathrm{e}^{c_1}x-3\,\mathrm{e}^{c_1}-x}{\mathrm{e}^{c_1}\left(2x-1\right)}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(y(x)-x*diff(y(x),x)=3-2*x^2*diff(y(x),x),y(x), singsol=all)
```

$$y(x) = \frac{c_1 x - 3}{2x - 1}$$

✓ Solution by Mathematica

Time used: 0.032 (sec). Leaf size: 24

```
DSolve[y[x]-x*y'[x]==3-2*x^2*y'[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{3 + c_1 x}{1 - 2x}$$
$$y(x) \to 3$$

2.8 problem Problem 8

Internal problem ID [2629]
Internal file name [`OUTPUT/2121_Sunday_June_05_2022_02_49_16_AM_95754766/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.4, Separable Differential Equations. page 43
Problem number: Problem 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y' - \frac{\cos(-y+x)}{\sin(x)\sin(y)} = -1}$$

2.8.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{\cos(x)\cot(y)}{\sin(x)} \end{aligned}$$

Where $f(x) = \frac{\cos(x)}{\sin(x)}$ and $g(y) = \cot(y)$. Integrating both sides gives

$$\frac{1}{\cot(y)}\,dy = \frac{\cos(x)}{\sin(x)}\,dx$$

$$\int \frac{1}{\cot(y)}dy = \int \frac{\cos(x)}{\sin(x)}dx$$
$$-\ln(\cos(y)) = \ln(\sin(x)) + c_1$$

Raising both side to exponential gives

$$\frac{1}{\cos(y)} = e^{\ln(\sin(x))+c_1}$$

Which simplifies to

$$\sec(y) = c_2 \sin(x)$$

Summary

The solution(s) found are the following

$$y = \operatorname{arcsec}(c_2 e^{c_1} \sin(x)) \tag{1}$$

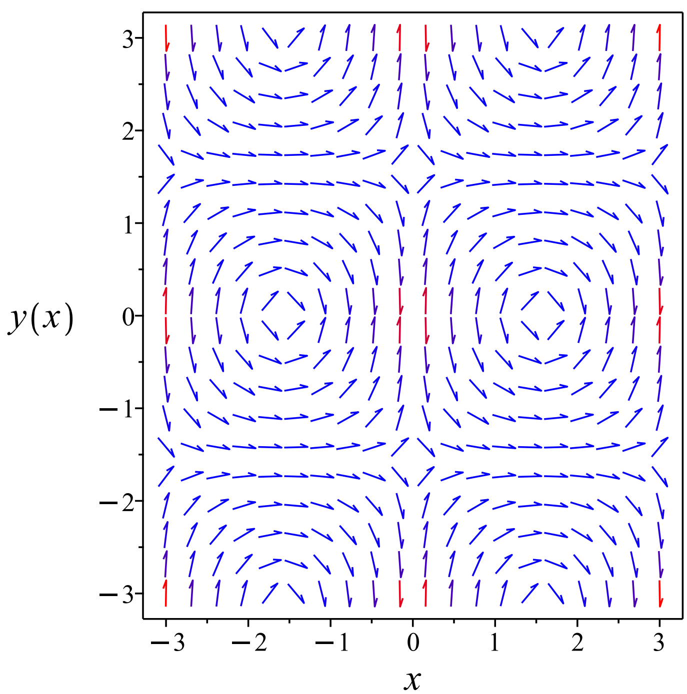


Figure 69: Slope field plot

Verification of solutions

$$y = \operatorname{arcsec}(c_2 e^{c_1} \sin(x))$$

Verified OK.

2.8.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{\sin(x)\sin(y) - \cos(-y+x)}{\sin(x)\sin(y)} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 62: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= \frac{\sin(x)}{\cos(x)} \\
\eta(x,y) &= 0
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{\frac{\sin(x)}{\cos(x)}} dx
\end{aligned}
$$

Which results in

$$
S = \ln(\sin(x))
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -\frac{\sin(x)\sin(y) - \cos(-y+x)}{\sin(x)\sin(y)}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= \cot(x) \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{\cos(x)\sin(y)}{-\sin(x)\sin(y) + \cos(-y + x)} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \tan(R)
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -\ln(\cos(R)) + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\ln(\sin(x)) = -\ln(\cos(y)) + c_1
$$

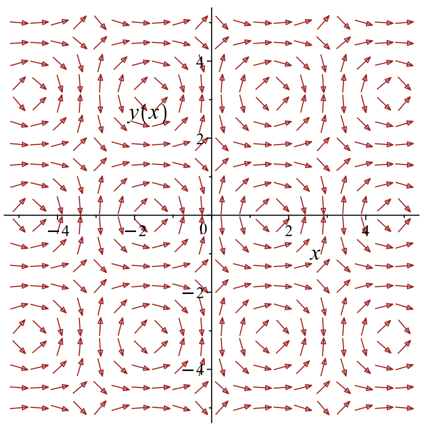
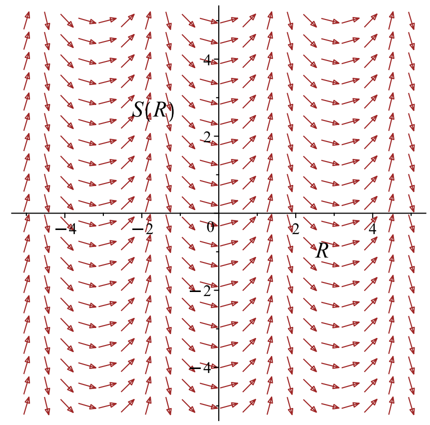
Which simplifies to

$$
\ln(\sin(x)) = -\ln(\cos(y)) + c_1
$$

Which gives

$$
y = \arccos\left(\frac{e^{c_1}}{\sin(x)}\right)
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{\sin(x)\sin(y)-\cos(-y+x)}{\sin(x)\sin(y)}$  	$R = y$ $S = \ln(\sin(x))$	$\frac{dS}{dR} = \tan(R)$ 

Summary

The solution(s) found are the following

$$y = \arccos\left(\frac{e^{c_1}}{\sin(x)}\right) \tag{1}$$

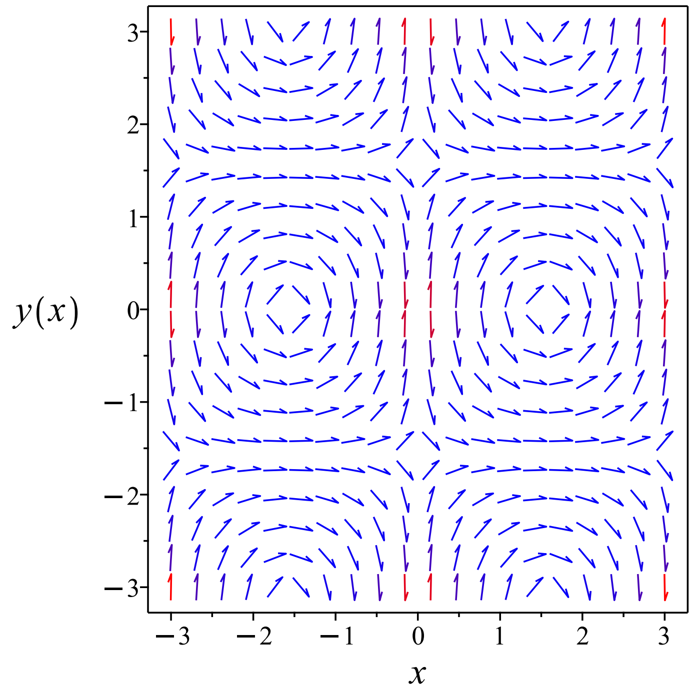


Figure 70: Slope field plot

Verification of solutions

$$y = \arccos\left(\frac{e^{c_1}}{\sin(x)}\right)$$

Verified OK.

2.8.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(\frac{\sin (y)}{\cos (y)}\right) \mathrm{d}y &= \left(\frac{\cos (x)}{\sin (x)}\right) \mathrm{d}x \\
\left(-\frac{\cos (x)}{\sin (x)}\right) \mathrm{d}x + \left(\frac{\sin (y)}{\cos (y)}\right) \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -\frac{\cos (x)}{\sin (x)} \\
N(x, y) &= \frac{\sin (y)}{\cos (y)}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{\cos (x)}{\sin (x)}\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{\sin(y)}{\cos(y)}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{\cos(x)}{\sin(x)}\,\mathrm{d}x \\
\phi &= -\ln(\sin(x)) + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{\sin(y)}{\cos(y)}$. Therefore equation (4) becomes

$$
\frac{\sin(y)}{\cos(y)} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
\begin{aligned}
f'(y) &= \frac{\sin(y)}{\cos(y)} \\
&= \tan(y)
\end{aligned}
$$

Integrating the above w.r.t y results in

$$\int f'(y)\,\mathrm{d}y = \int (\tan(y))\,\mathrm{d}y$$
$$f(y) = -\ln(\cos(y)) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(\sin(x)) - \ln(\cos(y)) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(\sin(x)) - \ln(\cos(y))$$

Summary
The solution(s) found are the following

$$-\ln(\sin(x)) - \ln(\cos(y)) = c_1 \tag{1}$$

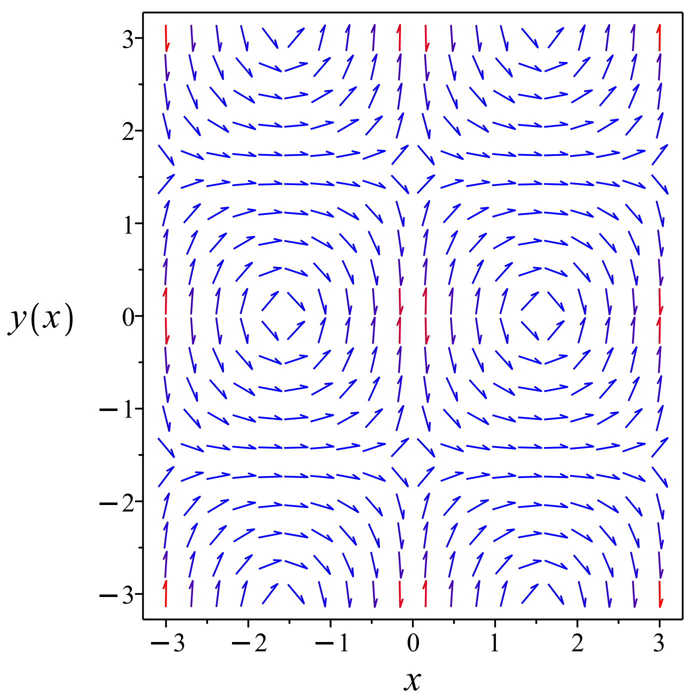


Figure 71: Slope field plot

Verification of solutions

$$-\ln(\sin(x)) - \ln(\cos(y)) = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.141 (sec). Leaf size: 11

```
dsolve(diff(y(x),x)=cos(x-y(x))/(sin(x)*sin(y(x)))-1,y(x), singsol=all)
```

$$y(x) = \arccos\left(\frac{\csc(x)}{c_1}\right)$$

✓ Solution by Mathematica

Time used: 5.812 (sec). Leaf size: 47

```
DSolve[y'[x]==Cos[x-y[x]]/(Sin[x]*Sin[y[x]])-1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\arccos\left(-\frac{1}{2}c_1\csc(x)\right)$$
$$y(x) \to \arccos\left(-\frac{1}{2}c_1\csc(x)\right)$$
$$y(x) \to -\frac{\pi}{2}$$
$$y(x) \to \frac{\pi}{2}$$

2.9 problem Problem 9

Internal problem ID [2630]
Internal file name [`OUTPUT/2122_Sunday_June_05_2022_02_49_20_AM_4206466/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.4, Separable Differential Equations. page 43
Problem number: Problem 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{x(y^2-1)}{2(x-2)(x-1)} = 0}$$

2.9.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{x\left(\frac{y^2}{2}-\frac{1}{2}\right)}{(x-2)(x-1)} \end{aligned}$$

Where $f(x) = \frac{x}{(x-2)(x-1)}$ and $g(y) = \frac{y^2}{2} - \frac{1}{2}$. Integrating both sides gives

$$\frac{1}{\frac{y^2}{2}-\frac{1}{2}}\,dy = \frac{x}{(x-2)(x-1)}\,dx$$

$$\int \frac{1}{\frac{y^2}{2}-\frac{1}{2}}\,dy = \int \frac{x}{(x-2)(x-1)}\,dx$$

$$-2\operatorname{arctanh}(y) = -\ln(x-1) + 2\ln(x-2) + c_1$$

Which results in

$$y = -\tanh\left(-\frac{\ln(x-1)}{2} + \ln(x-2) + \frac{c_1}{2}\right)$$

Summary

The solution(s) found are the following

$$y = -\tanh\left(-\frac{\ln(x-1)}{2} + \ln(x-2) + \frac{c_1}{2}\right) \tag{1}$$

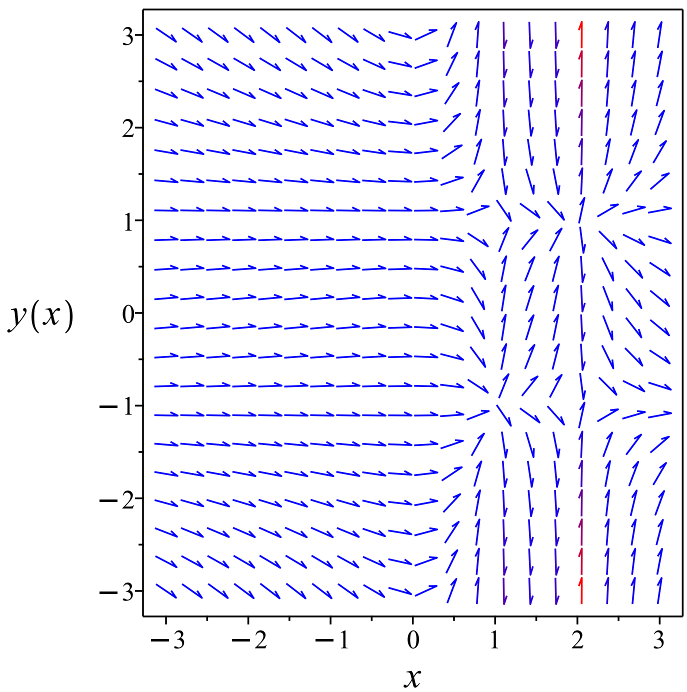


Figure 72: Slope field plot

Verification of solutions

$$y = -\tanh\left(-\frac{\ln(x-1)}{2} + \ln(x-2) + \frac{c_1}{2}\right)$$

Verified OK.

2.9.2 Maple step by step solution

Let's solve

$y' - \frac{x(y^2-1)}{2(x-2)(x-1)} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^2-1} = \frac{x}{2(x-2)(x-1)}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y^2-1} dx = \int \frac{x}{2(x-2)(x-1)} dx + c_1$

- Evaluate integral

  $-\mathrm{arctanh}(y) = -\frac{\ln(x-1)}{2} + \ln(x-2) + c_1$

- Solve for y

  $y = -\tanh\left(-\frac{\ln(x-1)}{2} + \ln(x-2) + c_1\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 21

```
dsolve(diff(y(x),x)=x*(y(x)^2-1)/(2*(x-2)*(x-1)),y(x), singsol=all)
```

$$y(x) = -\tanh\left(\ln(-2+x) - \frac{\ln(x-1)}{2} + \frac{c_1}{2}\right)$$

✓ Solution by Mathematica

Time used: 0.882 (sec). Leaf size: 51

```
DSolve[y'[x]==x*(y[x]^2-1)/(2*(x-2)*(x-1)),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{x + e^{2c_1}(x-2)^2 - 1}{-x + e^{2c_1}(x-2)^2 + 1}$$
$$y(x) \to -1$$
$$y(x) \to 1$$

2.10 problem Problem 10

Internal problem ID [2631]
Internal file name [`OUTPUT/2123_Sunday_June_05_2022_02_49_23_AM_6369853/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.4, Separable Differential Equations. page 43
Problem number: Problem 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{x^2 y - 32}{-x^2 + 16} = 2}$$

2.10.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{x^2(-y+2)}{x^2-16} \end{aligned}$$

Where $f(x) = \frac{x^2}{x^2-16}$ and $g(y) = -y+2$. Integrating both sides gives

$$\begin{aligned} \frac{1}{-y+2}\,dy &= \frac{x^2}{x^2-16}\,dx \\ \int \frac{1}{-y+2}\,dy &= \int \frac{x^2}{x^2-16}\,dx \\ -\ln(y-2) &= x - 2\ln(x+4) + 2\ln(x-4) + c_1 \end{aligned}$$

Raising both side to exponential gives

$$\frac{1}{y-2} = e^{x-2\ln(x+4)+2\ln(x-4)+c_1}$$

Which simplifies to

$$\frac{1}{y-2} = c_2 e^{x-2\ln(x+4)+2\ln(x-4)}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(2c_2 e^{x-2\ln(x+4)+2\ln(x-4)+c_1}+1\right)(x+4)^2 e^{-x-c_1}}{c_2 (x-4)^2} \tag{1}$$

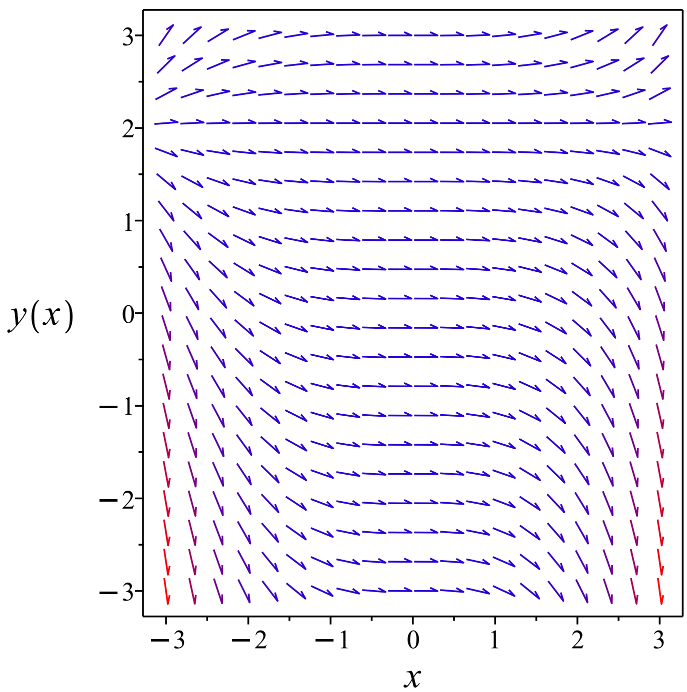


Figure 73: Slope field plot

Verification of solutions

$$y = \frac{\left(2c_2 e^{x-2\ln(x+4)+2\ln(x-4)+c_1}+1\right)(x+4)^2 e^{-x-c_1}}{c_2 (x-4)^2}$$

Verified OK.

2.10.2 Maple step by step solution

Let's solve

$y' - \frac{x^2 y-32}{-x^2+16} = 2$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y-2} = -\frac{x^2}{(x-4)(x+4)}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y-2}dx = \int -\frac{x^2}{(x-4)(x+4)}dx + c_1$

- Evaluate integral

  $\ln(y-2) = -x + 2\ln(x+4) - 2\ln(x-4) + c_1$

- Solve for y

  $y = \frac{e^{-x+c_1}x^2+8\,e^{-x+c_1}x+2x^2+16\,e^{-x+c_1}-16x+32}{x^2-8x+16}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 29

```
dsolve(diff(y(x),x)=(x^2*y(x)-32)/(16-x^2)+2,y(x), singsol=all)
```

$$y(x) = \frac{c_1(x+4)^2 e^{-x} + 2(x-4)^2}{(x-4)^2}$$

✓ Solution by Mathematica

Time used: 0.148 (sec). Leaf size: 40

```
DSolve[y'[x]==(x^2*y[x]-32)/(16-x^2)+2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{-x}(2e^x(x-4)^2 + c_1(x+4)^2)}{(x-4)^2}$$
$$y(x) \to 2$$

2.11 problem Problem 11

Internal problem ID [2632]
Internal file name [`OUTPUT/2124_Sunday_June_05_2022_02_49_25_AM_75605802/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.4, Separable Differential Equations. page 43
Problem number: Problem 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_separable]`

$$\boxed{(x-a)(x-b)\,y'-y=-c}$$

2.11.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{y-c}{(-x+a)(-x+b)} \end{aligned}$$

Where $f(x)=\frac{1}{(-x+a)(-x+b)}$ and $g(y)=y-c$. Integrating both sides gives

$$\begin{aligned} \frac{1}{y-c}\,dy &= \frac{1}{(-x+a)(-x+b)}\,dx \\ \int \frac{1}{y-c}\,dy &= \int \frac{1}{(-x+a)(-x+b)}\,dx \\ \ln(y-c) &= -\frac{\ln(x-b)}{-b+a}+\frac{\ln(x-a)}{-b+a}+c_1 \end{aligned}$$

Raising both side to exponential gives

$$y - c = \mathrm{e}^{-\frac{\ln(x-b)}{-b+a}+\frac{\ln(x-a)}{-b+a}+c_1}$$

Which simplifies to

$$y - c = c_2 \mathrm{e}^{-\frac{\ln(x-b)}{-b+a}+\frac{\ln(x-a)}{-b+a}}$$

Which simplifies to

$$y = c_2(x-b)^{-\frac{1}{-b+a}} (x-a)^{\frac{1}{-b+a}} \mathrm{e}^{c_1} + c$$

Summary

The solution(s) found are the following

$$y = c_2(x-b)^{-\frac{1}{-b+a}} (x-a)^{\frac{1}{-b+a}} \mathrm{e}^{c_1} + c \tag{1}$$

Verification of solutions

$$y = c_2(x-b)^{-\frac{1}{-b+a}} (x-a)^{\frac{1}{-b+a}} \mathrm{e}^{c_1} + c$$

Verified OK.

2.11.2 Maple step by step solution

Let's solve

$(x-a)(x-b)y' - y = -c$

- Highest derivative means the order of the ODE is 1

  y'
- Separate variables

  $\frac{y'}{y-c} = \frac{1}{(x-a)(x-b)}$
- Integrate both sides with respect to x

  $\int \frac{y'}{y-c}dx = \int \frac{1}{(x-a)(x-b)}dx + c_1$
- Evaluate integral

  $\ln(y-c) = -\frac{\ln(x-b)}{-b+a} + \frac{\ln(x-a)}{-b+a} + c_1$
- Solve for y

  $y = \mathrm{e}^{-\frac{-c_1 a + c_1 b + \ln\left(\frac{-x+b}{-x+a}\right)}{-b+a}} + c$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 36

```
dsolve((x-a)*(x-b)*diff(y(x),x)-(y(x)-c)=0,y(x), singsol=all)
```

$$y(x) = c + (x-b)^{-\frac{1}{a-b}} (x-a)^{\frac{1}{a-b}} c_1$$

✓ Solution by Mathematica

Time used: 0.102 (sec). Leaf size: 41

```
DSolve[(x-a)*(x-b)*y'[x]-(y[x]-c)==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\begin{aligned} y(x) &\to c + c_1(x-b)^{\frac{1}{b-a}}(x-a)^{\frac{1}{a-b}} \\ y(x) &\to c \end{aligned}$$

2.12 problem Problem 12

Internal problem ID [2633]
Internal file name [`OUTPUT/2125_Sunday_June_05_2022_02_49_28_AM_75197817/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.4, Separable Differential Equations. page 43
Problem number: Problem 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{(x^2+1)\,y'+y^2=-1}$$

With initial conditions

$$[y(0)=1]$$

2.12.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= -\frac{y^2+1}{x^2+1} \end{aligned}$$

The x domain of $f(x,y)$ when $y=1$ is

$$\{-\infty<x<\infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x, y)$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{y^2+1}{x^2+1}\right)\\ &= -\frac{2y}{x^2+1}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

2.12.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x, y)\\ &= f(x)g(y)\\ &= \frac{-y^2-1}{x^2+1}\end{aligned}$$

Where $f(x) = \frac{1}{x^2+1}$ and $g(y) = -y^2 - 1$. Integrating both sides gives

$$\begin{aligned}\frac{1}{-y^2-1}\,dy &= \frac{1}{x^2+1}\,dx\\ \int \frac{1}{-y^2-1}\,dy &= \int \frac{1}{x^2+1}\,dx\\ -\arctan(y) &= \arctan(x) + c_1\end{aligned}$$

Which results in

$$y = -\tan(\arctan(x) + c_1)$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\tan(c_1)$$

$$c_1 = -\frac{\pi}{4}$$

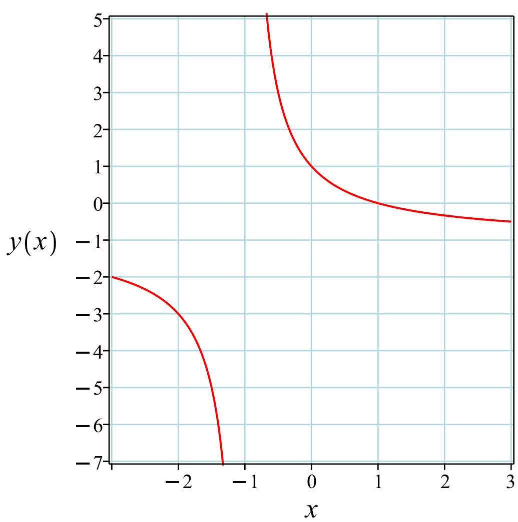
Substituting c_1 found above in the general solution gives

$$y = \frac{1-x}{x+1}$$

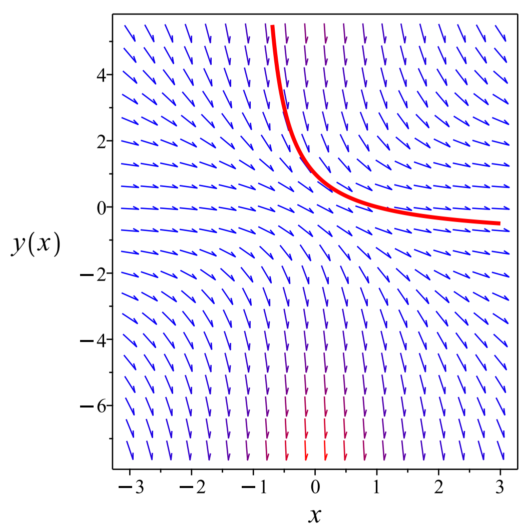
Summary

The solution(s) found are the following

$$y = \frac{1-x}{x+1} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{1-x}{x+1}$$

Verified OK.

2.12.3 Maple step by step solution

Let's solve

$[(x^2+1)\, y' + y^2 = -1, y(0) = 1]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{-y^2-1} = \frac{1}{x^2+1}$

- Integrate both sides with respect to x

  $\int \frac{y'}{-y^2-1} dx = \int \frac{1}{x^2+1} dx + c_1$

- Evaluate integral

  $-\arctan(y) = \arctan(x) + c_1$

- Solve for y

  $y = -\tan(\arctan(x) + c_1)$

- Use initial condition $y(0) = 1$

  $1 = -\tan(c_1)$

- Solve for c_1

  $c_1 = -\frac{\pi}{4}$

- Substitute $c_1 = -\frac{\pi}{4}$ into general solution and simplify

  $y = \cot\left(\arctan(x) + \frac{\pi}{4}\right)$

- Solution to the IVP

  $y = \cot\left(\arctan(x) + \frac{\pi}{4}\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 11

```
dsolve([(x^2+1)*diff(y(x),x)+y(x)^2=-1,y(0) = 1],y(x), singsol=all)
```

$$y(x) = \cot\left(\arctan(x) + \frac{\pi}{4}\right)$$

✓ Solution by Mathematica

Time used: 0.243 (sec). Leaf size: 14

```
DSolve[{(x^2+1)*y'[x]+y[x]^2==-1,{y[0]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \cot\left(\arctan(x) + \frac{\pi}{4}\right)$$

2.13 problem Problem 13

Internal problem ID [2634]
Internal file name [`OUTPUT/2126_Sunday_June_05_2022_02_49_30_AM_93577070/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.4, Separable Differential Equations. page 43
Problem number: Problem 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{\left(-x^{2}+1\right) y'+yx=ax}$$

With initial conditions

$$[y(0)=2a]$$

2.13.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{x}{x^2-1} \\ q(x) &= -\frac{ax}{x^2-1} \end{aligned}$$

Hence the ode is

$$y' - \frac{xy}{x^2-1} = -\frac{ax}{x^2-1}$$

The domain of $p(x) = -\frac{x}{x^2-1}$ is

$$\{-\infty \leq x < -1, -1 < x < 1, 1 < x \leq \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = -\frac{ax}{x^2-1}$ is

$$\{-\infty \leq x < -1, -1 < x < 1, 1 < x \leq \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

2.13.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{x(-a+y)}{x^2-1} \end{aligned}$$

Where $f(x) = \frac{x}{x^2-1}$ and $g(y) = -a + y$. Integrating both sides gives

$$\begin{aligned} \frac{1}{-a+y}\,dy &= \frac{x}{x^2-1}\,dx \\ \int \frac{1}{-a+y}\,dy &= \int \frac{x}{x^2-1}\,dx \\ \ln(-a+y) &= \frac{\ln(x-1)}{2} + \frac{\ln(x+1)}{2} + c_1 \end{aligned}$$

Raising both side to exponential gives

$$-a+y = \mathrm{e}^{\frac{\ln(x-1)}{2}+\frac{\ln(x+1)}{2}+c_1}$$

Which simplifies to

$$-a+y = c_2 \mathrm{e}^{\frac{\ln(x-1)}{2}+\frac{\ln(x+1)}{2}}$$

Which can be simplified to become

$$y = c_2\sqrt{x-1}\,\sqrt{x+1}\,\mathrm{e}^{c_1}+a$$

Initial conditions are used to solve for c_1. Substituting $x=0$ and $y=2a$ in the above solution gives an equation to solve for the constant of integration.

$$2a = i\mathrm{e}^{c_1}c_2+a$$

$$c_1 = \ln\left(-\frac{ia}{c_2}\right)$$

Substituting c_1 found above in the general solution gives

$$y = -i\sqrt{x-1}\,\sqrt{x+1}\,a+a$$

Summary

The solution(s) found are the following

$$y = -i\sqrt{x-1}\,\sqrt{x+1}\,a+a \tag{1}$$

Verification of solutions

$$y = -i\sqrt{x-1}\,\sqrt{x+1}\,a+a$$

Verified OK.

2.13.3 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int -\frac{x}{x^2-1}dx}\\ &= \mathrm{e}^{-\frac{\ln(x-1)}{2}-\frac{\ln(x+1)}{2}}\end{aligned}$$

Which simplifies to

$$\mu = \frac{1}{\sqrt{x-1}\,\sqrt{x+1}}$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(-\frac{ax}{x^2-1}\right) \\
\frac{d}{dx}\left(\frac{y}{\sqrt{x-1}\,\sqrt{x+1}}\right) &= \left(\frac{1}{\sqrt{x-1}\,\sqrt{x+1}}\right)\left(-\frac{ax}{x^2-1}\right) \\
d\left(\frac{y}{\sqrt{x-1}\,\sqrt{x+1}}\right) &= \left(-\frac{ax}{(x^2-1)\sqrt{x-1}\,\sqrt{x+1}}\right) dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{y}{\sqrt{x-1}\,\sqrt{x+1}} &= \int -\frac{ax}{(x^2-1)\sqrt{x-1}\,\sqrt{x+1}}\, dx \\
\frac{y}{\sqrt{x-1}\,\sqrt{x+1}} &= \frac{\sqrt{x-1}\,\sqrt{x+1}\,a}{x^2-1} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{\sqrt{x-1}\,\sqrt{x+1}}$ results in

$$
y = \frac{(x-1)(x+1)a}{x^2-1} + c_1\sqrt{x-1}\,\sqrt{x+1}
$$

which simplifies to

$$
y = a + c_1\sqrt{x-1}\,\sqrt{x+1}
$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 2a$ in the above solution gives an equation to solve for the constant of integration.

$$
2a = c_1 i + a
$$

$$
c_1 = -ia
$$

Substituting c_1 found above in the general solution gives

$$
y = -i\sqrt{x-1}\,\sqrt{x+1}\,a + a
$$

Summary

The solution(s) found are the following

$$
y = -i\sqrt{x-1}\,\sqrt{x+1}\,a + a \tag{1}
$$

Verification of solutions

$$
y = -i\sqrt{x-1}\,\sqrt{x+1}\,a + a
$$

Verified OK.

2.13.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{x(-a+y)}{x^2-1} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 68: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a+bx+cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \mathrm{e}^{\frac{\ln(x-1)}{2}+\frac{\ln(x+1)}{2}}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\mathrm{e}^{\frac{\ln(x-1)}{2}+\frac{\ln(x+1)}{2}}} dy
\end{aligned}
$$

Which results in

$$
S = \mathrm{e}^{\ln\left(\frac{1}{\sqrt{x-1}}\right)+\ln\left(\frac{1}{\sqrt{x+1}}\right)} y
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = \frac{x(-a+y)}{x^2-1}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{yx}{(x-1)^{\frac{3}{2}}(x+1)^{\frac{3}{2}}} \\
S_y &= \frac{1}{\sqrt{x-1}\,\sqrt{x+1}}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = -\frac{ax}{(x-1)^{\frac{3}{2}}(x+1)^{\frac{3}{2}}} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = -\frac{aR}{(R-1)^{\frac{3}{2}}(R+1)^{\frac{3}{2}}}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \frac{a}{\sqrt{R-1}\,\sqrt{R+1}} + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{y}{\sqrt{x-1}\,\sqrt{x+1}} = \frac{a}{\sqrt{x-1}\,\sqrt{x+1}} + c_1
$$

Which simplifies to

$$
\frac{y}{\sqrt{x-1}\,\sqrt{x+1}} = \frac{a}{\sqrt{x-1}\,\sqrt{x+1}} + c_1
$$

Which gives

$$
y = a + c_1\sqrt{x-1}\,\sqrt{x+1}
$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 2a$ in the above solution gives an equation to solve for the constant of integration.

$$
2a = c_1 i + a
$$

$$c_1 = -ia$$

Substituting c_1 found above in the general solution gives

$$y = -i\sqrt{x-1}\,\sqrt{x+1}\,a + a$$

Summary

The solution(s) found are the following

$$y = -i\sqrt{x-1}\,\sqrt{x+1}\,a + a \tag{1}$$

Verification of solutions

$$y = -i\sqrt{x-1}\,\sqrt{x+1}\,a + a$$

Verified OK.

2.13.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition

$\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{-a+y}\right)\mathrm{d}y = \left(\frac{x}{x^2-1}\right)\mathrm{d}x$$

$$\left(-\frac{x}{x^2-1}\right)\mathrm{d}x + \left(\frac{1}{-a+y}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x,y) &= -\frac{x}{x^2-1} \\ N(x,y) &= \frac{1}{-a+y} \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x}{x^2-1}\right) \\ &= 0 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{-a+y}\right) \\ &= 0 \end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int M \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int -\frac{x}{x^2-1} \mathrm{d}x \\
\phi &= -\frac{\ln(x-1)}{2} - \frac{\ln(x+1)}{2} + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{-a+y}$. Therefore equation (4) becomes

$$
\frac{1}{-a+y} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -\frac{1}{a-y}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \mathrm{d}y &= \int \left(-\frac{1}{a-y}\right) \mathrm{d}y \\
f(y) &= \ln(a-y) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$
\phi = -\frac{\ln(x-1)}{2} - \frac{\ln(x+1)}{2} + \ln(a-y) + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
c_1 = -\frac{\ln(x-1)}{2} - \frac{\ln(x+1)}{2} + \ln(a-y)
$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 2a$ in the above solution gives an equation to solve for the constant of integration.

$$-\frac{i\pi}{2} + \ln(-a) = c_1$$

$$c_1 = -\frac{i\pi}{2} + \ln(-a)$$

Substituting c_1 found above in the general solution gives

$$-\frac{\ln(x-1)}{2} - \frac{\ln(x+1)}{2} + \ln(a-y) = -\frac{i\pi}{2} + \ln(-a)$$

Solving for y from the above gives

$$y = \left(-i\sqrt{x-1}\,\sqrt{x+1}+1\right)a$$

Summary

The solution(s) found are the following

$$y = \left(-i\sqrt{x-1}\,\sqrt{x+1}+1\right)a \tag{1}$$

Verification of solutions

$$y = \left(-i\sqrt{x-1}\,\sqrt{x+1}+1\right)a$$

Verified OK. `{positive}`

2.13.6 Maple step by step solution

Let's solve

$[(-x^2+1)\,y' + yx = ax, y(0) = 2a]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{a-y} = -\frac{x}{(x-1)(x+1)}$

- Integrate both sides with respect to x

  $\int \frac{y'}{a-y}dx = \int -\frac{x}{(x-1)(x+1)}dx + c_1$

- Evaluate integral

$$-\ln(a-y) = -\frac{\ln((x-1)(x+1))}{2} + c_1$$

- Solve for y

$$y = -e^{\frac{\ln((x-1)(x+1))}{2} - c_1} + a$$

- Use initial condition $y(0) = 2a$

$$2a = -e^{\frac{I\pi}{2} - c_1} + a$$

- Solve for c_1

$$c_1 = \frac{I\pi}{2} - \ln(-a)$$

- Substitute $c_1 = \frac{I\pi}{2} - \ln(-a)$ into general solution and simplify

$$y = a\left(1 - I\sqrt{x^2 - 1}\right)$$

- Solution to the IVP

$$y = a\left(1 - I\sqrt{x^2 - 1}\right)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve([(1-x^2)*diff(y(x),x)+x*y(x)=a*x,y(0) = 2*a],y(x), singsol=all)
```

$$y(x) = a\left(1 - i\sqrt{x - 1}\,\sqrt{x + 1}\right)$$

✓ Solution by Mathematica

Time used: 0.04 (sec). Leaf size: 21

```
DSolve[{(1-x^2)*y'[x]+x*y[x]==a*x,{y[0]==2*a}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to a - ia\sqrt{x^2 - 1}$$

2.14 problem Problem 14

Internal problem ID [2635]
Internal file name [`OUTPUT/2127_Sunday_June_05_2022_02_49_32_AM_24557463/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.4, Separable Differential Equations. page 43
Problem number: Problem 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' + \frac{\sin(y+x)}{\sin(y)\cos(x)} = 1}$$

With initial conditions

$$\left[y\left(\frac{\pi}{4}\right) = \frac{\pi}{4}\right]$$

2.14.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= -\frac{-\cos(x)\sin(y) + \sin(y+x)}{\sin(y)\cos(x)} \end{aligned}$$

The x domain of $f(x,y)$ when $y=\frac{\pi}{4}$ is

$$\left\{x<\frac{1}{2}\pi+\pi\_Z142\vee\frac{1}{2}\pi+\pi\_Z142<x\right\}$$

And the point $x_0=\frac{\pi}{4}$ is inside this domain. The y domain of $f(x,y)$ when $x=\frac{\pi}{4}$ is

$$\{y<\pi\_Z143\vee\pi\_Z143<y\}$$

And the point $y_0=\frac{\pi}{4}$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y}&=\frac{\partial}{\partial y}\left(-\frac{-\cos(x)\sin(y)+\sin(y+x)}{\sin(y)\cos(x)}\right)\\&=-\frac{-\cos(x)\cos(y)+\cos(y+x)}{\sin(y)\cos(x)}+\frac{(-\cos(x)\sin(y)+\sin(y+x))\cos(y)}{\sin(y)^2\cos(x)}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y=\frac{\pi}{4}$ is

$$\left\{x<\frac{1}{2}\pi+\pi\_Z142\vee\frac{1}{2}\pi+\pi\_Z142<x\right\}$$

And the point $x_0=\frac{\pi}{4}$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x=\frac{\pi}{4}$ is

$$\{y<\pi\_Z143\vee\pi\_Z143<y\}$$

And the point $y_0=\frac{\pi}{4}$ is inside this domain. Therefore solution exists and is unique.

2.14.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y'&=F(x,y)\\&=f(x)g(y)\\&=-\frac{\sin(x)\cot(y)}{\cos(x)}\end{aligned}$$

Where $f(x)=-\frac{\sin(x)}{\cos(x)}$ and $g(y)=\cot(y)$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\cot(y)}\,dy&=-\frac{\sin(x)}{\cos(x)}\,dx\\\int\frac{1}{\cot(y)}\,dy&=\int-\frac{\sin(x)}{\cos(x)}\,dx\\-\ln(\cos(y))&=\ln(\cos(x))+c_1\end{aligned}$$

Raising both side to exponential gives

$$\frac{1}{\cos(y)} = \mathrm{e}^{\ln(\cos(x))+c_1}$$

Which simplifies to

$$\sec(y) = c_2 \cos(x)$$

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{4}$ and $y = \frac{\pi}{4}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{\pi}{4} = \frac{\pi}{2} - \arcsin\left(\frac{\sqrt{2}\,\mathrm{e}^{-c_1}}{c_2}\right)$$

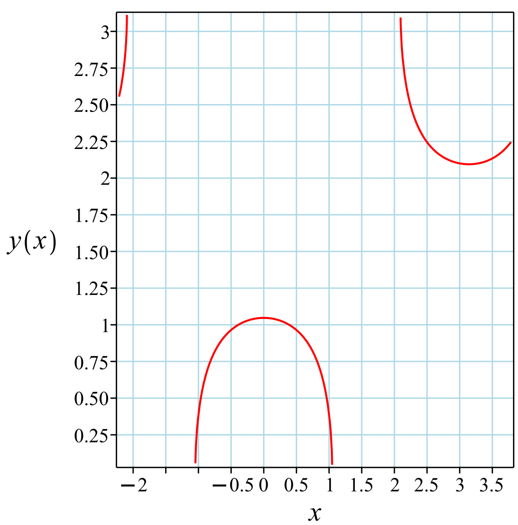
$$c_1 = -\ln\left(\frac{c_2}{2}\right)$$

Substituting c_1 found above in the general solution gives

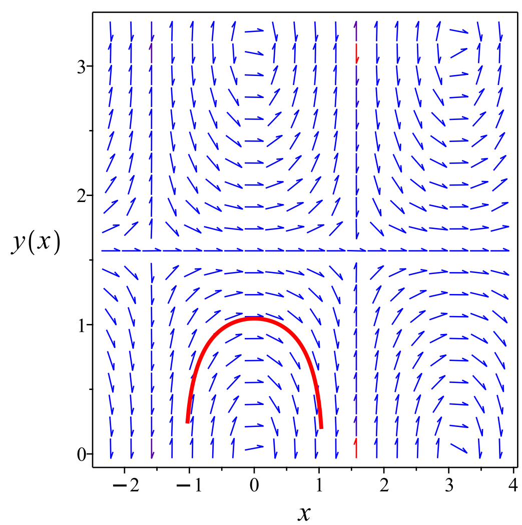
$$y = \frac{\pi}{2} - \arcsin\left(\frac{1}{2\cos(x)}\right)$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{\pi}{2} - \arcsin\left(\frac{1}{2\cos(x)}\right) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{\pi}{2} - \arcsin\left(\frac{1}{2\cos(x)}\right)$$

Verified OK.

2.14.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned}
y' &= -\frac{-\cos(x)\sin(y) + \sin(y+x)}{\sin(y)\cos(x)} \\
y' &= \omega(x,y)
\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 71: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= -\frac{\cos(x)}{\sin(x)} \\
\eta(x, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{-\frac{\cos(x)}{\sin(x)}} dx \end{aligned}$$

Which results in

$$S = \ln(\cos(x))$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{-\cos(x)\sin(y) + \sin(y+x)}{\sin(y)\cos(x)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= -\tan(x) \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{\sin(x)\sin(y)}{-\cos(x)\sin(y) + \sin(y+x)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \tan(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\ln(\cos(R)) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln(\cos(x)) = -\ln(\cos(y)) + c_1$$

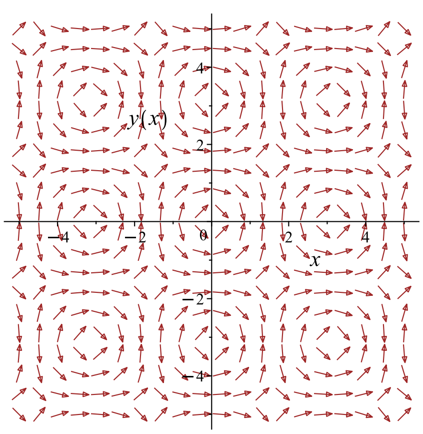
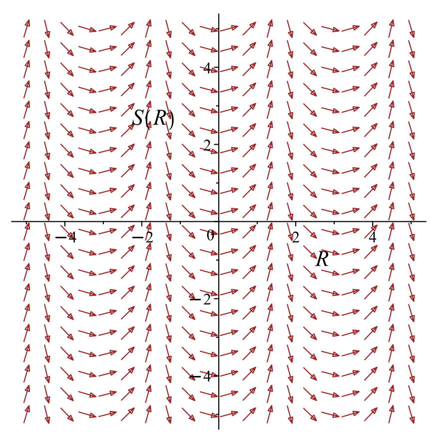
Which simplifies to

$$\ln(\cos(x)) = -\ln(\cos(y)) + c_1$$

Which gives

$$y = \arccos\left(\frac{e^{c_1}}{\cos(x)}\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{-\cos(x)\sin(y)+\sin(y+x)}{\sin(y)\cos(x)}$ 	$R = y$ $S = \ln(\cos(x))$	$\frac{dS}{dR} = \tan(R)$ 

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{4}$ and $y = \frac{\pi}{4}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{\pi}{4} = \frac{\pi}{2} - \arcsin\left(\sqrt{2}\, e^{c_1}\right)$$

$$c_1 = -\ln(2)$$

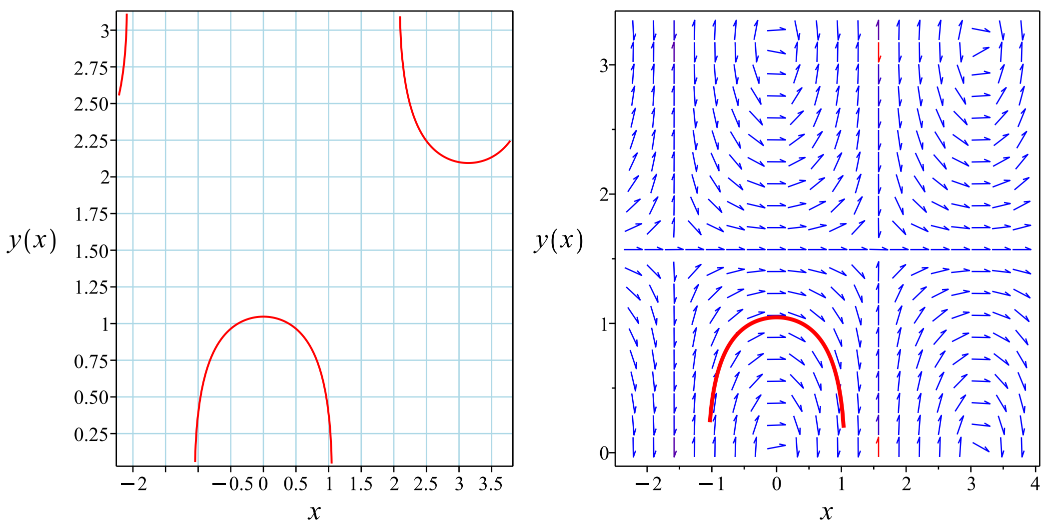
Substituting c_1 found above in the general solution gives

$$y = \frac{\pi}{2} - \arcsin\left(\frac{1}{2\cos(x)}\right)$$

Summary

The solution(s) found are the following

$$y = \frac{\pi}{2} - \arcsin\left(\frac{1}{2\cos(x)}\right) \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{\pi}{2} - \arcsin\left(\frac{1}{2\cos(x)}\right)$$

Verified OK.

2.14.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-\frac{\sin(y)}{\cos(y)}\right)\mathrm{d}y&=\left(\frac{\sin(x)}{\cos(x)}\right)\mathrm{d}x\\ \left(-\frac{\sin(x)}{\cos(x)}\right)\mathrm{d}x+\left(-\frac{\sin(y)}{\cos(y)}\right)\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= -\frac{\sin(x)}{\cos(x)} \\
N(x,y) &= -\frac{\sin(y)}{\cos(y)}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{\sin(x)}{\cos(x)}\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{\sin(y)}{\cos(y)}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int M \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int -\frac{\sin(x)}{\cos(x)} \,\mathrm{d}x
\end{aligned}
$$

$$
\phi = \ln(\cos(x)) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = -\frac{\sin(y)}{\cos(y)}$. Therefore equation (4) becomes

$$-\frac{\sin(y)}{\cos(y)} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$\begin{aligned} f'(y) &= -\frac{\sin(y)}{\cos(y)} \\ &= -\tan(y) \end{aligned}$$

Integrating the above w.r.t y results in

$$\begin{aligned} \int f'(y)\,\mathrm{d}y &= \int (-\tan(y))\,\mathrm{d}y \\ f(y) &= \ln(\cos(y)) + c_1 \end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \ln(\cos(x)) + \ln(\cos(y)) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \ln(\cos(x)) + \ln(\cos(y))$$

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{4}$ and $y = \frac{\pi}{4}$ in the above solution gives an equation to solve for the constant of integration.

$$-\ln(2) = c_1$$

$$c_1 = -\ln(2)$$

Substituting c_1 found above in the general solution gives

$$\ln(\cos(x)) + \ln(\cos(y)) = -\ln(2)$$

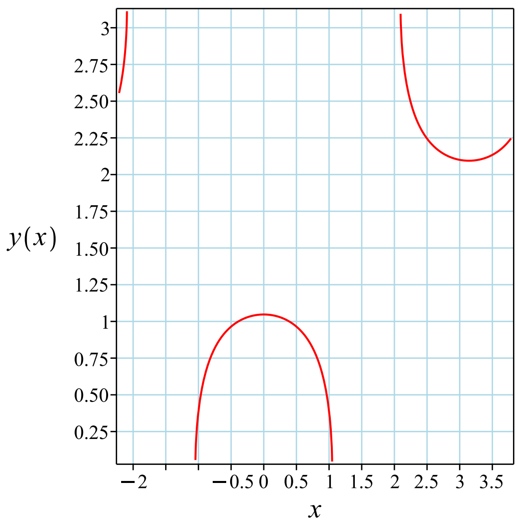
Solving for y from the above gives

$$y = \arccos\left(\frac{\sec(x)}{2}\right)$$

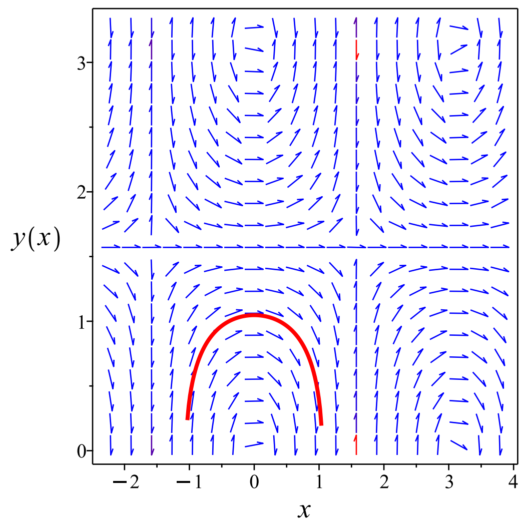
Summary

The solution(s) found are the following

$$y = \arccos\left(\frac{\sec(x)}{2}\right) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \arccos\left(\frac{\sec(x)}{2}\right)$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.172 (sec). Leaf size: 9

```
dsolve([diff(y(x),x)=1-(sin(x+y(x)))/(sin(y(x))*cos(x)),y(1/4*Pi) = 1/4*Pi],y(x), singsol=al
```

$$y(x) = \frac{\pi}{2} - \arcsin\left(\frac{\sec(x)}{2}\right)$$

✓ Solution by Mathematica

Time used: 6.259 (sec). Leaf size: 12

```
DSolve[{y'[x]==1-(Sin[x+y[x]])/(Sin[y[x]]*Cos[x]),{y[Pi/4]==Pi/4}},y[x],x,IncludeSingularSol
```

$$y(x) \to \arccos\left(\frac{\sec(x)}{2}\right)$$

2.15 problem Problem 15

Internal problem ID [2636]
Internal file name [`OUTPUT/2128_Sunday_June_05_2022_02_49_38_AM_92792261/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.4, Separable Differential Equations. page 43
Problem number: Problem 15.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y' - y^3 \sin(x) = 0}$$

With initial conditions

$$[y(0) = 0]$$

2.15.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= y^3 \sin(x) \end{aligned}$$

The x domain of $f(x, y)$ when $y = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x, y)$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(y^3 \sin(x)\right) \\ &= 3y^2 \sin(x)\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Therefore solution exists and is unique.

2.15.2 Solving as separable ode

In canonical form the ODE is

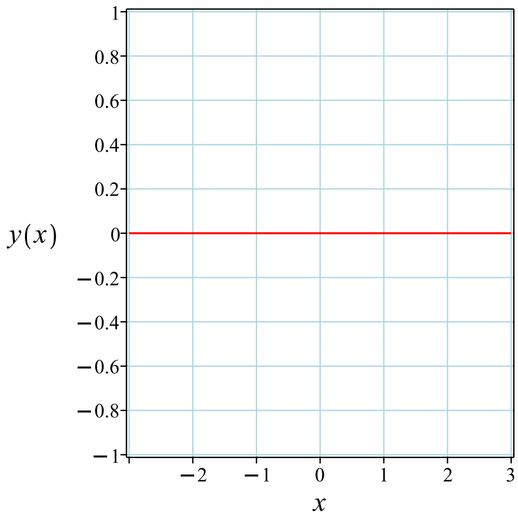
$$\begin{aligned}y' &= F(x, y) \\ &= f(x)g(y) \\ &= y^3 \sin(x)\end{aligned}$$

Where $f(x) = \sin(x)$ and $g(y) = y^3$. Since unique solution exists and $g(y)$ evaluated at $y_0 = 0$ is zero, then the solution is

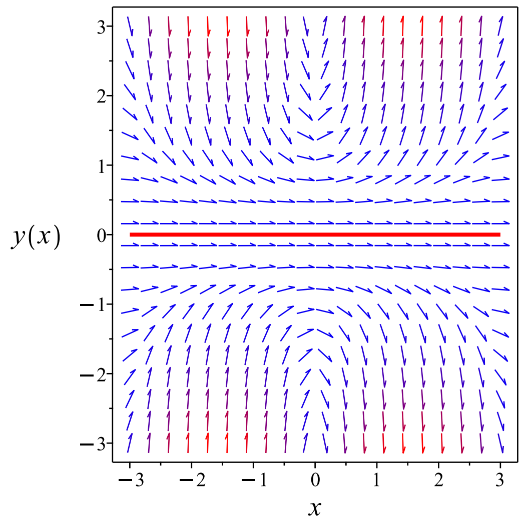
$$\begin{aligned}y &= y_0 \\ &= 0\end{aligned}$$

Summary
The solution(s) found are the following

$$y = 0 \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = 0$$

Verified OK.

2.15.3 Maple step by step solution

Let's solve

$[y' - y^3 \sin(x) = 0, y(0) = 0]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^3} = \sin(x)$

- Integrate both sides with respect to x

  $\int \frac{y'}{y^3} dx = \int \sin(x)\, dx + c_1$

- Evaluate integral

  $-\frac{1}{2y^2} = -\cos(x) + c_1$

- Solve for y

  $\left\{y = \frac{1}{\sqrt{-2c_1 + 2\cos(x)}}, y = -\frac{1}{\sqrt{-2c_1 + 2\cos(x)}}\right\}$

- Use initial condition $y(0) = 0$

  $0 = \frac{1}{\sqrt{-2c_1+2}}$
- Solution does not satisfy initial condition
- Use initial condition $y(0) = 0$

  $0 = -\frac{1}{\sqrt{-2c_1+2}}$
- Solution does not satisfy initial condition

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 5

```
dsolve([diff(y(x),x)=y(x)^3*sin(x),y(0) = 0],y(x), singsol=all)
```

$$y(x) = 0$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 6

```
DSolve[{y'[x]==y[x]^3*Sin[x],{y[0]==0}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 0$$

2.16 problem Problem 16

Internal problem ID [2637]
Internal file name [`OUTPUT/2129_Sunday_June_05_2022_02_49_40_AM_95114883/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.4, Separable Differential Equations. page 43
Problem number: Problem 16.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - \frac{2\sqrt{y-1}}{3} = 0}$$

With initial conditions

$$[y(1) = 1]$$

2.16.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= \frac{2\sqrt{y-1}}{3} \end{aligned}$$

The y domain of $f(x,y)$ when $x=1$ is

$$\{1 \leq y\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$
\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{2\sqrt{y-1}}{3}\right) \\
&= \frac{1}{3\sqrt{y-1}}
\end{aligned}
$$

The y domain of $\frac{\partial f}{\partial y}$ when $x = 1$ is

$$
\{1 < y\}
$$

But the point $y_0 = 1$ is not inside this domain. Hence existence and uniqueness theorem does not apply. Solution exists but no guarantee that unique solution exists.

2.16.2 Solving as separable ode

In canonical form the ODE is

$$
\begin{aligned}
y' &= F(x, y) \\
&= f(x)g(y) \\
&= \frac{2\sqrt{y-1}}{3}
\end{aligned}
$$

Where $f(x) = 1$ and $g(y) = \frac{2\sqrt{y-1}}{3}$. Integrating both sides gives

$$
\frac{1}{\frac{2\sqrt{y-1}}{3}}\,dy = 1\,dx
$$

$$
\int \frac{1}{\frac{2\sqrt{y-1}}{3}}\,dy = \int 1\,dx
$$

$$
3\sqrt{y-1} = x + c_1
$$

The solution is

$$
3\sqrt{y-1} - x - c_1 = 0
$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$
-1 - c_1 = 0
$$

$$
c_1 = -1
$$

Substituting c_1 found above in the general solution gives

$$3\sqrt{y-1}-x+1=0$$

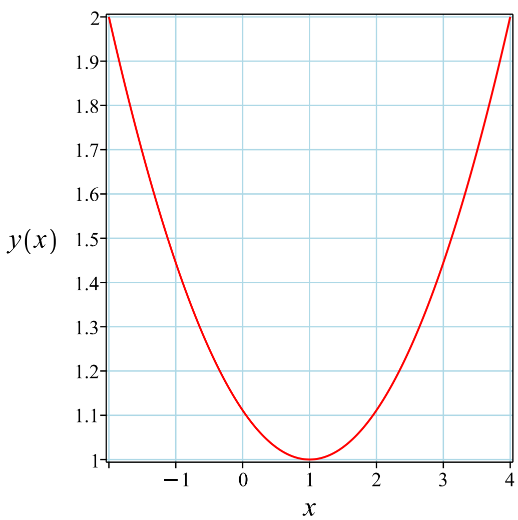
Solving for y from the above gives

$$y=\frac{1}{9}x^2-\frac{2}{9}x+\frac{10}{9}$$

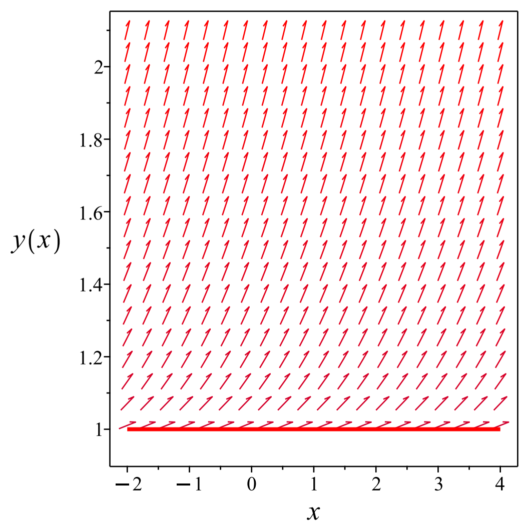
Summary

The solution(s) found are the following

$$y=\frac{1}{9}x^2-\frac{2}{9}x+\frac{10}{9} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y=\frac{1}{9}x^2-\frac{2}{9}x+\frac{10}{9}$$

Verified OK.

2.16.3 Maple step by step solution

Let's solve

$\left[y' - \frac{2\sqrt{y-1}}{3} = 0, y(1) = 1\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{\sqrt{y-1}} = \frac{2}{3}$

- Integrate both sides with respect to x

  $\int \frac{y'}{\sqrt{y-1}}dx = \int \frac{2}{3}dx + c_1$

- Evaluate integral

  $2\sqrt{y-1} = \frac{2x}{3} + c_1$

- Solve for y

  $y = \frac{1}{4}c_1^2 + \frac{1}{3}c_1 x + \frac{1}{9}x^2 + 1$

- Use initial condition $y(1) = 1$

  $1 = \frac{1}{4}c_1^2 + \frac{1}{3}c_1 + \frac{10}{9}$

- Solve for c_1

  $c_1 = \left(-\frac{2}{3}, -\frac{2}{3}\right)$

- Substitute $c_1 = \left(-\frac{2}{3}, -\frac{2}{3}\right)$ into general solution and simplify

  $y = \frac{1}{9}x^2 - \frac{2}{9}x - \frac{2}{9}$

- Solution to the IVP

  $y = \frac{1}{9}x^2 - \frac{2}{9}x - \frac{2}{9}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 5

```
dsolve([diff(y(x),x)=2/3*(y(x)-1)^(1/2),y(1) = 1],y(x), singsol=all)
```

$$y(x) = 1$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 17

```
DSolve[{y'[x]==1/3*(y[x]-1)^(1/2),{y[1]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{36}(x^2 - 2x + 37)$$

2.17 problem Problem 17

Internal problem ID [2638]
Internal file name [`OUTPUT/2130_Sunday_June_05_2022_02_49_44_AM_28871846/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.4, Separable Differential Equations. page 43
Problem number: Problem 17.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{mv' + kv^2 = mg}$$

With initial conditions

$$[v(0) = 0]$$

2.17.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} v' &= f(t, v) \\ &= -\frac{k\,v^2 - mg}{m} \end{aligned}$$

The v domain of $f(t, v)$ when $t = 0$ is

$$\{-\infty < v < \infty\}$$

And the point $v_0 = 0$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial v} &= \frac{\partial}{\partial v}\left(-\frac{k\,v^2 - mg}{m}\right)\\ &= -\frac{2kv}{m}\end{aligned}$$

The v domain of $\frac{\partial f}{\partial v}$ when $t = 0$ is

$$\{-\infty < v < \infty\}$$

And the point $v_0 = 0$ is inside this domain. Therefore solution exists and is unique.

2.17.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}v' &= F(t, v)\\ &= f(t)g(v)\\ &= \frac{-k\,v^2 + mg}{m}\end{aligned}$$

Where $f(t) = 1$ and $g(v) = \frac{-k\,v^2+mg}{m}$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\frac{-k\,v^2+mg}{m}}\,dv &= 1\,dt\\ \int \frac{1}{\frac{-k\,v^2+mg}{m}}\,dv &= \int 1\,dt\\ \frac{m\,\operatorname{arctanh}\left(\frac{kv}{\sqrt{mgk}}\right)}{\sqrt{mgk}} &= t + c_1\end{aligned}$$

Which results in

$$v = \frac{\tanh\left(\frac{\sqrt{mgk}\,(t+c_1)}{m}\right)\sqrt{mgk}}{k}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $v = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{\sqrt{mgk}\,\mathrm{e}^{\frac{2\sqrt{mgk}\,c_1}{m}} - \sqrt{mgk}}{k\,\mathrm{e}^{\frac{2\sqrt{mgk}\,c_1}{m}} + k}$$

$$
c_1 = 0
$$

Substituting c_1 found above in the general solution gives

$$
v = \lim_{c_1 \to 0} \frac{\tanh\left(\frac{\sqrt{mgk}\,(t+c_1)}{m}\right)\sqrt{mgk}}{k}
$$

Summary
The solution(s) found are the following

$$
v = \lim_{c_1 \to 0} \frac{\tanh\left(\frac{\sqrt{mgk}\,(t+c_1)}{m}\right)\sqrt{mgk}}{k} \tag{1}
$$

Verification of solutions

$$
v = \lim_{c_1 \to 0} \frac{\tanh\left(\frac{\sqrt{mgk}\,(t+c_1)}{m}\right)\sqrt{mgk}}{k}
$$

Verified OK.

2.17.3 Maple step by step solution

Let's solve

$[mv' + kv^2 = mg, v(0) = 0]$

- Highest derivative means the order of the ODE is 1

  v'

- Separate variables

  $\frac{v'}{mg-kv^2} = \frac{1}{m}$

- Integrate both sides with respect to t

  $\int \frac{v'}{mg-kv^2} dt = \int \frac{1}{m} dt + c_1$

- Evaluate integral

  $\frac{\operatorname{arctanh}\left(\frac{kv}{\sqrt{mgk}}\right)}{\sqrt{mgk}} = \frac{t}{m} + c_1$

- Solve for v

  $v = \frac{\tanh\left(\frac{\sqrt{mgk}\,(c_1 m+t)}{m}\right)\sqrt{mgk}}{k}$

- Use initial condition $v(0) = 0$

$$0 = \frac{\tanh(c_1\sqrt{mgk})\sqrt{mgk}}{k}$$

- Solve for c_1

$$c_1 = 0$$

- Substitute $c_1 = 0$ into general solution and simplify

$$v = \frac{\tanh\left(\frac{\sqrt{mgk}\,t}{m}\right)\sqrt{mgk}}{k}$$

- Solution to the IVP

$$v = \frac{\tanh\left(\frac{\sqrt{mgk}\,t}{m}\right)\sqrt{mgk}}{k}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 26

```
dsolve([m*diff(v(t),t)=m*g-k*v(t)^2,v(0) = 0],v(t), singsol=all)
```

$$v(t) = \frac{\tanh\left(\frac{\sqrt{mgk}\,t}{m}\right)\sqrt{mgk}}{k}$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 39

```
DSolve[{m*v'[t]==m*g-k*v[t]^2,{v[0]==0}},v[t],t,IncludeSingularSolutions -> True]
```

$$v(t) \to \frac{\sqrt{g}\sqrt{m}\tanh\left(\frac{\sqrt{g}\sqrt{k}t}{\sqrt{m}}\right)}{\sqrt{k}}$$

3 Chapter 1, First-Order Differential Equations. Section 1.6, First-Order Linear Differential Equations. page 59

3.1 problem Problem 1

Internal problem ID [2639]
Internal file name [`OUTPUT/2131_Sunday_June_05_2022_02_49_47_AM_87891120/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.6, First-Order Linear Differential Equations. page 59
Problem number: Problem 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' + y = 4\,\mathrm{e}^{x}}$$

3.1.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 4\,\mathrm{e}^{x} \end{aligned}$$

Hence the ode is

$$y' + y = 4\,\mathrm{e}^{x}$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int 1 dx} \\ &= \mathrm{e}^{x} \end{aligned}$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(4\,e^{x}\right) \\
\frac{d}{dx}\left(e^{x} y\right) &= \left(e^{x}\right)\left(4\,e^{x}\right) \\
d\left(e^{x} y\right) &= \left(4\,e^{2x}\right)\, dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
e^{x} y &= \int 4\,e^{2x}\, dx \\
e^{x} y &= 2\,e^{2x} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = e^{x}$ results in

$$
y = 2\,e^{-x} e^{2x} + c_1 e^{-x}
$$

which simplifies to

$$
y = 2\,e^{x} + c_1 e^{-x}
$$

Summary

The solution(s) found are the following

$$
y = 2\,e^{x} + c_1 e^{-x} \tag{1}
$$

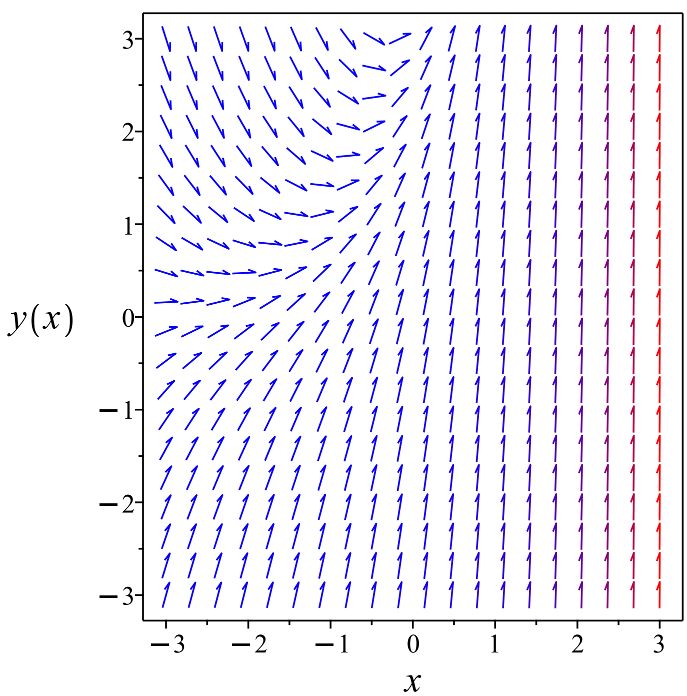


Figure 80: Slope field plot

Verification of solutions

$$y = 2\,\mathrm{e}^{x} + c_1 \mathrm{e}^{-x}$$

Verified OK.

3.1.2 Maple step by step solution

Let's solve

$y' + y = 4\,\mathrm{e}^{x}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -y + 4\,\mathrm{e}^{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + y = 4\,\mathrm{e}^{x}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\,(y' + y) = 4\mu(x)\,\mathrm{e}^{x}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\,(y' + y) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \mu(x)$

- Solve to find the integrating factor

  $\mu(x) = \mathrm{e}^{x}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int 4\mu(x)\,\mathrm{e}^{x} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int 4\mu(x)\,\mathrm{e}^{x} dx + c_1$

- Solve for y

  $y = \frac{\int 4\mu(x)\mathrm{e}^{x} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \mathrm{e}^{x}$

  $y = \frac{\int 4(\mathrm{e}^{x})^2 dx + c_1}{\mathrm{e}^{x}}$

- Evaluate the integrals on the rhs

  $y = \frac{2(e^x)^2 + c_1}{e^x}$

- Simplify

  $y = 2\,e^x + c_1 e^{-x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(diff(y(x),x)+y(x)=4*exp(x),y(x), singsol=all)
```

$$y(x) = 2\,e^x + e^{-x} c_1$$

✓ Solution by Mathematica

Time used: 0.041 (sec). Leaf size: 19

```
DSolve[y'[x]+y[x]==4*Exp[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 2e^x + c_1 e^{-x}$$

3.2 problem Problem 2

Internal problem ID [2640]
Internal file name [`OUTPUT/2132_Sunday_June_05_2022_02_49_48_AM_40991139/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.6, First-Order Linear Differential Equations. page 59
Problem number: Problem 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' + \frac{2y}{x} = 5x^2}$$

3.2.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{2}{x}$$
$$q(x) = 5x^2$$

Hence the ode is

$$y' + \frac{2y}{x} = 5x^2$$

The integrating factor μ is

$$\mu = e^{\int \frac{2}{x}dx}$$
$$= x^2$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(5x^2\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2 y\right) &= \left(x^2\right)\left(5x^2\right) \\
\mathrm{d}\left(x^2 y\right) &= \left(5x^4\right)\,\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
x^2 y &= \int 5x^4\,\mathrm{d}x \\
x^2 y &= x^5 + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = x^2$ results in

$$
y = x^3 + \frac{c_1}{x^2}
$$

Summary
The solution(s) found are the following

$$
y = x^3 + \frac{c_1}{x^2} \tag{1}
$$

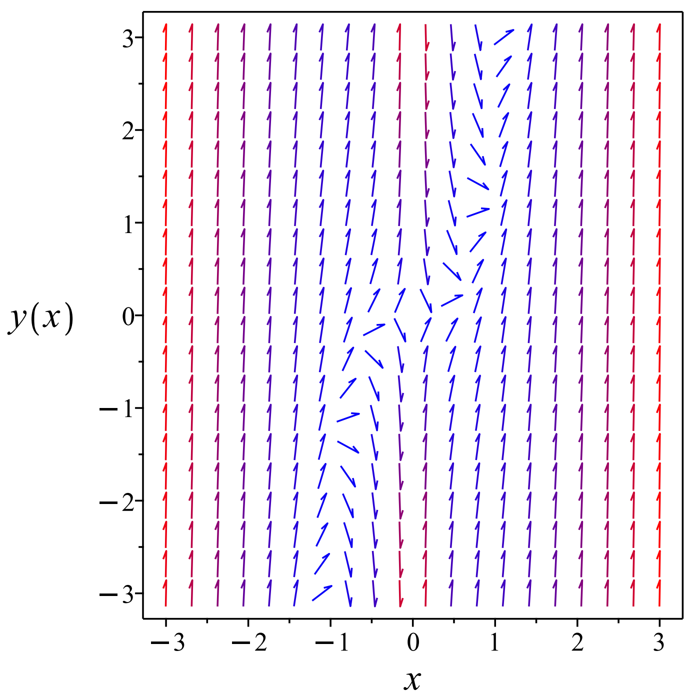


Figure 81: Slope field plot

Verification of solutions

$$y = x^3 + \frac{c_1}{x^2}$$

Verified OK.

3.2.2 Maple step by step solution

Let's solve

$y' + \frac{2y}{x} = 5x^2$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{2y}{x} + 5x^2$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{2y}{x} = 5x^2$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{2y}{x}\right) = 5\mu(x)\,x^2$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y' + \frac{2y}{x}\right) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{2\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = x^2$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int 5\mu(x)\,x^2 dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int 5\mu(x)\,x^2 dx + c_1$

- Solve for y

  $y = \frac{\int 5\mu(x)x^2 dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = x^2$

  $y = \frac{\int 5x^4 dx + c_1}{x^2}$

- Evaluate the integrals on the rhs

  $y = \frac{x^5+c_1}{x^2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve(diff(y(x),x)+2/x*y(x)=5*x^2,y(x), singsol=all)
```

$$y(x) = \frac{x^5 + c_1}{x^2}$$

✓ Solution by Mathematica

Time used: 0.031 (sec). Leaf size: 15

```
DSolve[y'[x]+2/x*y[x]==5*x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^5 + c_1}{x^2}$$

3.3 problem Problem 3

Internal problem ID [2641]
Internal file name [`OUTPUT/2133_Sunday_June_05_2022_02_49_50_AM_92282794/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.6, First-Order Linear Differential Equations. page 59
Problem number: Problem 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y'x^2 - 4yx = x^7 \sin(x)}$$

3.3.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{4}{x}$$
$$q(x) = x^5 \sin(x)$$

Hence the ode is

$$y' - \frac{4y}{x} = x^5 \sin(x)$$

The integrating factor μ is

$$\mu = e^{\int -\frac{4}{x}dx}$$
$$= \frac{1}{x^4}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(x^5 \sin(x)\right)$$
$$\frac{d}{dx}\left(\frac{y}{x^4}\right) = \left(\frac{1}{x^4}\right)\left(x^5 \sin(x)\right)$$
$$d\left(\frac{y}{x^4}\right) = (x \sin(x))\, dx$$

Integrating gives

$$\frac{y}{x^4} = \int x \sin(x)\, dx$$
$$\frac{y}{x^4} = \sin(x) - \cos(x)\, x + c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^4}$ results in

$$y = x^4(\sin(x) - \cos(x)\, x) + c_1 x^4$$

which simplifies to

$$y = x^4(\sin(x) - \cos(x)\, x + c_1)$$

Summary

The solution(s) found are the following

$$y = x^4(\sin(x) - \cos(x)\, x + c_1) \tag{1}$$

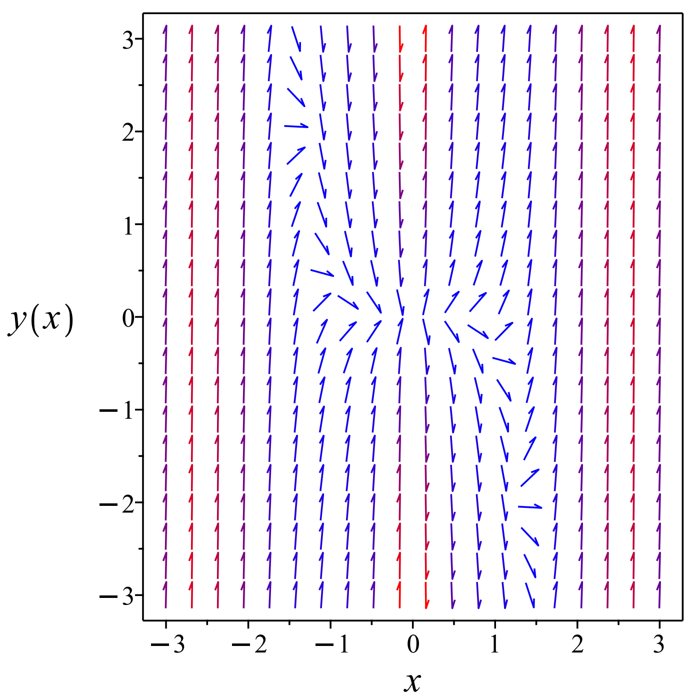


Figure 82: Slope field plot

Verification of solutions

$$y = x^4(\sin(x) - \cos(x)\,x + c_1)$$

Verified OK.

3.3.2 Maple step by step solution

Let's solve

$y'x^2 - 4yx = x^7 \sin(x)$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{4y}{x} + x^5 \sin(x)$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{4y}{x} = x^5 \sin(x)$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - \frac{4y}{x}\right) = \mu(x)\,x^5 \sin(x)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y' - \frac{4y}{x}\right) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\frac{4\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = \frac{1}{x^4}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int \mu(x)\,x^5 \sin(x)\,dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int \mu(x)\,x^5 \sin(x)\,dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x)x^5 \sin(x)dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{1}{x^4}$

  $y = x^4\left(\int x \sin(x)\,dx + c_1\right)$

- Evaluate the integrals on the rhs

  $y = x^4(\sin(x) - \cos(x)\, x + c_1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(x^2*diff(y(x),x)-4*x*y(x)=x^7*sin(x),y(x), singsol=all)
```

$$y(x) = (-x\cos(x) + \sin(x) + c_1)\, x^4$$

✓ Solution by Mathematica

Time used: 0.061 (sec). Leaf size: 19

```
DSolve[x^2*y'[x]-4*x*y[x]==x^7*Sin[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^4(\sin(x) - x\cos(x) + c_1)$$

3.4 problem Problem 4

Internal problem ID [2642]
Internal file name [`OUTPUT/2134_Sunday_June_05_2022_02_49_53_AM_32028512/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.6, First-Order Linear Differential Equations. page 59
Problem number: Problem 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' + 2yx = 2x^3}$$

3.4.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 2x$$
$$q(x) = 2x^3$$

Hence the ode is

$$y' + 2yx = 2x^3$$

The integrating factor μ is

$$\mu = e^{\int 2x dx}$$
$$= e^{x^2}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(2x^3\right)$$
$$\frac{d}{dx}\left(e^{x^2} y\right) = \left(e^{x^2}\right)\left(2x^3\right)$$
$$d\left(e^{x^2} y\right) = \left(2x^3 e^{x^2}\right) dx$$

Integrating gives

$$e^{x^2} y = \int 2x^3 e^{x^2}\, dx$$
$$e^{x^2} y = \left(x^2 - 1\right) e^{x^2} + c_1$$

Dividing both sides by the integrating factor $\mu = e^{x^2}$ results in

$$y = e^{-x^2}\left(x^2 - 1\right) e^{x^2} + c_1 e^{-x^2}$$

which simplifies to

$$y = x^2 - 1 + c_1 e^{-x^2}$$

Summary

The solution(s) found are the following

$$y = x^2 - 1 + c_1 e^{-x^2} \tag{1}$$

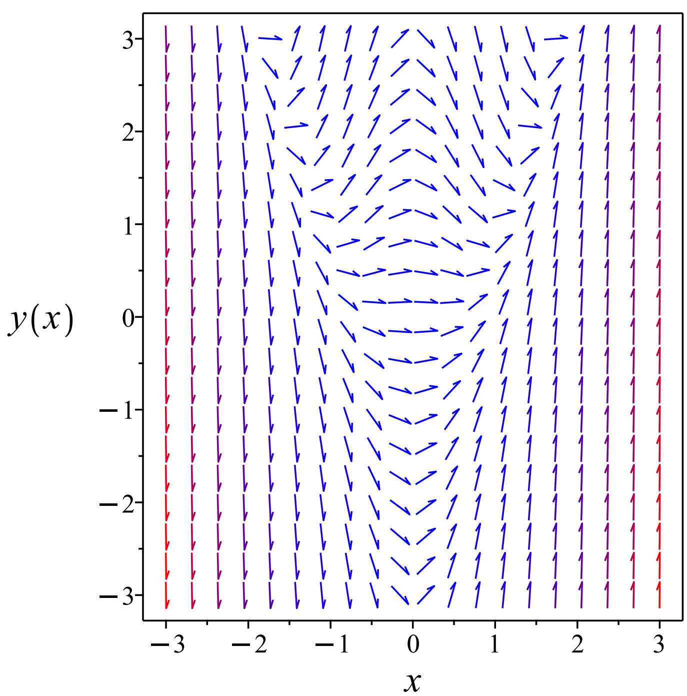


Figure 83: Slope field plot

Verification of solutions

$$
y = x^2 - 1 + c_1 e^{-x^2}
$$

Verified OK.

3.4.2 Maple step by step solution

Let's solve

$y' + 2yx = 2x^3$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -2yx + 2x^3$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + 2yx = 2x^3$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\,(y' + 2yx) = 2\mu(x)\,x^3$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\,(y' + 2yx) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = 2\mu(x)\,x$

- Solve to find the integrating factor

  $\mu(x) = e^{x^2}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int 2\mu(x)\,x^3 dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int 2\mu(x)\,x^3 dx + c_1$

- Solve for y

  $y = \frac{\int 2\mu(x)x^3 dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = e^{x^2}$

  $y = \frac{\int 2x^3 e^{x^2} dx + c_1}{e^{x^2}}$

- Evaluate the integrals on the rhs

  $y = \frac{(x^2-1)\,\mathrm{e}^{x^2}+c_1}{\mathrm{e}^{x^2}}$

- Simplify

  $y = x^2 - 1 + c_1 \mathrm{e}^{-x^2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(y(x),x)+2*x*y(x)=2*x^3,y(x), singsol=all)
```

$$y(x) = x^2 - 1 + c_1 \mathrm{e}^{-x^2}$$

✓ Solution by Mathematica

Time used: 0.032 (sec). Leaf size: 20

```
DSolve[y'[x]+2*x*y[x]==2*x^3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^2 + c_1 e^{-x^2} - 1$$

3.5 problem Problem 5

Internal problem ID [2643]
Internal file name [`OUTPUT/2135_Sunday_June_05_2022_02_49_55_AM_48257899/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.6, First-Order Linear Differential Equations. page 59
Problem number: Problem 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' + \frac{2xy}{-x^2+1} = 4x}$$

3.5.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{2x}{x^2-1}$$
$$q(x) = 4x$$

Hence the ode is

$$y' - \frac{2xy}{x^2-1} = 4x$$

The integrating factor μ is

$$\mu = e^{\int -\frac{2x}{x^2-1}dx}$$
$$= e^{-\ln(x-1)-\ln(x+1)}$$

Which simplifies to

$$\mu = \frac{1}{x^2 - 1}$$

The ode becomes

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)(4x) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x^2-1}\right) &= \left(\frac{1}{x^2-1}\right)(4x) \\
\mathrm{d}\left(\frac{y}{x^2-1}\right) &= \left(\frac{4x}{x^2-1}\right)\,\mathrm{d}x
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\frac{y}{x^2-1} &= \int \frac{4x}{x^2-1}\,\mathrm{d}x \\
\frac{y}{x^2-1} &= 2\ln(x-1) + 2\ln(x+1) + c_1
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2-1}$ results in

$$y = (x^2 - 1)(2\ln(x-1) + 2\ln(x+1)) + c_1(x^2 - 1)$$

which simplifies to

$$y = (x^2 - 1)(2\ln(x-1) + 2\ln(x+1) + c_1)$$

Summary

The solution(s) found are the following

$$y = (x^2 - 1)(2\ln(x-1) + 2\ln(x+1) + c_1) \tag{1}$$

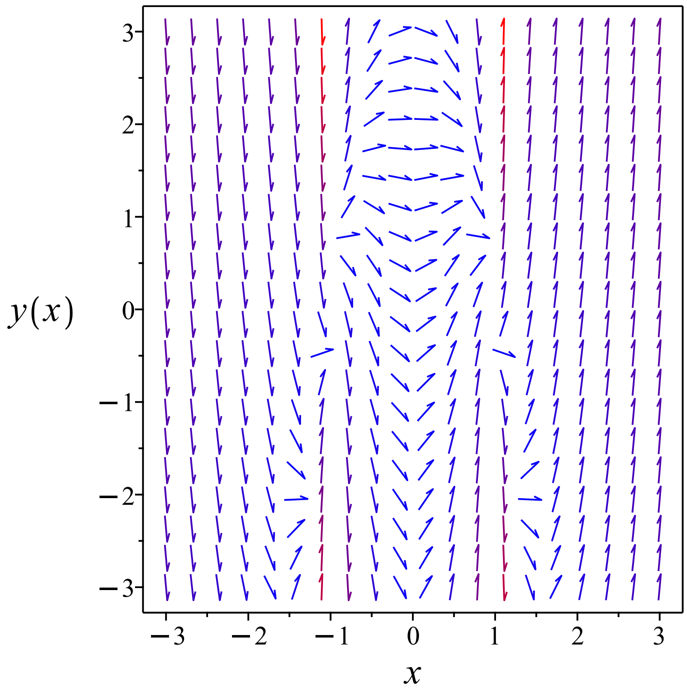


Figure 84: Slope field plot

Verification of solutions

$$y = (x^2 - 1)(2\ln(x-1) + 2\ln(x+1) + c_1)$$

Verified OK.

3.5.2 Maple step by step solution

Let's solve

$y' + \frac{2xy}{-x^2+1} = 4x$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{2xy}{x^2-1} + 4x$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{2xy}{x^2-1} = 4x$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$$\mu(x)\left(y' - \frac{2xy}{x^2-1}\right) = 4\mu(x)\, x$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

$$\mu(x)\left(y' - \frac{2xy}{x^2-1}\right) = \mu'(x)\, y + \mu(x)\, y'$$

- Isolate $\mu'(x)$

$$\mu'(x) = -\frac{2\mu(x)x}{x^2-1}$$

- Solve to find the integrating factor

$$\mu(x) = \frac{1}{(x+1)(x-1)}$$

- Integrate both sides with respect to x

$$\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int 4\mu(x)\, x dx + c_1$$

- Evaluate the integral on the lhs

$$\mu(x)\, y = \int 4\mu(x)\, x dx + c_1$$

- Solve for y

$$y = \frac{\int 4\mu(x)x dx + c_1}{\mu(x)}$$

- Substitute $\mu(x) = \frac{1}{(x+1)(x-1)}$

$$y = (x+1)(x-1)\left(\int \frac{4x}{(x-1)(x+1)} dx + c_1\right)$$

- Evaluate the integrals on the rhs

$$y = (x+1)(x-1)(2\ln((x-1)(x+1)) + c_1)$$

- Simplify

$$y = (2\ln(x^2-1) + c_1)(x^2-1)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
dsolve(diff(y(x),x)+2*x/(1-x^2)*y(x)=4*x,y(x), singsol=all)
```

$$y(x) = (2\ln(x-1) + 2\ln(x+1) + c_1)(x^2-1)$$

✓ Solution by Mathematica

Time used: 0.031 (sec). Leaf size: 22

```
DSolve[y'[x]+2*x/(1-x^2)*y[x]==4*x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to (x^2-1)\left(2\log(x^2-1) + c_1\right)$$

3.6 problem Problem 6

Internal problem ID [2644]
Internal file name [`OUTPUT/2136_Sunday_June_05_2022_02_49_57_AM_87142551/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.6, First-Order Linear Differential Equations. page 59
Problem number: Problem 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' + \frac{2xy}{x^2+1} = \frac{4}{(x^2+1)^2}}$$

3.6.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{2x}{x^2+1}$$
$$q(x) = \frac{4}{(x^2+1)^2}$$

Hence the ode is

$$y' + \frac{2xy}{x^2+1} = \frac{4}{(x^2+1)^2}$$

The integrating factor μ is

$$\mu = e^{\int \frac{2x}{x^2+1}dx}$$
$$= x^2 + 1$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{4}{(x^2+1)^2}\right)$$
$$\frac{d}{dx}\left((x^2+1)\, y\right) = (x^2+1)\left(\frac{4}{(x^2+1)^2}\right)$$
$$d\left((x^2+1)\, y\right) = \left(\frac{4}{x^2+1}\right) dx$$

Integrating gives

$$(x^2+1)\, y = \int \frac{4}{x^2+1}\, dx$$
$$(x^2+1)\, y = 4\arctan(x) + c_1$$

Dividing both sides by the integrating factor $\mu = x^2 + 1$ results in

$$y = \frac{4\arctan(x)}{x^2+1} + \frac{c_1}{x^2+1}$$

which simplifies to

$$y = \frac{4\arctan(x) + c_1}{x^2+1}$$

Summary

The solution(s) found are the following

$$y = \frac{4\arctan(x) + c_1}{x^2+1} \tag{1}$$

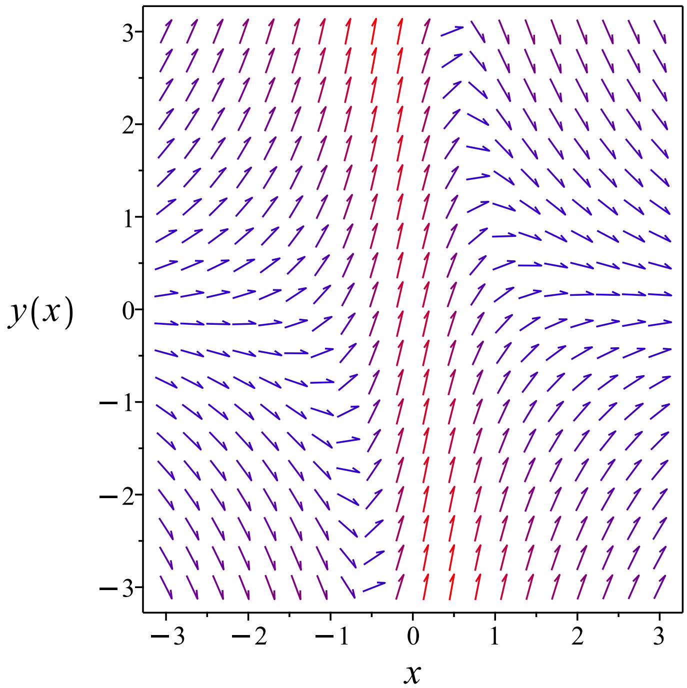


Figure 85: Slope field plot

Verification of solutions

$$y = \frac{4 \arctan(x) + c_1}{x^2 + 1}$$

Verified OK.

3.6.2 Maple step by step solution

Let's solve

$y' + \frac{2xy}{x^2+1} = \frac{4}{(x^2+1)^2}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{2xy}{x^2+1} + \frac{4}{(x^2+1)^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{2xy}{x^2+1} = \frac{4}{(x^2+1)^2}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y'+\frac{2xy}{x^2+1}\right)=\frac{4\mu(x)}{\left(x^2+1\right)^2}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y'+\frac{2xy}{x^2+1}\right)=\mu'(x)\,y+\mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x)=\frac{2\mu(x)x}{x^2+1}$

- Solve to find the integrating factor

  $\mu(x)=x^2+1$

- Integrate both sides with respect to x

  $\int\left(\frac{d}{dx}(\mu(x)\,y)\right)dx=\int\frac{4\mu(x)}{\left(x^2+1\right)^2}dx+c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y=\int\frac{4\mu(x)}{\left(x^2+1\right)^2}dx+c_1$

- Solve for y

  $y=\frac{\int\frac{4\mu(x)}{\left(x^2+1\right)^2}dx+c_1}{\mu(x)}$

- Substitute $\mu(x)=x^2+1$

  $y=\frac{\int\frac{4}{x^2+1}dx+c_1}{x^2+1}$

- Evaluate the integrals on the rhs

  $y=\frac{4\arctan(x)+c_1}{x^2+1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve(diff(y(x),x)+2*x/(1+x^2)*y(x)=4/(1+x^2)^2,y(x), singsol=all)
```

$$y(x) = \frac{4 \arctan(x) + c_1}{x^2 + 1}$$

✓ Solution by Mathematica

Time used: 0.038 (sec). Leaf size: 20

```
DSolve[y'[x]+2*x/(1+x^2)*y[x]==4/(1+x^2)^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{4 \arctan(x) + c_1}{x^2 + 1}$$

3.7 problem Problem 7

Internal problem ID [2645]
Internal file name [`OUTPUT/2137_Sunday_June_05_2022_02_49_59_AM_17371146/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.6, First-Order Linear Differential Equations. page 59
Problem number: Problem 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_linear]`

$$\boxed{2\cos(x)^2 y' + y\sin(2x) = 4\cos(x)^4}$$

3.7.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}p(x) &= \tan(x)\\ q(x) &= 2\cos(x)^2\end{aligned}$$

Hence the ode is

$$y' + y\tan(x) = 2\cos(x)^2$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int \tan(x)dx}\\ &= \frac{1}{\cos(x)}\end{aligned}$$

Which simplifies to

$$\mu = \sec(x)$$

The ode becomes

$$\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(2\cos(x)^2\right) \\
\frac{d}{dx}(\sec(x)\,y) &= (\sec(x))\left(2\cos(x)^2\right) \\
d(\sec(x)\,y) &= (2\cos(x))\,dx
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\sec(x)\,y &= \int 2\cos(x)\,dx \\
\sec(x)\,y &= 2\sin(x) + c_1
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \sec(x)$ results in

$$y = 2\sin(x)\cos(x) + c_1\cos(x)$$

which simplifies to

$$y = \cos(x)(2\sin(x) + c_1)$$

Summary

The solution(s) found are the following

$$y = \cos(x)(2\sin(x) + c_1) \tag{1}$$

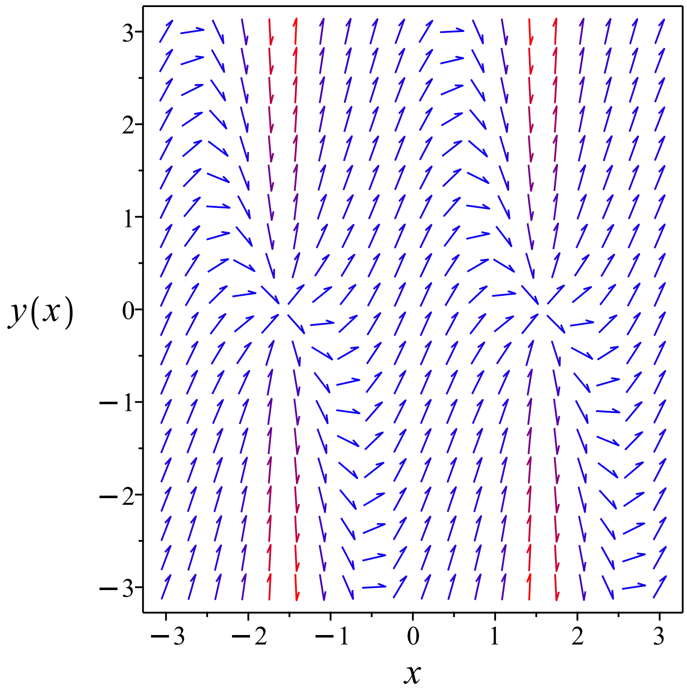


Figure 86: Slope field plot

Verification of solutions

$$y = \cos(x)(2\sin(x) + c_1)$$

Verified OK.

3.7.2 Maple step by step solution

Let's solve

$2\cos(x)^2 y' + y\sin(2x) = 4\cos(x)^4$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{\sin(2x)y}{2\cos(x)^2} + 2\cos(x)^2$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{\sin(2x)y}{2\cos(x)^2} = 2\cos(x)^2$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$$\mu(x)\left(y'+\frac{\sin(2x)y}{2\cos(x)^2}\right)=2\mu(x)\cos(x)^2$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

$$\mu(x)\left(y'+\frac{\sin(2x)y}{2\cos(x)^2}\right)=\mu'(x)\,y+\mu(x)\,y'$$

- Isolate $\mu'(x)$

$$\mu'(x)=\frac{\mu(x)\sin(2x)}{2\cos(x)^2}$$

- Solve to find the integrating factor

$$\mu(x)=\frac{1}{\cos(x)}$$

- Integrate both sides with respect to x

$$\int\left(\frac{d}{dx}(\mu(x)\,y)\right)dx=\int 2\mu(x)\cos(x)^2\,dx+c_1$$

- Evaluate the integral on the lhs

$$\mu(x)\,y=\int 2\mu(x)\cos(x)^2\,dx+c_1$$

- Solve for y

$$y=\frac{\int 2\mu(x)\cos(x)^2dx+c_1}{\mu(x)}$$

- Substitute $\mu(x)=\frac{1}{\cos(x)}$

$$y=\cos(x)\left(\int 2\cos(x)\,dx+c_1\right)$$

- Evaluate the integrals on the rhs

$$y=\cos(x)\,(2\sin(x)+c_1)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve(2*(cos(x)^2)*diff(y(x),x)+y(x)*sin(2*x)=4*cos(x)^4,y(x), singsol=all)
```

$$y(x) = (2\sin(x) + c_1)\cos(x)$$

✓ Solution by Mathematica

Time used: 0.058 (sec). Leaf size: 15

```
DSolve[2*(Cos[x]^2)*y'[x]+y[x]*Sin[2*x]==4*Cos[x]^4,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \cos(x)(2\sin(x) + c_1)$$

3.8 problem Problem 8

Internal problem ID [2646]
Internal file name [`OUTPUT/2138_Sunday_June_05_2022_02_50_01_AM_1763986/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.6, First-Order Linear Differential Equations. page 59
Problem number: Problem 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' + \frac{y}{\ln(x)\, x} = 9x^2}$$

3.8.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{1}{\ln(x)\, x}$$
$$q(x) = 9x^2$$

Hence the ode is

$$y' + \frac{y}{\ln(x)\, x} = 9x^2$$

The integrating factor μ is

$$\mu = e^{\int \frac{1}{\ln(x)x}dx}$$
$$= \ln(x)$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(9x^2\right) \\
\frac{d}{dx}(\ln(x)\,y) &= (\ln(x))\left(9x^2\right) \\
d(\ln(x)\,y) &= \left(9\ln(x)\,x^2\right)\,dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\ln(x)\,y &= \int 9\ln(x)\,x^2\,dx \\
\ln(x)\,y &= 3x^3\ln(x) - x^3 + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \ln(x)$ results in

$$
y = \frac{3x^3\ln(x) - x^3}{\ln(x)} + \frac{c_1}{\ln(x)}
$$

which simplifies to

$$
y = \frac{3x^3\ln(x) - x^3 + c_1}{\ln(x)}
$$

Summary

The solution(s) found are the following

$$
y = \frac{3x^3\ln(x) - x^3 + c_1}{\ln(x)} \tag{1}
$$

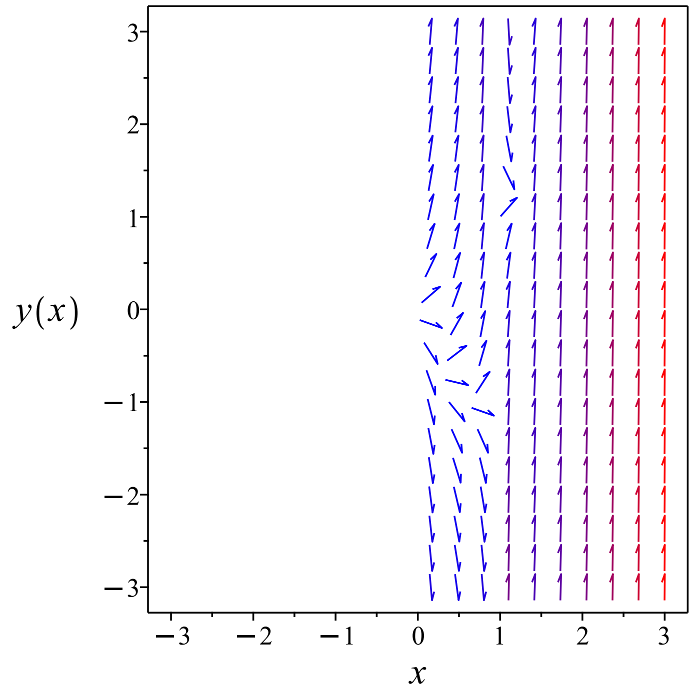


Figure 87: Slope field plot

Verification of solutions

$$y = \frac{3x^3 \ln(x) - x^3 + c_1}{\ln(x)}$$

Verified OK.

3.8.2 Maple step by step solution

Let's solve

$y' + \frac{y}{\ln(x)x} = 9x^2$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{y}{\ln(x)x} + 9x^2$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{y}{\ln(x)x} = 9x^2$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$\mu(x)\left(y'+\frac{y}{\ln(x)x}\right)=9\mu(x)\,x^2$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y'+\frac{y}{\ln(x)x}\right)=\mu'(x)\,y+\mu(x)\,y'$
- Isolate $\mu'(x)$

  $\mu'(x)=\frac{\mu(x)}{\ln(x)x}$
- Solve to find the integrating factor

  $\mu(x)=\ln(x)$
- Integrate both sides with respect to x

  $\int\left(\frac{d}{dx}(\mu(x)\,y)\right)dx=\int 9\mu(x)\,x^2dx+c_1$
- Evaluate the integral on the lhs

  $\mu(x)\,y=\int 9\mu(x)\,x^2dx+c_1$
- Solve for y

  $y=\frac{\int 9\mu(x)x^2dx+c_1}{\mu(x)}$
- Substitute $\mu(x)=\ln(x)$

  $y=\frac{\int 9\ln(x)x^2dx+c_1}{\ln(x)}$
- Evaluate the integrals on the rhs

  $y=\frac{3x^3\ln(x)-x^3+c_1}{\ln(x)}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(y(x),x)+1/(x*ln(x))*y(x)=9*x^2,y(x), singsol=all)
```

$$y(x) = \frac{3x^3 \ln(x) - x^3 + c_1}{\ln(x)}$$

✓ Solution by Mathematica

Time used: 0.034 (sec). Leaf size: 25

```
DSolve[y'[x]+1/(x*Log[x])*y[x]==9*x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{-x^3 + 3x^3 \log(x) + c_1}{\log(x)}$$

3.9 problem Problem 9

Internal problem ID [2647]
Internal file name [`OUTPUT/2139_Sunday_June_05_2022_02_50_04_AM_6156280/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.6, First-Order Linear Differential Equations. page 59
Problem number: Problem 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' - y\tan(x) = 8\sin(x)^3}$$

3.9.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\tan(x) \\ q(x) &= 8\sin(x)^3 \end{aligned}$$

Hence the ode is

$$y' - y\tan(x) = 8\sin(x)^3$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int -\tan(x)dx} \\ &= \cos(x) \end{aligned}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(8\sin(x)^3\right)$$
$$\frac{d}{dx}(\cos(x)\,y) = (\cos(x))\left(8\sin(x)^3\right)$$
$$d(\cos(x)\,y) = \left(8\sin(x)^3\cos(x)\right)\,dx$$

Integrating gives

$$\cos(x)\,y = \int 8\sin(x)^3\cos(x)\,dx$$
$$\cos(x)\,y = 2\sin(x)^4 + c_1$$

Dividing both sides by the integrating factor $\mu = \cos(x)$ results in

$$y = 2\sec(x)\sin(x)^4 + c_1\sec(x)$$

which simplifies to

$$y = \sec(x)\left(2\sin(x)^4 + c_1\right)$$

Summary

The solution(s) found are the following

$$y = \sec(x)\left(2\sin(x)^4 + c_1\right) \tag{1}$$

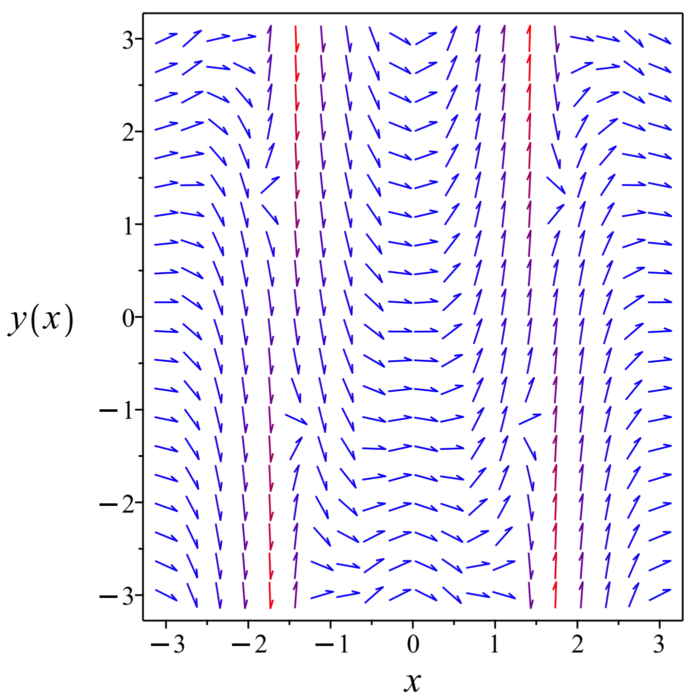


Figure 88: Slope field plot

Verification of solutions

$$y = \sec(x)\left(2\sin(x)^4 + c_1\right)$$

Verified OK.

3.9.2 Maple step by step solution

Let's solve

$y' - y\tan(x) = 8\sin(x)^3$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = y\tan(x) + 8\sin(x)^3$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - y\tan(x) = 8\sin(x)^3$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - y\tan(x)\right) = 8\mu(x)\sin(x)^3$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y' - y\tan(x)\right) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\mu(x)\tan(x)$

- Solve to find the integrating factor

  $\mu(x) = \cos(x)$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right)dx = \int 8\mu(x)\sin(x)^3\,dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int 8\mu(x)\sin(x)^3\,dx + c_1$

- Solve for y

  $y = \frac{\int 8\mu(x)\sin(x)^3dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \cos(x)$

  $y = \frac{\int 8\sin(x)^3\cos(x)dx + c_1}{\cos(x)}$

- Evaluate the integrals on the rhs

  $y = \frac{2\sin(x)^4 + c_1}{\cos(x)}$

- Simplify

  $y = \sec(x)\left(2\sin(x)^4 + c_1\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 24

```
dsolve(diff(y(x),x)-y(x)*tan(x)=8*sin(x)^3,y(x), singsol=all)
```

$$y(x) = 2\cos(x)^3 - 4\cos(x) + \frac{\sec(x)(4c_1 + 5)}{4}$$

✓ Solution by Mathematica

Time used: 0.046 (sec). Leaf size: 19

```
DSolve[y'[x]-y[x]*Tan[x]==8*Sin[x]^3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 2\sin^3(x)\tan(x) + c_1\sec(x)$$

3.10 problem Problem 10

Internal problem ID [2648]
Internal file name [OUTPUT/2140_Sunday_June_05_2022_02_50_06_AM_138995/index.tex]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.6, First-Order Linear Differential Equations. page 59
Problem number: Problem 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_linear]`

$$\boxed{x't + 2x = 4\,e^t}$$

3.10.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$x' + p(t)x = q(t)$$

Where here

$$p(t) = \frac{2}{t}$$
$$q(t) = \frac{4\,e^t}{t}$$

Hence the ode is

$$x' + \frac{2x}{t} = \frac{4\,e^t}{t}$$

The integrating factor μ is

$$\mu = e^{\int \frac{2}{t} dt}$$
$$= t^2$$

The ode becomes

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\mu x) &= (\mu)\left(\frac{4\,\mathrm{e}^{t}}{t}\right) \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(t^{2}x\right) &= \left(t^{2}\right)\left(\frac{4\,\mathrm{e}^{t}}{t}\right) \\
\mathrm{d}\left(t^{2}x\right) &= \left(4\,\mathrm{e}^{t}t\right)\,\mathrm{d}t
\end{aligned}$$

Integrating gives

$$\begin{aligned}
t^{2}x &= \int 4\,\mathrm{e}^{t}t\,\mathrm{d}t \\
t^{2}x &= 4(t-1)\,\mathrm{e}^{t}+c_{1}
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = t^{2}$ results in

$$x = \frac{4(t-1)\,\mathrm{e}^{t}}{t^{2}}+\frac{c_{1}}{t^{2}}$$

which simplifies to

$$x = \frac{(4t-4)\,\mathrm{e}^{t}+c_{1}}{t^{2}}$$

Summary
The solution(s) found are the following

$$x = \frac{(4t-4)\,\mathrm{e}^{t}+c_{1}}{t^{2}} \tag{1}$$

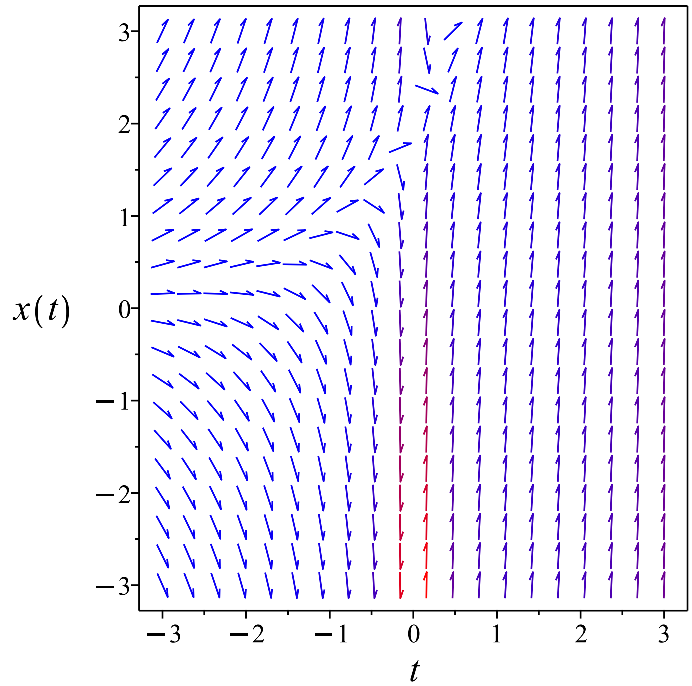


Figure 89: Slope field plot

Verification of solutions

$$x = \frac{(4t-4)\,\mathrm{e}^{t} + c_1}{t^2}$$

Verified OK.

3.10.2 Maple step by step solution

Let's solve

$x't + 2x = 4\,\mathrm{e}^{t}$

- Highest derivative means the order of the ODE is 1

  x'

- Isolate the derivative

  $x' = -\frac{2x}{t} + \frac{4\,\mathrm{e}^{t}}{t}$

- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE

  $x' + \frac{2x}{t} = \frac{4\,\mathrm{e}^{t}}{t}$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

$\mu(t)\left(x' + \frac{2x}{t}\right) = \frac{4\mu(t)\mathrm{e}^{t}}{t}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,x)$

  $\mu(t)\left(x' + \frac{2x}{t}\right) = \mu'(t)\,x + \mu(t)\,x'$

- Isolate $\mu'(t)$

  $\mu'(t) = \frac{2\mu(t)}{t}$

- Solve to find the integrating factor

  $\mu(t) = t^2$

- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\,x)\right) dt = \int \frac{4\mu(t)\mathrm{e}^{t}}{t} dt + c_1$

- Evaluate the integral on the lhs

  $\mu(t)\,x = \int \frac{4\mu(t)\mathrm{e}^{t}}{t} dt + c_1$

- Solve for x

  $x = \frac{\int \frac{4\mu(t)\mathrm{e}^{t}}{t} dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = t^2$

  $x = \frac{\int 4\,\mathrm{e}^{t} t dt + c_1}{t^2}$

- Evaluate the integrals on the rhs

  $x = \frac{4(t-1)\mathrm{e}^{t} + c_1}{t^2}$

- Simplify

  $x = \frac{(4t-4)\mathrm{e}^{t} + c_1}{t^2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve(t*diff(x(t),t)+2*x(t)=4*exp(t),x(t), singsol=all)
```

$$x(t) = \frac{(4t - 4)\,\mathrm{e}^{t} + c_1}{t^2}$$

✓ Solution by Mathematica

Time used: 0.049 (sec). Leaf size: 20

```
DSolve[t*x'[t]+2*x[t]==4*Exp[t],x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{4e^{t}(t-1) + c_1}{t^2}$$

3.11 problem Problem 11

Internal problem ID [2649]
Internal file name [`OUTPUT/2141_Sunday_June_05_2022_02_50_08_AM_17178087/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.6, First-Order Linear Differential Equations. page 59
Problem number: Problem 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' - \sin(x)(y \sec(x) - 2) = 0}$$

3.11.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\tan(x) \\ q(x) &= -2\sin(x) \end{aligned}$$

Hence the ode is

$$y' - y\tan(x) = -2\sin(x)$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int -\tan(x)dx} \\
&= \cos(x)
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)(-2\sin(x)) \\
\frac{d}{dx}(\cos(x)\,y) &= (\cos(x))(-2\sin(x)) \\
d(\cos(x)\,y) &= (-\sin(2x))\,dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\cos(x)\,y &= \int -\sin(2x)\,dx \\
\cos(x)\,y &= \frac{\cos(2x)}{2} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \cos(x)$ results in

$$
y = \frac{\sec(x)\cos(2x)}{2} + c_1 \sec(x)
$$

which simplifies to

$$
y = \cos(x) - \frac{\sec(x)}{2} + c_1 \sec(x)
$$

Summary

The solution(s) found are the following

$$
y = \cos(x) - \frac{\sec(x)}{2} + c_1 \sec(x) \tag{1}
$$

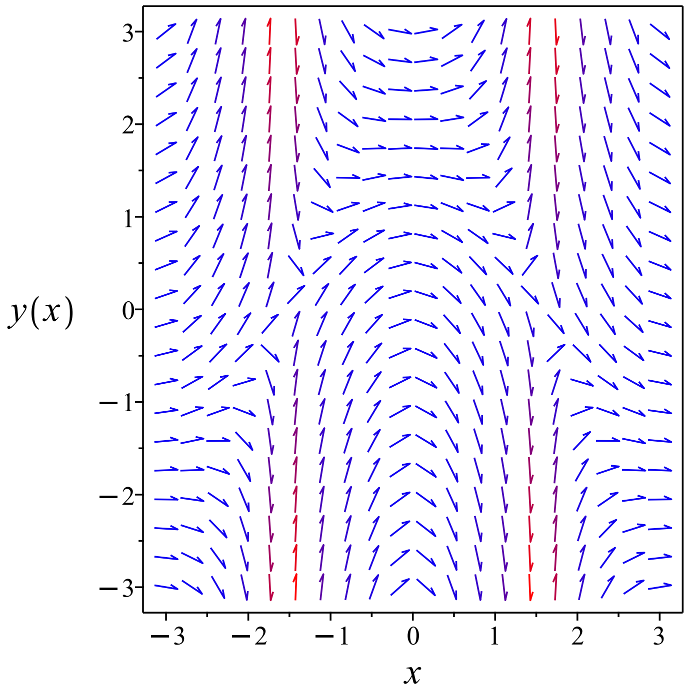


Figure 90: Slope field plot

Verification of solutions

$$y = \cos(x) - \frac{\sec(x)}{2} + c_1 \sec(x)$$

Verified OK.

3.11.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= \sin(x)(\sec(x)\,y - 2) \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 86: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{\cos(x)}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{\cos(x)}} dy \end{aligned}$$

Which results in

$$S = \cos(x)\, y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \sin(x)\,(\sec(x)\, y - 2)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\sin(x)\, y \\ S_y &= \cos(x) \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\sin(2x) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\sin(2R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{\cos(2R)}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\cos(x)\, y = \frac{\cos(2x)}{2} + c_1$$

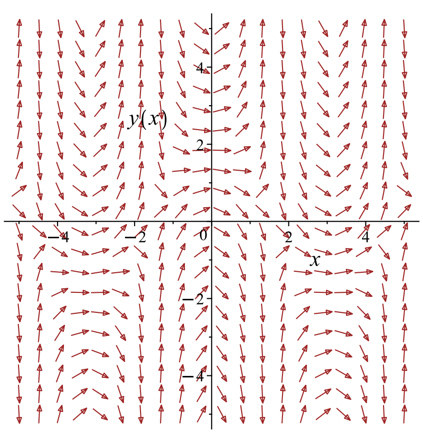
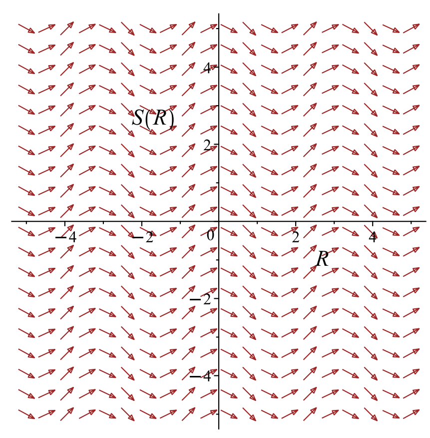
Which simplifies to

$$\cos(x)\, y = \frac{\cos(2x)}{2} + c_1$$

Which gives

$$y = \frac{\cos(2x) + 2c_1}{2\cos(x)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \sin(x)(\sec(x)\, y - 2)$ 	$R = x$ $S = \cos(x)\, y$	$\frac{dS}{dR} = -\sin(2R)$ 

Summary

The solution(s) found are the following

$$y = \frac{\cos(2x) + 2c_1}{2\cos(x)} \tag{1}$$

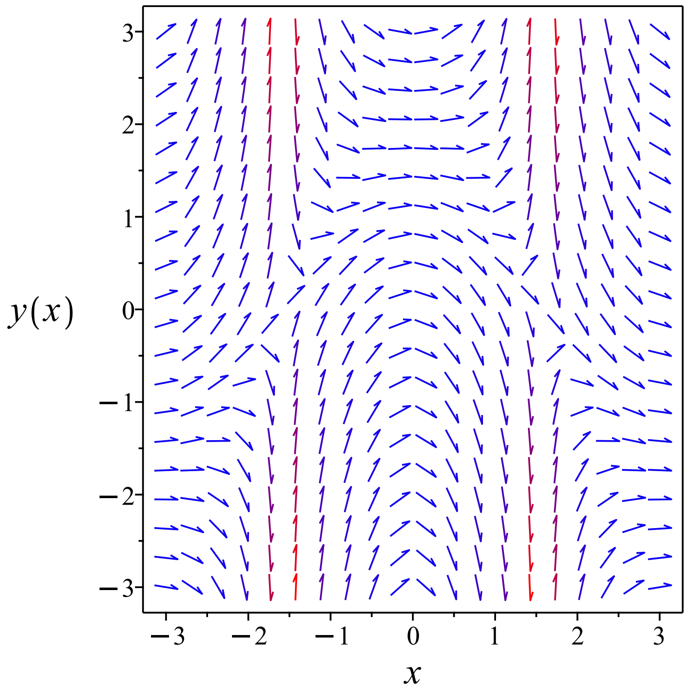


Figure 91: Slope field plot

Verification of solutions

$$y = \frac{\cos(2x) + 2c_1}{2\cos(x)}$$

Verified OK.

3.11.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\mathrm{d}y &= (\sin (x) (\sec (x)\, y - 2))\, \mathrm{d}x \\
(-\sin (x) (\sec (x)\, y - 2))\, \mathrm{d}x + \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -\sin (x) (\sec (x)\, y - 2) \\
N(x, y) &= 1
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-\sin (x) (\sec (x)\, y - 2)) \\
&= -\tan (x)
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1((-\sec(x)\sin(x)) - (0)) \\
&= -\tan(x)
\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int -\tan(x) \, \mathrm{d}x}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{\ln(\cos(x))} \\
&= \cos(x)
\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}
\overline{M} &= \mu M \\
&= \cos(x)(-\sin(x)(\sec(x)y - 2)) \\
&= -\sin(x)(y - 2\cos(x))
\end{aligned}$$

And

$$\begin{aligned}
\overline{N} &= \mu N \\
&= \cos(x)(1) \\
&= \cos(x)
\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
(-\sin(x)(y - 2\cos(x))) + (\cos(x))\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int \overline{M} \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int -\sin(x)(y - 2\cos(x)) \mathrm{d}x \\
\phi &= \cos(x)(-\cos(x) + y) + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \cos(x) + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \cos(x)$. Therefore equation (4) becomes

$$
\cos(x) = \cos(x) + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$
f(y) = c_1
$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$
\phi = \cos(x)(-\cos(x) + y) + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
c_1 = \cos(x)(-\cos(x) + y)
$$

The solution becomes

$$
y = \frac{\cos(x)^2 + c_1}{\cos(x)}
$$

Summary

The solution(s) found are the following

$$y = \frac{\cos(x)^2 + c_1}{\cos(x)} \tag{1}$$

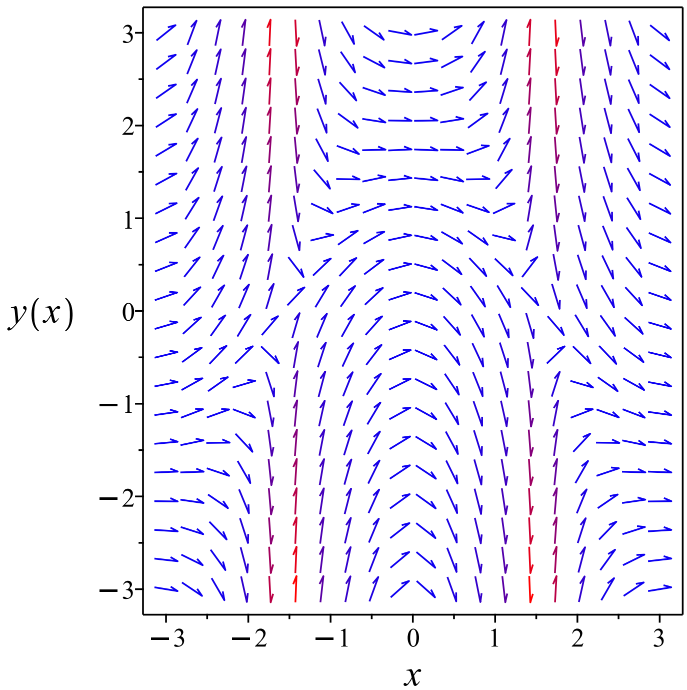


Figure 92: Slope field plot

Verification of solutions

$$y = \frac{\cos(x)^2 + c_1}{\cos(x)}$$

Verified OK.

3.11.4 Maple step by step solution

Let's solve

$y' - \sin(x)(y\sec(x) - 2) = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

$$y' = \sin(x) y \sec(x) - 2\sin(x)$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

$$-\sin(x) y \sec(x) + y' = -2\sin(x)$$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$$\mu(x) (-\sin(x) y \sec(x) + y') = -2\mu(x) \sin(x)$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x) y)$

$$\mu(x) (-\sin(x) y \sec(x) + y') = \mu'(x) y + \mu(x) y'$$

- Isolate $\mu'(x)$

$$\mu'(x) = -\mu(x) \sec(x) \sin(x)$$

- Solve to find the integrating factor

$$\mu(x) = \frac{1}{\sec(x)}$$

- Integrate both sides with respect to x

$$\int \left(\frac{d}{dx}(\mu(x) y)\right) dx = \int -2\mu(x) \sin(x) dx + c_1$$

- Evaluate the integral on the lhs

$$\mu(x) y = \int -2\mu(x) \sin(x) dx + c_1$$

- Solve for y

$$y = \frac{\int -2\mu(x) \sin(x) dx + c_1}{\mu(x)}$$

- Substitute $\mu(x) = \frac{1}{\sec(x)}$

$$y = \sec(x) \left(\int -\frac{2\sin(x)}{\sec(x)} dx + c_1\right)$$

- Evaluate the integrals on the rhs

$$y = \sec(x) \left(-\sin(x)^2 + c_1\right)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(diff(y(x),x)=sin(x)*(y(x)*sec(x)-2),y(x), singsol=all)
```

$$y(x) = \cos(x) - \frac{\sec(x)}{2} + \sec(x) c_1$$

✓ Solution by Mathematica

Time used: 0.045 (sec). Leaf size: 20

```
DSolve[y'[x]==Sin[x]*(y[x]*Sec[x]-2),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2} \sec(x)(\cos(2x) + 2c_1)$$

3.12 problem Problem 12

Internal problem ID [2650]
Internal file name [`OUTPUT/2142_Sunday_June_05_2022_02_50_11_AM_19325055/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.6, First-Order Linear Differential Equations. page 59
Problem number: Problem 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_linear]`

$$\boxed{-y\sin(x) - y'\cos(x) = -1}$$

3.12.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \tan(x)$$
$$q(x) = \sec(x)$$

Hence the ode is

$$y' + y\tan(x) = \sec(x)$$

The integrating factor μ is

$$\mu = e^{\int \tan(x)dx}$$
$$= \frac{1}{\cos(x)}$$

Which simplifies to

$$\mu = \sec(x)$$

The ode becomes

$$\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)(\sec(x)) \\
\frac{d}{dx}(\sec(x)\, y) &= (\sec(x))(\sec(x)) \\
d(\sec(x)\, y) &= \sec(x)^2\, dx
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\sec(x)\, y &= \int \sec(x)^2\, dx \\
\sec(x)\, y &= \tan(x) + c_1
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \sec(x)$ results in

$$y = \tan(x)\cos(x) + c_1\cos(x)$$

which simplifies to

$$y = c_1\cos(x) + \sin(x)$$

Summary

The solution(s) found are the following

$$y = c_1\cos(x) + \sin(x) \tag{1}$$

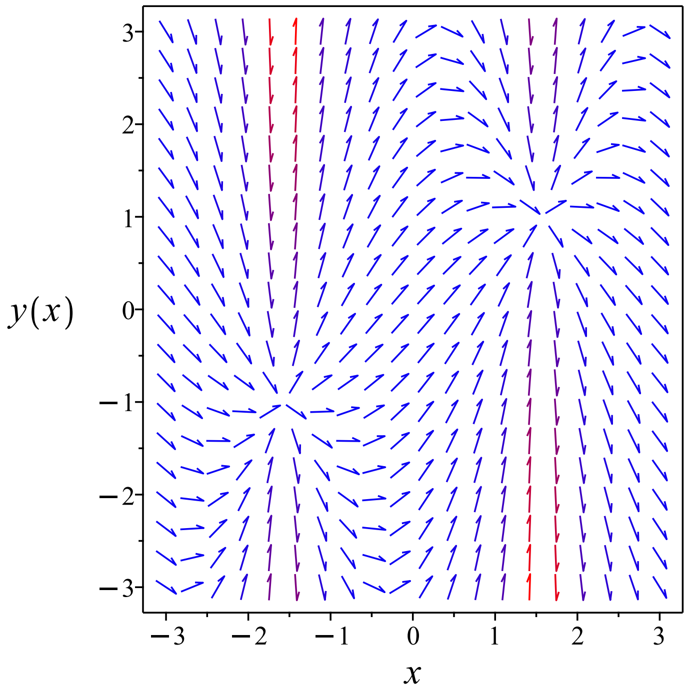


Figure 93: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + \sin(x)$$

Verified OK.

3.12.2 Maple step by step solution

Let's solve

$-y \sin(x) - y' \cos(x) = -1$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{\sin(x) y}{\cos(x)} + \frac{1}{\cos(x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{\sin(x) y}{\cos(x)} = \frac{1}{\cos(x)}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$$\mu(x)\left(y'+\frac{\sin(x)y}{\cos(x)}\right)=\frac{\mu(x)}{\cos(x)}$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

$$\mu(x)\left(y'+\frac{\sin(x)y}{\cos(x)}\right)=\mu'(x)\,y+\mu(x)\,y'$$

- Isolate $\mu'(x)$

$$\mu'(x)=\frac{\mu(x)\sin(x)}{\cos(x)}$$

- Solve to find the integrating factor

$$\mu(x)=\frac{1}{\cos(x)}$$

- Integrate both sides with respect to x

$$\int\left(\frac{d}{dx}(\mu(x)\,y)\right)dx=\int\frac{\mu(x)}{\cos(x)}dx+c_1$$

- Evaluate the integral on the lhs

$$\mu(x)\,y=\int\frac{\mu(x)}{\cos(x)}dx+c_1$$

- Solve for y

$$y=\frac{\int\frac{\mu(x)}{\cos(x)}dx+c_1}{\mu(x)}$$

- Substitute $\mu(x)=\frac{1}{\cos(x)}$

$$y=\cos\left(x\right)\left(\int\frac{1}{\cos(x)^2}dx+c_1\right)$$

- Evaluate the integrals on the rhs

$$y=\cos\left(x\right)\left(\tan\left(x\right)+c_1\right)$$

- Simplify

$$y=c_1\cos\left(x\right)+\sin\left(x\right)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 11

```
dsolve((1-y(x)*sin(x))-cos(x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \cos(x) c_1 + \sin(x)$$

✓ Solution by Mathematica

Time used: 0.04 (sec). Leaf size: 13

```
DSolve[(1-y[x]*Sin[x])-Cos[x]*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sin(x) + c_1 \cos(x)$$

3.13 problem Problem 13

Internal problem ID [2651]
Internal file name [`OUTPUT/2143_Sunday_June_05_2022_02_50_13_AM_9669064/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.6, First-Order Linear Differential Equations. page 59
Problem number: Problem 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' - \frac{y}{x} = 2\ln(x)\, x^2}$$

3.13.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{1}{x}$$
$$q(x) = 2\ln(x)\, x^2$$

Hence the ode is

$$y' - \frac{y}{x} = 2\ln(x)\, x^2$$

The integrating factor μ is

$$\mu = e^{\int -\frac{1}{x}dx}$$
$$= \frac{1}{x}$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(2\ln(x)\,x^2\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x}\right) &= \left(\frac{1}{x}\right)\left(2\ln(x)\,x^2\right) \\
\mathrm{d}\left(\frac{y}{x}\right) &= (2\ln(x)\,x)\ \mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{y}{x} &= \int 2\ln(x)\,x\,\mathrm{d}x \\
\frac{y}{x} &= \ln(x)\,x^2 - \frac{x^2}{2} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$
y = x\left(\ln(x)\,x^2 - \frac{x^2}{2}\right) + c_1 x
$$

which simplifies to

$$
y = x^3\ln(x) - \frac{x^3}{2} + c_1 x
$$

Summary

The solution(s) found are the following

$$
y = x^3\ln(x) - \frac{x^3}{2} + c_1 x \tag{1}
$$

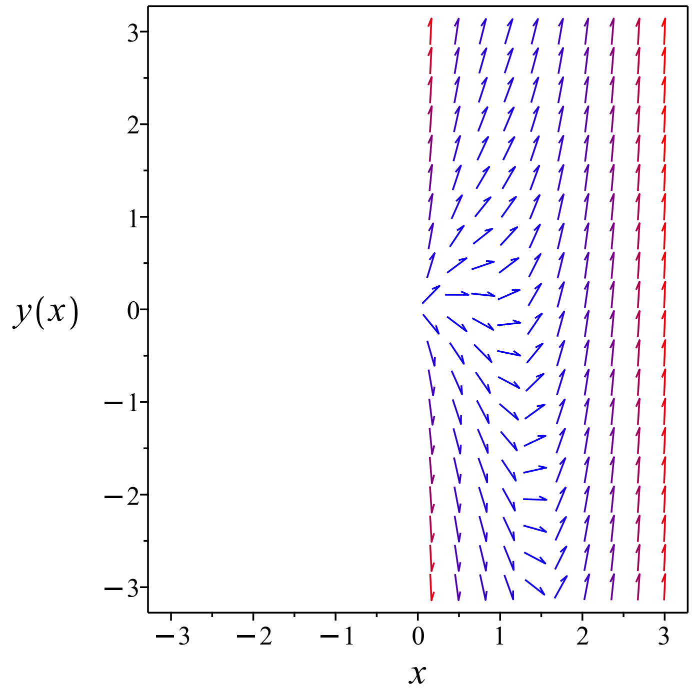


Figure 94: Slope field plot

Verification of solutions

$$y = x^3 \ln (x) - \frac{x^3}{2} + c_1 x$$

Verified OK.

3.13.2 Maple step by step solution

Let's solve

$y' - \frac{y}{x} = 2 \ln (x) x^2$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{y}{x} + 2 \ln (x) x^2$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{y}{x} = 2 \ln (x) x^2$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$\mu(x)\left(y'-\frac{y}{x}\right)=2\mu(x)\ln(x)\,x^2$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y'-\frac{y}{x}\right)=\mu'(x)\,y+\mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x)=-\frac{\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x)=\frac{1}{x}$

- Integrate both sides with respect to x

  $\int\left(\frac{d}{dx}(\mu(x)\,y)\right)dx=\int 2\mu(x)\ln(x)\,x^2dx+c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y=\int 2\mu(x)\ln(x)\,x^2dx+c_1$

- Solve for y

  $y=\frac{\int 2\mu(x)\ln(x)x^2dx+c_1}{\mu(x)}$

- Substitute $\mu(x)=\frac{1}{x}$

  $y=x\left(\int 2\ln(x)\,xdx+c_1\right)$

- Evaluate the integrals on the rhs

  $y=x\left(\ln(x)\,x^2-\frac{x^2}{2}+c_1\right)$

- Simplify

  $y=x^3\ln(x)-\frac{x^3}{2}+c_1x$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(diff(y(x),x)-1/x*y(x)=2*x^2*ln(x),y(x), singsol=all)
```

$$y(x) = x^3 \ln(x) - \frac{x^3}{2} + c_1 x$$

✓ Solution by Mathematica

Time used: 0.03 (sec). Leaf size: 23

```
DSolve[y'[x]-1/x*y[x]==2*x^2*Log[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{x^3}{2} + x^3 \log(x) + c_1 x$$

3.14 problem Problem 14

Internal problem ID [2652]
Internal file name [`OUTPUT/2144_Sunday_June_05_2022_02_50_15_AM_31038222/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.6, First-Order Linear Differential Equations. page 59
Problem number: Problem 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' + \alpha y = e^{\beta x}}$$

3.14.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= \alpha \\ q(x) &= e^{\beta x} \end{aligned}$$

Hence the ode is

$$y' + \alpha y = e^{\beta x}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int \alpha dx} \\ &= e^{\alpha x} \end{aligned}$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(e^{\beta x}\right) \\
\frac{d}{dx}(e^{\alpha x} y) &= (e^{\alpha x})\left(e^{\beta x}\right) \\
d(e^{\alpha x} y) &= e^{x(\alpha+\beta)}\, dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
e^{\alpha x} y &= \int e^{x(\alpha+\beta)}\, dx \\
e^{\alpha x} y &= \frac{e^{x(\alpha+\beta)}}{\alpha+\beta} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = e^{\alpha x}$ results in

$$
y = \frac{e^{-\alpha x} e^{x(\alpha+\beta)}}{\alpha+\beta} + c_1 e^{-\alpha x}
$$

which simplifies to

$$
y = \frac{c_1(\alpha+\beta)\, e^{-\alpha x} + e^{\beta x}}{\alpha+\beta}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1(\alpha+\beta)\, e^{-\alpha x} + e^{\beta x}}{\alpha+\beta} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1(\alpha+\beta)\, e^{-\alpha x} + e^{\beta x}}{\alpha+\beta}
$$

Verified OK.

3.14.2 Maple step by step solution

Let's solve

$y' + \alpha y = e^{\beta x}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\alpha y + \mathrm{e}^{\beta x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \alpha y = \mathrm{e}^{\beta x}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \alpha y\right) = \mu(x)\,\mathrm{e}^{\beta x}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y' + \alpha y\right) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \mu(x)\,\alpha$

- Solve to find the integrating factor

  $\mu(x) = \mathrm{e}^{\alpha x}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int \mu(x)\,\mathrm{e}^{\beta x} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int \mu(x)\,\mathrm{e}^{\beta x} dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x) \mathrm{e}^{\beta x} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \mathrm{e}^{\alpha x}$

  $y = \frac{\int \mathrm{e}^{\beta x} \mathrm{e}^{\alpha x} dx + c_1}{\mathrm{e}^{\alpha x}}$

- Evaluate the integrals on the rhs

  $y = \frac{\frac{\mathrm{e}^{\alpha x + \beta x}}{\alpha + \beta} + c_1}{\mathrm{e}^{\alpha x}}$

- Simplify

  $y = \frac{\mathrm{e}^{-\alpha x}\left(\mathrm{e}^{x(\alpha+\beta)} + c_1(\alpha+\beta)\right)}{\alpha+\beta}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(diff(y(x),x)+alpha*y(x)=exp(beta*x),y(x), singsol=all)
```

$$y(x) = \frac{e^{-\alpha x}\left(e^{x(\alpha+\beta)} + c_1(\alpha+\beta)\right)}{\alpha+\beta}$$

✓ Solution by Mathematica

Time used: 0.055 (sec). Leaf size: 31

```
DSolve[y'[x]+\[Alpha]*y[x]==Exp[\[Beta]*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{\alpha(-x)}\left(e^{x(\alpha+\beta)} + c_1(\alpha+\beta)\right)}{\alpha+\beta}$$

3.15 problem Problem 15

Internal problem ID [2653]
Internal file name [`OUTPUT/2145_Sunday_June_05_2022_02_50_17_AM_4708840/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.6, First-Order Linear Differential Equations. page 59
Problem number: Problem 15.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' + \frac{my}{x} = \ln(x)}$$

3.15.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{m}{x}$$
$$q(x) = \ln(x)$$

Hence the ode is

$$y' + \frac{my}{x} = \ln(x)$$

The integrating factor μ is

$$\mu = e^{\int \frac{m}{x} dx}$$
$$= e^{m \ln(x)}$$

Which simplifies to

$$\mu = x^m$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)(\ln(x)) \\ \frac{d}{dx}(x^m y) &= (x^m)(\ln(x)) \\ d(x^m y) &= (\ln(x)\,x^m)\,dx\end{aligned}$$

Integrating gives

$$\begin{aligned}x^m y &= \int \ln(x)\,x^m\,dx \\ x^m y &= \frac{x\ln(x)\,e^{m\ln(x)}}{m+1} - \frac{x\,e^{m\ln(x)}}{m^2+2m+1} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x^m$ results in

$$y = x^{-m}\left(\frac{x\ln(x)\,e^{m\ln(x)}}{m+1} - \frac{x\,e^{m\ln(x)}}{m^2+2m+1}\right) + c_1 x^{-m}$$

which simplifies to

$$y = \frac{c_1(m+1)^2\,x^{-m} + (-1+(m+1)\ln(x))\,x}{(m+1)^2}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1(m+1)^2\,x^{-m} + (-1+(m+1)\ln(x))\,x}{(m+1)^2} \tag{1}$$

Verification of solutions

$$y = \frac{c_1(m+1)^2\,x^{-m} + (-1+(m+1)\ln(x))\,x}{(m+1)^2}$$

Verified OK.

3.15.2 Maple step by step solution

Let's solve

$y' + \frac{my}{x} = \ln(x)$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{my}{x} + \ln(x)$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{my}{x} = \ln(x)$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{my}{x}\right) = \mu(x)\ln(x)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' + \frac{my}{x}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{\mu(x)m}{x}$

- Solve to find the integrating factor

  $\mu(x) = x^m$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x)\ln(x)\, dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \mu(x)\ln(x)\, dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x)\ln(x)dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = x^m$

  $y = \frac{\int \ln(x)x^m dx + c_1}{x^m}$

- Evaluate the integrals on the rhs

  $y = \frac{\frac{x\ln(x)e^{m\ln(x)}}{m+1} - \frac{x\, e^{m\ln(x)}}{m^2+2m+1} + c_1}{x^m}$

- Simplify

$$y = \frac{c_1 (m+1)^2 x^{-m} + (-1+(m+1)\ln(x))x}{(m+1)^2}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 33

```
dsolve(diff(y(x),x)+m/x*y(x)=ln(x),y(x), singsol=all)
```

$$y(x) = \frac{c_1 (m+1)^2 x^{-m} + x(-1+(m+1)\ln(x))}{(m+1)^2}$$

✓ Solution by Mathematica

Time used: 0.051 (sec). Leaf size: 29

```
DSolve[y'[x]+m/x*y[x]==Log[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x((m+1)\log(x)-1)}{(m+1)^2} + c_1 x^{-m}$$

3.16 problem Problem 16

Internal problem ID [2654]
Internal file name [`OUTPUT/2146_Sunday_June_05_2022_02_50_19_AM_68953379/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.6, First-Order Linear Differential Equations. page 59
Problem number: Problem 16.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' + \frac{2y}{x} = 4x}$$

With initial conditions

$$[y(1) = 2]$$

3.16.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{2}{x}$$
$$q(x) = 4x$$

Hence the ode is

$$y' + \frac{2y}{x} = 4x$$

The domain of $p(x) = \frac{2}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = 4x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

3.16.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{2}{x}dx} \\ &= x^2\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)(4x) \\ \frac{d}{dx}(x^2 y) &= (x^2)(4x) \\ d(x^2 y) &= (4x^3)\,dx\end{aligned}$$

Integrating gives

$$\begin{aligned}x^2 y &= \int 4x^3\,dx \\ x^2 y &= x^4 + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x^2$ results in

$$y = x^2 + \frac{c_1}{x^2}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = c_1 + 1$$

$$c_1 = 1$$

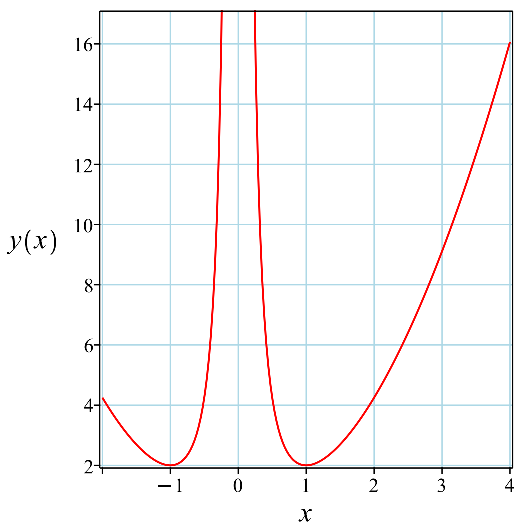
Substituting c_1 found above in the general solution gives

$$y = \frac{x^4 + 1}{x^2}$$

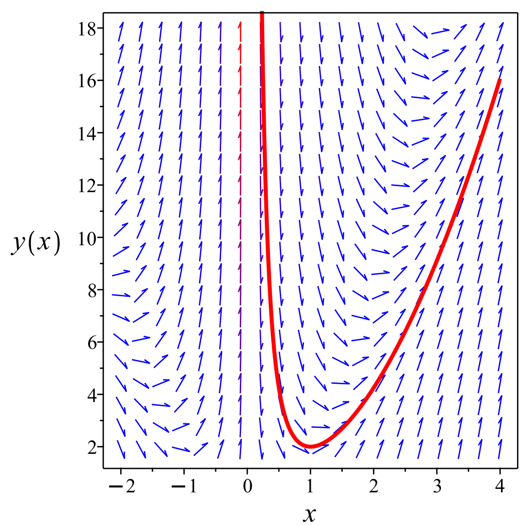
Summary

The solution(s) found are the following

$$y = \frac{x^4 + 1}{x^2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{x^4 + 1}{x^2}$$

Verified OK.

3.16.3 Maple step by step solution

Let's solve

$\left[y' + \frac{2y}{x} = 4x, y(1) = 2\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{2y}{x} + 4x$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{2y}{x} = 4x$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{2y}{x}\right) = 4\mu(x)\,x$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y' + \frac{2y}{x}\right) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{2\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = x^2$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int 4\mu(x)\,x dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int 4\mu(x)\,x dx + c_1$

- Solve for y

  $y = \frac{\int 4\mu(x)x dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = x^2$

  $y = \frac{\int 4x^3 dx + c_1}{x^2}$

- Evaluate the integrals on the rhs

  $y = \frac{x^4 + c_1}{x^2}$

- Use initial condition $y(1) = 2$

  $2 = c_1 + 1$

- Solve for c_1

  $c_1 = 1$
- Substitute $c_1 = 1$ into general solution and simplify

  $y = \frac{x^4+1}{x^2}$
- Solution to the IVP

  $y = \frac{x^4+1}{x^2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 13

```
dsolve([diff(y(x),x)+2/x*y(x)=4*x,y(1) = 2],y(x), singsol=all)
```

$$y(x) = \frac{x^4+1}{x^2}$$

✓ Solution by Mathematica

Time used: 0.028 (sec). Leaf size: 12

```
DSolve[{y'[x]+2/x*y[x]==4*x,{y[1]==2}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^2 + \frac{1}{x^2}$$

3.17 problem Problem 17

Internal problem ID [2655]
Internal file name [`OUTPUT/2147_Sunday_June_05_2022_02_50_22_AM_72927966/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.6, First-Order Linear Differential Equations. page 59
Problem number: Problem 17.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' \sin(x) - \cos(x) y = \sin(2x)}$$

With initial conditions

$$\left[y\left(\frac{\pi}{2}\right) = 2\right]$$

3.17.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\cot(x)$$
$$q(x) = 2\cos(x)$$

Hence the ode is

$$y' - y \cot(x) = 2\cos(x)$$

The domain of $p(x) = -\cot(x)$ is

$$\{x < \pi\_Z144 \lor \pi\_Z144 < x\}$$

And the point $x_0 = \frac{\pi}{2}$ is inside this domain. The domain of $q(x) = 2\cos(x)$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = \frac{\pi}{2}$ is also inside this domain. Hence solution exists and is unique.

3.17.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\cot(x)dx} \\ &= \frac{1}{\sin(x)}\end{aligned}$$

Which simplifies to

$$\mu = \csc(x)$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)(2\cos(x)) \\ \frac{d}{dx}(\csc(x)\, y) &= (\csc(x))(2\cos(x)) \\ d(\csc(x)\, y) &= (2\cot(x))\, dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\csc(x)\, y &= \int 2\cot(x)\, dx \\ \csc(x)\, y &= 2\ln(\sin(x)) + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \csc(x)$ results in

$$y = 2\sin(x)\ln(\sin(x)) + c_1 \sin(x)$$

which simplifies to

$$y = \sin(x)\left(2\ln(\sin(x)) + c_1\right)$$

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{2}$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = c_1$$

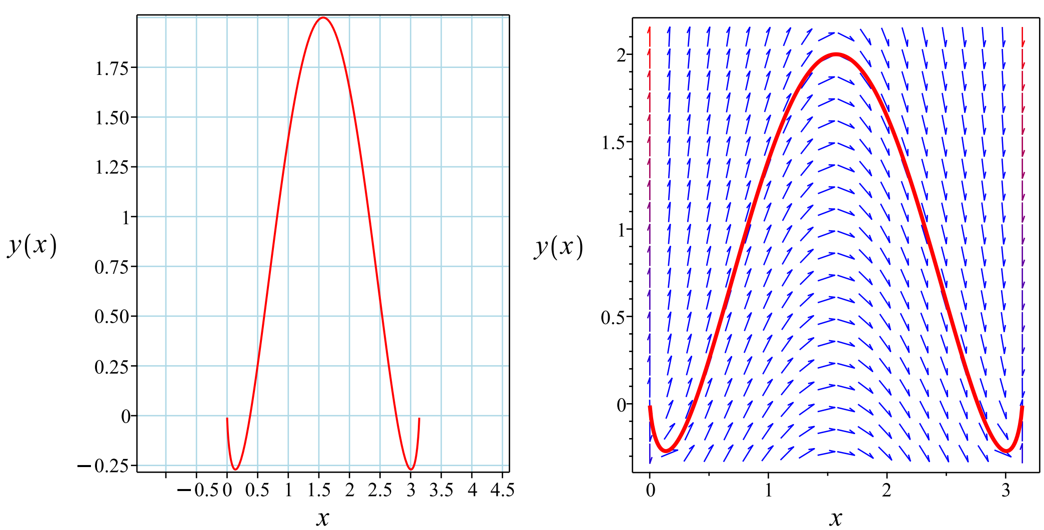
$$c_1 = 2$$

Substituting c_1 found above in the general solution gives

$$y = 2\sin(x)\ln(\sin(x)) + 2\sin(x)$$

Summary
The solution(s) found are the following

$$y = 2\sin(x)\ln(\sin(x)) + 2\sin(x) \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = 2\sin(x)\ln(\sin(x)) + 2\sin(x)$$

Verified OK.

3.17.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{\cos(x)\,y + \sin(2x)}{\sin(x)} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 94: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int b f(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \sin(x)
\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R,S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\sin(x)} dy
\end{aligned}$$

Which results in

$$S = \frac{y}{\sin(x)}$$

Now that R,S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{\cos(x)\,y + \sin(2x)}{\sin(x)}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\csc(x)\cot(x)\,y \\
S_y &= \csc(x)
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = 2\cot(x) \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = 2\cot(R)
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = 2\ln(\sin(R)) + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\csc(x)\,y = 2\ln(\sin(x)) + c_1
$$

Which simplifies to

$$
\csc(x)\,y = 2\ln(\sin(x)) + c_1
$$

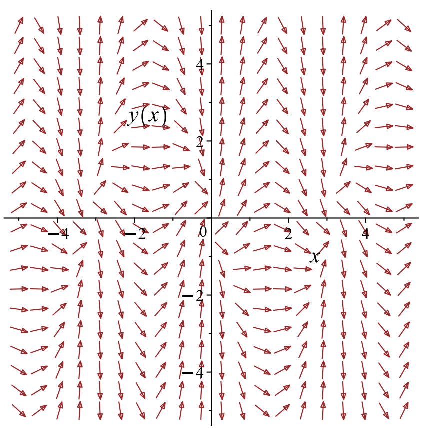
Which gives

$$
y = \frac{2\ln(\sin(x)) + c_1}{\csc(x)}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>

$\frac{dy}{dx} = \frac{\cos(x)y+\sin(2x)}{\sin(x)}$



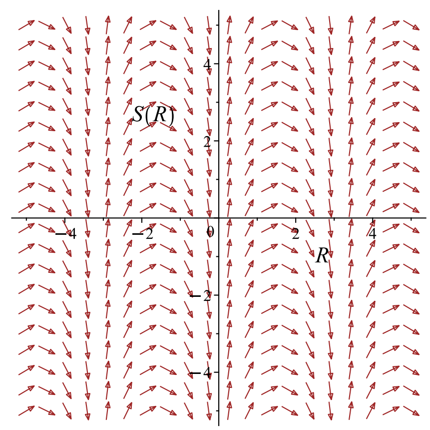
</td><td>

$R = x$

$S = \csc(x)\, y$

</td><td>

$\frac{dS}{dR} = 2\cot(R)$



</td></tr>
</table>

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{2}$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = c_1$$

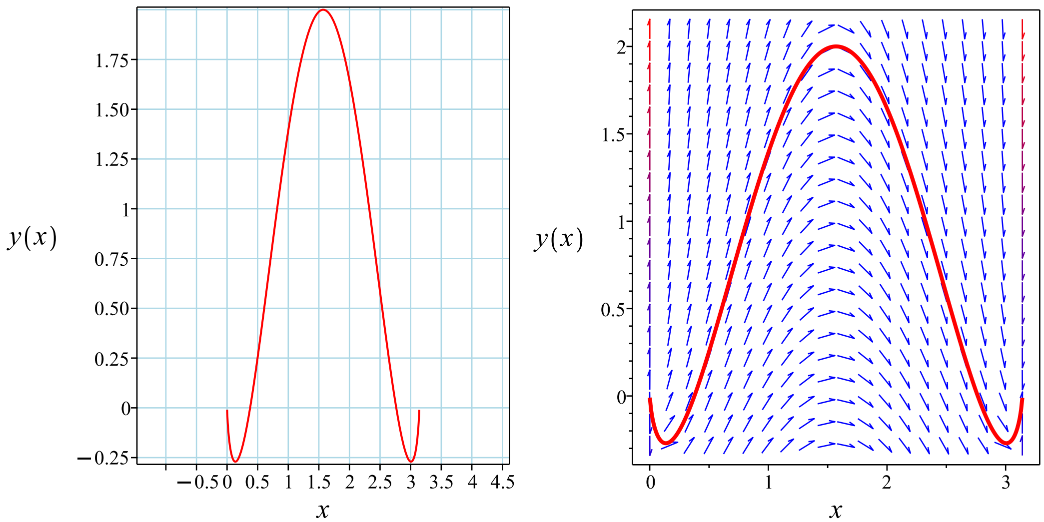
$$c_1 = 2$$

Substituting c_1 found above in the general solution gives

$$y = 2\sin(x)\ln(\sin(x)) + 2\sin(x)$$

Summary
The solution(s) found are the following

$$y = 2\sin(x)\ln(\sin(x)) + 2\sin(x) \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = 2\sin(x)\ln(\sin(x)) + 2\sin(x)$$

Verified OK.

3.17.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(\sin(x))\,\mathrm{d}y &= (\cos(x)\,y + \sin(2x))\,\mathrm{d}x\\ (-\cos(x)\,y - \sin(2x))\,\mathrm{d}x + (\sin(x))\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= -\cos(x)\,y - \sin(2x)\\ N(x, y) &= \sin(x) \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-\cos(x)\,y - \sin(2x))\\ &= -\cos(x)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(\sin(x))\\ &= \cos(x)\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \csc(x)\,((-\cos(x)) - (\cos(x)))\\ &= -2\cot(x)\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int -2\cot(x) \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-2\ln(\sin(x))} \\
&= \csc(x)^2
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \csc(x)^2 \left(-\cos(x)\, y - \sin(2x)\right) \\
&= -\cot(x) \left(\csc(x)\, y + 2\right)
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \csc(x)^2 \left(\sin(x)\right) \\
&= \csc(x)
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N} \frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(-\cot(x) \left(\csc(x)\, y + 2\right)\right) + \left(\csc(x)\right) \frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int \overline{M} \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -\cot(x) \left(\csc(x)\, y + 2\right) \mathrm{d}x
\end{aligned}
$$

$$
\phi = \csc(x)\, y + 2\ln\left(\csc(x)\right) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \csc(x) + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \csc(x)$. Therefore equation (4) becomes

$$\csc(x) = \csc(x) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \csc(x)\, y + 2\ln(\csc(x)) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \csc(x)\, y + 2\ln(\csc(x))$$

The solution becomes

$$y = -\frac{2\ln(\csc(x)) - c_1}{\csc(x)}$$

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{2}$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = c_1$$

$$c_1 = 2$$

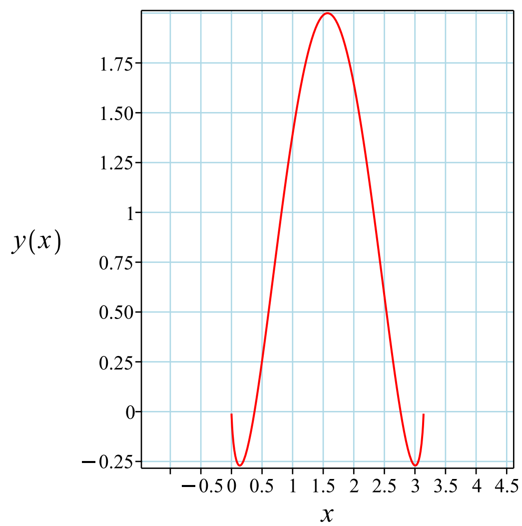
Substituting c_1 found above in the general solution gives

$$y = -2\ln(\csc(x))\sin(x) + 2\sin(x)$$

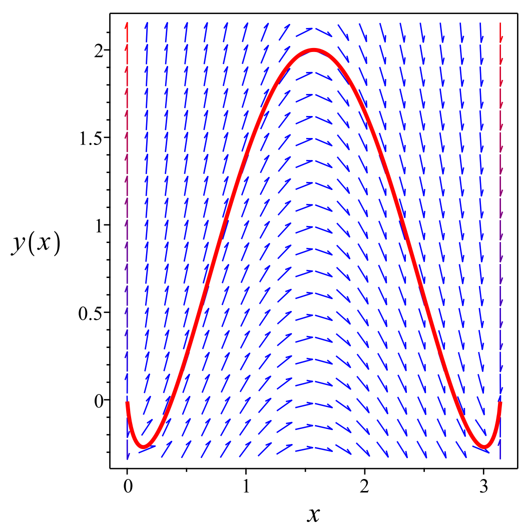
Summary

The solution(s) found are the following

$$y = -2\ln(\csc(x))\sin(x) + 2\sin(x) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -2\ln(\csc(x))\sin(x) + 2\sin(x)$$

Verified OK.

3.17.5 Maple step by step solution

Let's solve

$\left[y'\sin(x) - \cos(x)\,y = \sin(2x), y\left(\frac{\pi}{2}\right) = 2\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{\cos(x)y}{\sin(x)} + \frac{\sin(2x)}{\sin(x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{\cos(x)y}{\sin(x)} = \frac{\sin(2x)}{\sin(x)}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$$\mu(x)\left(y'-\frac{\cos(x)y}{\sin(x)}\right)=\frac{\mu(x)\sin(2x)}{\sin(x)}$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

$$\mu(x)\left(y'-\frac{\cos(x)y}{\sin(x)}\right)=\mu'(x)\,y+\mu(x)\,y'$$

- Isolate $\mu'(x)$

$$\mu'(x)=-\frac{\mu(x)\cos(x)}{\sin(x)}$$

- Solve to find the integrating factor

$$\mu(x)=\frac{1}{\sin(x)}$$

- Integrate both sides with respect to x

$$\int\left(\frac{d}{dx}(\mu(x)\,y)\right)dx=\int\frac{\mu(x)\sin(2x)}{\sin(x)}dx+c_1$$

- Evaluate the integral on the lhs

$$\mu(x)\,y=\int\frac{\mu(x)\sin(2x)}{\sin(x)}dx+c_1$$

- Solve for y

$$y=\frac{\int\frac{\mu(x)\sin(2x)}{\sin(x)}dx+c_1}{\mu(x)}$$

- Substitute $\mu(x)=\frac{1}{\sin(x)}$

$$y=\sin(x)\left(\int\frac{\sin(2x)}{\sin(x)^2}dx+c_1\right)$$

- Evaluate the integrals on the rhs

$$y=\sin(x)\,(2\ln(\sin(x))+c_1)$$

- Use initial condition $y\left(\frac{\pi}{2}\right)=2$

$$2=c_1$$

- Solve for c_1

$$c_1=2$$

- Substitute $c_1=2$ into general solution and simplify

$$y=(2\ln(\sin(x))+2)\sin(x)$$

- Solution to the IVP

$$y=(2\ln(\sin(x))+2)\sin(x)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve([sin(x)*diff(y(x),x)-y(x)*cos(x)=sin(2*x),y(1/2*Pi) = 2],y(x), singsol=all)
```

$$y(x) = (2\ln(\sin(x)) + 2)\sin(x)$$

✓ Solution by Mathematica

Time used: 0.048 (sec). Leaf size: 14

```
DSolve[{Sin[x]*y'[x]-y[x]*Cos[x]==Sin[2*x],{y[Pi/2]==2}},y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to 2\sin(x)(\log(\sin(x)) + 1)$$

3.18 problem Problem 18

Internal problem ID [2656]
Internal file name [`OUTPUT/2148_Sunday_June_05_2022_02_50_24_AM_55698290/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.6, First-Order Linear Differential Equations. page 59
Problem number: Problem 18.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "homogeneousTypeMapleC", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{x' + \frac{2x}{4-t} = 5}$$

With initial conditions

$$[x(0) = 4]$$

3.18.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x' + p(t)x = q(t)$$

Where here

$$\begin{aligned} p(t) &= -\frac{2}{-4+t} \\ q(t) &= 5 \end{aligned}$$

Hence the ode is

$$x' - \frac{2x}{-4+t} = 5$$

The domain of $p(t) = -\frac{2}{-4+t}$ is

$$\{t < 4 \lor 4 < t\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 5$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

3.18.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int -\frac{2}{-4+t}dt} \\ &= \frac{1}{(-4+t)^2} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dt}(\mu x) &= (\mu)(5) \\ \frac{d}{dt}\left(\frac{x}{(-4+t)^2}\right) &= \left(\frac{1}{(-4+t)^2}\right)(5) \\ d\left(\frac{x}{(-4+t)^2}\right) &= \left(\frac{5}{(-4+t)^2}\right) dt \end{aligned}$$

Integrating gives

$$
\begin{aligned}
\frac{x}{(-4+t)^2} &= \int \frac{5}{(-4+t)^2}\,\mathrm{d}t \\
\frac{x}{(-4+t)^2} &= -\frac{5}{-4+t} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{(-4+t)^2}$ results in

$$
x = 20 - 5t + c_1(-4+t)^2
$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $x = 4$ in the above solution gives an equation to solve for the constant of integration.

$$
4 = 20 + 16c_1
$$

$$
c_1 = -1
$$

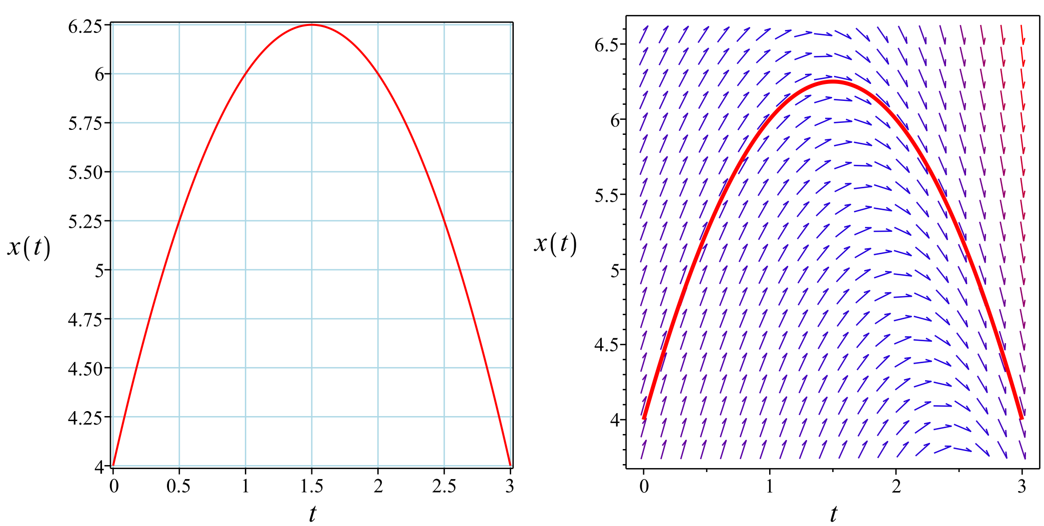
Substituting c_1 found above in the general solution gives

$$
x = -t^2 + 3t + 4
$$

Summary

The solution(s) found are the following

$$
x = -t^2 + 3t + 4 \tag{1}
$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$x = -t^2 + 3t + 4$$

Verified OK.

3.18.3 Solving as homogeneousTypeMapleC ode

Let $Y = x + y_0$ and $X = t + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = \frac{2Y(X) + 2y_0 - 20 + 5X + 5x_0}{-4 + X + x_0}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = 4$$
$$y_0 = 0$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = \frac{2Y(X) + 5X}{X}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X,Y) \\ &= \frac{2Y + 5X}{X} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X,Y)$$

In this case, it can be seen that both $M = 2Y + 5X$ and $N = X$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\frac{\mathrm{d}u}{\mathrm{d}X}X + u = 2u + 5$$
$$\frac{\mathrm{d}u}{\mathrm{d}X} = \frac{u(X) + 5}{X}$$

Or

$$\frac{d}{dX}u(X) - \frac{u(X)+5}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right)X - u(X) - 5 = 0$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$\begin{aligned}
u' &= F(X,u) \\
&= f(X)g(u) \\
&= \frac{u+5}{X}
\end{aligned}$$

Where $f(X) = \frac{1}{X}$ and $g(u) = u+5$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u+5}\, du &= \frac{1}{X}\, dX \\
\int \frac{1}{u+5}\, du &= \int \frac{1}{X}\, dX \\
\ln(u+5) &= \ln(X) + c_2
\end{aligned}$$

Raising both side to exponential gives

$$u + 5 = \mathrm{e}^{\ln(X)+c_2}$$

Which simplifies to

$$u + 5 = c_3 X$$

Which simplifies to

$$u(X) = c_3 X\,\mathrm{e}^{c_2} - 5$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$Y(X) = X(c_3 X\,\mathrm{e}^{c_2} - 5)$$

Using the solution for $Y(X)$

$$Y(X) = X(c_3 X\,\mathrm{e}^{c_2} - 5)$$

And replacing back terms in the above solution using

$$Y = x + y_0$$
$$X = t + x_0$$

Or

$$Y = x$$
$$X = t + 4$$

Then the solution in x becomes

$$x = (-4 + t)\left(c_3(-4 + t)\,\mathrm{e}^{c_2} - 5\right)$$

Initial conditions are used to solve for c_2. Substituting $t = 0$ and $x = 4$ in the above solution gives an equation to solve for the constant of integration.

$$4 = 20 + 16c_3\mathrm{e}^{c_2}$$

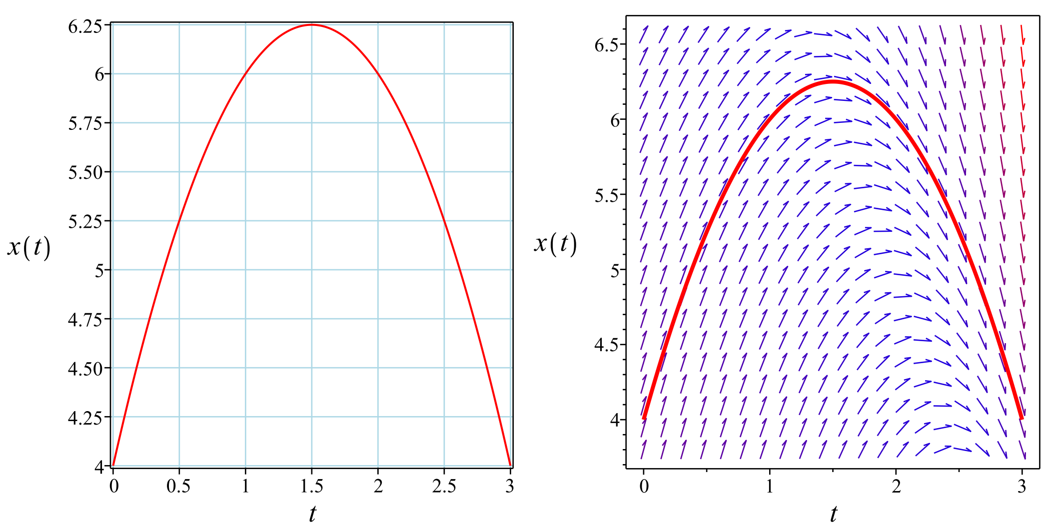
$$c_2 = \ln\left(-\frac{1}{c_3}\right)$$

Substituting c_2 found above in the general solution gives

$$x = -t^2 + 3t + 4$$

<u>Summary</u>
The solution(s) found are the following

$$x = -t^2 + 3t + 4 \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$x = -t^2 + 3t + 4$$

Verified OK.

3.18.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$x' = \frac{2x - 20 + 5t}{-4 + t}$$
$$x' = \omega(t, x)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_x - \xi_t) - \omega^2 \xi_x - \omega_t \xi - \omega_x \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 97: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(t, x) &= 0 \\ \eta(t, x) &= (-4 + t)^2 \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, x) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dx}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial x}\right) S(t, x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{(-4+t)^2} dy
\end{aligned}$$

Which results in

$$S = \frac{x}{(-4+t)^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,x) S_x}{R_t + \omega(t,x) R_x} \tag{2}$$

Where in the above R_t, R_x, S_t, S_x are all partial derivatives and $\omega(t,x)$ is the right hand side of the original ode given by

$$\omega(t,x) = \frac{2x - 20 + 5t}{-4+t}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_t &= 1 \\
R_x &= 0 \\
S_t &= -\frac{2x}{(-4+t)^3} \\
S_x &= \frac{1}{(-4+t)^2}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{5}{(-4+t)^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{5}{(-4+R)^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{5}{-4+R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, x coordinates. This results in

$$\frac{x}{(-4+t)^2} = -\frac{5}{-4+t} + c_1$$

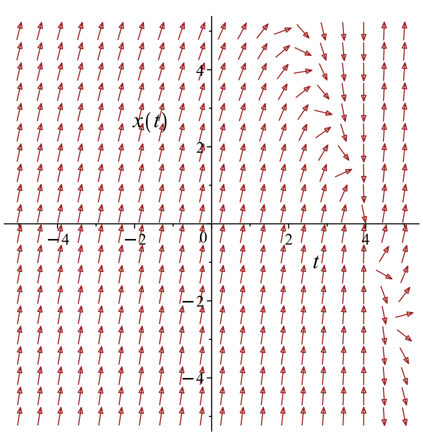
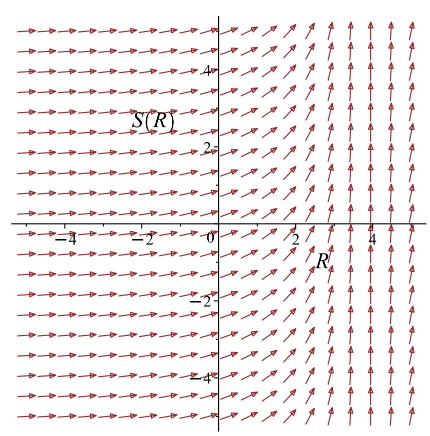
Which simplifies to

$$\frac{x}{(-4+t)^2} = -\frac{5}{-4+t} + c_1$$

Which gives

$$x = (c_1 t - 4c_1 - 5)(-4+t)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in t, x coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dx}{dt} = \frac{2x-20+5t}{-4+t}$

</td><td>$R = t$
$S = \frac{x}{(-4+t)^2}$</td><td>$\frac{dS}{dR} = \frac{5}{(-4+R)^2}$

</td></tr>
</table>

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $x = 4$ in the above solution gives an equation to solve for the constant of integration.

$$4 = 20 + 16c_1$$

$$c_1 = -1$$

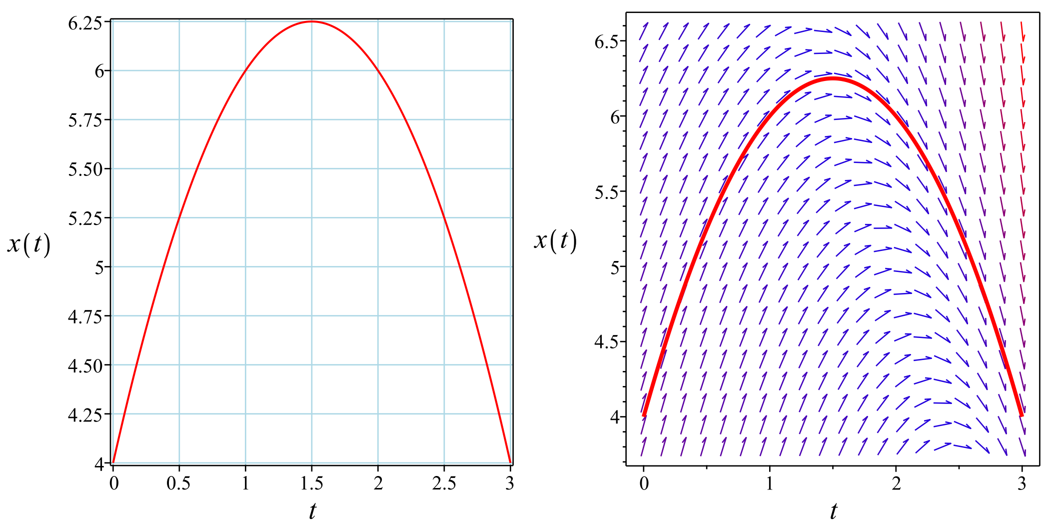
Substituting c_1 found above in the general solution gives

$$x = -(t + 1)(-4 + t)$$

Summary

The solution(s) found are the following

$$x = -(t + 1)(-4 + t) \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$x = -(t + 1)(-4 + t)$$

Verified OK.

3.18.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,x)\,\mathrm{d}t+N(t,x)\,\mathrm{d}x=0 \tag{1A}$$

Therefore

$$\mathrm{d}x=\left(-\frac{2x}{4-t}+5\right)\mathrm{d}t$$

$$\left(\frac{2x}{4-t}-5\right)\mathrm{d}t+\mathrm{d}x=0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(t,x) &= \frac{2x}{4-t} - 5 \\
N(t,x) &= 1
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{2x}{4-t} - 5\right) \\
&= -\frac{2}{-4+t}
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial x} \neq \frac{\partial N}{\partial t}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial t}\right) \\
&= 1\left(\left(\frac{2}{4-t}\right) - (0)\right) \\
&= -\frac{2}{-4+t}
\end{aligned}
$$

Since A does not depend on x, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}t} \\
&= e^{\int -\frac{2}{-4+t} \, \mathrm{d}t}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-2\ln(-4+t)} \\
&= \frac{1}{(-4+t)^2}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{(-4+t)^2}\left(\frac{2x}{4-t} - 5\right) \\
&= \frac{-2x+20-5t}{(-4+t)^3}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{(-4+t)^2}(1) \\
&= \frac{1}{(-4+t)^2}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}x}{\mathrm{d}t} &= 0 \\
\left(\frac{-2x+20-5t}{(-4+t)^3}\right) + \left(\frac{1}{(-4+t)^2}\right)\frac{\mathrm{d}x}{\mathrm{d}t} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(t,x)$

$$
\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial x} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. t gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int \overline{M}\,\mathrm{d}t \\
\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int \frac{-2x+20-5t}{(-4+t)^3}\,\mathrm{d}t
\end{aligned}
$$

$$
\phi = \frac{-20+5t+x}{(-4+t)^2} + f(x) \tag{3}
$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t x gives

$$
\frac{\partial \phi}{\partial x} = \frac{1}{(-4+t)^2} + f'(x) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial x} = \frac{1}{(-4+t)^2}$. Therefore equation (4) becomes

$$\frac{1}{(-4+t)^2} = \frac{1}{(-4+t)^2} + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = 0$$

Therefore

$$f(x) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(x)$ into equation (3) gives ϕ

$$\phi = \frac{-20 + 5t + x}{(-4+t)^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{-20 + 5t + x}{(-4+t)^2}$$

The solution becomes

$$x = (c_1 t - 4c_1 - 5)(-4 + t)$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $x = 4$ in the above solution gives an equation to solve for the constant of integration.

$$4 = 20 + 16c_1$$

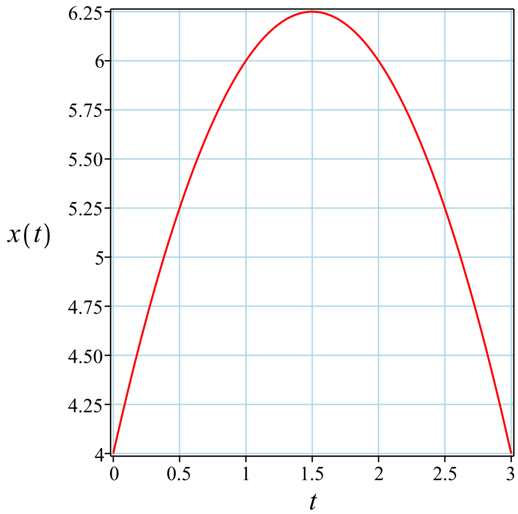
$$c_1 = -1$$

Substituting c_1 found above in the general solution gives

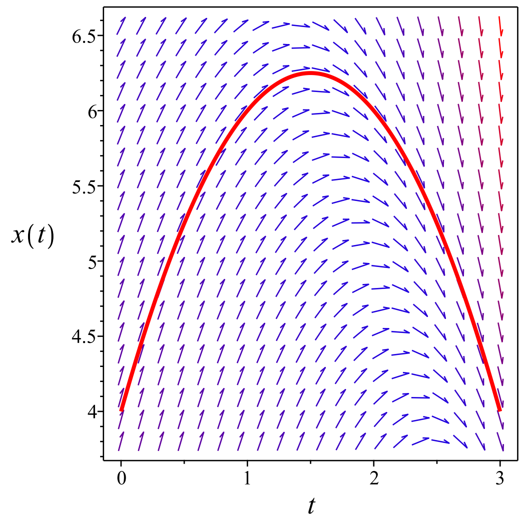
$$x = -(t + 1)(-4 + t)$$

Summary
The solution(s) found are the following

$$x = -(t + 1)(-4 + t) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = -(t+1)(-4+t)$$

Verified OK.

3.18.6 Maple step by step solution

Let's solve

$\left[x' + \frac{2x}{4-t} = 5, x(0) = 4\right]$

- Highest derivative means the order of the ODE is 1

  x'

- Isolate the derivative

  $x' = 5 + \frac{2x}{-4+t}$

- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE

  $x' - \frac{2x}{-4+t} = 5$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\left(x' - \frac{2x}{-4+t}\right) = 5\mu(t)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\, x)$

  $\mu(t)\left(x' - \frac{2x}{-4+t}\right) = \mu'(t)\, x + \mu(t)\, x'$

- Isolate $\mu'(t)$

  $\mu'(t) = -\frac{2\mu(t)}{-4+t}$
- Solve to find the integrating factor

  $\mu(t) = \frac{1}{(-4+t)^2}$
- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\, x)\right) dt = \int 5\mu(t)\, dt + c_1$
- Evaluate the integral on the lhs

  $\mu(t)\, x = \int 5\mu(t)\, dt + c_1$
- Solve for x

  $x = \frac{\int 5\mu(t)dt + c_1}{\mu(t)}$
- Substitute $\mu(t) = \frac{1}{(-4+t)^2}$

  $x = (-4+t)^2 \left(\int \frac{5}{(-4+t)^2} dt + c_1\right)$
- Evaluate the integrals on the rhs

  $x = (-4+t)^2 \left(-\frac{5}{-4+t} + c_1\right)$
- Simplify

  $x = (-4+t)\,(-5 + c_1(-4+t))$
- Use initial condition $x(0) = 4$

  $4 = 20 + 16c_1$
- Solve for c_1

  $c_1 = -1$
- Substitute $c_1 = -1$ into general solution and simplify

  $x = -(t+1)\,(-4+t)$
- Solution to the IVP

  $x = -(t+1)\,(-4+t)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 14

```
dsolve([diff(x(t),t)+2/(4-t)*x(t)=5,x(0) = 4],x(t), singsol=all)
```

$$x(t) = -t^2 + 3t + 4$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 15

```
DSolve[{x'[t]+2/(4-t)*x[t]==5,{x[0]==4}},x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to -t^2 + 3t + 4$$

3.19 problem Problem 19

Internal problem ID [2657]
Internal file name [`OUTPUT/2149_Sunday_June_05_2022_02_50_27_AM_10511743/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.6, First-Order Linear Differential Equations. page 59
Problem number: Problem 19.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' + y = e^x}$$

With initial conditions

$$[y(0) = 1]$$

3.19.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 1$$
$$q(x) = e^x$$

Hence the ode is

$$y' + y = e^x$$

The domain of $p(x) = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = e^x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

3.19.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int 1dx} \\ &= e^x\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(e^x\right) \\ \frac{d}{dx}\left(e^x y\right) &= \left(e^x\right)\left(e^x\right) \\ d\left(e^x y\right) &= e^{2x}\, dx\end{aligned}$$

Integrating gives

$$\begin{aligned}e^x y &= \int e^{2x}\, dx \\ e^x y &= \frac{e^{2x}}{2} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^x$ results in

$$y = \frac{e^{-x}e^{2x}}{2} + c_1 e^{-x}$$

which simplifies to

$$y = \frac{e^x}{2} + c_1 e^{-x}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{1}{2} + c_1$$

$$c_1 = \frac{1}{2}$$

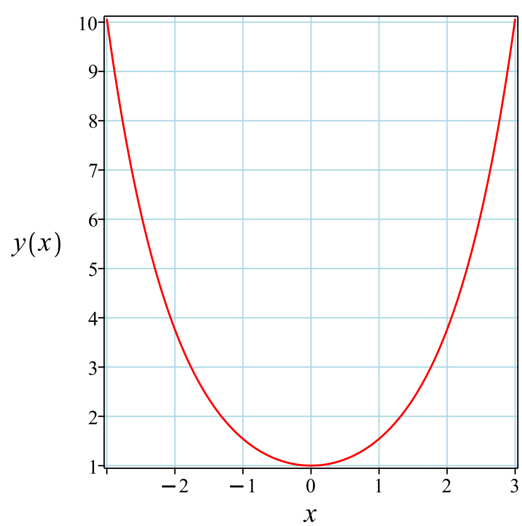
Substituting c_1 found above in the general solution gives

$$y = \frac{e^x}{2} + \frac{e^{-x}}{2}$$

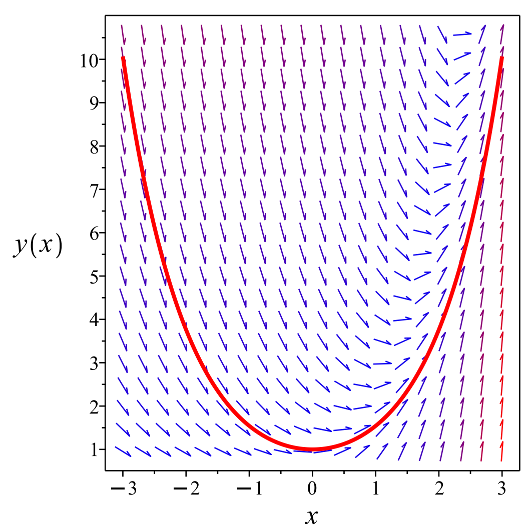
Summary

The solution(s) found are the following

$$y = \frac{e^x}{2} + \frac{e^{-x}}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{e^x}{2} + \frac{e^{-x}}{2}$$

Verified OK.

3.19.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -y + e^{x} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 100: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= 0 \\ \eta(x,y) &= \mathrm{e}^{-x} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{-x}} dy \end{aligned}$$

Which results in

$$S = \mathrm{e}^{x} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -y + \mathrm{e}^{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \mathrm{e}^{x} y \\ S_y &= \mathrm{e}^{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = e^{2x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = e^{2R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{e^{2R}}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^{x} y = \frac{e^{2x}}{2} + c_1$$

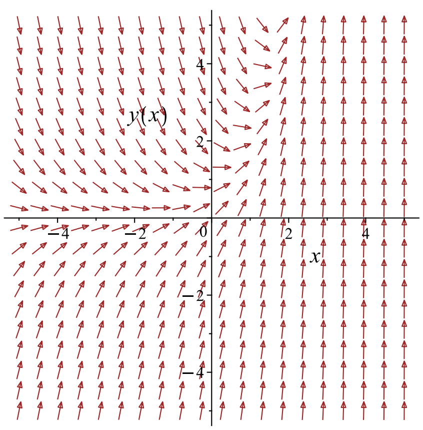
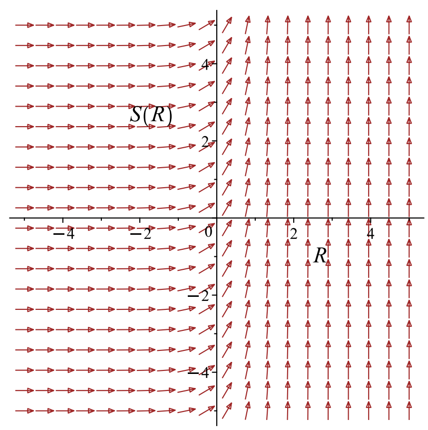
Which simplifies to

$$e^{x} y = \frac{e^{2x}}{2} + c_1$$

Which gives

$$y = \frac{(e^{2x} + 2c_1) e^{-x}}{2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -y + e^{x}$  	$R = x$ $S = e^{x} y$	$\frac{dS}{dR} = e^{2R}$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{1}{2} + c_1$$

$$c_1 = \frac{1}{2}$$

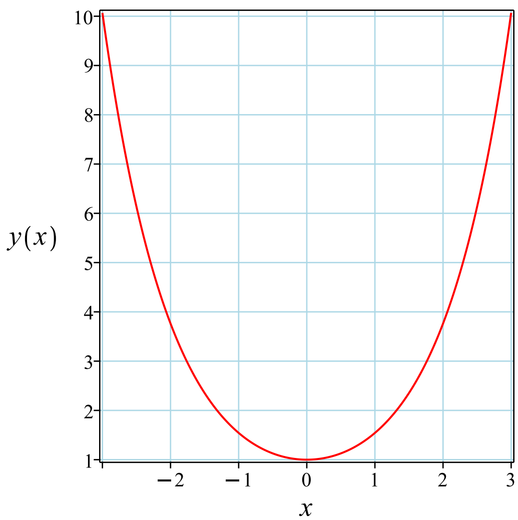
Substituting c_1 found above in the general solution gives

$$y = \frac{e^{x}}{2} + \frac{e^{-x}}{2}$$

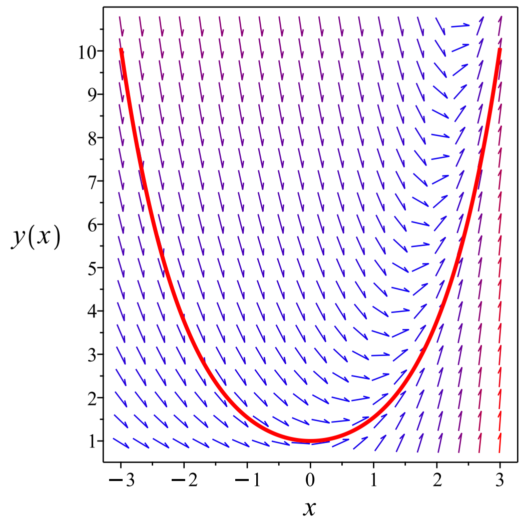
Summary

The solution(s) found are the following

$$y = \frac{e^{x}}{2} + \frac{e^{-x}}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{\mathrm{e}^{x}}{2} + \frac{\mathrm{e}^{-x}}{2}$$

Verified OK.

3.19.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y &= (-y + \mathrm{e}^x)\,\mathrm{d}x\\ (y - \mathrm{e}^x)\,\mathrm{d}x + \mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= y - \mathrm{e}^x\\ N(x, y) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y - \mathrm{e}^x)\\ &= 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= 1((1) - (0))\\ &= 1\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int 1 \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{x} \\ &= \mathrm{e}^{x}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \mathrm{e}^{x}(y - \mathrm{e}^{x}) \\ &= (y - \mathrm{e}^{x})\,\mathrm{e}^{x}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \mathrm{e}^{x}(1) \\ &= \mathrm{e}^{x}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ ((y - \mathrm{e}^{x})\,\mathrm{e}^{x}) + (\mathrm{e}^{x})\,\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int \overline{M} \, \mathrm{d}x \\ \int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int (y - \mathrm{e}^{x})\,\mathrm{e}^{x} \, \mathrm{d}x\end{aligned}$$

$$\phi = \mathrm{e}^{x} y - \frac{\mathrm{e}^{2x}}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{x}$. Therefore equation (4) becomes

$$\mathrm{e}^{x} = \mathrm{e}^{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \mathrm{e}^{x} y - \frac{\mathrm{e}^{2x}}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \mathrm{e}^{x} y - \frac{\mathrm{e}^{2x}}{2}$$

The solution becomes

$$y = \frac{(\mathrm{e}^{2x} + 2c_1)\,\mathrm{e}^{-x}}{2}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{1}{2} + c_1$$

$$c_1 = \frac{1}{2}$$

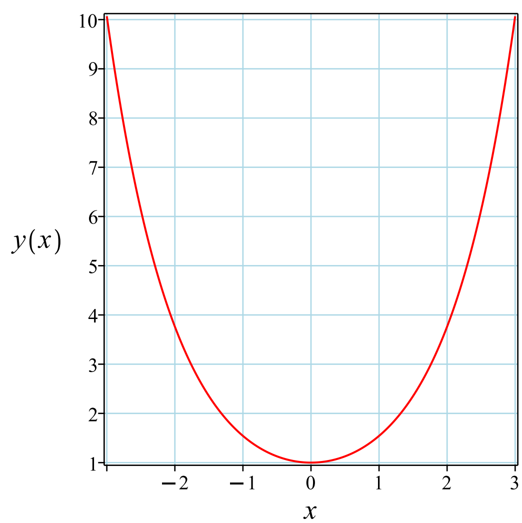
Substituting c_1 found above in the general solution gives

$$y = \frac{e^{x}}{2} + \frac{e^{-x}}{2}$$

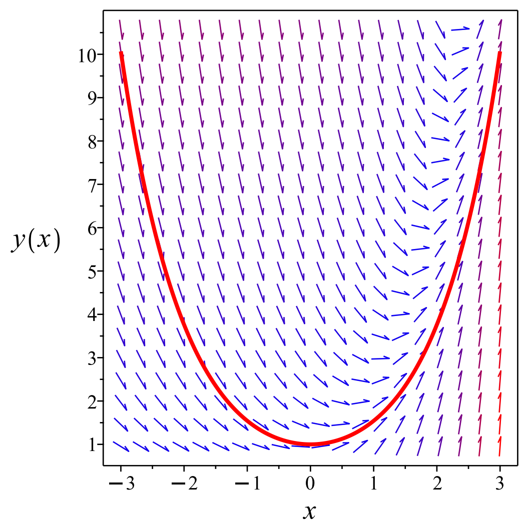
Summary

The solution(s) found are the following

$$y = \frac{e^{x}}{2} + \frac{e^{-x}}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{e^{x}}{2} + \frac{e^{-x}}{2}$$

Verified OK.

3.19.5 Maple step by step solution

Let's solve

$[y' + y = e^{x}, y(0) = 1]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

$y' = -y + \mathrm{e}^x$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + y = \mathrm{e}^x$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\,(y' + y) = \mu(x)\,\mathrm{e}^x$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\,(y' + y) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \mu(x)$

- Solve to find the integrating factor

  $\mu(x) = \mathrm{e}^x$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int \mu(x)\,\mathrm{e}^x dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int \mu(x)\,\mathrm{e}^x dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x)\mathrm{e}^x dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \mathrm{e}^x$

  $y = \frac{\int (\mathrm{e}^x)^2 dx + c_1}{\mathrm{e}^x}$

- Evaluate the integrals on the rhs

  $y = \frac{\frac{(\mathrm{e}^x)^2}{2} + c_1}{\mathrm{e}^x}$

- Simplify

  $y = \frac{\mathrm{e}^x}{2} + c_1 \mathrm{e}^{-x}$

- Use initial condition $y(0) = 1$

  $1 = \frac{1}{2} + c_1$

- Solve for c_1

  $c_1 = \frac{1}{2}$

- Substitute $c_1 = \frac{1}{2}$ into general solution and simplify

  $y = \frac{\mathrm{e}^x}{2} + \frac{\mathrm{e}^{-x}}{2}$

- Solution to the IVP

  $y = \frac{e^x}{2} + \frac{e^{-x}}{2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve([y(x)-exp(x)+diff(y(x),x)=0,y(0) = 1],y(x), singsol=all)
```

$$y(x) = \frac{e^x}{2} + \frac{e^{-x}}{2}$$

✓ Solution by Mathematica

Time used: 0.039 (sec). Leaf size: 21

```
DSolve[{y[x]-Exp[x]+y'[x]==0,{y[0]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}e^{-x}\left(e^{2x}+1\right)$$

3.20 problem Problem 20

Internal problem ID [2658]
Internal file name [`OUTPUT/2150_Sunday_June_05_2022_02_50_29_AM_36187375/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.6, First-Order Linear Differential Equations. page 59
Problem number: Problem 20.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' - 2y = \begin{cases} 1 & x \leq 1 \\ 0 & 1 < x \end{cases}}$$

With initial conditions

$$[y(0) = 3]$$

3.20.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$
\begin{aligned}
p(x) &= -2 \\
q(x) &= \left\{ \begin{array}{cc} 1 & x \leq 1 \\ 0 & 1 < x \end{array} \right.
\end{aligned}
$$

Hence the ode is

$$
y' - 2y = \left\{ \begin{array}{cc} 1 & x \leq 1 \\ 0 & 1 < x \end{array} \right.
$$

The domain of $p(x) = -2$ is

$$
\{-\infty < x < \infty\}
$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \left\{ \begin{array}{cc} 1 & x \leq 1 \\ 0 & 1 < x \end{array} \right.$ is

$$
\{x < 1 \lor 1 < x\}
$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

3.20.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int (-2)dx} \\
&= \mathrm{e}^{-2x}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu) \left(\left\{ \begin{array}{cc} 1 & x \leq 1 \\ 0 & 1 < x \end{array} \right. \right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{-2x} y\right) &= \left(\mathrm{e}^{-2x}\right) \left(\left\{ \begin{array}{cc} 1 & x \leq 1 \\ 0 & 1 < x \end{array} \right. \right) \\
\mathrm{d}\left(\mathrm{e}^{-2x} y\right) &= \left\{ \begin{array}{cc} \mathrm{e}^{-2x} & x \leq 1 \\ 0 & 1 < x \end{array} \right. \mathrm{d}x
\end{aligned}
$$

Integrating gives

$$e^{-2x}y = \int \left\{ \begin{array}{cc} e^{-2x} & x \leq 1 \\ 0 & 1 < x \end{array} \right. dx$$

$$e^{-2x}y = \left\{ \begin{array}{cc} -\frac{e^{-2x}}{2} & x \leq 1 \\ -\frac{e^{-2}}{2} & 1 < x \end{array} \right. + c_1$$

Dividing both sides by the integrating factor $\mu = e^{-2x}$ results in

$$y = e^{2x} \left(\left\{ \begin{array}{cc} -\frac{e^{-2x}}{2} & x \leq 1 \\ -\frac{e^{-2}}{2} & 1 < x \end{array} \right. \right) + e^{2x}c_1$$

which simplifies to

$$y = -\frac{e^{2x-2\left(\left\{ \begin{array}{cc} x & x \leq 1 \\ 1 & 1 < x \end{array} \right.\right)}}{2} + e^{2x}c_1$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = c_1 - \frac{1}{2}$$

$$c_1 = \frac{7}{2}$$

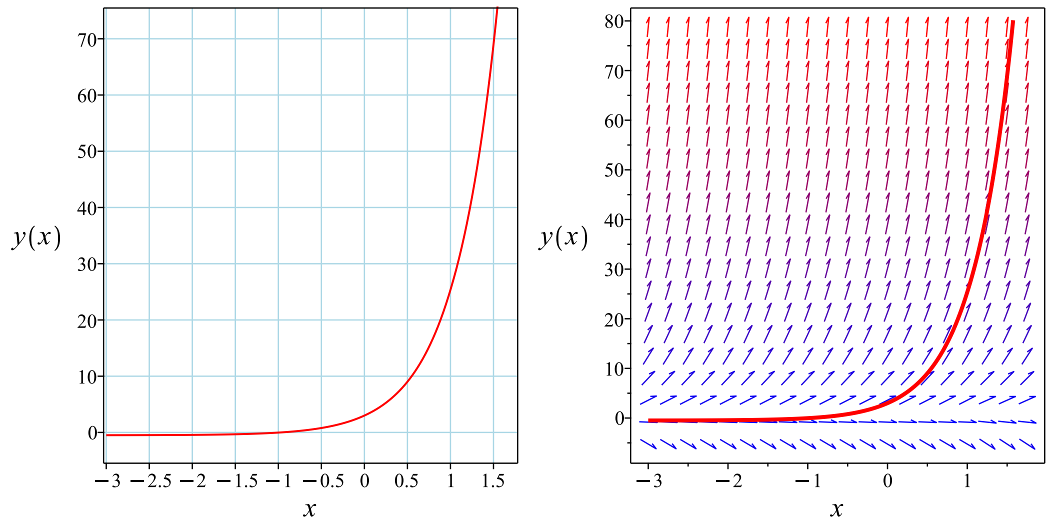
Substituting c_1 found above in the general solution gives

$$y = \left\{ \begin{array}{cc} -\frac{1}{2} + \frac{7\,e^{2x}}{2} & x \leq 1 \\ -\frac{e^{2x-2}}{2} + \frac{7\,e^{2x}}{2} & 1 < x \\ 3\,e^{2x} & \text{otherwise} \end{array} \right.$$

Summary

The solution(s) found are the following

$$y = \left\{ \begin{array}{cc} -\frac{1}{2} + \frac{7\,e^{2x}}{2} & x \leq 1 \\ -\frac{e^{2x-2}}{2} + \frac{7\,e^{2x}}{2} & 1 < x \\ 3\,e^{2x} & \text{otherwise} \end{array} \right. \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$
y = \begin{cases} -\frac{1}{2}+\frac{7\,\mathrm{e}^{2x}}{2} & x \le 1 \\ -\frac{\mathrm{e}^{2x-2}}{2}+\frac{7\,\mathrm{e}^{2x}}{2} & 1 < x \\ 3\,\mathrm{e}^{2x} & \text{otherwise} \end{cases}
$$

Verified OK.

3.20.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= 2y + \left(\begin{cases} 1 & x \le 1 \\ 0 & 1 < x \end{cases}\right) \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 103: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \mathrm{e}^{2x}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{2x}} dy \end{aligned}$$

Which results in

$$S = \mathrm{e}^{-2x} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = 2y + \left(\begin{cases} 1 & x \leq 1 \\ 0 & 1 < x \end{cases} \right)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -2\,\mathrm{e}^{-2x} y \\ S_y &= \mathrm{e}^{-2x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \begin{cases} \mathrm{e}^{-2x} & x \leq 1 \\ 0 & 1 < x \end{cases} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \begin{cases} \mathrm{e}^{-2R} & R \leq 1 \\ 0 & 1 < R \end{cases}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \begin{cases} -\frac{e^{-2R}}{2}+c_1 & R<1 \\ c_1-\frac{e^{-2}}{2} & 1\leq R \end{cases} \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^{-2x} y = \begin{cases} -\frac{e^{-2x}}{2}+c_1 & x<1 \\ c_1-\frac{e^{-2}}{2} & 1\leq x \end{cases}$$

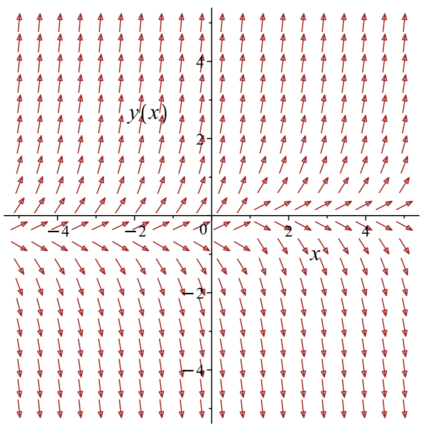
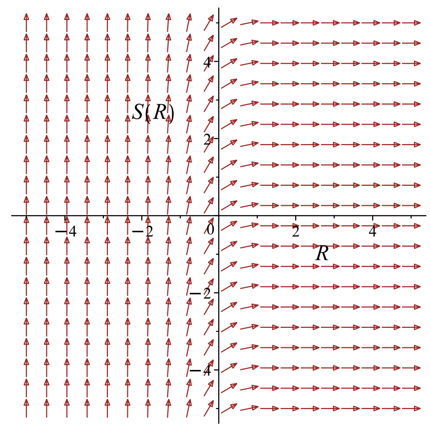
Which simplifies to

$$e^{-2x} y - c_1 + \frac{e^{-2\left(\begin{cases} x & x<1 \\ 1 & 1\leq x \end{cases}\right)}}{2} = 0$$

Which gives

$$y = \begin{cases} \left[e^{2x} c_1-\frac{1}{2}\right] & x<1 \\ \left[-\frac{(e^{-2}-2c_1)e^{2x}}{2}\right] & 1\leq x \end{cases}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = 2y + \left(\begin{cases} 1 & x \leq 1 \\ 0 & 1 < x \end{cases}\right)$ 	$R = x$ $S = e^{-2x} y$	$\frac{dS}{dR} = \begin{cases} e^{-2R} & R \leq 1 \\ 0 & 1 < R \end{cases}$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = \left[c_1 - \frac{1}{2}\right]$$

Unable to solve for constant of integration. Verification of solutions N/A

3.20.4 Maple step by step solution

Let's solve

$$\left[y' - 2y = \begin{cases} 1 & x \leq 1 \\ 0 & 1 < x \end{cases}, y(0) = 3\right]$$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

$$y' = 2y + \left\{ \begin{array}{cc} 1 & x \leq 1 \\ 0 & 1 < x \end{array} \right.$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

$$y' - 2y = \left\{ \begin{array}{cc} 1 & x \leq 1 \\ 0 & 1 < x \end{array} \right.$$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$$\mu(x) \left(y' - 2y\right) = \mu(x) \left(\left\{ \begin{array}{cc} 1 & x \leq 1 \\ 0 & 1 < x \end{array} \right.\right)$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}\left(\mu(x)\, y\right)$

$$\mu(x) \left(y' - 2y\right) = \mu'(x)\, y + \mu(x)\, y'$$

- Isolate $\mu'(x)$

$$\mu'(x) = -2\mu(x)$$

- Solve to find the integrating factor

$$\mu(x) = \mathrm{e}^{-2x}$$

- Integrate both sides with respect to x

$$\int \left(\frac{d}{dx}\left(\mu(x)\, y\right)\right) dx = \int \mu(x) \left(\left\{ \begin{array}{cc} 1 & x \leq 1 \\ 0 & 1 < x \end{array} \right.\right) dx + c_1$$

- Evaluate the integral on the lhs

$$\mu(x)\, y = \int \mu(x) \left(\left\{ \begin{array}{cc} 1 & x \leq 1 \\ 0 & 1 < x \end{array} \right.\right) dx + c_1$$

- Solve for y

$$y = \frac{\int \mu(x) \left(\left\{ \begin{array}{cc} 1 & x \leq 1 \\ 0 & 1 < x \end{array} \right.\right) dx + c_1}{\mu(x)}$$

- Substitute $\mu(x) = \mathrm{e}^{-2x}$

$$y = \frac{\int \left(\left\{ \begin{array}{cc} 1 & x \leq 1 \\ 0 & 1 < x \end{array} \right.\right) \mathrm{e}^{-2x} dx + c_1}{\mathrm{e}^{-2x}}$$

- Evaluate the integrals on the rhs

$$y = \frac{\begin{cases} -\frac{e^{-2x}}{2} & x \le 1 \\ -\frac{e^{-2}}{2} & 1 < x \end{cases} + c_1}{e^{-2x}}$$

- Simplify

$$y = -\frac{e^{2x}\left(e^{-2\left(\begin{cases} x & x \le 1 \\ 1 & 1 < x \end{cases}\right)} - 2c_1\right)}{2}$$

- Use initial condition $y(0) = 3$

$3 = c_1 - \frac{1}{2}$

- Solve for c_1

$c_1 = \frac{7}{2}$

- Substitute $c_1 = \frac{7}{2}$ into general solution and simplify

$$y = -\frac{e^{2x}\left(e^{-2\left(\begin{cases} x & x \le 1 \\ 1 & 1 < x \end{cases}\right)} - 7\right)}{2}$$

- Solution to the IVP

$$y = -\frac{e^{2x}\left(e^{-2\left(\begin{cases} x & x \le 1 \\ 1 & 1 < x \end{cases}\right)} - 7\right)}{2}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.172 (sec). Leaf size: 27

```
dsolve([diff(y(x),x)-2*y(x)=piecewise(x<=1,1,x>1,0),y(0) = 3],y(x), singsol=all)
```

$$y(x) = \frac{7\,\mathrm{e}^{2x}}{2} - \frac{\left(\begin{cases} 1 & x < 1 \\ \mathrm{e}^{2x-2} & 1 \leq x \end{cases}\right)}{2}$$

✓ Solution by Mathematica

Time used: 0.195 (sec). Leaf size: 42

```
DSolve[{y'[x] - 2*y[x] == Piecewise[{{1, x <= 1}, {0, x > 1}}],{y[0]==3}},y[x],x,IncludeSing
```

$$y(x) \to \begin{cases} \frac{1}{2}(-1+7e^{2x}) & x \leq 1 \\ \frac{1}{2}e^{2x-2}(-1+7e^{2}) & \text{True} \end{cases}$$

3.21 problem Problem 21

Internal problem ID [2659]
Internal file name [`OUTPUT/2151_Sunday_June_05_2022_02_50_34_AM_23091357/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.6, First-Order Linear Differential Equations. page 59
Problem number: Problem 21.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' - 2y = \begin{cases} 1-x & x<1 \\ 0 & 1\leq x \end{cases}}$$

With initial conditions

$$[y(0) = 1]$$

3.21.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$
\begin{aligned}
p(x) &= -2 \\
q(x) &= \left\{\begin{array}{cc} 1-x & x<1 \\ 0 & 1\leq x \end{array}\right.
\end{aligned}
$$

Hence the ode is

$$
y' - 2y = \left\{\begin{array}{cc} 1-x & x<1 \\ 0 & 1\leq x \end{array}\right.
$$

The domain of $p(x) = -2$ is

$$
\{-\infty < x < \infty\}
$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \left\{\begin{array}{cc} 1-x & x<1 \\ 0 & 1\leq x \end{array}\right.$ is

$$
\{-\infty < x < \infty\}
$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

3.21.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int (-2)dx} \\
&= e^{-2x}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(\left\{\begin{array}{cc} 1-x & x<1 \\ 0 & 1\leq x \end{array}\right.\right) \\
\frac{d}{dx}\left(e^{-2x} y\right) &= \left(e^{-2x}\right)\left(\left\{\begin{array}{cc} 1-x & x<1 \\ 0 & 1\leq x \end{array}\right.\right) \\
d\left(e^{-2x} y\right) &= \left(\left(\left\{\begin{array}{cc} 1-x & x<1 \\ 0 & 1\leq x \end{array}\right.\right) e^{-2x}\right) dx
\end{aligned}
$$

Integrating gives

$$\mathrm{e}^{-2x} y = \int \left(\begin{cases} 1-x & x<1 \\ 0 & 1\le x \end{cases} \right) \mathrm{e}^{-2x}\, dx$$

$$\mathrm{e}^{-2x} y = \begin{cases} \frac{\mathrm{e}^{-2x}(2x-1)}{4} & x\le 1 \\ \frac{\mathrm{e}^{-2}}{4} & 1<x \end{cases} + c_1$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-2x}$ results in

$$y = \mathrm{e}^{2x} \left(\begin{cases} \frac{\mathrm{e}^{-2x}(2x-1)}{4} & x\le 1 \\ \frac{\mathrm{e}^{-2}}{4} & 1<x \end{cases} \right) + \mathrm{e}^{2x} c_1$$

which simplifies to

$$y = \mathrm{e}^{2x} c_1 + \frac{\left(\begin{cases} 2x-1 & x\le 1 \\ \mathrm{e}^{2x-2} & 1<x \end{cases} \right)}{4}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1 - \frac{1}{4}$$

$$c_1 = \frac{5}{4}$$

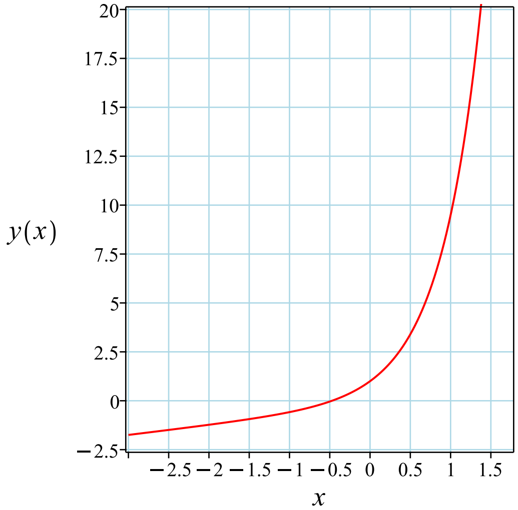
Substituting c_1 found above in the general solution gives

$$y = \begin{cases} \frac{5\,\mathrm{e}^{2x}}{4} + \frac{x}{2} - \frac{1}{4} & x\le 1 \\ \frac{5\,\mathrm{e}^{2x}}{4} + \frac{\mathrm{e}^{2x-2}}{4} & 1<x \\ \frac{5\,\mathrm{e}^{2x}}{4} & \text{otherwise} \end{cases}$$

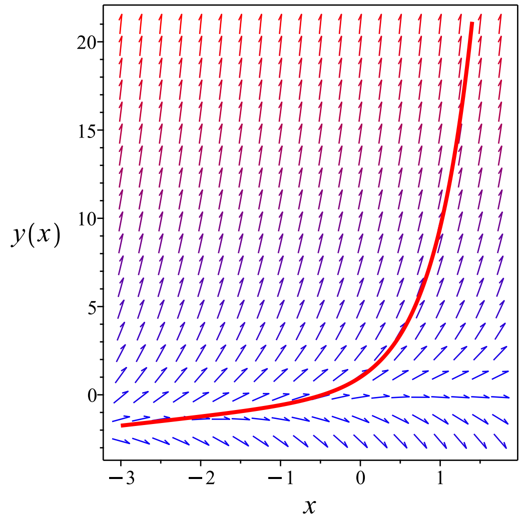
Summary

The solution(s) found are the following

$$y = \begin{cases} \frac{5\,\mathrm{e}^{2x}}{4} + \frac{x}{2} - \frac{1}{4} & x\le 1 \\ \frac{5\,\mathrm{e}^{2x}}{4} + \frac{\mathrm{e}^{2x-2}}{4} & 1<x \\ \frac{5\,\mathrm{e}^{2x}}{4} & \text{otherwise} \end{cases} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \begin{cases} \frac{5\,e^{2x}}{4} + \frac{x}{2} - \frac{1}{4} & x \leq 1 \\ \frac{5\,e^{2x}}{4} + \frac{e^{2x-2}}{4} & 1 < x \\ \frac{5\,e^{2x}}{4} & \text{otherwise} \end{cases}$$

Verified OK.

3.21.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = 2y + \left(\begin{cases} 1-x & x < 1 \\ 0 & 1 \leq x \end{cases} \right)$$

$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 106: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= e^{2x}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{2x}} dy \end{aligned}$$

Which results in

$$S = e^{-2x} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = 2y + \left(\begin{cases} 1-x & x<1 \\ 0 & 1 \leq x \end{cases} \right)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -2\,e^{-2x} y \\ S_y &= e^{-2x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \begin{cases} -(x-1)\,e^{-2x} & x<1 \\ 0 & 1 \leq x \end{cases} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \begin{cases} -(R-1)\,e^{-2R} & R<1 \\ 0 & 1 \leq R \end{cases}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \begin{cases} \frac{e^{-2R}R}{2} - \frac{e^{-2R}}{4} + c_1 & R < 1 \\ c_1 + \frac{e^{-2}}{4} & 1 \leq R \end{cases} \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^{-2x} y = \begin{cases} \frac{x\,e^{-2x}}{2} - \frac{e^{-2x}}{4} + c_1 & x < 1 \\ c_1 + \frac{e^{-2}}{4} & 1 \leq x \end{cases}$$

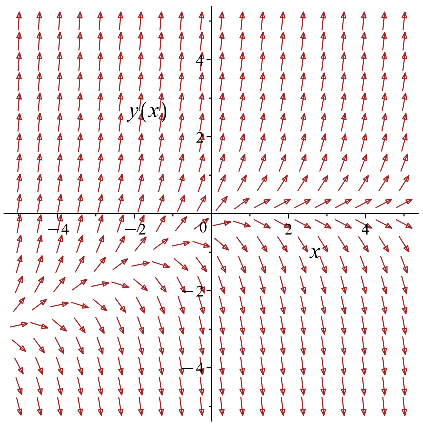
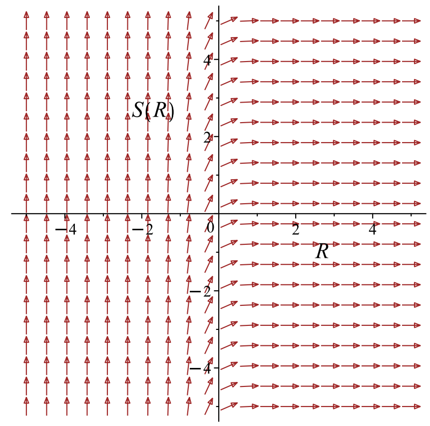
Which simplifies to

$$e^{-2x} y - c_1 - \frac{\left(\begin{cases} e^{-2x}(2x-1) & x < 1 \\ e^{-2} & 1 \leq x \end{cases}\right)}{4} = 0$$

Which gives

$$y = \begin{cases} \left[e^{2x} c_1 + \frac{x}{2} - \frac{1}{4}\right] & x < 1 \\ \left[\frac{(4c_1 + e^{-2})e^{2x}}{4}\right] & 1 \leq x \end{cases}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = 2y + \left(\begin{cases} 1-x & x<1 \\ 0 & 1\leq x \end{cases}\right)$ 	$R = x$ $S = \mathrm{e}^{-2x} y$	$\frac{dS}{dR} = \begin{cases} -(R-1)\,\mathrm{e}^{-2R} & R<1 \\ 0 & 1\leq R \end{cases}$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \left[c_1 - \frac{1}{4}\right]$$

Unable to solve for constant of integration. Verification of solutions N/A

3.21.4 Maple step by step solution

Let's solve

$$\left[y' - 2y = \begin{cases} 1-x & x<1 \\ 0 & 1\leq x \end{cases}, y(0) = 1\right]$$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

$$y' = 2y + \left\{\begin{array}{cc} 1-x & x<1 \\ 0 & 1\le x \end{array}\right.$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

$$y' - 2y = \left\{\begin{array}{cc} 1-x & x<1 \\ 0 & 1\le x \end{array}\right.$$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$$\mu(x)\left(y' - 2y\right) = \mu(x)\left(\left\{\begin{array}{cc} 1-x & x<1 \\ 0 & 1\le x \end{array}\right.\right)$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

$$\mu(x)\left(y' - 2y\right) = \mu'(x)\, y + \mu(x)\, y'$$

- Isolate $\mu'(x)$

$$\mu'(x) = -2\mu(x)$$

- Solve to find the integrating factor

$$\mu(x) = \mathrm{e}^{-2x}$$

- Integrate both sides with respect to x

$$\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x)\left(\left\{\begin{array}{cc} 1-x & x<1 \\ 0 & 1\le x \end{array}\right.\right) dx + c_1$$

- Evaluate the integral on the lhs

$$\mu(x)\, y = \int \mu(x)\left(\left\{\begin{array}{cc} 1-x & x<1 \\ 0 & 1\le x \end{array}\right.\right) dx + c_1$$

- Solve for y

$$y = \frac{\int \mu(x)\left(\left\{\begin{array}{cc} 1-x & x<1 \\ 0 & 1\le x \end{array}\right.\right) dx + c_1}{\mu(x)}$$

- Substitute $\mu(x) = \mathrm{e}^{-2x}$

$$y = \frac{\int \left(\left\{\begin{array}{cc} 1-x & x<1 \\ 0 & 1\le x \end{array}\right.\right)\mathrm{e}^{-2x} dx + c_1}{\mathrm{e}^{-2x}}$$

- Evaluate the integrals on the rhs

$$y = \frac{\left\{\begin{array}{cc} \frac{e^{-2x}(2x-1)}{4} & x \le 1 \\ \frac{e^{-2}}{4} & 1 < x \end{array}\right. + c_1}{e^{-2x}}$$

- Simplify

$$y = \frac{e^{2x}\left(\left\{\begin{array}{cc} e^{-2x}(2x-1) & x \le 1 \\ e^{-2} & 1 < x \end{array}\right. + 4c_1\right)}{4}$$

- Use initial condition $y(0) = 1$

$$1 = c_1 - \frac{1}{4}$$

- Solve for c_1

$$c_1 = \frac{5}{4}$$

- Substitute $c_1 = \frac{5}{4}$ into general solution and simplify

$$y = \frac{e^{2x}\left(\left\{\begin{array}{cc} e^{-2x}(2x-1) & x \le 1 \\ e^{-2} & 1 < x \end{array}\right. + 5\right)}{4}$$

- Solution to the IVP

$$y = \frac{e^{2x}\left(\left\{\begin{array}{cc} e^{-2x}(2x-1) & x \le 1 \\ e^{-2} & 1 < x \end{array}\right. + 5\right)}{4}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.172 (sec). Leaf size: 31

```
dsolve([diff(y(x),x)-2*y(x)=piecewise(x<1,1-x,x>=1,0),y(0) = 1],y(x), singsol=all)
```

$$y(x) = \frac{5\,\mathrm{e}^{2x}}{4} + \frac{\left(\left\{\begin{array}{cc} 2x-1 & x<1 \\ \mathrm{e}^{2x-2} & 1\le x \end{array}\right.\right)}{4}$$

✓ Solution by Mathematica

Time used: 0.084 (sec). Leaf size: 45

```
DSolve[{y'[x] - 2*y[x] == Piecewise[{{1-x, x < 1}, {0, x >= 1}}],{y[0]==1}},y[x],x,IncludeSi
```

$$y(x) \to \left\{\begin{array}{cc} \frac{1}{4}(2x+5e^{2x}-1) & x\le 1 \\ \frac{1}{4}e^{2x-2}(1+5e^2) & \text{True} \end{array}\right.$$

3.22 problem Problem 22

Internal problem ID [2660]
Internal file name [`OUTPUT/2152_Sunday_June_05_2022_02_50_37_AM_79437181/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.6, First-Order Linear Differential Equations. page 59
Problem number: Problem 22.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_ode_missing_y", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{y'' + \frac{y'}{x} = 9x}$$

3.22.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) + \frac{p(x)}{x} - 9x = 0$$

Which is now solve for $p(x)$ as first order ode.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$p'(x) + p(x)p(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= 9x \end{aligned}$$

Hence the ode is

$$p'(x) + \frac{p(x)}{x} = 9x$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int \frac{1}{x} dx} \\ &= x \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu p) &= (\mu)(9x) \\ \frac{\mathrm{d}}{\mathrm{d}x}(xp) &= (x)(9x) \\ \mathrm{d}(xp) &= (9x^2)\ \mathrm{d}x \end{aligned}$$

Integrating gives

$$\begin{aligned} xp &= \int 9x^2 \,\mathrm{d}x \\ xp &= 3x^3 + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = x$ results in

$$p(x) = 3x^2 + \frac{c_1}{x}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = 3x^2 + \frac{c_1}{x}$$

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{3x^3 + c_1}{x} \,\mathrm{d}x \\ &= x^3 + c_1 \ln(x) + c_2 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = x^3 + c_1 \ln (x) + c_2 \tag{1}$$

Verification of solutions

$$y = x^3 + c_1 \ln (x) + c_2$$

Verified OK.

3.22.2 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned} y' &= B'v + v'B \\ y'' &= B''v + B'v' + v''B + v'B' \\ &= v''B + 2v' + B' + B''v \end{aligned}$$

And now the original ode becomes

$$\begin{aligned} A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\ ABv'' + \left(2AB' + B^2\right) v' + \left(AB'' + BB' + CB\right) v &= 0 \end{aligned} \tag{1}$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + \left(2AB' + B^2\right) v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + \left(2AB' + B^2\right) u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The ODE is now normalized to

$$y''x + y' = 9x$$

Where now

$$
\begin{aligned}
A &= x \\
B &= 1 \\
C &= 0 \\
F &= 9x
\end{aligned}
$$

The above shows that for this ode

$$
\begin{aligned}
AB'' + BB' + CB &= (x)(0) + (1)(0) + (0)(1) \\
&= 0
\end{aligned}
$$

Hence the ode in v given in (1) now simplifies to

$$
xv'' + (1)v' = 0
$$

Now by applying $v' = u$ the above becomes

$$
xu'(x) + u(x) = 0
$$

Which is now solved for u. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= -\frac{u}{x}
\end{aligned}
$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\, du &= -\frac{1}{x}\, dx \\
\int \frac{1}{u}\, du &= \int -\frac{1}{x}\, dx \\
\ln(u) &= -\ln(x) + c_1 \\
u &= \mathrm{e}^{-\ln(x)+c_1} \\
&= \frac{c_1}{x}
\end{aligned}
$$

The ode for v now becomes

$$
\begin{aligned}
v' &= u \\
&= \frac{c_1}{x}
\end{aligned}
$$

Which is now solved for v. Integrating both sides gives

$$\begin{aligned} v(x) &= \int \frac{c_1}{x}\, \mathrm{d}x \\ &= c_1 \ln(x) + c_2 \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= Bv \\ &= (1)\,(c_1 \ln(x) + c_2) \\ &= c_1 \ln(x) + c_2 \end{aligned}$$

And now the particular solution $y_p(x)$ will be found. The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= 1 \\ y_2 &= \ln(x) \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} 1 & \ln(x) \\ \frac{d}{dx}(1) & \frac{d}{dx}(\ln(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} 1 & \ln(x) \\ 0 & \frac{1}{x} \end{vmatrix}$$

Therefore

$$W = (1)\left(\frac{1}{x}\right) - (\ln(x))(0)$$

Which simplifies to

$$W = \frac{1}{x}$$

Which simplifies to

$$W = \frac{1}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{9\ln(x)\, x}{1}\, dx$$

Which simplifies to

$$u_1 = -\int 9\ln(x)\, x dx$$

Hence

$$u_1 = -\frac{9\ln(x)\, x^2}{2} + \frac{9x^2}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{9x}{1}\, dx$$

Which simplifies to

$$u_2 = \int 9x dx$$

Hence

$$u_2 = \frac{9x^2}{2}$$

Which simplifies to

$$u_1 = -\frac{9x^2(-1+2\ln(x))}{4}$$

$$u_2 = \frac{9x^2}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{9x^2(-1+2\ln(x))}{4} + \frac{9\ln(x)\,x^2}{2}$$

Which simplifies to

$$y_p(x) = \frac{9x^2}{4}$$

Hence the complete solution is

$$\begin{aligned} y(x) &= y_h + y_p \\ &= (c_1 \ln(x) + c_2) + \left(\frac{9x^2}{4}\right) \\ &= c_1 \ln(x) + c_2 + \frac{9x^2}{4} \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \ln(x) + c_2 + \frac{9x^2}{4} \tag{1}$$

Verification of solutions

$$y = c_1 \ln(x) + c_2 + \frac{9x^2}{4}$$

Verified OK.

3.22.3 Solving using Kovacic algorithm

Writing the ode as

$$y''x + y' = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 1 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 109: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = -\frac{1}{4x^2}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = -\frac{1}{4x^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = -\frac{1}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_{\infty}^{-} = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_{\infty}^{-} - \left(\alpha_{c_1}^{-}\right) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_{\infty}$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (-)[\sqrt{r}]_{\infty} \\ &= \frac{1}{2x} + (-)(0) \\ &= \frac{1}{2x} \\ &= \frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{x}\,dx} \\
&= z_1 e^{-\frac{\ln(x)}{2}} \\
&= z_1\left(\frac{1}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = 1
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1}{x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(\ln(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(1) + c_2(1(\ln(x)))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y''x + y' = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 + c_2 \ln(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = 1$$
$$y_2 = \ln(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} 1 & \ln(x) \\ \frac{d}{dx}(1) & \frac{d}{dx}(\ln(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} 1 & \ln(x) \\ 0 & \frac{1}{x} \end{vmatrix}$$

Therefore

$$W = (1)\left(\frac{1}{x}\right) - (\ln(x))(0)$$

Which simplifies to

$$W = \frac{1}{x}$$

Which simplifies to

$$W = \frac{1}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{9 \ln(x) x}{1} dx$$

Which simplifies to

$$u_1 = -\int 9 \ln(x) x dx$$

Hence

$$u_1 = -\frac{9 \ln(x) x^2}{2} + \frac{9x^2}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{9x}{1} dx$$

Which simplifies to

$$u_2 = \int 9x dx$$

Hence

$$u_2 = \frac{9x^2}{2}$$

Which simplifies to

$$u_1 = -\frac{9x^2(-1 + 2\ln(x))}{4}$$

$$u_2 = \frac{9x^2}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{9x^2(-1 + 2\ln(x))}{4} + \frac{9\ln(x)\,x^2}{2}$$

Which simplifies to

$$y_p(x) = \frac{9x^2}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 + c_2 \ln(x)) + \left(\frac{9x^2}{4}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 + c_2 \ln(x) + \frac{9x^2}{4} \tag{1}$$

Verification of solutions

$$y = c_1 + c_2 \ln(x) + \frac{9x^2}{4}$$

Verified OK.

3.22.4 Maple step by step solution

Let's solve

$y''x + y' = 9x$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u = y'$ to reduce order of ODE

  $u'(x)\,x + u(x) = 9x$

- Isolate the derivative

  $u'(x) = 9 - \frac{u(x)}{x}$

- Group terms with $u(x)$ on the lhs of the ODE and the rest on the rhs of the ODE

$u'(x) + \frac{u(x)}{x} = 9$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x) \left(u'(x) + \frac{u(x)}{x}\right) = 9\mu(x)$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, u(x))$

  $\mu(x) \left(u'(x) + \frac{u(x)}{x}\right) = \mu'(x)\, u(x) + \mu(x)\, u'(x)$
- Isolate $\mu'(x)$

  $\mu'(x) = \frac{\mu(x)}{x}$
- Solve to find the integrating factor

  $\mu(x) = x$
- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, u(x))\right) dx = \int 9\mu(x)\, dx + c_1$
- Evaluate the integral on the lhs

  $\mu(x)\, u(x) = \int 9\mu(x)\, dx + c_1$
- Solve for $u(x)$

  $u(x) = \frac{\int 9\mu(x)dx + c_1}{\mu(x)}$
- Substitute $\mu(x) = x$

  $u(x) = \frac{\int 9x dx + c_1}{x}$
- Evaluate the integrals on the rhs

  $u(x) = \frac{\frac{9x^2}{2} + c_1}{x}$
- Solve 1st ODE for $u(x)$

  $u(x) = \frac{\frac{9x^2}{2} + c_1}{x}$
- Make substitution $u = y'$

  $y' = \frac{\frac{9x^2}{2} + c_1}{x}$
- Integrate both sides to solve for y

  $\int y' dx = \int \frac{\frac{9x^2}{2} + c_1}{x} dx + c_2$
- Compute integrals

  $y = c_1 \ln(x) + c_2 + \frac{9x^2}{4}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = -(-9*_a^2+_b(_a))/_a, _b(_a)`    *** Sub
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve(diff(y(x),x$2)+1/x*diff(y(x),x)=9*x,y(x), singsol=all)
```

$$y(x) = x^3 + \ln(x) c_1 + c_2$$

✓ Solution by Mathematica

Time used: 0.031 (sec). Leaf size: 16

```
DSolve[y''[x]+1/x*y'[x]==9*x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^3 + c_1 \log(x) + c_2$$

3.23 problem Problem 30

Internal problem ID [2661]
Internal file name [`OUTPUT/2153_Sunday_June_05_2022_02_50_40_AM_28462835/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.6, First-Order Linear Differential Equations. page 59
Problem number: Problem 30.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' + \frac{y}{x} = \cos(x)}$$

3.23.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{1}{x}$$
$$q(x) = \cos(x)$$

Hence the ode is

$$y' + \frac{y}{x} = \cos(x)$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{1}{x}dx} \\ &= x\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)(\cos(x)) \\ \frac{d}{dx}(xy) &= (x)(\cos(x)) \\ d(xy) &= (\cos(x)\,x)\;dx\end{aligned}$$

Integrating gives

$$\begin{aligned}xy &= \int \cos(x)\,x\,dx \\ xy &= x\sin(x) + \cos(x) + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x$ results in

$$y = \frac{x\sin(x) + \cos(x)}{x} + \frac{c_1}{x}$$

which simplifies to

$$y = \frac{x\sin(x) + \cos(x) + c_1}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{x\sin(x) + \cos(x) + c_1}{x} \tag{1}$$

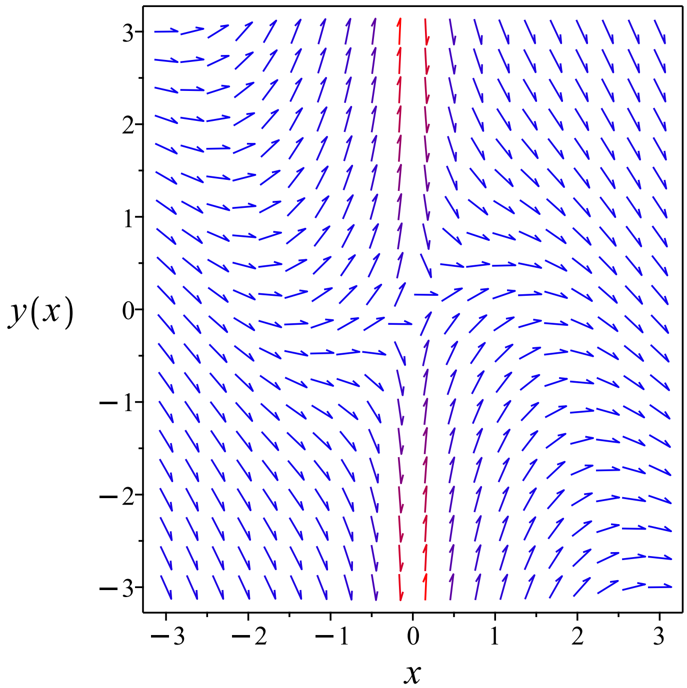


Figure 108: Slope field plot

Verification of solutions

$$y = \frac{x \sin(x) + \cos(x) + c_1}{x}$$

Verified OK.

3.23.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{-y + \cos(x)\, x}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 111: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{x}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{x}} dy \end{aligned}$$

Which results in

$$S = xy$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{-y + \cos(x) x}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= y \\ S_y &= x \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \cos(x) x \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \cos(R) R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \cos(R) + \sin(R) R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$yx = x \sin(x) + \cos(x) + c_1$$

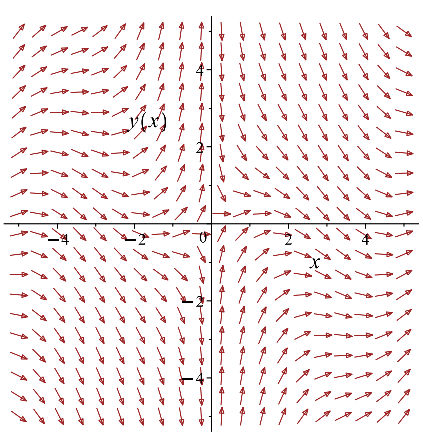
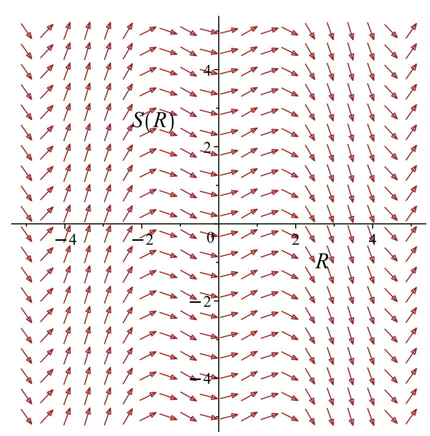
Which simplifies to

$$yx = x \sin(x) + \cos(x) + c_1$$

Which gives

$$y = \frac{x \sin(x) + \cos(x) + c_1}{x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{-y+\cos(x)x}{x}$ 	$R = x$ $S = xy$	$\frac{dS}{dR} = \cos(R) R$ 

Summary

The solution(s) found are the following

$$y = \frac{x \sin(x) + \cos(x) + c_1}{x} \tag{1}$$

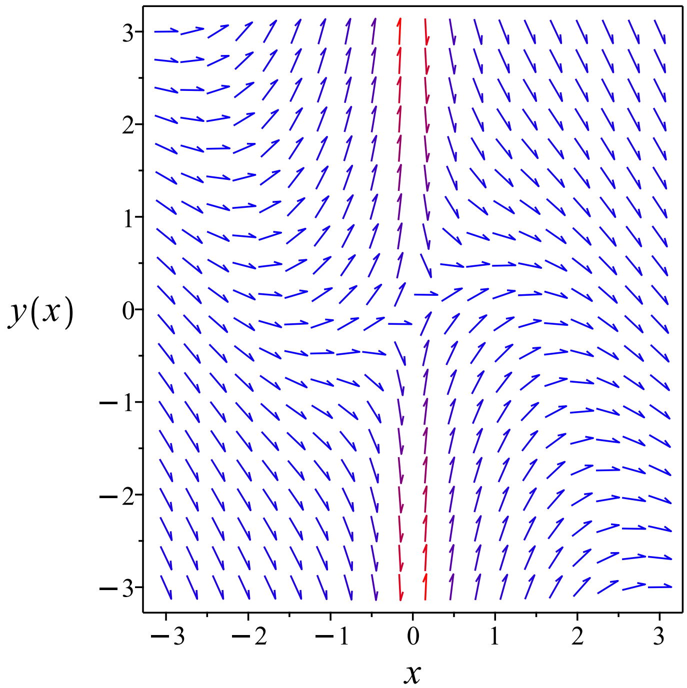


Figure 109: Slope field plot

Verification of solutions

$$y = \frac{x \sin(x) + \cos(x) + c_1}{x}$$

Verified OK.

3.23.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x)\,\mathrm{d}y &= (-y + \cos(x)\,x)\,\mathrm{d}x\\ (y - \cos(x)\,x)\,\mathrm{d}x + (x)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= y - \cos(x)\,x\\ N(x, y) &= x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y - \cos(x)\,x)\\ &= 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x)\\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int M \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int y - \cos(x) x \mathrm{d}x$$

$$\phi = xy - \cos(x) - x \sin(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x$. Therefore equation (4) becomes

$$x = x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = xy - \cos(x) - x \sin(x) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = xy - \cos(x) - x \sin(x)$$

The solution becomes

$$y = \frac{x \sin(x) + \cos(x) + c_1}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{x \sin(x) + \cos(x) + c_1}{x} \tag{1}$$

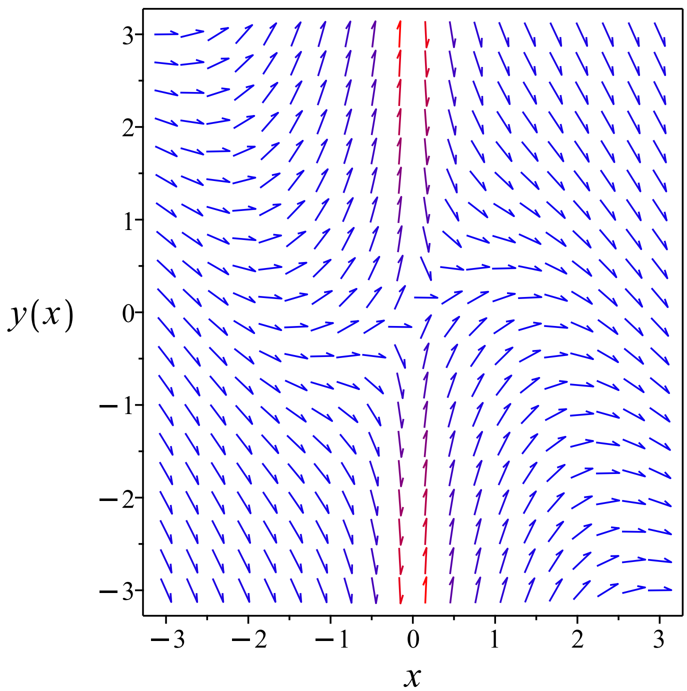


Figure 110: Slope field plot

Verification of solutions

$$y = \frac{x \sin(x) + \cos(x) + c_1}{x}$$

Verified OK.

3.23.4 Maple step by step solution

Let's solve

$y' + \frac{y}{x} = \cos(x)$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{y}{x} + \cos(x)$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{y}{x} = \cos(x)$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{y}{x}\right) = \mu(x)\cos(x)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' + \frac{y}{x}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = x$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x)\cos(x)\, dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \mu(x)\cos(x)\, dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x)\cos(x)dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = x$

  $y = \frac{\int \cos(x)x dx + c_1}{x}$

- Evaluate the integrals on the rhs

  $y = \frac{x\sin(x) + \cos(x) + c_1}{x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(diff(y(x),x)+1/x*y(x)=cos(x),y(x), singsol=all)
```

$$y(x) = \frac{x \sin(x) + \cos(x) + c_1}{x}$$

✓ Solution by Mathematica

Time used: 0.035 (sec). Leaf size: 18

```
DSolve[y'[x]+1/x*y[x]==Cos[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x \sin(x) + \cos(x) + c_1}{x}$$

3.24 problem Problem 31

Internal problem ID [2662]
Internal file name [`OUTPUT/2154_Sunday_June_05_2022_02_50_42_AM_84646342/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.6, First-Order Linear Differential Equations. page 59
Problem number: Problem 31.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' + y = e^{-2x}}$$

3.24.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= 1 \\ q(x) &= e^{-2x} \end{aligned}$$

Hence the ode is

$$y' + y = e^{-2x}$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int 1dx} \\
&= \mathrm{e}^{x}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\mathrm{e}^{-2x}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{x} y) &= (\mathrm{e}^{x})\left(\mathrm{e}^{-2x}\right) \\
\mathrm{d}(\mathrm{e}^{x} y) &= \mathrm{e}^{-x}\,\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\mathrm{e}^{x} y &= \int \mathrm{e}^{-x}\,\mathrm{d}x \\
\mathrm{e}^{x} y &= -\mathrm{e}^{-x} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{x}$ results in

$$
y = -\mathrm{e}^{-2x} + c_1 \mathrm{e}^{-x}
$$

which simplifies to

$$
y = -\mathrm{e}^{-2x} + c_1 \mathrm{e}^{-x}
$$

Summary

The solution(s) found are the following

$$
y = -\mathrm{e}^{-2x} + c_1 \mathrm{e}^{-x} \tag{1}
$$

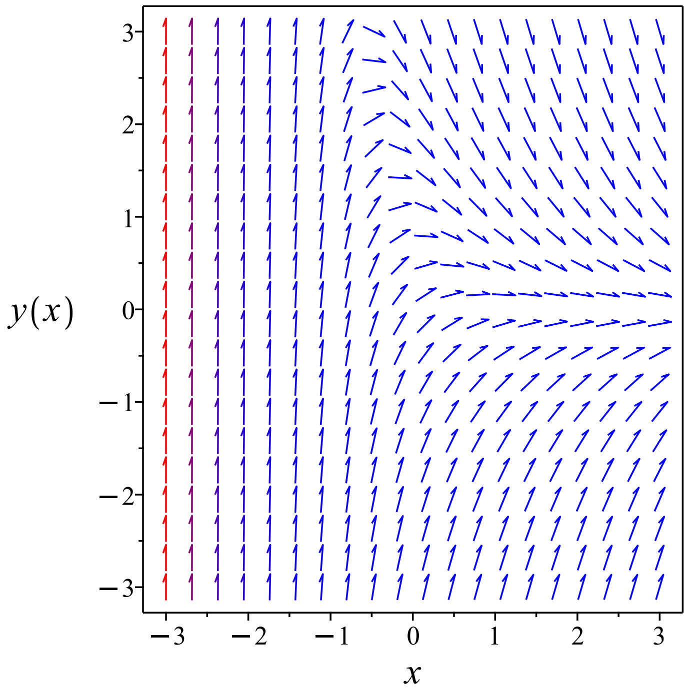


Figure 111: Slope field plot

Verification of solutions

$$y = -e^{-2x} + c_1 e^{-x}$$

Verified OK.

3.24.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -y + e^{-2x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 114: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \mathrm{e}^{-x}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{-x}} dy \end{aligned}$$

Which results in

$$S = \mathrm{e}^{x} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -y + \mathrm{e}^{-2x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \mathrm{e}^{x} y \\ S_y &= \mathrm{e}^{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \mathrm{e}^{-x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \mathrm{e}^{-R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\mathrm{e}^{-R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\mathrm{e}^{x} y = -\mathrm{e}^{-x} + c_1$$

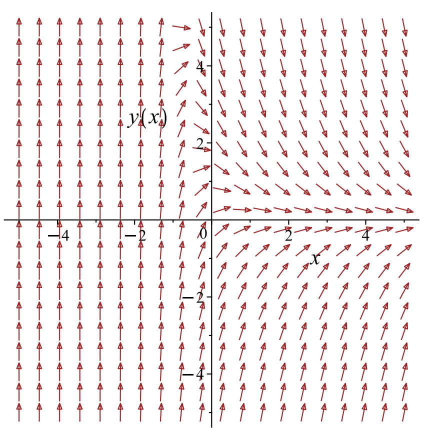
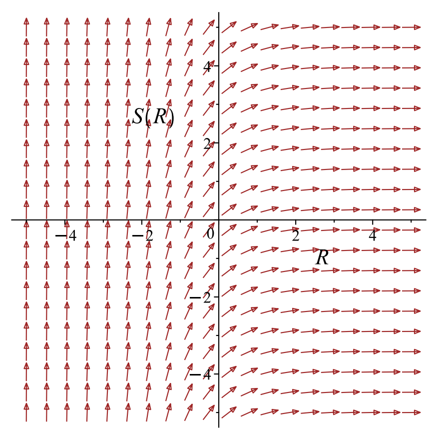
Which simplifies to

$$\mathrm{e}^{x} y = -\mathrm{e}^{-x} + c_1$$

Which gives

$$y = -\mathrm{e}^{-x} \left(\mathrm{e}^{-x} - c_1\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -y + \mathrm{e}^{-2x}$ 	$R = x$ $S = \mathrm{e}^{x} y$	$\frac{dS}{dR} = \mathrm{e}^{-R}$ 

Summary

The solution(s) found are the following

$$y = -\mathrm{e}^{-x} \left(\mathrm{e}^{-x} - c_1\right) \tag{1}$$

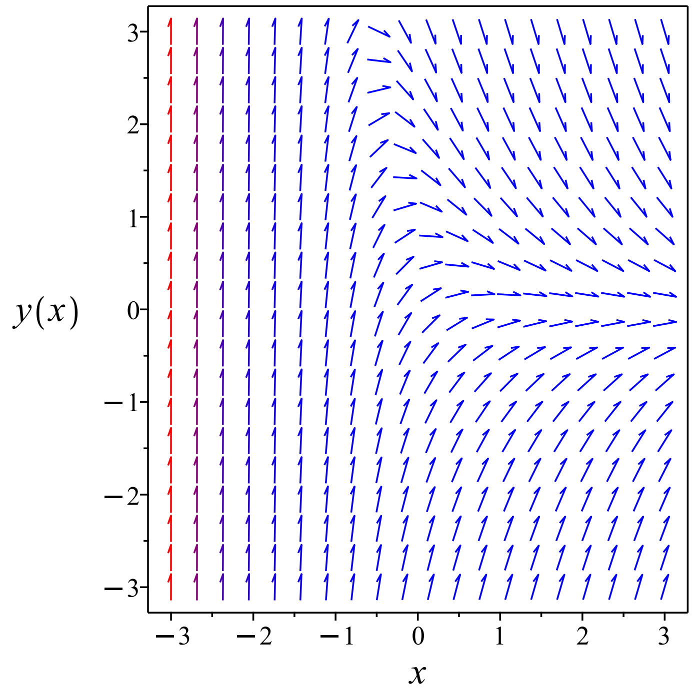


Figure 112: Slope field plot

Verification of solutions

$$y = -e^{-x}\left(e^{-x} - c_1\right)$$

Verified OK.

3.24.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\mathrm{d}y &= \left(-y + \mathrm{e}^{-2x}\right) \mathrm{d}x \\
\left(y - \mathrm{e}^{-2x}\right) \mathrm{d}x + \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= y - \mathrm{e}^{-2x} \\
N(x,y) &= 1
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(y - \mathrm{e}^{-2x}\right) \\
&= 1
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1((1) - (0)) \\
&= 1
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int 1 \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{x} \\
&= \mathrm{e}^{x}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{x}\left(y - \mathrm{e}^{-2x}\right) \\
&= \mathrm{e}^{x} y - \mathrm{e}^{-x}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{x}(1) \\
&= \mathrm{e}^{x}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\mathrm{e}^{x} y - \mathrm{e}^{-x}\right) + \left(\mathrm{e}^{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \mathrm{e}^{x} y - \mathrm{e}^{-x} \, \mathrm{d}x$$
$$\phi = \mathrm{e}^{x} y + \mathrm{e}^{-x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{x}$. Therefore equation (4) becomes

$$\mathrm{e}^{x} = \mathrm{e}^{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \mathrm{e}^{x} y + \mathrm{e}^{-x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \mathrm{e}^{x} y + \mathrm{e}^{-x}$$

The solution becomes

$$y = -\mathrm{e}^{-x} \left(\mathrm{e}^{-x} - c_1\right)$$

<u>Summary</u>
The solution(s) found are the following

$$y = -\mathrm{e}^{-x} \left(\mathrm{e}^{-x} - c_1\right) \tag{1}$$

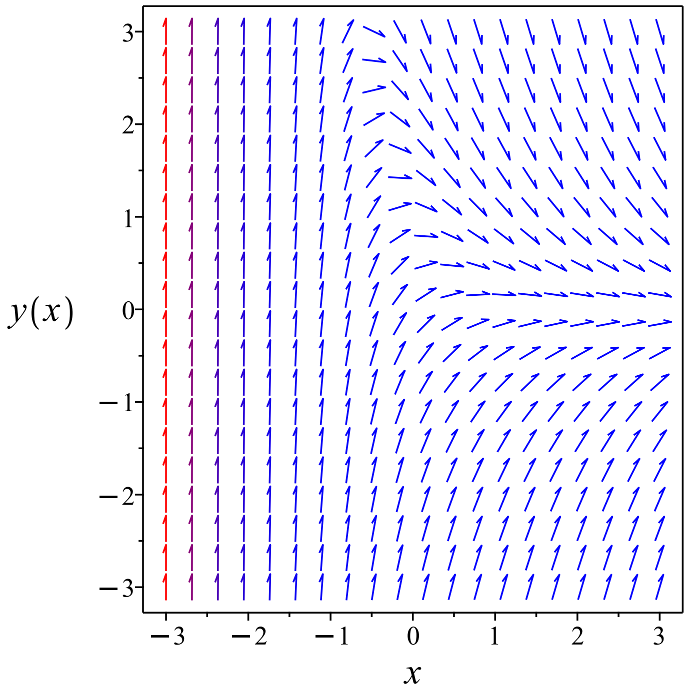


Figure 113: Slope field plot

Verification of solutions

$$y = -\mathrm{e}^{-x}\left(\mathrm{e}^{-x} - c_1\right)$$

Verified OK.

3.24.4 Maple step by step solution

Let's solve

$y' + y = \mathrm{e}^{-2x}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -y + \mathrm{e}^{-2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + y = \mathrm{e}^{-2x}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + y\right) = \mu(x)\,\mathrm{e}^{-2x}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\,(y'+y) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \mu(x)$

- Solve to find the integrating factor

  $\mu(x) = \mathrm{e}^{x}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x)\, \mathrm{e}^{-2x} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \mu(x)\, \mathrm{e}^{-2x} dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x)\mathrm{e}^{-2x} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \mathrm{e}^{x}$

  $y = \frac{\int \mathrm{e}^{-2x}\mathrm{e}^{x} dx + c_1}{\mathrm{e}^{x}}$

- Evaluate the integrals on the rhs

  $y = \frac{-\mathrm{e}^{-x} + c_1}{\mathrm{e}^{x}}$

- Simplify

  $y = (-\mathrm{e}^{-x} + c_1)\, \mathrm{e}^{-x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(y(x),x)+y(x)=exp(-2*x),y(x), singsol=all)
```

$$y(x) = \left(-\mathrm{e}^{-x} + c_1\right) \mathrm{e}^{-x}$$

✓ Solution by Mathematica

Time used: 0.055 (sec). Leaf size: 19

```
DSolve[y'[x]+y[x]==Exp[-2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-2x}(-1 + c_1 e^x)$$

3.25 problem Problem 32

Internal problem ID [2663]
Internal file name [`OUTPUT/2155_Sunday_June_05_2022_02_50_44_AM_2281112/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.6, First-Order Linear Differential Equations. page 59
Problem number: Problem 32.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' + y\cot(x) = 2\cos(x)}$$

3.25.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= \cot(x) \\ q(x) &= 2\cos(x) \end{aligned}$$

Hence the ode is

$$y' + y\cot(x) = 2\cos(x)$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \cot(x)dx} \\ &= \sin(x)\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)(2\cos(x)) \\ \frac{d}{dx}(\sin(x)\, y) &= (\sin(x))(2\cos(x)) \\ d(\sin(x)\, y) &= \sin(2x)\, dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\sin(x)\, y &= \int \sin(2x)\, dx \\ \sin(x)\, y &= -\frac{\cos(2x)}{2} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \sin(x)$ results in

$$y = -\frac{\csc(x)\cos(2x)}{2} + c_1 \csc(x)$$

which simplifies to

$$y = \csc(x)\left(-\cos(x)^2 + c_1 + \frac{1}{2}\right)$$

Summary

The solution(s) found are the following

$$y = \csc(x)\left(-\cos(x)^2 + c_1 + \frac{1}{2}\right) \tag{1}$$

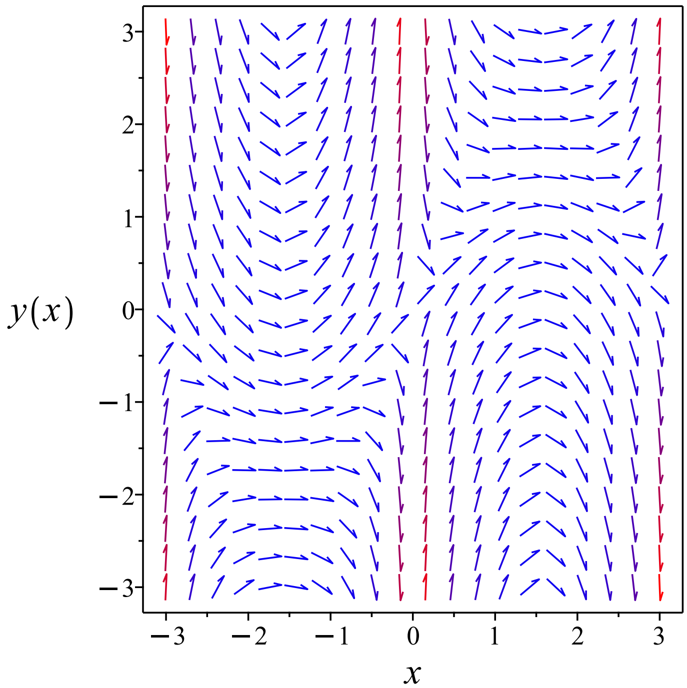


Figure 114: Slope field plot

Verification of solutions

$$y = \csc(x)\left(-\cos(x)^2 + c_1 + \frac{1}{2}\right)$$

Verified OK.

3.25.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\cot(x)\,y + 2\cos(x)$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 117: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{\sin(x)}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{\sin(x)}} dy \end{aligned}$$

Which results in

$$S = \sin(x) y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\cot(x) y + 2\cos(x)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \cos(x) y \\ S_y &= \sin(x) \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \sin(2x) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \sin(2R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{\cos(2R)}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$y\sin(x) = -\frac{\cos(2x)}{2} + c_1$$

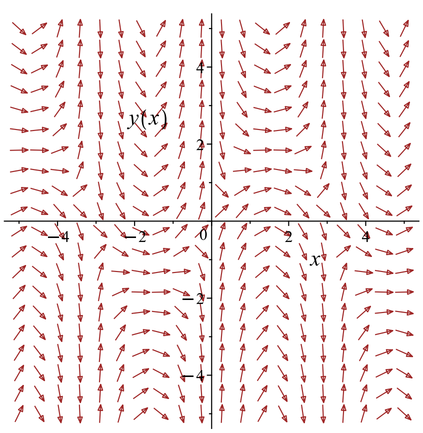
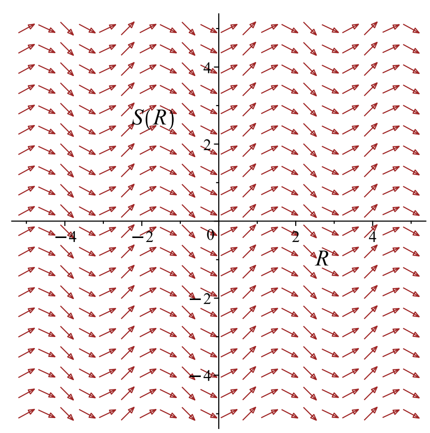
Which simplifies to

$$y\sin(x) = -\frac{\cos(2x)}{2} + c_1$$

Which gives

$$y = -\frac{\cos(2x) - 2c_1}{2\sin(x)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = -\cot(x)\, y + 2\cos(x)$

</td><td>$R = x$
$S = \sin(x)\, y$</td><td>$\frac{dS}{dR} = \sin(2R)$

</td></tr>
</table>

Summary

The solution(s) found are the following

$$y = -\frac{\cos(2x) - 2c_1}{2\sin(x)} \tag{1}$$

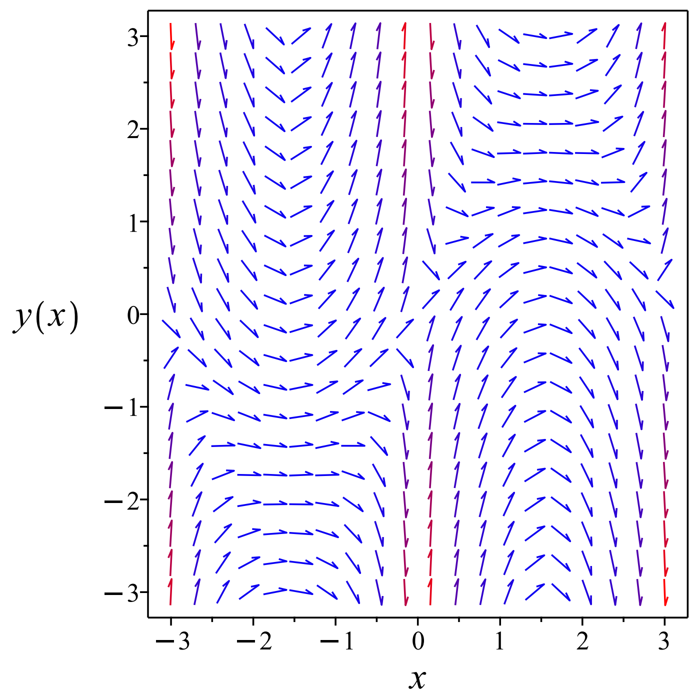


Figure 115: Slope field plot

<u>Verification of solutions</u>

$$
y = -\frac{\cos(2x) - 2c_1}{2\sin(x)}
$$

Verified OK.

3.25.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$
M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}
$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$
\frac{d}{dx}\phi(x,y) = 0
$$

Hence

$$
\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}
$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = (-\cot(x)\, y + 2\cos(x))\, \mathrm{d}x$$
$$(\cot(x)\, y - 2\cos(x))\, \mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = \cot(x)\, y - 2\cos(x)$$
$$N(x, y) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(\cot(x)\, y - 2\cos(x)) \\ &= \cot(x) \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\ &= 0 \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1((\cot(x)) - (0)) \\ &= \cot(x)\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int \cot(x) \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{\ln(\sin(x))} \\ &= \sin(x)\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \sin(x)(\cot(x)\, y - 2\cos(x)) \\ &= \cos(x)(-2\sin(x) + y)\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \sin(x)(1) \\ &= \sin(x)\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ (\cos(x)(-2\sin(x) + y)) + (\sin(x))\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int \overline{M} \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int \cos(x)\,(-2\sin(x)+y)\,\mathrm{d}x \\
\phi &= \sin(x)\,(y-\sin(x)) + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \sin(x) + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \sin(x)$. Therefore equation (4) becomes

$$
\sin(x) = \sin(x) + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$
f(y) = c_1
$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$
\phi = \sin(x)\,(y-\sin(x)) + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
c_1 = \sin(x)\,(y-\sin(x))
$$

The solution becomes

$$
y = \frac{\sin(x)^2 + c_1}{\sin(x)}
$$

Summary
The solution(s) found are the following

$$y = \frac{\sin(x)^2 + c_1}{\sin(x)} \tag{1}$$

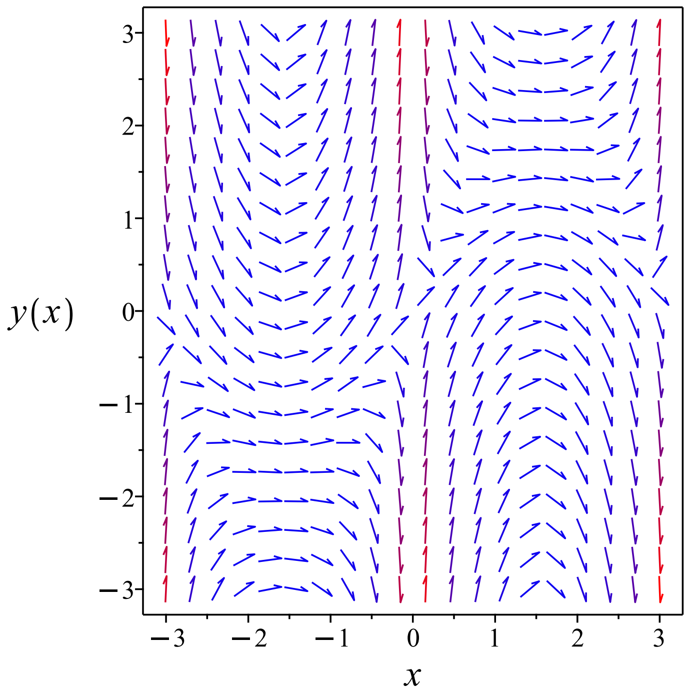


Figure 116: Slope field plot

Verification of solutions

$$y = \frac{\sin(x)^2 + c_1}{\sin(x)}$$

Verified OK.

3.25.4 Maple step by step solution

Let's solve

$y' + y \cot(x) = 2\cos(x)$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

$$y' = -y \cot(x) + 2\cos(x)$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

$$y' + y \cot(x) = 2\cos(x)$$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$$\mu(x) (y' + y \cot(x)) = 2\mu(x) \cos(x)$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

$$\mu(x) (y' + y \cot(x)) = \mu'(x)\, y + \mu(x)\, y'$$

- Isolate $\mu'(x)$

$$\mu'(x) = \mu(x) \cot(x)$$

- Solve to find the integrating factor

$$\mu(x) = \sin(x)$$

- Integrate both sides with respect to x

$$\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int 2\mu(x) \cos(x)\, dx + c_1$$

- Evaluate the integral on the lhs

$$\mu(x)\, y = \int 2\mu(x) \cos(x)\, dx + c_1$$

- Solve for y

$$y = \frac{\int 2\mu(x) \cos(x) dx + c_1}{\mu(x)}$$

- Substitute $\mu(x) = \sin(x)$

$$y = \frac{\int 2\sin(x) \cos(x) dx + c_1}{\sin(x)}$$

- Evaluate the integrals on the rhs

$$y = \frac{\sin(x)^2 + c_1}{\sin(x)}$$

- Simplify

$$y = \sin(x) + c_1 \csc(x)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(diff(y(x),x)+y(x)*cot(x)=2*cos(x),y(x), singsol=all)
```

$$y(x) = \csc(x)\left(-\cos(x)^2 + c_1 + \frac{1}{2}\right)$$

✓ Solution by Mathematica

Time used: 0.043 (sec). Leaf size: 20

```
DSolve[y'[x]+y[x]*Cot[x]==2*Cos[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{2}\csc(x)(\cos(2x) - 2c_1)$$

3.26 problem Problem 33

Internal problem ID [2664]
Internal file name [`OUTPUT/2156_Sunday_June_05_2022_02_50_47_AM_45393673/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.6, First-Order Linear Differential Equations. page 59
Problem number: Problem 33.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{xy' - y = \ln (x)\, x^2}$$

3.26.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{1}{x}$$
$$q(x) = \ln (x)\, x$$

Hence the ode is

$$y' - \frac{y}{x} = \ln (x)\, x$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int -\frac{1}{x}dx} \\
&= \frac{1}{x}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)(\ln(x)\, x) \\
\frac{d}{dx}\left(\frac{y}{x}\right) &= \left(\frac{1}{x}\right)(\ln(x)\, x) \\
d\left(\frac{y}{x}\right) &= \ln(x)\, dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{y}{x} &= \int \ln(x)\, dx \\
\frac{y}{x} &= \ln(x)\, x - x + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$
y = x(\ln(x)\, x - x) + c_1 x
$$

which simplifies to

$$
y = x(\ln(x)\, x + c_1 - x)
$$

Summary

The solution(s) found are the following

$$
y = x(\ln(x)\, x + c_1 - x) \tag{1}
$$

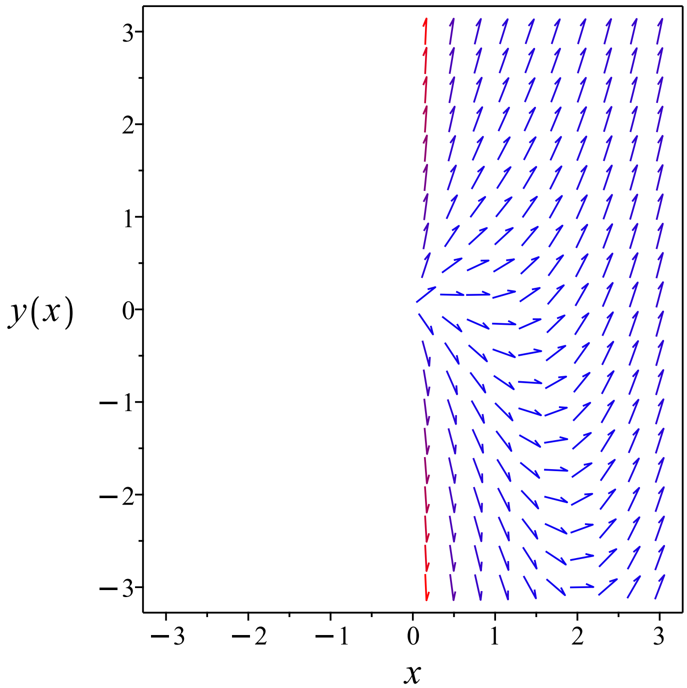


Figure 117: Slope field plot

Verification of solutions

$$y = x(\ln(x)x + c_1 - x)$$

Verified OK.

3.26.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)x$ on the above ode results in new ode in $u(x)$

$$x(u'(x)x + u(x)) - u(x)x = \ln(x)x^2$$

Integrating both sides gives

$$\begin{aligned} u(x) &= \int \ln(x)\,dx \\ &= \ln(x)x - x + c_2 \end{aligned}$$

Therefore the solution y is

$$\begin{aligned} y &= ux \\ &= x(\ln(x)x - x + c_2) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = x(\ln(x)\, x - x + c_2) \tag{1}$$

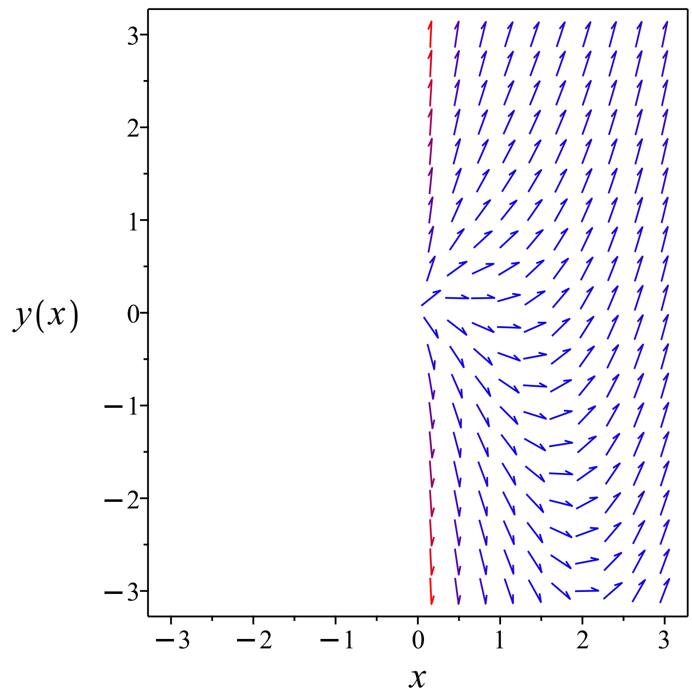


Figure 118: Slope field plot

Verification of solutions

$$y = x(\ln(x)\, x - x + c_2)$$

Verified OK.

3.26.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y + \ln(x)\, x^2}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 120: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= x
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x} dy \end{aligned}$$

Which results in

$$S = \frac{y}{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{y + \ln(x) x^2}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y}{x^2} \\ S_y &= \frac{1}{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \ln(x) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \ln(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = R\ln(R) - R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{x} = \ln(x)\, x + c_1 - x$$

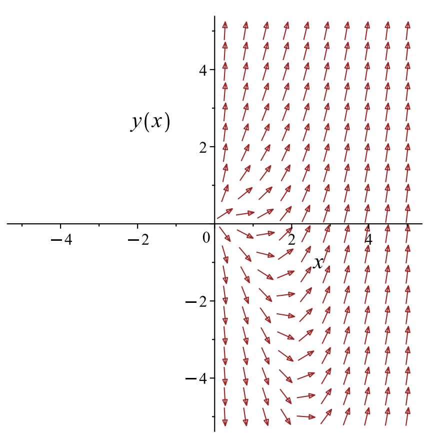
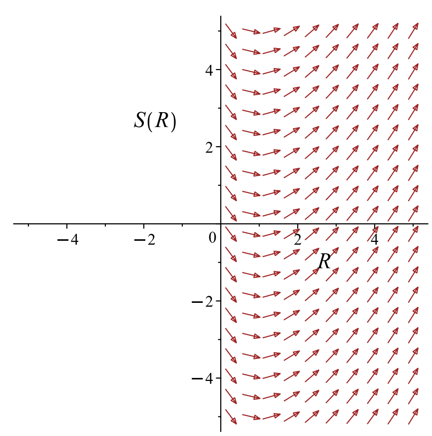
Which simplifies to

$$\frac{y}{x} = \ln(x)\, x + c_1 - x$$

Which gives

$$y = x(\ln(x)\, x + c_1 - x)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y+\ln(x)x^2}{x}$ 	$R = x$ $S = \frac{y}{x}$	$\frac{dS}{dR} = \ln(R)$ 

<u>Summary</u>
The solution(s) found are the following

$$y = x(\ln(x)\, x + c_1 - x) \tag{1}$$

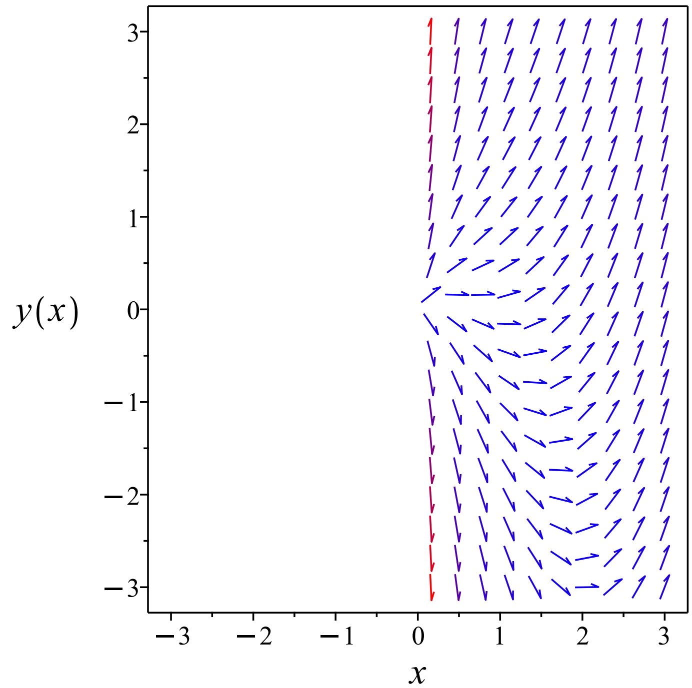


Figure 119: Slope field plot

Verification of solutions

$$y = x(\ln (x) x + c_1 - x)$$

Verified OK.

3.26.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
(x)\, \mathrm{d}y &= \left(y + \ln(x)\, x^2\right) \mathrm{d}x \\
\left(-y - \ln(x)\, x^2\right) \mathrm{d}x + (x)\, \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -y - \ln(x)\, x^2 \\
N(x, y) &= x
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-y - \ln(x)\, x^2\right) \\
&= -1
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x) \\
&= 1
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{x}((-1) - (1)) \\
&= -\frac{2}{x}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int -\frac{2}{x} \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-2\ln(x)} \\
&= \frac{1}{x^2}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{x^2}\left(-y - \ln(x)\, x^2\right) \\
&= \frac{-y - \ln(x)\, x^2}{x^2}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^2}(x) \\
&= \frac{1}{x}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{-y - \ln(x)\, x^2}{x^2}\right) + \left(\frac{1}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$

$$\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int \frac{-y - \ln(x)\,x^2}{x^2}\,\mathrm{d}x$$

$$\phi = -\ln(x)\,x + x + \frac{y}{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial\phi}{\partial y} = \frac{1}{x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = \frac{1}{x}$. Therefore equation (4) becomes

$$\frac{1}{x} = \frac{1}{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x)\,x + x + \frac{y}{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x)\,x + x + \frac{y}{x}$$

The solution becomes

$$y = x(\ln(x)\, x + c_1 - x)$$

Summary

The solution(s) found are the following

$$y = x(\ln(x)\, x + c_1 - x) \tag{1}$$

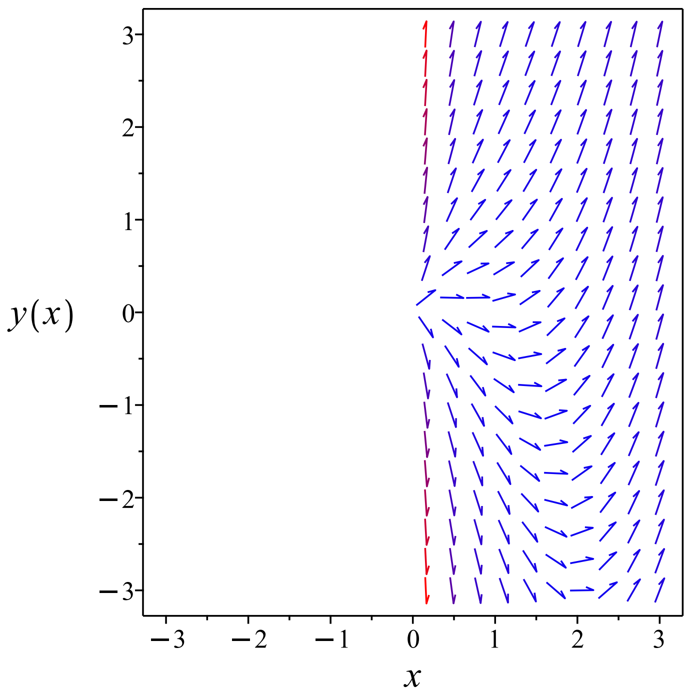


Figure 120: Slope field plot

Verification of solutions

$$y = x(\ln(x)\, x + c_1 - x)$$

Verified OK.

3.26.5 Maple step by step solution

Let's solve

$x y' - y = \ln(x)\, x^2$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{y}{x} + \ln(x)\, x$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{y}{x} = \ln(x)\, x$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - \frac{y}{x}\right) = \mu(x) \ln(x)\, x$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' - \frac{y}{x}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\frac{\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = \frac{1}{x}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x) \ln(x)\, x dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \mu(x) \ln(x)\, x dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x) \ln(x) x dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{1}{x}$

  $y = x\left(\int \ln(x)\, dx + c_1\right)$

- Evaluate the integrals on the rhs

  $y = x(\ln(x)\, x + c_1 - x)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(x*diff(y(x),x)-y(x)=x^2*ln(x),y(x), singsol=all)
```

$$y(x) = (x \ln (x) - x + c_1) x$$

✓ Solution by Mathematica

Time used: 0.03 (sec). Leaf size: 17

```
DSolve[x*y'[x]-y[x]==x^2*Log[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x(-x + x \log(x) + c_1)$$

4 Chapter 1, First-Order Differential Equations. Section 1.8, Change of Variables. page 79

4.1 problem Problem 9

Internal problem ID [2665]
Internal file name [`OUTPUT/2157_Sunday_June_05_2022_02_50_49_AM_69742392/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.8, Change of Variables. page 79
Problem number: Problem 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Riccati]
```

$$\boxed{y' - \frac{y^2 + yx + x^2}{x^2} = 0}$$

4.1.1 Solving as homogeneous ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{x^2 + xy + y^2}{x^2} \end{aligned} \tag{1}$$

An ode of the form $y' = \frac{M(x,y)}{N(x,y)}$ is called homogeneous if the functions $M(x,y)$ and $N(x,y)$ are both homogeneous functions and of the same order. Recall that a function $f(x,y)$ is homogeneous of order n if

$$f(t^n x, t^n y) = t^n f(x,y)$$

In this case, it can be seen that both $M = x^2+xy+y^2$ and $N = x^2$ are both homogeneous and of the same order $n = 2$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{y}{x}$, or $y = ux$. Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x}x + u$$

Applying the transformation $y = ux$ to the above ODE in (1) gives

$$\begin{aligned}\frac{du}{dx}x + u &= u^2 + u + 1\\ \frac{du}{dx} &= \frac{u(x)^2 + 1}{x}\end{aligned}$$

Or

$$u'(x) - \frac{u(x)^2 + 1}{x} = 0$$

Or

$$u'(x)\,x - u(x)^2 - 1 = 0$$

Which is now solved as separable in $u(x)$. Which is now solved in $u(x)$. In canonical form the ODE is

$$\begin{aligned}u' &= F(x,u)\\ &= f(x)g(u)\\ &= \frac{u^2+1}{x}\end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = u^2 + 1$. Integrating both sides gives

$$\begin{aligned}\frac{1}{u^2+1}\,du &= \frac{1}{x}\,dx\\ \int \frac{1}{u^2+1}\,du &= \int \frac{1}{x}\,dx\\ \arctan(u) &= \ln(x) + c_2\end{aligned}$$

The solution is

$$\arctan(u(x)) - \ln(x) - c_2 = 0$$

Now u in the above solution is replaced back by y using $u = \frac{y}{x}$ which results in the solution

$$\arctan\left(\frac{y}{x}\right) - \ln(x) - c_2 = 0$$

Summary

The solution(s) found are the following

$$\arctan\left(\frac{y}{x}\right) - \ln(x) - c_2 = 0 \tag{1}$$

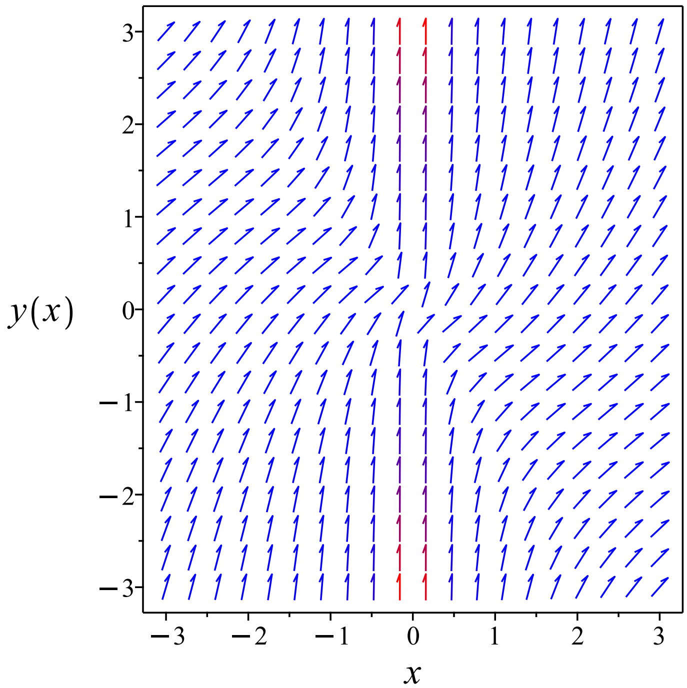


Figure 121: Slope field plot

Verification of solutions

$$\arctan\left(\frac{y}{x}\right) - \ln(x) - c_2 = 0$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 11

```
dsolve(diff(y(x),x)=(y(x)^2+x*y(x)+x^2)/x^2,y(x), singsol=all)
```

$$y(x) = \tan\left(\ln\left(x\right) + c_1\right) x$$

✓ Solution by Mathematica

Time used: 0.198 (sec). Leaf size: 13

```
DSolve[y'[x]==(y[x]^2+x*y[x]+x^2)/x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x \tan(\log(x) + c_1)$$

4.2 problem Problem 10

Internal problem ID [2666]
Internal file name [`OUTPUT/2158_Sunday_June_05_2022_02_50_52_AM_39559977/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.8, Change of Variables. page 79
Problem number: Problem 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{(3x-y)\,y'-3y=0}$$

4.2.1 Solving as homogeneous ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{3y}{-3x+y} \end{aligned} \tag{1}$$

An ode of the form $y'=\frac{M(x,y)}{N(x,y)}$ is called homogeneous if the functions $M(x,y)$ and $N(x,y)$ are both homogeneous functions and of the same order. Recall that a function $f(x,y)$ is homogeneous of order n if

$$f(t^n x, t^n y) = t^n f(x,y)$$

In this case, it can be seen that both $M=3y$ and $N=3x-y$ are both homogeneous and of the same order $n=1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u=\frac{y}{x}$, or $y=ux$. Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x}x + u$$

Applying the transformation $y = ux$ to the above ODE in (1) gives

$$
\begin{aligned}
\frac{du}{dx}x + u &= -\frac{3u}{u-3} \\
\frac{du}{dx} &= \frac{-\frac{3u(x)}{u(x)-3} - u(x)}{x}
\end{aligned}
$$

Or

$$
u'(x) - \frac{-\frac{3u(x)}{u(x)-3} - u(x)}{x} = 0
$$

Or

$$
u'(x)\, xu(x) - 3u'(x)\, x + u(x)^2 = 0
$$

Or

$$
x(u(x) - 3)\, u'(x) + u(x)^2 = 0
$$

Which is now solved as separable in $u(x)$. Which is now solved in $u(x)$. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x, u) \\
&= f(x)g(u) \\
&= -\frac{u^2}{x\,(u-3)}
\end{aligned}
$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u^2}{u-3}$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\frac{u^2}{u-3}}\, du &= -\frac{1}{x}\, dx \\
\int \frac{1}{\frac{u^2}{u-3}}\, du &= \int -\frac{1}{x}\, dx \\
\frac{3}{u} + \ln(u) &= -\ln(x) + c_2
\end{aligned}
$$

The solution is

$$
\frac{3}{u(x)} + \ln(u(x)) + \ln(x) - c_2 = 0
$$

Now u in the above solution is replaced back by y using $u = \frac{y}{x}$ which results in the solution

$$
\frac{3x}{y} + \ln\left(\frac{y}{x}\right) + \ln(x) - c_2 = 0
$$

Summary
The solution(s) found are the following

$$\frac{3x}{y}+\ln\left(\frac{y}{x}\right)+\ln(x)-c_2=0 \tag{1}$$

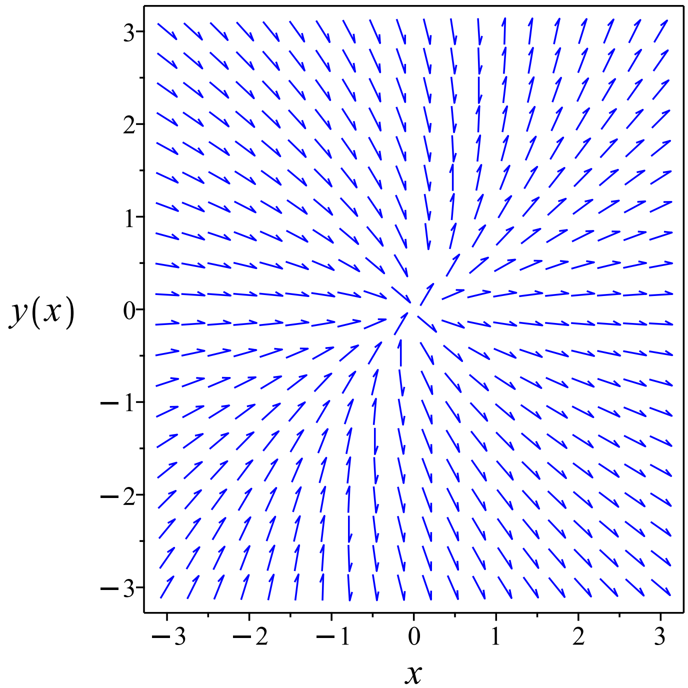


Figure 122: Slope field plot

Verification of solutions

$$\frac{3x}{y}+\ln\left(\frac{y}{x}\right)+\ln(x)-c_2=0$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 17

```
dsolve((3*x-y(x))*diff(y(x),x)=3*y(x),y(x), singsol=all)
```

$$y(x) = -\frac{3x}{\operatorname{LambertW}\left(-3x\,\mathrm{e}^{-3c_1}\right)}$$

✓ Solution by Mathematica

Time used: 6.103 (sec). Leaf size: 25

```
DSolve[(3*x-y[x])*y'[x]==3*y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{3x}{W\left(-3e^{-c_1}x\right)}$$
$$y(x) \to 0$$

4.3 problem Problem 11

Internal problem ID [2667]
Internal file name [`OUTPUT/2159_Sunday_June_05_2022_02_50_55_AM_84826546/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.8, Change of Variables. page 79
Problem number: Problem 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Riccati]
```

$$\boxed{y' - \frac{(y+x)^2}{2x^2} = 0}$$

4.3.1 Solving as homogeneous ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{(y+x)^2}{2x^2} \end{aligned} \tag{1}$$

An ode of the form $y' = \frac{M(x,y)}{N(x,y)}$ is called homogeneous if the functions $M(x,y)$ and $N(x,y)$ are both homogeneous functions and of the same order. Recall that a function $f(x,y)$ is homogeneous of order n if

$$f(t^n x, t^n y) = t^n f(x,y)$$

In this case, it can be seen that both $M = (y+x)^2$ and $N = 2x^2$ are both homogeneous and of the same order $n = 2$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{y}{x}$, or $y = ux$. Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x} x + u$$

Applying the transformation $y = ux$ to the above ODE in (1) gives

$$\begin{aligned}\frac{du}{dx}x + u &= \frac{1}{2} + u + \frac{1}{2}u^2 \\ \frac{du}{dx} &= \frac{\frac{1}{2} + \frac{u(x)^2}{2}}{x}\end{aligned}$$

Or

$$u'(x) - \frac{\frac{1}{2} + \frac{u(x)^2}{2}}{x} = 0$$

Or

$$2u'(x)\,x - u(x)^2 - 1 = 0$$

Which is now solved as separable in $u(x)$. Which is now solved in $u(x)$. In canonical form the ODE is

$$\begin{aligned}u' &= F(x, u) \\ &= f(x)g(u) \\ &= \frac{\frac{u^2}{2} + \frac{1}{2}}{x}\end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = \frac{u^2}{2} + \frac{1}{2}$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\frac{u^2}{2} + \frac{1}{2}}\,du &= \frac{1}{x}\,dx \\ \int \frac{1}{\frac{u^2}{2} + \frac{1}{2}}\,du &= \int \frac{1}{x}\,dx \\ 2\arctan(u) &= \ln(x) + c_2\end{aligned}$$

The solution is

$$2\arctan(u(x)) - \ln(x) - c_2 = 0$$

Now u in the above solution is replaced back by y using $u = \frac{y}{x}$ which results in the solution

$$2\arctan\left(\frac{y}{x}\right) - \ln(x) - c_2 = 0$$

<u>Summary</u>

The solution(s) found are the following

$$2\arctan\left(\frac{y}{x}\right) - \ln(x) - c_2 = 0 \tag{1}$$

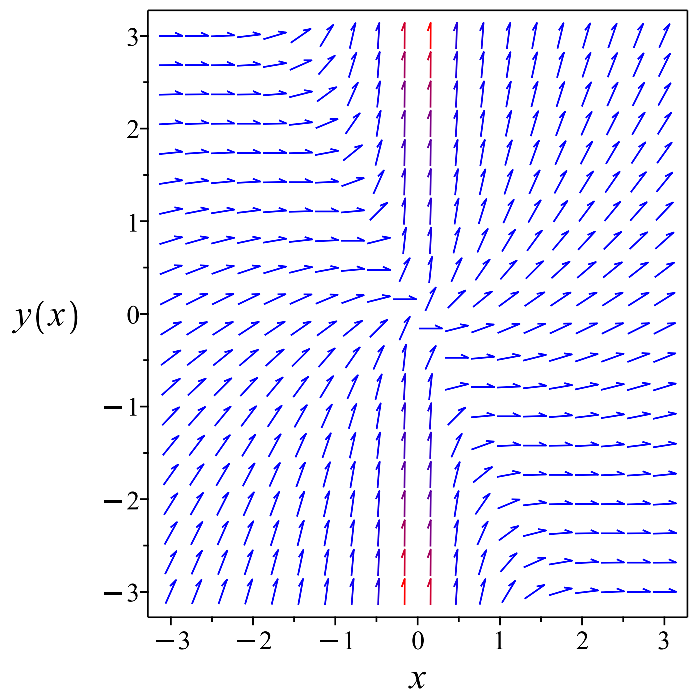


Figure 123: Slope field plot

Verification of solutions

$$2\arctan\left(\frac{y}{x}\right)-\ln\left(x\right)-c_2=0$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(diff(y(x),x)=(x+y(x))^2/(2*x^2),y(x), singsol=all)
```

$$y(x) = \tan\left(\frac{\ln(x)}{2} + \frac{c_1}{2}\right) x$$

✓ Solution by Mathematica

Time used: 0.236 (sec). Leaf size: 17

```
DSolve[y'[x]==(x+y[x])^2/(2*x^2),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x \tan\left(\frac{\log(x)}{2} + c_1\right)$$

4.4 problem Problem 12

Internal problem ID [2668]
Internal file name [`OUTPUT/2160_Sunday_June_05_2022_02_50_57_AM_39886309/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.8, Change of Variables. page 79
Problem number: Problem 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _dAlembert]
```

$$\boxed{\sin\left(\frac{y}{x}\right)(xy'-y)-x\cos\left(\frac{y}{x}\right)=0}$$

4.4.1 Solving as homogeneous ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y)\\ &= \frac{x\cos\left(\frac{y}{x}\right)+\sin\left(\frac{y}{x}\right)y}{\sin\left(\frac{y}{x}\right)x} \end{aligned} \tag{1}$$

An ode of the form $y'=\frac{M(x,y)}{N(x,y)}$ is called homogeneous if the functions $M(x,y)$ and $N(x,y)$ are both homogeneous functions and of the same order. Recall that a function $f(x,y)$ is homogeneous of order n if

$$f(t^n x, t^n y)=t^n f(x,y)$$

In this case, it can be seen that both $M=x\cos\left(\frac{y}{x}\right)+\sin\left(\frac{y}{x}\right)y$ and $N=\sin\left(\frac{y}{x}\right)x$ are both homogeneous and of the same order $n=1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u=\frac{y}{x}$, or $y=ux$. Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}u}{\mathrm{d}x}x+u$$

Applying the transformation $y = ux$ to the above ODE in (1) gives

$$
\begin{aligned}
\frac{du}{dx}x + u &= \frac{\cos(u)}{\sin(u)} + u \\
\frac{du}{dx} &= \frac{\cos(u(x))}{\sin(u(x))\,x}
\end{aligned}
$$

Or

$$
u'(x) - \frac{\cos(u(x))}{\sin(u(x))\,x} = 0
$$

Or

$$
u'(x)\sin(u(x))\,x - \cos(u(x)) = 0
$$

Which is now solved as separable in $u(x)$. Which is now solved in $u(x)$. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= \frac{\cot(u)}{x}
\end{aligned}
$$

Where $f(x) = \frac{1}{x}$ and $g(u) = \cot(u)$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\cot(u)}\,du &= \frac{1}{x}\,dx \\
\int \frac{1}{\cot(u)}\,du &= \int \frac{1}{x}\,dx \\
-\ln(\cos(u)) &= \ln(x) + c_2
\end{aligned}
$$

Raising both side to exponential gives

$$
\frac{1}{\cos(u)} = e^{\ln(x)+c_2}
$$

Which simplifies to

$$
\sec(u) = c_3 x
$$

Now u in the above solution is replaced back by y using $u = \frac{y}{x}$ which results in the solution

$$
y = x\ \mathrm{arcsec}\left(c_3 e^{c_2} x\right)
$$

Summary
The solution(s) found are the following

$$y = x \operatorname{arcsec}\left(c_3 e^{c_2} x\right) \tag{1}$$

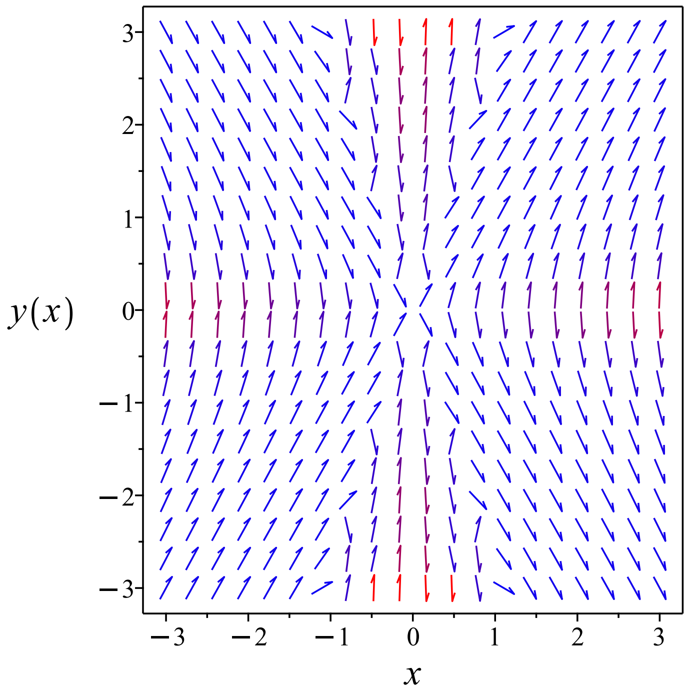


Figure 124: Slope field plot

Verification of solutions

$$y = x \operatorname{arcsec}\left(c_3 e^{c_2} x\right)$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 14

```
dsolve(sin(y(x)/x)*(x*diff(y(x),x)-y(x))=x*cos(y(x)/x),y(x), singsol=all)
```

$$y(x) = x \arccos\left(\frac{1}{c_1 x}\right)$$

✓ Solution by Mathematica

Time used: 25.367 (sec). Leaf size: 56

```
DSolve[Sin[y[x]/x]*(x*y'[x]-y[x])==x*Cos[y[x]/x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -x \arccos\left(\frac{e^{-c_1}}{x}\right)$$
$$y(x) \to x \arccos\left(\frac{e^{-c_1}}{x}\right)$$
$$y(x) \to -\frac{\pi x}{2}$$
$$y(x) \to \frac{\pi x}{2}$$

4.5 problem Problem 13

Internal problem ID [2669]
Internal file name [`OUTPUT/2161_Sunday_June_05_2022_02_51_01_AM_22761468/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.8, Change of Variables. page 79
Problem number: Problem 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{xy' - \sqrt{16x^2 - y^2} - y = 0}$$

4.5.1 Solving as homogeneous ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{\sqrt{16x^2 - y^2} + y}{x} \end{aligned} \tag{1}$$

An ode of the form $y' = \frac{M(x,y)}{N(x,y)}$ is called homogeneous if the functions $M(x, y)$ and $N(x, y)$ are both homogeneous functions and of the same order. Recall that a function $f(x, y)$ is homogeneous of order n if

$$f(t^n x, t^n y) = t^n f(x, y)$$

In this case, it can be seen that both $M = \sqrt{16x^2 - y^2} + y$ and $N = x$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{y}{x}$, or $y = ux$. Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x} x + u$$

Applying the transformation $y = ux$ to the above ODE in (1) gives

$$\frac{\mathrm{d}u}{\mathrm{d}x}x + u = \sqrt{-u^2 + 16} + u$$
$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\sqrt{-u(x)^2 + 16}}{x}$$

Or

$$u'(x) - \frac{\sqrt{-u(x)^2 + 16}}{x} = 0$$

Or

$$u'(x)\,x - \sqrt{-u(x)^2 + 16} = 0$$

Which is now solved as separable in $u(x)$. Which is now solved in $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{\sqrt{-u^2+16}}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = \sqrt{-u^2 + 16}$. Integrating both sides gives

$$\frac{1}{\sqrt{-u^2+16}}\,du = \frac{1}{x}\,dx$$
$$\int \frac{1}{\sqrt{-u^2+16}}\,du = \int \frac{1}{x}\,dx$$
$$\arcsin\left(\frac{u}{4}\right) = \ln(x) + c_2$$

The solution is

$$\arcsin\left(\frac{u(x)}{4}\right) - \ln(x) - c_2 = 0$$

Now u in the above solution is replaced back by y using $u = \frac{y}{x}$ which results in the solution

$$\arcsin\left(\frac{y}{4x}\right) - \ln(x) - c_2 = 0$$

Summary

The solution(s) found are the following

$$\arcsin\left(\frac{y}{4x}\right) - \ln(x) - c_2 = 0 \tag{1}$$

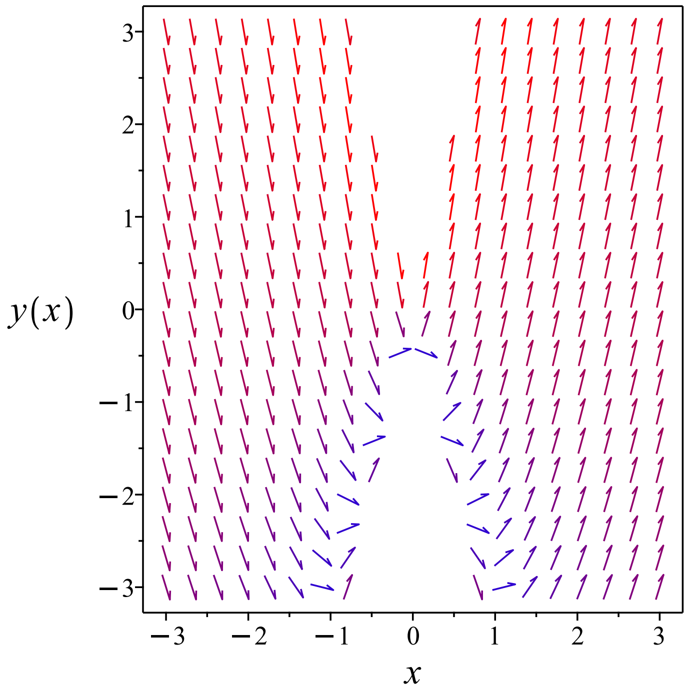


Figure 125: Slope field plot

Verification of solutions

$$\arcsin\left(\frac{y}{4x}\right) - \ln(x) - c_2 = 0$$

Verified OK. `{0 < x}`

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
trying homogeneous G
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 29

```
dsolve(x*diff(y(x),x)=sqrt(16*x^2-y(x)^2)+y(x),y(x), singsol=all)
```

$$-\arctan\left(\frac{y(x)}{\sqrt{16x^2-y(x)^2}}\right)+\ln(x)-c_1=0$$

✓ Solution by Mathematica

Time used: 0.398 (sec). Leaf size: 18

```
DSolve[x*y'[x]==Sqrt[16*x^2-y[x]^2]+y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to -4x\cosh(i\log(x)+c_1)$$

4.6 problem Problem 14

Internal problem ID [2670]
Internal file name [`OUTPUT/2162_Sunday_June_05_2022_02_51_06_AM_41940925/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.8, Change of Variables. page 79
Problem number: Problem 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{xy' - y - \sqrt{9x^2 + y^2} = 0}$$

4.6.1 Solving as homogeneous ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{y + \sqrt{9x^2 + y^2}}{x} \end{aligned} \tag{1}$$

An ode of the form $y' = \frac{M(x,y)}{N(x,y)}$ is called homogeneous if the functions $M(x,y)$ and $N(x,y)$ are both homogeneous functions and of the same order. Recall that a function $f(x,y)$ is homogeneous of order n if

$$f(t^n x, t^n y) = t^n f(x,y)$$

In this case, it can be seen that both $M = y + \sqrt{9x^2 + y^2}$ and $N = x$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{y}{x}$, or $y = ux$. Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x}x + u$$

Applying the transformation $y = ux$ to the above ODE in (1) gives

$$
\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}x}x + u &= u + \sqrt{u^2 + 9} \\
\frac{\mathrm{d}u}{\mathrm{d}x} &= \frac{\sqrt{u(x)^2 + 9}}{x}
\end{aligned}
$$

Or

$$
u'(x) - \frac{\sqrt{u(x)^2 + 9}}{x} = 0
$$

Or

$$
u'(x)\,x - \sqrt{u(x)^2 + 9} = 0
$$

Which is now solved as separable in $u(x)$. Which is now solved in $u(x)$. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= \frac{\sqrt{u^2+9}}{x}
\end{aligned}
$$

Where $f(x) = \frac{1}{x}$ and $g(u) = \sqrt{u^2 + 9}$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\sqrt{u^2+9}}\,du &= \frac{1}{x}\,dx \\
\int \frac{1}{\sqrt{u^2+9}}\,du &= \int \frac{1}{x}\,dx \\
\operatorname{arcsinh}\left(\frac{u}{3}\right) &= \ln(x) + c_2
\end{aligned}
$$

The solution is

$$
\operatorname{arcsinh}\left(\frac{u(x)}{3}\right) - \ln(x) - c_2 = 0
$$

Now u in the above solution is replaced back by y using $u = \frac{y}{x}$ which results in the solution

$$
\operatorname{arcsinh}\left(\frac{y}{3x}\right) - \ln(x) - c_2 = 0
$$

Summary

The solution(s) found are the following

$$
\operatorname{arcsinh}\left(\frac{y}{3x}\right) - \ln(x) - c_2 = 0 \tag{1}
$$

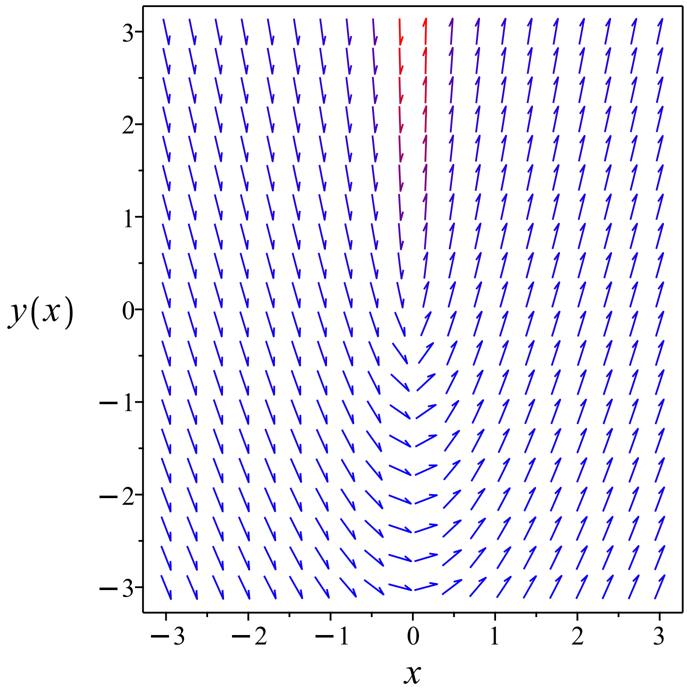


Figure 126: Slope field plot

Verification of solutions

$$\operatorname{arcsinh}\left(\frac{y}{3x}\right) - \ln(x) - c_2 = 0$$

Verified OK. {0 < x}

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
trying homogeneous G
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 28

```
dsolve(x*diff(y(x),x)-y(x)=sqrt(9*x^2+y(x)^2),y(x), singsol=all)
```

$$\frac{-c_1 x^2 + \sqrt{9x^2 + y(x)^2} + y(x)}{x^2} = 0$$

✓ Solution by Mathematica

Time used: 0.35 (sec). Leaf size: 27

```
DSolve[x*y'[x]-y[x]==Sqrt[9*x^2+y[x]^2],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{9e^{c_1}x^2}{2} - \frac{e^{-c_1}}{2}$$

4.7 problem Problem 15

Internal problem ID [2671]
Internal file name [`OUTPUT/2163_Sunday_June_05_2022_02_51_10_AM_21063386/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.8, Change of Variables. page 79
Problem number: Problem 15.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y(x^2 - y^2) - x(x^2 - y^2)\, y' = 0}$$

4.7.1 Solving as homogeneous ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{y}{x} \end{aligned} \tag{1}$$

An ode of the form $y' = \frac{M(x,y)}{N(x,y)}$ is called homogeneous if the functions $M(x,y)$ and $N(x,y)$ are both homogeneous functions and of the same order. Recall that a function $f(x,y)$ is homogeneous of order n if

$$f(t^n x, t^n y) = t^n f(x,y)$$

In this case, it can be seen that both $M = y$ and $N = x$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{y}{x}$, or $y = ux$. Hence

$$\frac{dy}{dx} = \frac{du}{dx}x + u$$

Applying the transformation $y = ux$ to the above ODE in (1) gives

$$\frac{\mathrm{d}u}{\mathrm{d}x}x + u = u$$
$$\frac{\mathrm{d}u}{\mathrm{d}x} = 0$$

Or

$$u'(x) = 0$$

Which is now solved as separable in $u(x)$. Which is now solved in $u(x)$. Integrating both sides gives

$$u(x) = \int 0 \,\mathrm{d}x$$
$$= c_2$$

Now u in the above solution is replaced back by y using $u = \frac{y}{x}$ which results in the solution

$$y = c_2 x$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_2 x \tag{1}$$

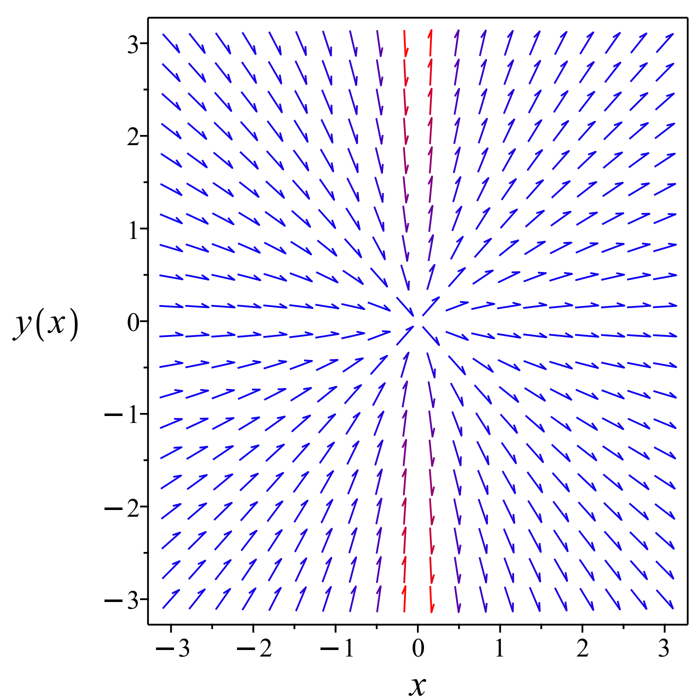


Figure 127: Slope field plot

Verification of solutions

$$y = c_2 x$$

Verified OK.

4.7.2 Maple step by step solution

Let's solve

$y(x^2 - y^2) - x(x^2 - y^2)\, y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = \frac{1}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int \frac{1}{x} dx + c_1$

- Evaluate integral

  $\ln(y) = \ln(x) + c_1$

- Solve for y

  $y = \mathrm{e}^{c_1} x$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(y(x)*(x^2-y(x)^2)-x*(x^2-y(x)^2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$
\begin{aligned}
y(x) &= -x \\
y(x) &= x \\
y(x) &= c_1 x
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 33

```
DSolve[y[x]*(x^2-y[x]^2)-x*(x^2-y[x]^2)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$
\begin{aligned}
y(x) &\to -x \\
y(x) &\to x \\
y(x) &\to c_1 x \\
y(x) &\to -x \\
y(x) &\to x
\end{aligned}
$$

4.8 problem Problem 16

Internal problem ID [2672]
Internal file name [`OUTPUT/2164_Sunday_June_05_2022_02_51_12_AM_44925127/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.8, Change of Variables. page 79
Problem number: Problem 16.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _dAlembert]
```

$$\boxed{xy' + y\ln(x) - y\ln(y) = 0}$$

4.8.1 Solving as homogeneous ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{y(-\ln(x)+\ln(y))}{x} \end{aligned} \tag{1}$$

An ode of the form $y' = \frac{M(x,y)}{N(x,y)}$ is called homogeneous if the functions $M(x,y)$ and $N(x,y)$ are both homogeneous functions and of the same order. Recall that a function $f(x,y)$ is homogeneous of order n if

$$f(t^n x, t^n y) = t^n f(x,y)$$

In this case, it can be seen that both $M = -y(\ln(x) - \ln(y))$ and $N = x$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{y}{x}$, or $y = ux$. Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x}x + u$$

Applying the transformation $y = ux$ to the above ODE in (1) gives

$$\frac{\mathrm{d}u}{\mathrm{d}x}x + u = u\ln(u)$$
$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{u(x)\ln(u(x)) - u(x)}{x}$$

Or

$$u'(x) - \frac{u(x)\ln(u(x)) - u(x)}{x} = 0$$

Or

$$u'(x)\,x - u(x)\ln(u(x)) + u(x) = 0$$

Which is now solved as separable in $u(x)$. Which is now solved in $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{u(\ln(u) - 1)}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = u(\ln(u) - 1)$. Integrating both sides gives

$$\frac{1}{u(\ln(u) - 1)}\,du = \frac{1}{x}\,dx$$
$$\int \frac{1}{u(\ln(u) - 1)}\,du = \int \frac{1}{x}\,dx$$
$$\ln(\ln(u) - 1) = \ln(x) + c_2$$

Raising both side to exponential gives

$$\ln(u) - 1 = \mathrm{e}^{\ln(x) + c_2}$$

Which simplifies to

$$\ln(u) - 1 = c_3 x$$

Now u in the above solution is replaced back by y using $u = \frac{y}{x}$ which results in the solution

$$y = x\,\mathrm{e}^{1 + c_3 \mathrm{e}^{c_2} x}$$

Summary

The solution(s) found are the following

$$y = x\,\mathrm{e}^{1 + c_3 \mathrm{e}^{c_2} x} \tag{1}$$

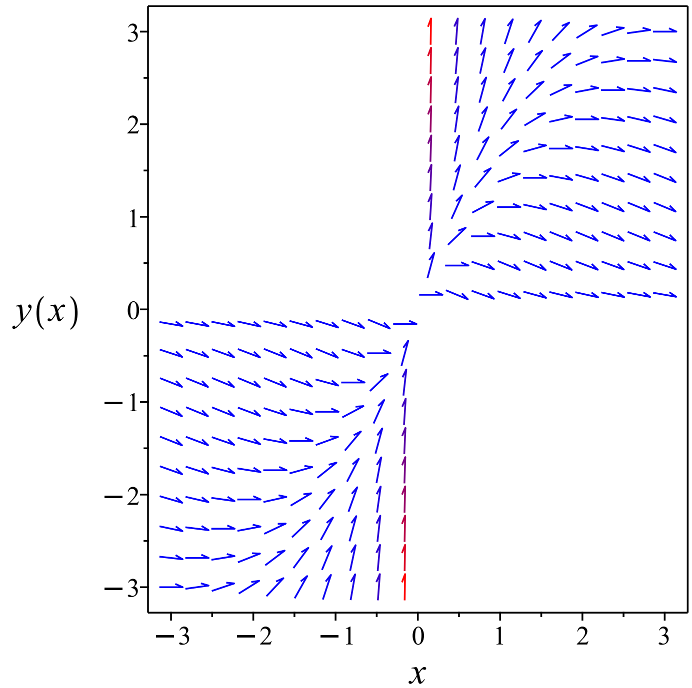


Figure 128: Slope field plot

Verification of solutions

$$y = x\,\mathrm{e}^{1+c_3 \mathrm{e}^{c_2} x}$$

Verified OK. `{0 < x}`

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 12

```
dsolve(x*diff(y(x),x)+y(x)*ln(x)=y(x)*ln(y(x)),y(x), singsol=all)
```

$$y(x) = x\,\mathrm{e}^{c_1 x+1}$$

✓ Solution by Mathematica

Time used: 0.257 (sec). Leaf size: 24

```
DSolve[x*y'[x]+y[x]*Log[x]==y[x]*Log[y[x]],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x e^{1+e^{c_1} x}$$
$$y(x) \to e x$$

4.9 problem Problem 17

Internal problem ID [2673]
Internal file name [`OUTPUT/2165_Sunday_June_05_2022_02_51_16_AM_34878343/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.8, Change of Variables. page 79
Problem number: Problem 17.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{y' - \frac{y^2 + 2yx - 2x^2}{x^2 - yx + y^2} = 0}$$

4.9.1 Solving as homogeneous ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{-2x^2 + 2xy + y^2}{x^2 - xy + y^2} \end{aligned} \tag{1}$$

An ode of the form $y' = \frac{M(x,y)}{N(x,y)}$ is called homogeneous if the functions $M(x,y)$ and $N(x,y)$ are both homogeneous functions and of the same order. Recall that a function $f(x,y)$ is homogeneous of order n if

$$f(t^n x, t^n y) = t^n f(x,y)$$

In this case, it can be seen that both $M = -2x^2 + 2xy + y^2$ and $N = x^2 - xy + y^2$ are both homogeneous and of the same order $n = 2$. Therefore this is a homogeneous ode.

Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{y}{x}$, or $y = ux$. Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x}x + u$$

Applying the transformation $y = ux$ to the above ODE in (1) gives

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}x}x + u &= \frac{u^2+2u-2}{u^2-u+1}\\ \frac{\mathrm{d}u}{\mathrm{d}x} &= \frac{\frac{u(x)^2+2u(x)-2}{u(x)^2-u(x)+1} - u(x)}{x}\end{aligned}$$

Or

$$u'(x) - \frac{\frac{u(x)^2+2u(x)-2}{u(x)^2-u(x)+1} - u(x)}{x} = 0$$

Or

$$u'(x)\,u(x)^2\,x - u'(x)\,u(x)\,x + u(x)^3 + u'(x)\,x - 2u(x)^2 - u(x) + 2 = 0$$

Or

$$x\left(u(x)^2 - u(x) + 1\right)u'(x) + u(x)^3 - 2u(x)^2 - u(x) + 2 = 0$$

Which is now solved as separable in $u(x)$. Which is now solved in $u(x)$. In canonical form the ODE is

$$\begin{aligned}u' &= F(x,u)\\ &= f(x)g(u)\\ &= -\frac{u^3-2u^2-u+2}{x\,(u^2-u+1)}\end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u^3-2u^2-u+2}{u^2-u+1}$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\frac{u^3-2u^2-u+2}{u^2-u+1}}\,du &= -\frac{1}{x}\,dx\\ \int \frac{1}{\frac{u^3-2u^2-u+2}{u^2-u+1}}\,du &= \int -\frac{1}{x}\,dx\\ \frac{\ln(u+1)}{2} - \frac{\ln(u-1)}{2} + \ln(u-2) &= -\ln(x) + c_2\end{aligned}$$

Raising both side to exponential gives

$$\mathrm{e}^{\frac{\ln(u+1)}{2} - \frac{\ln(u-1)}{2} + \ln(u-2)} = \mathrm{e}^{-\ln(x)+c_2}$$

Which simplifies to

$$
\frac{\sqrt{u+1}\,(u-2)}{\sqrt{u-1}} = \frac{c_3}{x}
$$

The solution is

$$
\frac{\sqrt{u(x)+1}\,(u(x)-2)}{\sqrt{u(x)-1}} = \frac{c_3}{x}
$$

Now u in the above solution is replaced back by y using $u = \frac{y}{x}$ which results in the solution

$$
\frac{\sqrt{\frac{y}{x}+1}\,\left(\frac{y}{x}-2\right)}{\sqrt{\frac{y}{x}-1}} = \frac{c_3}{x}
$$

Which simplifies to

$$
\frac{(y-2x)\sqrt{\frac{y+x}{x}}}{\sqrt{\frac{y-x}{x}}} = c_3
$$

Summary

The solution(s) found are the following

$$
\frac{(y-2x)\sqrt{\frac{y+x}{x}}}{\sqrt{\frac{y-x}{x}}} = c_3 \tag{1}
$$

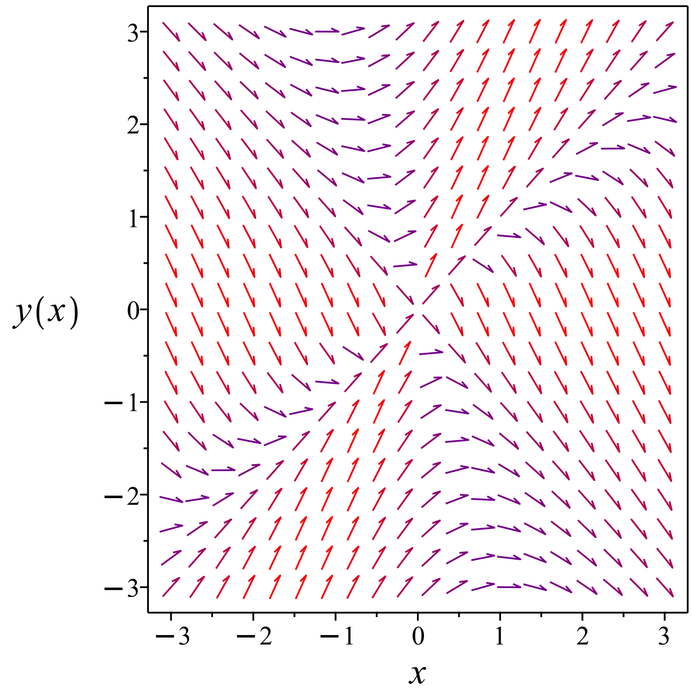


Figure 129: Slope field plot

Verification of solutions

$$\frac{(y-2x)\sqrt{\frac{y+x}{x}}}{\sqrt{\frac{y-x}{x}}} = c_3$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 80

```
dsolve(diff(y(x),x)=(y(x)^2+2*x*y(x)-2*x^2)/(x^2-x*y(x)+y(x)^2),y(x), singsol=all)
```

$$y(x) = \frac{x\left(-\operatorname{RootOf}\left(2\_Z^6+(9c_1x^2-1)\_Z^4-6x^2c_1\_Z^2+c_1x^2\right)^2+1\right)}{\operatorname{RootOf}\left(2\_Z^6+(9c_1x^2-1)\_Z^4-6x^2c_1\_Z^2+c_1x^2\right)^2}$$

✓ Solution by Mathematica

Time used: 60.187 (sec). Leaf size: 373

```
DSolve[y'[x]==(y[x]^2+2*x*y[x]-2*x^2)/(x^2-x*y[x]+y[x]^2),y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to \frac{\sqrt[3]{-54x^3+2\sqrt{729x^6+(-9x^2+3e^{2c_1})^3}}}{3\sqrt[3]{2}} - \frac{\sqrt[3]{2}(-3x^2+e^{2c_1})}{\sqrt[3]{-54x^3+2\sqrt{729x^6+(-9x^2+3e^{2c_1})^3}}} + x$$

$$y(x) \to \frac{(-1+i\sqrt{3})\sqrt[3]{-54x^3+2\sqrt{729x^6+(-9x^2+3e^{2c_1})^3}}}{6\sqrt[3]{2}} + \frac{(1+i\sqrt{3})(-3x^2+e^{2c_1})}{2^{2/3}\sqrt[3]{-54x^3+2\sqrt{729x^6+(-9x^2+3e^{2c_1})^3}}} + x$$

$$y(x) \to -\frac{(1+i\sqrt{3})\sqrt[3]{-54x^3+2\sqrt{729x^6+(-9x^2+3e^{2c_1})^3}}}{6\sqrt[3]{2}} + \frac{(1-i\sqrt{3})(-3x^2+e^{2c_1})}{2^{2/3}\sqrt[3]{-54x^3+2\sqrt{729x^6+(-9x^2+3e^{2c_1})^3}}} + x$$

4.10 problem Problem 18

Internal problem ID [2674]
Internal file name [`OUTPUT/2166_Sunday_June_05_2022_02_51_21_AM_90768622/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.8, Change of Variables. page 79
Problem number: Problem 18.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[[_homogeneous, `class A`]]
```

$$\boxed{2xyy' - x^2 \mathrm{e}^{-\frac{y^2}{x^2}} - 2y^2 = 0}$$

4.10.1 Solving as homogeneous ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{x^2 \mathrm{e}^{-\frac{y^2}{x^2}} + 2y^2}{2xy} \end{aligned} \tag{1}$$

An ode of the form $y' = \frac{M(x,y)}{N(x,y)}$ is called homogeneous if the functions $M(x, y)$ and $N(x, y)$ are both homogeneous functions and of the same order. Recall that a function $f(x, y)$ is homogeneous of order n if

$$f(t^n x, t^n y) = t^n f(x, y)$$

In this case, it can be seen that both $M = x^2 \mathrm{e}^{-\frac{y^2}{x^2}} + 2y^2$ and $N = 2xy$ are both homogeneous and of the same order $n = 2$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{y}{x}$, or $y = ux$. Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x} x + u$$

Applying the transformation $y = ux$ to the above ODE in (1) gives

$$
\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}x} x + u &= \frac{\mathrm{e}^{-u^2}}{2u} + u \\
\frac{\mathrm{d}u}{\mathrm{d}x} &= \frac{\mathrm{e}^{-u(x)^2}}{2u(x)\, x}
\end{aligned}
$$

Or

$$
u'(x) - \frac{\mathrm{e}^{-u(x)^2}}{2u(x)\, x} = 0
$$

Or

$$
2u'(x)\, u(x)\, \mathrm{e}^{u(x)^2} x - 1 = 0
$$

Which is now solved as separable in $u(x)$. Which is now solved in $u(x)$. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= \frac{\mathrm{e}^{-u^2}}{2ux}
\end{aligned}
$$

Where $f(x) = \frac{1}{2x}$ and $g(u) = \frac{\mathrm{e}^{-u^2}}{u}$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\frac{\mathrm{e}^{-u^2}}{u}}\, du &= \frac{1}{2x}\, dx \\
\int \frac{1}{\frac{\mathrm{e}^{-u^2}}{u}}\, du &= \int \frac{1}{2x}\, dx \\
\frac{\mathrm{e}^{u^2}}{2} &= \frac{\ln(x)}{2} + c_2
\end{aligned}
$$

The solution is

$$
\frac{\mathrm{e}^{u(x)^2}}{2} - \frac{\ln(x)}{2} - c_2 = 0
$$

Now u in the above solution is replaced back by y using $u = \frac{y}{x}$ which results in the solution

$$
\frac{\mathrm{e}^{\frac{y^2}{x^2}}}{2} - \frac{\ln(x)}{2} - c_2 = 0
$$

Summary
The solution(s) found are the following

$$\frac{e^{\frac{y^2}{x^2}}}{2} - \frac{\ln(x)}{2} - c_2 = 0 \tag{1}$$

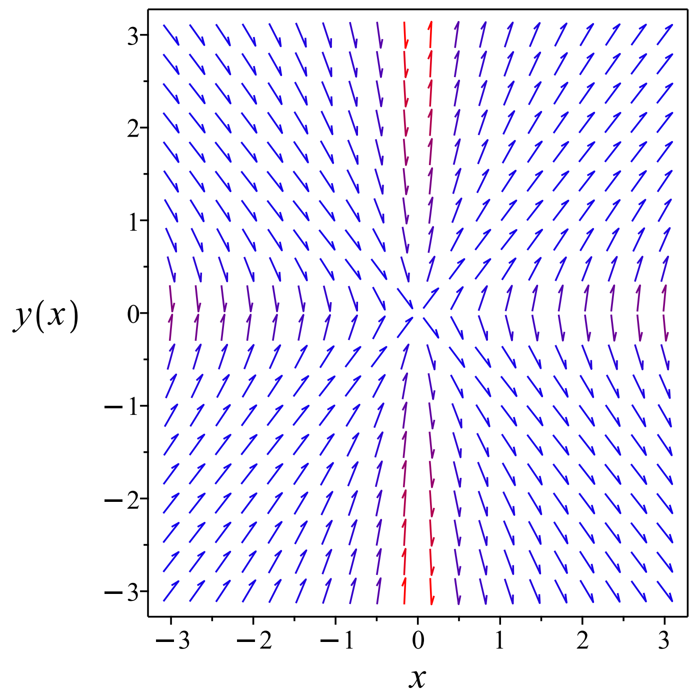


Figure 130: Slope field plot

Verification of solutions

$$\frac{e^{\frac{y^2}{x^2}}}{2} - \frac{\ln(x)}{2} - c_2 = 0$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve(2*x*y(x)*diff(y(x),x)-(x^2*exp(-y(x)^2/x^2)+2*y(x)^2)=0,y(x), singsol=all)
```

$$y(x) = \sqrt{\ln(\ln(x) + c_1)}\, x$$
$$y(x) = -\sqrt{\ln(\ln(x) + c_1)}\, x$$

✓ Solution by Mathematica

Time used: 2.17 (sec). Leaf size: 38

```
DSolve[2*x*y[x]*y'[x]-(x^2*Exp[-y[x]^2/x^2]+2*y[x]^2)==0,y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to -x\sqrt{\log(\log(x) + 2c_1)}$$
$$y(x) \to x\sqrt{\log(\log(x) + 2c_1)}$$

4.11 problem Problem 19

Internal problem ID [2675]
Internal file name [`OUTPUT/2167_Sunday_June_05_2022_02_51_25_AM_403017/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.8, Change of Variables. page 79
Problem number: Problem 19.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Riccati]
```

$$\boxed{y'x^2 - y^2 - 3yx = x^2}$$

4.11.1 Solving as homogeneous ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{x^2 + 3xy + y^2}{x^2} \end{aligned} \tag{1}$$

An ode of the form $y' = \frac{M(x,y)}{N(x,y)}$ is called homogeneous if the functions $M(x, y)$ and $N(x, y)$ are both homogeneous functions and of the same order. Recall that a function $f(x, y)$ is homogeneous of order n if

$$f(t^n x, t^n y) = t^n f(x, y)$$

In this case, it can be seen that both $M = x^2 + 3xy + y^2$ and $N = x^2$ are both homogeneous and of the same order $n = 2$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{y}{x}$, or $y = ux$. Hence

$$\frac{dy}{dx} = \frac{du}{dx} x + u$$

Applying the transformation $y = ux$ to the above ODE in (1) gives

$$
\begin{aligned}
\frac{du}{dx}x + u &= u^2 + 3u + 1 \\
\frac{du}{dx} &= \frac{u(x)^2 + 2u(x) + 1}{x}
\end{aligned}
$$

Or

$$
u'(x) - \frac{u(x)^2 + 2u(x) + 1}{x} = 0
$$

Or

$$
u'(x)\,x - u(x)^2 - 2u(x) - 1 = 0
$$

Or

$$
u'(x)\,x - (u(x) + 1)^2 = 0
$$

Which is now solved as separable in $u(x)$. Which is now solved in $u(x)$. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= \frac{(u+1)^2}{x}
\end{aligned}
$$

Where $f(x) = \frac{1}{x}$ and $g(u) = (u+1)^2$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{(u+1)^2}\,du &= \frac{1}{x}\,dx \\
\int \frac{1}{(u+1)^2}\,du &= \int \frac{1}{x}\,dx \\
-\frac{1}{u+1} &= \ln(x) + c_2
\end{aligned}
$$

The solution is

$$
-\frac{1}{u(x) + 1} - \ln(x) - c_2 = 0
$$

Now u in the above solution is replaced back by y using $u = \frac{y}{x}$ which results in the solution

$$
-\frac{1}{\frac{y}{x} + 1} - \ln(x) - c_2 = 0
$$

Which simplifies to

$$-\frac{y\ln(x)+c_2y+\ln(x)x+c_2x+x}{y+x}=0$$

Summary

The solution(s) found are the following

$$-\frac{y\ln(x)+c_2y+\ln(x)x+c_2x+x}{y+x}=0 \tag{1}$$

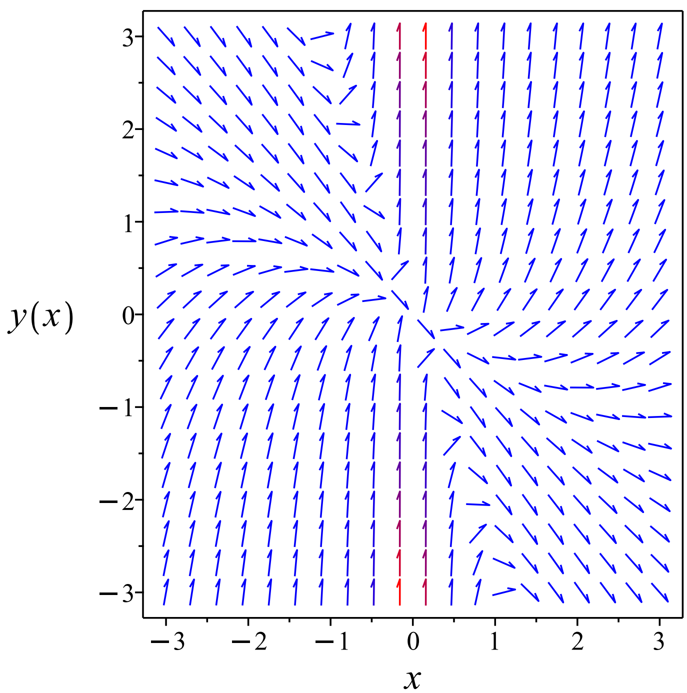


Figure 131: Slope field plot

Verification of solutions

$$-\frac{y\ln(x)+c_2y+\ln(x)x+c_2x+x}{y+x}=0$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve(x^2*diff(y(x),x)=y(x)^2+3*x*y(x)+x^2,y(x), singsol=all)
```

$$y(x) = -\frac{x(\ln(x) + c_1 + 1)}{\ln(x) + c_1}$$

✓ Solution by Mathematica

Time used: 0.146 (sec). Leaf size: 28

```
DSolve[x^2*y'[x]==y[x]^2+3*x*y[x]+x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{x(\log(x) + 1 + c_1)}{\log(x) + c_1}$$
$$y(x) \to -x$$

4.12 problem Problem 20

Internal problem ID [2676]
Internal file name [`OUTPUT/2168_Sunday_June_05_2022_02_51_28_AM_49626620/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.8, Change of Variables. page 79
Problem number: Problem 20.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{yy' - \sqrt{y^2 + x^2} = -x}$$

4.12.1 Solving as homogeneous ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{\sqrt{x^2+y^2}-x}{y} \end{aligned} \tag{1}$$

An ode of the form $y' = \frac{M(x,y)}{N(x,y)}$ is called homogeneous if the functions $M(x,y)$ and $N(x,y)$ are both homogeneous functions and of the same order. Recall that a function $f(x,y)$ is homogeneous of order n if

$$f(t^n x, t^n y) = t^n f(x,y)$$

In this case, it can be seen that both $M = \sqrt{x^2+y^2} - x$ and $N = y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{y}{x}$, or $y = ux$. Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x}x + u$$

Applying the transformation $y = ux$ to the above ODE in (1) gives

$$\frac{\mathrm{d}u}{\mathrm{d}x}x + u = \frac{\sqrt{u^2+1}-1}{u}$$
$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\frac{\sqrt{u(x)^2+1}-1}{u(x)} - u(x)}{x}$$

Or

$$u'(x) - \frac{\frac{\sqrt{u(x)^2+1}-1}{u(x)} - u(x)}{x} = 0$$

Or

$$u'(x)\,u(x)\,x + u(x)^2 - \sqrt{u\left(x\right)^2+1} + 1 = 0$$

Which is now solved as separable in $u(x)$. Which is now solved in $u(x)$. In canonical form the ODE is

$$\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= -\frac{u^2 - \sqrt{u^2+1}+1}{ux}
\end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u^2-\sqrt{u^2+1}+1}{u}$. Integrating both sides gives

$$\frac{1}{\frac{u^2-\sqrt{u^2+1}+1}{u}}\,du = -\frac{1}{x}\,dx$$
$$\int \frac{1}{\frac{u^2-\sqrt{u^2+1}+1}{u}}\,du = \int -\frac{1}{x}\,dx$$
$$-\operatorname{arctanh}\left(\frac{1}{\sqrt{u^2+1}}\right) + \ln\left(u\right) = -\ln\left(x\right) + c_2$$

The solution is

$$-\operatorname{arctanh}\left(\frac{1}{\sqrt{u\left(x\right)^2+1}}\right) + \ln\left(u(x)\right) + \ln\left(x\right) - c_2 = 0$$

Now u in the above solution is replaced back by y using $u = \frac{y}{x}$ which results in the solution

$$-\operatorname{arctanh}\left(\frac{1}{\sqrt{\frac{y^2}{x^2}+1}}\right) + \ln\left(\frac{y}{x}\right) + \ln\left(x\right) - c_2 = 0$$

Summary
The solution(s) found are the following

$$-\operatorname{arctanh}\left(\frac{1}{\sqrt{\frac{y^2+x^2}{x^2}}}\right)+\ln\left(\frac{y}{x}\right)+\ln\left(x\right)-c_2=0 \tag{1}$$

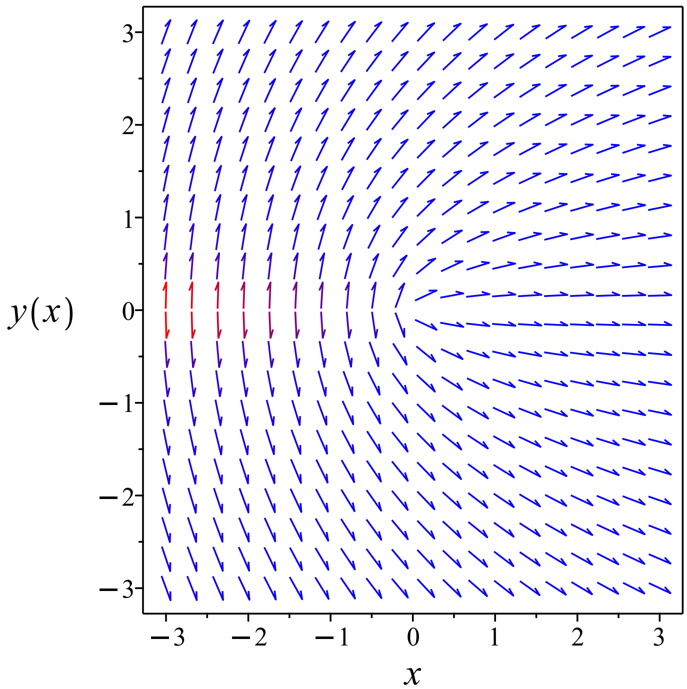


Figure 132: Slope field plot

Verification of solutions

$$-\operatorname{arctanh}\left(\frac{1}{\sqrt{\frac{y^2+x^2}{x^2}}}\right)+\ln\left(\frac{y}{x}\right)+\ln\left(x\right)-c_2=0$$

Verified OK. `{0 < x}`

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 27

```
dsolve(y(x)*diff(y(x),x)=sqrt(x^2+y(x)^2)-x,y(x), singsol=all)
```

$$\frac{-c_1 y(x)^2 + \sqrt{x^2 + y\left(x\right)^2} + x}{y\left(x\right)^2} = 0$$

✓ Solution by Mathematica

Time used: 0.409 (sec). Leaf size: 57

```
DSolve[y[x]*y'[x]==Sqrt[x^2+y[x]^2]-x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow -e^{\frac{c_1}{2}}\sqrt{2x + e^{c_1}}$$
$$y(x) \rightarrow e^{\frac{c_1}{2}}\sqrt{2x + e^{c_1}}$$
$$y(x) \rightarrow 0$$

4.13 problem Problem 21

Internal problem ID [2677]
Internal file name [`OUTPUT/2169_Sunday_June_05_2022_02_51_33_AM_71672999/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.8, Change of Variables. page 79
Problem number: Problem 21.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class B`]]
```

$$\boxed{2x(y+2x)\,y'-y(4x-y)=0}$$

4.13.1 Solving as homogeneous ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{y(-4x+y)}{2x\,(2x+y)} \end{aligned} \tag{1}$$

An ode of the form $y'=\frac{M(x,y)}{N(x,y)}$ is called homogeneous if the functions $M(x,y)$ and $N(x,y)$ are both homogeneous functions and of the same order. Recall that a function $f(x,y)$ is homogeneous of order n if

$$f(t^n x, t^n y) = t^n f(x,y)$$

In this case, it can be seen that both $M=y(4x-y)$ and $N=2x(2x+y)$ are both homogeneous and of the same order $n=2$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u=\frac{y}{x}$, or $y=ux$. Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}u}{\mathrm{d}x}x+u$$

Applying the transformation $y = ux$ to the above ODE in (1) gives

$$
\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}x}x + u &= -\frac{u(-4+u)}{2u+4} \\
\frac{\mathrm{d}u}{\mathrm{d}x} &= \frac{-\frac{u(x)(-4+u(x))}{2u(x)+4} - u(x)}{x}
\end{aligned}
$$

Or

$$
u'(x) - \frac{-\frac{u(x)(-4+u(x))}{2u(x)+4} - u(x)}{x} = 0
$$

Or

$$
2u'(x)\,xu(x) + 4u'(x)\,x + 3u(x)^2 = 0
$$

Or

$$
2x(u(x)+2)\,u'(x) + 3u(x)^2 = 0
$$

Which is now solved as separable in $u(x)$. Which is now solved in $u(x)$. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= -\frac{3u^2}{2x\,(u+2)}
\end{aligned}
$$

Where $f(x) = -\frac{3}{2x}$ and $g(u) = \frac{u^2}{u+2}$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\frac{u^2}{u+2}}\,du &= -\frac{3}{2x}\,dx \\
\int \frac{1}{\frac{u^2}{u+2}}\,du &= \int -\frac{3}{2x}\,dx \\
-\frac{2}{u} + \ln(u) &= -\frac{3\ln(x)}{2} + c_2
\end{aligned}
$$

The solution is

$$
-\frac{2}{u(x)} + \ln(u(x)) + \frac{3\ln(x)}{2} - c_2 = 0
$$

Now u in the above solution is replaced back by y using $u = \frac{y}{x}$ which results in the solution

$$
-\frac{2x}{y} + \ln\left(\frac{y}{x}\right) + \frac{3\ln(x)}{2} - c_2 = 0
$$

Summary
The solution(s) found are the following

$$-\frac{2x}{y}+\ln\left(\frac{y}{x}\right)+\frac{3\ln(x)}{2}-c_2=0 \tag{1}$$

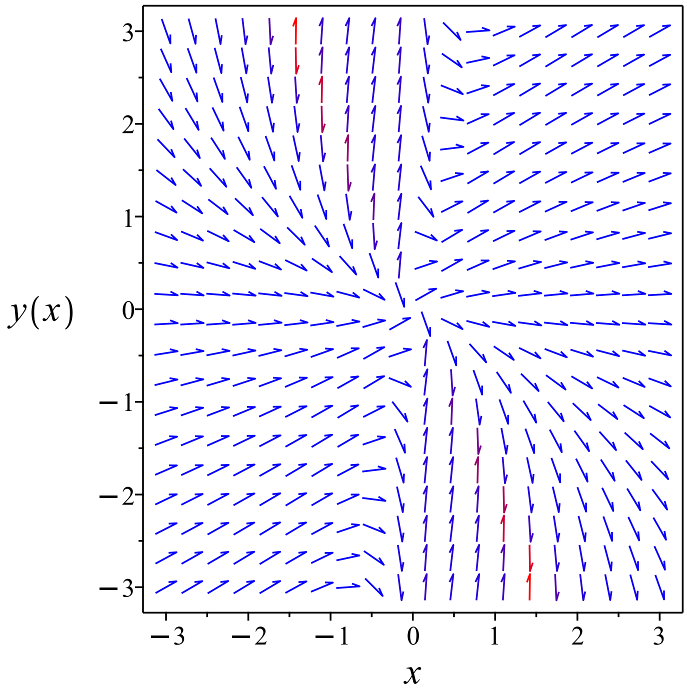


Figure 133: Slope field plot

Verification of solutions

$$-\frac{2x}{y}+\ln\left(\frac{y}{x}\right)+\frac{3\ln(x)}{2}-c_2=0$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(2*x*(y(x)+2*x)*diff(y(x),x)=y(x)*(4*x-y(x)),y(x), singsol=all)
```

$$y(x) = \frac{2x}{\text{LambertW}\left(2\,\mathrm{e}^{\frac{3c_1}{2}}x^{\frac{3}{2}}\right)}$$

✓ Solution by Mathematica

Time used: 5.346 (sec). Leaf size: 29

```
DSolve[2*x*(y[x]+2*x)*y'[x]==y[x]*(4*x-y[x]),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2x}{W\left(2e^{-c_1}x^{3/2}\right)}$$
$$y(x) \to 0$$

4.14 problem Problem 22

Internal problem ID [2678]
Internal file name [`OUTPUT/2170_Sunday_June_05_2022_02_51_36_AM_30877140/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.8, Change of Variables. page 79
Problem number: Problem 22.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _dAlembert]
```

$$\boxed{xy' - x\tan\left(\frac{y}{x}\right) - y = 0}$$

4.14.1 Solving as homogeneous ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{x\tan\left(\frac{y}{x}\right) + y}{x} \end{aligned} \tag{1}$$

An ode of the form $y' = \frac{M(x,y)}{N(x,y)}$ is called homogeneous if the functions $M(x,y)$ and $N(x,y)$ are both homogeneous functions and of the same order. Recall that a function $f(x,y)$ is homogeneous of order n if

$$f(t^n x, t^n y) = t^n f(x,y)$$

In this case, it can be seen that both $M = x\tan\left(\frac{y}{x}\right) + y$ and $N = x$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{y}{x}$, or $y = ux$. Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x}x + u$$

Applying the transformation $y = ux$ to the above ODE in (1) gives

$$\frac{\mathrm{d}u}{\mathrm{d}x}x + u = \tan(u) + u$$
$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\tan(u(x))}{x}$$

Or

$$u'(x) - \frac{\tan(u(x))}{x} = 0$$

Or

$$u'(x)\,x - \tan(u(x)) = 0$$

Which is now solved as separable in $u(x)$. Which is now solved in $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{\tan(u)}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = \tan(u)$. Integrating both sides gives

$$\frac{1}{\tan(u)}\,du = \frac{1}{x}\,dx$$
$$\int \frac{1}{\tan(u)}\,du = \int \frac{1}{x}\,dx$$
$$\ln(\sin(u)) = \ln(x) + c_2$$

Raising both side to exponential gives

$$\sin(u) = \mathrm{e}^{\ln(x)+c_2}$$

Which simplifies to

$$\sin(u) = c_3 x$$

Now u in the above solution is replaced back by y using $u = \frac{y}{x}$ which results in the solution

$$y = x\arcsin\left(c_3 \mathrm{e}^{c_2} x\right)$$

Summary
The solution(s) found are the following

$$y = x\arcsin\left(c_3 \mathrm{e}^{c_2} x\right) \tag{1}$$

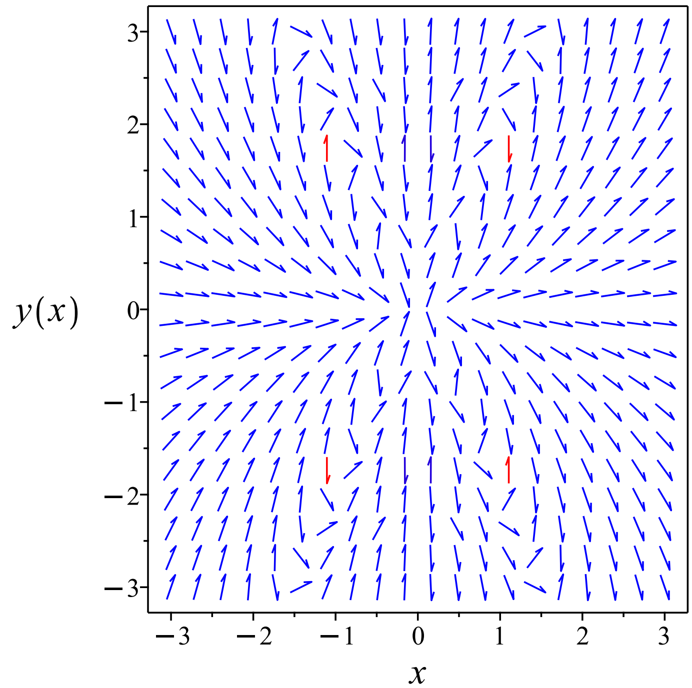


Figure 134: Slope field plot

Verification of solutions

$$y = x \arcsin\left(c_3 e^{c_2} x\right)$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 10

```
dsolve(x*diff(y(x),x)=x*tan(y(x)/x)+y(x),y(x), singsol=all)
```

$$y(x) = \arcsin(c_1 x)\, x$$

✓ Solution by Mathematica

Time used: 4.357 (sec). Leaf size: 19

```
DSolve[x*y'[x]==x*Tan[y[x]/x]+y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x \arcsin(e^{c_1} x)$$
$$y(x) \to 0$$

4.15 problem Problem 23

Internal problem ID [2679]
Internal file name [OUTPUT/2171_Sunday_June_05_2022_02_51_40_AM_38121038/index.tex]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.8, Change of Variables. page 79
Problem number: Problem 23.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _dAlembert]
```

$$\boxed{y' - \frac{x\sqrt{y^2+x^2}+y^2}{yx} = 0}$$

4.15.1 Solving as homogeneous ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{x\sqrt{x^2+y^2}+y^2}{yx} \end{aligned} \tag{1}$$

An ode of the form $y' = \frac{M(x,y)}{N(x,y)}$ is called homogeneous if the functions $M(x,y)$ and $N(x,y)$ are both homogeneous functions and of the same order. Recall that a function $f(x,y)$ is homogeneous of order n if

$$f(t^n x, t^n y) = t^n f(x,y)$$

In this case, it can be seen that both $M = x\sqrt{x^2+y^2}+y^2$ and $N = xy$ are both homogeneous and of the same order $n = 2$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{y}{x}$, or $y = ux$. Hence

$$\frac{dy}{dx} = \frac{du}{dx}x + u$$

Applying the transformation $y = ux$ to the above ODE in (1) gives

$$\frac{du}{dx}x + u = \frac{\sqrt{u^2+1}+u^2}{u}$$

$$\frac{du}{dx} = \frac{\frac{\sqrt{u(x)^2+1}+u(x)^2}{u(x)} - u(x)}{x}$$

Or

$$u'(x) - \frac{\frac{\sqrt{u(x)^2+1}+u(x)^2}{u(x)} - u(x)}{x} = 0$$

Or

$$u'(x)\,u(x)\,x - \sqrt{u(x)^2+1} = 0$$

Which is now solved as separable in $u(x)$. Which is now solved in $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{\sqrt{u^2+1}}{ux} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = \frac{\sqrt{u^2+1}}{u}$. Integrating both sides gives

$$\frac{1}{\frac{\sqrt{u^2+1}}{u}}\,du = \frac{1}{x}\,dx$$

$$\int \frac{1}{\frac{\sqrt{u^2+1}}{u}}\,du = \int \frac{1}{x}\,dx$$

$$\sqrt{u^2+1} = \ln(x) + c_2$$

The solution is

$$\sqrt{u(x)^2+1} - \ln(x) - c_2 = 0$$

Now u in the above solution is replaced back by y using $u = \frac{y}{x}$ which results in the solution

$$\sqrt{\frac{y^2}{x^2}+1} - \ln(x) - c_2 = 0$$

Summary
The solution(s) found are the following

$$\sqrt{\frac{y^2+x^2}{x^2}}-\ln(x)-c_2=0 \tag{1}$$

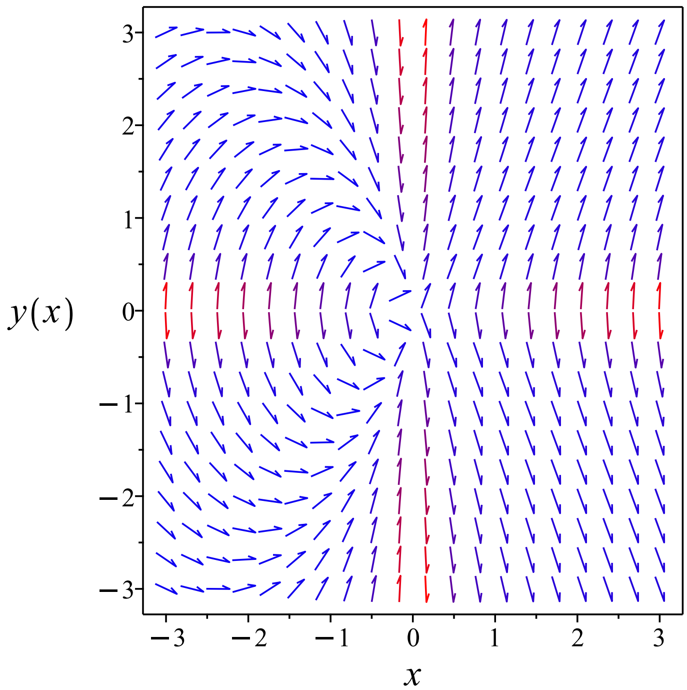


Figure 135: Slope field plot

Verification of solutions

$$\sqrt{\frac{y^2+x^2}{x^2}}-\ln(x)-c_2=0$$

Verified OK. `{0 < x}`

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
trying homogeneous G
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 28

```
dsolve(diff(y(x),x)=(x*sqrt(y(x)^2+x^2)+y(x)^2)/(x*y(x)),y(x), singsol=all)
```

$$\frac{x\ln\left(x\right)-c_1x-\sqrt{x^2+y\left(x\right)^2}}{x}=0$$

✓ Solution by Mathematica

Time used: 0.318 (sec). Leaf size: 54

```
DSolve[y'[x]==(x*Sqrt[y[x]^2+x^2]+y[x]^2)/(x*y[x]),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to -x\sqrt{\log^2(x)+2c_1\log(x)-1+{c_1}^2}$$
$$y(x)\to x\sqrt{\log^2(x)+2c_1\log(x)-1+{c_1}^2}$$

4.16 problem Problem 25

Internal problem ID [2680]
Internal file name [OUTPUT/2172_Sunday_June_05_2022_02_51_43_AM_66279645/index.tex]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.8, Change of Variables. page 79
Problem number: Problem 25.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y' - \frac{2(2y - x)}{y + x} = 0}$$

With initial conditions

$$[y(0) = 2]$$

4.16.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= \frac{4y - 2x}{y + x} \end{aligned}$$

The x domain of $f(x, y)$ when $y = 2$ is

$$\{x < -2 \lor -2 < x\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x, y)$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 2$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{4y - 2x}{y + x}\right) \\ &= \frac{4}{y + x} - \frac{2(2y - x)}{(y + x)^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 2$ is

$$\{x < -2 \lor -2 < x\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 2$ is inside this domain. Therefore solution exists and is unique.

4.16.2 Solving as homogeneous ode

In canonical form, the ODE is

$$\begin{aligned}y' &= F(x, y) \\ &= \frac{4y - 2x}{y + x}\end{aligned} \tag{1}$$

An ode of the form $y' = \frac{M(x,y)}{N(x,y)}$ is called homogeneous if the functions $M(x, y)$ and $N(x, y)$ are both homogeneous functions and of the same order. Recall that a function $f(x, y)$ is homogeneous of order n if

$$f(t^n x, t^n y) = t^n f(x, y)$$

In this case, it can be seen that both $M = 4y - 2x$ and $N = y + x$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{y}{x}$, or $y = ux$. Hence

$$\frac{dy}{dx} = \frac{du}{dx}x + u$$

Applying the transformation $y = ux$ to the above ODE in (1) gives

$$
\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}x}x + u &= \frac{4u-2}{u+1} \\
\frac{\mathrm{d}u}{\mathrm{d}x} &= \frac{\frac{4u(x)-2}{u(x)+1} - u(x)}{x}
\end{aligned}
$$

Or

$$
u'(x) - \frac{\frac{4u(x)-2}{u(x)+1} - u(x)}{x} = 0
$$

Or

$$
u'(x)\, xu(x) + u'(x)\, x + u(x)^2 - 3u(x) + 2 = 0
$$

Or

$$
x(u(x)+1)\, u'(x) + u(x)^2 - 3u(x) + 2 = 0
$$

Which is now solved as separable in $u(x)$. Which is now solved in $u(x)$. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= -\frac{u^2-3u+2}{x\,(u+1)}
\end{aligned}
$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u^2-3u+2}{u+1}$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\frac{u^2-3u+2}{u+1}}\, du &= -\frac{1}{x}\, dx \\
\int \frac{1}{\frac{u^2-3u+2}{u+1}}\, du &= \int -\frac{1}{x}\, dx \\
-2\ln(u-1) + 3\ln(u-2) &= -\ln(x) + c_2
\end{aligned}
$$

Raising both side to exponential gives

$$
\mathrm{e}^{-2\ln(u-1)+3\ln(u-2)} = \mathrm{e}^{-\ln(x)+c_2}
$$

Which simplifies to

$$
\frac{(u-2)^3}{(u-1)^2} = \frac{c_3}{x}
$$

The solution is

$$\frac{(u(x)-2)^3}{(u(x)-1)^2} = \frac{c_3}{x}$$

Now u in the above solution is replaced back by y using $u = \frac{y}{x}$ which results in the solution

$$\frac{\left(\frac{y}{x}-2\right)^3}{\left(\frac{y}{x}-1\right)^2} = \frac{c_3}{x}$$

Which simplifies to

$$-\frac{(2x-y)^3}{(-y+x)^2} = c_3$$

Substituting initial conditions and solving for c_3 gives $c_3 = 2$. Hence the solution be-
Summary
The solution(s) found are the following
comes

$$-\frac{(2x-y)^3}{(-y+x)^2} = 2 \tag{1}$$

Verification of solutions

$$-\frac{(2x-y)^3}{(-y+x)^2} = 2$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.657 (sec). Leaf size: 273

```
dsolve([diff(y(x),x)=2*(2*y(x)-x)/(x+y(x)),y(0) = 2],y(x), singsol=all)
```

$$y(x) = \frac{\left(3\sqrt{3}\,x\sqrt{x\,(27x+8)}+27x^2+36x+8\right)^{\frac{1}{3}}}{3} + \frac{4x+\frac{4}{3}}{\left(3\sqrt{3}\,x\sqrt{x\,(27x+8)}+27x^2+36x+8\right)^{\frac{1}{3}}} + 2x + \frac{2}{3}$$

✓ Solution by Mathematica

Time used: 60.289 (sec). Leaf size: 121

```
DSolve[{y'[x]==2*(2*y[x]-x)/(x+y[x]),{y[0]==2}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{3}\left(x\left(\frac{12}{\sqrt[3]{3\sqrt{3}\sqrt{x^3(27x+8)}+27x^2+36x+8}}+6\right) + \sqrt[3]{3\sqrt{3}\sqrt{x^3(27x+8)}+27x^2+36x+8} + \frac{4}{\sqrt[3]{3\sqrt{3}\sqrt{x^3(27x+8)}+27x^2+36x+8}}+2\right)$$

4.17 problem Problem 26

Internal problem ID [2681]
Internal file name [OUTPUT/2173_Sunday_June_05_2022_02_51_49_AM_57005676/index.tex]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.8, Change of Variables. page 79
Problem number: Problem 26.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "differentialType", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y' - \frac{2x - y}{4y + x} = 0}$$

With initial conditions

$$[y(1) = 1]$$

4.17.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= -\frac{-2x + y}{x + 4y} \end{aligned}$$

The x domain of $f(x, y)$ when $y = 1$ is

$$\{x < -4 \lor -4 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $f(x, y)$ when $x = 1$ is

$$\left\{y < -\frac{1}{4} \vee -\frac{1}{4} < y\right\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{-2x+y}{x+4y}\right)\\ &= -\frac{1}{x+4y} + \frac{-8x+4y}{(x+4y)^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 1$ is

$$\{x < -4 \vee -4 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 1$ is

$$\left\{y < -\frac{1}{4} \vee -\frac{1}{4} < y\right\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

4.17.2 Solving as homogeneous ode

In canonical form, the ODE is

$$\begin{aligned}y' &= F(x, y)\\ &= -\frac{-2x+y}{x+4y}\end{aligned} \tag{1}$$

An ode of the form $y' = \frac{M(x,y)}{N(x,y)}$ is called homogeneous if the functions $M(x, y)$ and $N(x, y)$ are both homogeneous functions and of the same order. Recall that a function $f(x, y)$ is homogeneous of order n if

$$f(t^n x, t^n y) = t^n f(x, y)$$

In this case, it can be seen that both $M = 2x - y$ and $N = x + 4y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{y}{x}$, or $y = ux$. Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x}x + u$$

Applying the transformation $y = ux$ to the above ODE in (1) gives

$$\frac{du}{dx}x + u = \frac{-u+2}{4u+1}$$
$$\frac{du}{dx} = \frac{\frac{-u(x)+2}{4u(x)+1} - u(x)}{x}$$

Or

$$u'(x) - \frac{\frac{-u(x)+2}{4u(x)+1} - u(x)}{x} = 0$$

Or

$$4u'(x)\, xu(x) + u'(x)\, x + 4u(x)^2 + 2u(x) - 2 = 0$$

Or

$$-2 + x(4u(x)+1)\, u'(x) + 4u(x)^2 + 2u(x) = 0$$

Which is now solved as separable in $u(x)$. Which is now solved in $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{2(2u^2+u-1)}{x\,(4u+1)} \end{aligned}$$

Where $f(x) = -\frac{2}{x}$ and $g(u) = \frac{2u^2+u-1}{4u+1}$. Integrating both sides gives

$$\frac{1}{\frac{2u^2+u-1}{4u+1}}\, du = -\frac{2}{x}\, dx$$
$$\int \frac{1}{\frac{2u^2+u-1}{4u+1}}\, du = \int -\frac{2}{x}\, dx$$
$$\ln\left(2u^2+u-1\right) = -2\ln(x) + c_2$$

Raising both side to exponential gives

$$2u^2 + u - 1 = \mathrm{e}^{-2\ln(x)+c_2}$$

Which simplifies to

$$2u^2 + u - 1 = \frac{c_3}{x^2}$$

Which simplifies to

$$2u(x)^2 + u(x) - 1 = \frac{c_3 e^{c_2}}{x^2}$$

The solution is

$$2u(x)^2 + u(x) - 1 = \frac{c_3 e^{c_2}}{x^2}$$

Now u in the above solution is replaced back by y using $u = \frac{y}{x}$ which results in the solution

$$\frac{2y^2}{x^2} + \frac{y}{x} - 1 = \frac{c_3 e^{c_2}}{x^2}$$

Which simplifies to

$$-(y + x)(x - 2y) = c_3 e^{c_2}$$

Substituting initial conditions and solving for c_2 gives $c_2 = \ln\left(\frac{2}{c_3}\right)$. Hence the solution becomes

Summary

The solution(s) found are the following

$$-(y + x)(x - 2y) = 2 \tag{1}$$

Verification of solutions

$$-(y + x)(x - 2y) = 2$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.141 (sec). Leaf size: 19

```
dsolve([diff(y(x),x)=(2*x-y(x))/(x+4*y(x)),y(1) = 1],y(x), singsol=all)
```

$$y(x) = -\frac{x}{4} + \frac{\sqrt{9x^2+16}}{4}$$

✓ Solution by Mathematica

Time used: 0.472 (sec). Leaf size: 24

```
DSolve[{y'[x]==(2*x-y[x])/(x+4*y[x]),{y[1]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4}\left(\sqrt{9x^2+16} - x\right)$$

4.18 problem Problem 27

Internal problem ID [2682]
Internal file name [OUTPUT/2174_Sunday_June_05_2022_02_51_53_AM_7818457/index.tex]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.8, Change of Variables. page 79
Problem number: Problem 27.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _dAlembert]
```

$$\boxed{y' - \frac{y - \sqrt{y^2 + x^2}}{x} = 0}$$

With initial conditions

$$[y(3) = 4]$$

4.18.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= \frac{y - \sqrt{x^2 + y^2}}{x} \end{aligned}$$

The x domain of $f(x, y)$ when $y = 4$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 3$ is inside this domain. The y domain of $f(x,y)$ when $x = 3$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 4$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{y-\sqrt{x^2+y^2}}{x}\right)\\ &= \frac{1-\frac{y}{\sqrt{x^2+y^2}}}{x}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 4$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 3$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 3$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 4$ is inside this domain. Therefore solution exists and is unique.

4.18.2 Solving as homogeneous ode

In canonical form, the ODE is

$$\begin{aligned}y' &= F(x,y)\\ &= \frac{y-\sqrt{x^2+y^2}}{x}\end{aligned} \tag{1}$$

An ode of the form $y' = \frac{M(x,y)}{N(x,y)}$ is called homogeneous if the functions $M(x,y)$ and $N(x,y)$ are both homogeneous functions and of the same order. Recall that a function $f(x,y)$ is homogeneous of order n if

$$f(t^n x, t^n y) = t^n f(x,y)$$

In this case, it can be seen that both $M = y-\sqrt{x^2+y^2}$ and $N = x$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{y}{x}$, or $y = ux$. Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x}x + u$$

Applying the transformation $y = ux$ to the above ODE in (1) gives

$$
\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}x}x + u &= u - \sqrt{u^2+1} \\
\frac{\mathrm{d}u}{\mathrm{d}x} &= -\frac{\sqrt{u(x)^2+1}}{x}
\end{aligned}
$$

Or

$$
u'(x) + \frac{\sqrt{u(x)^2+1}}{x} = 0
$$

Or

$$
u'(x)\,x + \sqrt{u(x)^2+1} = 0
$$

Which is now solved as separable in $u(x)$. Which is now solved in $u(x)$. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= -\frac{\sqrt{u^2+1}}{x}
\end{aligned}
$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \sqrt{u^2+1}$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\sqrt{u^2+1}}\,du &= -\frac{1}{x}\,dx \\
\int \frac{1}{\sqrt{u^2+1}}\,du &= \int -\frac{1}{x}\,dx \\
\operatorname{arcsinh}(u) &= -\ln(x) + c_2
\end{aligned}
$$

The solution is

$$
\operatorname{arcsinh}(u(x)) + \ln(x) - c_2 = 0
$$

Now u in the above solution is replaced back by y using $u = \frac{y}{x}$ which results in the solution

$$
\operatorname{arcsinh}\left(\frac{y}{x}\right) + \ln(x) - c_2 = 0
$$

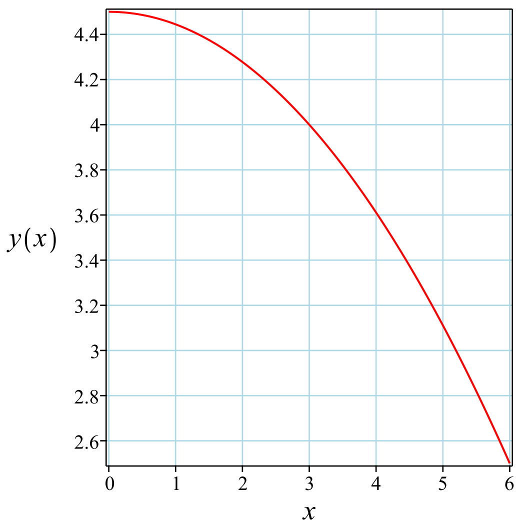
Substituting initial conditions and solving for c_2 gives $c_2 = \operatorname{arcsinh}\left(\frac{4}{3}\right) + \ln(3)$. Hence the solution becomes Solving for y from the above gives

$$
y = -\sinh\left(\ln(x) - \operatorname{arcsinh}\left(\frac{4}{3}\right) - \ln(3)\right)x
$$

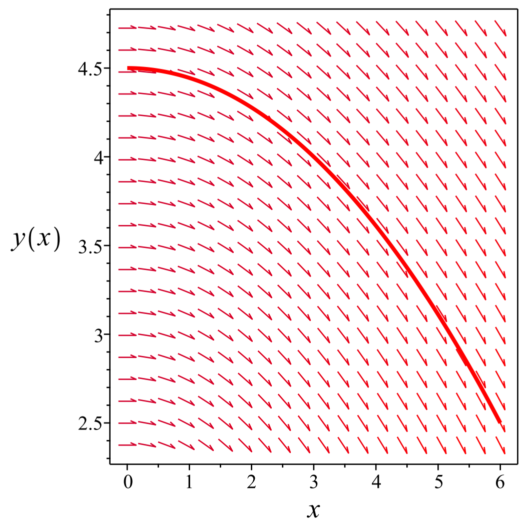
Summary

The solution(s) found are the following

$$y = -\sinh\left(\ln(x) - \operatorname{arcsinh}\left(\frac{4}{3}\right) - \ln(3)\right) x \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\sinh\left(\ln(x) - \operatorname{arcsinh}\left(\frac{4}{3}\right) - \ln(3)\right) x$$

Verified OK. `{0 < x}`

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
trying homogeneous G
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.375 (sec). Leaf size: 21

```
dsolve([diff(y(x),x)=(y(x)-sqrt(x^2+y(x)^2))/x,y(3) = 4],y(x), singsol=all)
```

$$y(x) = \frac{x^2}{2} - \frac{1}{2}$$
$$y(x) = -\frac{x^2}{18} + \frac{9}{2}$$

✓ Solution by Mathematica

Time used: 0.248 (sec). Leaf size: 29

```
DSolve[{y'[x]==(y[x]-Sqrt[x^2+y[x]^2])/x,{y[3]==4}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{9}{2} - \frac{x^2}{18}$$
$$y(x) \to \frac{1}{2}(x^2 - 1)$$

4.19 problem Problem 28

Internal problem ID [2683]
Internal file name [`OUTPUT/2175_Sunday_June_05_2022_02_52_00_AM_11057609/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.8, Change of Variables. page 79
Problem number: Problem 28.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{xy' - y - \sqrt{-y^2 + 4x^2} = 0}$$

4.19.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= \frac{y + \sqrt{4x^2 - y^2}}{x} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 + \frac{\left(y + \sqrt{4x^2 - y^2}\right)(b_3 - a_2)}{x} - \frac{\left(y + \sqrt{4x^2 - y^2}\right)^2 a_3}{x^2} \\ &- \left(\frac{4}{\sqrt{4x^2 - y^2}} - \frac{y + \sqrt{4x^2 - y^2}}{x^2}\right)(x a_2 + y a_3 + a_1) \\ &- \frac{\left(1 - \frac{y}{\sqrt{4x^2 - y^2}}\right)(x b_2 + y b_3 + b_1)}{x} = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{(4x^2 - y^2)^{\frac{3}{2}} a_3 + 4x^3 a_2 - 4x^3 b_3 + 8x^2 y a_3 - x^2 y b_2 - y^3 a_3 + \sqrt{4x^2 - y^2}\, x b_1 - \sqrt{4x^2 - y^2}\, y a_1 - x y b_1 + y^2}{\sqrt{4x^2 - y^2}\, x^2}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} &-(4x^2 - y^2)^{\frac{3}{2}} a_3 - 4x^3 a_2 + 4x^3 b_3 - 8x^2 y a_3 + x^2 y b_2 + y^3 a_3 \\ &- \sqrt{4x^2 - y^2}\, x b_1 + \sqrt{4x^2 - y^2}\, y a_1 + x y b_1 - y^2 a_1 = 0 \end{aligned} \tag{6E}$$

Simplifying the above gives

$$\begin{aligned} &-(4x^2 - y^2)^{\frac{3}{2}} a_3 + (4x^2 - y^2)\, x b_3 - (4x^2 - y^2)\, y a_3 - 4x^3 a_2 - 4x^2 y a_3 + x^2 y b_2 \\ &+ x\, y^2 b_3 + (4x^2 - y^2)\, a_1 - \sqrt{4x^2 - y^2}\, x b_1 + \sqrt{4x^2 - y^2}\, y a_1 - 4x^2 a_1 + x y b_1 = 0 \end{aligned} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$\begin{aligned} &-4x^3 a_2 + 4x^3 b_3 - 4x^2 \sqrt{4x^2 - y^2}\, a_3 - 8x^2 y a_3 + x^2 y b_2 + \sqrt{4x^2 - y^2}\, y^2 a_3 \\ &+ y^3 a_3 - \sqrt{4x^2 - y^2}\, x b_1 + x y b_1 + \sqrt{4x^2 - y^2}\, y a_1 - y^2 a_1 = 0 \end{aligned}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \sqrt{4x^2 - y^2}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \sqrt{4x^2 - y^2} = v_3\right\}$$

The above PDE (6E) now becomes

$$-4v_1^3 a_2 - 8v_1^2 v_2 a_3 - 4v_1^2 v_3 a_3 + v_2^3 a_3 + v_3 v_2^2 a_3 + v_1^2 v_2 b_2 \\ + 4v_1^3 b_3 - v_2^2 a_1 + v_3 v_2 a_1 + v_1 v_2 b_1 - v_3 v_1 b_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$(-4a_2 + 4b_3) v_1^3 + (-8a_3 + b_2) v_1^2 v_2 - 4v_1^2 v_3 a_3 + v_1 v_2 b_1 \\ - v_3 v_1 b_1 + v_2^3 a_3 + v_3 v_2^2 a_3 - v_2^2 a_1 + v_3 v_2 a_1 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} a_1 &= 0 \\ a_3 &= 0 \\ b_1 &= 0 \\ -a_1 &= 0 \\ -4a_3 &= 0 \\ -b_1 &= 0 \\ -4a_2 + 4b_3 &= 0 \\ -8a_3 + b_2 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = x$$
$$\eta = y$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(\frac{y + \sqrt{4x^2 - y^2}}{x}\right)(x) \\
&= -\sqrt{4x^2 - y^2} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{-\sqrt{4x^2 - y^2}} dy
\end{aligned}$$

Which results in

$$S = -\arctan\left(\frac{y}{\sqrt{4x^2 - y^2}}\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y+\sqrt{4x^2-y^2}}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{y}{\sqrt{4x^2-y^2}\, x} \\
S_y &= -\frac{1}{\sqrt{4x^2-y^2}}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\arctan\left(\frac{y}{\sqrt{-y^2+4x^2}}\right) = -\ln(x) + c_1$$

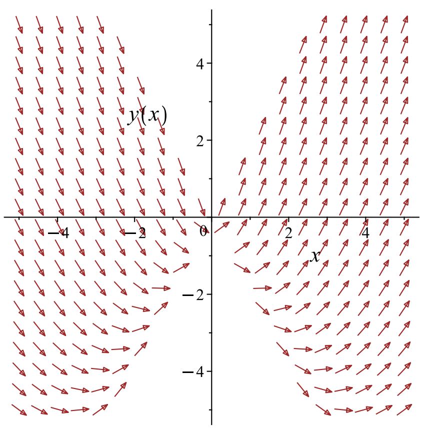
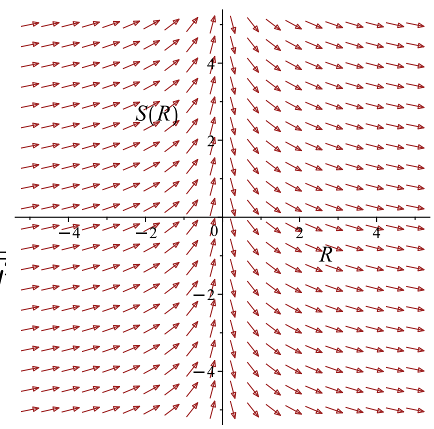
Which simplifies to

$$-\arctan\left(\frac{y}{\sqrt{-y^2+4x^2}}\right) = -\ln(x) + c_1$$

Which gives

$$y = -2\tan(-\ln(x)+c_1)\sqrt{\frac{x^2}{\tan(-\ln(x)+c_1)^2+1}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = \frac{y+\sqrt{4x^2-y^2}}{x}$

</td><td>$R = x$
$S = -\arctan\left(\frac{y}{\sqrt{4x^2-y^2}}\right)$</td><td>$\frac{dS}{dR} = -\frac{1}{R}$

</td></tr>
</table>

Summary

The solution(s) found are the following

$$
y = -2\tan\left(-\ln\left(x\right)+c_1\right)\sqrt{\frac{x^2}{\tan\left(-\ln\left(x\right)+c_1\right)^2+1}} \tag{1}
$$

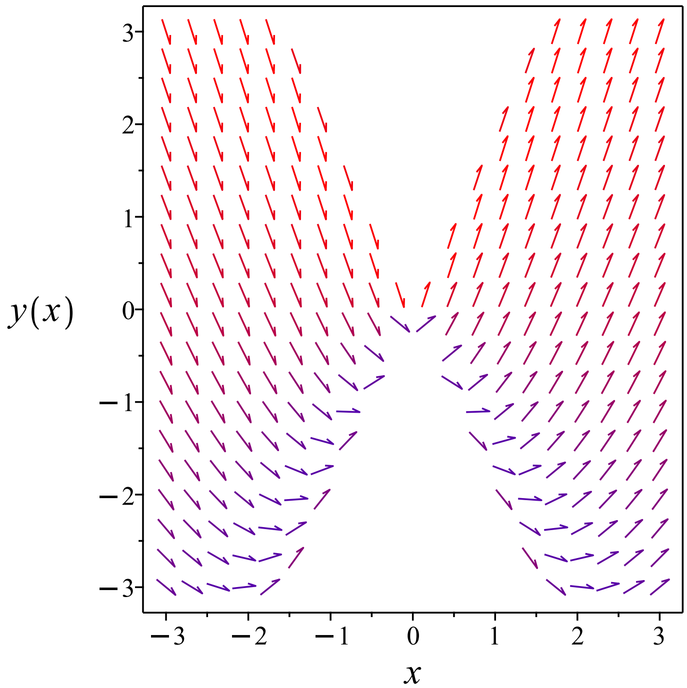


Figure 137: Slope field plot

Verification of solutions

$$y = -2\tan(-\ln(x) + c_1)\sqrt{\frac{x^2}{\tan(-\ln(x) + c_1)^2 + 1}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
trying homogeneous G
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 29

```
dsolve(x*diff(y(x),x)-y(x)=sqrt(4*x^2-y(x)^2),y(x), singsol=all)
```

$$-\arctan\left(\frac{y(x)}{\sqrt{4x^2-y(x)^2}}\right)+\ln(x)-c_1=0$$

✓ Solution by Mathematica

Time used: 0.416 (sec). Leaf size: 18

```
DSolve[x*y'[x]-y[x]==Sqrt[4*x^2-y[x]^2],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to -2x\cosh(i\log(x)+c_1)$$

4.20 problem Problem 29(a)

Internal problem ID [2684]
Internal file name [`OUTPUT/2176_Sunday_June_05_2022_02_52_06_AM_56666401/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.8, Change of Variables. page 79
Problem number: Problem 29(a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y' - \frac{x + ya}{ax - y} = 0}$$

4.20.1 Solving as homogeneous ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\frac{ya + x}{-ax + y} \end{aligned} \tag{1}$$

An ode of the form $y' = \frac{M(x,y)}{N(x,y)}$ is called homogeneous if the functions $M(x, y)$ and $N(x, y)$ are both homogeneous functions and of the same order. Recall that a function $f(x, y)$ is homogeneous of order n if

$$f(t^n x, t^n y) = t^n f(x, y)$$

In this case, it can be seen that both $M = ya+x$ and $N = ax-y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{y}{x}$, or $y = ux$. Hence

$$\frac{dy}{dx} = \frac{du}{dx}x + u$$

Applying the transformation $y = ux$ to the above ODE in (1) gives

$$\begin{aligned}\frac{du}{dx}x + u &= \frac{au+1}{a-u}\\ \frac{du}{dx} &= \frac{\frac{au(x)+1}{a-u(x)} - u(x)}{x}\end{aligned}$$

Or

$$u'(x) - \frac{\frac{au(x)+1}{a-u(x)} - u(x)}{x} = 0$$

Or

$$u'(x)\,xu(x) - u'(x)\,xa + u(x)^2 + 1 = 0$$

Or

$$-x(a - u(x))\,u'(x) + u(x)^2 + 1 = 0$$

Which is now solved as separable in $u(x)$. Which is now solved in $u(x)$. In canonical form the ODE is

$$\begin{aligned}u' &= F(x,u)\\ &= f(x)g(u)\\ &= -\frac{-u^2-1}{x\,(a-u)}\end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{-u^2-1}{a-u}$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\frac{-u^2-1}{a-u}}\,du &= -\frac{1}{x}\,dx\\ \int \frac{1}{\frac{-u^2-1}{a-u}}\,du &= \int -\frac{1}{x}\,dx\\ \frac{\ln(u^2+1)}{2} - a\arctan(u) &= -\ln(x) + c_2\end{aligned}$$

The solution is

$$\frac{\ln\left(u(x)^2+1\right)}{2} - a\arctan(u(x)) + \ln(x) - c_2 = 0$$

Now u in the above solution is replaced back by y using $u = \frac{y}{x}$ which results in the solution

$$\frac{\ln\left(\frac{y^2}{x^2}+1\right)}{2} - a\arctan\left(\frac{y}{x}\right) + \ln(x) - c_2 = 0$$

Summary

The solution(s) found are the following

$$\frac{\ln\left(\frac{y^2}{x^2}+1\right)}{2} - a\arctan\left(\frac{y}{x}\right) + \ln(x) - c_2 = 0 \tag{1}$$

Verification of solutions

$$\frac{\ln\left(\frac{y^2}{x^2}+1\right)}{2} - a\arctan\left(\frac{y}{x}\right) + \ln(x) - c_2 = 0$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.359 (sec). Leaf size: 25

```
dsolve(diff(y(x),x)=(x+a*y(x))/(a*x-y(x)),y(x), singsol=all)
```

$$y(x) = \tan\left(\operatorname{RootOf}\left(-2a\_Z + \ln\left(\sec(\_Z)^2 x^2\right) + 2c_1\right)\right) x$$

✓ Solution by Mathematica

Time used: 0.039 (sec). Leaf size: 34

```
DSolve[y'[x]==(x+a*y[x])/(a*x-y[x]),y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[a\arctan\left(\frac{y(x)}{x}\right)-\frac{1}{2}\log\left(\frac{y(x)^2}{x^2}+1\right)=\log(x)+c_1,y(x)\right]$$

4.21 problem Problem 29(b)

Internal problem ID [2685]
Internal file name [OUTPUT/2177_Sunday_June_05_2022_02_52_11_AM_56933570/index.tex]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.8, Change of Variables. page 79
Problem number: Problem 29(b).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y' - \frac{x + \frac{y}{2}}{\frac{x}{2} - y} = 0}$$

With initial conditions

$$[y(1) = 1]$$

4.21.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= -\frac{2x + y}{2y - x} \end{aligned}$$

The x domain of $f(x, y)$ when $y = 1$ is

$$\{x < 2 \lor 2 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $f(x, y)$ when $x = 1$ is

$$\left\{y < \frac{1}{2} \vee \frac{1}{2} < y\right\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{2x + y}{2y - x}\right) \\
&= -\frac{1}{2y - x} + \frac{2y + 4x}{(2y - x)^2}
\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 1$ is

$$\{x < 2 \vee 2 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 1$ is

$$\left\{y < \frac{1}{2} \vee \frac{1}{2} < y\right\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

4.21.2 Solving as homogeneous ode

In canonical form, the ODE is

$$\begin{aligned}
y' &= F(x, y) \\
&= -\frac{2x + y}{2y - x}
\end{aligned} \tag{1}$$

An ode of the form $y' = \frac{M(x,y)}{N(x,y)}$ is called homogeneous if the functions $M(x, y)$ and $N(x, y)$ are both homogeneous functions and of the same order. Recall that a function $f(x, y)$ is homogeneous of order n if

$$f(t^n x, t^n y) = t^n f(x, y)$$

In this case, it can be seen that both $M = 2x + y$ and $N = x - 2y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{y}{x}$, or $y = ux$. Hence

$$\frac{dy}{dx} = \frac{du}{dx}x + u$$

Applying the transformation $y = ux$ to the above ODE in (1) gives

$$\frac{du}{dx}x + u = \frac{-u-2}{2u-1}$$
$$\frac{du}{dx} = \frac{\frac{-u(x)-2}{2u(x)-1} - u(x)}{x}$$

Or

$$u'(x) - \frac{\frac{-u(x)-2}{2u(x)-1} - u(x)}{x} = 0$$

Or

$$2u'(x)\,xu(x) - u'(x)\,x + 2u(x)^2 + 2 = 0$$

Or

$$2 + x(2u(x) - 1)\,u'(x) + 2u(x)^2 = 0$$

Which is now solved as separable in $u(x)$. Which is now solved in $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{2(u^2+1)}{x\,(2u-1)} \end{aligned}$$

Where $f(x) = -\frac{2}{x}$ and $g(u) = \frac{u^2+1}{2u-1}$. Integrating both sides gives

$$\frac{1}{\frac{u^2+1}{2u-1}}\,du = -\frac{2}{x}\,dx$$
$$\int \frac{1}{\frac{u^2+1}{2u-1}}\,du = \int -\frac{2}{x}\,dx$$
$$\ln\left(u^2+1\right) - \arctan(u) = -2\ln(x) + c_2$$

The solution is

$$\ln\left(u(x)^2+1\right) - \arctan(u(x)) + 2\ln(x) - c_2 = 0$$

Now u in the above solution is replaced back by y using $u = \frac{y}{x}$ which results in the solution

$$\ln\left(\frac{y^2}{x^2}+1\right) - \arctan\left(\frac{y}{x}\right) + 2\ln(x) - c_2 = 0$$

Substituting initial conditions and solving for c_2 gives $c_2 = \ln(2) - \frac{\pi}{4}$. Hence the solution

Summary

The solution(s) found are the following

becomes

$$\ln\left(\frac{y^2}{x^2}+1\right) - \arctan\left(\frac{y}{x}\right) + 2\ln(x) - \ln(2) + \frac{\pi}{4} = 0 \tag{1}$$

Verification of solutions

$$\ln\left(\frac{y^2}{x^2}+1\right) - \arctan\left(\frac{y}{x}\right) + 2\ln(x) - \ln(2) + \frac{\pi}{4} = 0$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.188 (sec). Leaf size: 30

```
dsolve([diff(y(x),x)=(x+1/2*y(x))/(1/2*x-y(x)),y(1) = 1],y(x), singsol=all)
```

$$y(x) = \tan\left(\operatorname{RootOf}\left(4\_Z - 4\ln\left(\sec\left(\_Z\right)^2\right) - 8\ln(x) + 4\ln(2) - \pi\right)\right) x$$

✓ Solution by Mathematica

Time used: 0.046 (sec). Leaf size: 42

```
DSolve[{y'[x]==(x+1/2*y[x])/(1/2*x-y[x]),{y[1]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\log\left(\frac{y(x)^2}{x^2}+1\right) - \arctan\left(\frac{y(x)}{x}\right) = \frac{1}{4}(4\log(2)-\pi) - 2\log(x), y(x)\right]$$

4.22 problem Problem 38

Internal problem ID [2686]
Internal file name [`OUTPUT/2178_Sunday_June_05_2022_02_52_15_AM_83503958/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.8, Change of Variables. page 79
Problem number: Problem 38.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class D`], _Bernoulli]
```

$$\boxed{y' - \frac{y}{x} - \frac{4x^2 \cos(x)}{y} = 0}$$

4.22.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$u'(x)\, x - \frac{4x \cos(x)}{u(x)} = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= \frac{4\cos(x)}{u} \end{aligned}$$

Where $f(x) = 4\cos(x)$ and $g(u) = \frac{1}{u}$. Integrating both sides gives

$$\frac{1}{\frac{1}{u}}\,du = 4\cos(x)\,dx$$

$$\int \frac{1}{\frac{1}{u}}\,du = \int 4\cos(x)\,dx$$

$$\frac{u^2}{2} = 4\sin(x) + c_2$$

The solution is

$$\frac{u(x)^2}{2} - 4\sin(x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{y^2}{2x^2} - 4\sin(x) - c_2 = 0$$
$$\frac{y^2}{2x^2} - 4\sin(x) - c_2 = 0$$

Summary

The solution(s) found are the following

$$\frac{y^2}{2x^2} - 4\sin(x) - c_2 = 0 \tag{1}$$

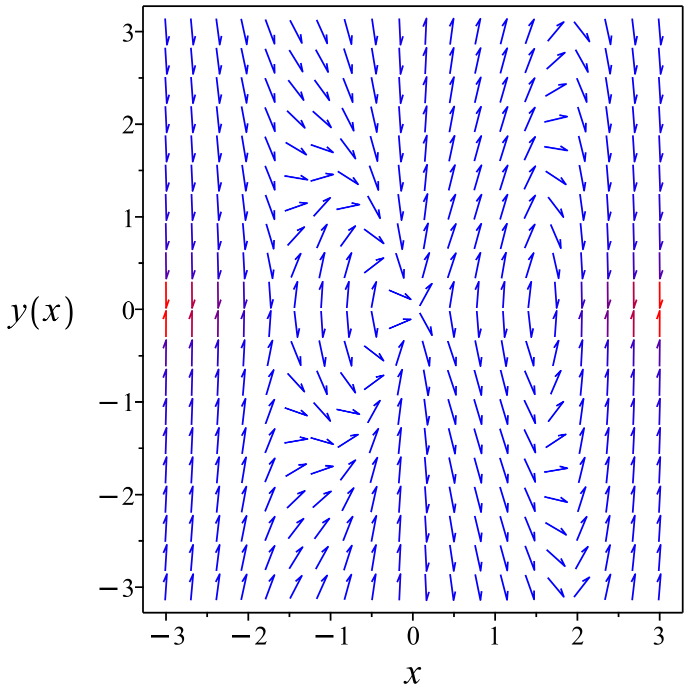


Figure 138: Slope field plot

Verification of solutions

$$\frac{y^2}{2x^2} - 4\sin(x) - c_2 = 0$$

Verified OK.

4.22.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y^2 + 4\cos(x)\,x^3}{xy}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 124: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \frac{x^2}{y}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{x^2}{y}} dy \end{aligned}$$

Which results in

$$S = \frac{y^2}{2x^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y^2 + 4\cos(x)\, x^3}{xy}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y^2}{x^3} \\ S_y &= \frac{y}{x^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 4\cos(x) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 4\cos(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = 4\sin(R) + c_1 \tag{4}$$

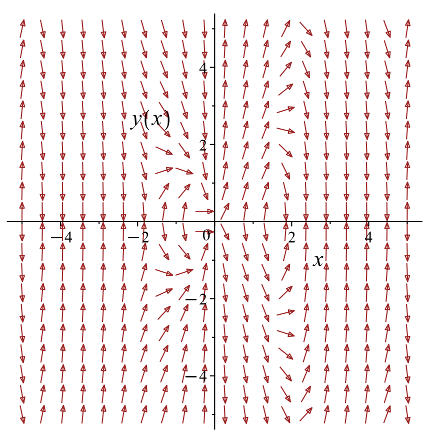
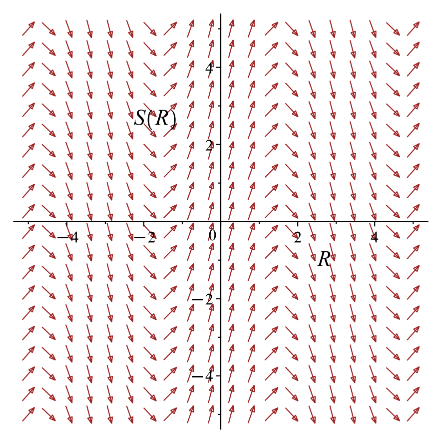
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y^2}{2x^2} = 4\sin(x) + c_1$$

Which simplifies to

$$\frac{y^2}{2x^2} = 4\sin(x) + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y^2+4\cos(x)x^3}{xy}$ 	$R = x$ $S = \frac{y^2}{2x^2}$	$\frac{dS}{dR} = 4\cos(R)$ 

Summary
The solution(s) found are the following

$$\frac{y^2}{2x^2} = 4\sin(x) + c_1 \tag{1}$$

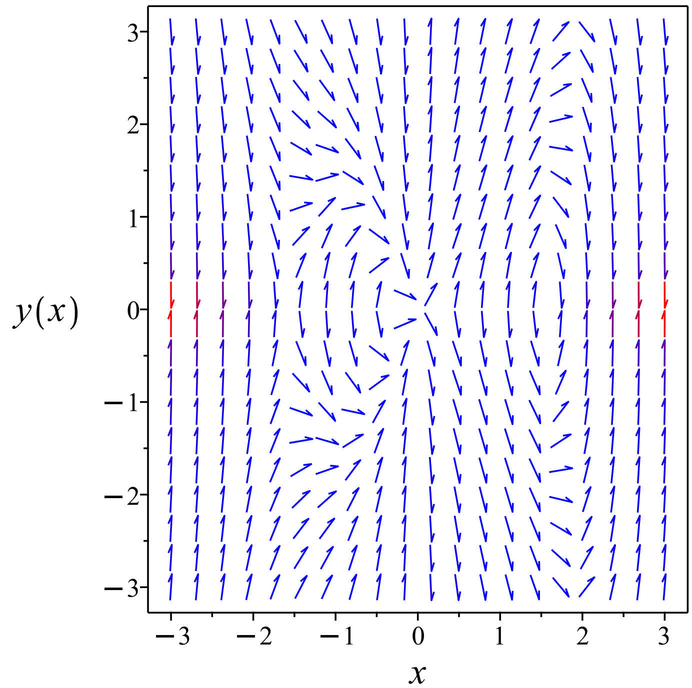


Figure 139: Slope field plot

Verification of solutions

$$\frac{y^2}{2x^2} = 4\sin(x) + c_1$$

Verified OK.

4.22.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{y^2 + 4\cos(x)\, x^3}{xy} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{1}{x} y + 4x^2 \cos(x) \frac{1}{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x) y + f_1(x) y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x) y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= \frac{1}{x} \\
f_1(x) &= 4x^2 \cos(x) \\
n &= -1
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$
y'y = \frac{y^2}{x} + 4x^2 \cos(x) \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= y^2 \qquad (5)
\end{aligned}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = 2yy' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
\frac{w'(x)}{2} &= \frac{w(x)}{x} + 4x^2 \cos(x) \\
w' &= \frac{2w}{x} + 8x^2 \cos(x) \qquad (7)
\end{aligned}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{2}{x} \\
q(x) &= 8x^2 \cos(x)
\end{aligned}
$$

Hence the ode is

$$w'(x) - \frac{2w(x)}{x} = 8x^2 \cos(x)$$

The integrating factor μ is

$$\mu = e^{\int -\frac{2}{x}dx}$$
$$= \frac{1}{x^2}$$

The ode becomes

$$\frac{d}{dx}(\mu w) = (\mu)\left(8x^2 \cos(x)\right)$$
$$\frac{d}{dx}\left(\frac{w}{x^2}\right) = \left(\frac{1}{x^2}\right)\left(8x^2 \cos(x)\right)$$
$$d\left(\frac{w}{x^2}\right) = (8\cos(x))\ dx$$

Integrating gives

$$\frac{w}{x^2} = \int 8\cos(x)\ dx$$
$$\frac{w}{x^2} = 8\sin(x) + c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2}$ results in

$$w(x) = 8x^2 \sin(x) + c_1 x^2$$

which simplifies to

$$w(x) = x^2(8\sin(x) + c_1)$$

Replacing w in the above by y^2 using equation (5) gives the final solution.

$$y^2 = x^2(8\sin(x) + c_1)$$

Solving for y gives

$$y(x) = \sqrt{8\sin(x) + c_1}\, x$$
$$y(x) = -\sqrt{8\sin(x) + c_1}\, x$$

Summary

The solution(s) found are the following

$$y = \sqrt{8 \sin (x) + c_1}\, x \tag{1}$$
$$y = -\sqrt{8 \sin (x) + c_1}\, x \tag{2}$$

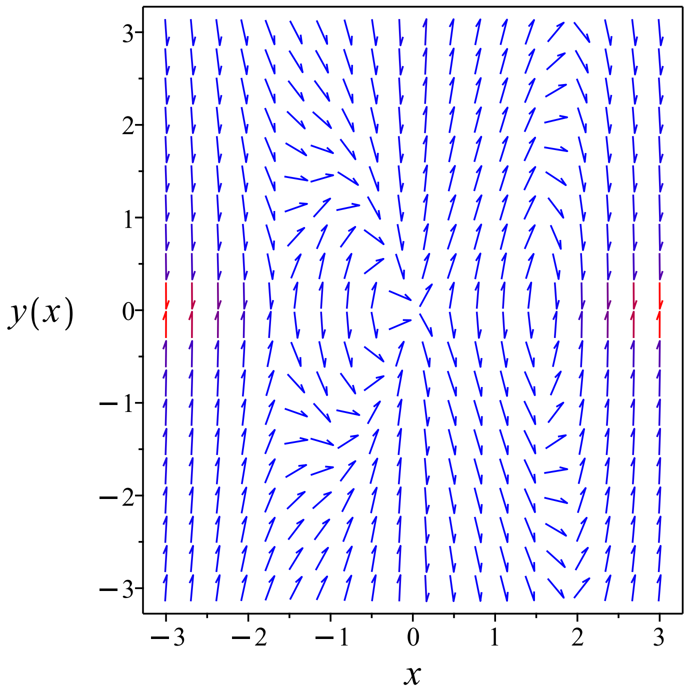


Figure 140: Slope field plot

Verification of solutions

$$y = \sqrt{8 \sin (x) + c_1}\, x$$

Verified OK.

$$y = -\sqrt{8 \sin (x) + c_1}\, x$$

Verified OK.

4.22.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(xy)\,\mathrm{d}y &= \left(y^2 + 4\cos(x)\,x^3\right)\mathrm{d}x\\ \left(-y^2 - 4\cos(x)\,x^3\right)\mathrm{d}x + (xy)\,\mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -y^2 - 4\cos(x)\,x^3\\ N(x,y) &= xy\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-y^2 - 4\cos(x)\, x^3\right)\\ &= -2y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(xy)\\ &= y\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{xy}((-2y) - (y))\\ &= -\frac{3}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \,\mathrm{d}x}\\ &= e^{\int -\frac{3}{x} \,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-3\ln(x)}\\ &= \frac{1}{x^3}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{x^3}\left(-y^2 - 4\cos(x)\, x^3\right)\\ &= \frac{-y^2 - 4\cos(x)\, x^3}{x^3}\end{aligned}$$

And

$$\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^3}(xy) \\
&= \frac{y}{x^2}
\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{-y^2 - 4\cos(x)\,x^3}{x^3}\right) + \left(\frac{y}{x^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{-y^2 - 4\cos(x)\,x^3}{x^3}\,\mathrm{d}x
\end{aligned}$$

$$\phi = -4\sin(x) + \frac{y^2}{2x^2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{y}{x^2} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{y}{x^2}$. Therefore equation (4) becomes

$$\frac{y}{x^2} = \frac{y}{x^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -4\sin(x) + \frac{y^2}{2x^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -4\sin(x) + \frac{y^2}{2x^2}$$

Summary
The solution(s) found are the following

$$\frac{y^2}{2x^2} - 4\sin(x) = c_1 \tag{1}$$

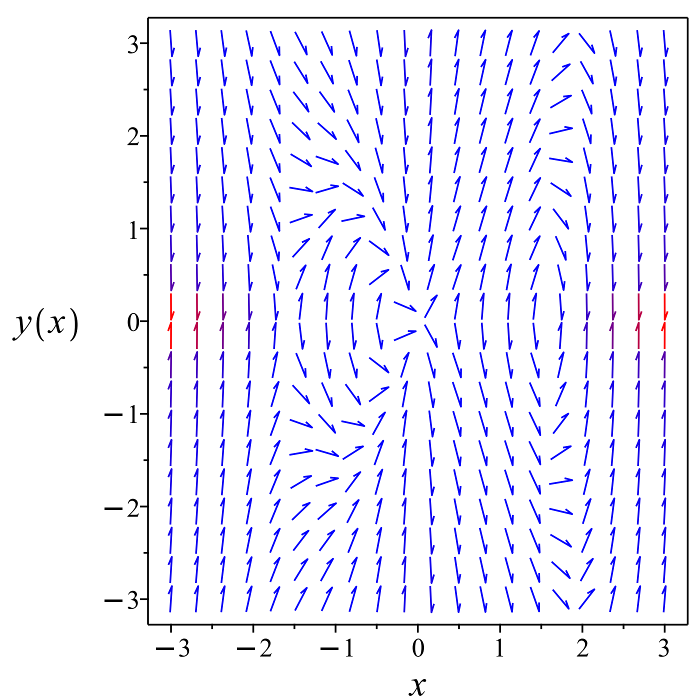


Figure 141: Slope field plot

Verification of solutions

$$\frac{y^2}{2x^2} - 4\sin(x) = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 28

```
dsolve(diff(y(x),x)-1/x*y(x)=4*x^2/y(x)*cos(x),y(x), singsol=all)
```

$$y(x) = \sqrt{8\sin(x) + c_1}\, x$$
$$y(x) = -\sqrt{8\sin(x) + c_1}\, x$$

✓ Solution by Mathematica

Time used: 0.298 (sec). Leaf size: 36

```
DSolve[y'[x]-1/x*y[x]==4*x^2/y[x]*Cos[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -x\sqrt{8\sin(x) + c_1}$$
$$y(x) \to x\sqrt{8\sin(x) + c_1}$$

4.23 problem Problem 39

Internal problem ID [2687]
Internal file name [`OUTPUT/2179_Sunday_June_05_2022_02_52_19_AM_84147090/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.8, Change of Variables. page 79
Problem number: Problem 39.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_Bernoulli]`

$$\boxed{y' + \frac{y\tan(x)}{2} - 2y^3\sin(x) = 0}$$

4.23.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{\tan(x)\, y}{2} + 2y^3\sin(x) \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 126: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{y^3}{\cos(x)}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{y^3}{\cos(x)}} dy
\end{aligned}$$

Which results in

$$S = -\frac{\cos(x)}{2y^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{\tan(x) y}{2} + 2y^3 \sin(x)$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{\sin(x)}{2y^2} \\
S_y &= \frac{\cos(x)}{y^3}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \sin(2x) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \sin(2R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{\cos(2R)}{2} + c_1 \tag{4}$$

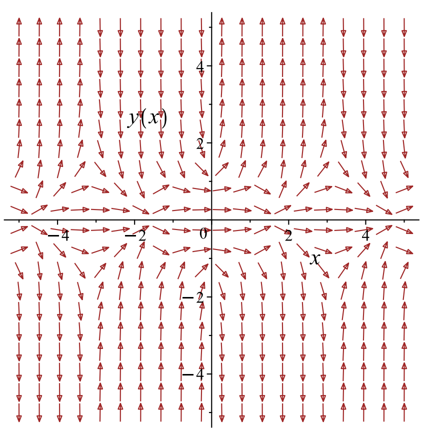
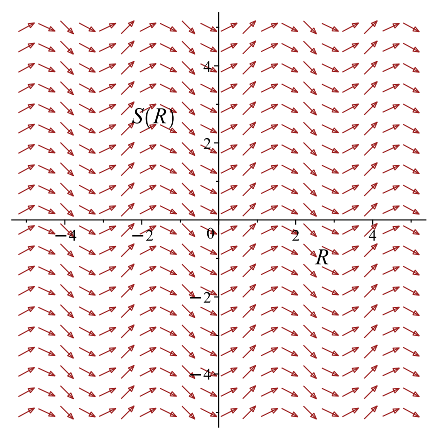
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{\cos(x)}{2y^2} = -\frac{\cos(2x)}{2} + c_1$$

Which simplifies to

$$-\frac{\cos(x)}{2y^2} = -\frac{\cos(2x)}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{\tan(x)y}{2} + 2y^3\sin(x)$ 	$R = x$ $S = -\frac{\cos(x)}{2y^2}$	$\frac{dS}{dR} = \sin(2R)$ 

Summary

The solution(s) found are the following

$$-\frac{\cos(x)}{2y^2} = -\frac{\cos(2x)}{2} + c_1 \tag{1}$$

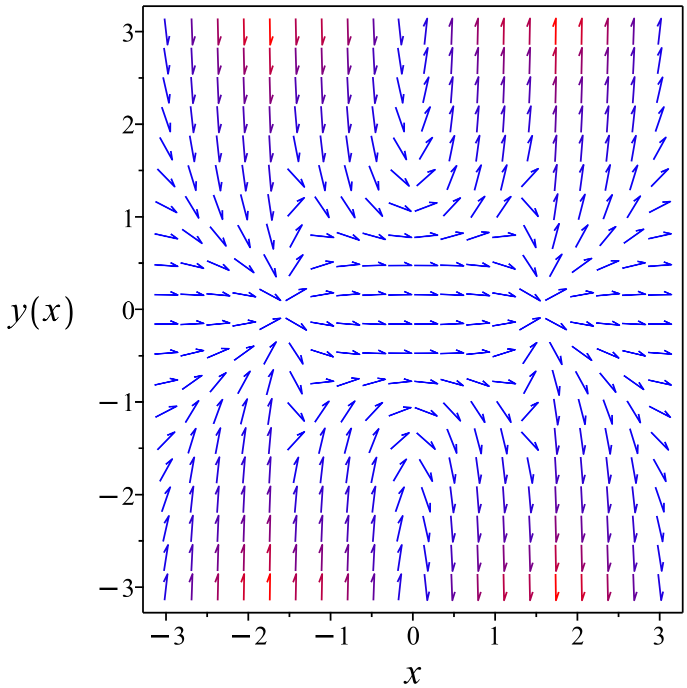


Figure 142: Slope field plot

Verification of solutions

$$-\frac{\cos(x)}{2y^2} = -\frac{\cos(2x)}{2} + c_1$$

Verified OK.

4.23.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{\tan(x)\,y}{2} + 2y^3\sin(x) \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{\tan(x)}{2}y + 2\sin(x)\,y^3 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -\frac{\tan(x)}{2} \\ f_1(x) &= 2\sin(x) \\ n &= 3 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^3$ gives

$$y'\frac{1}{y^3} = -\frac{\tan(x)}{2y^2} + 2\sin(x) \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \frac{1}{y^2} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{2}{y^3}y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -\frac{w'(x)}{2} &= -\frac{\tan(x)\,w(x)}{2} + 2\sin(x) \\ w' &= \tan(x)\,w - 4\sin(x) \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$
\begin{aligned}
p(x) &= -\tan(x) \\
q(x) &= -4\sin(x)
\end{aligned}
$$

Hence the ode is

$$
w'(x) - \tan(x)\, w(x) = -4\sin(x)
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int -\tan(x)dx} \\
&= \cos(x)
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)\,(-4\sin(x)) \\
\frac{\mathrm{d}}{\mathrm{d}x}(\cos(x)\, w) &= (\cos(x))\,(-4\sin(x)) \\
\mathrm{d}(\cos(x)\, w) &= (-2\sin(2x))\ \mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\cos(x)\, w &= \int -2\sin(2x)\ \mathrm{d}x \\
\cos(x)\, w &= \cos(2x) + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \cos(x)$ results in

$$
w(x) = \sec(x)\cos(2x) + c_1 \sec(x)
$$

which simplifies to

$$
w(x) = (c_1 - 1)\sec(x) + 2\cos(x)
$$

Replacing w in the above by $\frac{1}{y^2}$ using equation (5) gives the final solution.

$$
\frac{1}{y^2} = (c_1 - 1)\sec(x) + 2\cos(x)
$$

Solving for y gives

$$
\begin{aligned}
y(x) &= \frac{\sqrt{\left(2\cos(x)^2 - 1 + c_1\right)\cos(x)}}{2\cos(x)^2 - 1 + c_1} \\
y(x) &= -\frac{\sqrt{\left(2\cos(x)^2 - 1 + c_1\right)\cos(x)}}{2\cos(x)^2 - 1 + c_1}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = \frac{\sqrt{\left(2 \cos \left(x\right)^{2}-1+c_{1}\right) \cos \left(x\right)}}{2 \cos \left(x\right)^{2}-1+c_{1}} \tag{1}$$

$$y = -\frac{\sqrt{\left(2 \cos \left(x\right)^{2}-1+c_{1}\right) \cos \left(x\right)}}{2 \cos \left(x\right)^{2}-1+c_{1}} \tag{2}$$

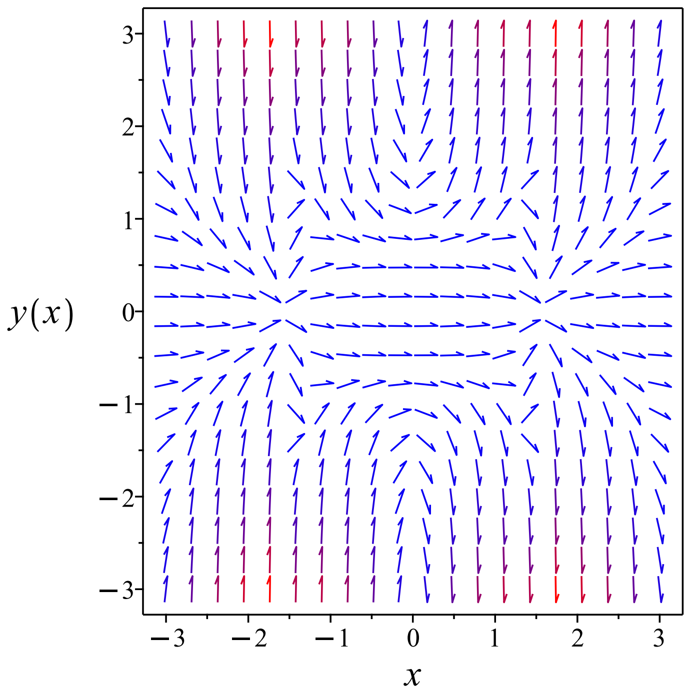


Figure 143: Slope field plot

Verification of solutions

$$y = \frac{\sqrt{\left(2 \cos \left(x\right)^{2}-1+c_{1}\right) \cos \left(x\right)}}{2 \cos \left(x\right)^{2}-1+c_{1}}$$

Verified OK.

$$y = -\frac{\sqrt{\left(2 \cos \left(x\right)^{2}-1+c_{1}\right) \cos \left(x\right)}}{2 \cos \left(x\right)^{2}-1+c_{1}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 56

```
dsolve(diff(y(x),x)+1/2*tan(x)*y(x)=2*y(x)^3*sin(x),y(x), singsol=all)
```

$$y(x) = -\frac{\sqrt{\left(-2\sin\left(x\right)^{2}+c_{1}\right)\cos\left(x\right)}}{-2\sin\left(x\right)^{2}+c_{1}}$$
$$y(x) = \frac{\sqrt{\left(-2\sin\left(x\right)^{2}+c_{1}\right)\cos\left(x\right)}}{-2\sin\left(x\right)^{2}+c_{1}}$$

✓ Solution by Mathematica

Time used: 5.32 (sec). Leaf size: 227

```
DSolve[y'[x]+1/2*Tan(x)*y[x]==2*y[x]^3*Sin[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{e^{\frac{1}{4}/\text{Tan}}\sqrt[4]{\text{Tan}}}{\sqrt{e^{\frac{\text{Tan}x^2}{2}}\left(-i\sqrt{2\pi}\text{erf}\left(\frac{\text{Tan}x+i}{\sqrt{2}\sqrt{\text{Tan}}}\right)+\sqrt{2\pi}\text{erfi}\left(\frac{1+i\text{Tan}x}{\sqrt{2}\sqrt{\text{Tan}}}\right)+c_1e^{\frac{1}{2}/\text{Tan}}\sqrt{\text{Tan}}\right)}}$$
$$y(x) \to \frac{e^{\frac{1}{4}/\text{Tan}}\sqrt[4]{\text{Tan}}}{\sqrt{e^{\frac{\text{Tan}x^2}{2}}\left(-i\sqrt{2\pi}\text{erf}\left(\frac{\text{Tan}x+i}{\sqrt{2}\sqrt{\text{Tan}}}\right)+\sqrt{2\pi}\text{erfi}\left(\frac{1+i\text{Tan}x}{\sqrt{2}\sqrt{\text{Tan}}}\right)+c_1e^{\frac{1}{2}/\text{Tan}}\sqrt{\text{Tan}}\right)}}$$
$$y(x) \to 0$$

4.24 problem Problem 40

Internal problem ID [2688]
Internal file name [`OUTPUT/2180_Sunday_June_05_2022_02_52_27_AM_46090535/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.8, Change of Variables. page 79
Problem number: Problem 40.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_Bernoulli]`

$$\boxed{y' - \frac{3y}{2x} - 6y^{\frac{1}{3}}x^2 \ln(x) = 0}$$

4.24.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{6y^{\frac{1}{3}}x^3 \ln(x) + \frac{3y}{2}}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 128: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x, y) = 0$$
$$\eta(x, y) = y^{\frac{1}{3}} x \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{y^{\frac{1}{3}} x} dy \end{aligned}$$

Which results in

$$S = \frac{3y^{\frac{2}{3}}}{2x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{6y^{\frac{1}{3}} x^3 \ln(x) + \frac{3y}{2}}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{3y^{\frac{2}{3}}}{2x^2} \\ S_y &= \frac{1}{y^{\frac{1}{3}} x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 6 \ln(x) x \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 6 \ln(R) R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = 3R^2 \ln(R) - \frac{3R^2}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{3y^{\frac{2}{3}}}{2x} = 3\ln(x)\, x^2 - \frac{3x^2}{2} + c_1$$

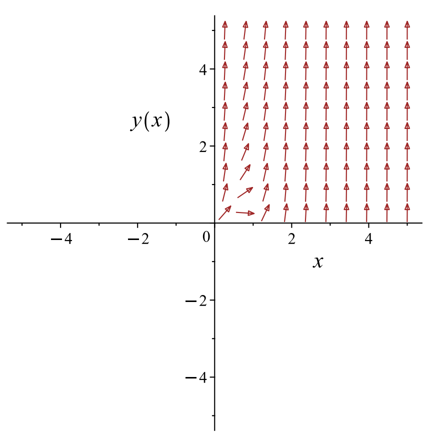
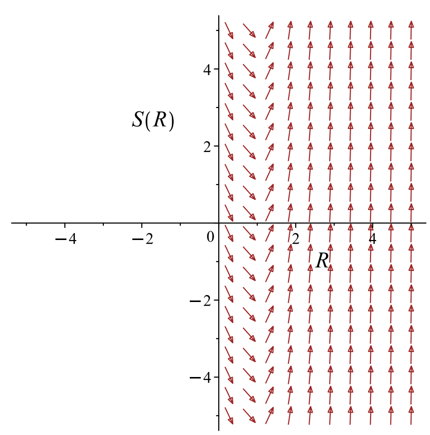
Which simplifies to

$$\frac{3y^{\frac{2}{3}}}{2x} = 3\ln(x)\, x^2 - \frac{3x^2}{2} + c_1$$

Which gives

$$y = \frac{\left(18x^3 \ln(x) - 9x^3 + 6c_1 x\right)^{\frac{3}{2}}}{27}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{6y^{\frac{1}{3}}x^3\ln(x)+\frac{3y}{2}}{x}$ 	$R = x$ $S = \frac{3y^{\frac{2}{3}}}{2x}$	$\frac{dS}{dR} = 6\ln(R)\, R$ 

Summary

The solution(s) found are the following

$$y = \frac{(18x^3 \ln(x) - 9x^3 + 6c_1 x)^{\frac{3}{2}}}{27} \tag{1}$$

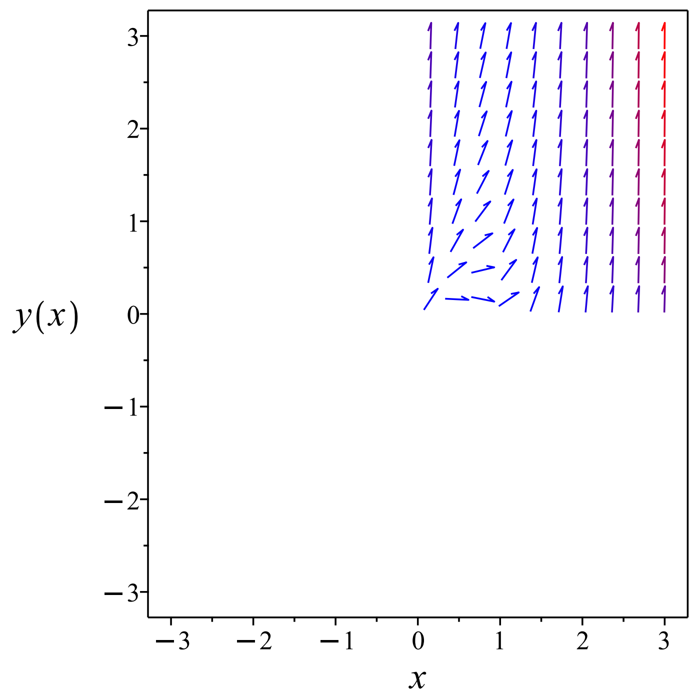


Figure 144: Slope field plot

Verification of solutions

$$y = \frac{(18x^3 \ln(x) - 9x^3 + 6c_1 x)^{\frac{3}{2}}}{27}$$

Verified OK.

4.24.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{6y^{\frac{1}{3}} x^3 \ln(x) + \frac{3y}{2}}{x} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{3}{2x} y + 6 \ln(x)\, x^2 y^{\frac{1}{3}} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= \frac{3}{2x} \\ f_1(x) &= 6 \ln (x) x^2 \\ n &= \frac{1}{3} \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^{\frac{1}{3}}$ gives

$$y' \frac{1}{y^{\frac{1}{3}}} = \frac{3y^{\frac{2}{3}}}{2x} + 6 \ln (x) x^2 \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= y^{\frac{2}{3}} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = \frac{2}{3y^{\frac{1}{3}}} y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} \frac{3w'(x)}{2} &= \frac{3w(x)}{2x} + 6 \ln (x) x^2 \\ w' &= \frac{w}{x} + 4 \ln (x) x^2 \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned}
p(x) &= -\frac{1}{x} \\
q(x) &= 4\ln(x)\, x^2
\end{aligned}$$

Hence the ode is

$$w'(x) - \frac{w(x)}{x} = 4\ln(x)\, x^2$$

The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int -\frac{1}{x}dx} \\
&= \frac{1}{x}
\end{aligned}$$

The ode becomes

$$\begin{aligned}
\frac{d}{dx}(\mu w) &= (\mu)\left(4\ln(x)\, x^2\right) \\
\frac{d}{dx}\left(\frac{w}{x}\right) &= \left(\frac{1}{x}\right)\left(4\ln(x)\, x^2\right) \\
d\left(\frac{w}{x}\right) &= (4\ln(x)\, x)\ dx
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\frac{w}{x} &= \int 4\ln(x)\, x\, dx \\
\frac{w}{x} &= 2\ln(x)\, x^2 - x^2 + c_1
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$w(x) = x\left(2\ln(x)\, x^2 - x^2\right) + c_1 x$$

which simplifies to

$$w(x) = 2x^3 \ln(x) - x^3 + c_1 x$$

Replacing w in the above by $y^{\frac{2}{3}}$ using equation (5) gives the final solution.

$$y^{\frac{2}{3}} = 2x^3 \ln(x) - x^3 + c_1 x$$

Summary

The solution(s) found are the following

$$y^{\frac{2}{3}} = 2x^3 \ln(x) - x^3 + c_1 x \tag{1}$$

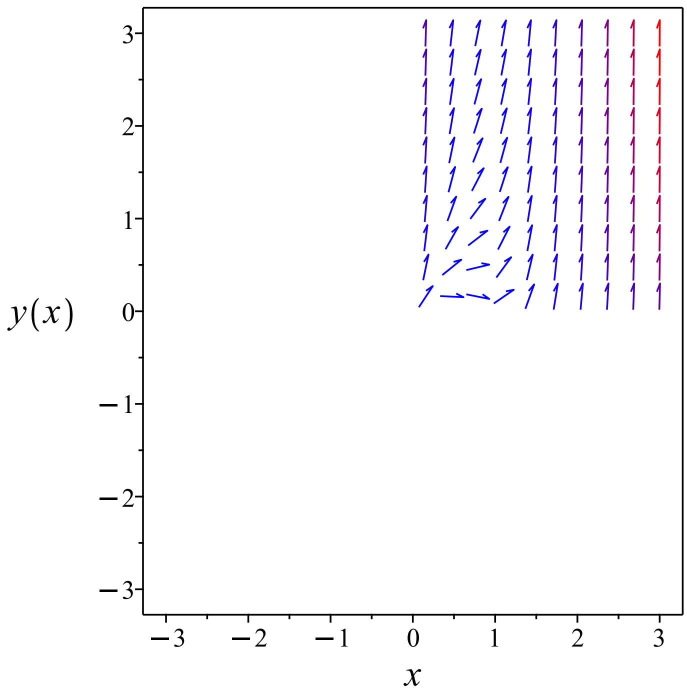


Figure 145: Slope field plot

Verification of solutions

$$y^{\frac{2}{3}} = 2x^3 \ln(x) - x^3 + c_1 x$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 22

```
dsolve(diff(y(x),x)-3/(2*x)*y(x)=6*y(x)^(1/3)*x^2*ln(x),y(x), singsol=all)
```

$$-2x^3 \ln(x) + x^3 + y(x)^{\frac{2}{3}} - c_1 x = 0$$

✓ Solution by Mathematica

Time used: 0.795 (sec). Leaf size: 26

```
DSolve[y'[x]-3/(2*x)*y[x]==6*y[x]^(1/3)*x^2*Log[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \left(x\left(-x^2 + 2x^2 \log(x) + c_1\right)\right)^{3/2}$$

4.25 problem Problem 41

Internal problem ID [2689]
Internal file name [`OUTPUT/2181_Sunday_June_05_2022_02_52_32_AM_14695870/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.8, Change of Variables. page 79
Problem number: Problem 41.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_Bernoulli]`

$$\boxed{y' + \frac{2y}{x} - 6\sqrt{x^2+1}\,\sqrt{y} = 0}$$

4.25.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= \frac{6\sqrt{x^2+1}\,\sqrt{y}\,x - 2y}{x} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 130: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{\sqrt{y}}{x}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{\sqrt{y}}{x}} dy \end{aligned}$$

Which results in

$$S = 2\sqrt{y}\, x$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{6\sqrt{x^2+1}\,\sqrt{y}\, x - 2y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 2\sqrt{y} \\ S_y &= \frac{x}{\sqrt{y}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 6x\sqrt{x^2+1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 6R\sqrt{R^2+1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = 2\left(R^2+1\right)^{\frac{3}{2}} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$2\sqrt{y}\, x = 2\left(x^2+1\right)^{\frac{3}{2}} + c_1$$

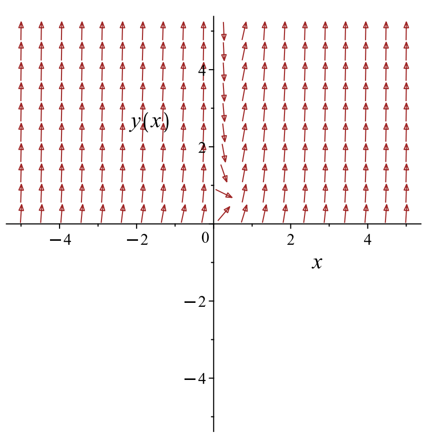
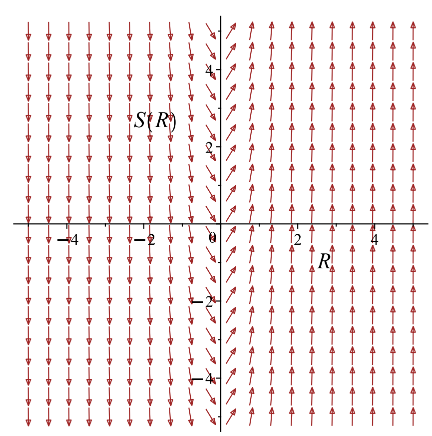
Which simplifies to

$$2\sqrt{y}\, x = 2\left(x^2+1\right)^{\frac{3}{2}} + c_1$$

Which gives

$$y = \frac{\left(2(x^2+1)^{\frac{3}{2}} + c_1\right)^2}{4x^2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{6\sqrt{x^2+1}\,\sqrt{y}\,x-2y}{x}$ 	$R = x$ $S = 2\sqrt{y}\, x$	$\frac{dS}{dR} = 6R\sqrt{R^2+1}$ 

Summary
The solution(s) found are the following

$$
y = \frac{\left(2(x^2+1)^{\frac{3}{2}} + c_1\right)^2}{4x^2} \tag{1}
$$

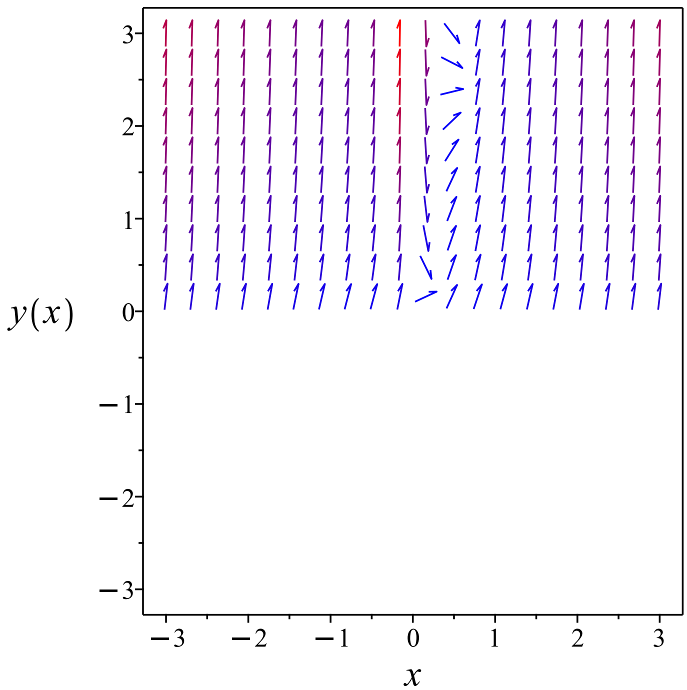


Figure 146: Slope field plot

Verification of solutions

$$
y = \frac{\left(2(x^2+1)^{\frac{3}{2}} + c_1\right)^2}{4x^2}
$$

Verified OK.

4.25.2 Solving as bernoulli ode

In canonical form, the ODE is

$$
\begin{aligned}
y' &= F(x, y) \\
&= \frac{6\sqrt{x^2+1}\,\sqrt{y}\,x - 2y}{x}
\end{aligned}
$$

This is a Bernoulli ODE.

$$
y' = -\frac{2}{x}y + 6\sqrt{x^2+1}\sqrt{y} \tag{1}
$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -\frac{2}{x} \\ f_1(x) &= 6\sqrt{x^2+1} \\ n &= \frac{1}{2} \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \sqrt{y}$ gives

$$y'\frac{1}{\sqrt{y}} = -\frac{2\sqrt{y}}{x} + 6\sqrt{x^2+1} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \sqrt{y} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = \frac{1}{2\sqrt{y}}y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} 2w'(x) &= -\frac{2w(x)}{x} + 6\sqrt{x^2+1} \\ w' &= -\frac{w}{x} + 3\sqrt{x^2+1} \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned}p(x) &= \frac{1}{x}\\ q(x) &= 3\sqrt{x^2+1}\end{aligned}$$

Hence the ode is

$$w'(x) + \frac{w(x)}{x} = 3\sqrt{x^2+1}$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int \frac{1}{x}dx}\\ &= x\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)\left(3\sqrt{x^2+1}\right)\\ \frac{\mathrm{d}}{\mathrm{d}x}(xw) &= (x)\left(3\sqrt{x^2+1}\right)\\ \mathrm{d}(xw) &= \left(3x\sqrt{x^2+1}\right)\,\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}xw &= \int 3x\sqrt{x^2+1}\,\mathrm{d}x\\ xw &= \left(x^2+1\right)^{\frac{3}{2}} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x$ results in

$$w(x) = \frac{(x^2+1)^{\frac{3}{2}}}{x} + \frac{c_1}{x}$$

Replacing w in the above by $\sqrt{y}$ using equation (5) gives the final solution.

$$\sqrt{y} = \frac{(x^2+1)^{\frac{3}{2}}}{x} + \frac{c_1}{x}$$

Summary

The solution(s) found are the following

$$\sqrt{y} = \frac{(x^2+1)^{\frac{3}{2}}}{x} + \frac{c_1}{x} \tag{1}$$

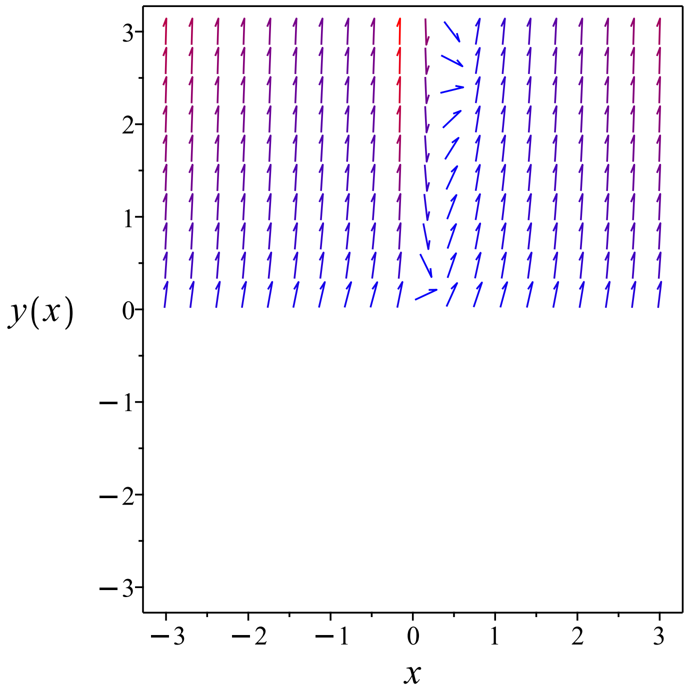


Figure 147: Slope field plot

Verification of solutions

$$\sqrt{y} = \frac{(x^2+1)^{\frac{3}{2}}}{x} + \frac{c_1}{x}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 38

```
dsolve(diff(y(x),x)+2/x*y(x)=6*sqrt(1+x^2)*sqrt(y(x)),y(x), singsol=all)
```

$$\frac{-x^2\sqrt{x^2+1}+x\sqrt{y\left(x\right)}-c_1-\sqrt{x^2+1}}{x}=0$$

✓ Solution by Mathematica

Time used: 0.228 (sec). Leaf size: 55

```
DSolve[y'[x]+2/x*y[x]==6*Sqrt[1+x^2]*Sqrt[y[x]],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to \frac{x^6+3x^4+x^2\left(3+2c_1\sqrt{x^2+1}\right)+2c_1\sqrt{x^2+1}+1+{c_1}^2}{x^2}$$

4.26 problem Problem 42

Internal problem ID [2690]
Internal file name [`OUTPUT/2182_Sunday_June_05_2022_02_52_36_AM_3516020/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.8, Change of Variables. page 79
Problem number: Problem 42.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "bernoulli", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, _Bernoulli]
```

$$\boxed{y' + \frac{2y}{x} - 6y^2x^4 = 0}$$

4.26.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned}
y' &= \frac{2y(3x^5y - 1)}{x} \\
y' &= \omega(x, y)
\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 132: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= y^2x^2
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{y^2 x^2} dy \end{aligned}$$

Which results in

$$S = -\frac{1}{x^2 y}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{2y(3x^5 y - 1)}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{2}{x^3 y} \\ S_y &= \frac{1}{x^2 y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 6x^2 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 6R^2$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = 2R^3 + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{1}{x^2 y} = 2x^3 + c_1$$

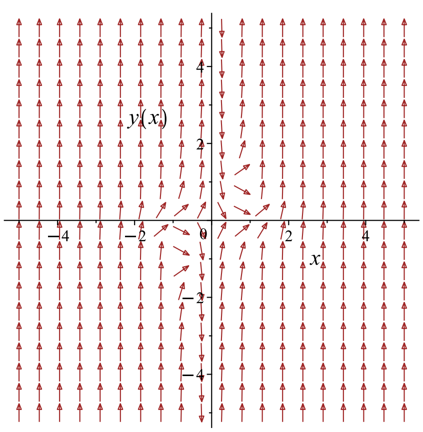
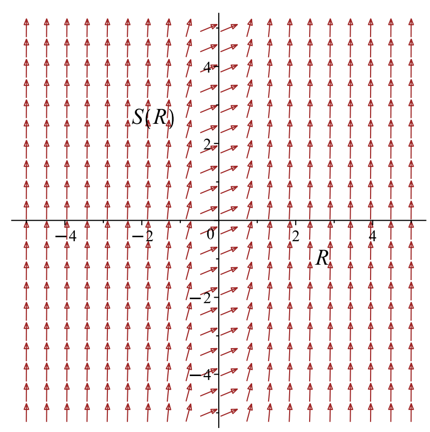
Which simplifies to

$$-\frac{1}{x^2 y} = 2x^3 + c_1$$

Which gives

$$y = -\frac{1}{x^2 (2x^3 + c_1)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{2y(3x^5y-1)}{x}$ 	$R = x$ $S = -\frac{1}{x^2 y}$	$\frac{dS}{dR} = 6R^2$ 

Summary

The solution(s) found are the following

$$y = -\frac{1}{x^2 (2x^3 + c_1)} \tag{1}$$

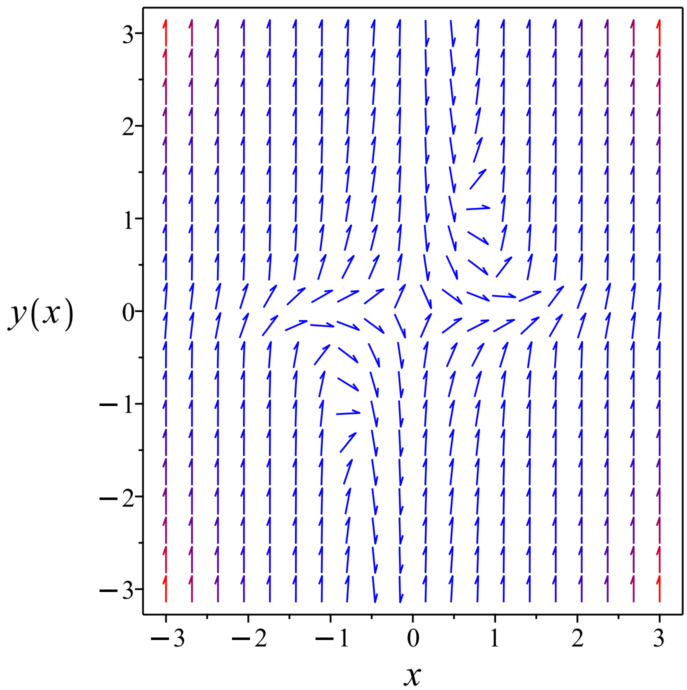


Figure 148: Slope field plot

Verification of solutions

$$y = -\frac{1}{x^2 (2x^3 + c_1)}$$

Verified OK.

4.26.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{2y(3x^5 y - 1)}{x} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{2}{x} y + 6x^4 y^2 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -\frac{2}{x} \\ f_1(x) &= 6x^4 \\ n &= 2 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^2$ gives

$$y'\frac{1}{y^2} = -\frac{2}{xy} + 6x^4 \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \frac{1}{y} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{1}{y^2}y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -w'(x) &= -\frac{2w(x)}{x} + 6x^4 \\ w' &= \frac{2w}{x} - 6x^4 \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned}
p(x) &= -\frac{2}{x} \\
q(x) &= -6x^4
\end{aligned}$$

Hence the ode is

$$w'(x) - \frac{2w(x)}{x} = -6x^4$$

The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int -\frac{2}{x}dx} \\
&= \frac{1}{x^2}
\end{aligned}$$

The ode becomes

$$\begin{aligned}
\frac{d}{dx}(\mu w) &= (\mu)\left(-6x^4\right) \\
\frac{d}{dx}\left(\frac{w}{x^2}\right) &= \left(\frac{1}{x^2}\right)\left(-6x^4\right) \\
d\left(\frac{w}{x^2}\right) &= \left(-6x^2\right) dx
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\frac{w}{x^2} &= \int -6x^2 \, dx \\
\frac{w}{x^2} &= -2x^3 + c_1
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2}$ results in

$$w(x) = -2x^5 + c_1 x^2$$

Replacing w in the above by $\frac{1}{y}$ using equation (5) gives the final solution.

$$\frac{1}{y} = -2x^5 + c_1 x^2$$

Or

$$y = \frac{1}{-2x^5 + c_1 x^2}$$

Which is simplified to

$$y = \frac{1}{x^2 \left(-2x^3 + c_1\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{1}{x^2 \left(-2x^3 + c_1\right)} \tag{1}$$

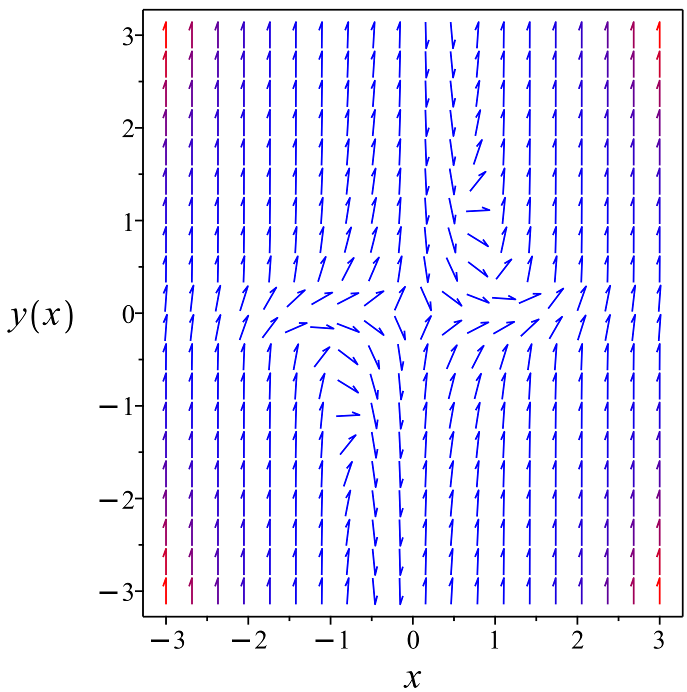


Figure 149: Slope field plot

Verification of solutions

$$y = \frac{1}{x^2 \left(-2x^3 + c_1\right)}$$

Verified OK.

4.26.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\mathrm{d}y=\left(-\frac{2y}{x}+6x^4y^2\right)\mathrm{d}x$$

$$\left(\frac{2y}{x}-6x^4y^2\right)\mathrm{d}x+\mathrm{d}y=0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= \frac{2y}{x} - 6x^4y^2 \\
N(x,y) &= 1
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{2y}{x} - 6x^4y^2\right) \\
&= \frac{2}{x} - 12x^4y
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1\left(\left(\frac{2}{x} - 12x^4y\right) - (0)\right) \\
&= \frac{2}{x} - 12x^4y
\end{aligned}
$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= -\frac{x}{6x^5y^2 - 2y}\left((0) - \left(\frac{2}{x} - 12x^4y\right)\right) \\
&= \frac{-6x^5y + 1}{3x^5y^2 - y}
\end{aligned}
$$

Since B depends on x, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$\begin{aligned} R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} \\ &= \frac{(0) - \left(\frac{2}{x} - 12x^4y\right)}{x\left(\frac{2y}{x} - 6x^4y^2\right) - y\,(1)} \\ &= -\frac{2}{xy} \end{aligned}$$

Replacing all powers of terms xy by t gives

$$R = -\frac{2}{t}$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$\begin{aligned} \mu &= e^{\int R \,\mathrm{d}t} \\ &= e^{\int \left(-\frac{2}{t}\right) \mathrm{d}t} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-2\ln(t)} \\ &= \frac{1}{t^2} \end{aligned}$$

Now t is replaced back with xy giving

$$\mu = \frac{1}{x^2y^2}$$

Multiplying M and N by this integrating factor gives new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N

$$\begin{aligned} \overline{M} &= \mu M \\ &= \frac{1}{x^2y^2}\left(\frac{2y}{x} - 6x^4y^2\right) \\ &= \frac{-6x^5y + 2}{x^3y} \end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^2 y^2}(1) \\
&= \frac{1}{x^2 y^2}
\end{aligned}
$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{-6x^5 y + 2}{x^3 y}\right) + \left(\frac{1}{x^2 y^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int \overline{M} \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int \frac{-6x^5 y + 2}{x^3 y} \,\mathrm{d}x
\end{aligned}
$$

$$
\phi = \frac{-2x^5 y - 1}{x^2 y} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = -\frac{2x^3}{y} - \frac{-2x^5 y - 1}{x^2 y^2} + f'(y) \tag{4}
$$

$$
= \frac{1}{x^2 y^2} + f'(y)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{x^2 y^2}$. Therefore equation (4) becomes

$$
\frac{1}{x^2 y^2} = \frac{1}{x^2 y^2} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{-2x^5y - 1}{x^2y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{-2x^5y - 1}{x^2y}$$

The solution becomes

$$y = -\frac{1}{x^2 (2x^3 + c_1)}$$

Summary

The solution(s) found are the following

$$y = -\frac{1}{x^2 (2x^3 + c_1)} \tag{1}$$

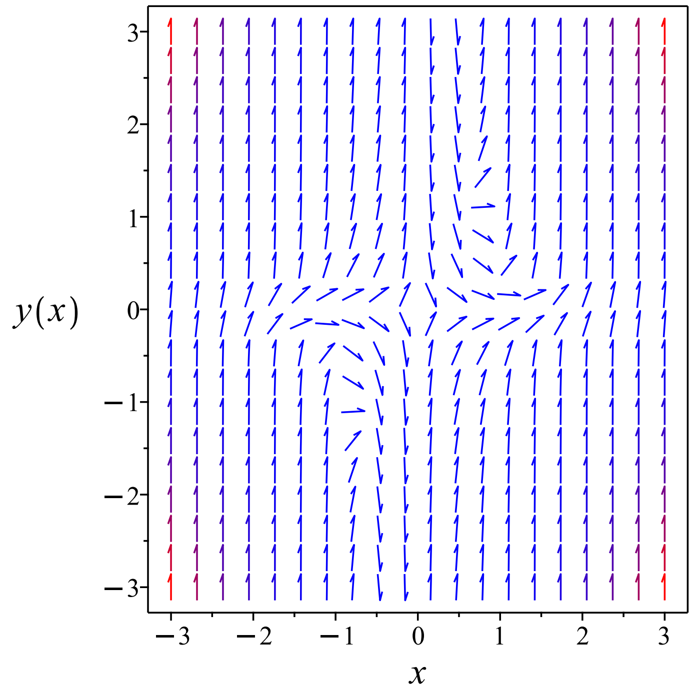


Figure 150: Slope field plot

<u>Verification of solutions</u>

$$y = -\frac{1}{x^2 (2x^3 + c_1)}$$

Verified OK.

4.26.4 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{2y(3x^5y - 1)}{x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -\frac{2y}{x} + 6x^4y^2$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = -\frac{2}{x}$ and $f_2(x) = 6x^4$. Let

$$
\begin{aligned}
y &= \frac{-u'}{f_2 u} \\
&= \frac{-u'}{6x^4 u}
\end{aligned}
\tag{1}
$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$
f_2 u''(x) - \left(f_2' + f_1 f_2\right) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}
$$

But

$$
\begin{aligned}
f_2' &= 24x^3 \\
f_1 f_2 &= -12x^3 \\
f_2^2 f_0 &= 0
\end{aligned}
$$

Substituting the above terms back in equation (2) gives

$$
6x^4 u''(x) - 12x^3 u'(x) = 0
$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$
u(x) = c_2 x^3 + c_1
$$

The above shows that

$$
u'(x) = 3c_2 x^2
$$

Using the above in (1) gives the solution

$$
y = -\frac{c_2}{2x^2 \left(c_2 x^3 + c_1\right)}
$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$
y = -\frac{1}{2x^2 \left(x^3 + c_3\right)}
$$

Summary

The solution(s) found are the following

$$y = -\frac{1}{2x^2 (x^3 + c_3)} \tag{1}$$

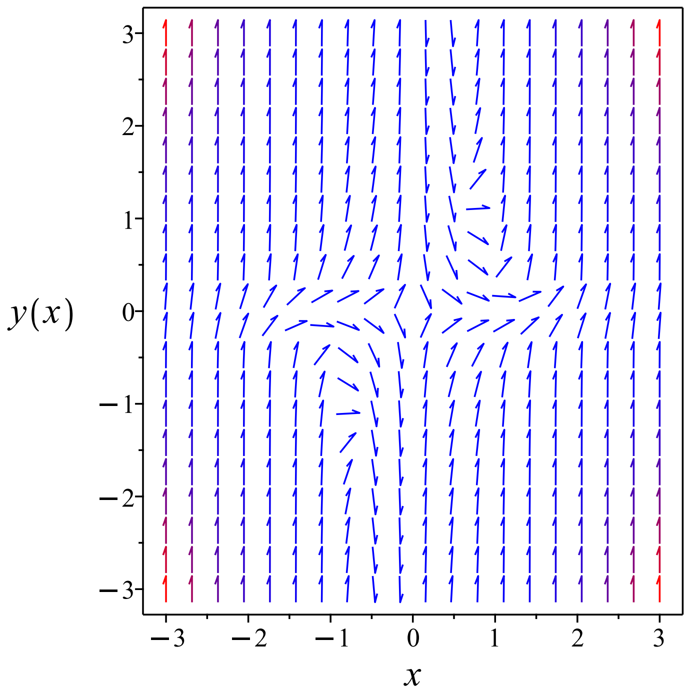


Figure 151: Slope field plot

Verification of solutions

$$y = -\frac{1}{2x^2 (x^3 + c_3)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(y(x),x)+2/x*y(x)=6*y(x)^2*x^4,y(x), singsol=all)
```

$$y(x) = \frac{1}{(-2x^3 + c_1)\, x^2}$$

✓ Solution by Mathematica

Time used: 0.131 (sec). Leaf size: 24

```
DSolve[y'[x]+2/x*y[x]==6*y[x]^2*x^4,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{-2x^5 + c_1 x^2}$$
$$y(x) \to 0$$

4.27 problem Problem 43

Internal problem ID [2691]
Internal file name [`OUTPUT/2183_Sunday_June_05_2022_02_52_38_AM_40107222/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.8, Change of Variables. page 79
Problem number: Problem 43.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, _Bernoulli]
```

$$\boxed{2x(y'+y^3x^2)+y=0}$$

4.27.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{y(2x^3y^2+1)}{2x}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 134: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= 0 \\ \eta(x,y) &= x\, y^3 \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x\,y^3} dy \end{aligned}$$

Which results in

$$S = -\frac{1}{2x\,y^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{y(2x^3y^2 + 1)}{2x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{2x^2y^2} \\ S_y &= \frac{1}{x\,y^3} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -x \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{R^2}{2} + c_1 \tag{4}$$

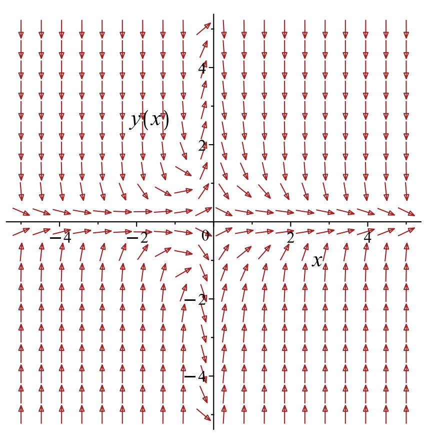
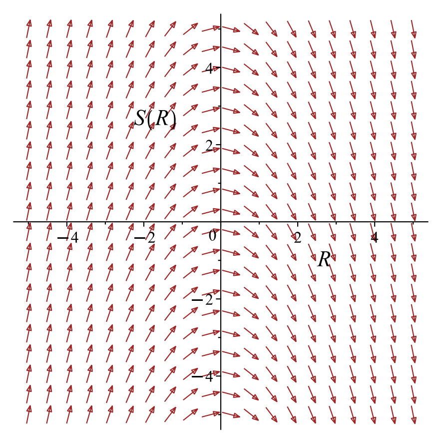
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{1}{2xy^2} = -\frac{x^2}{2} + c_1$$

Which simplifies to

$$-\frac{1}{2xy^2} = -\frac{x^2}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y(2x^3y^2+1)}{2x}$ 	$R = x$ $S = -\frac{1}{2x\,y^2}$	$\frac{dS}{dR} = -R$ 

Summary

The solution(s) found are the following

$$-\frac{1}{2xy^2} = -\frac{x^2}{2} + c_1 \tag{1}$$

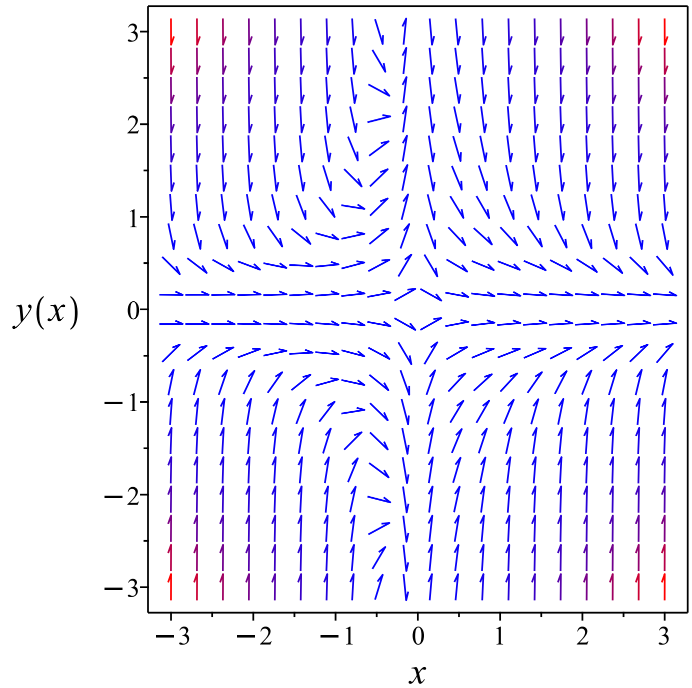


Figure 152: Slope field plot

Verification of solutions

$$-\frac{1}{2x y^{2}} = -\frac{x^{2}}{2}+c_{1}$$

Verified OK.

4.27.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{y(2x^{3}y^{2}+1)}{2x} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{1}{2x}y - x^{2}y^{3} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= -\frac{1}{2x} \\
f_1(x) &= -x^2 \\
n &= 3
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = y^3$ gives

$$
y' \frac{1}{y^3} = -\frac{1}{2x\, y^2} - x^2 \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= \frac{1}{y^2}
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = -\frac{2}{y^3} y' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
-\frac{w'(x)}{2} &= -\frac{w(x)}{2x} - x^2 \\
w' &= \frac{w}{x} + 2x^2
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{1}{x} \\
q(x) &= 2x^2
\end{aligned}
$$

Hence the ode is

$$w'(x) - \frac{w(x)}{x} = 2x^2$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{1}{x}dx} \\ &= \frac{1}{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)\left(2x^2\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{w}{x}\right) &= \left(\frac{1}{x}\right)\left(2x^2\right) \\ \mathrm{d}\left(\frac{w}{x}\right) &= (2x)\,\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{w}{x} &= \int 2x\,\mathrm{d}x \\ \frac{w}{x} &= x^2 + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$w(x) = x^3 + c_1 x$$

Replacing w in the above by $\frac{1}{y^2}$ using equation (5) gives the final solution.

$$\frac{1}{y^2} = x^3 + c_1 x$$

Solving for y gives

$$\begin{aligned}y(x) &= \frac{1}{\sqrt{x\left(x^2+c_1\right)}} \\ y(x) &= -\frac{1}{\sqrt{x\left(x^2+c_1\right)}}\end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{1}{\sqrt{x\left(x^{2}+c_{1}\right)}} \tag{1}$$

$$y = -\frac{1}{\sqrt{x\left(x^{2}+c_{1}\right)}} \tag{2}$$

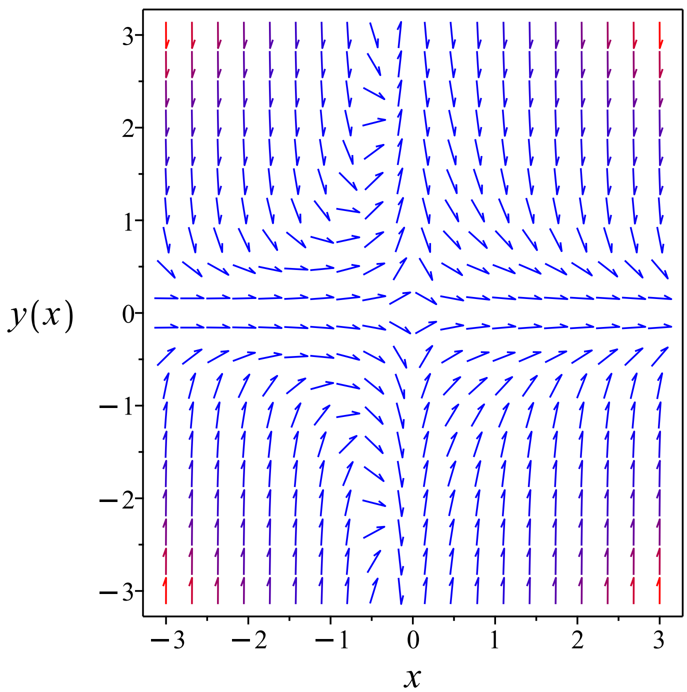


Figure 153: Slope field plot

Verification of solutions

$$y = \frac{1}{\sqrt{x\left(x^{2}+c_{1}\right)}}$$

Verified OK.

$$y = -\frac{1}{\sqrt{x\left(x^{2}+c_{1}\right)}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(2*x*(diff(y(x),x)+y(x)^3*x^2)+y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{1}{\sqrt{x\,(x^2+c_1)}}$$
$$y(x) = -\frac{1}{\sqrt{x\,(x^2+c_1)}}$$

✓ Solution by Mathematica

Time used: 0.326 (sec). Leaf size: 40

```
DSolve[2*x*(y'[x]+y[x]^3*x^2)+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{\sqrt{x\,(x^2+c_1)}}$$
$$y(x) \to \frac{1}{\sqrt{x\,(x^2+c_1)}}$$
$$y(x) \to 0$$

4.28 problem Problem 44

Internal problem ID [2692]
Internal file name [`OUTPUT/2184_Sunday_June_05_2022_02_52_44_AM_88096937/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.8, Change of Variables. page 79
Problem number: Problem 44.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_rational, _Bernoulli]
```

$$\boxed{(x-a)(x-b)\left(y'-\sqrt{y}\right)-2(b-a)\,y=0}$$

4.28.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{-\sqrt{y}\,ab+\sqrt{y}\,ax+\sqrt{y}\,bx-\sqrt{y}\,x^2+2ya-2yb}{(-x+a)(-x+b)} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ,η

Table 136: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \sqrt{y}\, \mathrm{e}^{-\frac{(2a-2b)\left(-\frac{\ln(x-b)}{-b+a}+\frac{\ln(x-a)}{-b+a}\right)}{2}}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\sqrt{y}\, \mathrm{e}^{-\frac{(2a-2b)\left(-\frac{\ln(x-b)}{-b+a}+\frac{\ln(x-a)}{-b+a}\right)}{2}}} dy \end{aligned}$$

Which results in

$$S = \frac{2\sqrt{y}\,(x-a)}{x-b}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{-\sqrt{y}\, ab + \sqrt{y}\, ax + \sqrt{y}\, bx - \sqrt{y}\, x^2 + 2ya - 2yb}{(-x+a)\,(-x+b)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{2\sqrt{y}\,(-b+a)}{(-x+b)^2} \\ S_y &= \frac{-x+a}{\sqrt{y}\,(-x+b)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{-x+a}{-x+b} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{-R+a}{-R+b}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = R + (b - a) \ln (R - b) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{2\sqrt{y}\,(-x + a)}{-x + b} = x + (b - a) \ln (x - b) + c_1$$

Which simplifies to

$$\frac{2\sqrt{y}\,(-x + a)}{-x + b} = x + (b - a) \ln (x - b) + c_1$$

Which gives

$$y = \frac{(\ln (x - b)\, ab - \ln (x - b)\, ax - \ln (x - b)\, b^2 + \ln (x - b)\, bx - c_1 b + c_1 x - bx + x^2)^2}{4\,(-x + a)^2}$$

Summary

The solution(s) found are the following

$$y = \frac{(\ln (x - b)\, ab - \ln (x - b)\, ax - \ln (x - b)\, b^2 + \ln (x - b)\, bx - c_1 b + c_1 x - bx + x^2)^2}{4\,(-x + a)^2} \tag{1}$$

Verification of solutions

$$y = \frac{(\ln (x - b)\, ab - \ln (x - b)\, ax - \ln (x - b)\, b^2 + \ln (x - b)\, bx - c_1 b + c_1 x - bx + x^2)^2}{4\,(-x + a)^2}$$

Verified OK.

4.28.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\frac{-\sqrt{y}\, ab + \sqrt{y}\, ax + \sqrt{y}\, bx - \sqrt{y}\, x^2 + 2ya - 2yb}{(-x + a)\,(-x + b)} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{2a-2b}{(-x+a)(-x+b)}y - \frac{-ab+ax+bx-x^2}{(-x+a)(-x+b)}\sqrt{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned}
f_0(x) &= -\frac{2a-2b}{(-x+a)(-x+b)} \\
f_1(x) &= -\frac{-ab+ax+bx-x^2}{(-x+a)(-x+b)} \\
n &= \frac{1}{2}
\end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \sqrt{y}$ gives

$$y'\frac{1}{\sqrt{y}} = -\frac{(2a-2b)\sqrt{y}}{(-x+a)(-x+b)} - \frac{-ab+ax+bx-x^2}{(-x+a)(-x+b)} \tag{4}$$

Let

$$\begin{aligned}
w &= y^{1-n} \\
&= \sqrt{y}
\end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = \frac{1}{2\sqrt{y}}y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned}
2w'(x) &= -\frac{(2a-2b)\,w(x)}{(-x+a)(-x+b)} - \frac{-ab+ax+bx-x^2}{(-x+a)(-x+b)} \\
w' &= -\frac{(2a-2b)\,w}{2(-x+a)(-x+b)} - \frac{-ab+ax+bx-x^2}{2(-x+a)(-x+b)}
\end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned}
p(x) &= -\frac{b-a}{(-x+a)(-x+b)} \\
q(x) &= \frac{1}{2}
\end{aligned}$$

Hence the ode is

$$w'(x) - \frac{(b-a)\, w(x)}{(-x+a)(-x+b)} = \frac{1}{2}$$

The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int -\frac{b-a}{(-x+a)(-x+b)}dx} \\
&= e^{(-b+a)\left(-\frac{\ln(x-b)}{-b+a}+\frac{\ln(x-a)}{-b+a}\right)}
\end{aligned}$$

Which simplifies to

$$\mu = \frac{-x+a}{-x+b}$$

The ode becomes

$$\begin{aligned}
\frac{d}{dx}(\mu w) &= (\mu)\left(\frac{1}{2}\right) \\
\frac{d}{dx}\left(\frac{(-x+a)\, w}{-x+b}\right) &= \left(\frac{-x+a}{-x+b}\right)\left(\frac{1}{2}\right) \\
d\left(\frac{(-x+a)\, w}{-x+b}\right) &= \left(\frac{-x+a}{2b-2x}\right) dx
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\frac{(-x+a)\, w}{-x+b} &= \int \frac{-x+a}{2b-2x}\, dx \\
\frac{(-x+a)\, w}{-x+b} &= \frac{x}{2} + \frac{(b-a)\ln(x-b)}{2} + c_1
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{-x+a}{-x+b}$ results in

$$w(x) = \frac{(-x+b)\left(\frac{x}{2} + \frac{(b-a)\ln(x-b)}{2}\right)}{-x+a} + \frac{(-x+b)\,c_1}{-x+a}$$

which simplifies to

$$w(x) = -\frac{(-x+b)\left(\ln(x-b)(-b+a) - x - 2c_1\right)}{2a-2x}$$

Replacing w in the above by $\sqrt{y}$ using equation (5) gives the final solution.

$$\sqrt{y} = -\frac{(-x+b)\left(\ln(x-b)(-b+a) - x - 2c_1\right)}{2a-2x}$$

Summary

The solution(s) found are the following

$$\sqrt{y} = -\frac{(-x+b)\left(\ln(x-b)(-b+a) - x - 2c_1\right)}{2a-2x} \tag{1}$$

Verification of solutions

$$\sqrt{y} = -\frac{(-x+b)\left(\ln(x-b)(-b+a) - x - 2c_1\right)}{2a-2x}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 55

```
dsolve((x-a)*(x-b)*(diff(y(x),x)-sqrt(y(x)))=2*(b-a)*y(x),y(x), singsol=all)
```

$$\frac{(-x+b)(a-b)\ln(x-b) + (2a-2x)\sqrt{y(x)} - (x+2c_1)(-x+b)}{2a-2x} = 0$$

✓ Solution by Mathematica

Time used: 0.478 (sec). Leaf size: 43

```
DSolve[(x-a)*(x-b)*(y'[x]-Sqrt[y[x]])==2*(b-a)*y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{(b-x)^2((b-a)\log(x-b)+x+2c_1)^2}{4(a-x)^2}$$

4.29 problem Problem 45

Internal problem ID [2693]
Internal file name [`OUTPUT/2185_Sunday_June_05_2022_02_52_47_AM_85575651/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.8, Change of Variables. page 79
Problem number: Problem 45.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_Bernoulli]
```

$$\boxed{y' + \frac{6y}{x} - \frac{3y^{\frac{2}{3}} \cos(x)}{x} = 0}$$

4.29.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{-6y + 3y^{\frac{2}{3}} \cos(x)}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 138: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \frac{y^{\frac{2}{3}}}{x^2}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{y^{\frac{2}{3}}}{x^2}} dy \end{aligned}$$

Which results in

$$S = 3x^2 y^{\frac{1}{3}}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{-6y + 3y^{\frac{2}{3}} \cos(x)}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 6y^{\frac{1}{3}} x \\ S_y &= \frac{x^2}{y^{\frac{2}{3}}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 3\cos(x)\, x \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 3\cos(R)\, R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = 3\cos(R) + 3\sin(R)\,R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$3x^2 y^{\frac{1}{3}} = 3\cos(x) + 3x\sin(x) + c_1$$

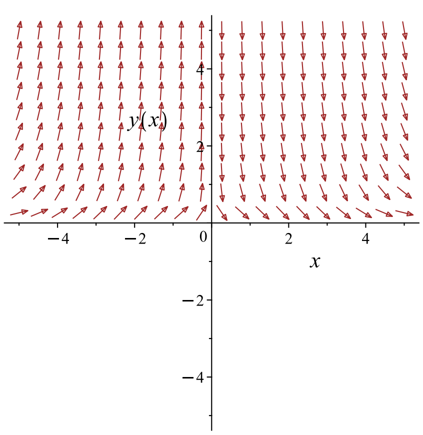
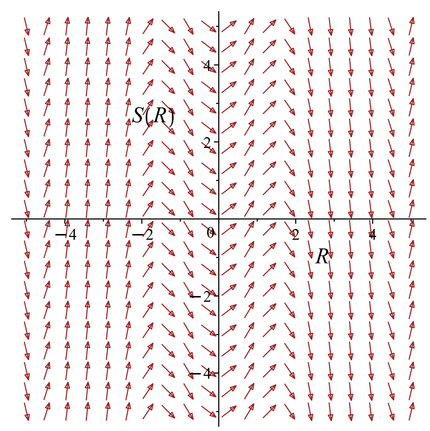
Which simplifies to

$$3x^2 y^{\frac{1}{3}} = 3\cos(x) + 3x\sin(x) + c_1$$

Which gives

$$y = \frac{27x^3\sin(x)^3 + 27c_1x^2\sin(x)^2 + 81x^2\sin(x)^2\cos(x) + 9c_1^2x\sin(x) + 54c_1x\sin(x)\cos(x) - 81\sin(x)}{27x^6}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{-6y+3y^{\frac{2}{3}}\cos(x)}{x}$ 	$R = x$ $S = 3x^2y^{\frac{1}{3}}$	$\frac{dS}{dR} = 3\cos(R)\,R$ 

Summary

The solution(s) found are the following

$$y = \frac{27x^3 \sin(x)^3 + 27c_1 x^2 \sin(x)^2 + 81x^2 \sin(x)^2 \cos(x) + 9c_1^2 x \sin(x) + 54c_1 x \sin(x)\cos(x) - 81\sin(x)^3 x}{27x^6} \tag{1}$$

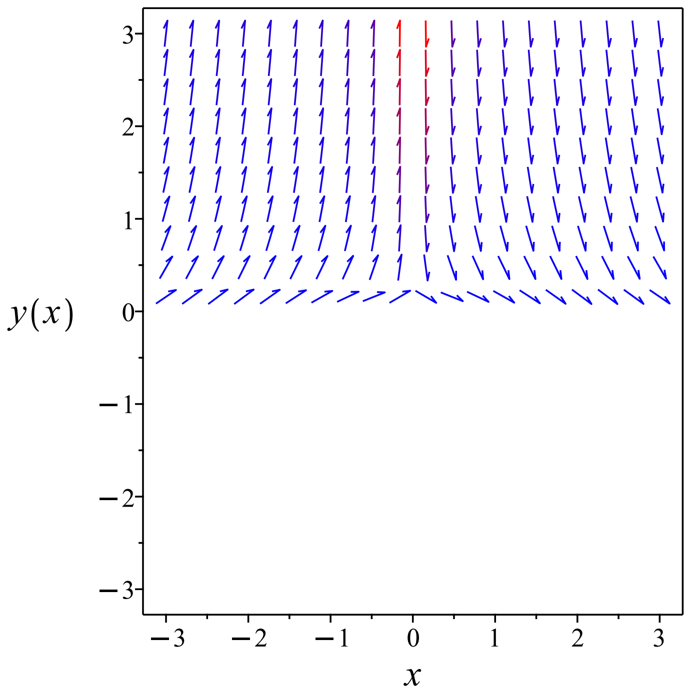


Figure 154: Slope field plot

Verification of solutions

$$y = \frac{27x^3 \sin(x)^3 + 27c_1 x^2 \sin(x)^2 + 81x^2 \sin(x)^2 \cos(x) + 9c_1^2 x \sin(x) + 54c_1 x \sin(x)\cos(x) - 81\sin(x)^3 x}{27x^6}$$

Verified OK.

4.29.2 Solving as bernoulli ode

In canonical form, the ODE is

$$
\begin{aligned}
y' &= F(x,y) \\
&= \frac{-6y + 3y^{\frac{2}{3}} \cos(x)}{x}
\end{aligned}
$$

This is a Bernoulli ODE.

$$
y' = -\frac{6}{x}y + \frac{3\cos(x)}{x}y^{\frac{2}{3}} \tag{1}
$$

The standard Bernoulli ODE has the form

$$
y' = f_0(x)y + f_1(x)y^n \tag{2}
$$

The first step is to divide the above equation by y^n which gives

$$
\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}
$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= -\frac{6}{x} \\
f_1(x) &= \frac{3\cos(x)}{x} \\
n &= \frac{2}{3}
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = y^{\frac{2}{3}}$ gives

$$
y'\frac{1}{y^{\frac{2}{3}}} = -\frac{6y^{\frac{1}{3}}}{x} + \frac{3\cos(x)}{x} \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= y^{\frac{1}{3}}
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = \frac{1}{3y^{\frac{2}{3}}}y' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
3w'(x) &= -\frac{6w(x)}{x} + \frac{3\cos(x)}{x} \\
w' &= -\frac{2w}{x} + \frac{\cos(x)}{x}
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= \frac{2}{x} \\
q(x) &= \frac{\cos(x)}{x}
\end{aligned}
$$

Hence the ode is

$$
w'(x) + \frac{2w(x)}{x} = \frac{\cos(x)}{x}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int \frac{2}{x} dx} \\
&= x^2
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)\left(\frac{\cos(x)}{x}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}(x^2 w) &= (x^2)\left(\frac{\cos(x)}{x}\right) \\
\mathrm{d}(x^2 w) &= (\cos(x)\, x)\, \mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
x^2 w &= \int \cos(x)\, x \,\mathrm{d}x \\
x^2 w &= x\sin(x) + \cos(x) + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = x^2$ results in

$$
w(x) = \frac{x\sin(x) + \cos(x)}{x^2} + \frac{c_1}{x^2}
$$

which simplifies to

$$w(x) = \frac{x \sin(x) + \cos(x) + c_1}{x^2}$$

Replacing w in the above by $y^{\frac{1}{3}}$ using equation (5) gives the final solution.

$$y^{\frac{1}{3}} = \frac{x \sin(x) + \cos(x) + c_1}{x^2}$$

Summary

The solution(s) found are the following

$$y^{\frac{1}{3}} = \frac{x \sin(x) + \cos(x) + c_1}{x^2} \tag{1}$$

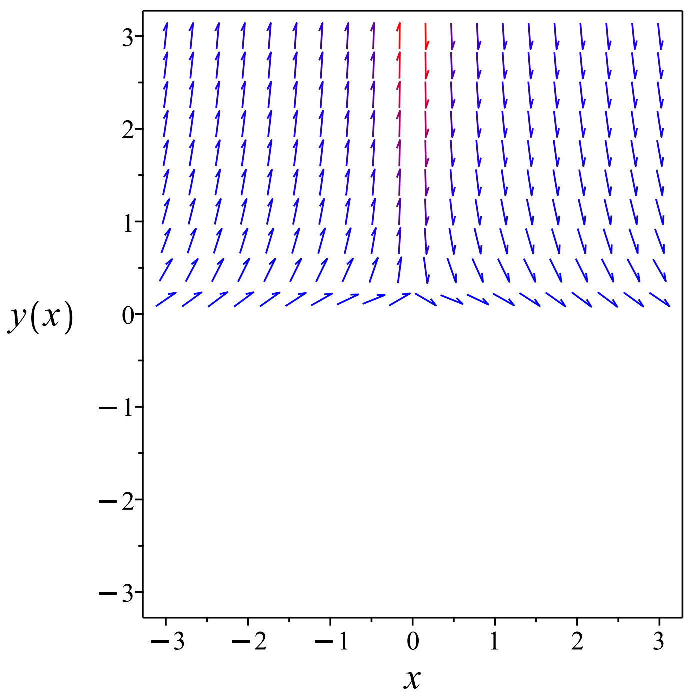


Figure 155: Slope field plot

Verification of solutions

$$y^{\frac{1}{3}} = \frac{x \sin(x) + \cos(x) + c_1}{x^2}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 28

```
dsolve(diff(y(x),x)+6/x*y(x)=3/x*y(x)^(2/3)*cos(x),y(x), singsol=all)
```

$$\frac{y(x)^{\frac{1}{3}} x^2 - x \sin(x) - \cos(x) - c_1}{x^2} = 0$$

✓ Solution by Mathematica

Time used: 0.194 (sec). Leaf size: 20

```
DSolve[y'[x]+6/x*y[x]==3/x*y[x]^(2/3)*Cos[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{(x\sin(x) + \cos(x) + c_1)^3}{x^6}$$

4.30 problem Problem 46

Internal problem ID [2694]
Internal file name [`OUTPUT/2186_Sunday_June_05_2022_02_52_53_AM_36459872/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.8, Change of Variables. page 79
Problem number: Problem 46.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_Bernoulli]
```

$$\boxed{y' + 4yx - 4\sqrt{y}\,x^3 = 0}$$

4.30.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -4xy + 4\sqrt{y}\,x^3$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 140: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \sqrt{y}\, \mathrm{e}^{-x^2}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\sqrt{y}\, \mathrm{e}^{-x^2}} dy \end{aligned}$$

Which results in

$$S = 2\sqrt{y}\, \mathrm{e}^{x^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -4xy + 4\sqrt{y}\, x^3$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 4\sqrt{y}\, x\, \mathrm{e}^{x^2} \\ S_y &= \frac{\mathrm{e}^{x^2}}{\sqrt{y}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 4x^3 \mathrm{e}^{x^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 4R^3 \mathrm{e}^{R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = 2\left(R^2 - 1\right) e^{R^2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$2\sqrt{y}\, e^{x^2} = 2\left(x^2 - 1\right) e^{x^2} + c_1$$

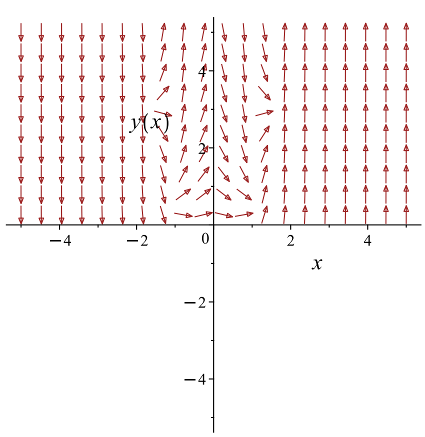
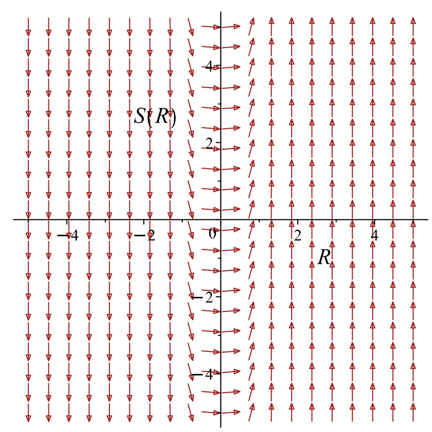
Which simplifies to

$$2\sqrt{y}\, e^{x^2} = 2\left(x^2 - 1\right) e^{x^2} + c_1$$

Which gives

$$y = \frac{\left(2x^2 e^{x^2} - 2\, e^{x^2} + c_1\right)^2 e^{-2x^2}}{4}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = -4xy + 4\sqrt{y}\, x^3$
</td><td>$R = x$
$S = 2\sqrt{y}\, e^{x^2}$</td><td>$\frac{dS}{dR} = 4R^3 e^{R^2}$
</td></tr>
</table>

Summary
The solution(s) found are the following

$$y = \frac{\left(2x^2 e^{x^2} - 2\,e^{x^2} + c_1\right)^2 e^{-2x^2}}{4} \tag{1}$$

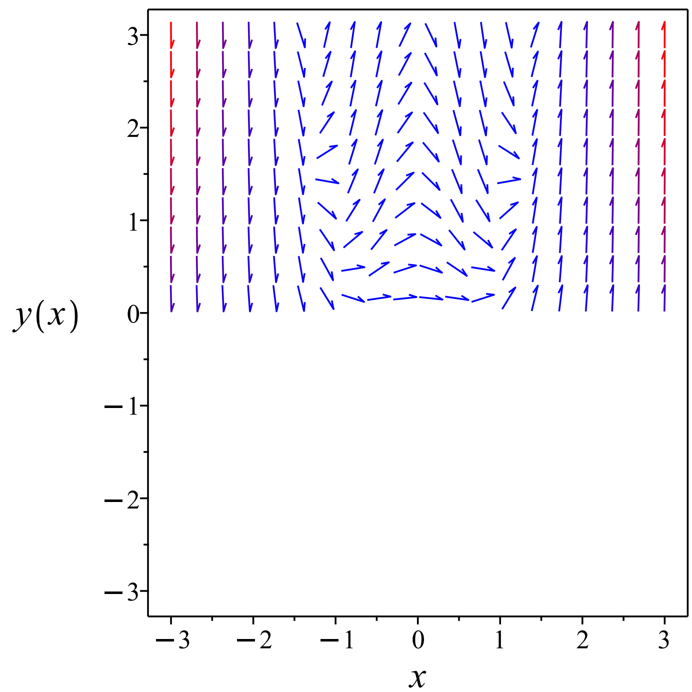


Figure 156: Slope field plot

Verification of solutions

$$y = \frac{\left(2x^2 e^{x^2} - 2\,e^{x^2} + c_1\right)^2 e^{-2x^2}}{4}$$

Verified OK.

4.30.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -4xy + 4\sqrt{y}\, x^3 \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -4xy + 4x^3\sqrt{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -4x \\ f_1(x) &= 4x^3 \\ n &= \frac{1}{2} \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \sqrt{y}$ gives

$$y'\frac{1}{\sqrt{y}} = -4\sqrt{y}\,x + 4x^3 \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \sqrt{y} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = \frac{1}{2\sqrt{y}}y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} 2w'(x) &= -4w(x)\,x + 4x^3 \\ w' &= 2x^3 - 2xw \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$
\begin{aligned}
p(x) &= 2x \\
q(x) &= 2x^3
\end{aligned}
$$

Hence the ode is

$$
w'(x) + 2w(x)\, x = 2x^3
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int 2x dx} \\
&= \mathrm{e}^{x^2}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)\left(2x^3\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{x^2} w\right) &= \left(\mathrm{e}^{x^2}\right)\left(2x^3\right) \\
\mathrm{d}\left(\mathrm{e}^{x^2} w\right) &= \left(2x^3 \mathrm{e}^{x^2}\right) \mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\mathrm{e}^{x^2} w &= \int 2x^3 \mathrm{e}^{x^2} \,\mathrm{d}x \\
\mathrm{e}^{x^2} w &= \left(x^2 - 1\right) \mathrm{e}^{x^2} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{x^2}$ results in

$$
w(x) = \mathrm{e}^{-x^2}\left(x^2 - 1\right) \mathrm{e}^{x^2} + c_1 \mathrm{e}^{-x^2}
$$

which simplifies to

$$
w(x) = x^2 - 1 + c_1 \mathrm{e}^{-x^2}
$$

Replacing w in the above by $\sqrt{y}$ using equation (5) gives the final solution.

$$
\sqrt{y} = x^2 - 1 + c_1 \mathrm{e}^{-x^2}
$$

Summary

The solution(s) found are the following

$$
\sqrt{y} = x^2 - 1 + c_1 \mathrm{e}^{-x^2} \tag{1}
$$

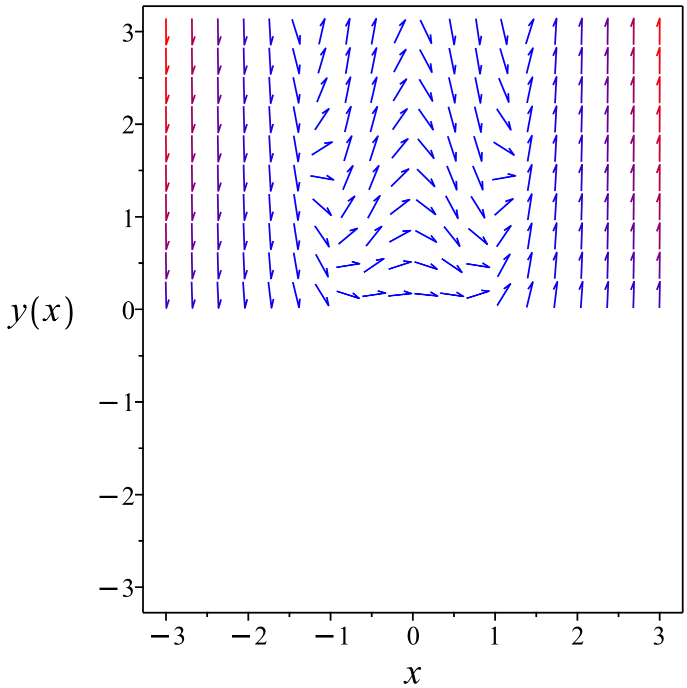


Figure 157: Slope field plot

Verification of solutions

$$\sqrt{y} = x^2 - 1 + c_1 e^{-x^2}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 23

```
dsolve(diff(y(x),x)+4*x*y(x)=4*x^3*sqrt(y(x)),y(x), singsol=all)
```

$$-x^2 + 1 - c_1 e^{-x^2} + \sqrt{y(x)} = 0$$

✓ Solution by Mathematica

Time used: 0.159 (sec). Leaf size: 29

```
DSolve[y'[x]+4*x*y[x]==4*x^3*Sqrt[y[x]],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-2x^2}\left(e^{x^2}\left(x^2-1\right)+c_1\right)^2$$

4.31 problem Problem 47

Internal problem ID [2695]
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Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.8, Change of Variables. page 79
Problem number: Problem 47.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_Bernoulli]`

$$\boxed{y' - \frac{y}{2\ln(x)\, x} - 2xy^3 = 0}$$

4.31.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y(4x^2y^2\ln(x) + 1)}{2\ln(x)\, x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 142: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \frac{y^3}{\ln(x)}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{y^3}{\ln(x)}} dy \end{aligned}$$

Which results in

$$S = -\frac{\ln(x)}{2y^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{y(4x^2y^2 \ln(x) + 1)}{2 \ln(x) x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{1}{2x\, y^2} \\ S_y &= \frac{\ln(x)}{y^3} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 2 \ln(x) x \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 2 \ln(R) R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = R^2 \ln(R) - \frac{R^2}{2} + c_1 \tag{4}$$

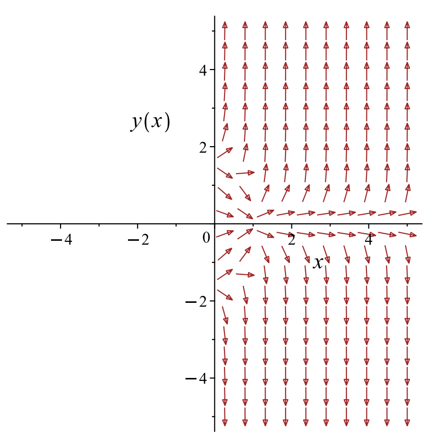
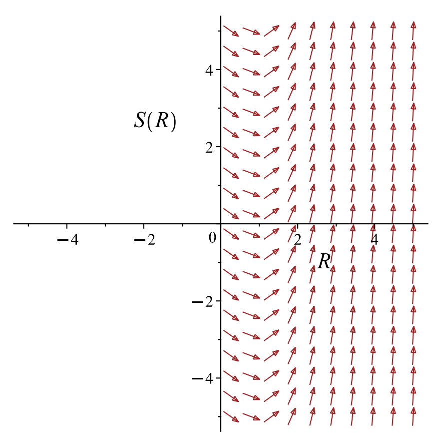
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{\ln(x)}{2y^2} = \ln(x)\, x^2 - \frac{x^2}{2} + c_1$$

Which simplifies to

$$-\frac{\ln(x)}{2y^2} = \ln(x)\, x^2 - \frac{x^2}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y(4x^2y^2\ln(x)+1)}{2\ln(x)x}$ 	$R = x$ $S = -\frac{\ln(x)}{2y^2}$	$\frac{dS}{dR} = 2\ln(R)\, R$ 

Summary

The solution(s) found are the following

$$-\frac{\ln(x)}{2y^2} = \ln(x)\, x^2 - \frac{x^2}{2} + c_1 \tag{1}$$

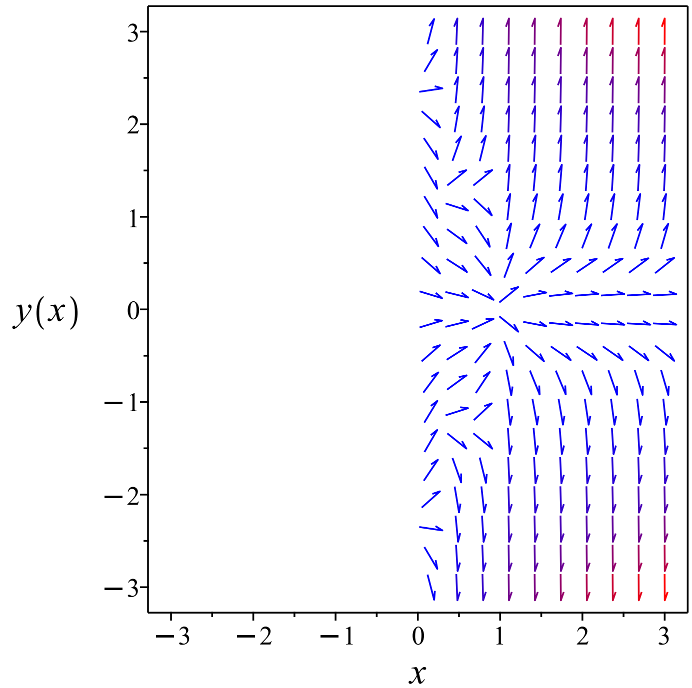


Figure 158: Slope field plot

Verification of solutions

$$-\frac{\ln(x)}{2y^2} = \ln(x)\,x^2 - \frac{x^2}{2} + c_1$$

Verified OK.

4.31.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{y(4x^2y^2\ln(x)+1)}{2\ln(x)\,x} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{1}{2\ln(x)\,x}y + 2xy^3 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= \frac{1}{2\ln(x)\,x} \\
f_1(x) &= 2x \\
n &= 3
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = y^3$ gives

$$
y'\frac{1}{y^3} = \frac{1}{2y^2\ln(x)\,x} + 2x \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= \frac{1}{y^2}
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = -\frac{2}{y^3}y' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
-\frac{w'(x)}{2} &= \frac{w(x)}{2\ln(x)\,x} + 2x \\
w' &= -\frac{w}{\ln(x)\,x} - 4x
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= \frac{1}{\ln(x)\,x} \\
q(x) &= -4x
\end{aligned}
$$

Hence the ode is

$$w'(x) + \frac{w(x)}{\ln(x)\,x} = -4x$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int \frac{1}{\ln(x)x}dx} \\ &= \ln(x)\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)(-4x) \\ \frac{\mathrm{d}}{\mathrm{d}x}(\ln(x)\,w) &= (\ln(x))(-4x) \\ \mathrm{d}(\ln(x)\,w) &= (-4\ln(x)\,x)\ \mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\ln(x)\,w &= \int -4\ln(x)\,x\,\mathrm{d}x \\ \ln(x)\,w &= -2\ln(x)\,x^2 + x^2 + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \ln(x)$ results in

$$w(x) = \frac{-2\ln(x)\,x^2 + x^2}{\ln(x)} + \frac{c_1}{\ln(x)}$$

which simplifies to

$$w(x) = \frac{-2\ln(x)\,x^2 + x^2 + c_1}{\ln(x)}$$

Replacing w in the above by $\frac{1}{y^2}$ using equation (5) gives the final solution.

$$\frac{1}{y^2} = \frac{-2\ln(x)\,x^2 + x^2 + c_1}{\ln(x)}$$

Solving for y gives

$$\begin{aligned}y(x) &= -\frac{\sqrt{-2\ln(x)^2\,x^2 + (x^2 + c_1)\ln(x)}}{2\ln(x)\,x^2 - x^2 - c_1} \\ y(x) &= \frac{\sqrt{-2\ln(x)^2\,x^2 + (x^2 + c_1)\ln(x)}}{2\ln(x)\,x^2 - x^2 - c_1}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = -\frac{\sqrt{-2\ln(x)^2 x^2 + (x^2 + c_1)\ln(x)}}{2\ln(x)x^2 - x^2 - c_1} \tag{1}$$

$$y = \frac{\sqrt{-2\ln(x)^2 x^2 + (x^2 + c_1)\ln(x)}}{2\ln(x)x^2 - x^2 - c_1} \tag{2}$$

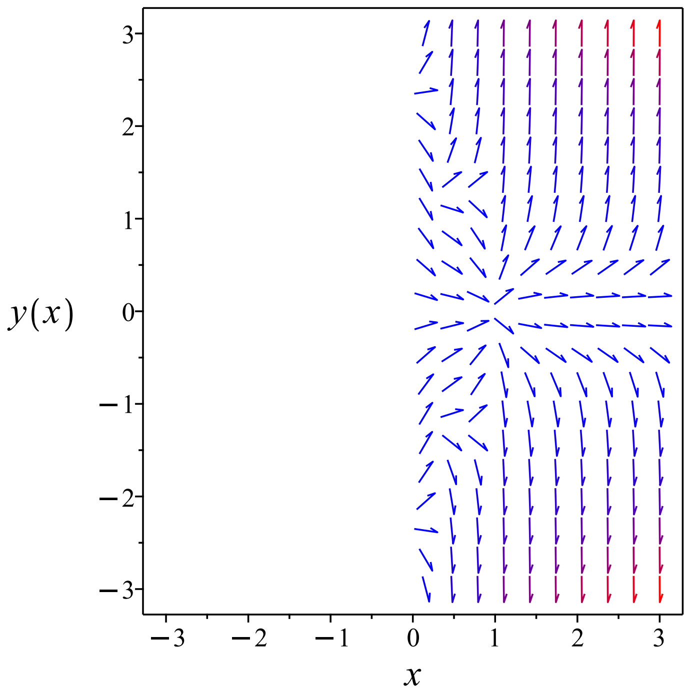


Figure 159: Slope field plot

Verification of solutions

$$y = -\frac{\sqrt{-2\ln(x)^2 x^2 + (x^2 + c_1)\ln(x)}}{2\ln(x)x^2 - x^2 - c_1}$$

Verified OK.

$$y = \frac{\sqrt{-2\ln(x)^2 x^2 + (x^2 + c_1)\ln(x)}}{2\ln(x)x^2 - x^2 - c_1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 86

```
dsolve(diff(y(x),x)-1/(2*x*ln(x))*y(x)=2*x*y(x)^3,y(x), singsol=all)
```

$$y(x) = \frac{\sqrt{-2\ln\left(x\right)^{2} x^{2}+\left(x^{2}+c_{1}\right)\ln\left(x\right)}}{2\ln\left(x\right)x^{2}-x^{2}-c_{1}}$$

$$y(x) = -\frac{\sqrt{-2\ln\left(x\right)^{2} x^{2}+\left(x^{2}+c_{1}\right)\ln\left(x\right)}}{2\ln\left(x\right)x^{2}-x^{2}-c_{1}}$$

✓ Solution by Mathematica

Time used: 0.274 (sec). Leaf size: 63

```
DSolve[y'[x]-1/(2*x*Log[x])*y[x]==2*x*y[x]^3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sqrt{\log(x)}}{\sqrt{x^2-2x^2\log(x)+c_1}}$$

$$y(x) \to \frac{\sqrt{\log(x)}}{\sqrt{x^2-2x^2\log(x)+c_1}}$$

$$y(x) \to 0$$

4.32 problem Problem 48

Internal problem ID [2696]
Internal file name [`OUTPUT/2188_Sunday_June_05_2022_02_53_02_AM_57356492/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.8, Change of Variables. page 79
Problem number: Problem 48.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, _Bernoulli]
```

$$\boxed{y' - \frac{y}{(\pi - 1)\,x} - \frac{3xy^{\pi}}{1-\pi} = 0}$$

4.32.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{3x^2y^{\pi} - y}{(\pi - 1)\,x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 144: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{y^{\pi}}{x}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{y^{\pi}}{x}} dy \end{aligned}$$

Which results in

$$S = -\frac{y\, y^{-\pi} x}{\pi - 1}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{3x^2 y^{\pi} - y}{(\pi - 1)\, x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y^{1-\pi}}{\pi - 1} \\ S_y &= x\, y^{-\pi} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{3x^2}{\pi - 1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{3R^2}{\pi - 1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{R^3}{\pi - 1} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{x y^{1-\pi}}{\pi - 1} = -\frac{x^3}{\pi - 1} + c_1$$

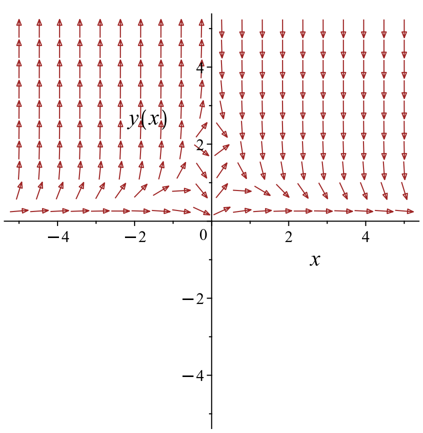
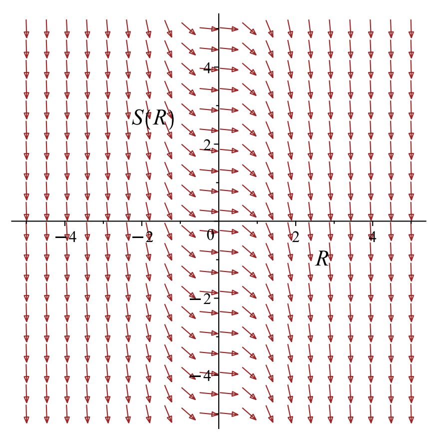
Which simplifies to

$$\frac{x^3 - c_1 \pi - x y^{1-\pi} + c_1}{\pi - 1} = 0$$

Which gives

$$y = \mathrm{e}^{-\frac{\ln\left(-\frac{-x^3 + c_1 \pi - c_1}{x}\right)}{\pi - 1}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{3x^2 y^{\pi} - y}{(\pi-1)x}$ 	$R = x$ $S = -\frac{x\, y^{1-\pi}}{\pi - 1}$	$\frac{dS}{dR} = -\frac{3R^2}{\pi-1}$ 

Summary
The solution(s) found are the following

$$y = e^{-\frac{\ln\left(-\frac{-x^3+c_1\pi-c_1}{x}\right)}{\pi-1}} \tag{1}$$

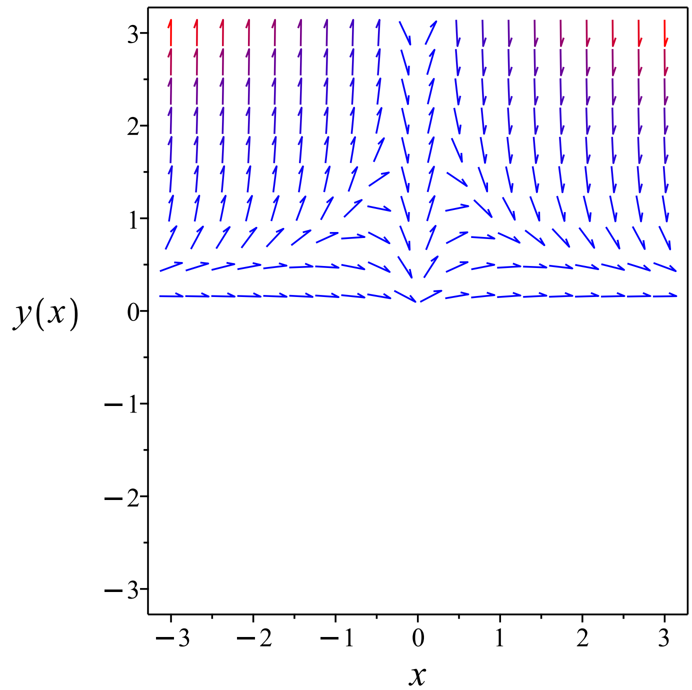


Figure 160: Slope field plot

Verification of solutions

$$y = e^{-\frac{\ln\left(-\frac{-x^3+c_1\pi-c_1}{x}\right)}{\pi-1}}$$

Verified OK.

4.32.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{3x^2y^\pi - y}{(\pi-1)\,x} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{1}{(\pi-1)\,x}y - \frac{3x}{\pi-1}y^\pi \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= \frac{1}{(\pi - 1)\,x} \\ f_1(x) &= -\frac{3x}{\pi - 1} \\ n &= \pi \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^\pi$ gives

$$y'y^{-\pi} = \frac{y^{1-\pi}}{(\pi - 1)\,x} - \frac{3x}{\pi - 1} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= y^{1-\pi} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = (1 - \pi)\, y^{-\pi} y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} \frac{w'(x)}{1 - \pi} &= \frac{w(x)}{(\pi - 1)\,x} - \frac{3x}{\pi - 1} \\ w' &= \frac{(1 - \pi)\,w}{(\pi - 1)\,x} - \frac{3(1 - \pi)\,x}{\pi - 1} \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned}p(x) &= \frac{1}{x}\\ q(x) &= 3x\end{aligned}$$

Hence the ode is

$$w'(x) + \frac{w(x)}{x} = 3x$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int \frac{1}{x}dx}\\ &= x\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)(3x)\\ \frac{\mathrm{d}}{\mathrm{d}x}(xw) &= (x)(3x)\\ \mathrm{d}(xw) &= (3x^2)\,\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}xw &= \int 3x^2\,\mathrm{d}x\\ xw &= x^3 + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x$ results in

$$w(x) = x^2 + \frac{c_1}{x}$$

Replacing w in the above by $y^{1-\pi}$ using equation (5) gives the final solution.

$$y^{1-\pi} = x^2 + \frac{c_1}{x}$$

Summary

The solution(s) found are the following

$$y^{1-\pi} = x^2 + \frac{c_1}{x} \tag{1}$$

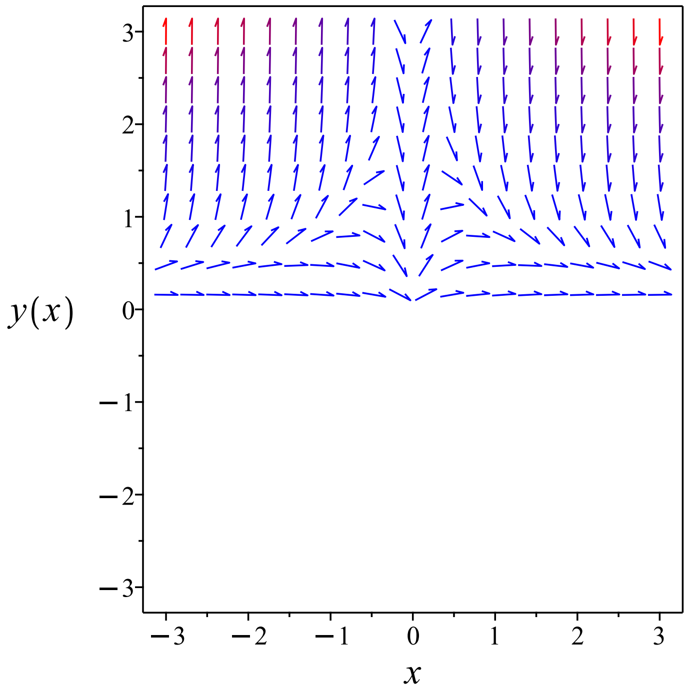


Figure 161: Slope field plot

Verification of solutions

$$y^{1-\pi} = x^2 + \frac{c_1}{x}$$

Verified OK.

4.32.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\, \mathrm{d}x + N(x,y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}((\pi-1)\, x)\, \mathrm{d}y &= \left(-3x^2 y^\pi + y\right) \mathrm{d}x \\ \left(3x^2 y^\pi - y\right) \mathrm{d}x + ((\pi-1)\, x)\, \mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= 3x^2 y^\pi - y \\ N(x,y) &= (\pi-1)\, x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(3x^2 y^\pi - y\right) \\ &= 3x^2 \pi\, y^{\pi-1} - 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}((\pi-1)\, x) \\ &= \pi - 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{(\pi - 1)\, x}\left(\left(\frac{3x^2\pi\, y^\pi}{y} - 1\right) - (\pi - 1)\right) \\
&= \frac{\pi(-1 + 3x^2 y^{\pi-1})}{(\pi - 1)\, x}
\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= \frac{1}{3x^2 y^\pi - y}\left((\pi - 1) - \left(\frac{3x^2\pi\, y^\pi}{y} - 1\right)\right) \\
&= -\frac{\pi}{y}
\end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned}
\mu &= e^{\int B \, \mathrm{d}y} \\
&= e^{\int -\frac{\pi}{y} \, \mathrm{d}y}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{-\pi \ln(y)} \\
&= y^{-\pi}
\end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned}
\overline{M} &= \mu M \\
&= y^{-\pi}(3x^2 y^\pi - y) \\
&= 3x^2 - y^{1-\pi}
\end{aligned}$$

And

$$\begin{aligned}
\overline{N} &= \mu N \\
&= y^{-\pi}((\pi - 1)\, x) \\
&= (\pi - 1)\, x\, y^{-\pi}
\end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(3x^2 - y^{1-\pi}\right) + \left((\pi - 1)\, x\, y^{-\pi}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int 3x^2 - y^{1-\pi}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = x\left(x^2 - y^{1-\pi}\right) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = -\frac{x\, y^{1-\pi}(1-\pi)}{y} + f'(y) \tag{4}
$$

$$
= (\pi - 1)\, x\, y^{-\pi} + f'(y)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = (\pi - 1)\, x\, y^{-\pi}$. Therefore equation (4) becomes

$$
(\pi - 1)\, x\, y^{-\pi} = (\pi - 1)\, x\, y^{-\pi} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$
f(y) = c_1
$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = x\left(x^2 - y^{1-\pi}\right) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x\left(x^2 - y^{1-\pi}\right)$$

The solution becomes

$$y = \mathrm{e}^{-\frac{\ln\left(-\frac{-x^3+c_1}{x}\right)}{\pi-1}}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-\frac{\ln\left(-\frac{-x^3+c_1}{x}\right)}{\pi-1}} \tag{1}$$

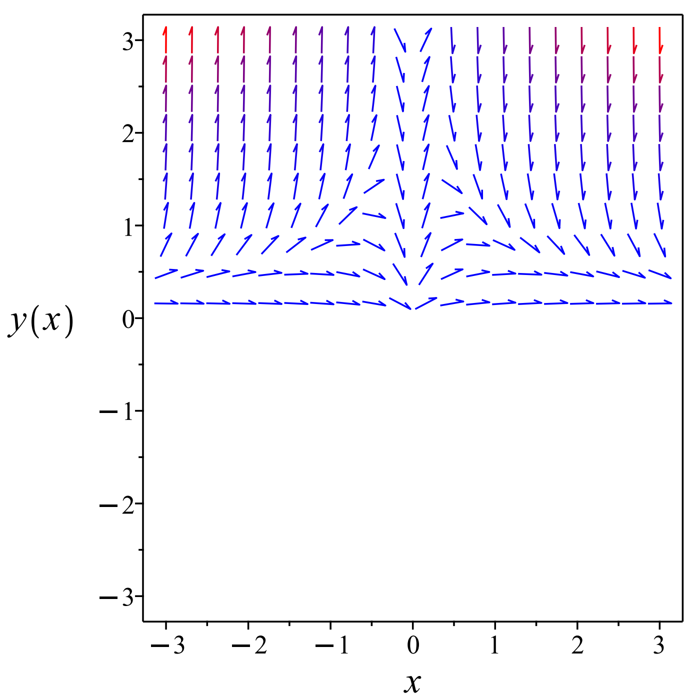


Figure 162: Slope field plot

Verification of solutions

$$y = \mathrm{e}^{-\frac{\ln\left(-\frac{-x^3+c_1}{x}\right)}{\pi-1}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 21

```
dsolve(diff(y(x),x)-1/( (Pi-1)*x)*y(x)=3/(1-Pi)*x*y(x)^Pi,y(x), singsol=all)
```

$$y(x) = \left(\frac{x^3+c_1}{x}\right)^{-\frac{1}{\pi-1}}$$

✓ Solution by Mathematica

Time used: 1.02 (sec). Leaf size: 28

```
DSolve[y'[x]-1/( (Pi-1)*x)*y[x]==3/(1-Pi)*x*y[x]^Pi,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \left(\frac{x^3+c_1}{x}\right)^{\frac{1}{1-\pi}}$$
$$y(x) \to 0$$

4.33 problem Problem 49

Internal problem ID [2697]
Internal file name [`OUTPUT/2189_Sunday_June_05_2022_02_53_06_AM_8005125/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.8, Change of Variables. page 79
Problem number: Problem 49.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_Bernoulli]
```

$$\boxed{2y' + y\cot(x) - \frac{8\cos(x)^3}{y} = 0}$$

4.33.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned}
y' &= \frac{8\cos(x)^3 - \cot(x)\, y^2}{2y} \\
y' &= \omega(x,y)
\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 146: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{y \sin(x)}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{y \sin(x)}} dy \end{aligned}$$

Which results in

$$S = \frac{y^2 \sin(x)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{8 \cos(x)^3 - \cot(x) y^2}{2y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{\cos(x) y^2}{2} \\ S_y &= \sin(x) y \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 4 \cos(x)^3 \sin(x) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 4 \cos(R)^3 \sin(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\cos(R)^4 + c_1 \tag{4}$$

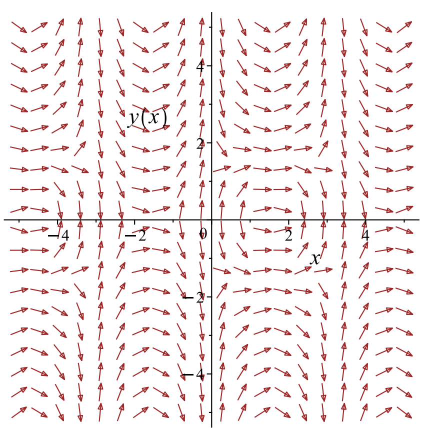
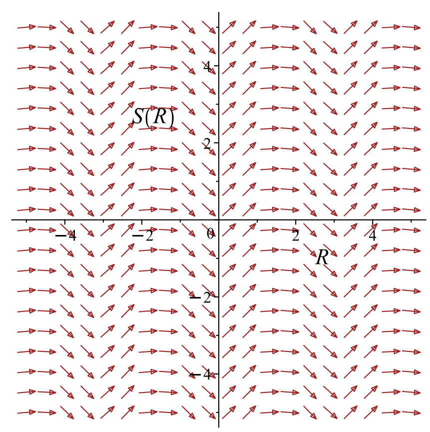
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y^2 \sin(x)}{2} = -\cos(x)^4 + c_1$$

Which simplifies to

$$\frac{y^2 \sin(x)}{2} = -\cos(x)^4 + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{8\cos(x)^3 - \cot(x)y^2}{2y}$ 	$R = x$ $S = \frac{y^2 \sin(x)}{2}$	$\frac{dS}{dR} = 4\cos(R)^3 \sin(R)$ 

Summary

The solution(s) found are the following

$$\frac{y^2 \sin(x)}{2} = -\cos(x)^4 + c_1 \tag{1}$$

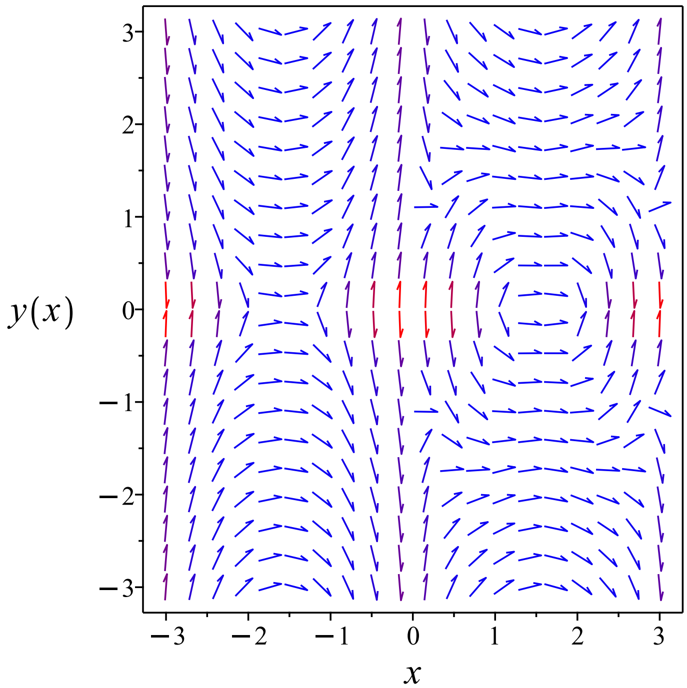


Figure 163: Slope field plot

Verification of solutions

$$\frac{y^2 \sin(x)}{2} = -\cos(x)^4 + c_1$$

Verified OK.

4.33.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{8\cos(x)^3 - \cot(x)\, y^2}{2y} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{\cot(x)}{2} y + 4\cos(x)^3 \frac{1}{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x) y + f_1(x) y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned}
f_0(x) &= -\frac{\cot(x)}{2} \\
f_1(x) &= 4\cos(x)^3 \\
n &= -1
\end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$y'y = -\frac{\cot(x)\, y^2}{2} + 4\cos(x)^3 \tag{4}$$

Let

$$\begin{aligned}
w &= y^{1-n} \\
&= y^2 \qquad (5)
\end{aligned}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = 2yy' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned}
\frac{w'(x)}{2} &= -\frac{\cot(x)\, w(x)}{2} + 4\cos(x)^3 \\
w' &= -\cot(x)\, w + 8\cos(x)^3 \qquad (7)
\end{aligned}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned}
p(x) &= \cot(x) \\
q(x) &= 8\cos(x)^3
\end{aligned}$$

Hence the ode is

$$w'(x) + \cot(x) w(x) = 8\cos(x)^3$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \cot(x)dx}\\ &= \sin(x)\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)\left(8\cos(x)^3\right)\\ \frac{d}{dx}(\sin(x) w) &= (\sin(x))\left(8\cos(x)^3\right)\\ d(\sin(x) w) &= \left(8\cos(x)^3 \sin(x)\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\sin(x) w &= \int 8\cos(x)^3 \sin(x) dx\\ \sin(x) w &= -2\cos(x)^4 + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \sin(x)$ results in

$$w(x) = -2\csc(x)\cos(x)^4 + c_1 \csc(x)$$

which simplifies to

$$w(x) = \csc(x)\left(-2\cos(x)^4 + c_1\right)$$

Replacing w in the above by y^2 using equation (5) gives the final solution.

$$y^2 = \csc(x)\left(-2\cos(x)^4 + c_1\right)$$

Solving for y gives

$$\begin{aligned}y(x) &= \csc(x)\sqrt{\sin(x)\left(-2\cos(x)^4 + c_1\right)}\\ y(x) &= -\csc(x)\sqrt{\sin(x)\left(-2\cos(x)^4 + c_1\right)}\end{aligned}$$

Summary
The solution(s) found are the following

$$y = \csc(x)\sqrt{\sin(x)\left(-2\cos(x)^4 + c_1\right)} \tag{1}$$

$$y = -\csc(x)\sqrt{\sin(x)\left(-2\cos(x)^4 + c_1\right)} \tag{2}$$

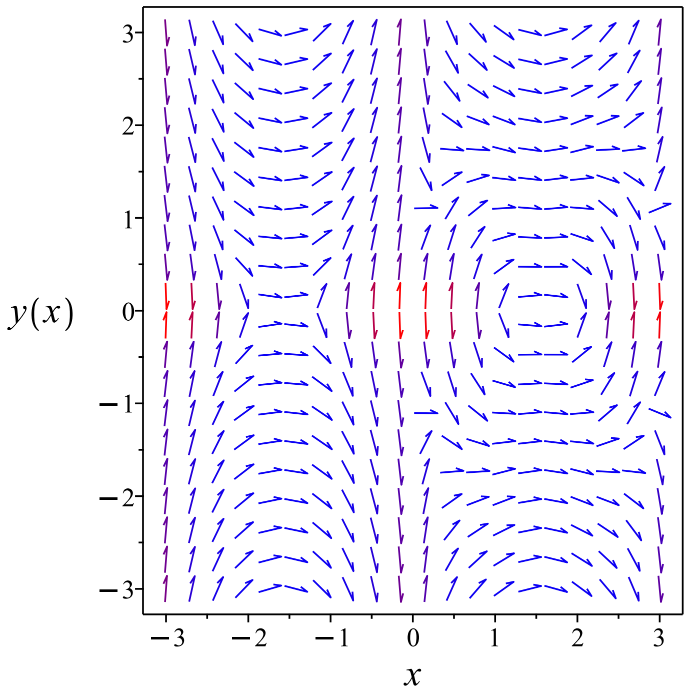


Figure 164: Slope field plot

Verification of solutions

$$y = \csc(x)\sqrt{\sin(x)\left(-2\cos(x)^4 + c_1\right)}$$

Verified OK.

$$y = -\csc(x)\sqrt{\sin(x)\left(-2\cos(x)^4 + c_1\right)}$$

Verified OK.

4.33.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(2y)\,\mathrm{d}y&=\left(8\cos(x)^3-\cot(x)\,y^2\right)\mathrm{d}x\\ \left(-8\cos(x)^3+\cot(x)\,y^2\right)\mathrm{d}x+(2y)\,\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-8\cos(x)^3+\cot(x)\,y^2\\ N(x,y)&=2y\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-8\cos\left(x\right)^{3} + \cot\left(x\right) y^{2}\right)\\ &= 2\cot\left(x\right) y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(2y)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{2y}((2\cot\left(x\right) y) - (0))\\ &= \cot\left(x\right)\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int \cot(x)\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{\ln(\sin(x))}\\ &= \sin\left(x\right)\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \sin\left(x\right)\left(-8\cos\left(x\right)^{3} + \cot\left(x\right) y^{2}\right)\\ &= \cos\left(x\right)\left(-8\cos\left(x\right)^{2}\sin\left(x\right) + y^{2}\right)\end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \sin(x)(2y) \\
&= 2\sin(x)\, y
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\cos(x)\left(-8\cos(x)^2 \sin(x) + y^2\right)\right) + (2\sin(x)\, y)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \cos(x)\left(-8\cos(x)^2\sin(x) + y^2\right)\mathrm{d}x
\end{aligned}
$$

$$
\phi = \sin(x)\left(2\sin(x)^3 + y^2 - 4\sin(x)\right) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 2\sin(x)\, y + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 2\sin(x)\, y$. Therefore equation (4) becomes

$$
2\sin(x)\, y = 2\sin(x)\, y + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \sin(x)\left(2\sin(x)^3 + y^2 - 4\sin(x)\right) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \sin(x)\left(2\sin(x)^3 + y^2 - 4\sin(x)\right)$$

Summary

The solution(s) found are the following

$$\sin(x)\left(2\sin(x)^3 + y^2 - 4\sin(x)\right) = c_1 \tag{1}$$

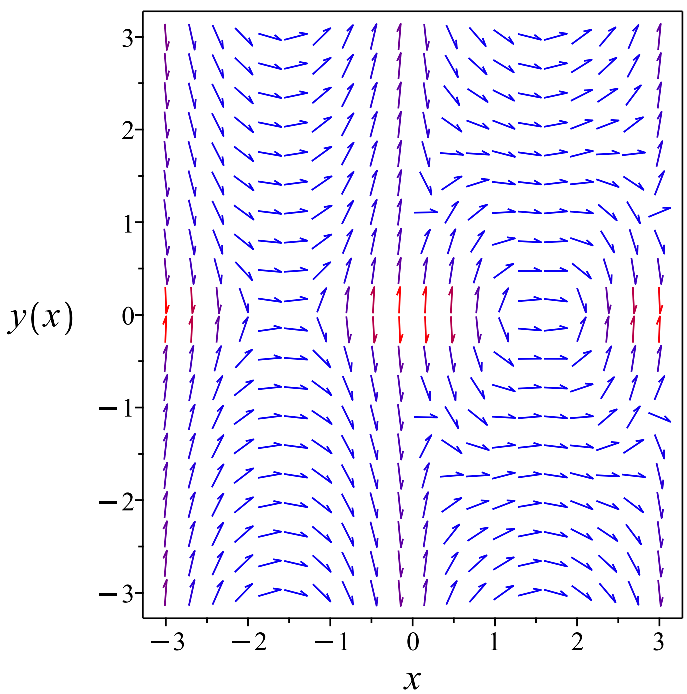


Figure 165: Slope field plot

Verification of solutions

$$\sin(x)\left(2\sin(x)^3 + y^2 - 4\sin(x)\right) = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 40

```
dsolve(2*diff(y(x),x)+y(x)*cot(x)=8/y(x)*cos(x)^3,y(x), singsol=all)
```

$$y(x) = \csc(x)\sqrt{\sin(x)\left(-2\cos(x)^4 + c_1\right)}$$
$$y(x) = -\csc(x)\sqrt{\sin(x)\left(-2\cos(x)^4 + c_1\right)}$$

✓ Solution by Mathematica

Time used: 3.971 (sec). Leaf size: 47

```
DSolve[2*y'[x]+y[x]*Cot[x]==8/y[x]*Cos[x]^3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{-2\cos^3(x)\cot(x) + c_1\csc(x)}$$
$$y(x) \to \sqrt{-2\cos^3(x)\cot(x) + c_1\csc(x)}$$

4.34 problem Problem 50

Internal problem ID [2698]
Internal file name [`OUTPUT/2190_Sunday_June_05_2022_02_53_11_AM_12770465/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.8, Change of Variables. page 79
Problem number: Problem 50.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "bernoulli", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{\left(1-\sqrt{3}\right) y'+y \sec \left(x \right)-y^{\sqrt{3}} \sec \left(x \right)=0}$$

4.34.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}
y' &= F(x,y) \\
&= f(x)g(y) \\
&= \frac{\sec \left(x \right) \left(-y^{\sqrt{3}}+y \right)}{\sqrt{3}-1}
\end{aligned}$$

Where $f(x) = \frac{\sec(x)}{\sqrt{3}-1}$ and $g(y) = -y^{\sqrt{3}} + y$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{-y^{\sqrt{3}} + y}\, dy &= \frac{\sec(x)}{\sqrt{3}-1}\, dx \\
\int \frac{1}{-y^{\sqrt{3}} + y}\, dy &= \int \frac{\sec(x)}{\sqrt{3}-1}\, dx \\
\left(\frac{3}{2} + \frac{\sqrt{3}}{2}\right) \ln(y) + \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}\right) \ln\left(-e^{\sqrt{3}\ln(y)} + y\right) \\
&= \left(\frac{1}{2} + \frac{\sqrt{3}}{2}\right) \ln(\sec(x) + \tan(x)) + c_1
\end{aligned}
$$

The above can be written as

$$
\begin{aligned}
\frac{\left(\sqrt{3}+3\right)\ln(y) - \left(1+\sqrt{3}\right)\ln\left(-e^{\sqrt{3}\ln(y)} + y\right)}{2} &= \left(\frac{1}{2} + \frac{\sqrt{3}}{2}\right) \ln(\sec(x) + \tan(x)) + c_1 \\
\left(\sqrt{3}+3\right)\ln(y) - \left(1+\sqrt{3}\right)\ln\left(-e^{\sqrt{3}\ln(y)} + y\right) &= (2)\left(\left(\frac{1}{2} + \frac{\sqrt{3}}{2}\right) \ln(\sec(x) + \tan(x)) + c_1\right) \\
&= 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}\right) \ln(\sec(x) + \tan(x)) + 2c_1
\end{aligned}
$$

Raising both side to exponential gives

$$
e^{\left(\sqrt{3}+3\right)\ln(y) - \left(1+\sqrt{3}\right)\ln\left(-e^{\sqrt{3}\ln(y)} + y\right)} = e^{2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}\right) \ln(\sec(x) + \tan(x)) + 2c_1}
$$

Which simplifies to

$$
\begin{aligned}
y^{\sqrt{3}+3}\left(-y^{\sqrt{3}} + y\right)^{-1-\sqrt{3}} &= 2c_1 e^{2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}\right) \ln(\sec(x) + \tan(x))} \\
&= c_2 e^{2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}\right) \ln(\sec(x) + \tan(x))}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = \operatorname{RootOf}\Bigg(& c_2 \sin(x) \_Z e^{2c_1} \left(-\frac{\cos(x)}{\sin(x)-1}\right)^{\sqrt{3}} \\
& - c_2 \sin(x) e^{2c_1} \_Z^{\sqrt{3}} \left(-\frac{\cos(x)}{\sin(x)-1}\right)^{\sqrt{3}} + c_2 \_Z e^{2c_1} \left(-\frac{\cos(x)}{\sin(x)-1}\right)^{\sqrt{3}} \\
& - c_2 e^{2c_1} \_Z^{\sqrt{3}} \left(-\frac{\cos(x)}{\sin(x)-1}\right)^{\sqrt{3}} - \_Z^{\sqrt{3}} \_Z^{3} \left(-\_Z^{\sqrt{3}} + \_Z\right)^{-\sqrt{3}} \cos(x)\Bigg)
\end{aligned} \tag{1}
$$

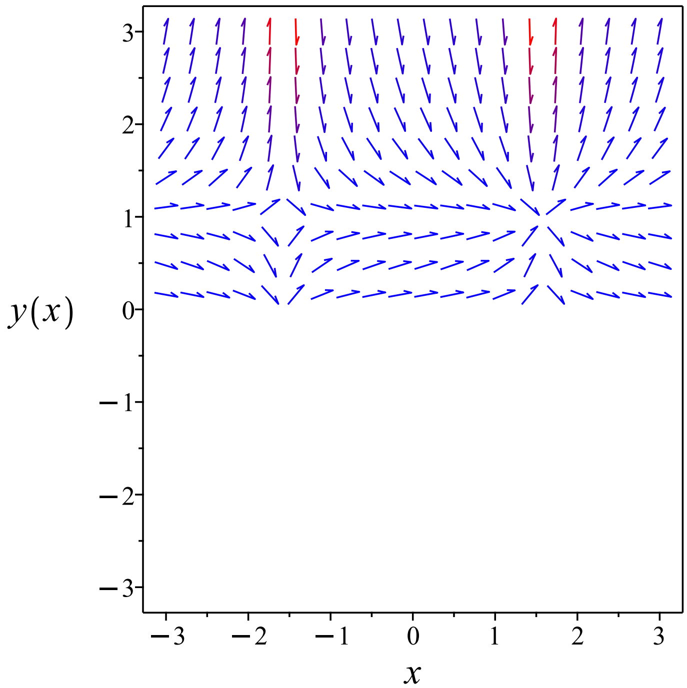


Figure 166: Slope field plot

Verification of solutions

$$
y = \operatorname{RootOf}\left(c_2 \sin(x)\_Z\, e^{2c_1}\left(-\frac{\cos(x)}{\sin(x)-1}\right)^{\sqrt{3}} - c_2 \sin(x)\, e^{2c_1}\_Z^{\sqrt{3}}\left(-\frac{\cos(x)}{\sin(x)-1}\right)^{\sqrt{3}} + c_2\_Z\, e^{2c_1}\left(-\frac{\cos(x)}{\sin(x)-1}\right)^{\sqrt{3}} - c_2 e^{2c_1}\_Z^{\sqrt{3}}\left(-\frac{\cos(x)}{\sin(x)-1}\right)^{\sqrt{3}} - \_Z^{\sqrt{3}}\_Z^{3}\left(-\_Z^{\sqrt{3}} + \_Z\right)^{-\sqrt{3}}\cos(x)\right)
$$

Warning, solution could not be verified

4.34.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
y' = -\frac{\sec(x)\left(y^{\sqrt{3}} - y\right)}{\sqrt{3}-1}
$$
$$
y' = \omega(x, y)
$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 148: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, y) &= \frac{\sqrt{3} - 1}{\sec(x)} \\ \eta(x, y) &= 0 \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{\sqrt{3}-1}{\sec(x)}} dx \end{aligned}$$

Which results in

$$S = \frac{\ln (\sec (x) + \tan (x))}{\sqrt{3} - 1}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{\sec (x) \left(y^{\sqrt{3}} - y\right)}{\sqrt{3} - 1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= \frac{\sec (x)}{\sqrt{3} - 1} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{-y^{\sqrt{3}} + y} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{-R^{\sqrt{3}} + R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \left(\frac{3}{2} + \frac{\sqrt{3}}{2}\right) \ln(R) + \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}\right) \ln\left(-R^{\sqrt{3}} + R\right) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(\sec(x) + \tan(x))}{\sqrt{3} - 1} = \left(\frac{3}{2} + \frac{\sqrt{3}}{2}\right) \ln(y) + \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}\right) \ln\left(y - y^{\sqrt{3}}\right) + c_1$$

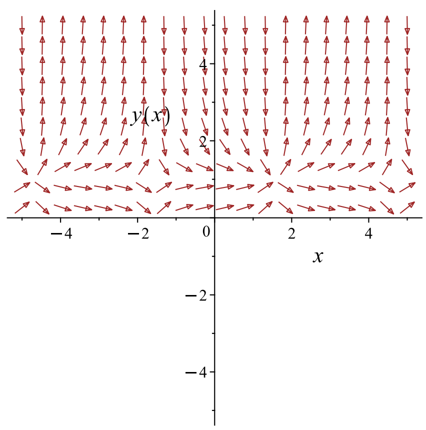
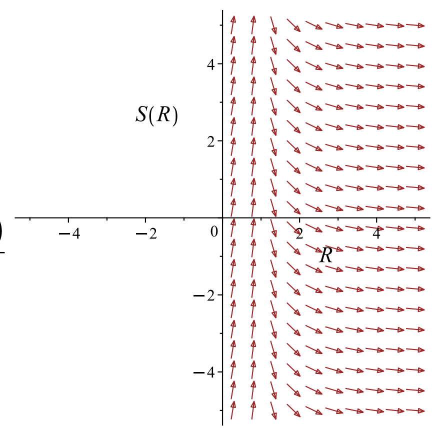
Which simplifies to

$$\frac{\ln\left(y - y^{\sqrt{3}}\right) + \ln(\sec(x) + \tan(x)) + (-\ln(y) - c_1)\sqrt{3} + c_1}{\sqrt{3} - 1} = 0$$

Which gives

$$y = \mathrm{e}^{\frac{\left(\ln\left(\frac{1+\sin(x)}{\sin(x)\mathrm{e}^{c_1} + \cos(x)\mathrm{e}^{c_1\sqrt{3}} + \mathrm{e}^{c_1}}\right) + c_1\right)\left(1+\sqrt{3}\right)}{2}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{\sec(x)\left(y^{\sqrt{3}}-y\right)}{\sqrt{3}-1}$ 	$R = y$ $S = \frac{\ln(\sec(x)+\tan(x))}{\sqrt{3}-1}$	$\frac{dS}{dR} = \frac{1}{-R^{\sqrt{3}}+R}$ 

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{\frac{\left(\ln\left(\frac{1+\sin(x)}{\sin(x)\mathrm{e}^{c_1}+\cos(x)\mathrm{e}^{c_1\sqrt{3}}+\mathrm{e}^{c_1}}\right)+c_1\right)\left(1+\sqrt{3}\right)}{2}} \tag{1}$$

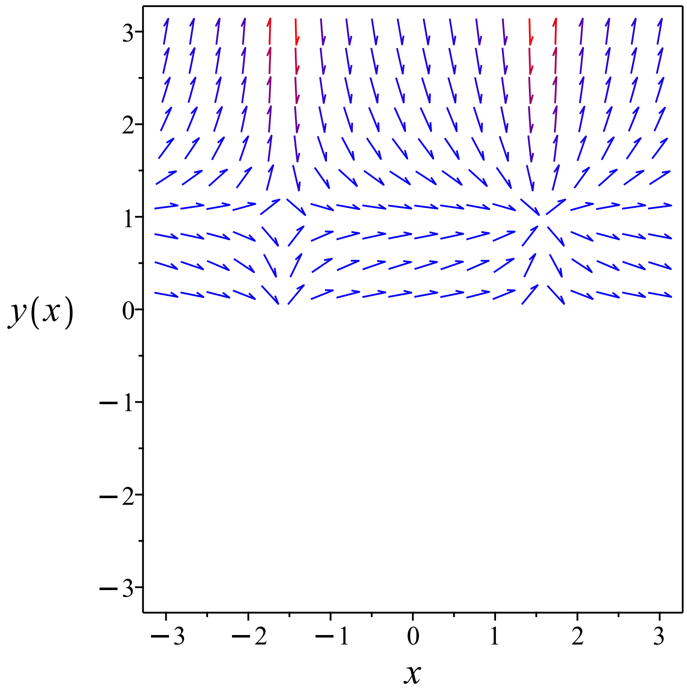


Figure 167: Slope field plot

Verification of solutions

$$y = e^{\frac{\left(\ln\left(\frac{1+\sin(x)}{\sin(x)e^{c_1}+\cos(x)e^{c_1}\sqrt{3}+e^{c_1}}\right)+c_1\right)(1+\sqrt{3})}{2}}$$

Verified OK.

4.34.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{\sec(x)\left(-y^{\sqrt{3}}+y\right)}{\sqrt{3}-1} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{\sec(x)}{\sqrt{3}-1}y - \frac{\sec(x)}{\sqrt{3}-1}y^{\sqrt{3}} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= \frac{\sec(x)}{\sqrt{3}-1} \\ f_1(x) &= -\frac{\sec(x)}{\sqrt{3}-1} \\ n &= \sqrt{3} \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^{\sqrt{3}}$ gives

$$y' y^{-\sqrt{3}} = \frac{\sec(x)\, y^{1-\sqrt{3}}}{\sqrt{3}-1} - \frac{\sec(x)}{\sqrt{3}-1} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= y^{1-\sqrt{3}} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = \left(1-\sqrt{3}\right) y^{-\sqrt{3}} y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} \frac{w'(x)}{1-\sqrt{3}} &= \frac{\sec(x)\, w(x)}{\sqrt{3}-1} - \frac{\sec(x)}{\sqrt{3}-1} \\ w' &= \frac{\left(1-\sqrt{3}\right)\sec(x)\, w}{\sqrt{3}-1} - \frac{\left(1-\sqrt{3}\right)\sec(x)}{\sqrt{3}-1} \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$
\begin{aligned}
p(x) &= \sec(x) \\
q(x) &= \sec(x)
\end{aligned}
$$

Hence the ode is

$$
w'(x) + \sec(x)\, w(x) = \sec(x)
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int \sec(x) dx} \\
&= \sec(x) + \tan(x)
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu w) &= (\mu)\,(\sec(x)) \\
\frac{d}{dx}((\sec(x) + \tan(x))\, w) &= (\sec(x) + \tan(x))\,(\sec(x)) \\
d((\sec(x) + \tan(x))\, w) &= \left(-\frac{1}{\sin(x) - 1}\right) dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
(\sec(x) + \tan(x))\, w &= \int -\frac{1}{\sin(x) - 1}\, dx \\
(\sec(x) + \tan(x))\, w &= -\frac{2}{-1 + \tan\left(\frac{x}{2}\right)} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \sec(x) + \tan(x)$ results in

$$
w(x) = -\frac{2}{(\sec(x) + \tan(x))\left(-1 + \tan\left(\frac{x}{2}\right)\right)} + \frac{c_1}{\sec(x) + \tan(x)}
$$

Replacing w in the above by $y^{1-\sqrt{3}}$ using equation (5) gives the final solution.

$$
y^{1-\sqrt{3}} = -\frac{2}{(\sec(x) + \tan(x))\left(-1 + \tan\left(\frac{x}{2}\right)\right)} + \frac{c_1}{\sec(x) + \tan(x)}
$$

Summary

The solution(s) found are the following

$$
y^{1-\sqrt{3}} = -\frac{2}{(\sec(x) + \tan(x))\left(-1 + \tan\left(\frac{x}{2}\right)\right)} + \frac{c_1}{\sec(x) + \tan(x)} \tag{1}
$$

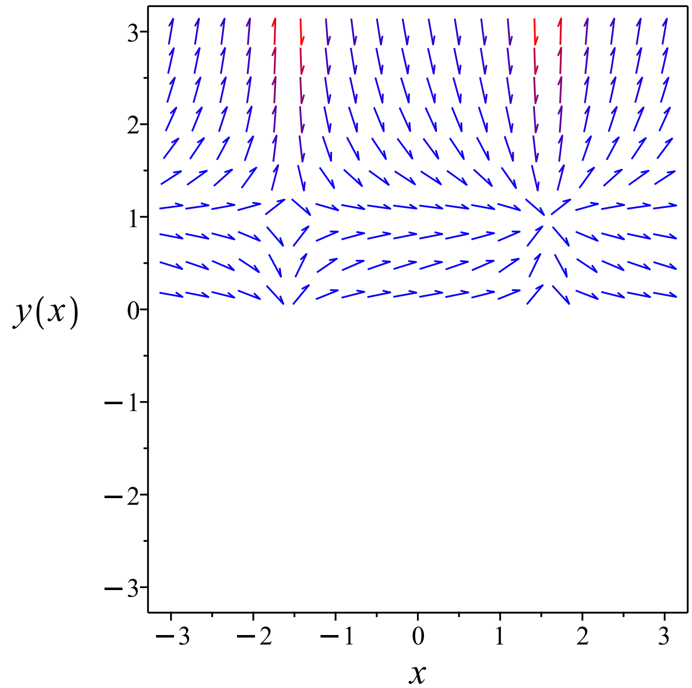


Figure 168: Slope field plot

Verification of solutions

$$y^{1-\sqrt{3}} = -\frac{2}{(\sec(x)+\tan(x))\left(-1+\tan\left(\frac{x}{2}\right)\right)} + \frac{c_1}{\sec(x)+\tan(x)}$$

Verified OK.

4.34.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(\frac{\sqrt{3}-1}{-y^{\sqrt{3}}+y}\right)\mathrm{d}y &= (\sec(x))\,\mathrm{d}x \\
(-\sec(x))\,\mathrm{d}x + \left(\frac{\sqrt{3}-1}{-y^{\sqrt{3}}+y}\right)\mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -\sec(x) \\
N(x, y) &= \frac{\sqrt{3}-1}{-y^{\sqrt{3}}+y}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-\sec(x)) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{\sqrt{3}-1}{-y^{\sqrt{3}}+y}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \qquad (1)\\
\frac{\partial \phi}{\partial y} &= N \qquad (2)
\end{aligned}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int M \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int -\sec(x) \,\mathrm{d}x \\
\phi &= -\ln(\sec(x)+\tan(x)) + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y}=\frac{\sqrt{3}-1}{-y^{\sqrt{3}}+y}$. Therefore equation (4) becomes

$$
\frac{\sqrt{3}-1}{-y^{\sqrt{3}}+y} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
\begin{aligned}
f'(y) &= -\frac{\sqrt{3}-1}{y^{\sqrt{3}}-y} \\
&= \frac{\sqrt{3}-1}{-y^{\sqrt{3}}+y}
\end{aligned}
$$

Integrating the above w.r.t y results in

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{\sqrt{3}-1}{-y^{\sqrt{3}}+y}\right)\mathrm{d}y$$

$$f(y) = \sqrt{3}\,\ln(y) - \ln\left(-\mathrm{e}^{\sqrt{3}\,\ln(y)} + y\right) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(\sec(x) + \tan(x)) + \sqrt{3}\,\ln(y) - \ln\left(-\mathrm{e}^{\sqrt{3}\,\ln(y)} + y\right) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(\sec(x) + \tan(x)) + \sqrt{3}\,\ln(y) - \ln\left(-\mathrm{e}^{\sqrt{3}\,\ln(y)} + y\right)$$

The solution becomes

$$y = \mathrm{e}^{\frac{\ln\left(\frac{1+\sin(x)}{\sin(x)\mathrm{e}^{c_1}+\cos(x)+\mathrm{e}^{c_1}}\right)}{2}+\frac{\ln\left(\frac{1+\sin(x)}{\sin(x)\mathrm{e}^{c_1}+\cos(x)+\mathrm{e}^{c_1}}\right)\sqrt{3}}{2}+\frac{c_1}{2}+\frac{c_1\sqrt{3}}{2}}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{\frac{\ln\left(\frac{1+\sin(x)}{\sin(x)\mathrm{e}^{c_1}+\cos(x)+\mathrm{e}^{c_1}}\right)}{2}+\frac{\ln\left(\frac{1+\sin(x)}{\sin(x)\mathrm{e}^{c_1}+\cos(x)+\mathrm{e}^{c_1}}\right)\sqrt{3}}{2}+\frac{c_1}{2}+\frac{c_1\sqrt{3}}{2}} \tag{1}$$

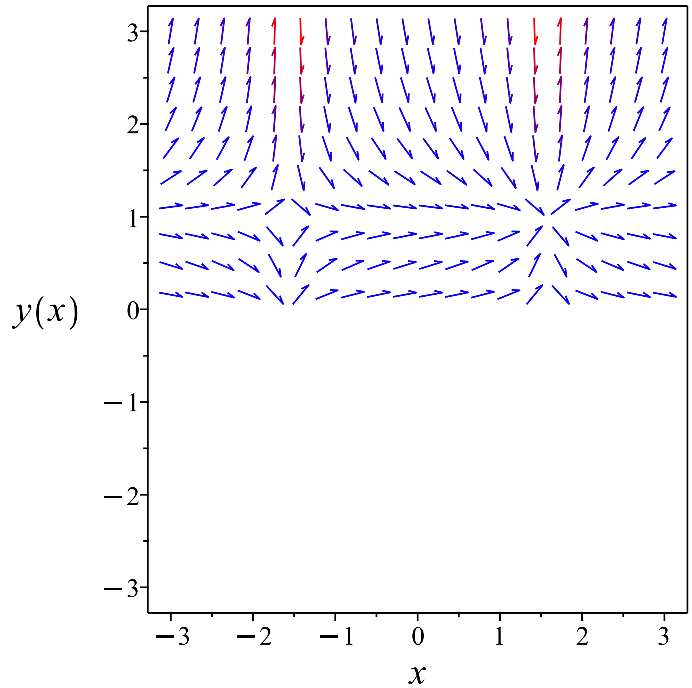


Figure 169: Slope field plot

Verification of solutions

$$y = \mathrm{e}^{\frac{\ln\left(\frac{1+\sin(x)}{\sin(x)\mathrm{e}^{c_1}+\cos(x)+\mathrm{e}^{c_1}}\right)}{2}+\frac{\ln\left(\frac{1+\sin(x)}{\sin(x)\mathrm{e}^{c_1}+\cos(x)+\mathrm{e}^{c_1}}\right)\sqrt{3}}{2}+\frac{c_1}{2}+\frac{c_1\sqrt{3}}{2}}$$

Verified OK.

4.34.5 Maple step by step solution

Let's solve

$\left(1-\sqrt{3}\right) y' + y \sec\left(x\right) - y^{\sqrt{3}} \sec\left(x\right) = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^{\sqrt{3}}-y} = -\frac{\left(1+\sqrt{3}\right)\sec(x)}{2}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y^{\sqrt{3}}-y}dx = \int -\frac{\left(1+\sqrt{3}\right)\sec(x)}{2}dx + c_1$

- Evaluate integral

$$\left(-\frac{3}{2}-\frac{\sqrt{3}}{2}\right)\ln\left(y\right)+\left(\frac{1}{2}+\frac{\sqrt{3}}{2}\right)\ln\left(-\mathrm{e}^{\sqrt{3}\,\ln(y)}+y\right)=\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}\right)\ln\left(\sec\left(x\right)+\tan\left(x\right)\right)+c_1$$

- Solve for y

$$y=\mathrm{e}^{\frac{\left(\sqrt{3}\,\ln\left(\frac{\cos(x)}{\cos(x)\mathrm{e}^{-c_1\sqrt{3}+c_1}-\sin(x)+1}\right)+c_1\sqrt{3}-3c_1\right)\left(\sqrt{3}+3\right)}{6}}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.219 (sec). Leaf size: 23

```
dsolve((1-sqrt(3))*diff(y(x),x)+y(x)*sec(x)=y(x)^sqrt(3)*sec(x),y(x), singsol=all)
```

$$y(x)=\left(-c_1\tan\left(x\right)+1+\sec\left(x\right)c_1\right)^{-\frac{1}{2}-\frac{\sqrt{3}}{2}}$$

✓ Solution by Mathematica

Time used: 0.608 (sec). Leaf size: 76

```
DSolve[(1-Sqrt[3])*y'[x]+y[x]*Sec[x]==y[x]^Sqrt[3]*Sec[x],y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to \text{InverseFunction}\left[\frac{\log\left(1-\#1^{\sqrt{3}-1}\right)-\left(\sqrt{3}-1\right)\log(\#1)}{\sqrt{3}-1}\&\right]\left[-\frac{2\text{arctanh}\left(\tan\left(\frac{x}{2}\right)\right)}{\sqrt{3}-1}+c_1\right]$$

$$y(x) \to 0$$

$$y(x) \to 1$$

4.35 problem Problem 51

Internal problem ID [2699]
Internal file name [`OUTPUT/2191_Sunday_June_05_2022_02_53_15_AM_62391903/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.8, Change of Variables. page 79
Problem number: Problem 51.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "bernoulli", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_rational, _Bernoulli]
```

$$\boxed{y' + \frac{2xy}{x^2+1} - y^2x = 0}$$

With initial conditions

$$[y(0) = 1]$$

4.35.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= \frac{xy(x^2y + y - 2)}{x^2+1} \end{aligned}$$

The x domain of $f(x,y)$ when $y = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x,y)$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{xy(x^2y+y-2)}{x^2+1}\right)\\ &= \frac{x(x^2y+y-2)}{x^2+1} + xy\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

4.35.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned}y' &= \frac{xy(x^2y+y-2)}{x^2+1}\\ y' &= \omega(x,y)\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 151: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, y) &= 0 \\ \eta(x, y) &= \left(x^2 + 1\right) y^2 \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\left(x^2+1\right) y^2} dy \end{aligned}$$

Which results in

$$S = -\frac{1}{\left(x^2+1\right) y}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{xy(x^2 y + y - 2)}{x^2+1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{2x}{\left(x^2+1\right)^2 y} \\ S_y &= \frac{1}{\left(x^2+1\right) y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{x}{x^2+1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{R}{R^2+1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{\ln(R^2+1)}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{1}{(x^2+1)\,y} = \frac{\ln(x^2+1)}{2} + c_1$$

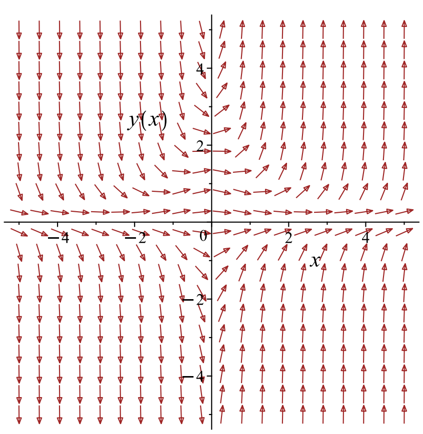
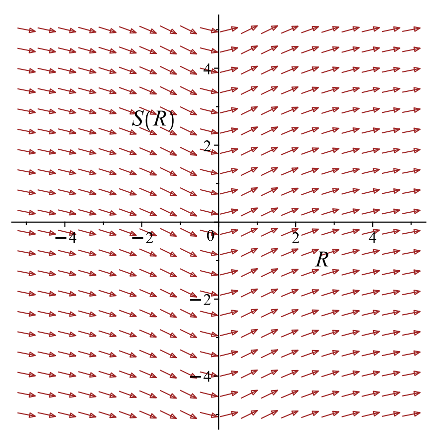
Which simplifies to

$$-\frac{1}{(x^2+1)\,y} = \frac{\ln(x^2+1)}{2} + c_1$$

Which gives

$$y = -\frac{2}{\ln(x^2+1)\,x^2 + 2c_1x^2 + \ln(x^2+1) + 2c_1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{xy(x^2y+y-2)}{x^2+1}$ 	$R = x$ $S = -\frac{1}{(x^2+1)\,y}$	$\frac{dS}{dR} = \frac{R}{R^2+1}$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\frac{1}{c_1}$$

$$c_1 = -1$$

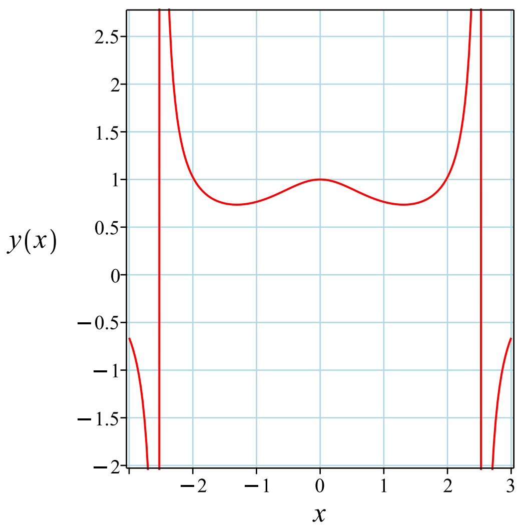
Substituting c_1 found above in the general solution gives

$$y = -\frac{2}{\ln\left(x^2+1\right)x^2 - 2x^2 + \ln\left(x^2+1\right) - 2}$$

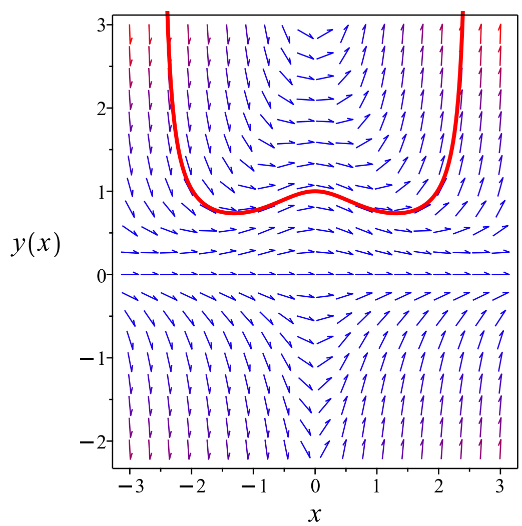
Summary

The solution(s) found are the following

$$y = -\frac{2}{\ln\left(x^2+1\right)x^2 - 2x^2 + \ln\left(x^2+1\right) - 2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{2}{\ln\left(x^2+1\right)x^2 - 2x^2 + \ln\left(x^2+1\right) - 2}$$

Verified OK.

4.35.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{xy(x^2y+y-2)}{x^2+1} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{2x}{x^2+1}y + xy^2 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -\frac{2x}{x^2+1} \\ f_1(x) &= x \\ n &= 2 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^2$ gives

$$y'\frac{1}{y^2} = -\frac{2x}{(x^2+1)\,y} + x \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \frac{1}{y} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{1}{y^2}y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
-w'(x) &= -\frac{2xw(x)}{x^2+1} + x \\
w' &= \frac{2xw}{x^2+1} - x
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{2x}{x^2+1} \\
q(x) &= -x
\end{aligned}
$$

Hence the ode is

$$
w'(x) - \frac{2xw(x)}{x^2+1} = -x
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int -\frac{2x}{x^2+1}dx} \\
&= \frac{1}{x^2+1}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu w) &= (\mu)(-x) \\
\frac{d}{dx}\left(\frac{w}{x^2+1}\right) &= \left(\frac{1}{x^2+1}\right)(-x) \\
d\left(\frac{w}{x^2+1}\right) &= \left(-\frac{x}{x^2+1}\right) dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{w}{x^2+1} &= \int -\frac{x}{x^2+1}\, dx \\
\frac{w}{x^2+1} &= -\frac{\ln(x^2+1)}{2} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2+1}$ results in

$$w(x) = -\frac{(x^2+1)\ln(x^2+1)}{2} + (x^2+1)\, c_1$$

which simplifies to

$$w(x) = (x^2+1)\left(-\frac{\ln(x^2+1)}{2} + c_1\right)$$

Replacing w in the above by $\frac{1}{y}$ using equation (5) gives the final solution.

$$\frac{1}{y} = (x^2+1)\left(-\frac{\ln(x^2+1)}{2} + c_1\right)$$

Or

$$y = \frac{1}{(x^2+1)\left(-\frac{\ln(x^2+1)}{2} + c_1\right)}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{1}{c_1}$$

$$c_1 = 1$$

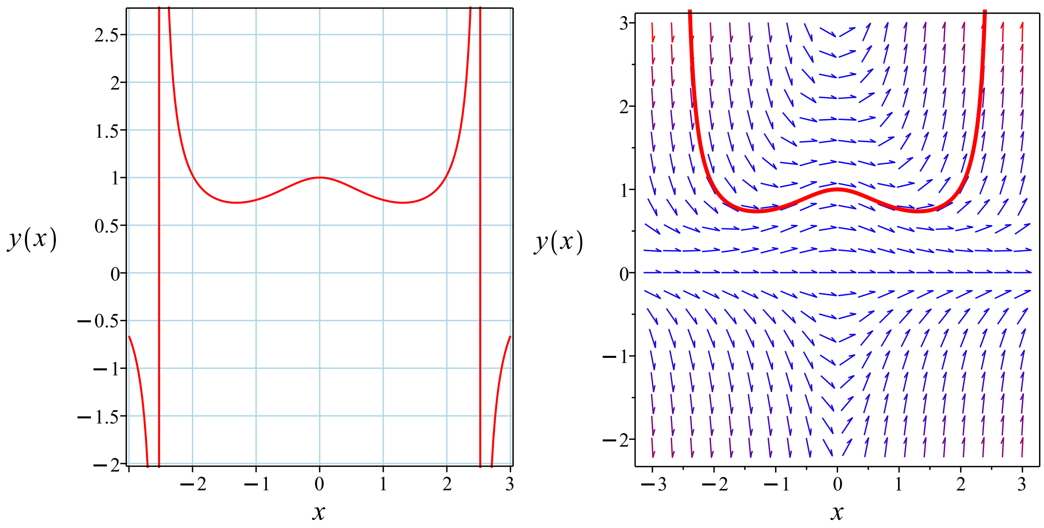
Substituting c_1 found above in the general solution gives

$$y = -\frac{2}{\ln(x^2+1)\, x^2 - 2x^2 + \ln(x^2+1) - 2}$$

Summary

The solution(s) found are the following

$$y = -\frac{2}{\ln(x^2+1)\, x^2 - 2x^2 + \ln(x^2+1) - 2} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = -\frac{2}{\ln(x^2+1)\,x^2 - 2x^2 + \ln(x^2+1) - 2}$$

Verified OK.

4.35.4 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{xy(x^2y + y - 2)}{x^2+1} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{x^3y^2}{x^2+1} + \frac{x\,y^2}{x^2+1} - \frac{2xy}{x^2+1}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = -\frac{2x}{x^2+1}$ and $f_2(x) = x$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{xu} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 1 \\ f_1 f_2 &= -\frac{2x^2}{x^2+1} \\ f_2^2 f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$x u''(x) - \left(1 - \frac{2x^2}{x^2+1}\right) u'(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 + \ln\left(x^2+1\right) c_2$$

The above shows that

$$u'(x) = \frac{2x c_2}{x^2+1}$$

Using the above in (1) gives the solution

$$y = -\frac{2c_2}{(x^2+1)\,(c_1 + \ln(x^2+1)\, c_2)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{2}{(x^2+1)\,(c_3 + \ln(x^2+1))}$$

Initial conditions are used to solve for c_3. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\frac{2}{c_3}$$

$$c_3 = -2$$

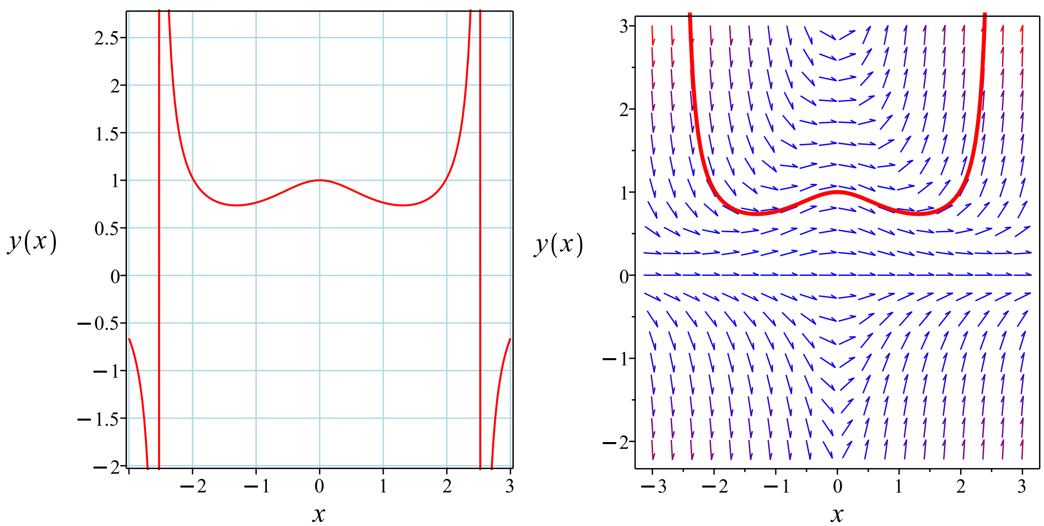
Substituting c_3 found above in the general solution gives

$$y = -\frac{2}{\ln(x^2+1)\,x^2 - 2x^2 + \ln(x^2+1) - 2}$$

Summary

The solution(s) found are the following

$$y = -\frac{2}{\ln(x^2+1)\,x^2 - 2x^2 + \ln(x^2+1) - 2} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = -\frac{2}{\ln(x^2+1)\,x^2 - 2x^2 + \ln(x^2+1) - 2}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 23

```
dsolve([diff(y(x),x)+2*x/(1+x^2)*y(x)=x*y(x)^2,y(0) = 1],y(x), singsol=all)
```

$$y(x) = -\frac{2}{(x^2+1)(\ln(x^2+1)-2)}$$

✓ Solution by Mathematica

Time used: 0.196 (sec). Leaf size: 24

```
DSolve[{y'[x]+2*x/(1+x^2)*y[x]==x*y[x]^2,{y[0]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{2}{(x^2+1)(\log(x^2+1)-2)}$$

4.36 problem Problem 52

Internal problem ID [2700]
Internal file name [`OUTPUT/2192_Sunday_June_05_2022_02_53_19_AM_81078817/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.8, Change of Variables. page 79
Problem number: Problem 52.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_Bernoulli]
```

$$\boxed{y' + y\cot(x) - y^3 \sin(x)^3 = 0}$$

With initial conditions

$$\left[y\left(\frac{\pi}{2}\right) = 1\right]$$

4.36.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= -\cot(x)\,y + y^3 \sin(x)^3 \end{aligned}$$

The x domain of $f(x,y)$ when $y=1$ is

$$\{x < \pi\_Z144 \lor \pi\_Z144 < x\}$$

And the point $x_0 = \frac{\pi}{2}$ is inside this domain. The y domain of $f(x, y)$ when $x = \frac{\pi}{2}$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\cot(x)\, y + y^3 \sin(x)^3\right) \\
&= 3\sin(x)^3 y^2 - \cot(x)
\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 1$ is

$$\{x < \pi\_Z144 \vee \pi\_Z144 < x\}$$

And the point $x_0 = \frac{\pi}{2}$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = \frac{\pi}{2}$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

4.36.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned}
y' &= -\cot(x)\, y + y^3 \sin(x)^3 \\
y' &= \omega(x, y)
\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 153: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= y^3 \sin(x)^2
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{y^3 \sin(x)^2} dy \end{aligned}$$

Which results in

$$S = -\frac{1}{2\sin(x)^2 y^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\cot(x)\, y + y^3 \sin(x)^3$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{\csc(x)^2 \cot(x)}{y^2} \\ S_y &= \frac{\csc(x)^2}{y^3} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \sin(x) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \sin(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\cos(R) + c_1 \tag{4}$$

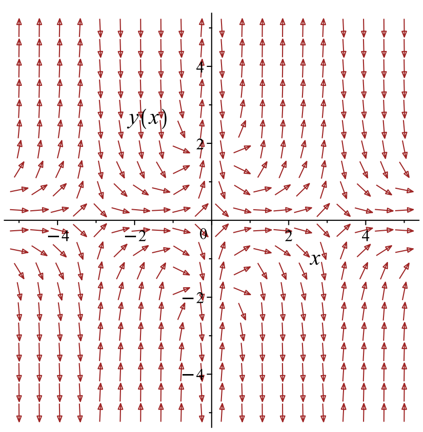
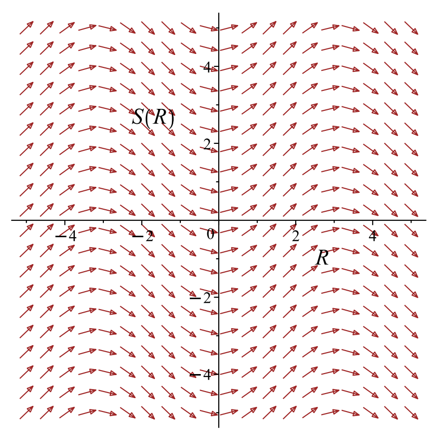
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{\csc(x)^2}{2y^2} = -\cos(x) + c_1$$

Which simplifies to

$$-\frac{\csc(x)^2}{2y^2} = -\cos(x) + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\cot(x) y + y^3 \sin(x)^3$ 	$R = x$ $S = -\frac{\csc(x)^2}{2y^2}$	$\frac{dS}{dR} = \sin(R)$ 

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{2}$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$-\frac{1}{2} = c_1$$

$$c_1 = -\frac{1}{2}$$

Substituting c_1 found above in the general solution gives

$$-\frac{\csc(x)^2}{2y^2} = -\cos(x) - \frac{1}{2}$$

The above simplifies to

$$2\cos(x)\, y^2 - \csc(x)^2 + y^2 = 0$$

Summary

The solution(s) found are the following

$$2y^2 \cos(x) - \csc(x)^2 + y^2 = 0 \tag{1}$$

Verification of solutions

$$2y^2 \cos(x) - \csc(x)^2 + y^2 = 0$$

Verified OK.

4.36.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\cot(x)\, y + y^3 \sin(x)^3 \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\cot(x)\, y + \sin(x)^3\, y^3 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -\cot(x) \\ f_1(x) &= \sin(x)^3 \\ n &= 3 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^3$ gives

$$y'\frac{1}{y^3} = -\frac{\cot(x)}{y^2} + \sin(x)^3 \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \frac{1}{y^2} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{2}{y^3}y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -\frac{w'(x)}{2} &= -\cot(x)\,w(x) + \sin(x)^3 \\ w' &= 2\cot(x)\,w - 2\sin(x)^3 \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= -2\cot(x) \\ q(x) &= -2\sin(x)^3 \end{aligned}$$

Hence the ode is

$$w'(x) - 2\cot(x)\,w(x) = -2\sin(x)^3$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int -2\cot(x)dx} \\
&= \frac{1}{\sin(x)^2}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)\left(-2\sin(x)^3\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{w}{\sin(x)^2}\right) &= \left(\frac{1}{\sin(x)^2}\right)\left(-2\sin(x)^3\right) \\
\mathrm{d}\left(\frac{w}{\sin(x)^2}\right) &= (-2\sin(x))\ \mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{w}{\sin(x)^2} &= \int -2\sin(x)\ \mathrm{d}x \\
\frac{w}{\sin(x)^2} &= 2\cos(x) + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{\sin(x)^2}$ results in

$$
w(x) = 2\sin(x)^2\cos(x) + c_1\sin(x)^2
$$

which simplifies to

$$
w(x) = \sin(x)^2(2\cos(x) + c_1)
$$

Replacing w in the above by $\frac{1}{y^2}$ using equation (5) gives the final solution.

$$
\frac{1}{y^2} = \sin(x)^2(2\cos(x) + c_1)
$$

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{2}$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$
1 = c_1
$$

$$
c_1 = 1
$$

Substituting c_1 found above in the general solution gives

$$\frac{1}{y^2} = \sin(x)^2 (2\cos(x) + 1)$$

The above simplifies to

$$-2y^2 \sin(x)^2 \cos(x) - y^2 \sin(x)^2 + 1 = 0$$

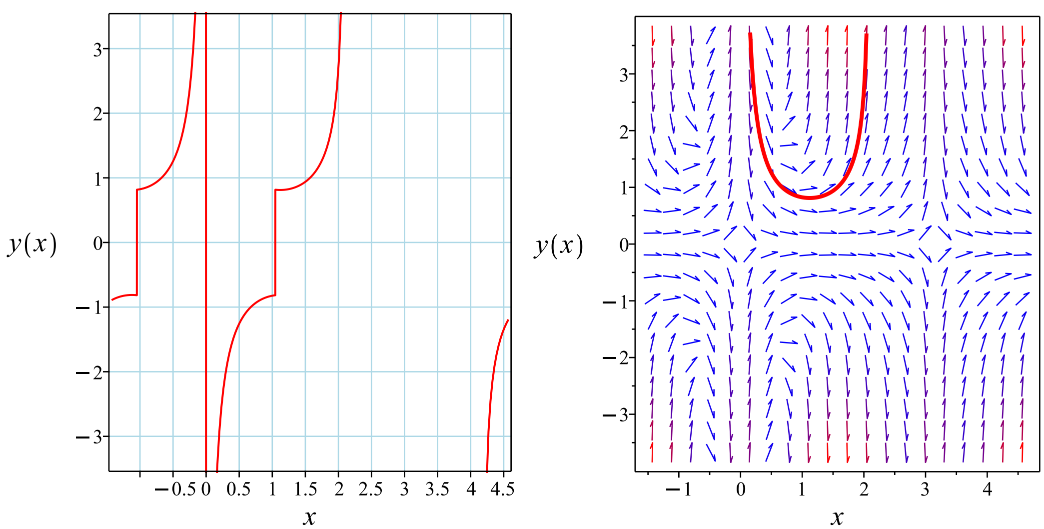
Solving for y from the above gives

$$y = \frac{\csc(x)\sqrt{(2\cos(x) - 1)^2 (2\cos(x) + 1)}}{-4\cos(x)^2 + 1}$$

<u>Summary</u>

The solution(s) found are the following

$$y = \frac{\csc(x)\sqrt{(2\cos(x) - 1)^2 (2\cos(x) + 1)}}{-4\cos(x)^2 + 1} \tag{1}$$



(a) Solution plot

(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{\csc(x)\sqrt{(2\cos(x) - 1)^2 (2\cos(x) + 1)}}{-4\cos(x)^2 + 1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 1.375 (sec). Leaf size: 34

```
dsolve([diff(y(x),x)+y(x)*cot(x)=y(x)^3*sin(x)^3,y(1/2*Pi) = 1],y(x), singsol=all)
```

$$y(x) = \frac{\csc(x)\sqrt{(2\cos(x)-1)^2(1+2\cos(x))}}{1-4\cos(x)^2}$$

✓ Solution by Mathematica

Time used: 0.933 (sec). Leaf size: 20

```
DSolve[{y'[x]+y[x]*Cot[x]==y[x]^3*Sin[x]^3,{y[Pi/2]==1}},y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to \frac{1}{\sqrt{\sin^2(x)(2\cos(x)+1)}}$$

4.37 problem Problem 54

Internal problem ID [2701]
Internal file name [`OUTPUT/2193_Sunday_June_05_2022_02_53_26_AM_44867446/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.8, Change of Variables. page 79
Problem number: Problem 54.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "homogeneousTypeC", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _Riccati]
```

$$\boxed{y' - (9x - y)^2 = 0}$$

With initial conditions

$$[y(0) = 0]$$

4.37.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= (-9x + y)^2 \end{aligned}$$

The x domain of $f(x,y)$ when $y = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x, y)$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left((-9x + y)^2\right) \\ &= -18x + 2y\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Therefore solution exists and is unique.

4.37.2 Solving as homogeneousTypeC ode

Let

$$z = 9x - y \tag{1}$$

Then

$$z'(x) = 9 - y'$$

Therefore

$$y' = -z'(x) + 9$$

Hence the given ode can now be written as

$$-z'(x) + 9 = z^2$$

This is separable first order ode. Integrating

$$\begin{aligned}\int dx &= \int \frac{1}{-z^2 + 9} dz \\ x + c_1 &= -\frac{\ln(z - 3)}{6} + \frac{\ln(z + 3)}{6}\end{aligned}$$

Replacing z back by its value from (1) then the above gives the solution as

$$y = \frac{(9x-3)\,e^{6x+6c_1} - 9x - 3}{e^{6x+6c_1} - 1}$$

$$y = \frac{(9x-3)\,e^{6x+6c_1} - 9x - 3}{e^{6x+6c_1} - 1}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{-3\,e^{6c_1} - 3}{e^{6c_1} - 1}$$

$$c_1 = \frac{i\pi}{6}$$

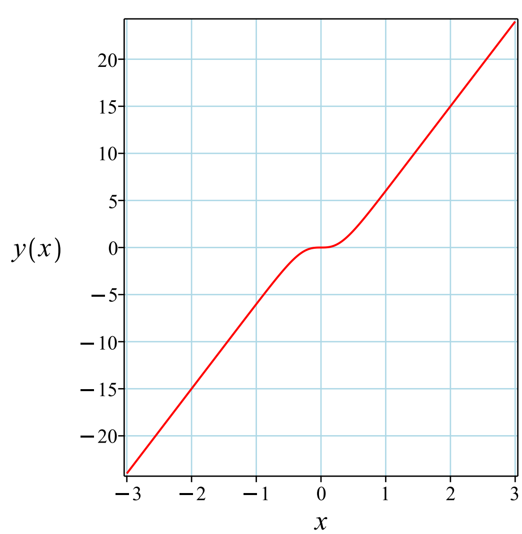
Substituting c_1 found above in the general solution gives

$$y = \frac{9x\,e^{6x} - 3\,e^{6x} + 9x + 3}{e^{6x} + 1}$$

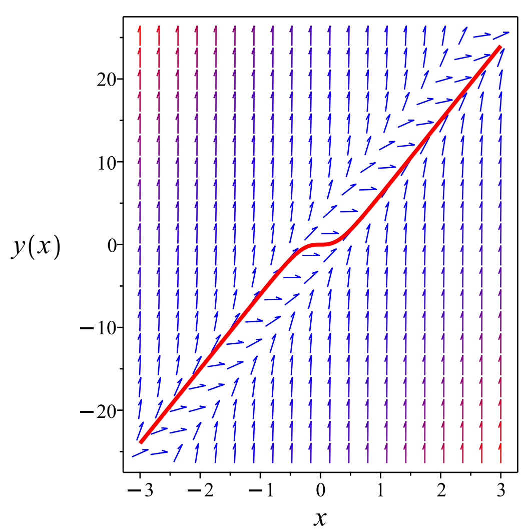
Summary

The solution(s) found are the following

$$y = \frac{9x\,e^{6x} - 3\,e^{6x} + 9x + 3}{e^{6x} + 1} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{9x\,\mathrm{e}^{6x} - 3\,\mathrm{e}^{6x} + 9x + 3}{\mathrm{e}^{6x} + 1}$$

Verified OK.

4.37.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= (-9x + y)^2 \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `homogeneous Type C`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 155: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 1 \\
\eta(x, y) &= 9
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Therefore

$$\begin{aligned}\frac{dy}{dx} &= \frac{\eta}{\xi}\\ &= \frac{9}{1}\\ &= 9\end{aligned}$$

This is easily solved to give

$$y = 9x + c_1$$

Where now the coordinate R is taken as the constant of integration. Hence

$$R = -9x + y$$

And S is found from

$$\begin{aligned}dS &= \frac{dx}{\xi}\\ &= \frac{dx}{1}\end{aligned}$$

Integrating gives

$$\begin{aligned}S &= \int \frac{dx}{T}\\ &= x\end{aligned}$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = (-9x+y)^2$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x &= -9\\ R_y &= 1\\ S_x &= 1\\ S_y &= 0\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{-9 + (-9x + y)^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^2 - 9}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{\ln (R - 3)}{6} - \frac{\ln (R + 3)}{6} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$x = \frac{\ln (-9x + y - 3)}{6} - \frac{\ln (-9x + y + 3)}{6} + c_1$$

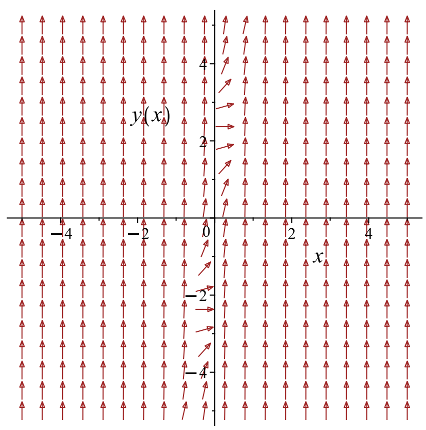
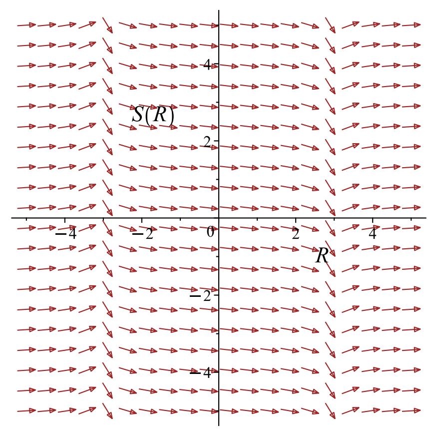
Which simplifies to

$$x = \frac{\ln (-9x + y - 3)}{6} - \frac{\ln (-9x + y + 3)}{6} + c_1$$

Which gives

$$y = \frac{9x\, \mathrm{e}^{-6x+6c_1} + 3\, \mathrm{e}^{-6x+6c_1} - 9x + 3}{\mathrm{e}^{-6x+6c_1} - 1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = (-9x+y)^2$ 	$R = -9x + y$ $S = x$	$\frac{dS}{dR} = \frac{1}{R^2-9}$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{3\,\mathrm{e}^{6c_1} + 3}{\mathrm{e}^{6c_1} - 1}$$

$$c_1 = \frac{i\pi}{6}$$

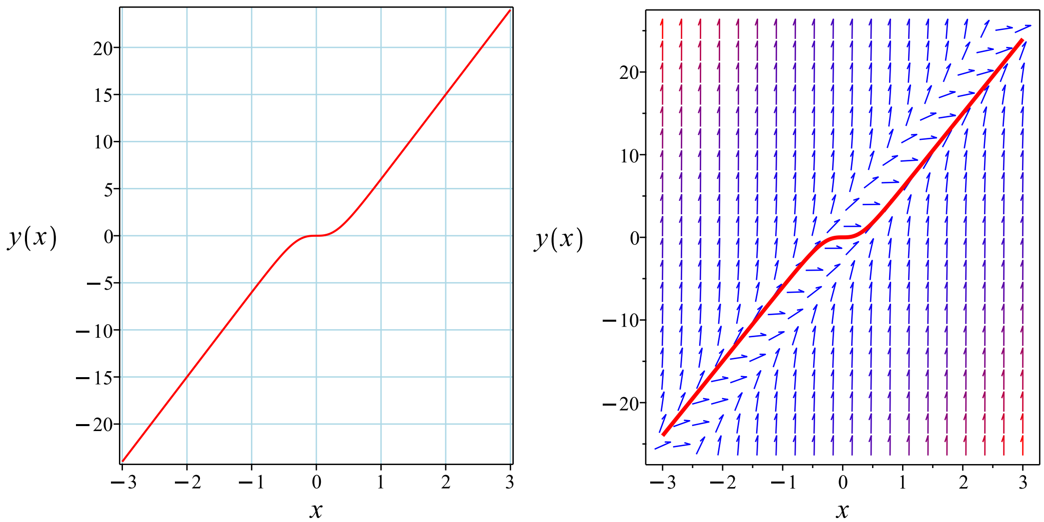
Substituting c_1 found above in the general solution gives

$$y = \frac{9x\,\mathrm{e}^{-6x} + 3\,\mathrm{e}^{-6x} + 9x - 3}{\mathrm{e}^{-6x} + 1}$$

Summary

The solution(s) found are the following

$$y = \frac{9x\,\mathrm{e}^{-6x} + 3\,\mathrm{e}^{-6x} + 9x - 3}{\mathrm{e}^{-6x} + 1} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{9x\,\mathrm{e}^{-6x} + 3\,\mathrm{e}^{-6x} + 9x - 3}{\mathrm{e}^{-6x} + 1}$$

Verified OK.

4.37.4 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= (-9x + y)^2 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = 81x^2 - 18xy + y^2$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 81x^2$, $f_1(x) = -18x$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1f_2 &= -18x \\ f_2^2f_0 &= 81x^2 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) + 18xu'(x) + 81x^2u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1\mathrm{e}^{-\frac{9}{2}x^2+3x} + c_2\mathrm{e}^{-3x-\frac{9}{2}x^2}$$

The above shows that

$$u'(x) = c_2(-3-9x)\,\mathrm{e}^{-3x-\frac{9}{2}x^2} - 9c_1\mathrm{e}^{-\frac{9}{2}x^2+3x}\left(x-\frac{1}{3}\right)$$

Using the above in (1) gives the solution

$$y = -\frac{c_2(-3-9x)\,\mathrm{e}^{-3x-\frac{9}{2}x^2} - 9c_1\mathrm{e}^{-\frac{9}{2}x^2+3x}\left(x-\frac{1}{3}\right)}{c_1\mathrm{e}^{-\frac{9}{2}x^2+3x} + c_2\mathrm{e}^{-3x-\frac{9}{2}x^2}}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{(3+9x)\,\mathrm{e}^{-3x-\frac{9}{2}x^2} + 9c_3\mathrm{e}^{-\frac{9}{2}x^2+3x}\left(x-\frac{1}{3}\right)}{c_3\mathrm{e}^{-\frac{9}{2}x^2+3x} + \mathrm{e}^{-3x-\frac{9}{2}x^2}}$$

Initial conditions are used to solve for c_3. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{-3c_3+3}{c_3+1}$$

$$c_3 = 1$$

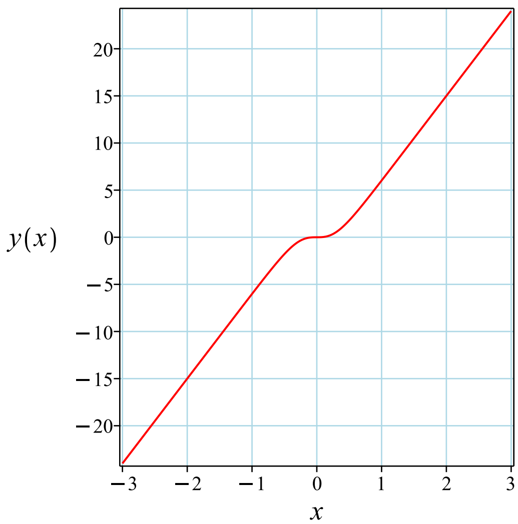
Substituting c_3 found above in the general solution gives

$$y = \frac{9\,\mathrm{e}^{-\frac{3x(3x-2)}{2}}x + 9\,\mathrm{e}^{-\frac{3x(2+3x)}{2}}x - 3\,\mathrm{e}^{-\frac{3x(3x-2)}{2}} + 3\,\mathrm{e}^{-\frac{3x(2+3x)}{2}}}{\mathrm{e}^{-\frac{3x(3x-2)}{2}} + \mathrm{e}^{-\frac{3x(2+3x)}{2}}}$$

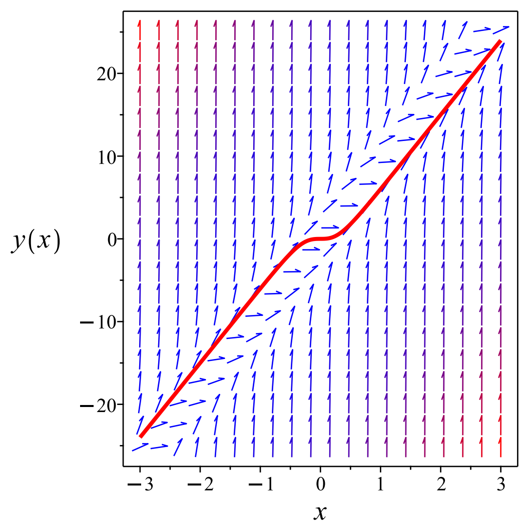
Summary

The solution(s) found are the following

$$y = \frac{9\,\mathrm{e}^{-\frac{3x(3x-2)}{2}}x + 9\,\mathrm{e}^{-\frac{3x(2+3x)}{2}}x - 3\,\mathrm{e}^{-\frac{3x(3x-2)}{2}} + 3\,\mathrm{e}^{-\frac{3x(2+3x)}{2}}}{\mathrm{e}^{-\frac{3x(3x-2)}{2}} + \mathrm{e}^{-\frac{3x(2+3x)}{2}}} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{9\,\mathrm{e}^{-\frac{3x(3x-2)}{2}}x + 9\,\mathrm{e}^{-\frac{3x(2+3x)}{2}}x - 3\,\mathrm{e}^{-\frac{3x(3x-2)}{2}} + 3\,\mathrm{e}^{-\frac{3x(2+3x)}{2}}}{\mathrm{e}^{-\frac{3x(3x-2)}{2}} + \mathrm{e}^{-\frac{3x(2+3x)}{2}}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
   -> Calling odsolve with the ODE`, diff(y(x), x) = 9, y(x)`      *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.11 (sec). Leaf size: 28

```
dsolve([diff(y(x),x)=(9*x-y(x))^2,y(0) = 0],y(x), singsol=all)
```

$$y(x) = \frac{(9x-3)\,\mathrm{e}^{6x}+9x+3}{1+\mathrm{e}^{6x}}$$

✓ Solution by Mathematica

Time used: 0.15 (sec). Leaf size: 31

```
DSolve[{y'[x]==(9*x-y[x])^2,{y[0]==0}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{9x+e^{6x}(9x-3)+3}{e^{6x}+1}$$

4.38 problem Problem 55

Internal problem ID [2702]
Internal file name [`OUTPUT/2194_Sunday_June_05_2022_02_53_30_AM_80305114/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.8, Change of Variables. page 79
Problem number: Problem 55.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "homogeneousTypeC", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _Riccati]
```

$$\boxed{y' - (4x + y + 2)^2 = 0}$$

4.38.1 Solving as homogeneousTypeC ode

Let

$$z = 4x + y + 2 \tag{1}$$

Then

$$z'(x) = 4 + y'$$

Therefore

$$y' = z'(x) - 4$$

Hence the given ode can now be written as

$$z'(x) - 4 = z^2$$

This is separable first order ode. Integrating

$$\int dx = \int \frac{1}{z^2 + 4} dz$$

$$x + c_1 = \frac{\arctan\left(\frac{z}{2}\right)}{2}$$

Replacing z back by its value from (1) then the above gives the solution as

$$y = -4x - 2 + 2\tan(2x + 2c_1)$$

$$y = -4x - 2 + 2\tan(2x + 2c_1)$$

Summary

The solution(s) found are the following

$$y = -4x - 2 + 2\tan(2x + 2c_1) \tag{1}$$

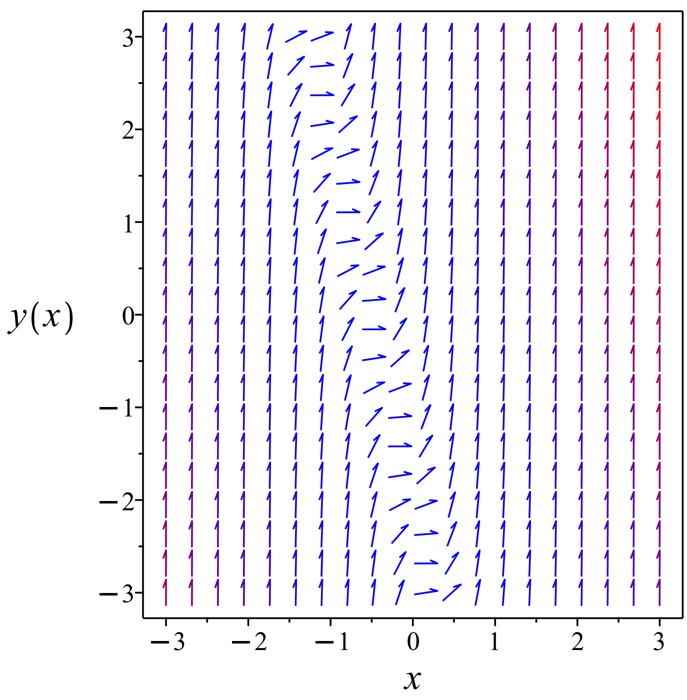


Figure 177: Slope field plot

Verification of solutions

$$y = -4x - 2 + 2\tan(2x + 2c_1)$$

Verified OK.

4.38.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= (4x+y+2)^2 \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `homogeneous Type C`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 157: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a+bx+cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 1 \\
\eta(x,y) &= -4
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Therefore

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{\eta}{\xi} \\
&= \frac{-4}{1} \\
&= -4
\end{aligned}
$$

This is easily solved to give

$$
y = -4x + c_1
$$

Where now the coordinate R is taken as the constant of integration. Hence

$$
R = 4x + y
$$

And S is found from

$$
\begin{aligned}
dS &= \frac{dx}{\xi} \\
&= \frac{dx}{1}
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
S &= \int \frac{dx}{T} \\
&= x
\end{aligned}
$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = (4x+y+2)^2$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 4 \\ R_y &= 1 \\ S_x &= 1 \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{4+(4x+y+2)^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{4+(R+2)^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{\arctan\left(\frac{R}{2}+1\right)}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$x = \frac{\arctan\left(2x+\frac{y}{2}+1\right)}{2} + c_1$$

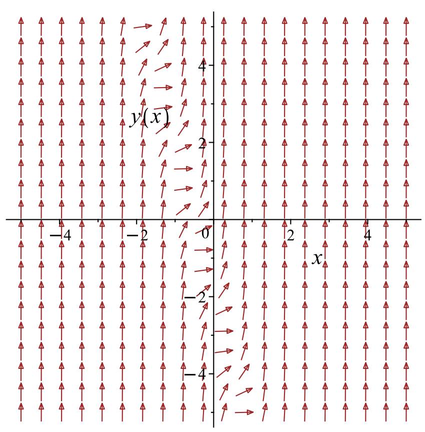
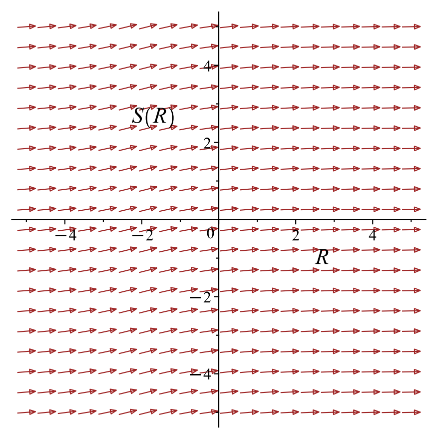
Which simplifies to

$$x = \frac{\arctan\left(2x + \frac{y}{2} + 1\right)}{2} + c_1$$

Which gives

$$y = -4x - 2 - 2\tan\left(-2x + 2c_1\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = (4x + y + 2)^2$ 	$R = 4x + y$ $S = x$	$\frac{dS}{dR} = \frac{1}{4+(R+2)^2}$ 

Summary

The solution(s) found are the following

$$y = -4x - 2 - 2\tan\left(-2x + 2c_1\right) \tag{1}$$

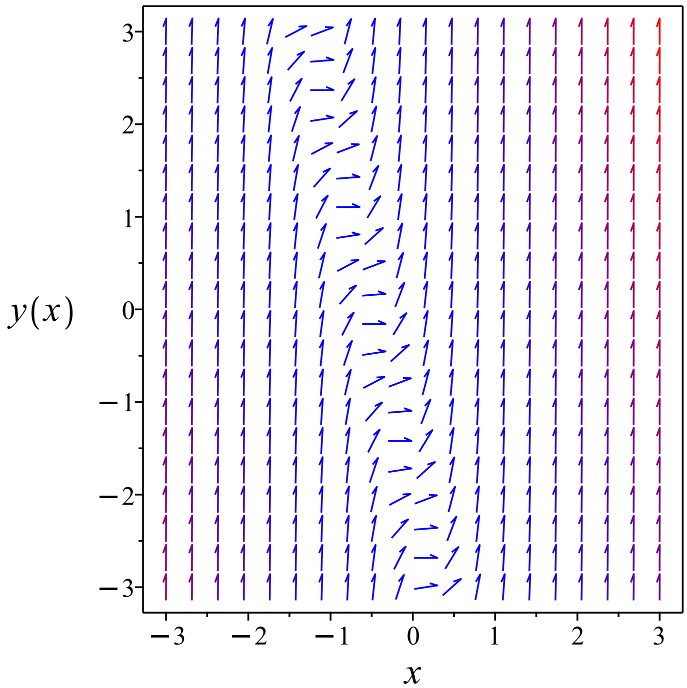


Figure 178: Slope field plot

Verification of solutions

$$y = -4x - 2 - 2\tan(-2x + 2c_1)$$

Verified OK.

4.38.3 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= (4x + y + 2)^2 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = 16x^2 + 8xy + y^2 + 16x + 4y + 4$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = (4x+2)^2$, $f_1(x) = 8x+4$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= 8x+4 \\ f_2^2 f_0 &= (4x+2)^2 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) - (8x+4) u'(x) + (4x+2)^2 u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = e^{2x(x+1)}(c_1 \cos(2x) + c_2 \sin(2x))$$

The above shows that

$$u'(x) = 4\,e^{2x(x+1)}\left(\left(\left(x+\frac{1}{2}\right)c_1 + \frac{c_2}{2}\right)\cos(2x) + \sin(2x)\left(-\frac{c_1}{2} + \left(x+\frac{1}{2}\right)c_2\right)\right)$$

Using the above in (1) gives the solution

$$y = -\frac{4\left(\left(\left(x+\frac{1}{2}\right)c_1 + \frac{c_2}{2}\right)\cos(2x) + \sin(2x)\left(-\frac{c_1}{2} + \left(x+\frac{1}{2}\right)c_2\right)\right)}{c_1 \cos(2x) + c_2 \sin(2x)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{(-2 + (-4x-2)c_3)\cos(2x) - 4\sin(2x)\left(-\frac{c_3}{2} + x + \frac{1}{2}\right)}{c_3 \cos(2x) + \sin(2x)}$$

Summary

The solution(s) found are the following

$$y = \frac{(-2 + (-4x - 2) c_3) \cos(2x) - 4 \sin(2x) \left(-\frac{c_3}{2} + x + \frac{1}{2}\right)}{c_3 \cos(2x) + \sin(2x)} \tag{1}$$

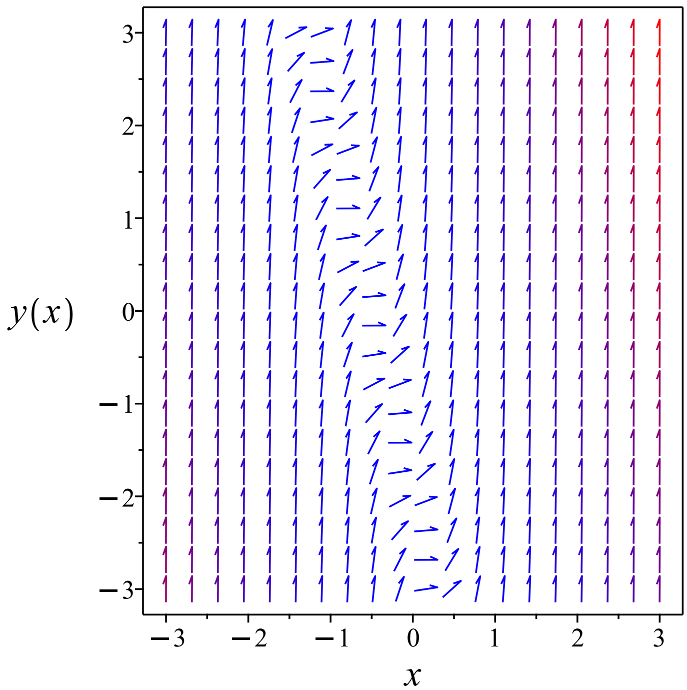


Figure 179: Slope field plot

Verification of solutions

$$y = \frac{(-2 + (-4x - 2) c_3) \cos(2x) - 4 \sin(2x) \left(-\frac{c_3}{2} + x + \frac{1}{2}\right)}{c_3 \cos(2x) + \sin(2x)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(diff(y(x),x)=(4*x+y(x)+2)^2,y(x), singsol=all)
```

$$y(x) = -4x - 2 - 2\tan(-2x + 2c_1)$$

✓ Solution by Mathematica

Time used: 0.16 (sec). Leaf size: 41

```
DSolve[y'[x]==(4*x+y[x]+2)^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -4x + \frac{1}{c_1 e^{4ix} - \frac{i}{4}} - (2+2i)$$
$$y(x) \to -4x - (2+2i)$$

4.39 problem Problem 56

Internal problem ID [2703]
Internal file name [`OUTPUT/2195_Sunday_June_05_2022_02_53_33_AM_80948642/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.8, Change of Variables. page 79
Problem number: Problem 56.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _dAlembert]
```

$$\boxed{y' - \sin(3x - 3y + 1)^2 = 0}$$

4.39.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= \sin(3x - 3y + 1)^2 \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}
&b_2 + \sin(3x-3y+1)^2(b_3-a_2) - \sin(3x-3y+1)^4 a_3 \\
&- 6\sin(3x-3y+1)\cos(3x-3y+1)(xa_2+ya_3+a_1) \\
&+ 6\sin(3x-3y+1)\cos(3x-3y+1)(xb_2+yb_3+b_1) = 0
\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\begin{aligned}
&-\sin(3x-3y+1)^4 a_3 - 6\sin(3x-3y+1)\cos(3x-3y+1)\,xa_2 \\
&+6\sin(3x-3y+1)\cos(3x-3y+1)\,xb_2 - 6\sin(3x-3y+1)\cos(3x-3y+1)\,ya_3 \\
&+6\sin(3x-3y+1)\cos(3x-3y+1)\,yb_3 - \sin(3x-3y+1)^2 a_2 \\
&+\sin(3x-3y+1)^2 b_3 - 6\sin(3x-3y+1)\cos(3x-3y+1)\,a_1 \\
&+6\sin(3x-3y+1)\cos(3x-3y+1)\,b_1 + b_2 = 0
\end{aligned}$$

Setting the numerator to zero gives

$$\begin{aligned}
&-\sin(3x-3y+1)^4 a_3 - 6\sin(3x-3y+1)\cos(3x-3y+1)\,xa_2 \\
&+6\sin(3x-3y+1)\cos(3x-3y+1)\,xb_2 \\
&-6\sin(3x-3y+1)\cos(3x-3y+1)\,ya_3 \\
&+6\sin(3x-3y+1)\cos(3x-3y+1)\,yb_3 - \sin(3x-3y+1)^2 a_2 \\
&+\sin(3x-3y+1)^2 b_3 - 6\sin(3x-3y+1)\cos(3x-3y+1)\,a_1 \\
&+6\sin(3x-3y+1)\cos(3x-3y+1)\,b_1 + b_2 = 0
\end{aligned} \tag{6E}$$

Simplifying the above gives

$$\begin{aligned}
&b_2 - \frac{3a_3}{8} - \frac{a_2}{2} + \frac{b_3}{2} - \frac{a_3\cos(12x-12y+4)}{8} + \frac{a_3\cos(6x-6y+2)}{2} \\
&- 3xa_2\sin(6x-6y+2) + 3xb_2\sin(6x-6y+2) - 3ya_3\sin(6x-6y+2) \\
&+ 3yb_3\sin(6x-6y+2) + \frac{a_2\cos(6x-6y+2)}{2} - \frac{b_3\cos(6x-6y+2)}{2} \\
&- 3a_1\sin(6x-6y+2) + 3b_1\sin(6x-6y+2) = 0
\end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y, \cos(6x-6y+2), \cos(12x-12y+4), \sin(6x-6y+2)\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2, \cos(6x - 6y + 2) = v_3, \cos(12x - 12y + 4) = v_4, \sin(6x - 6y + 2) = v_5\}$$

The above PDE (6E) now becomes

$$\begin{aligned} b_2 - \frac{3}{8}a_3 - \frac{1}{2}a_2 + \frac{1}{2}b_3 - \frac{1}{8}a_3v_4 + \frac{1}{2}a_3v_3 - 3v_1a_2v_5 + 3v_1b_2v_5 \\ - 3v_2a_3v_5 + 3v_2b_3v_5 + \frac{1}{2}a_2v_3 - \frac{1}{2}b_3v_3 - 3a_1v_5 + 3b_1v_5 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4, v_5\}$$

Equation (7E) now becomes

$$\begin{aligned} b_2 - \frac{3a_3}{8} - \frac{a_2}{2} + \frac{b_3}{2} + (-3a_2 + 3b_2)\,v_5v_1 + (-3a_3 + 3b_3)\,v_5v_2 \\ + \left(\frac{a_3}{2} + \frac{a_2}{2} - \frac{b_3}{2}\right)v_3 - \frac{a_3v_4}{8} + (-3a_1 + 3b_1)\,v_5 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -\frac{a_3}{8} &= 0 \\ -3a_1 + 3b_1 &= 0 \\ -3a_2 + 3b_2 &= 0 \\ -3a_3 + 3b_3 &= 0 \\ \frac{a_3}{2} + \frac{a_2}{2} - \frac{b_3}{2} &= 0 \\ b_2 - \frac{3a_3}{8} - \frac{a_2}{2} + \frac{b_3}{2} &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= b_1 \\ a_2 &= 0 \\ a_3 &= 0 \\ b_1 &= b_1 \\ b_2 &= 0 \\ b_3 &= 0 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = 1$$
$$\eta = 1$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Therefore

$$\begin{aligned}\frac{dy}{dx} &= \frac{\eta}{\xi} \\ &= \frac{1}{1} \\ &= 1\end{aligned}$$

This is easily solved to give

$$y = x + c_1$$

Where now the coordinate R is taken as the constant of integration. Hence

$$R = y - x$$

And S is found from

$$\begin{aligned}dS &= \frac{dx}{\xi} \\ &= \frac{dx}{1}\end{aligned}$$

Integrating gives

$$\begin{aligned}S &= \int \frac{dx}{T} \\ &= x\end{aligned}$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \sin(3x - 3y + 1)^2$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= -1 \\ R_y &= 1 \\ S_x &= 1 \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\sec(3x - 3y + 1)^2 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\sec(3R - 1)^2$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{\tan(3R - 1)}{3} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$x = \frac{\tan(3x - 3y + 1)}{3} + c_1$$

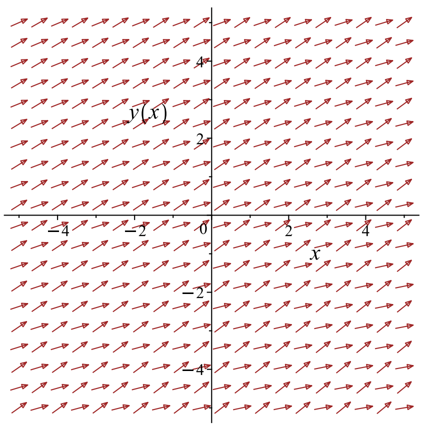
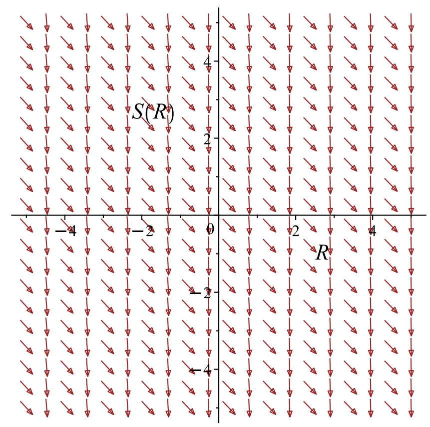
Which simplifies to

$$x = \frac{\tan(3x - 3y + 1)}{3} + c_1$$

Which gives

$$y = x + \frac{1}{3} + \frac{\arctan(-3x + 3c_1)}{3}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \sin(3x - 3y + 1)^2$ 	$R = y - x$ $S = x$	$\frac{dS}{dR} = -\sec(3R - 1)^2$ 

Summary
The solution(s) found are the following

$$y = x + \frac{1}{3} + \frac{\arctan(-3x + 3c_1)}{3} \tag{1}$$

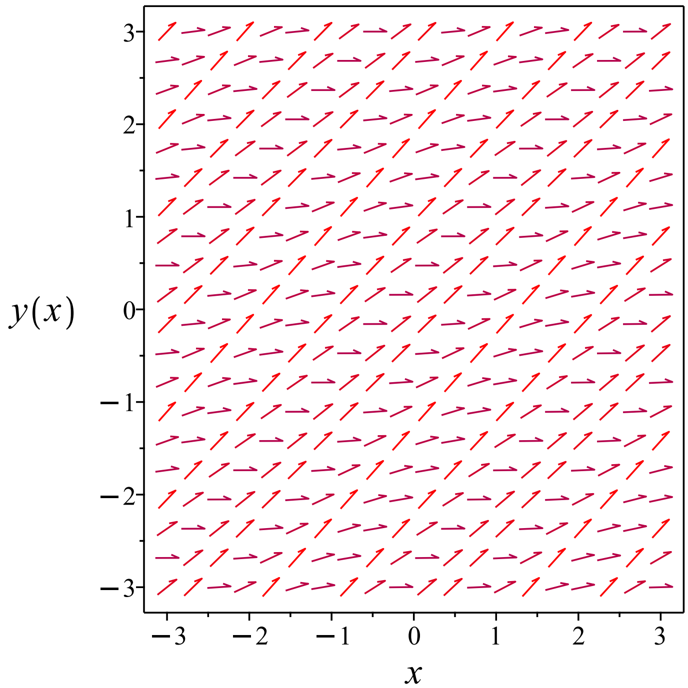


Figure 180: Slope field plot

Verification of solutions

$$y = x + \frac{1}{3} + \frac{\arctan(-3x + 3c_1)}{3}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 17

```
dsolve(diff(y(x),x)=(sin(3*x-3*y(x)+1))^2,y(x), singsol=all)
```

$$y(x) = x + \frac{1}{3} + \frac{\arctan(-3x + 3c_1)}{3}$$

✓ Solution by Mathematica

Time used: 0.599 (sec). Leaf size: 43

```
DSolve[y'[x]==(Sin[3*x-3*y[x]+1])^2,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[2y(x) - 2\left(\frac{1}{3}\tan(-3y(x) + 3x + 1) - \frac{1}{3}\arctan(\tan(-3y(x) + 3x + 1))\right) = c_1, y(x)\right]$$

4.40 problem Problem 58

Internal problem ID [2704]
Internal file name [`OUTPUT/2196_Sunday_June_05_2022_02_54_04_AM_48229966/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.8, Change of Variables. page 79
Problem number: Problem 58.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`]]
```

$$\boxed{y' - \frac{y(\ln(yx) - 1)}{x} = 0}$$

4.40.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= \frac{y(\ln(xy) - 1)}{x} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}
&b_2 + \frac{y(\ln(xy) - 1)(b_3 - a_2)}{x} - \frac{y^2(\ln(xy) - 1)^2 a_3}{x^2} \\
&- \left(-\frac{y(\ln(xy) - 1)}{x^2} + \frac{y}{x^2}\right)(xa_2 + ya_3 + a_1) \\
&- \left(\frac{\ln(xy) - 1}{x} + \frac{1}{x}\right)(xb_2 + yb_3 + b_1) = 0
\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{\ln(xy)^2 y^2 a_3 + \ln(xy) x^2 b_2 - 3\ln(xy) y^2 a_3 + \ln(xy) xb_1 - \ln(xy) ya_1 - b_2 x^2 + xya_2 + xyb_3 + 3y^2 a_3 +}{x^2}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned}
&-\ln(xy)^2 y^2 a_3 - \ln(xy) x^2 b_2 + 3\ln(xy) y^2 a_3 - \ln(xy) xb_1 \\
&+ \ln(xy) ya_1 + b_2 x^2 - xya_2 - xyb_3 - 3y^2 a_3 - 2ya_1 = 0
\end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y, \ln(xy)\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2, \ln(xy) = v_3\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&-v_3^2 v_2^2 a_3 + 3v_3 v_2^2 a_3 - v_3 v_1^2 b_2 + v_3 v_2 a_1 - v_1 v_2 a_2 \\
&- 3v_2^2 a_3 - v_3 v_1 b_1 + b_2 v_1^2 - v_1 v_2 b_3 - 2v_2 a_1 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$\begin{aligned}&-v_3 v_1^2 b_2 + b_2 v_1^2 + (-a_2 - b_3) v_1 v_2 - v_3 v_1 b_1 - v_3^2 v_2^2 a_3 \\ &+ 3 v_3 v_2^2 a_3 - 3 v_2^2 a_3 + v_3 v_2 a_1 - 2 v_2 a_1 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}a_1 &= 0\\ b_2 &= 0\\ -2a_1 &= 0\\ -3a_3 &= 0\\ -a_3 &= 0\\ 3a_3 &= 0\\ -b_1 &= 0\\ -b_2 &= 0\\ -a_2 - b_3 &= 0\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1 &= 0\\ a_2 &= -b_3\\ a_3 &= 0\\ b_1 &= 0\\ b_2 &= 0\\ b_3 &= b_3\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}\xi &= -x\\ \eta &= y\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(\frac{y(\ln(xy) - 1)}{x}\right)(-x) \\
&= y\ln(xy) \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{y\ln(xy)} dy
\end{aligned}$$

Which results in

$$S = \ln(\ln(xy))$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y(\ln(xy) - 1)}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{1}{x\left(\ln\left(x\right)+\ln\left(y\right)\right)} \\
S_y &= \frac{1}{y\left(\ln\left(x\right)+\ln\left(y\right)\right)}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{\ln\left(xy\right)}{x\left(\ln\left(x\right)+\ln\left(y\right)\right)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \ln\left(R\right) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln\left(\ln\left(x\right)+\ln\left(y\right)\right) = \ln\left(x\right) + c_1$$

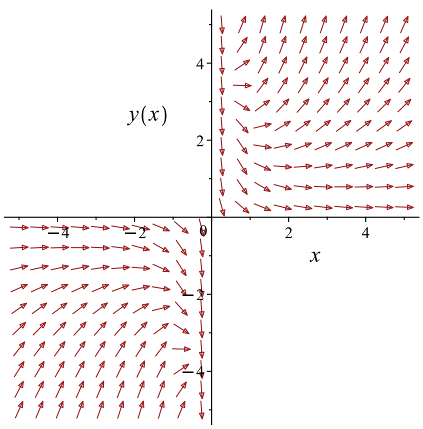
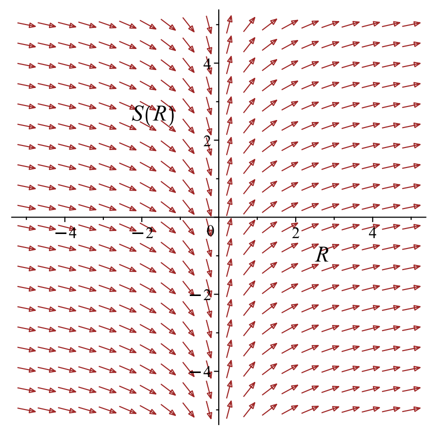
Which simplifies to

$$\ln\left(\ln\left(x\right)+\ln\left(y\right)\right) = \ln\left(x\right) + c_1$$

Which gives

$$y = \frac{e^{e^{c_1} x}}{x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y(\ln(xy)-1)}{x}$ 	$R = x$ $S = \ln(\ln(x) + \ln(y))$	$\frac{dS}{dR} = \frac{1}{R}$ 

Summary

The solution(s) found are the following

$$y = \frac{e^{e^{c_1} x}}{x} \tag{1}$$

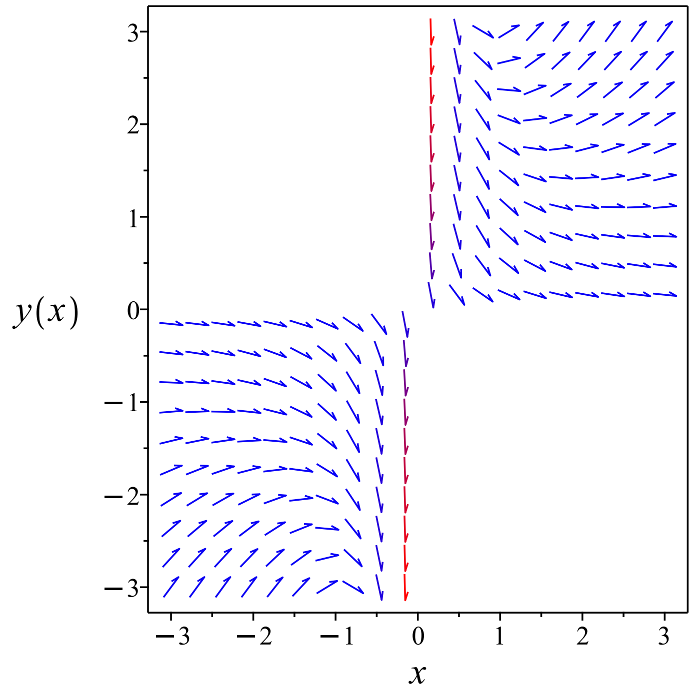


Figure 181: Slope field plot

Verification of solutions

$$y = \frac{e^{e^{c_1} x}}{x}$$

Verified OK.

4.40.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\mathrm{d}y = \left(\frac{y(\ln(xy) - 1)}{x}\right) \mathrm{d}x
$$

$$
\left(-\frac{y(\ln(xy) - 1)}{x}\right) \mathrm{d}x + \mathrm{d}y = 0 \tag{2A}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -\frac{y(\ln(xy) - 1)}{x} \\
N(x, y) &= 1
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{y(\ln(xy) - 1)}{x}\right) \\
&= -\frac{\ln(xy)}{x}
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1\left(\left(-\frac{\ln(xy) - 1}{x} - \frac{1}{x}\right) - (0)\right) \\
&= -\frac{\ln(xy)}{x}
\end{aligned}
$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= -\frac{x}{y(\ln(xy) - 1)}\left((0) - \left(-\frac{\ln(xy) - 1}{x} - \frac{1}{x}\right)\right) \\
&= -\frac{\ln(xy)}{y(\ln(xy) - 1)}
\end{aligned}
$$

Since B depends on x, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$
R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}
$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$
\begin{aligned}
R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} \\
&= \frac{(0) - \left(-\frac{\ln(xy)-1}{x} - \frac{1}{x}\right)}{x\left(-\frac{y(\ln(xy)-1)}{x}\right) - y(1)} \\
&= -\frac{1}{xy}
\end{aligned}
$$

Replacing all powers of terms xy by t gives

$$
R = -\frac{1}{t}
$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$\begin{aligned}\mu &= e^{\int R\,\mathrm{d}t} \\ &= e^{\int \left(-\frac{1}{t}\right)\mathrm{d}t}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\ln(t)} \\ &= \frac{1}{t}\end{aligned}$$

Now t is replaced back with xy giving

$$\mu = \frac{1}{xy}$$

Multiplying M and N by this integrating factor gives new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N

$$\begin{aligned}\overline{M} &= \mu M \\ &= \frac{1}{xy}\left(-\frac{y(\ln(xy)-1)}{x}\right) \\ &= \frac{-\ln(xy)+1}{x^2}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \frac{1}{xy}(1) \\ &= \frac{1}{xy}\end{aligned}$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\frac{-\ln(xy)+1}{x^2}\right) + \left(\frac{1}{xy}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int \frac{-\ln(xy)+1}{x^2}\,\mathrm{d}x$$
$$\phi = \frac{\ln(xy)}{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial\phi}{\partial y} = \frac{1}{xy} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = \frac{1}{xy}$. Therefore equation (4) becomes

$$\frac{1}{xy} = \frac{1}{xy} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{\ln(xy)}{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{\ln(xy)}{x}$$

The solution becomes

$$y = \frac{e^{c_1 x}}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{e^{c_1 x}}{x} \tag{1}$$

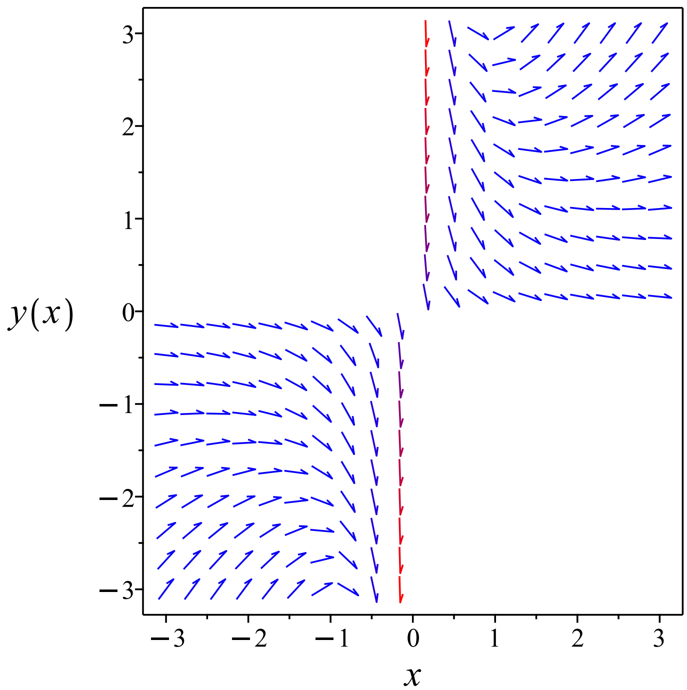


Figure 182: Slope field plot

Verification of solutions

$$y = \frac{e^{c_1 x}}{x}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 14

```
dsolve(diff(y(x),x)=y(x)/x*(ln(x*y(x))-1),y(x), singsol=all)
```

$$y(x) = \frac{\mathrm{e}^{\frac{x}{c_1}}}{x}$$

✓ Solution by Mathematica

Time used: 0.233 (sec). Leaf size: 24

```
DSolve[y'[x]==y[x]/x*(Log[x*y[x]]-1),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{e^{c_1}x}}{x}$$
$$y(x) \to \frac{1}{x}$$

4.41 problem Problem 59

Internal problem ID [2705]
Internal file name [`OUTPUT/2197_Sunday_June_05_2022_02_54_09_AM_11715771/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.8, Change of Variables. page 79
Problem number: Problem 59.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries], _Riccati]
```

$$\boxed{y' - 2x(y+x)^2 = -1}$$

With initial conditions

$$[y(0) = 1]$$

4.41.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= 2x^3 + 4x^2y + 2x\,y^2 - 1 \end{aligned}$$

The x domain of $f(x,y)$ when $y = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x, y)$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(2x^3 + 4x^2 y + 2x\, y^2 - 1\right) \\
&= 4x^2 + 4xy
\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

4.41.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned}
y' &= 2x^3 + 4x^2 y + 2x\, y^2 - 1 \\
y' &= \omega(x, y)
\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + (2x^3 + 4x^2y + 2x\,y^2 - 1)(b_3 - a_2) - (2x^3 + 4x^2y + 2x\,y^2 - 1)^2 a_3 \\ - (6x^2 + 8xy + 2y^2)(xa_2 + ya_3 + a_1) - (4x^2 + 4xy)(xb_2 + yb_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-4x^6a_3 - 16x^5ya_3 - 24x^4y^2a_3 - 16x^3y^3a_3 - 4x^2y^4a_3 - 8x^3a_2 + 4x^3a_3 \\ - 4x^3b_2 + 2x^3b_3 - 12x^2ya_2 + 2x^2ya_3 - 4x^2yb_2 - 4x\,y^2a_2 - 4x\,y^2a_3 - 2x\,y^2b_3 \\ - 2y^3a_3 - 6x^2a_1 - 4x^2b_1 - 8xya_1 - 4xyb_1 - 2y^2a_1 + a_2 - a_3 + b_2 - b_3 = 0$$

Setting the numerator to zero gives

$$-4x^6a_3 - 16x^5ya_3 - 24x^4y^2a_3 - 16x^3y^3a_3 - 4x^2y^4a_3 - 8x^3a_2 + 4x^3a_3 \\ - 4x^3b_2 + 2x^3b_3 - 12x^2ya_2 + 2x^2ya_3 - 4x^2yb_2 - 4x\,y^2a_2 - 4x\,y^2a_3 - 2x\,y^2b_3 \\ - 2y^3a_3 - 6x^2a_1 - 4x^2b_1 - 8xya_1 - 4xyb_1 - 2y^2a_1 + a_2 - a_3 + b_2 - b_3 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-4a_3v_1^6 - 16a_3v_1^5v_2 - 24a_3v_1^4v_2^2 - 16a_3v_1^3v_2^3 - 4a_3v_1^2v_2^4 - 8a_2v_1^3 - 12a_2v_1^2v_2 \\ - 4a_2v_1v_2^2 + 4a_3v_1^3 + 2a_3v_1^2v_2 - 4a_3v_1v_2^2 - 2a_3v_2^3 - 4b_2v_1^3 - 4b_2v_1^2v_2 + 2b_3v_1^3 \\ - 2b_3v_1v_2^2 - 6a_1v_1^2 - 8a_1v_1v_2 - 2a_1v_2^2 - 4b_1v_1^2 - 4b_1v_1v_2 + a_2 - a_3 + b_2 - b_3 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&-4a_3v_1^6 - 16a_3v_1^5v_2 - 24a_3v_1^4v_2^2 - 16a_3v_1^3v_2^3 + (-8a_2 + 4a_3 - 4b_2 + 2b_3)\,v_1^3 \\
&- 4a_3v_1^2v_2^4 + (-12a_2 + 2a_3 - 4b_2)\,v_1^2v_2 + (-6a_1 - 4b_1)\,v_1^2 \\
&+ (-4a_2 - 4a_3 - 2b_3)\,v_1v_2^2 + (-8a_1 - 4b_1)\,v_1v_2 \\
&- 2a_3v_2^3 - 2a_1v_2^2 + a_2 - a_3 + b_2 - b_3 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-2a_1 &= 0 \\
-24a_3 &= 0 \\
-16a_3 &= 0 \\
-4a_3 &= 0 \\
-2a_3 &= 0 \\
-8a_1 - 4b_1 &= 0 \\
-6a_1 - 4b_1 &= 0 \\
-12a_2 + 2a_3 - 4b_2 &= 0 \\
-4a_2 - 4a_3 - 2b_3 &= 0 \\
-8a_2 + 4a_3 - 4b_2 + 2b_3 &= 0 \\
a_2 - a_3 + b_2 - b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= a_2 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= -3a_2 \\
b_3 &= -2a_2
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= -3x - 2y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= -3x - 2y - \left(2x^{3} + 4x^{2}y + 2x\,y^{2} - 1\right)(x) \\
&= -2x^{4} - 4x^{3}y - 2y^{2}x^{2} - 2x - 2y \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{-2x^{4} - 4x^{3}y - 2y^{2}x^{2} - 2x - 2y} dy
\end{aligned}$$

Which results in

$$S = \frac{\ln\left(x^{3} + x^{2}y + 1\right)}{2} - \frac{\ln\left(y + x\right)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = 2x^{3} + 4x^{2}y + 2x\,y^{2} - 1$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{x(3x+2y)}{2x^3+2x^2y+2} - \frac{1}{2x+2y} \\
S_y &= -\frac{1}{2\left(x^3+x^2y+1\right)\left(y+x\right)}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = 0 \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = 0
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{\ln\left(x^3+x^2y+1\right)}{2} - \frac{\ln\left(y+x\right)}{2} = c_1
$$

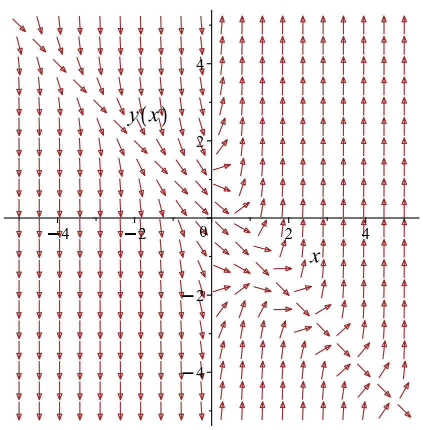
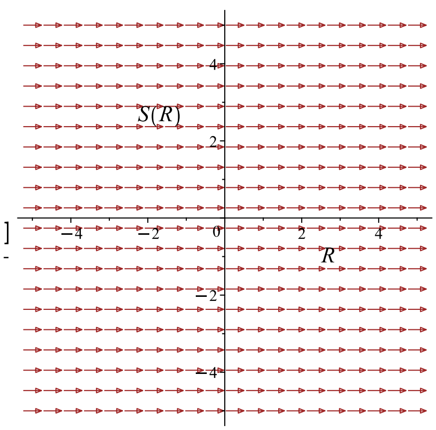
Which simplifies to

$$
\frac{\ln\left(x^3+x^2y+1\right)}{2} - \frac{\ln\left(y+x\right)}{2} = c_1
$$

Which gives

$$
y = -\frac{-x^3+x\,\mathrm{e}^{2c_1}-1}{-x^2+\mathrm{e}^{2c_1}}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = 2x^3 + 4x^2y + 2x\,y^2 - 1$ 	$R = x$ $S = \frac{\ln(x^3 + x^2y + 1)}{2} -$	$\frac{dS}{dR} = 0$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = e^{-2c_1}$$

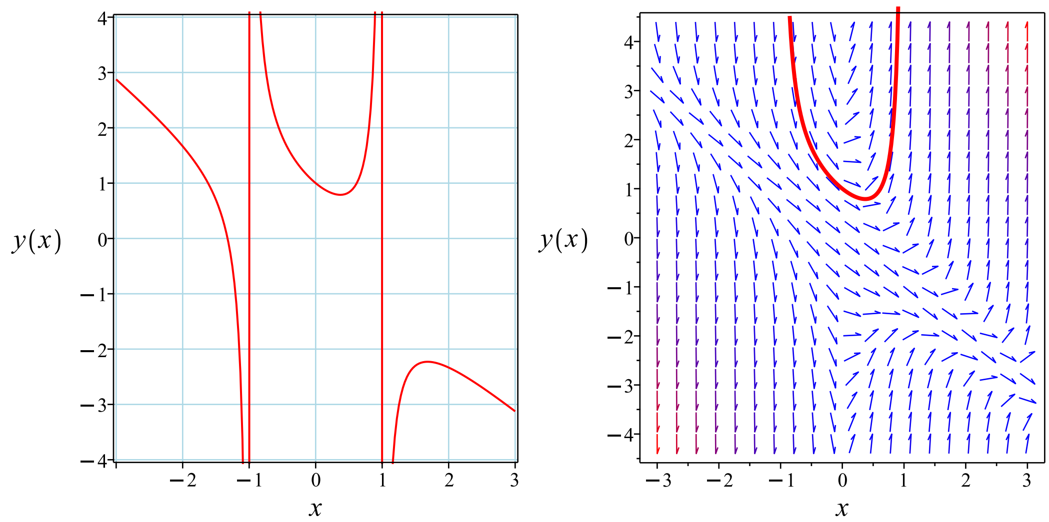
$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

$$y = \frac{-x^3 + x - 1}{x^2 - 1}$$

Summary
The solution(s) found are the following

$$y = \frac{-x^3 + x - 1}{x^2 - 1} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{-x^3 + x - 1}{x^2 - 1}$$

Verified OK.

4.41.3 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= 2x^3 + 4x^2y + 2x\,y^2 - 1 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = 2x^3 + 4x^2y + 2x\,y^2 - 1$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 2x^3 - 1$, $f_1(x) = 4x^2$ and $f_2(x) = 2x$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{2xu} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 2 \\ f_1f_2 &= 8x^3 \\ f_2^2f_0 &= 4x^2(2x^3-1) \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$2xu''(x) - (8x^3+2)\,u'(x) + 4x^2(2x^3-1)\,u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{e}^{\frac{2x^3}{3}}(c_2x^2+c_1)$$

The above shows that

$$u'(x) = 2\,\mathrm{e}^{\frac{2x^3}{3}}x(c_2x^3+c_1x+c_2)$$

Using the above in (1) gives the solution

$$y = -\frac{c_2x^3+c_1x+c_2}{c_2x^2+c_1}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{-x^3-c_3x-1}{x^2+c_3}$$

Initial conditions are used to solve for c_3. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\frac{1}{c_3}$$

$$c_3 = -1$$

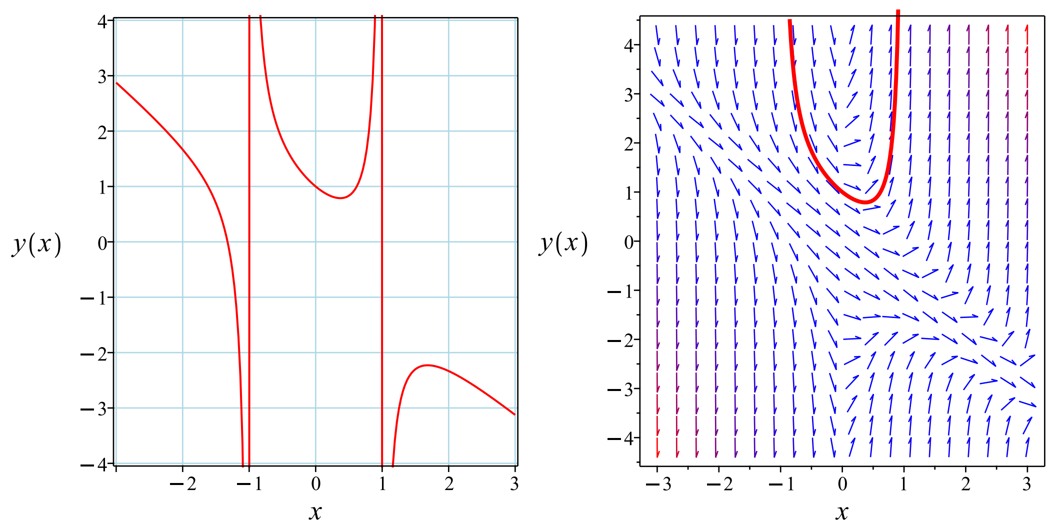
Substituting c_3 found above in the general solution gives

$$y = -\frac{x^3 - x + 1}{x^2 - 1}$$

Summary

The solution(s) found are the following

$$y = -\frac{x^3 - x + 1}{x^2 - 1} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -\frac{x^3 - x + 1}{x^2 - 1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
differential order: 1; found: 1 linear symmetries. Trying reduction of order
1st order, trying the canonical coordinates of the invariance group
   -> Calling odsolve with the ODE`, diff(y(x), x) = (-3*x-2*y(x))/x, y(x)`      *** Subleve
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
<- 1st order, canonical coordinates successful`
```

✓ Solution by Maple

Time used: 0.141 (sec). Leaf size: 20

```
dsolve([diff(y(x),x)=2*x*(x+y(x))^2-1,y(0) = 1],y(x), singsol=all)
```

$$y(x)=\frac{-x^3+x-1}{x^2-1}$$

✓ Solution by Mathematica

Time used: 0.146 (sec). Leaf size: 21

```
DSolve[{y'[x]==2*x*(x+y[x])^2-1,{y[0]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to \frac{-x^3+x-1}{x^2-1}$$

4.42 problem Problem 60

Internal problem ID [2706]
Internal file name [`OUTPUT/2198_Sunday_June_05_2022_02_54_12_AM_12631337/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.8, Change of Variables. page 79
Problem number: Problem 60.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeMapleC", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y' - \frac{x + 2y - 1}{2x - y + 3} = 0}$$

4.42.1 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = -\frac{X + x_0 + 2Y(X) + 2y_0 - 1}{-2X - 2x_0 + Y(X) + y_0 - 3}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = -1$$
$$y_0 = 1$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = -\frac{X + 2Y(X)}{-2X + Y(X)}$$

In canonical form, the ODE is

$$\begin{aligned}
Y' &= F(X, Y) \\
&= -\frac{X + 2Y}{-2X + Y}
\end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X, Y)$$

In this case, it can be seen that both $M = X + 2Y$ and $N = 2X - Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}X}X + u &= \frac{-2u - 1}{u - 2} \\
\frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\frac{-2u(X)-1}{u(X)-2} - u(X)}{X}
\end{aligned}$$

Or

$$\frac{d}{dX}u(X) - \frac{\frac{-2u(X)-1}{u(X)-2} - u(X)}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right) X u(X) - 2\left(\frac{d}{dX}u(X)\right) X + u(X)^2 + 1 = 0$$

Or

$$X(u(X) - 2)\left(\frac{d}{dX}u(X)\right) + u(X)^2 + 1 = 0$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$\begin{aligned}
u' &= F(X, u) \\
&= f(X)g(u) \\
&= -\frac{u^2 + 1}{X(u - 2)}
\end{aligned}$$

Where $f(X) = -\frac{1}{X}$ and $g(u) = \frac{u^2+1}{u-2}$. Integrating both sides gives

$$\frac{1}{\frac{u^2+1}{u-2}}\,du = -\frac{1}{X}\,dX$$

$$\int \frac{1}{\frac{u^2+1}{u-2}}\,du = \int -\frac{1}{X}\,dX$$

$$\frac{\ln(u^2+1)}{2} - 2\arctan(u) = -\ln(X) + c_2$$

The solution is

$$\frac{\ln\left(u(X)^2+1\right)}{2} - 2\arctan(u(X)) + \ln(X) - c_2 = 0$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$\frac{\ln\left(\frac{Y(X)^2}{X^2}+1\right)}{2} - 2\arctan\left(\frac{Y(X)}{X}\right) + \ln(X) - c_2 = 0$$

Using the solution for $Y(X)$

$$\frac{\ln\left(\frac{Y(X)^2}{X^2}+1\right)}{2} - 2\arctan\left(\frac{Y(X)}{X}\right) + \ln(X) - c_2 = 0$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y + 1$$
$$X = x - 1$$

Then the solution in y becomes

$$\frac{\ln\left(\frac{(y-1)^2}{(x+1)^2}+1\right)}{2} - 2\arctan\left(\frac{y-1}{x+1}\right) + \ln(x+1) - c_2 = 0$$

Summary

The solution(s) found are the following

$$
\frac{\ln\left(\frac{(y-1)^2}{(x+1)^2}+1\right)}{2}-2\arctan\left(\frac{y-1}{x+1}\right)+\ln(x+1)-c_2=0 \tag{1}
$$

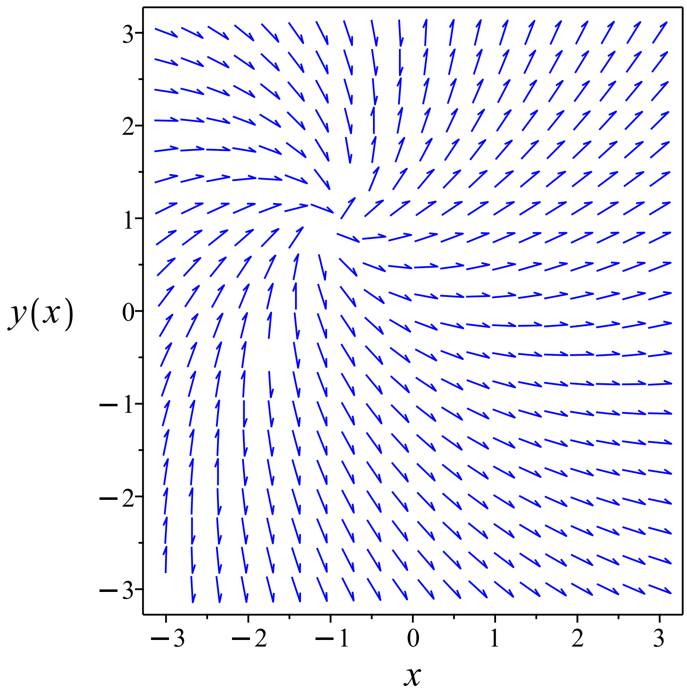


Figure 185: Slope field plot

Verification of solutions

$$
\frac{\ln\left(\frac{(y-1)^2}{(x+1)^2}+1\right)}{2}-2\arctan\left(\frac{y-1}{x+1}\right)+\ln(x+1)-c_2=0
$$

Verified OK.

4.42.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{x+2y-1}{-2x+y-3} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 - \frac{(x+2y-1)(b_3-a_2)}{-2x+y-3} - \frac{(x+2y-1)^2 a_3}{(-2x+y-3)^2} \\ & - \left(-\frac{1}{-2x+y-3} - \frac{2(x+2y-1)}{(-2x+y-3)^2}\right)(xa_2+ya_3+a_1) \\ & - \left(-\frac{2}{-2x+y-3} + \frac{x+2y-1}{(-2x+y-3)^2}\right)(xb_2+yb_3+b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{2x^2a_2 + x^2a_3 + x^2b_2 - 2x^2b_3 - 2xya_2 + 4xya_3 + 4xyb_2 + 2xyb_3 - 2y^2a_2 - y^2a_3 - y^2b_2 + 2y^2b_3 + 6xa_2 -}{(2x - y +}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} & -2x^2a_2 - x^2a_3 - x^2b_2 + 2x^2b_3 + 2xya_2 - 4xya_3 - 4xyb_2 - 2xyb_3 + 2y^2a_2 \\ & + y^2a_3 + y^2b_2 - 2y^2b_3 - 6xa_2 + 2xa_3 - 5xb_1 + 7xb_2 + xb_3 + 5ya_1 \\ & - 7ya_2 - ya_3 - 6yb_2 + 2yb_3 - 5a_1 + 3a_2 - a_3 - 5b_1 + 9b_2 - 3b_3 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-2a_2v_1^2 + 2a_2v_1v_2 + 2a_2v_2^2 - a_3v_1^2 - 4a_3v_1v_2 + a_3v_2^2 - b_2v_1^2 - 4b_2v_1v_2 + b_2v_2^2 \\ + 2b_3v_1^2 - 2b_3v_1v_2 - 2b_3v_2^2 + 5a_1v_2 - 6a_2v_1 - 7a_2v_2 + 2a_3v_1 - a_3v_2 - 5b_1v_1 \\ + 7b_2v_1 - 6b_2v_2 + b_3v_1 + 2b_3v_2 - 5a_1 + 3a_2 - a_3 - 5b_1 + 9b_2 - 3b_3 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$(-2a_2 - a_3 - b_2 + 2b_3)\, v_1^2 + (2a_2 - 4a_3 - 4b_2 - 2b_3)\, v_1v_2 \\ + (-6a_2 + 2a_3 - 5b_1 + 7b_2 + b_3)\, v_1 + (2a_2 + a_3 + b_2 - 2b_3)\, v_2^2 \\ + (5a_1 - 7a_2 - a_3 - 6b_2 + 2b_3)\, v_2 - 5a_1 + 3a_2 - a_3 - 5b_1 + 9b_2 - 3b_3 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -2a_2 - a_3 - b_2 + 2b_3 &= 0 \\ 2a_2 - 4a_3 - 4b_2 - 2b_3 &= 0 \\ 2a_2 + a_3 + b_2 - 2b_3 &= 0 \\ 5a_1 - 7a_2 - a_3 - 6b_2 + 2b_3 &= 0 \\ -6a_2 + 2a_3 - 5b_1 + 7b_2 + b_3 &= 0 \\ -5a_1 + 3a_2 - a_3 - 5b_1 + 9b_2 - 3b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= b_2 + b_3 \\ a_2 &= b_3 \\ a_3 &= -b_2 \\ b_1 &= b_2 - b_3 \\ b_2 &= b_2 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= 1 - y \\
\eta &= x + 1
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= x + 1 - \left(-\frac{x + 2y - 1}{-2x + y - 3}\right)(1 - y) \\
&= \frac{2x^2 + 2y^2 + 4x - 4y + 4}{2x - y + 3} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{2x^2 + 2y^2 + 4x - 4y + 4}{2x - y + 3}} dy
\end{aligned}
$$

Which results in

$$
S = -\frac{\ln(x^2 + y^2 + 2x - 2y + 2)}{4} + \arctan\left(\frac{2y - 2}{2 + 2x}\right)
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{x + 2y - 1}{-2x + y - 3}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{-x - 2y + 1}{2x^2 + 2y^2 + 4x - 4y + 4} \\
S_y &= \frac{2x - y + 3}{2x^2 + 2y^2 + 4x - 4y + 4}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

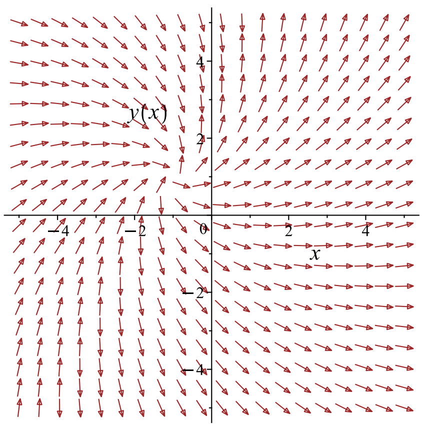
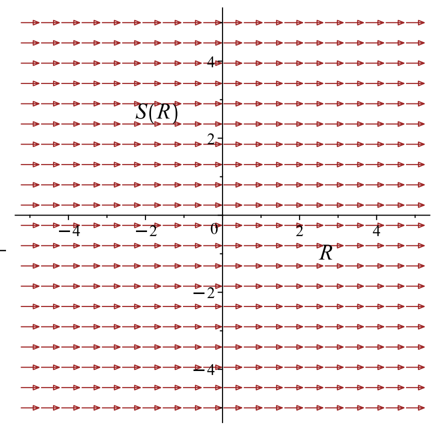
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{\ln\left(y^2 + x^2 - 2y + 2x + 2\right)}{4} + \arctan\left(\frac{y - 1}{x + 1}\right) = c_1$$

Which simplifies to

$$-\frac{\ln\left(y^2 + x^2 - 2y + 2x + 2\right)}{4} + \arctan\left(\frac{y - 1}{x + 1}\right) = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = -\frac{x+2y-1}{-2x+y-3}$
</td><td>$R = x$
$S = -\frac{\ln(x^2+y^2+2x-}{4}$</td><td>$\frac{dS}{dR} = 0$
</td></tr>
</table>

Summary

The solution(s) found are the following

$$
-\frac{\ln(y^2+x^2-2y+2x+2)}{4} + \arctan\left(\frac{y-1}{x+1}\right) = c_1 \tag{1}
$$

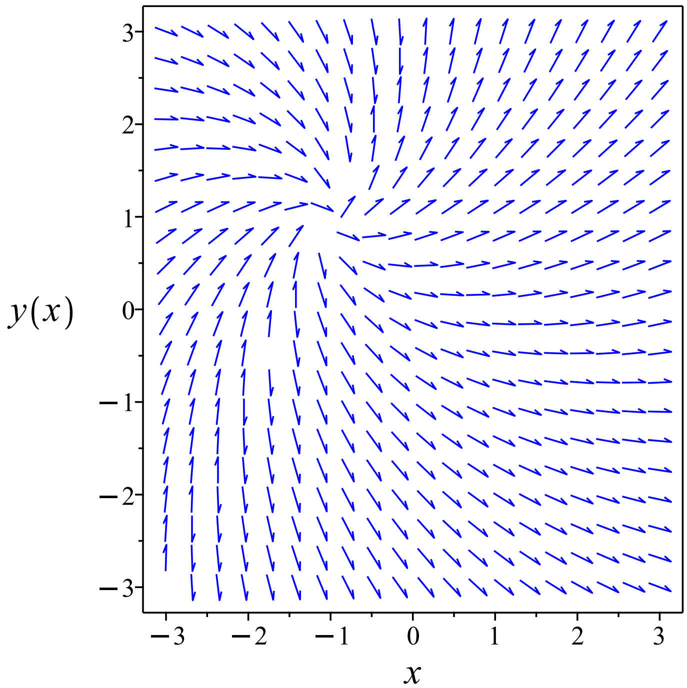


Figure 186: Slope field plot

Verification of solutions

$$
-\frac{\ln \left(y^{2}+x^{2}-2 y+2 x+2\right)}{4}+\arctan \left(\frac{y-1}{x+1}\right)=c_{1}
$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 32

```
dsolve(diff(y(x),x)=(x+2*y(x)-1)/(2*x-y(x)+3),y(x), singsol=all)
```

$$y(x) = 1 + \tan\left(\operatorname{RootOf}\left(4\_Z + \ln\left(\sec\left(\_Z\right)^2\right) + 2\ln\left(x+1\right) + 2c_1\right)\right)(-x-1)$$

✓ Solution by Mathematica

Time used: 0.064 (sec). Leaf size: 68

```
DSolve[y'[x]==(x+2*y[x]-1)/(2*x-y[x]+3),y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[32\arctan\left(\frac{-2y(x)-x+1}{-y(x)+2x+3}\right) + 8\log\left(\frac{x^2+y(x)^2-2y(x)+2x+2}{5(x+1)^2}\right) + 16\log(x+1) + 5c_1 = 0, y(x)\right]$$

4.43 problem Problem 61

Internal problem ID [2707]
Internal file name [`OUTPUT/2199_Sunday_June_05_2022_02_54_17_AM_90629347/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.8, Change of Variables. page 79
Problem number: Problem 61.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' + p(x)\, y + q(x)\, y^2 = r(x)}$$

4.43.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -p(x)\, y - q(x)\, y^2 + r(x) \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -p(x)\, y - q(x)\, y^2 + r(x)$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = r(x)$, $f_1(x) = -p(x)$ and $f_2(x) = -q(x)$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-q\,(x)\, u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -q'(x) \\ f_1 f_2 &= q(x)\, p(x) \\ f_2^2 f_0 &= r(x)\, q(x)^2 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-q(x)\, u''(x) - (q(x)\, p(x) - q'(x))\, u'(x) + r(x)\, q(x)^2\, u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{DESol}\left(\left\{-r(x)\, q(x)\_Y(x) + \frac{(q(x)\, p(x) - q'(x))\_Y'(x)}{q(x)} + \_Y''(x)\right\}, \{\_Y(x)\}\right)$$

The above shows that

$$u'(x) = \frac{d}{dx}\mathrm{DESol}\left(\left\{-r(x)\, q(x)\_Y(x) + \frac{(q(x)\, p(x) - q'(x))\_Y'(x)}{q(x)} + \_Y''(x)\right\}, \{\_Y(x)\}\right)$$

Using the above in (1) gives the solution

$$y = \frac{\frac{d}{dx}\mathrm{DESol}\left(\left\{-r(x)\, q(x)\_Y(x) + \frac{(q(x)p(x)-q'(x))\_Y'(x)}{q(x)} + \_Y''(x)\right\}, \{\_Y(x)\}\right)}{q(x)\,\mathrm{DESol}\left(\left\{-r(x)\, q(x)\_Y(x) + \frac{(q(x)p(x)-q'(x))\_Y'(x)}{q(x)} + \_Y''(x)\right\}, \{\_Y(x)\}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\frac{d}{dx}\mathrm{DESol}\left(\left\{-r(x)\, q(x)\_Y(x) + \frac{(q(x)p(x)-q'(x))\_Y'(x)}{q(x)} + \_Y''(x)\right\}, \{\_Y(x)\}\right)}{q(x)\,\mathrm{DESol}\left(\left\{-r(x)\, q(x)\_Y(x) + \frac{(q(x)p(x)-q'(x))\_Y'(x)}{q(x)} + \_Y''(x)\right\}, \{\_Y(x)\}\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{\frac{d}{dx}\,\mathrm{DESol}\left(\left\{-r(x)\,q(x)\_Y(x) + \frac{(q(x)p(x)-q'(x))\_Y'(x)}{q(x)} + \_Y''(x)\right\}, \{\_Y(x)\}\right)}{q(x)\,\mathrm{DESol}\left(\left\{-r(x)\,q(x)\_Y(x) + \frac{(q(x)p(x)-q'(x))\_Y'(x)}{q(x)} + \_Y''(x)\right\}, \{\_Y(x)\}\right)} \tag{1}$$

Verification of solutions

$$y = \frac{\frac{d}{dx}\,\mathrm{DESol}\left(\left\{-r(x)\,q(x)\_Y(x) + \frac{(q(x)p(x)-q'(x))\_Y'(x)}{q(x)} + \_Y''(x)\right\}, \{\_Y(x)\}\right)}{q(x)\,\mathrm{DESol}\left(\left\{-r(x)\,q(x)\_Y(x) + \frac{(q(x)p(x)-q'(x))\_Y'(x)}{q(x)} + \_Y''(x)\right\}, \{\_Y(x)\}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = -(p(x)*q(x)-(diff(q(x), x)))*(
      Methods for second order ODEs:
   -> Trying a change of variables to reduce to Bernoulli
   -> Calling odsolve with the ODE`, diff(y(x), x)-(-q(x)*y(x)^2+y(x)-p(x)*y(x)*x+x^2*r(x))/
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      trying Bernoulli
      trying separable
      trying inverse linear
      trying homogeneous types:
      trying Chini
      differential order: 1; looking for linear symmetries
      trying exact
      Looking for potential symmetries
      trying Riccati
      trying Riccati sub-methods:
         trying Riccati_symmetries
      trying inverse_Riccati
      trying 1st order ODE linearizable_by_differentiation
   -> trying a symmetry pattern of the form [F(x)*G(y), 0]
   -> trying a symmetry pattern of the form [0, F(x)*G(y)]
   -> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
trying inverse_Riccati
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 4
```

✗ Solution by Maple

```
dsolve(diff(y(x),x)+p(x)*y(x)+q(x)*y(x)^2=r(x),y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]+p[x]*y[x]+q[x]*y[x]^2==r[x],y[x],x,IncludeSingularSolutions -> True]
```

Not solved

4.44 problem Problem 62

Internal problem ID [2708]
Internal file name [`OUTPUT/2200_Sunday_June_05_2022_02_54_20_AM_5311040/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.8, Change of Variables. page 79
Problem number: Problem 62.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, _Riccati]
```

$$\boxed{y' + \frac{2y}{x} - y^2 = -\frac{2}{x^2}}$$

4.44.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{y^2x^2 - 2xy - 2}{x^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}
&b_2 + \frac{(y^2x^2 - 2xy - 2)(b_3 - a_2)}{x^2} - \frac{(y^2x^2 - 2xy - 2)^2 a_3}{x^4} \\
&- \left(\frac{2x\,y^2 - 2y}{x^2} - \frac{2(y^2x^2 - 2xy - 2)}{x^3}\right)(xa_2 + ya_3 + a_1) \\
&- \frac{(2x^2y - 2x)(xb_2 + yb_3 + b_1)}{x^2} = 0
\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{x^4y^4a_3 + 2x^5yb_2 + x^4y^2a_2 + x^4y^2b_3 - 4x^3y^3a_3 + 2x^4yb_1 - 3b_2x^4 + 2x^2y^2a_3 - 2x^3b_1 + 2x^2ya_1 + 2x^2a_2 +}{x^4}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned}
&-x^4y^4a_3 - 2x^5yb_2 - x^4y^2a_2 - x^4y^2b_3 + 4x^3y^3a_3 - 2x^4yb_1 + 3b_2x^4 \\
&- 2x^2y^2a_3 + 2x^3b_1 - 2x^2ya_1 - 2x^2a_2 - 2x^2b_3 - 12xya_3 - 4xa_1 - 4a_3 = 0
\end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&-a_3v_1^4v_2^4 - a_2v_1^4v_2^2 + 4a_3v_1^3v_2^3 - 2b_2v_1^5v_2 - b_3v_1^4v_2^2 - 2b_1v_1^4v_2 - 2a_3v_1^2v_2^2 \\
&+ 3b_2v_1^4 - 2a_1v_1^2v_2 + 2b_1v_1^3 - 2a_2v_1^2 - 12a_3v_1v_2 - 2b_3v_1^2 - 4a_1v_1 - 4a_3 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} &-2b_2v_1^5v_2 - a_3v_1^4v_2^4 + (-a_2 - b_3)\, v_1^4v_2^2 - 2b_1v_1^4v_2 + 3b_2v_1^4 + 4a_3v_1^3v_2^3 + 2b_1v_1^3 \\ &- 2a_3v_1^2v_2^2 - 2a_1v_1^2v_2 + (-2a_2 - 2b_3)\, v_1^2 - 12a_3v_1v_2 - 4a_1v_1 - 4a_3 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -4a_1 &= 0 \\ -2a_1 &= 0 \\ -12a_3 &= 0 \\ -4a_3 &= 0 \\ -2a_3 &= 0 \\ -a_3 &= 0 \\ 4a_3 &= 0 \\ -2b_1 &= 0 \\ 2b_1 &= 0 \\ -2b_2 &= 0 \\ 3b_2 &= 0 \\ -2a_2 - 2b_3 &= 0 \\ -a_2 - b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= -b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= -x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(\frac{y^2x^2 - 2xy - 2}{x^2}\right)(-x) \\
&= \frac{y^2x^2 - xy - 2}{x} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{y^2x^2 - xy - 2}{x}} dy
\end{aligned}
$$

Which results in

$$
S = -\frac{\ln(xy + 1)}{3} + \frac{\ln(xy - 2)}{3}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$
\omega(x, y) = \frac{y^2x^2 - 2xy - 2}{x^2}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{y}{(xy+1)(xy-2)} \\
S_y &= \frac{x}{(xy+1)(xy-2)}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{1}{x} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{1}{R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \ln(R) + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
-\frac{\ln(1+yx)}{3} + \frac{\ln(yx-2)}{3} = \ln(x) + c_1
$$

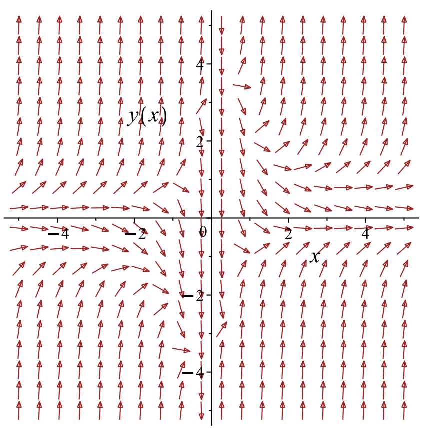
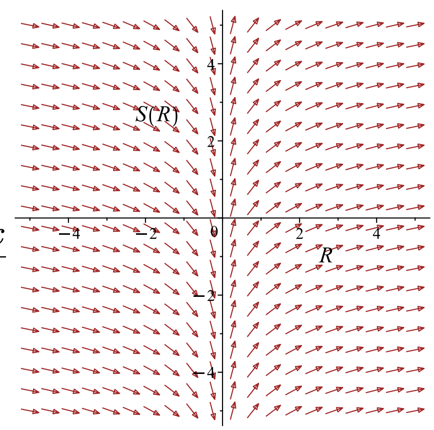
Which simplifies to

$$
-\frac{\ln(1+yx)}{3} + \frac{\ln(yx-2)}{3} = \ln(x) + c_1
$$

Which gives

$$
y = -\frac{x^3 e^{3c_1} + 2}{x\,(x^3 e^{3c_1} - 1)}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = \frac{y^2x^2-2xy-2}{x^2}$

</td><td>$R = x$
$S = -\frac{\ln(xy+1)}{3} + \frac{\ln(x}{}$</td><td>$\frac{dS}{dR} = \frac{1}{R}$

</td></tr>
</table>

Summary

The solution(s) found are the following

$$y = -\frac{x^3 e^{3c_1} + 2}{x\left(x^3 e^{3c_1} - 1\right)} \tag{1}$$

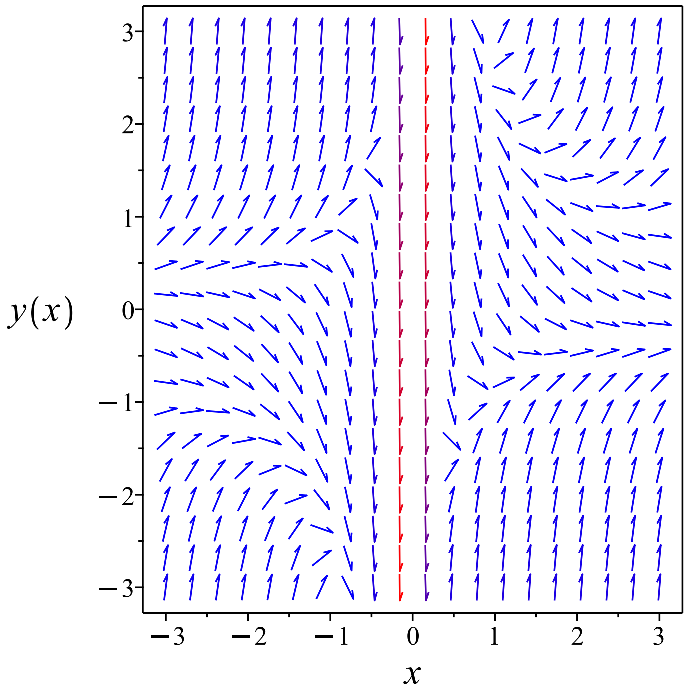


Figure 187: Slope field plot

Verification of solutions

$$y = -\frac{x^3 e^{3c_1} + 2}{x\left(x^3 e^{3c_1} - 1\right)}$$

Verified OK.

4.44.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y) \,\mathrm{d}x + N(x, y) \,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\mathrm{d}y &= \left(-\frac{2y}{x} + y^2 - \frac{2}{x^2}\right) \mathrm{d}x \\
\left(\frac{2y}{x} - y^2 + \frac{2}{x^2}\right) \mathrm{d}x + \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= \frac{2y}{x} - y^2 + \frac{2}{x^2} \\
N(x, y) &= 1
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{2y}{x} - y^2 + \frac{2}{x^2}\right) \\
&= \frac{-2xy + 2}{x}
\end{aligned}
$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= 1\left(\left(\frac{2}{x} - 2y\right) - (0)\right)\\ &= \frac{-2xy + 2}{x}\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\\ &= \frac{1}{\frac{2y}{x} - y^2 + \frac{2}{x^2}}\left((0) - \left(\frac{2}{x} - 2y\right)\right)\\ &= -\frac{2x(xy-1)}{y^2x^2 - 2xy - 2}\end{aligned}$$

Since B depends on x, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$\begin{aligned}R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}\\ &= \frac{(0) - \left(\frac{2}{x} - 2y\right)}{x\left(\frac{2y}{x} - y^2 + \frac{2}{x^2}\right) - y\,(1)}\\ &= \frac{-2xy + 2}{y^2x^2 - xy - 2}\end{aligned}$$

Replacing all powers of terms xy by t gives

$$R = \frac{-2t + 2}{t^2 - t - 2}$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$\begin{aligned}\mu &= e^{\int R \, \mathrm{d}t}\\ &= e^{\int \left(\frac{-2t+2}{t^2-t-2}\right) \mathrm{d}t}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\frac{4\ln(t+1)}{3}-\frac{2\ln(t-2)}{3}}\\ &= \frac{1}{(t+1)^{\frac{4}{3}}(t-2)^{\frac{2}{3}}}\end{aligned}$$

Now t is replaced back with xy giving

$$\mu = \frac{1}{(xy+1)^{\frac{4}{3}}(xy-2)^{\frac{2}{3}}}$$

Multiplying M and N by this integrating factor gives new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{(xy+1)^{\frac{4}{3}}(xy-2)^{\frac{2}{3}}}\left(\frac{2y}{x}-y^2+\frac{2}{x^2}\right)\\ &= -\frac{y^2x^2-2xy-2}{x^2(xy+1)^{\frac{4}{3}}(xy-2)^{\frac{2}{3}}}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \frac{1}{(xy+1)^{\frac{4}{3}}(xy-2)^{\frac{2}{3}}}(1)\\ &= \frac{1}{(xy+1)^{\frac{4}{3}}(xy-2)^{\frac{2}{3}}}\end{aligned}$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$\begin{aligned}\overline{M}+\overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \left(-\frac{y^2x^2-2xy-2}{x^2(xy+1)^{\frac{4}{3}}(xy-2)^{\frac{2}{3}}}\right)+\left(\frac{1}{(xy+1)^{\frac{4}{3}}(xy-2)^{\frac{2}{3}}}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int -\frac{y^2x^2 - 2xy - 2}{x^2 (xy+1)^{\frac{4}{3}} (xy-2)^{\frac{2}{3}}} \mathrm{d}x$$

$$\phi = \frac{(xy-2)^{\frac{1}{3}}}{x (xy+1)^{\frac{1}{3}}} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\begin{aligned} \frac{\partial \phi}{\partial y} &= -\frac{(xy-2)^{\frac{1}{3}}}{3 (xy+1)^{\frac{4}{3}}} + \frac{1}{3 (xy+1)^{\frac{1}{3}} (xy-2)^{\frac{2}{3}}} + f'(y) \qquad (4) \\ &= \frac{1}{(xy+1)^{\frac{4}{3}} (xy-2)^{\frac{2}{3}}} + f'(y) \end{aligned}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{(xy+1)^{\frac{4}{3}} (xy-2)^{\frac{2}{3}}}$. Therefore equation (4) becomes

$$\frac{1}{(xy+1)^{\frac{4}{3}} (xy-2)^{\frac{2}{3}}} = \frac{1}{(xy+1)^{\frac{4}{3}} (xy-2)^{\frac{2}{3}}} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{(xy-2)^{\frac{1}{3}}}{x (xy+1)^{\frac{1}{3}}} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{(xy-2)^{\frac{1}{3}}}{x\,(xy+1)^{\frac{1}{3}}}$$

The solution becomes

$$y = -\frac{c_1^3 x^3 + 2}{(c_1^3 x^3 - 1)\,x}$$

Summary

The solution(s) found are the following

$$y = -\frac{c_1^3 x^3 + 2}{(c_1^3 x^3 - 1)\,x} \tag{1}$$

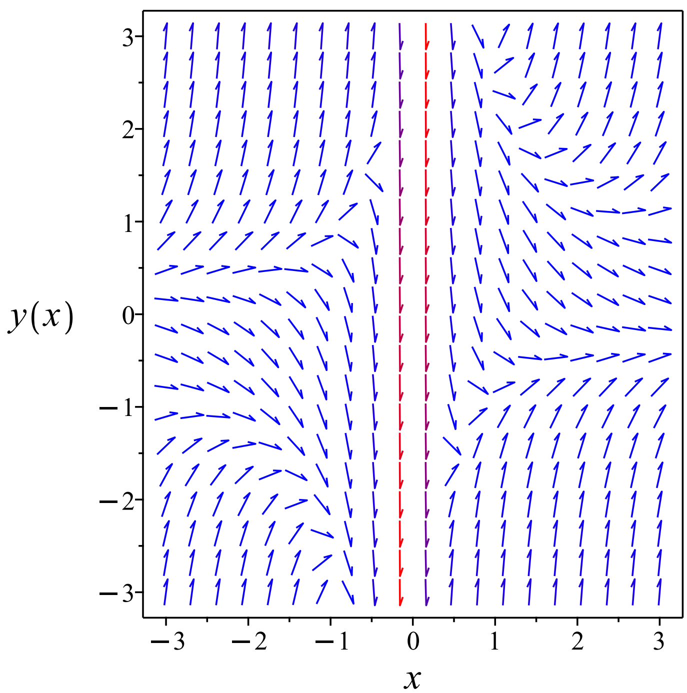


Figure 188: Slope field plot

Verification of solutions

$$y = -\frac{c_1^3 x^3 + 2}{(c_1^3 x^3 - 1)\,x}$$

Verified OK.

4.44.3 Solving as riccati ode

In canonical form the ODE is

$$
\begin{aligned}
y' &= F(x, y) \\
&= \frac{y^2 x^2 - 2xy - 2}{x^2}
\end{aligned}
$$

This is a Riccati ODE. Comparing the ODE to solve

$$
y' = -\frac{2y}{x} + y^2 - \frac{2}{x^2}
$$

With Riccati ODE standard form

$$
y' = f_0(x) + f_1(x)y + f_2(x)y^2
$$

Shows that $f_0(x) = -\frac{2}{x^2}$, $f_1(x) = -\frac{2}{x}$ and $f_2(x) = 1$. Let

$$
\begin{aligned}
y &= \frac{-u'}{f_2 u} \\
&= \frac{-u'}{u} \qquad (1)
\end{aligned}
$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$
f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \qquad (2)
$$

But

$$
\begin{aligned}
f_2' &= 0 \\
f_1 f_2 &= -\frac{2}{x} \\
f_2^2 f_0 &= -\frac{2}{x^2}
\end{aligned}
$$

Substituting the above terms back in equation (2) gives

$$
u''(x) + \frac{2u'(x)}{x} - \frac{2u(x)}{x^2} = 0
$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$
u(x) = \frac{c_1 x^3 + c_2}{x^2}
$$

The above shows that

$$u'(x) = \frac{c_1 x^3 - 2c_2}{x^3}$$

Using the above in (1) gives the solution

$$y = -\frac{c_1 x^3 - 2c_2}{x (c_1 x^3 + c_2)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{-c_3 x^3 + 2}{x (c_3 x^3 + 1)}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{-c_3 x^3 + 2}{x (c_3 x^3 + 1)} \tag{1}$$

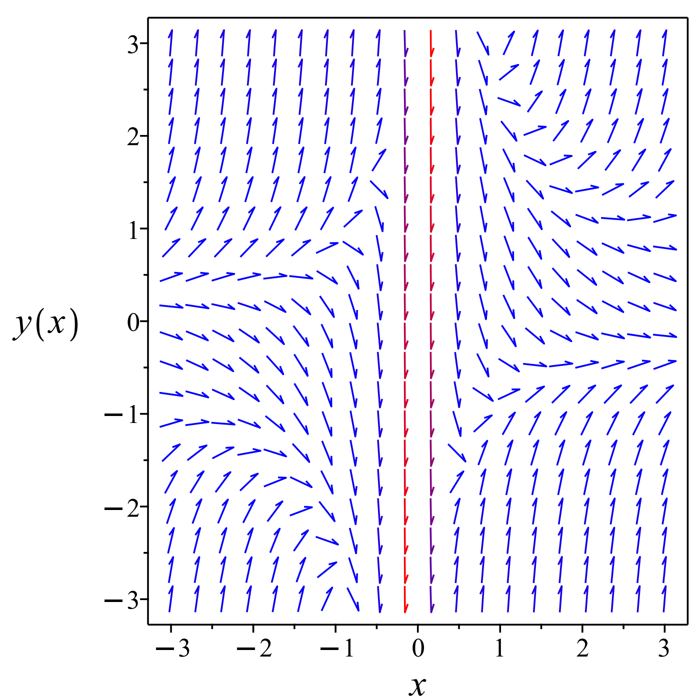


Figure 189: Slope field plot

Verification of solutions

$$
y = \frac{-c_3 x^3 + 2}{x (c_3 x^3 + 1)}
$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.531 (sec). Leaf size: 24

```
dsolve(diff(y(x),x)+2/x*y(x)-y(x)^2=-2/x^2,y(x), singsol=all)
```

$$
y(x) = \frac{x^3 + 2c_1}{(-x^3 + c_1)\, x}
$$

✓ Solution by Mathematica

Time used: 0.175 (sec). Leaf size: 35

```
DSolve[y'[x]+2/x*y[x]-y[x]^2==-2/x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$
y(x) \to \frac{2 + 3c_1 x^3}{x - 3c_1 x^4}
$$

$$
y(x) \to -\frac{1}{x}
$$

4.45 problem Problem 63

Internal problem ID [2709]
Internal file name [`OUTPUT/2201_Sunday_June_05_2022_02_54_24_AM_25723167/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.8, Change of Variables. page 79
Problem number: Problem 63.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, _Riccati]
```

$$\boxed{y' + \frac{7y}{x} - 3y^2 = \frac{3}{x^2}}$$

4.45.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{3y^2x^2 - 7xy + 3}{x^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 + \frac{(3y^2x^2 - 7xy + 3)(b_3 - a_2)}{x^2} - \frac{(3y^2x^2 - 7xy + 3)^2 a_3}{x^4} \\ & - \left(\frac{6x\,y^2 - 7y}{x^2} - \frac{2(3y^2x^2 - 7xy + 3)}{x^3}\right)(xa_2 + ya_3 + a_1) \\ & - \frac{(6x^2y - 7x)(xb_2 + yb_3 + b_1)}{x^2} = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{9x^4y^4a_3 + 6x^5yb_2 + 3x^4y^2a_2 + 3x^4y^2b_3 - 42x^3y^3a_3 + 6x^4yb_1 - 8b_2x^4 + 74x^2y^2a_3 - 7x^3b_1 + 7x^2ya_1 - 3x}{x^4}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} & -9x^4y^4a_3 - 6x^5yb_2 - 3x^4y^2a_2 - 3x^4y^2b_3 + 42x^3y^3a_3 - 6x^4yb_1 + 8b_2x^4 \\ & - 74x^2y^2a_3 + 7x^3b_1 - 7x^2ya_1 + 3x^2a_2 + 3x^2b_3 + 48xya_3 + 6xa_1 - 9a_3 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} & -9a_3v_1^4v_2^4 - 3a_2v_1^4v_2^2 + 42a_3v_1^3v_2^3 - 6b_2v_1^5v_2 - 3b_3v_1^4v_2^2 - 6b_1v_1^4v_2 - 74a_3v_1^2v_2^2 \\ & + 8b_2v_1^4 - 7a_1v_1^2v_2 + 7b_1v_1^3 + 3a_2v_1^2 + 48a_3v_1v_2 + 3b_3v_1^2 + 6a_1v_1 - 9a_3 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$-6b_2v_1^5v_2 - 9a_3v_1^4v_2^4 + (-3a_2 - 3b_3)\, v_1^4v_2^2 - 6b_1v_1^4v_2 + 8b_2v_1^4 + 42a_3v_1^3v_2^3 \\ + 7b_1v_1^3 - 74a_3v_1^2v_2^2 - 7a_1v_1^2v_2 + (3a_2 + 3b_3)\, v_1^2 + 48a_3v_1v_2 + 6a_1v_1 - 9a_3 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -7a_1 &= 0 \\ 6a_1 &= 0 \\ -74a_3 &= 0 \\ -9a_3 &= 0 \\ 42a_3 &= 0 \\ 48a_3 &= 0 \\ -6b_1 &= 0 \\ 7b_1 &= 0 \\ -6b_2 &= 0 \\ 8b_2 &= 0 \\ -3a_2 - 3b_3 &= 0 \\ 3a_2 + 3b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= -b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= -x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(\frac{3y^2x^2 - 7xy + 3}{x^2}\right)(-x) \\
&= \frac{3y^2x^2 - 6xy + 3}{x} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{3y^2x^2 - 6xy + 3}{x}} dy
\end{aligned}$$

Which results in

$$S = -\frac{1}{3(xy - 1)}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{3y^2x^2 - 7xy + 3}{x^2}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{y}{3\left(xy-1\right)^2} \\
S_y &= \frac{x}{3\left(xy-1\right)^2}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \ln\left(R\right) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{1}{3yx-3} = \ln\left(x\right) + c_1$$

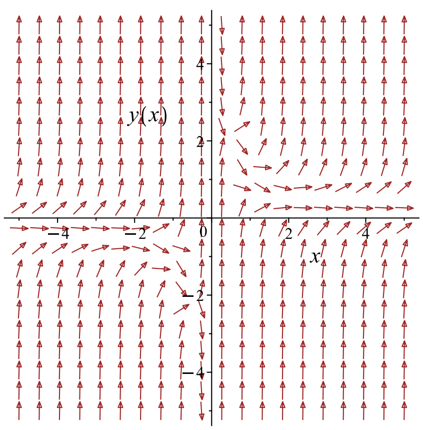
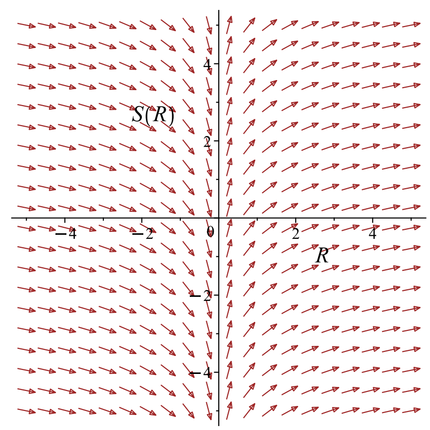
Which simplifies to

$$-\frac{1}{3yx-3} = \ln\left(x\right) + c_1$$

Which gives

$$y = \frac{3\ln\left(x\right) + 3c_1 - 1}{3x\left(\ln\left(x\right) + c_1\right)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{3y^2x^2-7xy+3}{x^2}$ 	$R = x$ $S = -\frac{1}{3xy-3}$	$\frac{dS}{dR} = \frac{1}{R}$ 

Summary

The solution(s) found are the following

$$y = \frac{3\ln(x) + 3c_1 - 1}{3x(\ln(x) + c_1)} \tag{1}$$

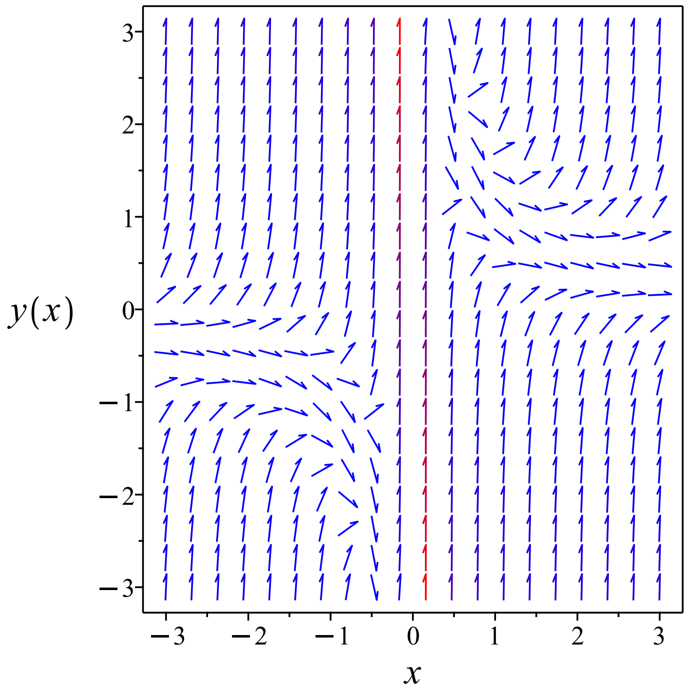


Figure 190: Slope field plot

Verification of solutions

$$y = \frac{3\ln(x) + 3c_1 - 1}{3x(\ln(x) + c_1)}$$

Verified OK.

4.45.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\mathrm{d}y = \left(-\frac{7y}{x} + 3y^2 + \frac{3}{x^2}\right) \mathrm{d}x
$$

$$
\left(\frac{7y}{x} - 3y^2 - \frac{3}{x^2}\right) \mathrm{d}x + \mathrm{d}y = 0 \tag{2A}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= \frac{7y}{x} - 3y^2 - \frac{3}{x^2} \\
N(x,y) &= 1
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{7y}{x} - 3y^2 - \frac{3}{x^2}\right) \\
&= \frac{7}{x} - 6y
\end{aligned}
$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= 1\left(\left(\frac{7}{x} - 6y\right) - (0)\right)\\ &= \frac{7}{x} - 6y\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\\ &= \frac{1}{\frac{7y}{x} - 3y^2 - \frac{3}{x^2}}\left((0) - \left(\frac{7}{x} - 6y\right)\right)\\ &= \frac{-6x^2y + 7x}{3y^2x^2 - 7xy + 3}\end{aligned}$$

Since B depends on x, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$\begin{aligned}R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}\\ &= \frac{(0) - \left(\frac{7}{x} - 6y\right)}{x\left(\frac{7y}{x} - 3y^2 - \frac{3}{x^2}\right) - y\,(1)}\\ &= \frac{-6xy + 7}{3\,(xy - 1)^2}\end{aligned}$$

Replacing all powers of terms xy by t gives

$$R = \frac{-6t + 7}{3\,(t - 1)^2}$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$
\begin{aligned}
\mu &= e^{\int R \,\mathrm{d}t} \\
&= e^{\int \left(\frac{-6t+7}{3(t-1)^2}\right) \mathrm{d}t}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-2\ln(t-1)-\frac{1}{3(t-1)}} \\
&= \frac{\mathrm{e}^{-\frac{1}{3t-3}}}{(t-1)^2}
\end{aligned}
$$

Now t is replaced back with xy giving

$$
\mu = \frac{\mathrm{e}^{-\frac{1}{3xy-3}}}{(xy-1)^2}
$$

Multiplying M and N by this integrating factor gives new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{\mathrm{e}^{-\frac{1}{3xy-3}}}{(xy-1)^2}\left(\frac{7y}{x} - 3y^2 - \frac{3}{x^2}\right) \\
&= -\frac{3\left(y^2x^2 - \frac{7}{3}xy + 1\right)\mathrm{e}^{-\frac{1}{3xy-3}}}{x^2\,(xy-1)^2}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{\mathrm{e}^{-\frac{1}{3xy-3}}}{(xy-1)^2}(1) \\
&= \frac{\mathrm{e}^{-\frac{1}{3xy-3}}}{(xy-1)^2}
\end{aligned}
$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(-\frac{3\left(y^2x^2 - \frac{7}{3}xy + 1\right)\mathrm{e}^{-\frac{1}{3xy-3}}}{x^2\,(xy-1)^2}\right) + \left(\frac{\mathrm{e}^{-\frac{1}{3xy-3}}}{(xy-1)^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int -\frac{3\left(y^2x^2 - \frac{7}{3}xy + 1\right)\mathrm{e}^{-\frac{1}{3xy-3}}}{x^2\,(xy-1)^2}\,\mathrm{d}x$$
$$\phi = \frac{3\,\mathrm{e}^{-\frac{1}{3xy-3}}}{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial\phi}{\partial y} = \frac{9\,\mathrm{e}^{-\frac{1}{3xy-3}}}{(3xy-3)^2} + f'(y) \tag{4}$$
$$= \frac{\mathrm{e}^{-\frac{1}{3xy-3}}}{(xy-1)^2} + f'(y)$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = \frac{\mathrm{e}^{-\frac{1}{3xy-3}}}{(xy-1)^2}$. Therefore equation (4) becomes

$$\frac{\mathrm{e}^{-\frac{1}{3xy-3}}}{(xy-1)^2} = \frac{\mathrm{e}^{-\frac{1}{3xy-3}}}{(xy-1)^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{3\,\mathrm{e}^{-\frac{1}{3xy-3}}}{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{3\,\mathrm{e}^{-\frac{1}{3xy-3}}}{x}$$

The solution becomes

$$y = \frac{3\ln\left(\frac{c_1 x}{3}\right) - 1}{3x\ln\left(\frac{c_1 x}{3}\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{3\ln\left(\frac{c_1 x}{3}\right) - 1}{3x\ln\left(\frac{c_1 x}{3}\right)} \tag{1}$$

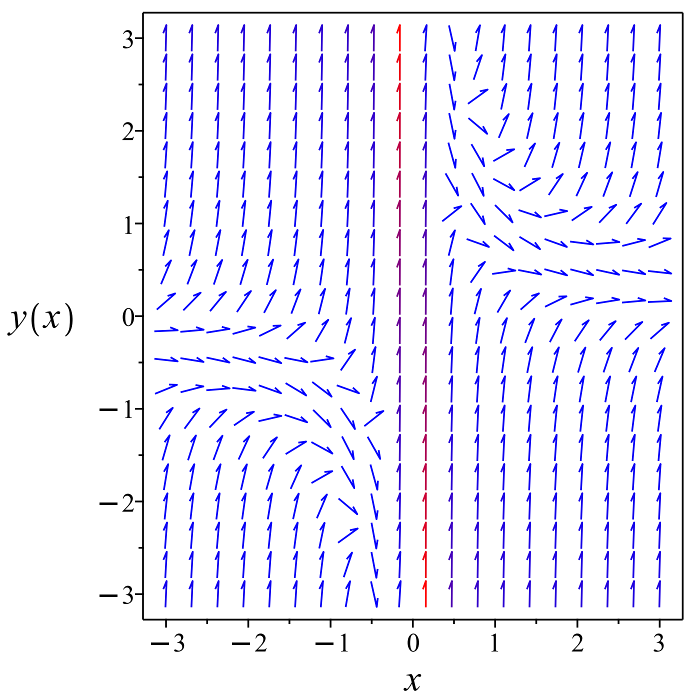


Figure 191: Slope field plot

Verification of solutions

$$y = \frac{3\ln\left(\frac{c_1 x}{3}\right) - 1}{3x\ln\left(\frac{c_1 x}{3}\right)}$$

Verified OK.

4.45.3 Solving as riccati ode

In canonical form the ODE is

$$
\begin{aligned}
y' &= F(x,y)\\
&= \frac{3y^2x^2 - 7xy + 3}{x^2}
\end{aligned}
$$

This is a Riccati ODE. Comparing the ODE to solve

$$
y' = -\frac{7y}{x} + 3y^2 + \frac{3}{x^2}
$$

With Riccati ODE standard form

$$
y' = f_0(x) + f_1(x)y + f_2(x)y^2
$$

Shows that $f_0(x) = \frac{3}{x^2}$, $f_1(x) = -\frac{7}{x}$ and $f_2(x) = 3$. Let

$$
\begin{aligned}
y &= \frac{-u'}{f_2 u}\\
&= \frac{-u'}{3u} \qquad (1)
\end{aligned}
$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$
f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \qquad (2)
$$

But

$$
\begin{aligned}
f_2' &= 0\\
f_1f_2 &= -\frac{21}{x}\\
f_2^2f_0 &= \frac{27}{x^2}
\end{aligned}
$$

Substituting the above terms back in equation (2) gives

$$
3u''(x) + \frac{21u'(x)}{x} + \frac{27u(x)}{x^2} = 0
$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$
u(x) = \frac{c_1 + c_2\ln(x)}{x^3}
$$

The above shows that

$$u'(x) = \frac{-3c_1 - 3c_2 \ln(x) + c_2}{x^4}$$

Using the above in (1) gives the solution

$$y = -\frac{-3c_1 - 3c_2 \ln(x) + c_2}{3x(c_1 + c_2 \ln(x))}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{c_3 + \ln(x) - \frac{1}{3}}{x(c_3 + \ln(x))}$$

Summary

The solution(s) found are the following

$$y = \frac{c_3 + \ln(x) - \frac{1}{3}}{x(c_3 + \ln(x))} \tag{1}$$

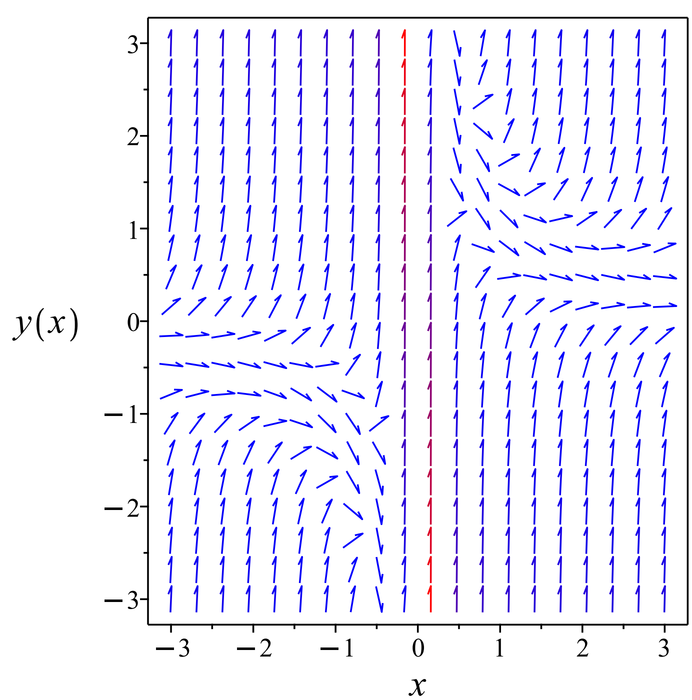


Figure 192: Slope field plot

Verification of solutions

$$y = \frac{c_3 + \ln(x) - \frac{1}{3}}{x(c_3 + \ln(x))}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve(diff(y(x),x)+7/x*y(x)-3*y(x)^2=3/x^2,y(x), singsol=all)
```

$$y(x) = \frac{3\ln(x) - 3c_1 - 1}{3x(\ln(x) - c_1)}$$

✓ Solution by Mathematica

Time used: 0.157 (sec). Leaf size: 15

```
DSolve[y'[x]+7/x*y[x]-3*y[x]^2==3/x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{x}$$
$$y(x) \to \frac{1}{x}$$

4.46 problem Problem 64

Internal problem ID [2710]
Internal file name [`OUTPUT/2202_Sunday_June_05_2022_02_54_26_AM_90493069/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.8, Change of Variables. page 79
Problem number: Problem 64.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[[_1st_order, `_with_symmetry_[F(x),G(x)*y+H(x)]`]]
```

$$\boxed{p(x)\ln(y) = -\frac{y'}{y} + q(x)}$$

4.46.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{y}\right)\mathrm{d}y &= (-p(x)\ln(y) + q(x))\,\mathrm{d}x\\ (p(x)\ln(y) - q(x))\,\mathrm{d}x + \left(\frac{1}{y}\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= p(x)\ln(y) - q(x)\\ N(x, y) &= \frac{1}{y}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(p(x)\ln(y) - q(x))\\ &= \frac{p(x)}{y}\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= y\left(\left(\frac{p(x)}{y}\right) - (0)\right) \\
&= p(x)
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int p(x) \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{\int p(x) dx} \\
&= \mathrm{e}^{\int p(x) dx}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{\int p(x) dx}(p(x) \ln (y) - q(x)) \\
&= (p(x) \ln (y) - q(x)) \, \mathrm{e}^{\int p(x) dx}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{\int p(x) dx}\left(\frac{1}{y}\right) \\
&= \frac{\mathrm{e}^{\int p(x) dx}}{y}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\left(p(x)\ln\left(y\right) - q(x)\right)\mathrm{e}^{\int p(x)dx}\right) + \left(\frac{\mathrm{e}^{\int p(x)dx}}{y}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int \left(p(x)\ln\left(y\right) - q(x)\right)\mathrm{e}^{\int p(x)dx}\,\mathrm{d}x \\
\phi &= \int^{x} \left(p(\_a)\ln\left(y\right) - q(\_a)\right)\mathrm{e}^{\int p(\_a)d\_a} d\_a + f(y)
\end{aligned} \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial\phi}{\partial y} = \frac{\mathrm{e}^{\int^{x} p(\_a)d\_a}}{y} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = \frac{\mathrm{e}^{\int p(x)dx}}{y}$. Therefore equation (4) becomes

$$
\frac{\mathrm{e}^{\int p(x)dx}}{y} = \frac{\mathrm{e}^{\int^{x} p(\_a)d\_a}}{y} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{-\mathrm{e}^{\int^{x} p(\_a)d\_a} + \mathrm{e}^{\int p(x)dx}}{y}
$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{-\mathrm{e}^{\int^x p(\_a)d\_a} + \mathrm{e}^{\int p(x)dx}}{y}\right) \mathrm{d}y$$

$$f(y) = \int_0^y \frac{-\mathrm{e}^{\int^x p(\_a)d\_a} + \mathrm{e}^{\int p(x)dx}}{\_a} d\_a + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \int^x (p(\_a)\ln(y) - q(\_a))\,\mathrm{e}^{\int p(\_a)d\_a} d\_a + \int_0^y \frac{-\mathrm{e}^{\int^x p(\_a)d\_a} + \mathrm{e}^{\int p(x)dx}}{\_a} d\_a + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \int^x (p(\_a)\ln(y) - q(\_a))\,\mathrm{e}^{\int p(\_a)d\_a} d\_a + \int_0^y \frac{-\mathrm{e}^{\int^x p(\_a)d\_a} + \mathrm{e}^{\int p(x)dx}}{\_a} d\_a$$

Summary

The solution(s) found are the following

$$\int^x (p(\_a)\ln(y) - q(\_a))\,\mathrm{e}^{\int p(\_a)d\_a} d\_a + \int_0^y \frac{-\mathrm{e}^{\int^x p(\_a)d\_a} + \mathrm{e}^{\int p(x)dx}}{\_a} d\_a = c_1 \quad (1)$$

Verification of solutions

$$\int^x (p(\_a)\ln(y) - q(\_a))\,\mathrm{e}^{\int p(\_a)d\_a} d\_a + \int_0^y \frac{-\mathrm{e}^{\int^x p(\_a)d\_a} + \mathrm{e}^{\int p(x)dx}}{\_a} d\_a = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.218 (sec). Leaf size: 27

```
dsolve(diff(y(x),x)/y(x)+p(x)*ln(y(x))=q(x),y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{\mathrm{e}^{-\left(\int p(x)dx\right)}\left(\int \mathrm{e}^{\int p(x)dx} q(x)dx - c_1\right)}$$

✓ Solution by Mathematica

Time used: 0.201 (sec). Leaf size: 109

```
DSolve[y'[x]/y[x]+p[x]*Log[y[x]]==q[x],y[x],x,IncludeSingularSolutions -> True]
```

$$\begin{aligned} \text{Solve}\Bigg[&\int_1^x \exp\left(-\int_1^{K[2]} -p(K[1])dK[1]\right)(\log(y(x))p(K[2]) - q(K[2]))dK[2] \\ &+ \int_1^{y(x)} \left(\frac{\exp\left(-\int_1^x -p(K[1])dK[1]\right)}{K[3]}\right. \\ &\left. - \int_1^x \frac{\exp\left(-\int_1^{K[2]} -p(K[1])dK[1]\right)p(K[2])}{K[3]}dK[2]\right)dK[3] = c_1, y(x)\Bigg] \end{aligned}$$

4.47 problem Problem 65

Internal problem ID [2711]
Internal file name [`OUTPUT/2203_Sunday_June_05_2022_02_54_32_AM_48815743/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.8, Change of Variables. page 79
Problem number: Problem 65.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _dAlembert]
```

$$\boxed{-\frac{2\ln(y)}{x} = -\frac{y'}{y} + \frac{1-2\ln(x)}{x}}$$

With initial conditions

$$[y(1) = \mathrm{e}]$$

4.47.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= -\frac{y(2\ln(x) - 2\ln(y) - 1)}{x} \end{aligned}$$

The x domain of $f(x,y)$ when $y = \mathrm{e}$ is

$$\{0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $f(x,y)$ when $x = 1$ is

$$\{0 < y\}$$

And the point $y_0 = \mathrm{e}$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{y(2\ln(x) - 2\ln(y) - 1)}{x}\right)\\ &= -\frac{2\ln(x) - 2\ln(y) - 1}{x} + \frac{2}{x}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = \mathrm{e}$ is

$$\{0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 1$ is

$$\{0 < y\}$$

And the point $y_0 = \mathrm{e}$ is inside this domain. Therefore solution exists and is unique.

4.47.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned}y' &= -\frac{y(2\ln(x) - 2\ln(y) - 1)}{x}\\ y' &= \omega(x,y)\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
& b_2-\frac{y(2\ln(x)-2\ln(y)-1)(b_3-a_2)}{x}-\frac{y^2(2\ln(x)-2\ln(y)-1)^2 a_3}{x^2} \\
& -\left(-\frac{2y}{x^2}+\frac{y(2\ln(x)-2\ln(y)-1)}{x^2}\right)(xa_2+ya_3+a_1) \\
& -\left(-\frac{2\ln(x)-2\ln(y)-1}{x}+\frac{2}{x}\right)(xb_2+yb_3+b_1)=0
\end{aligned}
\tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{4\ln(x)^2 y^2a_3-8\ln(x)\ln(y)\,y^2a_3+4\ln(y)^2 y^2a_3-2\ln(x)\,x^2b_2-2\ln(x)\,y^2a_3+2\ln(y)\,x^2b_2+2\ln(y)}{}
$$

$$
=0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
& -4\ln(x)^2 y^2a_3+8\ln(x)\ln(y)\,y^2a_3-4\ln(y)^2 y^2a_3+2\ln(x)\,x^2b_2 \\
& +2\ln(x)\,y^2a_3-2\ln(y)\,x^2b_2-2\ln(y)\,y^2a_3+2\ln(x)\,xb_1-2\ln(x)\,ya_1 \\
& -2\ln(y)\,xb_1+2\ln(y)\,ya_1-2b_2x^2+2xya_2-2xyb_3+2y^2a_3-3xb_1+3ya_1=0
\end{aligned}
\tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x,y\}$ in them.

$$
\{x,y,\ln(x),\ln(y)\}
$$

The following substitution is now made to be able to collect on all terms with $\{x,y\}$ in them

$$
\{x=v_1,y=v_2,\ln(x)=v_3,\ln(y)=v_4\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
& -4v_3^2v_2^2a_3+8v_3v_4v_2^2a_3-4v_4^2v_2^2a_3+2v_3v_2^2a_3-2v_4v_2^2a_3 \\
& +2v_3v_1^2b_2-2v_4v_1^2b_2-2v_3v_2a_1+2v_4v_2a_1+2v_1v_2a_2+2v_2^2a_3 \\
& +2v_3v_1b_1-2v_4v_1b_1-2b_2v_1^2-2v_1v_2b_3+3v_2a_1-3v_1b_1=0
\end{aligned}
\tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1,v_2,v_3,v_4\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&2v_3v_1^2b_2 - 2v_4v_1^2b_2 - 2b_2v_1^2 + (2a_2 - 2b_3)\, v_1v_2 + 2v_3v_1b_1 \\
&\quad - 2v_4v_1b_1 - 3v_1b_1 - 4v_3^2v_2^2a_3 + 8v_3v_4v_2^2a_3 + 2v_3v_2^2a_3 - 4v_4^2v_2^2a_3 \\
&\quad - 2v_4v_2^2a_3 + 2v_2^2a_3 - 2v_3v_2a_1 + 2v_4v_2a_1 + 3v_2a_1 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-2a_1 &= 0 \\
2a_1 &= 0 \\
3a_1 &= 0 \\
-4a_3 &= 0 \\
-2a_3 &= 0 \\
2a_3 &= 0 \\
8a_3 &= 0 \\
-3b_1 &= 0 \\
-2b_1 &= 0 \\
2b_1 &= 0 \\
-2b_2 &= 0 \\
2b_2 &= 0 \\
2a_2 - 2b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(-\frac{y(2\ln(x) - 2\ln(y) - 1)}{x}\right)(x) \\
&= 2\ln(x)\,y - 2\ln(y)\,y \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{2\ln(x)\,y - 2\ln(y)\,y} dy
\end{aligned}$$

Which results in

$$S = -\frac{\ln(\ln(x) - \ln(y))}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{y(2\ln(x) - 2\ln(y) - 1)}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{1}{2x(\ln(x) - \ln(y))} \\
S_y &= \frac{1}{2y(\ln(x) - \ln(y))}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{\ln(\ln(x) - \ln(y))}{2} = -\ln(x) + c_1$$

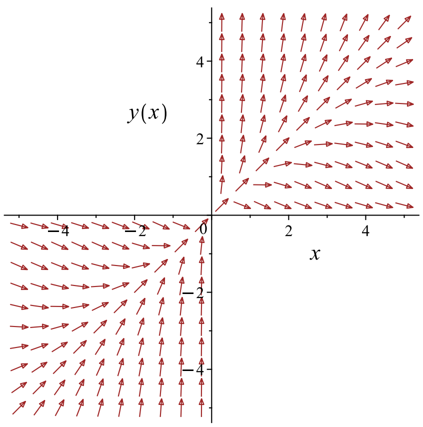
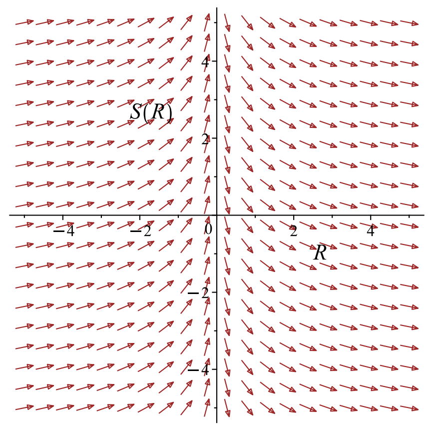
Which simplifies to

$$-\frac{\ln(\ln(x) - \ln(y))}{2} = -\ln(x) + c_1$$

Which gives

$$y = \mathrm{e}^{-\mathrm{e}^{-2c_1}x^2} x$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y(2\ln(x)-2\ln(y)-1)}{x}$ 	$R = x$ $S = -\frac{\ln(\ln(x)-\ln(y))}{2}$	$\frac{dS}{dR} = -\frac{1}{R}$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = \mathrm{e}$ in the above solution gives an equation to solve for the constant of integration.

$$\mathrm{e} = \mathrm{e}^{-\mathrm{e}^{-2c_1}}$$

$$c_1 = -\frac{i\pi}{2}$$

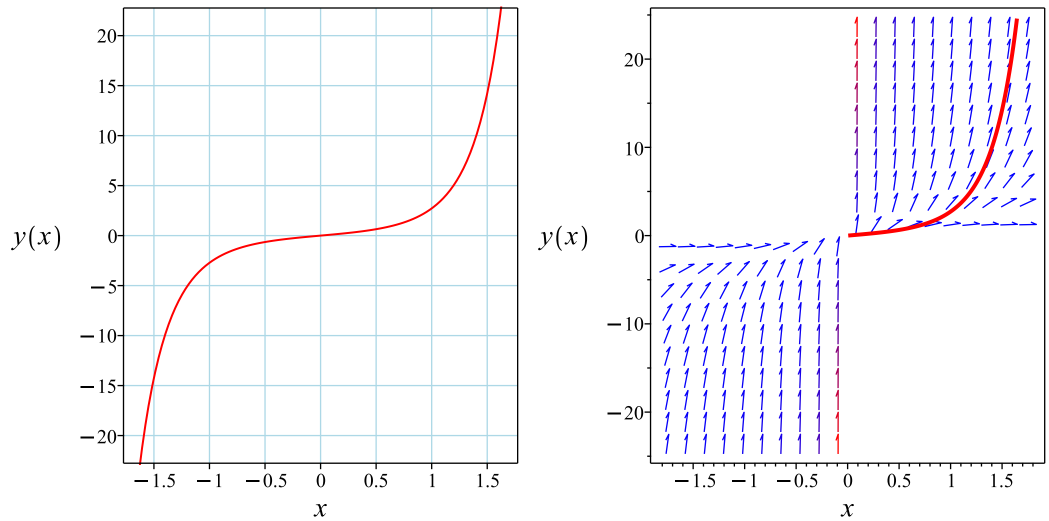
Substituting c_1 found above in the general solution gives

$$y = x\,\mathrm{e}^{x^2}$$

Summary

The solution(s) found are the following

$$y = x\,\mathrm{e}^{x^2} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = x\,\mathrm{e}^{x^2}$$

Verified OK.

4.47.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{y}\right)\mathrm{d}y &= \left(\frac{2\ln(y)}{x} + \frac{1 - 2\ln(x)}{x}\right)\mathrm{d}x\\ \left(-\frac{2\ln(y)}{x} - \frac{1 - 2\ln(x)}{x}\right)\mathrm{d}x + \left(\frac{1}{y}\right)\mathrm{d}y &= 0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -\frac{2\ln(y)}{x} - \frac{1 - 2\ln(x)}{x}\\ N(x, y) &= \frac{1}{y}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{2\ln(y)}{x} - \frac{1 - 2\ln(x)}{x}\right)\\ &= -\frac{2}{xy}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= y\left(\left(-\frac{2}{xy}\right) - (0)\right) \\
&= -\frac{2}{x}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int -\frac{2}{x} \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-2\ln(x)} \\
&= \frac{1}{x^2}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{x^2}\left(-\frac{2\ln(y)}{x} - \frac{1 - 2\ln(x)}{x}\right) \\
&= \frac{2\ln(x) - 2\ln(y) - 1}{x^3}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^2}\left(\frac{1}{y}\right) \\
&= \frac{1}{x^2 y}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{2\ln(x) - 2\ln(y) - 1}{x^3}\right) + \left(\frac{1}{x^2 y}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \frac{2\ln(x) - 2\ln(y) - 1}{x^3} \mathrm{d}x$$
$$\phi = \frac{-\ln(x) + \ln(y)}{x^2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{1}{x^2 y} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{x^2 y}$. Therefore equation (4) becomes

$$\frac{1}{x^2 y} = \frac{1}{x^2 y} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{-\ln(x) + \ln(y)}{x^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{-\ln(x) + \ln(y)}{x^2}$$

The solution becomes

$$y = e^{c_1 x^2} x$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = e$ in the above solution gives an equation to solve for the constant of integration.

$$e = e^{c_1}$$

$$c_1 = 1$$

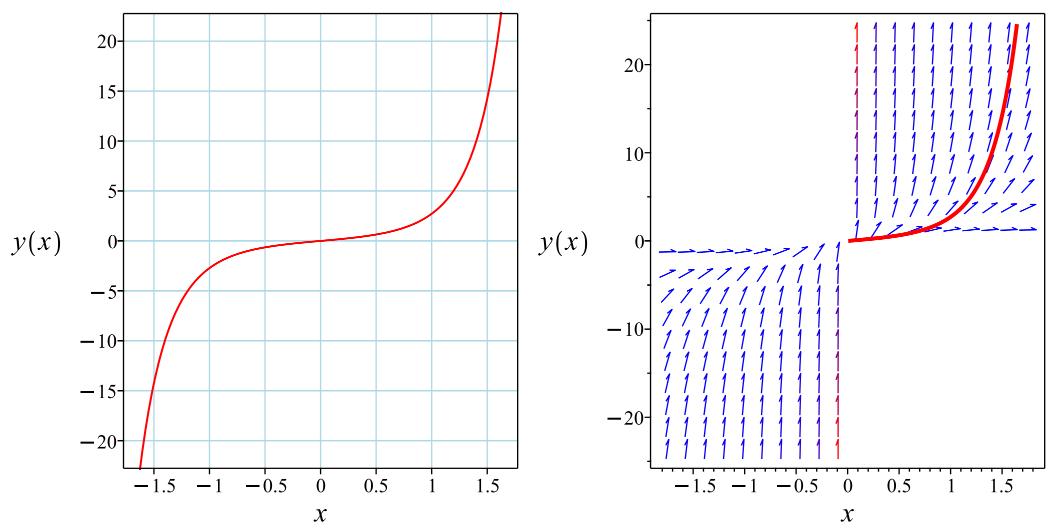
Substituting c_1 found above in the general solution gives

$$y = x\,e^{x^2}$$

Summary

The solution(s) found are the following

$$y = x\,e^{x^2} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = x\,e^{x^2}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 10

```
dsolve([diff(y(x),x)/y(x)-2/x*ln(y(x))=1/x*(1-2*ln(x)),y(1) = exp(1)],y(x), singsol=all)
```

$$y(x) = x\,\mathrm{e}^{x^2}$$

✓ Solution by Mathematica

Time used: 0.215 (sec). Leaf size: 12

```
DSolve[{y'[x]/y[x]-2/x*Log[y[x]]==1/x*(1-2*Log[x]),{y[1]==Exp[1]}},y[x],x,IncludeSingularSol
```

$$y(x) \to e^{x^2} x$$

4.48 problem Problem 67

Internal problem ID [2712]
Internal file name [`OUTPUT/2204_Sunday_June_05_2022_02_54_36_AM_5765779/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.8, Change of Variables. page 79
Problem number: Problem 67.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{\sec(y)^2 y' + \frac{\tan(y)}{2\sqrt{x+1}} = \frac{1}{2\sqrt{x+1}}}$$

4.48.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{\cos(y)^2 (\tan(y)-1)}{2\sqrt{x+1}} \end{aligned}$$

Where $f(x) = -\frac{1}{2\sqrt{x+1}}$ and $g(y) = \cos(y)^2 (\tan(y)-1)$. Integrating both sides gives

$$\frac{1}{\cos(y)^2 (\tan(y)-1)}\, dy = -\frac{1}{2\sqrt{x+1}}\, dx$$

$$\int \frac{1}{\cos(y)^2 (\tan(y)-1)} dy = \int -\frac{1}{2\sqrt{x+1}} dx$$

$$\ln(\tan(y)-1) = -\sqrt{x+1} + c_1$$

Raising both side to exponential gives

$$\tan(y) - 1 = e^{-\sqrt{x+1}+c_1}$$

Which simplifies to

$$\tan(y) - 1 = c_2 e^{-\sqrt{x+1}}$$

Summary

The solution(s) found are the following

$$y = \arctan\left(1 + c_2 e^{-\sqrt{x+1}+c_1}\right) \tag{1}$$

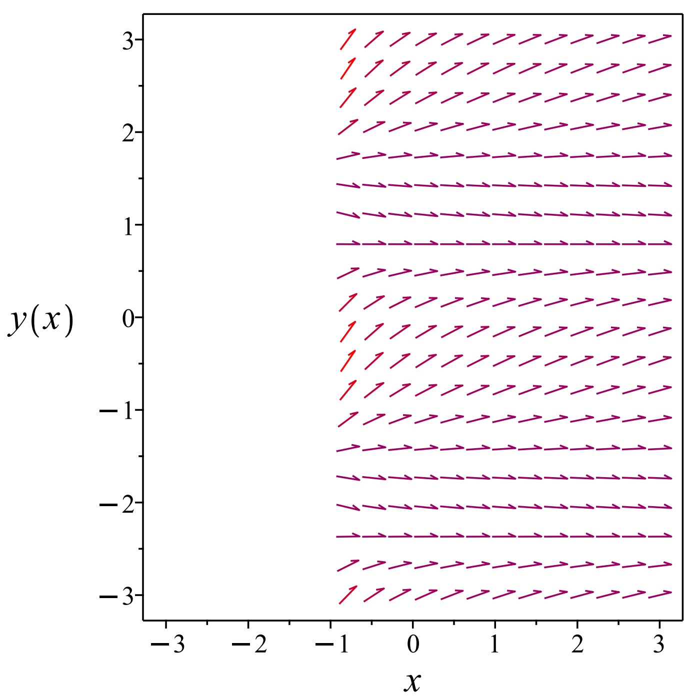


Figure 195: Slope field plot

Verification of solutions

$$y = \arctan\left(1 + c_2 e^{-\sqrt{x+1}+c_1}\right)$$

Verified OK.

4.48.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{\tan(y)-1}{2\sec(y)^2\sqrt{x+1}} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ,η

Table 159: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y'=f(x)y(x)+g(x)$	0	$e^{\int f dx}$
separable ode	$y'=f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y'=f(x)$	0	1
quadrature ode	$y'=g(y)$	1	0
homogeneous ODEs of Class A	$y'=f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y'=(a+bx+cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y'=\frac{y}{x}+g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y'=g(x)\,e^{h(x)+by}+f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y'=\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y'=f(x)\,y+g(x)\,y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y'=f_1(x)\,y+f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= -2\sqrt{x+1} \\
\eta(x,y) &= 0
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{-2\sqrt{x+1}} dx
\end{aligned}
$$

Which results in

$$
S = -\sqrt{x+1}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -\frac{\tan(y) - 1}{2\sec(y)^2 \sqrt{x+1}}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= -\frac{1}{2\sqrt{x+1}} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{\sec(y)^2}{\tan(y) - 1} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{\sec(R)^2}{\tan(R) - 1}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \ln(\tan(R) - 1) + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
-\sqrt{x+1} = \ln(\tan(y) - 1) + c_1
$$

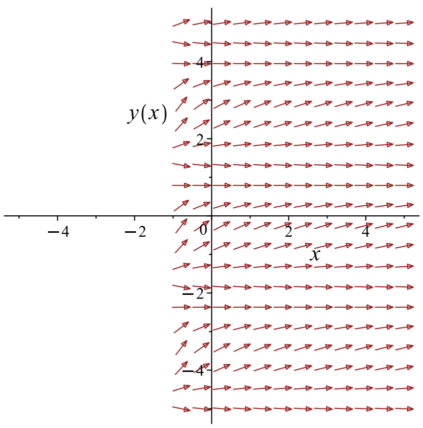
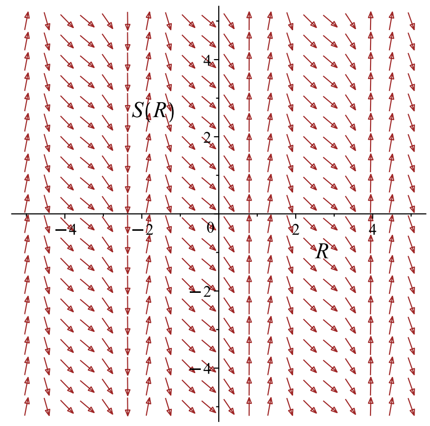
Which simplifies to

$$
-\sqrt{x+1} = \ln(\tan(y) - 1) + c_1
$$

Which gives

$$
y = \arctan\left(\mathrm{e}^{-\sqrt{x+1}-c_1} + 1\right)
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{\tan(y)-1}{2\sec(y)^2\sqrt{x+1}}$ 	$R = y$ $S = -\sqrt{x+1}$	$\frac{dS}{dR} = \frac{\sec(R)^2}{\tan(R)-1}$ 

Summary

The solution(s) found are the following

$$y = \arctan\left(\mathrm{e}^{-\sqrt{x+1}-c_1} + 1\right) \tag{1}$$

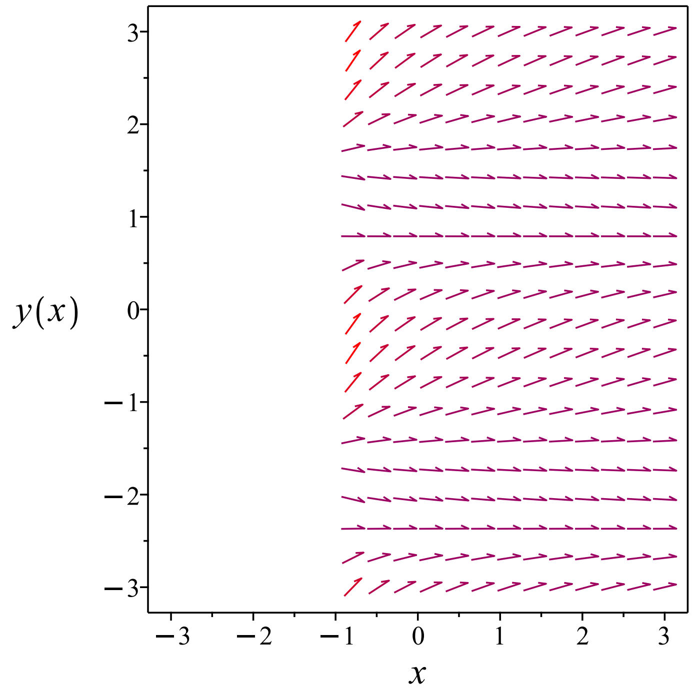


Figure 196: Slope field plot

Verification of solutions

$$y = \arctan\left(\mathrm{e}^{-\sqrt{x+1}-c_1} + 1\right)$$

Verified OK.

4.48.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(-\frac{2 \sec(y)^2}{\tan(y) - 1}\right) \mathrm{d}y &= \left(\frac{1}{\sqrt{x+1}}\right) \mathrm{d}x \\
\left(-\frac{1}{\sqrt{x+1}}\right) \mathrm{d}x + \left(-\frac{2 \sec(y)^2}{\tan(y) - 1}\right) \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -\frac{1}{\sqrt{x+1}} \\
N(x, y) &= -\frac{2 \sec(y)^2}{\tan(y) - 1}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{\sqrt{x+1}}\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{2\sec(y)^2}{\tan(y)-1}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{1}{\sqrt{x+1}}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -2\sqrt{x+1} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{2\sec(y)^2}{\tan(y)-1}$. Therefore equation (4) becomes

$$
-\frac{2\sec(y)^2}{\tan(y)-1} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -\frac{2\sec(y)^2}{\tan(y)-1}
$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\frac{2\sec(y)^2}{\tan(y)-1}\right)\mathrm{d}y$$

$$f(y) = -2\ln(\tan(y)-1) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -2\sqrt{x+1} - 2\ln(\tan(y)-1) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -2\sqrt{x+1} - 2\ln(\tan(y)-1)$$

Summary

The solution(s) found are the following

$$-2\sqrt{x+1} - 2\ln(\tan(y)-1) = c_1 \tag{1}$$

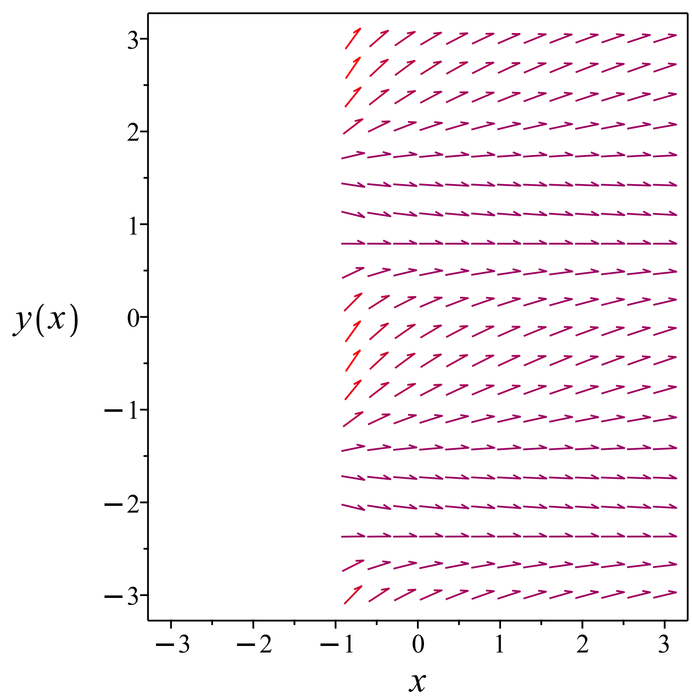


Figure 197: Slope field plot

Verification of solutions

$$-2\sqrt{x+1}-2\ln\left(\tan\left(y\right)-1\right)=c_1$$

Verified OK.

4.48.4 Maple step by step solution

Let's solve

$\sec\left(y\right)^2 y'+\frac{\tan(y)}{2\sqrt{x+1}}=\frac{1}{2\sqrt{x+1}}$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'\sec(y)^2}{\tan(y)-1}=-\frac{1}{2\sqrt{x+1}}$

- Integrate both sides with respect to x

  $\int\frac{y'\sec(y)^2}{\tan(y)-1}dx=\int-\frac{1}{2\sqrt{x+1}}dx+c_1$

- Evaluate integral

  $\ln\left(\tan\left(y\right)-1\right)=-\sqrt{x+1}+c_1$

- Solve for y

  $y=\arctan\left(\mathrm{e}^{-\sqrt{x+1}+c_1}+1\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 17

```
dsolve(sec(y(x))^2*diff(y(x),x)+1/(2*sqrt(1+x))*tan(y(x))=1/(2*sqrt(1+x)),y(x), singsol=all)
```

$$y(x) = \arctan\left(e^{-\sqrt{x+1}}c_1 + 1\right)$$

✓ Solution by Mathematica

Time used: 60.288 (sec). Leaf size: 247

```
DSolve[Sec[y[x]]^2*y'[x]+1/(2*Sqrt[1+x])*Tan[y[x]]==1/(2*Sqrt[1+x]),y[x],x,IncludeSingularSo
```

$$y(x) \to -\arccos\left(-\frac{e^{\sqrt{x+1}+2c_1}}{\sqrt{-2e^{\sqrt{x+1}+2c_1} + 2e^{2\sqrt{x+1}+4c_1} + 1}}\right)$$

$$y(x) \to \arccos\left(-\frac{e^{\sqrt{x+1}+2c_1}}{\sqrt{-2e^{\sqrt{x+1}+2c_1} + 2e^{2\sqrt{x+1}+4c_1} + 1}}\right)$$

$$y(x) \to -\arccos\left(\frac{e^{\sqrt{x+1}+2c_1}}{\sqrt{-2e^{\sqrt{x+1}+2c_1} + 2e^{2\sqrt{x+1}+4c_1} + 1}}\right)$$

$$y(x) \to \arccos\left(\frac{e^{\sqrt{x+1}+2c_1}}{\sqrt{-2e^{\sqrt{x+1}+2c_1} + 2e^{2\sqrt{x+1}+4c_1} + 1}}\right)$$

5 Chapter 1, First-Order Differential Equations. Section 1.9, Exact Differential Equations. page 91

5.1 problem Problem 1

Internal problem ID [2713]
Internal file name [`OUTPUT/2205_Sunday_June_05_2022_02_54_42_AM_86103940/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.9, Exact Differential Equations. page 91
Problem number: Problem 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`` [`x=_G(y,y')`] ``

Unable to solve or complete the solution.

$$\boxed{y\,\mathrm{e}^{yx}+(2y-x\,\mathrm{e}^{yx})\,y'=0}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 5
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type`
```

✗ Solution by Maple

```
dsolve(y(x)*exp(x*y(x))+(2*y(x)-x*exp(x*y(x)))*diff(y(x),x)=0,y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*Exp[x*y[x]]+(2*y[x]-x*Exp[x*y[x]])*y'[x]==0,y[x],x,IncludeSingularSolutions -> T
```

Not solved

5.2 problem Problem 2

Internal problem ID [2714]
Internal file name [`OUTPUT/2206_Sunday_June_05_2022_02_54_44_AM_7544482/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.9, Exact Differential Equations. page 91
Problem number: Problem 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _exact]
```

$$\boxed{\cos(yx) - xy\sin(yx) - x^2\sin(yx)\,y' = 0}$$

5.2.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-x^2\sin(xy)\right)\mathrm{d}y &= \left(-\cos(xy) + xy\sin(xy)\right)\mathrm{d}x\\ \left(-xy\sin(xy) + \cos(xy)\right)\mathrm{d}x + \left(-x^2\sin(xy)\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -xy\sin(xy) + \cos(xy)\\ N(x,y) &= -x^2\sin(xy)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-xy\sin(xy) + \cos(xy)\right)\\ &= -x(y\cos(xy)\,x + 2\sin(xy))\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-x^2\sin(xy)\right)\\ &= -x(y\cos(xy)\,x + 2\sin(xy))\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int M\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int -xy\sin(xy) + \cos(xy)\,\mathrm{d}x$$
$$\phi = x\cos(xy) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -x^2\sin(xy) + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -x^2\sin(xy)$. Therefore equation (4) becomes

$$-x^2\sin(xy) = -x^2\sin(xy) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = x\cos(xy) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x\cos(xy)$$

Summary

The solution(s) found are the following

$$x \cos(yx) = c_1 \tag{1}$$

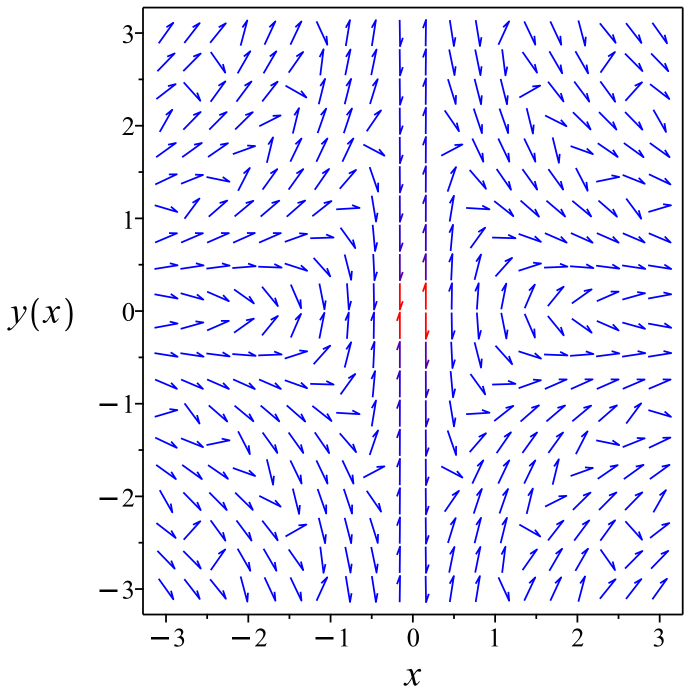


Figure 198: Slope field plot

Verification of solutions

$$x \cos(yx) = c_1$$

Verified OK.

5.2.2 Maple step by step solution

Let's solve

$\cos(yx) - xy\sin(yx) - x^2\sin(yx)\, y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  ◦ ODE is exact if the lhs is the total derivative of a C^2 function

$F'(x,y) = 0$

- Compute derivative of lhs

  $F'(x,y) + \left(\frac{\partial}{\partial y}F(x,y)\right)y' = 0$

- Evaluate derivatives

  $-2x\sin(xy) - x^2y\cos(xy) = -2x\sin(xy) - x^2y\cos(xy)$

- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y) = c_1, M(x,y) = F'(x,y), N(x,y) = \frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y) = \int(-xy\sin(xy) + \cos(xy))\,dx + f_1(y)$

- Evaluate integral

  $F(x,y) = -\frac{\sin(xy)-y\cos(xy)x}{y} + \frac{\sin(xy)}{y} + f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y) = \frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $-x^2\sin(xy) = \frac{\sin(xy)-y\cos(xy)x}{y^2} - x^2\sin(xy) - \frac{\sin(xy)}{y^2} + \frac{x\cos(xy)}{y} + \frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y) = -\frac{\sin(xy)-y\cos(xy)x}{y^2} + \frac{\sin(xy)}{y^2} - \frac{x\cos(xy)}{y}$

- Solve for $f_1(y)$

  $f_1(y) = 0$

- Substitute $f_1(y)$ into equation for $F(x,y)$

  $F(x,y) = -\frac{\sin(xy)-y\cos(xy)x}{y} + \frac{\sin(xy)}{y}$

- Substitute $F(x,y)$ into the solution of the ODE

  $-\frac{\sin(xy)-y\cos(xy)x}{y} + \frac{\sin(xy)}{y} = c_1$

- Solve for y

  $y = \frac{\arccos\left(\frac{c_1}{x}\right)}{x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 14

```
dsolve((cos(x*y(x))-x*y(x)*sin(x*y(x)))-x^2*sin(x*y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{\arccos\left(\frac{c_1}{x}\right)}{x}$$

✓ Solution by Mathematica

Time used: 5.673 (sec). Leaf size: 34

```
DSolve[(Cos[x*y[x]]-x*y[x]*Sin[x*y[x]])-x^2*Sin[x*y[x]]*y'[x]==0,y[x],x,IncludeSingularSolut
```

$$y(x) \to -\frac{\arccos\left(-\frac{c_1}{x}\right)}{x}$$
$$y(x) \to \frac{\arccos\left(-\frac{c_1}{x}\right)}{x}$$

5.3 problem Problem 3

Internal problem ID [2715]
Internal file name [`OUTPUT/2207_Sunday_June_05_2022_02_54_48_AM_75051852/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.9, Exact Differential Equations. page 91
Problem number: Problem 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y + xy' = -3x^2}$$

5.3.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
(x)\,\mathrm{d}y &= \left(-3x^2 - y\right)\mathrm{d}x \\
\left(3x^2 + y\right)\mathrm{d}x + (x)\,\mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= 3x^2 + y \\
N(x, y) &= x
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(3x^2 + y\right) \\
&= 1
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x) \\
&= 1
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int M \,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int 3x^2 + y \,\mathrm{d}x$$
$$\phi = x(x^2 + y) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x$. Therefore equation (4) becomes

$$x = x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = x(x^2 + y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x(x^2 + y)$$

The solution becomes

$$y = \frac{-x^3 + c_1}{x}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{-x^3 + c_1}{x} \tag{1}$$

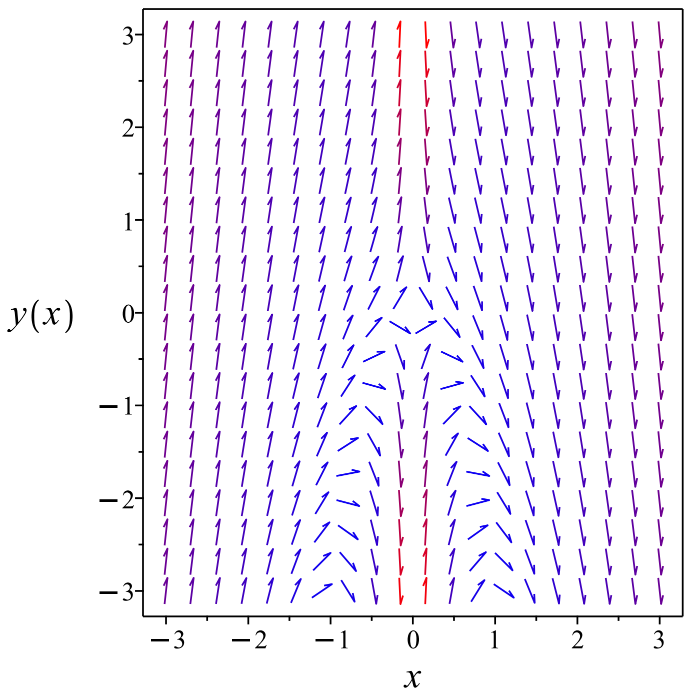


Figure 199: Slope field plot

<u>Verification of solutions</u>

$$y = \frac{-x^3 + c_1}{x}$$

Verified OK.

5.3.2 Maple step by step solution

Let's solve

$y + xy' = -3x^2$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{y}{x} - 3x$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{y}{x} = -3x$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{y}{x}\right) = -3\mu(x)\, x$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' + \frac{y}{x}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = x$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int -3\mu(x)\, x dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int -3\mu(x)\, x dx + c_1$

- Solve for y

  $y = \frac{\int -3\mu(x) x dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = x$

  $y = \frac{\int -3x^2 dx + c_1}{x}$

- Evaluate the integrals on the rhs

  $y = \frac{-x^3 + c_1}{x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve((y(x)+3*x^2)+x*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{-x^3 + c_1}{x}$$

✓ Solution by Mathematica

Time used: 0.028 (sec). Leaf size: 17

```
DSolve[(y[x]+3*x^2)+x*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{-x^3 + c_1}{x}$$

5.4 problem Problem 4

Internal problem ID [2716]
Internal file name [`OUTPUT/2208_Sunday_June_05_2022_02_54_50_AM_5743854/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.9, Exact Differential Equations. page 91
Problem number: Problem 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[_exact, [_1st_order, `_with_symmetry_[F(x)*G(y),0]`]]
```

$$\boxed{2x\,\mathrm{e}^{y}+\left(3y^{2}+x^{2}\mathrm{e}^{y}\right)y'=0}$$

5.4.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\,\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(3y^2 + x^2 \mathrm{e}^y\right) \mathrm{d}y &= \left(-2x\,\mathrm{e}^y\right) \mathrm{d}x\\ \left(2x\,\mathrm{e}^y\right) \mathrm{d}x + \left(3y^2 + x^2 \mathrm{e}^y\right) \mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= 2x\,\mathrm{e}^y\\ N(x,y) &= 3y^2 + x^2 \mathrm{e}^y\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(2x\,\mathrm{e}^y\right)\\ &= 2x\,\mathrm{e}^y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(3y^2 + x^2 \mathrm{e}^y\right)\\ &= 2x\,\mathrm{e}^y\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int 2x \, \mathrm{e}^y \, \mathrm{d}x$$

$$\phi = x^2 \mathrm{e}^y + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x^2 \mathrm{e}^y + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 3y^2 + x^2 \mathrm{e}^y$. Therefore equation (4) becomes

$$3y^2 + x^2 \mathrm{e}^y = x^2 \mathrm{e}^y + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 3y^2$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (3y^2) \, \mathrm{d}y$$

$$f(y) = y^3 + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = x^2 \mathrm{e}^y + y^3 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x^2 e^y + y^3$$

Summary

The solution(s) found are the following

$$x^2 e^y + y^3 = c_1 \tag{1}$$

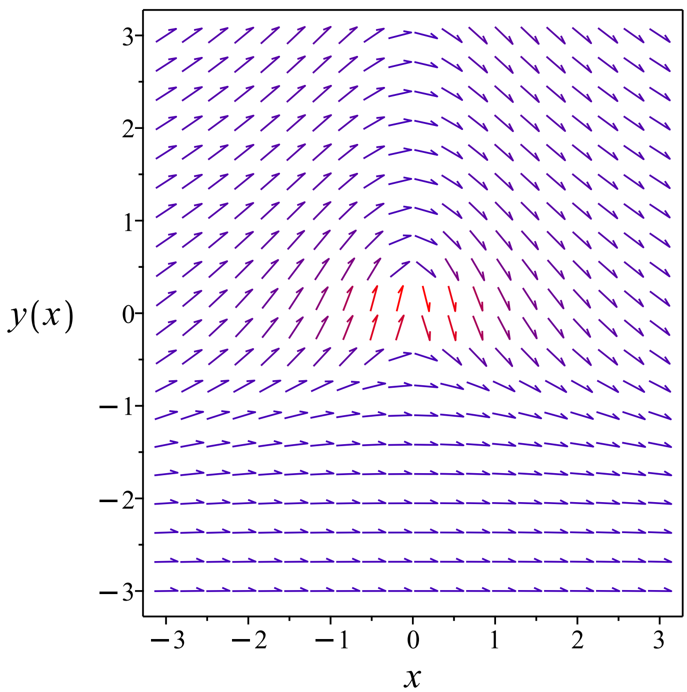


Figure 200: Slope field plot

Verification of solutions

$$x^2 e^y + y^3 = c_1$$

Verified OK.

5.4.2 Maple step by step solution

Let's solve

$2x\,e^{y}+\left(3y^{2}+x^{2}e^{y}\right)y'=0$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x,y)=0$

  - Compute derivative of lhs

    $F'(x,y)+\left(\frac{\partial}{\partial y}F(x,y)\right)y'=0$

  - Evaluate derivatives

    $2x\,e^{y}=2x\,e^{y}$

  - Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y)=c_1, M(x,y)=F'(x,y), N(x,y)=\frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y)=\int 2x\,e^{y}dx+f_1(y)$

- Evaluate integral

  $F(x,y)=x^{2}e^{y}+f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y)=\frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $3y^{2}+x^{2}e^{y}=x^{2}e^{y}+\frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y)=3y^{2}$

- Solve for $f_1(y)$

  $f_1(y)=y^{3}$

- Substitute $f_1(y)$ into equation for $F(x,y)$

$$F(x,y) = x^2 e^y + y^3$$

- Substitute $F(x,y)$ into the solution of the ODE

  $x^2 e^y + y^3 = c_1$

- Solve for y

  $y = RootOf\left(-e^{\_Z} x^2 - \_Z^3 + c_1\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(2*x*exp(y(x))+(3*y(x)^2+x^2*exp(y(x)))*diff(y(x),x)=0,y(x), singsol=all)
```

$$x^2 e^{y(x)} + y(x)^3 + c_1 = 0$$

✓ Solution by Mathematica

Time used: 0.258 (sec). Leaf size: 19

```
DSolve[2*x*Exp[y[x]]+(3*y[x]^2+x^2*Exp[y[x]])*y'[x]==0,y[x],x,IncludeSingularSolutions -> Tr
```

$$\text{Solve}\left[x^2 e^{y(x)} + y(x)^3 = c_1, y(x)\right]$$

5.5 problem Problem 5

Internal problem ID [2717]
Internal file name [`OUTPUT/2209_Sunday_June_05_2022_02_54_54_AM_50811869/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.9, Exact Differential Equations. page 91
Problem number: Problem 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{2yx + (x^2 + 1)\, y' = 0}$$

5.5.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(-\frac{1}{2y}\right)\mathrm{d}y = \left(\frac{x}{x^2+1}\right)\mathrm{d}x$$
$$\left(-\frac{x}{x^2+1}\right)\mathrm{d}x + \left(-\frac{1}{2y}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = -\frac{x}{x^2+1}$$
$$N(x,y) = -\frac{1}{2y}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x}{x^2+1}\right)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{1}{2y}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int M\, \mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int -\frac{x}{x^2+1}\, \mathrm{d}x\end{aligned}$$

$$\phi = -\frac{\ln(x^2+1)}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{2y}$. Therefore equation (4) becomes

$$-\frac{1}{2y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{2y}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\frac{1}{2y}\right)\mathrm{d}y$$
$$f(y) = -\frac{\ln(y)}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\ln(x^2+1)}{2} - \frac{\ln(y)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\ln(x^2+1)}{2} - \frac{\ln(y)}{2}$$

The solution becomes

$$y = \frac{\mathrm{e}^{-2c_1}}{x^2+1}$$

Summary

The solution(s) found are the following

$$y = \frac{\mathrm{e}^{-2c_1}}{x^2+1} \tag{1}$$

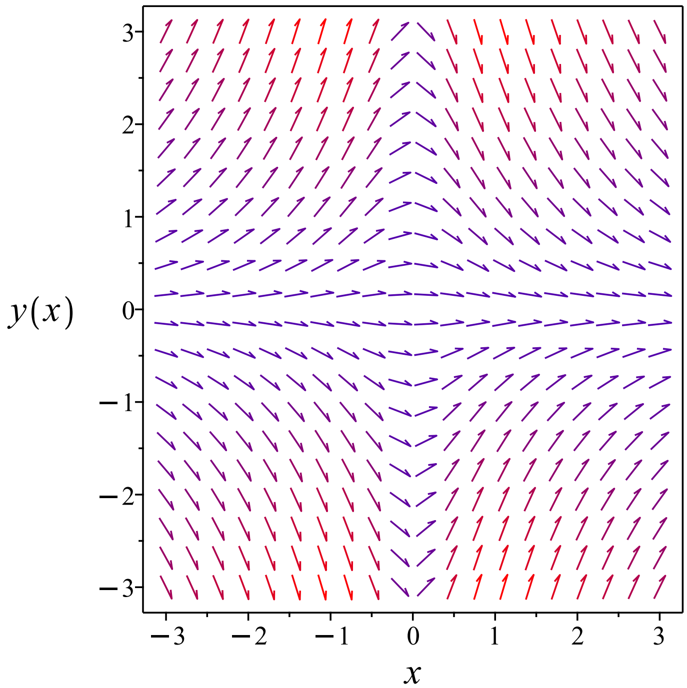


Figure 201: Slope field plot

Verification of solutions

$$y = \frac{e^{-2c_1}}{x^2 + 1}$$

Verified OK.

5.5.2 Maple step by step solution

Let's solve

$2yx + (x^2 + 1)\, y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int (2yx + (x^2 + 1)\, y')\, dx = \int 0 dx + c_1$

- Evaluate integral

  $(x^2 + 1)\, y = c_1$

- Solve for y

$$y = \frac{c_1}{x^2+1}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve(2*x*y(x)+(x^2+1)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1}{x^2+1}$$

✓ Solution by Mathematica

Time used: 0.028 (sec). Leaf size: 20

```
DSolve[2*x*y[x]+(x^2+1)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_1}{x^2+1}$$
$$y(x) \to 0$$

5.6 problem Problem 6

Internal problem ID [2718]
Internal file name [`OUTPUT/2210_Sunday_June_05_2022_02_54_55_AM_25370470/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.9, Exact Differential Equations. page 91
Problem number: Problem 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _exact, _rational, _Bernoulli]
```

$$\boxed{y^2 + 2xyy' = 2x}$$

5.6.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
(2xy)\,\mathrm{d}y &= \left(-y^2 + 2x\right)\mathrm{d}x \\
\left(y^2 - 2x\right)\mathrm{d}x + (2xy)\,\mathrm{d}y &= 0 \qquad\qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= y^2 - 2x \\
N(x,y) &= 2xy
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(y^2 - 2x\right) \\
&= 2y
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(2xy) \\
&= 2y
\end{aligned}
$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\begin{align*}
\frac{\partial \phi}{\partial x} &= M \tag{1} \\
\frac{\partial \phi}{\partial y} &= N \tag{2}
\end{align*}
$$

Integrating (1) w.r.t. x gives

$$
\begin{align*}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int M \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int y^2 - 2x \mathrm{d}x \\
\phi &= x\,y^2 - x^2 + f(y) \tag{3}
\end{align*}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 2xy + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 2xy$. Therefore equation (4) becomes

$$
2xy = 2xy + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$
f(y) = c_1
$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$
\phi = x\,y^2 - x^2 + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
c_1 = x\,y^2 - x^2
$$

Summary

The solution(s) found are the following

$$y^2x - x^2 = c_1 \tag{1}$$

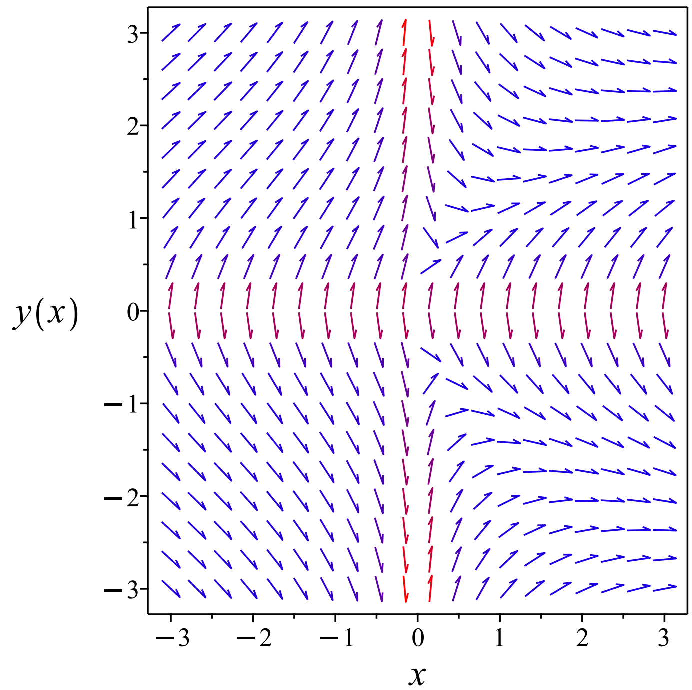


Figure 202: Slope field plot

Verification of solutions

$$y^2x - x^2 = c_1$$

Verified OK.

5.6.2 Maple step by step solution

Let's solve

$y^2 + 2xyy' = 2x$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  ◦ ODE is exact if the lhs is the total derivative of a C^2 function

$F'(x,y)=0$

- Compute derivative of lhs

  $F'(x,y)+\left(\frac{\partial}{\partial y}F(x,y)\right)y'=0$

- Evaluate derivatives

  $2y=2y$

- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y)=c_1, M(x,y)=F'(x,y), N(x,y)=\frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y)=\int(y^2-2x)\,dx+f_1(y)$

- Evaluate integral

  $F(x,y)=x\,y^2-x^2+f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y)=\frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $2xy=2xy+\frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y)=0$

- Solve for $f_1(y)$

  $f_1(y)=0$

- Substitute $f_1(y)$ into equation for $F(x,y)$

  $F(x,y)=x\,y^2-x^2$

- Substitute $F(x,y)$ into the solution of the ODE

  $x\,y^2-x^2=c_1$

- Solve for y

  $\left\{y=\frac{\sqrt{x(x^2+c_1)}}{x}, y=-\frac{\sqrt{x(x^2+c_1)}}{x}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
dsolve((y(x)^2-2*x)+2*x*y(x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{\sqrt{x\,(x^2 + c_1)}}{x}$$
$$y(x) = -\frac{\sqrt{x\,(x^2 + c_1)}}{x}$$

✓ Solution by Mathematica

Time used: 0.207 (sec). Leaf size: 42

```
DSolve[(y[x]^2-2*x)+2*x*y[x]*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sqrt{x^2 + c_1}}{\sqrt{x}}$$
$$y(x) \to \frac{\sqrt{x^2 + c_1}}{\sqrt{x}}$$

5.7 problem Problem 7

Internal problem ID [2719]
Internal file name [`OUTPUT/2211_Sunday_June_05_2022_02_54_58_AM_62966589/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.9, Exact Differential Equations. page 91
Problem number: Problem 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[_exact, [_1st_order, `_with_symmetry_[F(x),G(x)]`]]
```

$$\boxed{2yx - y^2 + (-y + x)^2 y' = -4\,\mathrm{e}^{2x}}$$

5.7.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M\\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left((-y+x)^2\right)\mathrm{d}y &= \left(-4\,\mathrm{e}^{2x} - 2xy + y^2\right)\mathrm{d}x\\
\left(2xy - y^2 + 4\,\mathrm{e}^{2x}\right)\mathrm{d}x + \left((-y+x)^2\right)\mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= 2xy - y^2 + 4\,\mathrm{e}^{2x}\\
N(x, y) &= (-y+x)^2
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(2xy - y^2 + 4\,\mathrm{e}^{2x}\right)\\
&= 2x - 2y
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left((-y+x)^2\right)\\
&= 2x - 2y
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int 2xy - y^2 + 4\,\mathrm{e}^{2x} \, \mathrm{d}x$$
$$\phi = x^2 y - x\,y^2 + 2\,\mathrm{e}^{2x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x^2 - 2xy + f'(y) \tag{4}$$
$$= x(x - 2y) + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = (-y + x)^2$. Therefore equation (4) becomes

$$(-y + x)^2 = x(x - 2y) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = y^2$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (y^2) \, \mathrm{d}y$$
$$f(y) = \frac{y^3}{3} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = x^2y - x\,y^2 + 2\,\mathrm{e}^{2x} + \frac{y^3}{3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x^2y - x\,y^2 + 2\,\mathrm{e}^{2x} + \frac{y^3}{3}$$

Summary

The solution(s) found are the following

$$x^2y - y^2x + 2\,\mathrm{e}^{2x} + \frac{y^3}{3} = c_1 \tag{1}$$

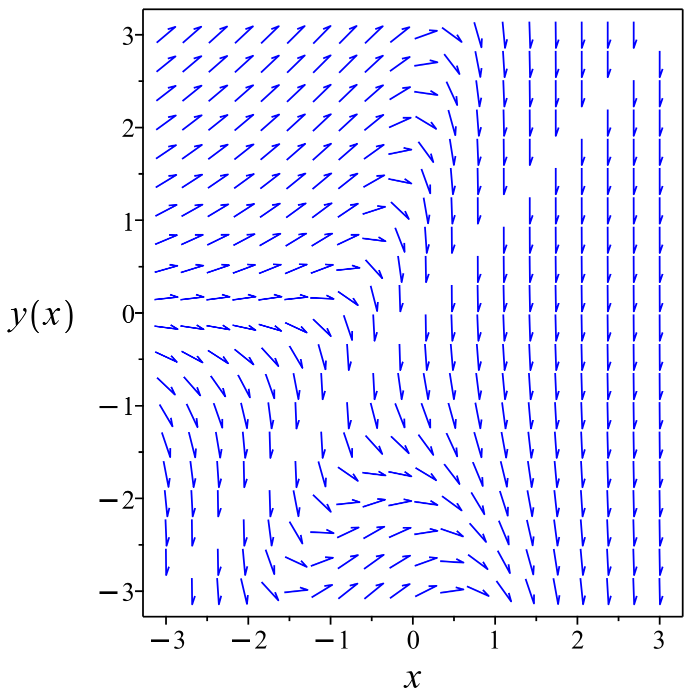


Figure 203: Slope field plot

Verification of solutions

$$x^2y - y^2x + 2\,\mathrm{e}^{2x} + \frac{y^3}{3} = c_1$$

Verified OK.

5.7.2 Maple step by step solution

Let's solve

$2yx - y^2 + (-y+x)^2 y' = -4\,e^{2x}$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x,y) = 0$

  - Compute derivative of lhs

    $F'(x,y) + \left(\frac{\partial}{\partial y}F(x,y)\right) y' = 0$

  - Evaluate derivatives

    $2x - 2y = 2x - 2y$

  - Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y) = c_1, M(x,y) = F'(x,y), N(x,y) = \frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y) = \int (2xy - y^2 + 4\,e^{2x})\, dx + f_1(y)$

- Evaluate integral

  $F(x,y) = x^2 y - x\,y^2 + 2\,e^{2x} + f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y) = \frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $(-y+x)^2 = x^2 - 2xy + \frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y) = (-y+x)^2 - x^2 + 2xy$

- Solve for $f_1(y)$

  $f_1(y) = \frac{y^3}{3}$

- Substitute $f_1(y)$ into equation for $F(x,y)$

$F(x,y) = x^2y - x\,y^2 + 2\,\mathrm{e}^{2x} + \frac{y^3}{3}$

- Substitute $F(x,y)$ into the solution of the ODE

  $x^2y - x\,y^2 + 2\,\mathrm{e}^{2x} + \frac{y^3}{3} = c_1$

- Solve for y

  $\left\{y = \left(-x^3 - 6\,\mathrm{e}^{2x} + 3c_1\right)^{\frac{1}{3}} + x, y = -\frac{\left(-x^3-6\,\mathrm{e}^{2x}+3c_1\right)^{\frac{1}{3}}}{2} - \frac{\mathrm{I}\sqrt{3}\,\left(-x^3-6\,\mathrm{e}^{2x}+3c_1\right)^{\frac{1}{3}}}{2} + x, y = -\frac{\left(-x^3-6\,\mathrm{e}^{2x}+\right.}{2}\right.$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 117

```
dsolve((4*exp(2*x)+2*x*y(x)-y(x)^2)+(x-y(x))^2*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \left(-x^3 - 6\,\mathrm{e}^{2x} - 3c_1\right)^{\frac{1}{3}} + x$$
$$y(x) = -\frac{\left(-x^3 - 6\,\mathrm{e}^{2x} - 3c_1\right)^{\frac{1}{3}}}{2} - \frac{i\sqrt{3}\,\left(-x^3 - 6\,\mathrm{e}^{2x} - 3c_1\right)^{\frac{1}{3}}}{2} + x$$
$$y(x) = -\frac{\left(-x^3 - 6\,\mathrm{e}^{2x} - 3c_1\right)^{\frac{1}{3}}}{2} + \frac{i\sqrt{3}\,\left(-x^3 - 6\,\mathrm{e}^{2x} - 3c_1\right)^{\frac{1}{3}}}{2} + x$$

✓ Solution by Mathematica

Time used: 1.472 (sec). Leaf size: 112

```
DSolve[(4*Exp[2*x]+2*x*y[x]-y[x]^2)+(x-y[x])^2*y'[x]==0,y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to x + \sqrt[3]{-x^3 - 6e^{2x} + 3c_1}$$
$$y(x) \to x + \frac{1}{2}i\left(\sqrt{3} + i\right)\sqrt[3]{-x^3 - 6e^{2x} + 3c_1}$$
$$y(x) \to x - \frac{1}{2}\left(1 + i\sqrt{3}\right)\sqrt[3]{-x^3 - 6e^{2x} + 3c_1}$$

5.8 problem Problem 8

Internal problem ID [2720]
Internal file name [`OUTPUT/2212_Sunday_June_05_2022_02_55_00_AM_89385217/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.9, Exact Differential Equations. page 91
Problem number: Problem 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _exact, _rational, _Riccati]
```

$$\boxed{-\frac{y}{y^2+x^2}+\frac{xy'}{y^2+x^2}=-\frac{1}{x}}$$

5.8.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\,\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(\frac{x}{x^2 + y^2}\right) \mathrm{d}y &= \left(-\frac{1}{x} + \frac{y}{x^2 + y^2}\right) \mathrm{d}x \\
\left(\frac{1}{x} - \frac{y}{x^2 + y^2}\right) \mathrm{d}x + \left(\frac{x}{x^2 + y^2}\right) \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= \frac{1}{x} - \frac{y}{x^2 + y^2} \\
N(x, y) &= \frac{x}{x^2 + y^2}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{1}{x} - \frac{y}{x^2 + y^2}\right) \\
&= \frac{-x^2 + y^2}{\left(x^2 + y^2\right)^2}
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) \\
&= \frac{-x^2+y^2}{\left(x^2+y^2\right)^2}
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \qquad (1) \\
\frac{\partial \phi}{\partial y} &= N \qquad (2)
\end{aligned}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int M \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int \frac{1}{x} - \frac{y}{x^2+y^2} \mathrm{d}x \\
\phi &= \ln(x) - \arctan\left(\frac{x}{y}\right) + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\begin{aligned}
\frac{\partial \phi}{\partial y} &= \frac{x}{y^2\left(\frac{x^2}{y^2}+1\right)} + f'(y) \qquad (4) \\
&= \frac{x}{x^2+y^2} + f'(y)
\end{aligned}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{x}{x^2+y^2}$. Therefore equation (4) becomes

$$
\frac{x}{x^2+y^2} = \frac{x}{x^2+y^2} + f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \ln(x) - \arctan\left(\frac{x}{y}\right) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \ln(x) - \arctan\left(\frac{x}{y}\right)$$

The solution becomes

$$y = -\frac{x}{\tan(-\ln(x) + c_1)}$$

Summary

The solution(s) found are the following

$$y = -\frac{x}{\tan(-\ln(x) + c_1)} \tag{1}$$

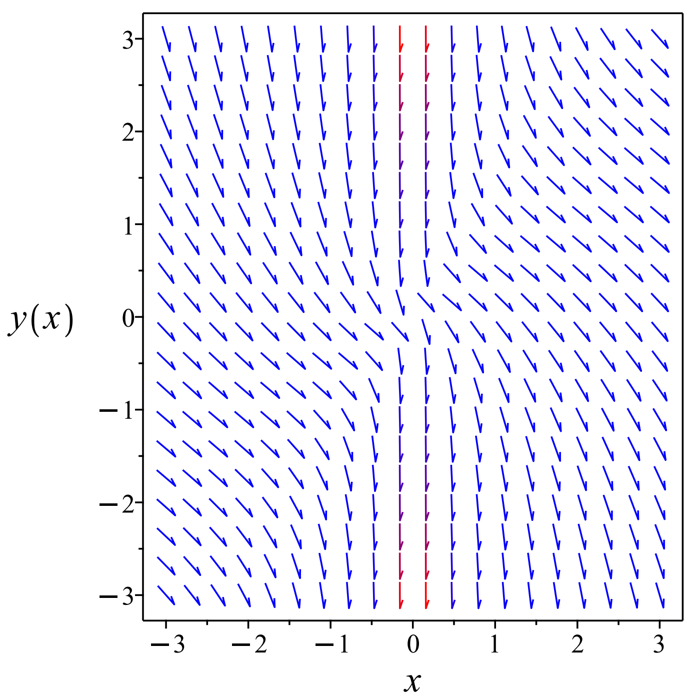


Figure 204: Slope field plot

Verification of solutions

$$y = -\frac{x}{\tan(-\ln(x) + c_1)}$$

Verified OK.

5.8.2 Maple step by step solution

Let's solve

$-\frac{y}{y^2+x^2} + \frac{xy'}{y^2+x^2} = -\frac{1}{x}$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact
  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x, y) = 0$
  - Compute derivative of lhs

    $F'(x, y) + \left(\frac{\partial}{\partial y} F(x, y)\right) y' = 0$
  - Evaluate derivatives

    $-\frac{1}{x^2+y^2} + \frac{2y^2}{(x^2+y^2)^2} = -\frac{2x^2}{(x^2+y^2)^2} + \frac{1}{x^2+y^2}$
  - Simplify

    $\frac{-x^2+y^2}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}$
  - Condition met, ODE is exact
- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y), N(x, y) = \frac{\partial}{\partial y} F(x, y)\right]$
- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int \left(\frac{1}{x} - \frac{y}{x^2+y^2}\right) dx + f_1(y)$
- Evaluate integral

  $F(x, y) = \ln(x) - \arctan\left(\frac{x}{y}\right) + f_1(y)$
- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y} F(x, y)$

- Compute derivative

  $\frac{x}{x^2+y^2} = \frac{x}{y^2\left(\frac{x^2}{y^2}+1\right)} + \frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y) = \frac{x}{x^2+y^2} - \frac{x}{y^2\left(\frac{x^2}{y^2}+1\right)}$

- Solve for $f_1(y)$

  $f_1(y) = 0$

- Substitute $f_1(y)$ into equation for $F(x,y)$

  $F(x,y) = \ln(x) - \arctan\left(\frac{x}{y}\right)$

- Substitute $F(x,y)$ into the solution of the ODE

  $\ln(x) - \arctan\left(\frac{x}{y}\right) = c_1$

- Solve for y

  $y = -\frac{x}{\tan(-\ln(x)+c_1)}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 12

```
dsolve((1/x-y(x)/(x^2+y(x)^2))+x/(x^2+y(x)^2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\tan(\ln(x)+c_1)\,x$$

✓ Solution by Mathematica

Time used: 0.205 (sec). Leaf size: 15

```
DSolve[(1/x-y[x]/(x^2+y[x]^2))+x/(x^2+y[x]^2)*y'[x]==0,y[x],x,IncludeSingularSolutions -> Tr
```

$$y(x) \to x \tan(-\log(x) + c_1)$$

5.9 problem Problem 9

Internal problem ID [2721]
Internal file name [`OUTPUT/2213_Sunday_June_05_2022_02_55_02_AM_44629639/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.9, Exact Differential Equations. page 91
Problem number: Problem 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[_exact, [_1st_order, `_with_symmetry_[F(x),G(x)*y+H(x)]`]]
```

$$\boxed{y\cos(yx) + x\cos(yx)\, y' = \sin(x)}$$

5.9.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x\cos(xy))\,\mathrm{d}y &= (-y\cos(xy) + \sin(x))\,\mathrm{d}x \\ (y\cos(xy) - \sin(x))\,\mathrm{d}x + (x\cos(xy))\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= y\cos(xy) - \sin(x) \\ N(x,y) &= x\cos(xy)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y\cos(xy) - \sin(x)) \\ &= -xy\sin(xy) + \cos(xy)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x\cos(xy)) \\ &= -xy\sin(xy) + \cos(xy)\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x}=M \tag{1}$$
$$\frac{\partial \phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int M\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int y\cos(xy)-\sin(x)\,\mathrm{d}x$$
$$\phi=\sin(xy)+\cos(x)+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y}=x\cos(xy)+f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y}=x\cos(xy)$. Therefore equation (4) becomes

$$x\cos(xy)=x\cos(xy)+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y)=0$$

Therefore

$$f(y)=c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi=\sin(xy)+\cos(x)+c_1$$

But since ϕ itself is a constant function, then let $\phi=c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1=\sin(xy)+\cos(x)$$

Summary

The solution(s) found are the following

$$\sin(yx) + \cos(x) = c_1 \tag{1}$$

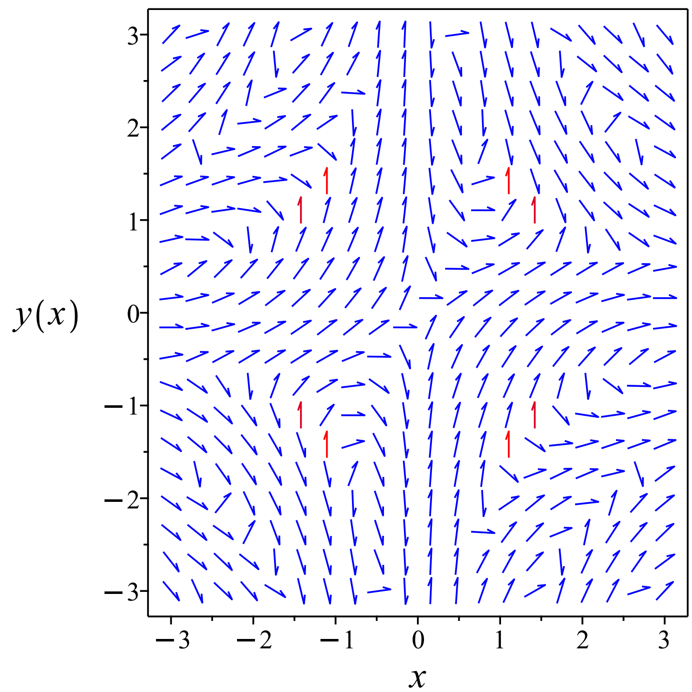


Figure 205: Slope field plot

Verification of solutions

$$\sin(yx) + \cos(x) = c_1$$

Verified OK.

5.9.2 Maple step by step solution

Let's solve

$y\cos(yx) + x\cos(yx)\, y' = \sin(x)$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact
  - ∘ ODE is exact if the lhs is the total derivative of a C^2 function

$F'(x, y) = 0$

- Compute derivative of lhs

  $F'(x, y) + \left(\frac{\partial}{\partial y}F(x, y)\right) y' = 0$
- Evaluate derivatives

  $-xy \sin(xy) + \cos(xy) = -xy \sin(xy) + \cos(xy)$
- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y), N(x, y) = \frac{\partial}{\partial y}F(x, y)\right]$
- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int (y \cos(xy) - \sin(x))\, dx + f_1(y)$
- Evaluate integral

  $F(x, y) = \sin(xy) + \cos(x) + f_1(y)$
- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y}F(x, y)$
- Compute derivative

  $x \cos(xy) = x \cos(xy) + \frac{d}{dy}f_1(y)$
- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y) = 0$
- Solve for $f_1(y)$

  $f_1(y) = 0$
- Substitute $f_1(y)$ into equation for $F(x, y)$

  $F(x, y) = \sin(xy) + \cos(x)$
- Substitute $F(x, y)$ into the solution of the ODE

  $\sin(xy) + \cos(x) = c_1$
- Solve for y

  $y = \frac{\arcsin(-\cos(x)+c_1)}{x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve((y(x)*cos(x*y(x))-sin(x))+x*cos(x*y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{\arcsin(\cos(x) + c_1)}{x}$$

✓ Solution by Mathematica

Time used: 0.628 (sec). Leaf size: 17

```
DSolve[(y[x]*Cos[x*y[x]]-Sin[x])+x*Cos[x*y[x]]*y'[x]==0,y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to \frac{\arcsin(-\cos(x) + c_1)}{x}$$

5.10 problem Problem 10

Internal problem ID [2722]
Internal file name [`OUTPUT/2214_Sunday_June_05_2022_02_55_05_AM_93691168/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.9, Exact Differential Equations. page 91
Problem number: Problem 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[_exact, _Bernoulli]
```

$$\boxed{2y^2 e^{2x} + 2y\, e^{2x} y' = -3x^2}$$

5.10.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(2\,\mathrm{e}^{2x} y\right) \mathrm{d}y &= \left(-2y^2 \mathrm{e}^{2x} - 3x^2\right) \mathrm{d}x \\
\left(2y^2 \mathrm{e}^{2x} + 3x^2\right) \mathrm{d}x + \left(2\,\mathrm{e}^{2x} y\right) \mathrm{d}y &= 0 \qquad (2\mathrm{A})
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= 2y^2 \mathrm{e}^{2x} + 3x^2 \\
N(x, y) &= 2\,\mathrm{e}^{2x} y
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(2y^2 \mathrm{e}^{2x} + 3x^2\right) \\
&= 4\,\mathrm{e}^{2x} y
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(2\,\mathrm{e}^{2x} y\right) \\
&= 4\,\mathrm{e}^{2x} y
\end{aligned}
$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x}=M \tag{1}$$
$$\frac{\partial \phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int M\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int 2y^2\mathrm{e}^{2x}+3x^2\,\mathrm{d}x$$
$$\phi=x^3+y^2\mathrm{e}^{2x}+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y}=2\,\mathrm{e}^{2x}y+f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y}=2\,\mathrm{e}^{2x}y$. Therefore equation (4) becomes

$$2\,\mathrm{e}^{2x}y=2\,\mathrm{e}^{2x}y+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y)=0$$

Therefore

$$f(y)=c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi=x^3+y^2\mathrm{e}^{2x}+c_1$$

But since ϕ itself is a constant function, then let $\phi=c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1=x^3+y^2\mathrm{e}^{2x}$$

Summary

The solution(s) found are the following

$$x^3 + y^2 e^{2x} = c_1 \tag{1}$$

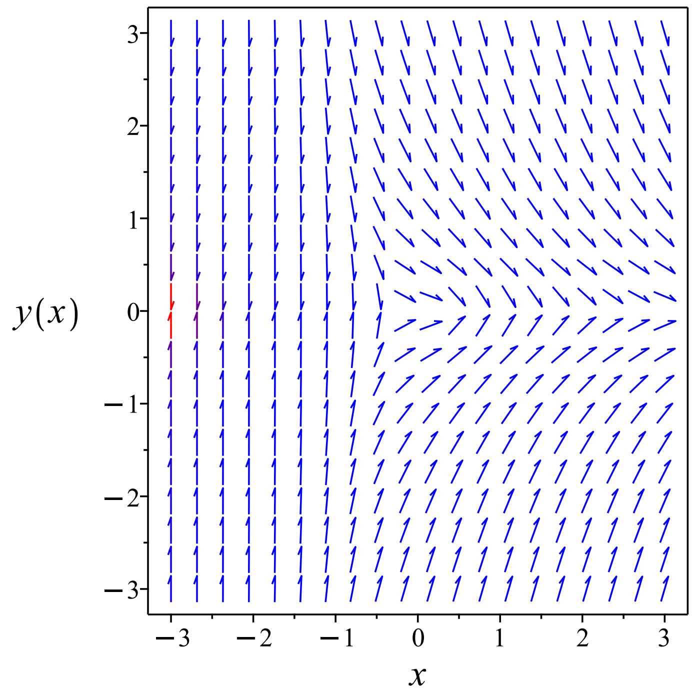


Figure 206: Slope field plot

Verification of solutions

$$x^3 + y^2 e^{2x} = c_1$$

Verified OK.

5.10.2 Maple step by step solution

Let's solve

$2y^2 e^{2x} + 2y\, e^{2x} y' = -3x^2$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  ◦ ODE is exact if the lhs is the total derivative of a C^2 function

$F'(x, y) = 0$

- Compute derivative of lhs

  $F'(x, y) + \left(\frac{\partial}{\partial y} F(x, y)\right) y' = 0$

- Evaluate derivatives

  $4\,\mathrm{e}^{2x} y = 4\,\mathrm{e}^{2x} y$

- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y), N(x, y) = \frac{\partial}{\partial y} F(x, y)\right]$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int (2y^2 \mathrm{e}^{2x} + 3x^2)\, dx + f_1(y)$

- Evaluate integral

  $F(x, y) = x^3 + y^2 \mathrm{e}^{2x} + f_1(y)$

- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y} F(x, y)$

- Compute derivative

  $2\,\mathrm{e}^{2x} y = 2\,\mathrm{e}^{2x} y + \frac{d}{dy} f_1(y)$

- Isolate for $\frac{d}{dy} f_1(y)$

  $\frac{d}{dy} f_1(y) = 0$

- Solve for $f_1(y)$

  $f_1(y) = 0$

- Substitute $f_1(y)$ into equation for $F(x, y)$

  $F(x, y) = x^3 + y^2 \mathrm{e}^{2x}$

- Substitute $F(x, y)$ into the solution of the ODE

  $x^3 + y^2 \mathrm{e}^{2x} = c_1$

- Solve for y

  $\left\{y = \frac{\sqrt{\mathrm{e}^{2x}(-x^3 + c_1)}}{\mathrm{e}^{2x}}, y = -\frac{\sqrt{\mathrm{e}^{2x}(-x^3 + c_1)}}{\mathrm{e}^{2x}}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 46

```
dsolve((2*y(x)^2*exp(2*x)+3*x^2)+2*y(x)*exp(2*x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = e^{-2x}\sqrt{e^{2x}\left(-x^3+c_1\right)}$$
$$y(x) = -e^{-2x}\sqrt{e^{2x}\left(-x^3+c_1\right)}$$

✓ Solution by Mathematica

Time used: 7.702 (sec). Leaf size: 47

```
DSolve[(2*y[x]^2*Exp[2*x]+3*x^2)+2*y[x]*Exp[2*x]*y'[x]==0,y[x],x,IncludeSingularSolutions ->
```

$$y(x) \rightarrow -\sqrt{e^{-2x}\left(-x^3+c_1\right)}$$
$$y(x) \rightarrow \sqrt{e^{-2x}\left(-x^3+c_1\right)}$$

5.11 problem Problem 11

Internal problem ID [2723]
Internal file name [`OUTPUT/2215_Sunday_June_05_2022_02_55_07_AM_95970603/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.9, Exact Differential Equations. page 91
Problem number: Problem 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_exact]`

$$\boxed{y^2 + (2yx + \sin(y))\, y' = -\cos(x)}$$

5.11.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(2xy + \sin(y))\,\mathrm{d}y &= \left(-y^2 - \cos(x)\right)\mathrm{d}x \\ \left(y^2 + \cos(x)\right)\mathrm{d}x + (2xy + \sin(y))\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= y^2 + \cos(x) \\ N(x, y) &= 2xy + \sin(y) \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(y^2 + \cos(x)\right) \\ &= 2y \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(2xy + \sin(y)) \\ &= 2y \end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int y^2 + \cos(x) \, \mathrm{d}x$$
$$\phi = x\, y^2 + \sin(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 2xy + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 2xy + \sin(y)$. Therefore equation (4) becomes

$$2xy + \sin(y) = 2xy + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \sin(y)$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (\sin(y)) \, \mathrm{d}y$$
$$f(y) = -\cos(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = x\, y^2 + \sin(x) - \cos(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x\,y^2 + \sin(x) - \cos(y)$$

Summary
The solution(s) found are the following

$$y^2x + \sin(x) - \cos(y) = c_1 \tag{1}$$

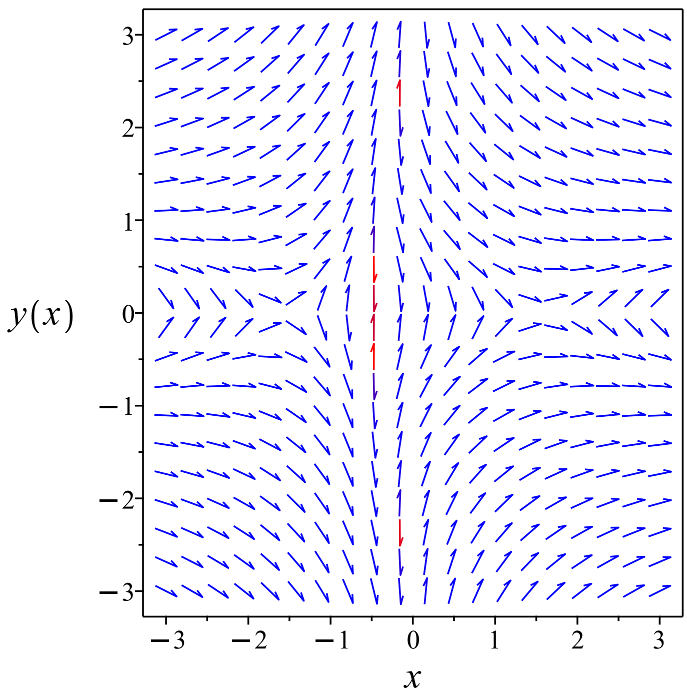


Figure 207: Slope field plot

Verification of solutions

$$y^2x + \sin(x) - \cos(y) = c_1$$

Verified OK.

5.11.2 Maple step by step solution

Let's solve

$y^2 + (2yx + \sin(y))\, y' = -\cos(x)$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x, y) = 0$

  - Compute derivative of lhs

    $F'(x, y) + \left(\frac{\partial}{\partial y}F(x, y)\right) y' = 0$

  - Evaluate derivatives

    $2y = 2y$

  - Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y), N(x, y) = \frac{\partial}{\partial y}F(x, y)\right]$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int (y^2 + \cos(x))\, dx + f_1(y)$

- Evaluate integral

  $F(x, y) = x\, y^2 + \sin(x) + f_1(y)$

- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y}F(x, y)$

- Compute derivative

  $2xy + \sin(y) = 2xy + \frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y) = \sin(y)$

- Solve for $f_1(y)$

  $f_1(y) = -\cos(y)$

- Substitute $f_1(y)$ into equation for $F(x, y)$

$$F(x, y) = x\,y^2 + \sin(x) - \cos(y)$$

- Substitute $F(x, y)$ into the solution of the ODE

$$x\,y^2 + \sin(x) - \cos(y) = c_1$$

- Solve for y

$$y = RootOf\left(-x\_Z^2 + c_1 + \cos(\_Z) - \sin(x)\right)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 18

```
dsolve((y(x)^2+cos(x))+(2*x*y(x)+sin(y(x)))*diff(y(x),x)=0,y(x), singsol=all)
```

$$xy(x)^2 + \sin(x) - \cos(y(x)) + c_1 = 0$$

✓ Solution by Mathematica

Time used: 0.227 (sec). Leaf size: 20

```
DSolve[(y[x]^2+Cos[x])+(2*x*y[x]+Sin[y[x]])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True
```

$$\text{Solve}\left[xy(x)^2 - \cos(y(x)) + \sin(x) = c_1, y(x)\right]$$

5.12 problem Problem 12

Internal problem ID [2724]
Internal file name [`OUTPUT/2216_Sunday_June_05_2022_02_55_11_AM_23277851/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 1, First-Order Differential Equations. Section 1.9, Exact Differential Equations. page 91
Problem number: Problem 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[_exact]
```

$$\boxed{\sin(y)+\cos(x)\,y+(x\cos(y)+\sin(x))\,y'=0}$$

5.12.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x\cos(y) + \sin(x))\,\mathrm{d}y &= (-\cos(x)\,y - \sin(y))\,\mathrm{d}x \\ (\sin(y) + \cos(x)\,y)\,\mathrm{d}x + (x\cos(y) + \sin(x))\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= \sin(y) + \cos(x)\,y \\ N(x, y) &= x\cos(y) + \sin(x)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(\sin(y) + \cos(x)\,y) \\ &= \cos(y) + \cos(x)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x\cos(y) + \sin(x)) \\ &= \cos(y) + \cos(x)\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x}=M \tag{1}$$
$$\frac{\partial\phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x=\int M\,\mathrm{d}x$$
$$\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x=\int\sin(y)+\cos(x)\,y\,\mathrm{d}x$$
$$\phi=\sin(x)\,y+x\sin(y)+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial\phi}{\partial y}=x\cos(y)+\sin(x)+f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial y}=x\cos(y)+\sin(x)$. Therefore equation (4) becomes

$$x\cos(y)+\sin(x)=x\cos(y)+\sin(x)+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y)=0$$

Therefore

$$f(y)=c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi=\sin(x)\,y+x\sin(y)+c_1$$

But since ϕ itself is a constant function, then let $\phi=c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1=\sin(x)\,y+x\sin(y)$$

Summary
The solution(s) found are the following

$$y \sin(x) + x \sin(y) = c_1 \tag{1}$$

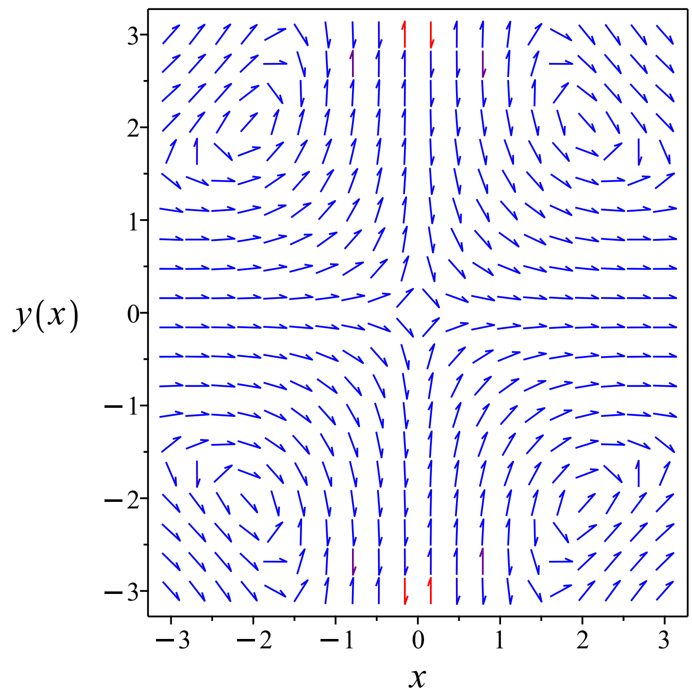


Figure 208: Slope field plot

Verification of solutions

$$y \sin(x) + x \sin(y) = c_1$$

Verified OK.

5.12.2 Maple step by step solution

Let's solve

$\sin(y) + \cos(x) y + (x \cos(y) + \sin(x)) y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact
  - ∘ ODE is exact if the lhs is the total derivative of a C^2 function

$F'(x, y) = 0$

- Compute derivative of lhs

  $F'(x, y) + \left(\frac{\partial}{\partial y} F(x, y)\right) y' = 0$

- Evaluate derivatives

  $\cos(y) + \cos(x) = \cos(y) + \cos(x)$

- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y), N(x, y) = \frac{\partial}{\partial y} F(x, y)\right]$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int (\sin(y) + \cos(x) y) dx + f_1(y)$

- Evaluate integral

  $F(x, y) = \sin(x) y + x \sin(y) + f_1(y)$

- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y} F(x, y)$

- Compute derivative

  $x \cos(y) + \sin(x) = \sin(x) + x \cos(y) + \frac{d}{dy} f_1(y)$

- Isolate for $\frac{d}{dy} f_1(y)$

  $\frac{d}{dy} f_1(y) = 0$

- Solve for $f_1(y)$

  $f_1(y) = 0$

- Substitute $f_1(y)$ into equation for $F(x, y)$

  $F(x, y) = \sin(x) y + x \sin(y)$

- Substitute $F(x, y)$ into the solution of the ODE

  $\sin(x) y + x \sin(y) = c_1$

- Solve for y

  $y = \mathit{RootOf}(-\_Z \sin(x) - x \sin(\_Z) + c_1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 15

```
dsolve((sin(y(x))+y(x)*cos(x))+(x*cos(y(x))+sin(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x)\sin(x) + x\sin(y(x)) + c_1 = 0$$

✓ Solution by Mathematica

Time used: 0.146 (sec). Leaf size: 17

```
DSolve[(Sin[y[x]]+y[x]*Cos[x])+(x*Cos[y[x]]+Sin[x])*y'[x]==0,y[x],x,IncludeSingularSolutions
```

$$\text{Solve}[x\sin(y(x)) + y(x)\sin(x) = c_1, y(x)]$$

6 Chapter 8, Linear differential equations of order n. Section 8.1, General Theory for Linear Differential Equations. page 502

6.1 problem Problem 23

Internal problem ID [2725]
Internal file name [`OUTPUT/2217_Sunday_June_05_2022_02_55_15_AM_7655955/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.1, General Theory for Linear Differential Equations. page 502
Problem number: Problem 23.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 2y' - 3y = 0}$$

6.1.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = -3$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 2\lambda\, \mathrm{e}^{\lambda x} - 3\, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda - 3 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = -3$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-2^2 - (4)(1)(-3)} \\ &= 1 \pm 2\end{aligned}$$

Hence

$$\lambda_1 = 1 + 2$$
$$\lambda_2 = 1 - 2$$

Which simplifies to

$$\lambda_1 = 3$$
$$\lambda_2 = -1$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(3)x} + c_2 e^{(-1)x}$$

Or

$$y = c_1 \mathrm{e}^{3x} + c_2 \mathrm{e}^{-x}$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1 \mathrm{e}^{3x} + c_2 \mathrm{e}^{-x} \tag{1}$$

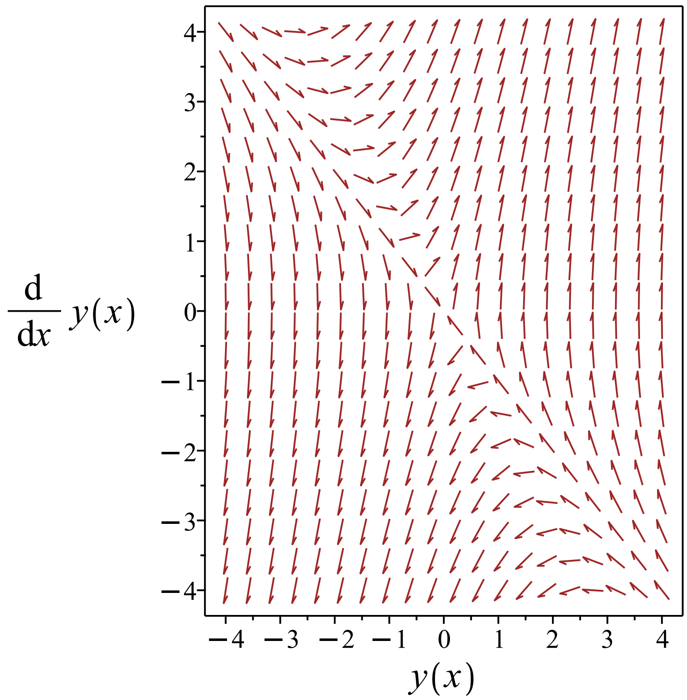


Figure 209: Slope field plot

Verification of solutions

$$y = c_1 e^{3x} + c_2 e^{-x}$$

Verified OK.

6.1.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' - 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \\ C &= -3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 173: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-2x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2}{1}\,dx} \\
&= z_1 e^{x} \\
&= z_1(\mathrm{e}^{x})
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2x}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{\mathrm{e}^{4x}}{4}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-x}\right) + c_2\left(\mathrm{e}^{-x}\left(\frac{\mathrm{e}^{4x}}{4}\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{3x}}{4} \tag{1}
$$

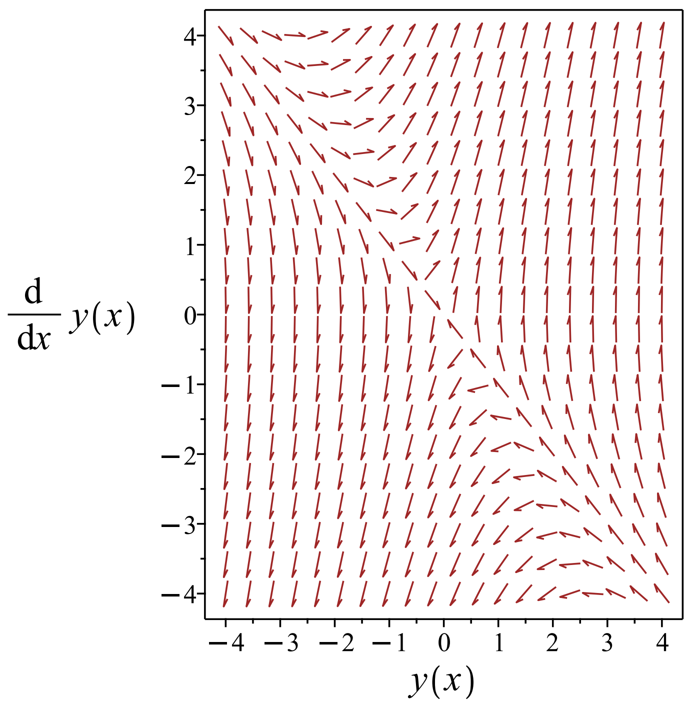


Figure 210: Slope field plot

Verification of solutions

$$
y = c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{3x}}{4}
$$

Verified OK.

6.1.3 Maple step by step solution

Let's solve

$y'' - 2y' - 3y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 2r - 3 = 0$

- Factor the characteristic polynomial

  $(r+1)(r-3) = 0$

- Roots of the characteristic polynomial

  $r = (-1, 3)$

- 1st solution of the ODE

  $y_1(x) = e^{-x}$

- 2nd solution of the ODE

  $y_2(x) = e^{3x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{-x} + c_2 e^{3x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(y(x),x$2)-2*diff(y(x),x)-3*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{3x} + e^{-x} c_2$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 22

```
DSolve[y''[x]-2*y'[x]-3*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x} \left(c_2 e^{4x} + c_1\right)$$

6.2 problem Problem 24

Internal problem ID [2726]
Internal file name [`OUTPUT/2218_Sunday_June_05_2022_02_55_16_AM_57616816/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.1, General Theory for Linear Differential Equations. page 502
Problem number: Problem 24.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 7y' + 10y = 0}$$

6.2.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 7, C = 10$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 7\lambda\,\mathrm{e}^{\lambda x} + 10\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 7\lambda + 10 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 7, C = 10$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{-7}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{7^2 - (4)(1)(10)} \\
&= -\frac{7}{2} \pm \frac{3}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= -\frac{7}{2} + \frac{3}{2} \\
\lambda_2 &= -\frac{7}{2} - \frac{3}{2}
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= -2 \\
\lambda_2 &= -5
\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}
y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\
y &= c_1 e^{(-2)x} + c_2 e^{(-5)x}
\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-5x}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-5x} \tag{1}$$

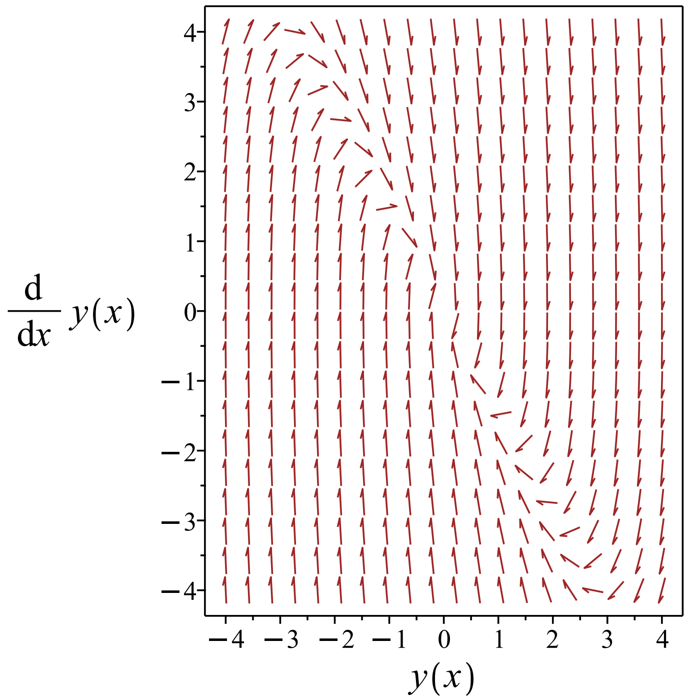


Figure 211: Slope field plot

Verification of solutions

$$y = c_1 e^{-2x} + c_2 e^{-5x}$$

Verified OK.

6.2.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 7y' + 10y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 7 \\ C &= 10 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 9 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{9z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 175: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{9}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{3x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{7}{1}\,dx} \\ &= z_1 e^{-\frac{7x}{2}} \\ &= z_1\left(\mathrm{e}^{-\frac{7x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-5x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{7}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-7x}}{(y_1)^2}\,dx \\ &= y_1\left(\frac{\mathrm{e}^{3x}}{3}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(e^{-5x}\right) + c_2 \left(e^{-5x} \left(\frac{e^{3x}}{3}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 e^{-5x} + \frac{c_2 e^{-2x}}{3} \tag{1}
$$

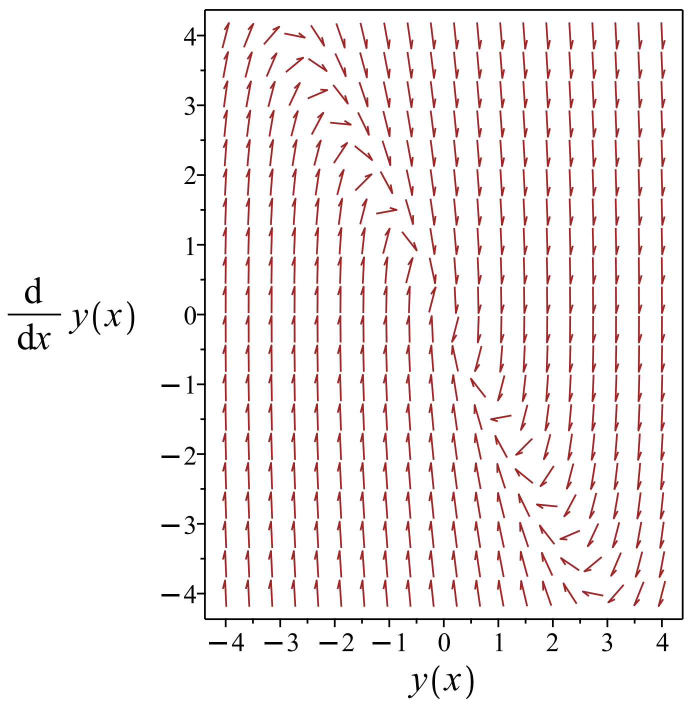


Figure 212: Slope field plot

Verification of solutions

$$
y = c_1 e^{-5x} + \frac{c_2 e^{-2x}}{3}
$$

Verified OK.

6.2.3 Maple step by step solution

Let's solve

$y'' + 7y' + 10y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + 7r + 10 = 0$

- Factor the characteristic polynomial

  $(r + 5)(r + 2) = 0$

- Roots of the characteristic polynomial

  $r = (-5, -2)$

- 1st solution of the ODE

  $y_1(x) = e^{-5x}$

- 2nd solution of the ODE

  $y_2(x) = e^{-2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{-5x} + c_2 e^{-2x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(y(x),x$2)+7*diff(y(x),x)+10*y(x)=0,y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{-5x}c_1 + \mathrm{e}^{-2x}c_2$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 22

```
DSolve[y''[x]+7*y'[x]+10*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-5x}\left(c_2e^{3x} + c_1\right)$$

6.3 problem Problem 25

Internal problem ID [2727]
Internal file name [`OUTPUT/2219_Sunday_June_05_2022_02_55_18_AM_29793156/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.1, General Theory for Linear Differential Equations. page 502
Problem number: Problem 25.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 36y = 0}$$

6.3.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -36$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 36 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 36 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -36$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-36)} \\ &= \pm 6\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +6 \\ \lambda_2 &= -6\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 6 \\ \lambda_2 &= -6\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(6)x} + c_2 e^{(-6)x}\end{aligned}$$

Or

$$y = e^{6x} c_1 + c_2 e^{-6x}$$

Summary

The solution(s) found are the following

$$y = e^{6x} c_1 + c_2 e^{-6x} \tag{1}$$

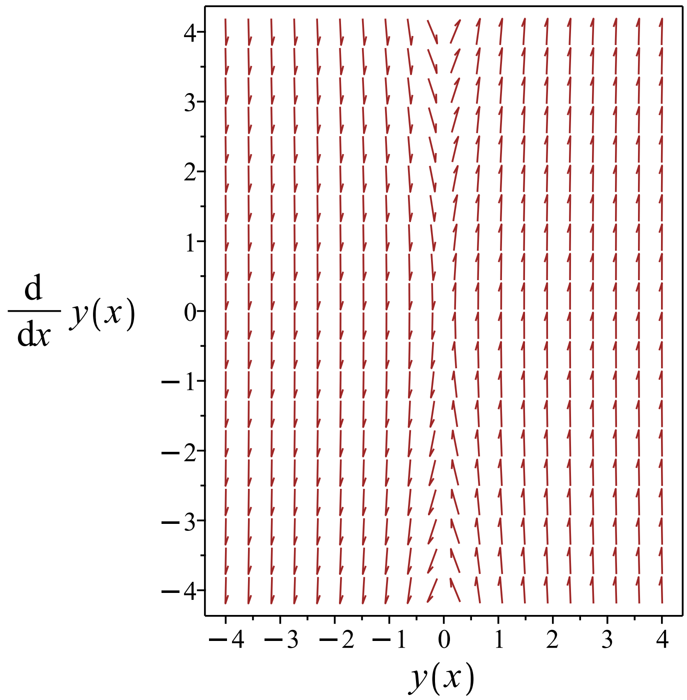


Figure 213: Slope field plot

Verification of solutions

$$y = e^{6x}c_1 + c_2e^{-6x}$$

Verified OK.

6.3.2 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$y'y'' - 36yy' = 0$$

Integrating the above w.r.t x gives

$$\int (y'y'' - 36yy')\,dx = 0$$
$$\frac{y'^2}{2} - 18y^2 = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{36y^2 + 2c_1} \tag{1}$$
$$y' = -\sqrt{36y^2 + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{36y^2+2c_1}}dy = \int dx$$
$$\frac{\ln\left(y\sqrt{36}+\sqrt{36y^2+2c_1}\right)\sqrt{36}}{36} = x + c_2$$

Raising both side to exponential gives

$$\mathrm{e}^{\frac{\ln\left(y\sqrt{36}+\sqrt{36y^2+2c_1}\right)\sqrt{36}}{36}} = \mathrm{e}^{x+c_2}$$

Which simplifies to

$$\left(6y+\sqrt{36y^2+2c_1}\right)^{\frac{1}{6}} = c_3\mathrm{e}^{x}$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{36y^2+2c_1}}dy = \int dx$$
$$-\frac{\ln\left(y\sqrt{36}+\sqrt{36y^2+2c_1}\right)\sqrt{36}}{36} = x + c_4$$

Raising both side to exponential gives

$$\mathrm{e}^{-\frac{\ln\left(y\sqrt{36}+\sqrt{36y^2+2c_1}\right)\sqrt{36}}{36}} = \mathrm{e}^{x+c_4}$$

Which simplifies to

$$\frac{1}{\left(6y+\sqrt{36y^2+2c_1}\right)^{\frac{1}{6}}} = c_5\mathrm{e}^{x}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(\mathrm{e}^{12x}c_3^{12}-2c_1\right)\mathrm{e}^{-6x}}{12c_3^{6}} \tag{1}$$
$$y = -\frac{\left(2c_1c_5^{12}\mathrm{e}^{12x}-1\right)\mathrm{e}^{-6x}}{12c_5^{6}} \tag{2}$$

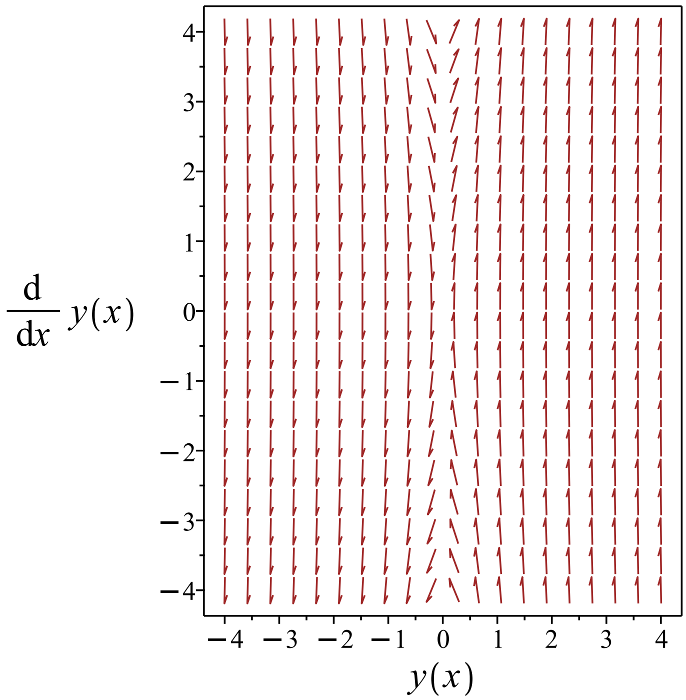


Figure 214: Slope field plot

Verification of solutions

$$y = \frac{(e^{12x}c_3^{12} - 2c_1)\, e^{-6x}}{12c_3^6}$$

Verified OK.

$$y = -\frac{(2c_1c_5^{12}e^{12x} - 1)\, e^{-6x}}{12c_5^6}$$

Verified OK.

6.3.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 36y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -36 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{36}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 36 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 36 z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 177: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 36$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-6x}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \mathrm{e}^{-6x}
\end{aligned}
$$

Which simplifies to

$$y_1 = \mathrm{e}^{-6x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= \mathrm{e}^{-6x} \int \frac{1}{\mathrm{e}^{-12x}}\, dx \\
&= \mathrm{e}^{-6x} \left(\frac{\mathrm{e}^{12x}}{12} \right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-6x}\right) + c_2 \left(\mathrm{e}^{-6x} \left(\frac{\mathrm{e}^{12x}}{12}\right)\right)
\end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-6x} + \frac{c_2 \mathrm{e}^{6x}}{12} \tag{1}$$

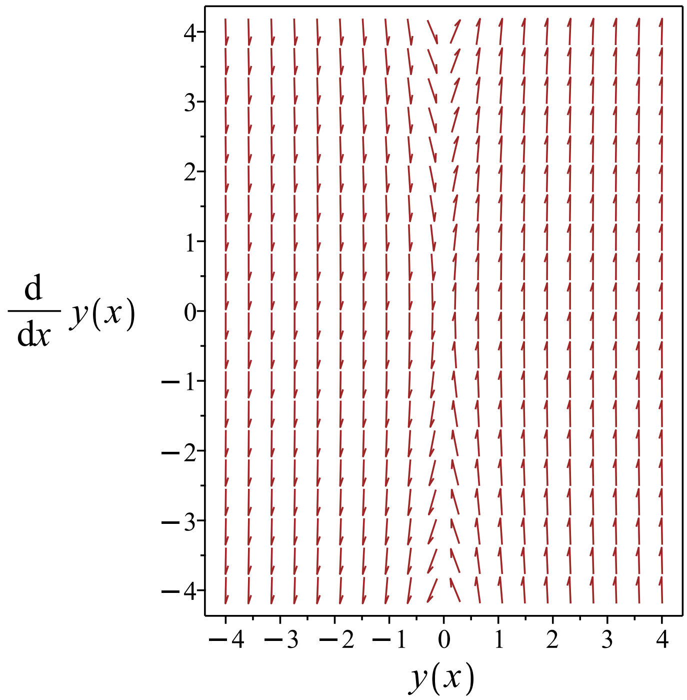


Figure 215: Slope field plot

Verification of solutions

$$y = c_1 e^{-6x} + \frac{c_2 e^{6x}}{12}$$

Verified OK.

6.3.4 Maple step by step solution

Let's solve

$y'' - 36y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 36 = 0$

- Factor the characteristic polynomial

  $(r - 6)(r + 6) = 0$

- Roots of the characteristic polynomial

$$r = (-6, 6)$$

- 1st solution of the ODE

$$y_1(x) = \mathrm{e}^{-6x}$$

- 2nd solution of the ODE

$$y_2(x) = \mathrm{e}^{6x}$$

- General solution of the ODE

$$y = c_1 y_1(x) + c_2 y_2(x)$$

- Substitute in solutions

$$y = c_1 \mathrm{e}^{-6x} + c_2 \mathrm{e}^{6x}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(y(x),x$2)-36*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \mathrm{e}^{-6x} + \mathrm{e}^{6x} c_2$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 22

```
DSolve[y''[x]-36*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{6x} + c_2 e^{-6x}$$

6.4 problem Problem 26

Internal problem ID [2728]
Internal file name [`OUTPUT/2220_Sunday_June_05_2022_02_55_20_AM_74449003/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.1, General Theory for Linear Differential Equations. page 502
Problem number: Problem 26.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 4y' = 0}$$

6.4.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 4, C = 0$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 4\lambda\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4\lambda = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 4, C = 0$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-4}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{4^2 - (4)\,(1)\,(0)}\\ &= -2 \pm 2\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -2 + 2\\ \lambda_2 &= -2 - 2\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 0\\ \lambda_2 &= -4\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(0)x} + c_2 e^{(-4)x}\end{aligned}$$

Or

$$y = c_1 + c_2 \mathrm{e}^{-4x}$$

Summary

The solution(s) found are the following

$$y = c_1 + c_2 \mathrm{e}^{-4x} \tag{1}$$

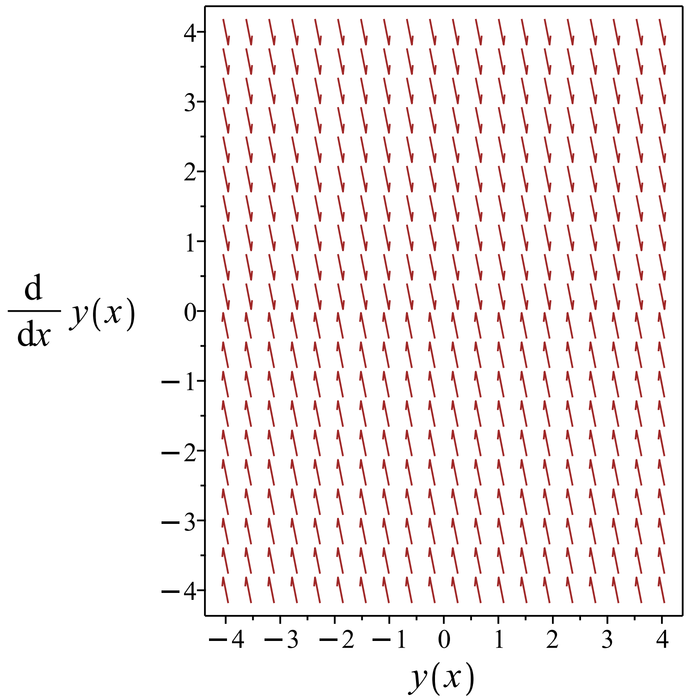


Figure 216: Slope field plot

Verification of solutions

$$y = c_1 + c_2 e^{-4x}$$

Verified OK.

6.4.2 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + 4y')\,dx = 0$$
$$4y + y' = c_1$$

Which is now solved for y. Integrating both sides gives

$$\int \frac{1}{-4y + c_1} dy = \int dx$$
$$-\frac{\ln(-4y + c_1)}{4} = x + c_2$$

Raising both side to exponential gives

$$\frac{1}{(-4y + c_1)^{\frac{1}{4}}} = e^{x + c_2}$$

Which simplifies to

$$\frac{1}{(-4y+c_1)^{\frac{1}{4}}} = c_3 \mathrm{e}^{x}$$

Summary

The solution(s) found are the following

$$y = -\frac{\mathrm{e}^{-4x}}{4c_3^4} + \frac{c_1}{4} \tag{1}$$

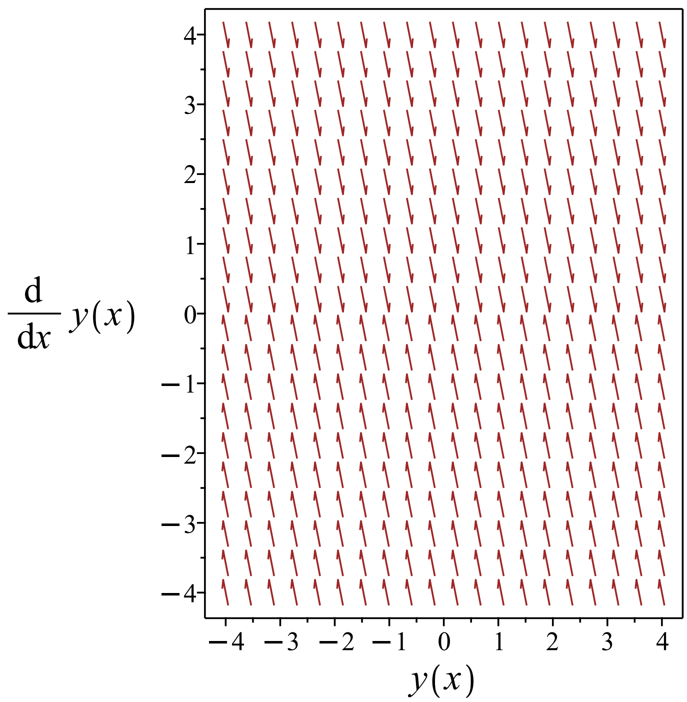


Figure 217: Slope field plot

Verification of solutions

$$y = -\frac{\mathrm{e}^{-4x}}{4c_3^4} + \frac{c_1}{4}$$

Verified OK.

6.4.3 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) + 4p(x) = 0$$

Which is now solve for $p(x)$ as first order ode. Integrating both sides gives

$$\int -\frac{1}{4p}dp = \int dx$$
$$-\frac{\ln(p)}{4} = x + c_1$$

Raising both side to exponential gives

$$\frac{1}{p^{\frac{1}{4}}} = \mathrm{e}^{x+c_1}$$

Which simplifies to

$$\frac{1}{p^{\frac{1}{4}}} = c_2\mathrm{e}^{x}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = \frac{\mathrm{e}^{-4x}}{c_2^4}$$

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{\mathrm{e}^{-4x}}{c_2^4}\, \mathrm{d}x \\ &= -\frac{\mathrm{e}^{-4x}}{4c_2^4} + c_3 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -\frac{\mathrm{e}^{-4x}}{4c_2^4} + c_3 \tag{1}$$

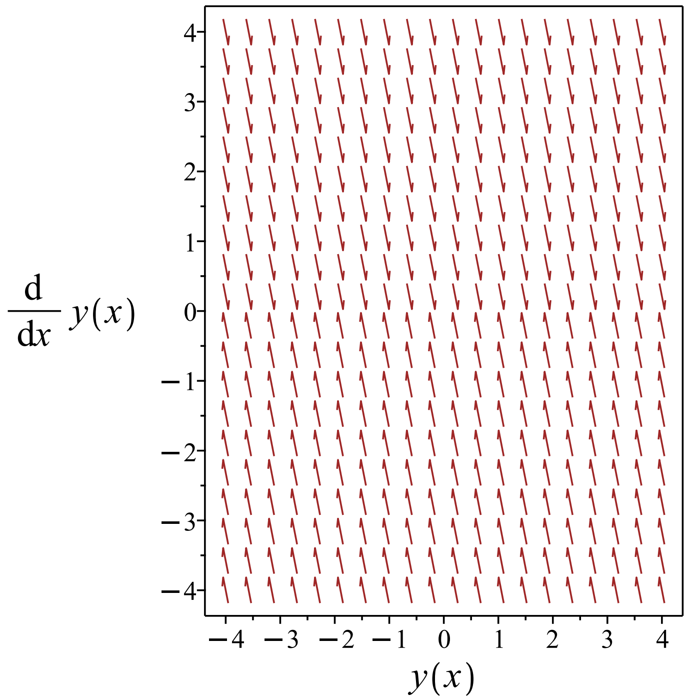


Figure 218: Slope field plot

Verification of solutions

$$y = -\frac{e^{-4x}}{4c_2^4} + c_3$$

Verified OK.

6.4.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y'' + 4y' = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + 4y')\,dx = 0$$

$$4y + y' = c_1$$

Which is now solved for y. Integrating both sides gives

$$\int \frac{1}{-4y+c_1}dy = \int dx$$
$$-\frac{\ln(-4y+c_1)}{4} = x + c_2$$

Raising both side to exponential gives

$$\frac{1}{(-4y+c_1)^{\frac{1}{4}}} = e^{x+c_2}$$

Which simplifies to

$$\frac{1}{(-4y+c_1)^{\frac{1}{4}}} = c_3 e^{x}$$

Summary

The solution(s) found are the following

$$y = -\frac{e^{-4x}}{4c_3^4} + \frac{c_1}{4} \tag{1}$$

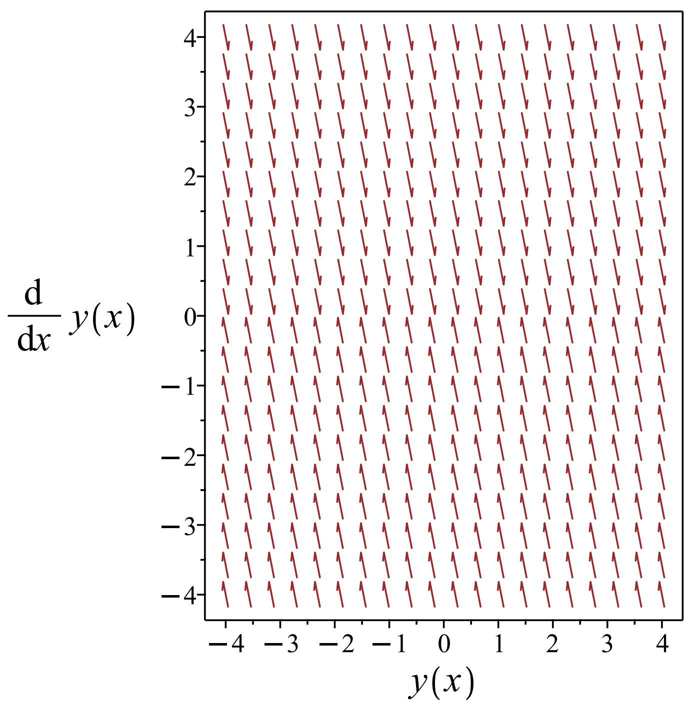


Figure 219: Slope field plot

Verification of solutions

$$y = -\frac{e^{-4x}}{4c_3^4} + \frac{c_1}{4}$$

Verified OK.

6.4.5 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 179: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-2x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4}{1}\,dx} \\
&= z_1 e^{-2x} \\
&= z_1 \left(\mathrm{e}^{-2x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-4x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-4x}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{\mathrm{e}^{4x}}{4}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-4x}\right) + c_2 \left(\mathrm{e}^{-4x} \left(\frac{\mathrm{e}^{4x}}{4}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-4x} + \frac{c_2}{4} \tag{1}
$$

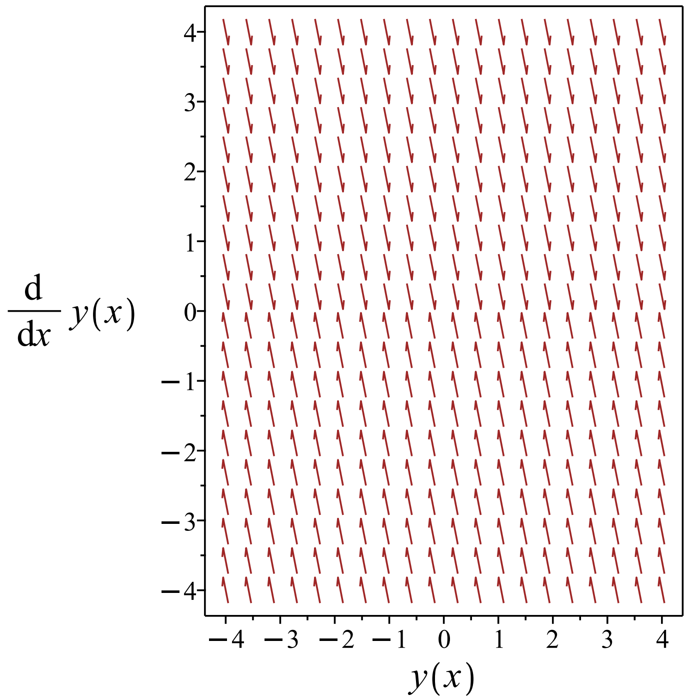


Figure 220: Slope field plot

Verification of solutions

$$y = c_1 e^{-4x} + \frac{c_2}{4}$$

Verified OK.

6.4.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 4 \\ r(x) &= 0 \\ s(x) &= 0 \end{aligned}$$

Hence

$$
\begin{aligned}
p''(x) &= 0 \\
q'(x) &= 0
\end{aligned}
$$

Therefore (1) becomes

$$
0 - (0) + (0) = 0
$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$
(p(x)\,y' + (q(x) - p'(x))\,y)' = s(x)
$$

Integrating gives

$$
p(x)\,y' + (q(x) - p'(x))\,y = \int s(x)\;dx
$$

Substituting the above values for p, q, r, s gives

$$
4y + y' = c_1
$$

We now have a first order ode to solve which is

$$
4y + y' = c_1
$$

Integrating both sides gives

$$
\begin{aligned}
\int \frac{1}{-4y + c_1}dy &= \int dx \\
-\frac{\ln(-4y + c_1)}{4} &= x + c_2
\end{aligned}
$$

Raising both side to exponential gives

$$
\frac{1}{(-4y + c_1)^{\frac{1}{4}}} = \mathrm{e}^{x+c_2}
$$

Which simplifies to

$$
\frac{1}{(-4y + c_1)^{\frac{1}{4}}} = c_3\mathrm{e}^{x}
$$

Summary

The solution(s) found are the following

$$
y = -\frac{\mathrm{e}^{-4x}}{4c_3^4} + \frac{c_1}{4} \tag{1}
$$

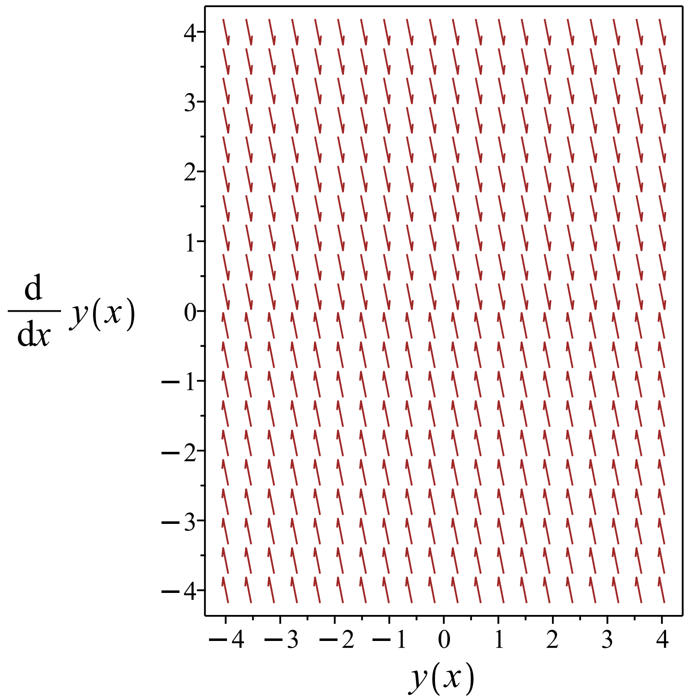


Figure 221: Slope field plot

Verification of solutions

$$y = -\frac{e^{-4x}}{4c_3^4} + \frac{c_1}{4}$$

Verified OK.

6.4.7 Maple step by step solution

Let's solve

$y'' + 4y' = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + 4r = 0$

- Factor the characteristic polynomial

  $r(r + 4) = 0$

- Roots of the characteristic polynomial

$$r = (-4, 0)$$

- 1st solution of the ODE

$$y_1(x) = \mathrm{e}^{-4x}$$

- 2nd solution of the ODE

$$y_2(x) = 1$$

- General solution of the ODE

$$y = c_1 y_1(x) + c_2 y_2(x)$$

- Substitute in solutions

$$y = c_1 \mathrm{e}^{-4x} + c_2$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(diff(y(x),x$2)+4*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = c_1 + c_2 \mathrm{e}^{-4x}$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 19

```
DSolve[y''[x]+4*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 - \frac{1}{4} c_1 e^{-4x}$$

6.5 problem Problem 27

Internal problem ID [2729]
Internal file name [`OUTPUT/2221_Sunday_June_05_2022_02_55_22_AM_3391712/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.1, General Theory for Linear Differential Equations. page 502
Problem number: Problem 27.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' - 3y'' - y' + 3y = 0}$$

The characteristic equation is

$$\lambda^3 - 3\lambda^2 - \lambda + 3 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = 3$$
$$\lambda_3 = -1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + c_2 e^{x} + e^{3x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-x}$$
$$y_2 = e^{x}$$
$$y_3 = e^{3x}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} + c_2 e^{x} + e^{3x} c_3 \tag{1}$$

Verification of solutions

$$y = c_1 e^{-x} + c_2 e^{x} + e^{3x} c_3$$

Verified OK.

6.5.1 Maple step by step solution

Let's solve

$y''' - 3y'' - y' + 3y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$
  - Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = 3y_3(x) + y_2(x) - 3y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = 3y_3(x) + y_2(x) - 3y_1(x)]$

- Define vector

  $$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & 1 & 3 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & 1 & 3 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[3, \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^x \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[3, \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{3x} \cdot \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{y} = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_2 e^x \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + e^{3x} c_3 \cdot \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = c_1 e^{-x} + c_2 e^x + \frac{e^{3x} c_3}{9}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(diff(y(x),x$3)-3*diff(y(x),x$2)-diff(y(x),x)+3*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \mathrm{e}^{3x} + \mathrm{e}^{-x} c_2 + c_3 \mathrm{e}^{x}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 28

```
DSolve[y'''[x]-3*y''[x]-y'[x]+3*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{-x} + c_2 e^{x} + c_3 e^{3x}$$

6.6 problem Problem 28

Internal problem ID [2730]
Internal file name [OUTPUT/2222_Sunday_June_05_2022_02_55_24_AM_30528737/index.tex]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.1, General Theory for Linear Differential Equations. page 502
Problem number: Problem 28.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' + 3y'' - 4y' - 12y = 0}$$

The characteristic equation is

$$\lambda^3 + 3\lambda^2 - 4\lambda - 12 = 0$$

The roots of the above equation are

$$\lambda_1 = 2$$
$$\lambda_2 = -3$$
$$\lambda_3 = -2$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-2x} + c_2 e^{-3x} + c_3 e^{2x}$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-2x}$$
$$y_2 = e^{-3x}$$
$$y_3 = e^{2x}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-2x} + c_2 e^{-3x} + c_3 e^{2x} \tag{1}$$

Verification of solutions

$$y = c_1 e^{-2x} + c_2 e^{-3x} + c_3 e^{2x}$$

Verified OK.

6.6.1 Maple step by step solution

Let's solve

$y''' + 3y'' - 4y' - 12y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$

  ◦ Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = -3y_3(x) + 4y_2(x) + 12y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = -3y_3(x) + 4y_2(x) + 12y_1(x)]$

- Define vector

  $$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\overrightarrow{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 12 & 4 & -3 \end{bmatrix} \cdot \overrightarrow{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 12 & 4 & -3 \end{bmatrix}$$

- Rewrite the system as

$$\overrightarrow{y}'(x) = A \cdot \overrightarrow{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-3, \begin{bmatrix} \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix}\right], \left[-2, \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-3, \begin{bmatrix} \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = \mathrm{e}^{-3x} \cdot \begin{bmatrix} \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[-2, \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^{-2x} \cdot \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{2x} \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-3x} \cdot \begin{bmatrix} \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix} + c_2 e^{-2x} \cdot \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix} + c_3 e^{2x} \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{(9c_3 e^{5x} + 9c_2 e^{x} + 4c_1) e^{-3x}}{36}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(diff(y(x),x$3)+3*diff(y(x),x$2)-4*diff(y(x),x)-12*y(x)=0,y(x), singsol=all)
```

$$y(x) = \left(c_3 \mathrm{e}^{5x} + \mathrm{e}^{x} c_1 + c_2\right) \mathrm{e}^{-3x}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 28

```
DSolve[y'''[x]+3*y''[x]-4*y'[x]-12*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-3x}\left(c_2 e^{x} + c_3 e^{5x} + c_1\right)$$

6.7 problem Problem 29

Internal problem ID [2731]
Internal file name [`OUTPUT/2223_Sunday_June_05_2022_02_55_25_AM_42386429/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.1, General Theory for Linear Differential Equations. page 502
Problem number: Problem 29.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' + 3y'' - 18y' - 40y = 0}$$

The characteristic equation is

$$\lambda^3 + 3\lambda^2 - 18\lambda - 40 = 0$$

The roots of the above equation are

$$\lambda_1 = -5$$
$$\lambda_2 = -2$$
$$\lambda_3 = 4$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-2x} + e^{4x} c_2 + e^{-5x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-2x}$$
$$y_2 = e^{4x}$$
$$y_3 = e^{-5x}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-2x} + e^{4x} c_2 + e^{-5x} c_3 \tag{1}$$

Verification of solutions

$$y = c_1 e^{-2x} + e^{4x} c_2 + e^{-5x} c_3$$

Verified OK.

6.7.1 Maple step by step solution

Let's solve

$y''' + 3y'' - 18y' - 40y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

- ☐ Convert linear ODE into a system of first order ODEs

  - Define new variable $y_1(x)$

    $y_1(x) = y$

  - Define new variable $y_2(x)$

    $y_2(x) = y'$

  - Define new variable $y_3(x)$

    $y_3(x) = y''$

  - Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = -3y_3(x) + 18y_2(x) + 40y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = -3y_3(x) + 18y_2(x) + 40y_1(x)]$

- Define vector

  $$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 40 & 18 & -3 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 40 & 18 & -3 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-5, \begin{bmatrix} \frac{1}{25} \\ -\frac{1}{5} \\ 1 \end{bmatrix}\right], \left[-2, \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right], \left[4, \begin{bmatrix} \frac{1}{16} \\ \frac{1}{4} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-5, \begin{bmatrix} \frac{1}{25} \\ -\frac{1}{5} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{-5x} \cdot \begin{bmatrix} \frac{1}{25} \\ -\frac{1}{5} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[-2, \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^{-2x} \cdot \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[4, \begin{bmatrix} \frac{1}{16} \\ \frac{1}{4} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{4x} \cdot \begin{bmatrix} \frac{1}{16} \\ \frac{1}{4} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{y} = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-5x} \cdot \begin{bmatrix} \frac{1}{25} \\ -\frac{1}{5} \\ 1 \end{bmatrix} + c_2 e^{-2x} \cdot \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix} + c_3 e^{4x} \cdot \begin{bmatrix} \frac{1}{16} \\ \frac{1}{4} \\ 1 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{(25c_3 e^{9x} + 100c_2 e^{3x} + 16c_1) e^{-5x}}{400}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(y(x),x$3)+3*diff(y(x),x$2)-18*diff(y(x),x)-40*y(x)=0,y(x), singsol=all)
```

$$y(x) = \left(c_3 \mathrm{e}^{9x} + c_2 \mathrm{e}^{3x} + c_1\right) \mathrm{e}^{-5x}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 30

```
DSolve[y'''[x]+3*y''[x]-18*y'[x]-40*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-5x}\left(c_2 e^{3x} + c_3 e^{9x} + c_1\right)$$

6.8 problem Problem 30

Internal problem ID [2732]
Internal file name [`OUTPUT/2224_Sunday_June_05_2022_02_55_27_AM_35086072/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.1, General Theory for Linear Differential Equations. page 502
Problem number: Problem 30.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' - y'' - 2y' = 0}$$

The characteristic equation is

$$\lambda^3 - \lambda^2 - 2\lambda = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 2$$
$$\lambda_3 = -1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-x} + c_2 + c_3 \mathrm{e}^{2x}$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = \mathrm{e}^{-x}$$
$$y_2 = 1$$
$$y_3 = \mathrm{e}^{2x}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} + c_2 + c_3 e^{2x} \tag{1}$$

Verification of solutions

$$y = c_1 e^{-x} + c_2 + c_3 e^{2x}$$

Verified OK.

6.8.1 Maple step by step solution

Let's solve

$y''' - y'' - 2y' = 0$

- Highest derivative means the order of the ODE is 3

  y'''

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$

  ◦ Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = y_3(x) + 2y_2(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = y_3(x) + 2y_2(x)]$

- Define vector

  $\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = \mathrm{e}^{2x} \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{y} = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 \mathrm{e}^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_3 \mathrm{e}^{2x} \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} + \begin{bmatrix} c_2 \\ 0 \\ 0 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = c_1 \mathrm{e}^{-x} + \frac{c_3 \mathrm{e}^{2x}}{4} + c_2$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve(diff(y(x),x$3)-diff(y(x),x$2)-2*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = c_1 + e^{-x} c_2 + c_3 e^{2x}$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 28

```
DSolve[y'''[x]-y''[x]-2*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1(-e^{-x}) + \frac{1}{2} c_2 e^{2x} + c_3$$

6.9 problem Problem 31

Internal problem ID [2733]
Internal file name [`OUTPUT/2225_Sunday_June_05_2022_02_55_29_AM_63189436/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.1, General Theory for Linear Differential Equations. page 502
Problem number: Problem 31.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' + y'' - 10y' + 8y = 0}$$

The characteristic equation is

$$\lambda^3 + \lambda^2 - 10\lambda + 8 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = 2$$
$$\lambda_3 = -4$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^x + c_2 e^{2x} + e^{-4x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^x$$
$$y_2 = e^{2x}$$
$$y_3 = e^{-4x}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{x} + c_2 e^{2x} + e^{-4x} c_3 \tag{1}$$

Verification of solutions

$$y = c_1 e^{x} + c_2 e^{2x} + e^{-4x} c_3$$

Verified OK.

6.9.1 Maple step by step solution

Let's solve

$y''' + y'' - 10y' + 8y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$

  ◦ Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = -y_3(x) + 10y_2(x) - 8y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = -y_3(x) + 10y_2(x) - 8y_1(x)]$

- Define vector

  $\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -8 & 10 & -1 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -8 & 10 & -1 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-4, \begin{bmatrix} \frac{1}{16} \\ -\frac{1}{4} \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-4, \begin{bmatrix} \frac{1}{16} \\ -\frac{1}{4} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{-4x} \cdot \begin{bmatrix} \frac{1}{16} \\ -\frac{1}{4} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{2x} \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{y} = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-4x} \cdot \begin{bmatrix} \frac{1}{16} \\ -\frac{1}{4} \\ 1 \end{bmatrix} + c_2 e^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_3 e^{2x} \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{(4c_3 e^{6x} + 16c_2 e^{5x} + c_1) e^{-4x}}{16}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(y(x),x$3)+diff(y(x),x$2)-10*diff(y(x),x)+8*y(x)=0,y(x), singsol=all)
```

$$y(x) = \left(\mathrm{e}^{6x} c_2 + c_1 \mathrm{e}^{5x} + c_3\right) \mathrm{e}^{-4x}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 28

```
DSolve[y'''[x]+y''[x]-10*y'[x]+8*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{-4x} + c_2 e^x + c_3 e^{2x}$$

6.10 problem Problem 32

Internal problem ID [2734]
Internal file name [`OUTPUT/2226_Sunday_June_05_2022_02_55_30_AM_40969575/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.1, General Theory for Linear Differential Equations. page 502
Problem number: Problem 32.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' - 2y''' - y'' + 2y' = 0}$$

The characteristic equation is

$$\lambda^4 - 2\lambda^3 - \lambda^2 + 2\lambda = 0$$

The roots of the above equation are

$$\begin{aligned}
\lambda_1 &= 0 \\
\lambda_2 &= 1 \\
\lambda_3 &= 2 \\
\lambda_4 &= -1
\end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-x} + c_2 + c_3 \mathrm{e}^{x} + \mathrm{e}^{2x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= \mathrm{e}^{-x} \\
y_2 &= 1 \\
y_3 &= \mathrm{e}^{x} \\
y_4 &= \mathrm{e}^{2x}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-x} + c_2 + c_3 \mathrm{e}^{x} + \mathrm{e}^{2x} c_4 \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{-x} + c_2 + c_3 \mathrm{e}^{x} + \mathrm{e}^{2x} c_4$$

Verified OK.

6.10.1 Maple step by step solution

Let's solve

$y'''' - 2y''' - y'' + 2y' = 0$

- Highest derivative means the order of the ODE is 4

  y''''

□ Convert linear ODE into a system of first order ODEs

  - Define new variable $y_1(x)$

    $y_1(x) = y$

  - Define new variable $y_2(x)$

    $y_2(x) = y'$

  - Define new variable $y_3(x)$

    $y_3(x) = y''$

  - Define new variable $y_4(x)$

    $y_4(x) = y'''$

  - Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = 2y_4(x) + y_3(x) - 2y_2(x)$

  Convert linear ODE into a system of first order ODEs

$[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = 2y_4(x) + y_3(x) - 2y_2(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -2 & 1 & 2 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -2 & 1 & 2 \end{bmatrix}$$

- Rewrite the system as

$\vec{y}'(x) = A \cdot \vec{y}(x)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \left[\begin{array}{c} -1 \\ 1 \\ -1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = \mathrm{e}^{-x} \cdot \left[\begin{array}{c} -1 \\ 1 \\ -1 \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[0, \left[\begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_2 = \left[\begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \end{array}\right]$$

- Consider eigenpair

$$\left[1, \left[\begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_4 = e^{2x} \cdot \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{y} = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3 + c_4 \vec{y}_4$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-x} \cdot \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} + c_3 e^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + e^{2x} c_4 \cdot \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} + \begin{bmatrix} c_2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = -c_1 e^{-x} + c_3 e^{x} + \frac{e^{2x} c_4}{8} + c_2$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 22

```
dsolve(diff(y(x),x$4)-2*diff(y(x),x$3)-diff(y(x),x$2)+2*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = c_1 + e^{-x}c_2 + c_3e^x + c_4e^{2x}$$

✓ Solution by Mathematica

Time used: 0.016 (sec). Leaf size: 34

```
DSolve[y''''[x]-2*y'''[x]-y''[x]+2*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1\left(-e^{-x}\right) + c_2e^x + \frac{1}{2}c_3e^{2x} + c_4$$

6.11 problem Problem 33

Internal problem ID [2735]
Internal file name [`OUTPUT/2227_Sunday_June_05_2022_02_55_32_AM_56122593/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.1, General Theory for Linear Differential Equations. page 502
Problem number: Problem 33.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' - 13y'' + 36y = 0}$$

The characteristic equation is

$$\lambda^4 - 13\lambda^2 + 36 = 0$$

The roots of the above equation are

$$\lambda_1 = 2$$
$$\lambda_2 = -2$$
$$\lambda_3 = -3$$
$$\lambda_4 = 3$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-3x} + c_3 \mathrm{e}^{2x} + \mathrm{e}^{3x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= \mathrm{e}^{-2x} \\
y_2 &= \mathrm{e}^{-3x} \\
y_3 &= \mathrm{e}^{2x} \\
y_4 &= \mathrm{e}^{3x}
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-3x} + c_3 \mathrm{e}^{2x} + \mathrm{e}^{3x} c_4 \tag{1}
$$

Verification of solutions

$$
y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-3x} + c_3 \mathrm{e}^{2x} + \mathrm{e}^{3x} c_4
$$

Verified OK.

6.11.1 Maple step by step solution

Let's solve

$y'''' - 13y'' + 36y = 0$

- Highest derivative means the order of the ODE is 4

  y''''

- □ Convert linear ODE into a system of first order ODEs
  - ○ Define new variable $y_1(x)$

    $y_1(x) = y$
  - ○ Define new variable $y_2(x)$

    $y_2(x) = y'$
  - ○ Define new variable $y_3(x)$

    $y_3(x) = y''$
  - ○ Define new variable $y_4(x)$

    $y_4(x) = y'''$
  - ○ Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = 13y_3(x) - 36y_1(x)$

  Convert linear ODE into a system of first order ODEs

$$[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = 13y_3(x) - 36y_1(x)]$$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -36 & 0 & 13 & 0 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -36 & 0 & 13 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-3, \begin{bmatrix} -\frac{1}{27} \\ \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix}\right], \left[-2, \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right], \left[3, \begin{bmatrix} \frac{1}{27} \\ \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-3, \left[\begin{array}{c} -\frac{1}{27} \\ \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = e^{-3x} \cdot \left[\begin{array}{c} -\frac{1}{27} \\ \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[-2, \left[\begin{array}{c} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^{-2x} \cdot \left[\begin{array}{c} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[2, \left[\begin{array}{c} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{2x} \cdot \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[3, \begin{bmatrix} \frac{1}{27} \\ \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_4 = e^{3x} \cdot \begin{bmatrix} \frac{1}{27} \\ \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{y} = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3 + c_4 \vec{y}_4$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-3x} \cdot \begin{bmatrix} -\frac{1}{27} \\ \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix} + c_2 e^{-2x} \cdot \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix} + c_3 e^{2x} \cdot \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} + e^{3x} c_4 \cdot \begin{bmatrix} \frac{1}{27} \\ \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{\left(8\, e^{6x} c_4 + 27 c_3 e^{5x} - 27 c_2 e^{x} - 8 c_1\right) e^{-3x}}{216}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(diff(y(x),x$4)-13*diff(y(x),x$2)+36*y(x)=0,y(x), singsol=all)
```

$$y(x) = \left(c_1 e^{6x} + c_4 e^{5x} + c_2 e^{x} + c_3\right) e^{-3x}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 35

```
DSolve[y''''[x]-13*y''[x]+36*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-3x}\left(c_2 e^{x} + e^{5x}(c_4 e^{x} + c_3) + c_1\right)$$

6.12 problem Problem 34

Internal problem ID [2736]
Internal file name [`OUTPUT/2228_Sunday_June_05_2022_02_55_34_AM_22159462/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.1, General Theory for Linear Differential Equations. page 502
Problem number: Problem 34.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler], [_2nd_order, _linear, `_with_symmetry_[0,F(
   x)]`]]
```

$$\boxed{x^2y'' + 3xy' - 8y = 0}$$

6.12.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} + 3xrx^{r-1} - 8x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r + 3r\,x^r - 8x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) + 3r - 8 = 0$$

Or

$$r^2 + 2r - 8 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -4$$
$$r_2 = 2$$

Since the roots are real and distinct, then the general solution is

$$y = c_1y_1 + c_2y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = \frac{c_1}{x^4} + c_2x^2$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{x^4} + c_2x^2 \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x^4} + c_2x^2$$

Verified OK.

6.12.2 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$x^2y'' + 3xy' - 8y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$p(x) = \frac{3}{x}$$
$$q(x) = -\frac{8}{x^2}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\
&= \int \mathrm{e}^{-\left(\int \frac{3}{x}dx\right)}dx \\
&= \int e^{-3\ln(x)}\,dx \\
&= \int \frac{1}{x^3}dx \\
&= -\frac{1}{2x^2}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{-\frac{8}{x^2}}{\frac{1}{x^6}} \\
&= -8x^4
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0$$
$$\frac{d^2}{d\tau^2}y(\tau) - 8x^4 y(\tau) = 0$$

But in terms of τ

$$-8x^4 = -\frac{2}{\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) - \frac{2y(\tau)}{\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 - 2y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} - 2\tau^r = 0$$

Simplifying gives

$$r(r-1)\,\tau^r + 0\,\tau^r - 2\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$r(r-1) + 0 - 2 = 0$$

Or

$$r^2 - r - 2 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= -1 \\ r_2 &= 2 \end{aligned}$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1 y_1 + c_2 y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = \frac{c_1}{\tau} + c_2\tau^2$$

The above solution is now transformed back to y using (6) which results in

$$y = -2c_1x^2 + \frac{c_2}{4x^4}$$

Summary

The solution(s) found are the following

$$y = -2c_1x^2 + \frac{c_2}{4x^4} \tag{1}$$

Verification of solutions

$$y = -2c_1x^2 + \frac{c_2}{4x^4}$$

Verified OK.

6.12.3 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$x^2y'' + 3xy' - 8y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{3}{x} \\ q(x) &= -\frac{8}{x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{3n}{x^2} - \frac{8}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 2 \tag{6}$$

Substituting this value in (3) gives

$$
\begin{aligned}
v''(x) + \frac{7v'(x)}{x} &= 0 \\
v''(x) + \frac{7v'(x)}{x} &= 0 \qquad (7)
\end{aligned}
$$

Using the substitution

$$
u(x) = v'(x)
$$

Then (7) becomes

$$
u'(x) + \frac{7u(x)}{x} = 0 \qquad (8)
$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= -\frac{7u}{x}
\end{aligned}
$$

Where $f(x) = -\frac{7}{x}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= -\frac{7}{x}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{7}{x}\,dx \\
\ln(u) &= -7\ln(x) + c_1 \\
u &= \mathrm{e}^{-7\ln(x)+c_1} \\
&= \frac{c_1}{x^7}
\end{aligned}
$$

Now that $u(x)$ is known, then

$$
\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\,dx + c_2 \\
&= -\frac{c_1}{6x^6} + c_2
\end{aligned}
$$

Hence

$$
\begin{aligned}
y &= v(x)\, x^n \\
&= \left(-\frac{c_1}{6x^6} + c_2\right) x^2 \\
&= \frac{6c_2x^6 - c_1}{6x^4}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \left(-\frac{c_1}{6x^6} + c_2\right) x^2 \tag{1}
$$

Verification of solutions

$$
y = \left(-\frac{c_1}{6x^6} + c_2\right) x^2
$$

Verified OK.

6.12.4 Solving using Kovacic algorithm

Writing the ode as

$$
x^2y'' + 3xy' - 8y = 0 \tag{1}
$$

$$
Ay'' + By' + Cy = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= x^2 \\
B &= 3x \\
C &= -8
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = ye^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$
z''(x) = rz(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{35}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 35 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{35}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 188: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{35}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{35}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{35}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{35}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{5}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -\frac{5}{2} - \left(-\frac{5}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{5}{2x} + (-)\,(0) \\
&= -\frac{5}{2x} \\
&= -\frac{5}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{5}{2x}\right)(0) + \left(\left(\frac{5}{2x^2}\right) + \left(-\frac{5}{2x}\right)^2 - \left(\frac{35}{4x^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int -\frac{5}{2x} dx} \\ &= \frac{1}{x^{\frac{5}{2}}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{3x}{x^2} \, dx} \\ &= z_1 e^{-\frac{3\ln(x)}{2}} \\ &= z_1 \left(\frac{1}{x^{\frac{3}{2}}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x^4}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{3x}{x^2} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-3\ln(x)}}{(y_1)^2} \, dx \\ &= y_1 \left(\frac{x^6}{6}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{1}{x^4}\right) + c_2\left(\frac{1}{x^4}\left(\frac{x^6}{6}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{x^4} + \frac{c_2 x^2}{6} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x^4} + \frac{c_2 x^2}{6}$$

Verified OK.

6.12.5 Maple step by step solution

Let's solve

$x^2 y'' + 3xy' - 8y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{3y'}{x} + \frac{8y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{3y'}{x} - \frac{8y}{x^2} = 0$

- Multiply by denominators of the ODE

  $x^2 y'' + 3xy' - 8y = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  ◦ Calculate the 1st derivative of y with respect to x , using the chain rule

  $y' = \left(\frac{d}{dt} y(t)\right) t'(x)$

- Compute derivative

  $y' = \frac{\frac{d}{dt}y(t)}{x}$

- Calculate the 2nd derivative of y with respect to x , using the chain rule

  $y'' = \left(\frac{d^2}{dt^2}y(t)\right)t'(x)^2 + t''(x)\left(\frac{d}{dt}y(t)\right)$

- Compute derivative

  $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) + 3\frac{d}{dt}y(t) - 8y(t) = 0$

- Simplify

  $\frac{d^2}{dt^2}y(t) + 2\frac{d}{dt}y(t) - 8y(t) = 0$

- Characteristic polynomial of ODE

  $r^2 + 2r - 8 = 0$

- Factor the characteristic polynomial

  $(r + 4)(r - 2) = 0$

- Roots of the characteristic polynomial

  $r = (-4, 2)$

- 1st solution of the ODE

  $y_1(t) = e^{-4t}$

- 2nd solution of the ODE

  $y_2(t) = e^{2t}$

- General solution of the ODE

  $y(t) = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y(t) = c_1 e^{-4t} + c_2 e^{2t}$

- Change variables back using $t = \ln(x)$

  $y = \frac{c_1}{x^4} + c_2 x^2$

- Simplify

  $y = \frac{c_1}{x^4} + c_2 x^2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(x^2*diff(y(x),x$2)+3*x*diff(y(x),x)-8*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 x^6 + c_2}{x^4}$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 18

```
DSolve[x^2*y''[x]+3*x*y'[x]-8*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_2 x^6 + c_1}{x^4}$$

6.13 problem Problem 35

Internal problem ID [2737]
Internal file name [`OUTPUT/2229_Sunday_June_05_2022_02_55_35_AM_94114452/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.1, General Theory for Linear Differential Equations. page 502
Problem number: Problem 35.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "exact linear second order ode", "second_order_integrable_as_is", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{2x^2y'' + 5xy' + y = 0}$$

6.13.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$2x^2(r(r-1))x^{r-2} + 5xrx^{r-1} + x^r = 0$$

Simplifying gives

$$2r(r-1)\,x^r + 5r\,x^r + x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$2r(r-1) + 5r + 1 = 0$$

Or

$$2r^2 + 3r + 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= -1 \\ r_2 &= -\frac{1}{2} \end{aligned}$$

Since the roots are real and distinct, then the general solution is

$$y = c_1y_1 + c_2y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = \frac{c_1}{x} + \frac{c_2}{\sqrt{x}}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1}{x} + \frac{c_2}{\sqrt{x}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x} + \frac{c_2}{\sqrt{x}}$$

Verified OK.

6.13.2 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$2x^2y'' + 5xy' + y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{5}{2x} \\ q(x) &= \frac{1}{2x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\ &= \int \mathrm{e}^{-\left(\int \frac{5}{2x}dx\right)}dx \\ &= \int e^{-\frac{5\ln(x)}{2}}\,dx \\ &= \int \frac{1}{x^{\frac{5}{2}}}dx \\ &= -\frac{2}{3x^{\frac{3}{2}}} \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$
\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{1}{2x^2}}{\frac{1}{x^5}} \\
&= \frac{x^3}{2}
\end{aligned} \tag{7}
$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$
\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + \frac{x^3 y(\tau)}{2} &= 0
\end{aligned}
$$

But in terms of τ

$$
\frac{x^3}{2} = \frac{2}{9\tau^2}
$$

Hence the above ode becomes

$$
\frac{d^2}{d\tau^2}y(\tau) + \frac{2y(\tau)}{9\tau^2} = 0
$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$
9\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + 2y(\tau) = 0
$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$
9\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 2\tau^r = 0
$$

Simplifying gives

$$
9r(r-1)\,\tau^r + 0\,\tau^r + 2\tau^r = 0
$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$
9r(r-1) + 0 + 2 = 0
$$

Or

$$
9r^2 - 9r + 2 = 0 \tag{1}
$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{3}$$
$$r_2 = \frac{2}{3}$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1 y_1 + c_2 y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = c_1 \tau^{\frac{1}{3}} + c_2 \tau^{\frac{2}{3}}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{18^{\frac{1}{3}} \left(-\frac{1}{x^{\frac{3}{2}}}\right)^{\frac{1}{3}} \left(c_2 18^{\frac{1}{3}} \left(-\frac{1}{x^{\frac{3}{2}}}\right)^{\frac{1}{3}} + 3c_1\right)}{9}$$

Summary

The solution(s) found are the following

$$y = \frac{18^{\frac{1}{3}} \left(-\frac{1}{x^{\frac{3}{2}}}\right)^{\frac{1}{3}} \left(c_2 18^{\frac{1}{3}} \left(-\frac{1}{x^{\frac{3}{2}}}\right)^{\frac{1}{3}} + 3c_1\right)}{9} \tag{1}$$

Verification of solutions

$$y = \frac{18^{\frac{1}{3}} \left(-\frac{1}{x^{\frac{3}{2}}}\right)^{\frac{1}{3}} \left(c_2 18^{\frac{1}{3}} \left(-\frac{1}{x^{\frac{3}{2}}}\right)^{\frac{1}{3}} + 3c_1\right)}{9}$$

Verified OK.

6.13.3 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$2x^2 y'' + 5xy' + y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$
\begin{aligned}
p(x) &= \frac{5}{2x} \\
q(x) &= \frac{1}{2x^2}
\end{aligned}
$$

Applying change of variables $\tau = g(x)$ to (2) results

$$
\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}
$$

Where τ is the new independent variable, and

$$
p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}
$$

$$
q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}
$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$
\begin{aligned}
\tau' &= \frac{1}{c}\sqrt{q} \\
&= \frac{\sqrt{2}\,\sqrt{\frac{1}{x^2}}}{2c} \qquad (6) \\
\tau'' &= -\frac{\sqrt{2}}{2c\sqrt{\frac{1}{x^2}}\,x^3}
\end{aligned}
$$

Substituting the above into (4) results in

$$
\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{-\frac{\sqrt{2}}{2c\sqrt{\frac{1}{x^2}}\,x^3} + \frac{5}{2x}\frac{\sqrt{2}\sqrt{\frac{1}{x^2}}}{2c}}{\left(\frac{\sqrt{2}\sqrt{\frac{1}{x^2}}}{2c}\right)^2} \\
&= \frac{3c\sqrt{2}}{2}
\end{aligned}
$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + \frac{3c\sqrt{2}\left(\frac{d}{d\tau}y(\tau)\right)}{2} + c^2 y(\tau) &= 0 \qquad (7)
\end{aligned}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = \mathrm{e}^{-\frac{3\sqrt{2}\,c\tau}{4}} \left(c_1 \cosh\left(\frac{\sqrt{2}\,c\tau}{4}\right) + i c_2 \sinh\left(\frac{\sqrt{2}\,c\tau}{4}\right)\right)$$

Now from (6)

$$\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int \frac{\sqrt{2}\sqrt{\frac{1}{x^2}}}{2} dx}{c} \\
&= \frac{\sqrt{2}\sqrt{\frac{1}{x^2}}\,x\ln(x)}{2c}
\end{aligned}$$

Substituting the above into the solution obtained gives

$$y = \frac{c_1 \cosh\left(\frac{\ln(x)}{4}\right) + i c_2 \sinh\left(\frac{\ln(x)}{4}\right)}{x^{\frac{3}{4}}}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 \cosh\left(\frac{\ln(x)}{4}\right) + i c_2 \sinh\left(\frac{\ln(x)}{4}\right)}{x^{\frac{3}{4}}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \cosh\left(\frac{\ln(x)}{4}\right) + i c_2 \sinh\left(\frac{\ln(x)}{4}\right)}{x^{\frac{3}{4}}}$$

Verified OK.

6.13.4 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$2x^2y'' + 5xy' + y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$
\begin{aligned}
p(x) &= \frac{5}{2x} \\
q(x) &= \frac{1}{2x^2}
\end{aligned}
$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$
v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}
$$

Let the coefficient of $v(x)$ above be zero. Hence

$$
\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}
$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$
\frac{n(n-1)}{x^2} + \frac{5n}{2x^2} + \frac{1}{2x^2} = 0 \tag{5}
$$

Solving (5) for n gives

$$
n = -\frac{1}{2} \tag{6}
$$

Substituting this value in (3) gives

$$
\begin{aligned}
v''(x) + \frac{3v'(x)}{2x} &= 0 \\
v''(x) + \frac{3v'(x)}{2x} &= 0
\end{aligned} \tag{7}
$$

Using the substitution

$$
u(x) = v'(x)
$$

Then (7) becomes

$$
u'(x) + \frac{3u(x)}{2x} = 0 \tag{8}
$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= -\frac{3u}{2x}
\end{aligned}
$$

Where $f(x) = -\frac{3}{2x}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\, du &= -\frac{3}{2x}\, dx \\
\int \frac{1}{u}\, du &= \int -\frac{3}{2x}\, dx \\
\ln\left(u\right) &= -\frac{3\ln\left(x\right)}{2} + c_1 \\
u &= \mathrm{e}^{-\frac{3\ln(x)}{2}+c_1} \\
&= \frac{c_1}{x^{\frac{3}{2}}}
\end{aligned}
$$

Now that $u(x)$ is known, then

$$
\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\, dx + c_2 \\
&= -\frac{2c_1}{\sqrt{x}} + c_2
\end{aligned}
$$

Hence

$$
\begin{aligned}
y &= v(x)\, x^n \\
&= \frac{-\frac{2c_1}{\sqrt{x}} + c_2}{\sqrt{x}} \\
&= \frac{\sqrt{x}\, c_2 - 2c_1}{x}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{-\frac{2c_1}{\sqrt{x}} + c_2}{\sqrt{x}} \tag{1}
$$

Verification of solutions

$$y = \frac{-\frac{2c_1}{\sqrt{x}} + c_2}{\sqrt{x}}$$

Verified OK.

6.13.5 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int \left(2x^2y'' + 5xy' + y\right) dx = 0$$
$$yx + 2y'x^2 = c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{1}{2x}$$
$$q(x) = \frac{c_1}{2x^2}$$

Hence the ode is

$$y' + \frac{y}{2x} = \frac{c_1}{2x^2}$$

The integrating factor μ is

$$\mu = \mathrm{e}^{\int \frac{1}{2x}dx}$$
$$= \sqrt{x}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = (\mu)\left(\frac{c_1}{2x^2}\right)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{x}\, y\right) = \left(\sqrt{x}\right)\left(\frac{c_1}{2x^2}\right)$$
$$\mathrm{d}\left(\sqrt{x}\, y\right) = \left(\frac{c_1}{2x^{\frac{3}{2}}}\right) \mathrm{d}x$$

Integrating gives

$$
\begin{aligned}
\sqrt{x}\, y &= \int \frac{c_1}{2x^{\frac{3}{2}}}\, \mathrm{d}x \\
\sqrt{x}\, y &= -\frac{c_1}{\sqrt{x}} + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \sqrt{x}$ results in

$$
y = -\frac{c_1}{x} + \frac{c_2}{\sqrt{x}}
$$

Summary

The solution(s) found are the following

$$
y = -\frac{c_1}{x} + \frac{c_2}{\sqrt{x}} \tag{1}
$$

Verification of solutions

$$
y = -\frac{c_1}{x} + \frac{c_2}{\sqrt{x}}
$$

Verified OK.

6.13.6 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$
2x^2y'' + 5xy' + y = 0
$$

Integrating both sides of the ODE w.r.t x gives

$$
\begin{aligned}
\int \left(2x^2y'' + 5xy' + y\right) dx &= 0 \\
yx + 2y'x^2 &= c_1
\end{aligned}
$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
y' + p(x)y = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= \frac{1}{2x} \\
q(x) &= \frac{c_1}{2x^2}
\end{aligned}
$$

Hence the ode is

$$y' + \frac{y}{2x} = \frac{c_1}{2x^2}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{1}{2x}dx} \\ &= \sqrt{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{c_1}{2x^2}\right) \\ \frac{d}{dx}\left(\sqrt{x}\, y\right) &= \left(\sqrt{x}\right)\left(\frac{c_1}{2x^2}\right) \\ d\left(\sqrt{x}\, y\right) &= \left(\frac{c_1}{2x^{\frac{3}{2}}}\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\sqrt{x}\, y &= \int \frac{c_1}{2x^{\frac{3}{2}}} \, dx \\ \sqrt{x}\, y &= -\frac{c_1}{\sqrt{x}} + c_2\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \sqrt{x}$ results in

$$y = -\frac{c_1}{x} + \frac{c_2}{\sqrt{x}}$$

Summary

The solution(s) found are the following

$$y = -\frac{c_1}{x} + \frac{c_2}{\sqrt{x}} \tag{1}$$

Verification of solutions

$$y = -\frac{c_1}{x} + \frac{c_2}{\sqrt{x}}$$

Verified OK.

6.13.7 Solving using Kovacic algorithm

Writing the ode as

$$
\begin{align*}
2x^2y'' + 5xy' + y &= 0 \tag{1} \\
Ay'' + By' + Cy &= 0 \tag{2}
\end{align*}
$$

Comparing (1) and (2) shows that

$$
\begin{align*}
A &= 2x^2 \\
B &= 5x \tag{3} \\
C &= 1
\end{align*}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = ye^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$
z''(x) = rz(x) \tag{4}
$$

Where r is given by

$$
\begin{align*}
r &= \frac{s}{t} \tag{5} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{align*}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-3}{16x^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{align*}
s &= -3 \\
t &= 16x^2
\end{align*}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(-\frac{3}{16x^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 190: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16x^2}$$

For the pole at $x=0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b=-\frac{3}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{3}{4} \\
\alpha_c^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{4}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r=\frac{s}{t}=-\frac{3}{16x^2}
$$

Since the $\gcd(s,t)=1$. This gives $b=-\frac{3}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{3}{4} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{4}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=-\frac{3}{16x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_{\infty}^{-} = \frac{1}{4}$ then

$$
\begin{aligned}
d &= \alpha_{\infty}^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= \frac{1}{4} - \left(\frac{1}{4}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_{\infty}
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{1}{4x} + (-)(0) \\
&= \frac{1}{4x} \\
&= \frac{1}{4x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{1}{4x}\right)(0) + \left(\left(-\frac{1}{4x^2}\right) + \left(\frac{1}{4x}\right)^2 - \left(-\frac{3}{16x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \frac{1}{4x} dx} \\
&= x^{\frac{1}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{5x}{2x^2}\,dx} \\
&= z_1 e^{-\frac{5\ln(x)}{4}} \\
&= z_1 \left(\frac{1}{x^{\frac{5}{4}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{1}{x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{5x}{2x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{5\ln(x)}{2}}}{(y_1)^2}\,dx \\
&= y_1 (2\sqrt{x})
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{x}\right) + c_2 \left(\frac{1}{x}(2\sqrt{x})\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1}{x} + \frac{2c_2}{\sqrt{x}} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1}{x} + \frac{2c_2}{\sqrt{x}}
$$

Verified OK.

6.13.8 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned}
p(x) &= 2x^2 \\
q(x) &= 5x \\
r(x) &= 1 \\
s(x) &= 0
\end{aligned}$$

Hence

$$\begin{aligned}
p''(x) &= 4 \\
q'(x) &= 5
\end{aligned}$$

Therefore (1) becomes

$$4 - (5) + (1) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\,y' + (q(x) - p'(x))\,y)' = s(x)$$

Integrating gives

$$p(x)\,y' + (q(x) - p'(x))\,y = \int s(x)\,dx$$

Substituting the above values for p, q, r, s gives

$$yx + 2y'x^2 = c_1$$

We now have a first order ode to solve which is

$$yx + 2y'x^2 = c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$
\begin{aligned}
p(x) &= \frac{1}{2x} \\
q(x) &= \frac{c_1}{2x^2}
\end{aligned}
$$

Hence the ode is

$$
y' + \frac{y}{2x} = \frac{c_1}{2x^2}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int \frac{1}{2x} dx} \\
&= \sqrt{x}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{c_1}{2x^2}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{x}\, y\right) &= \left(\sqrt{x}\right)\left(\frac{c_1}{2x^2}\right) \\
\mathrm{d}\left(\sqrt{x}\, y\right) &= \left(\frac{c_1}{2x^{\frac{3}{2}}}\right) \mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\sqrt{x}\, y &= \int \frac{c_1}{2x^{\frac{3}{2}}} \,\mathrm{d}x \\
\sqrt{x}\, y &= -\frac{c_1}{\sqrt{x}} + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \sqrt{x}$ results in

$$
y = -\frac{c_1}{x} + \frac{c_2}{\sqrt{x}}
$$

Summary

The solution(s) found are the following

$$
y = -\frac{c_1}{x} + \frac{c_2}{\sqrt{x}} \tag{1}
$$

Verification of solutions

$$
y = -\frac{c_1}{x} + \frac{c_2}{\sqrt{x}}
$$

Verified OK.

6.13.9 Maple step by step solution

Let's solve

$2x^2y'' + 5xy' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{5y'}{2x} - \frac{y}{2x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{5y'}{2x} + \frac{y}{2x^2} = 0$

- Multiply by denominators of the ODE

  $2x^2y'' + 5xy' + y = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  ◦ Calculate the 1st derivative of y with respect to x , using the chain rule

  $y' = \left(\frac{d}{dt}y(t)\right)t'(x)$

  ◦ Compute derivative

  $y' = \frac{\frac{d}{dt}y(t)}{x}$

  ◦ Calculate the 2nd derivative of y with respect to x , using the chain rule

  $y'' = \left(\frac{d^2}{dt^2}y(t)\right)t'(x)^2 + t''(x)\left(\frac{d}{dt}y(t)\right)$

  ◦ Compute derivative

  $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $2x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) + 5\frac{d}{dt}y(t) + y(t) = 0$

- Simplify

  $2\frac{d^2}{dt^2}y(t) + 3\frac{d}{dt}y(t) + y(t) = 0$

- Isolate 2nd derivative

$\frac{d^2}{dt^2}y(t) = -\frac{3\frac{d}{dt}y(t)}{2} - \frac{y(t)}{2}$

- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dt^2}y(t) + \frac{3\frac{d}{dt}y(t)}{2} + \frac{y(t)}{2} = 0$
- Characteristic polynomial of ODE

  $r^2 + \frac{3}{2}r + \frac{1}{2} = 0$
- Factor the characteristic polynomial

  $\frac{(r+1)(2r+1)}{2} = 0$
- Roots of the characteristic polynomial

  $r = \left(-1, -\frac{1}{2}\right)$
- 1st solution of the ODE

  $y_1(t) = \mathrm{e}^{-t}$
- 2nd solution of the ODE

  $y_2(t) = \mathrm{e}^{-\frac{t}{2}}$
- General solution of the ODE

  $y(t) = c_1 y_1(t) + c_2 y_2(t)$
- Substitute in solutions

  $y(t) = c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{-\frac{t}{2}}$
- Change variables back using $t = \ln(x)$

  $y = \frac{c_1}{x} + \frac{c_2}{\sqrt{x}}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(2*x^2*diff(y(x),x$2)+5*x*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1}{\sqrt{x}} + \frac{c_2}{x}$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 20

```
DSolve[2*x^2*y''[x]+5*x*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_2\sqrt{x} + c_1}{x}$$

6.14 problem Problem 36

Internal problem ID [2738]
Internal file name [`OUTPUT/2230_Sunday_June_05_2022_02_55_37_AM_78723603/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.1, General Theory for Linear Differential Equations. page 502
Problem number: Problem 36.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_ODE_non_constant_coefficients_of_type_Euler"**

Maple gives the following as the ode type

```
[[_3rd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{x^3 y''' + x^2 y'' - 2xy' + 2y = 0}$$

This is Euler ODE of higher order. Let $y = x^\lambda$. Hence

$$\begin{aligned} y' &= \lambda\, x^{\lambda-1} \\ y'' &= \lambda(\lambda-1)\, x^{\lambda-2} \\ y''' &= \lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} \end{aligned}$$

Substituting these back into

$$x^3 y''' + x^2 y'' - 2xy' + 2y = 0$$

gives

$$-2x\lambda\, x^{\lambda-1} + x^2\lambda(\lambda-1)\, x^{\lambda-2} + x^3\lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} + 2x^\lambda = 0$$

Which simplifies to

$$-2\lambda\, x^\lambda + \lambda(\lambda-1)\, x^\lambda + \lambda(\lambda-1)\,(\lambda-2)\, x^\lambda + 2x^\lambda = 0$$

And since $x^\lambda \neq 0$ then dividing through by x^λ, the above becomes

$$-2\lambda + \lambda(\lambda - 1) + \lambda(\lambda - 1)(\lambda - 2) + 2 = 0$$

Simplifying gives the characteristic equation as

$$\lambda^3 - 2\lambda^2 - \lambda + 2 = 0$$

Solving the above gives the following roots

$$\lambda_1 = 1$$
$$\lambda_2 = 2$$
$$\lambda_3 = -1$$

This table summarises the result

root	multiplicity	type of root
-1	1	real root
1	1	real root
2	1	real root

The solution is generated by going over the above table. For each real root λ of multiplicity one generates a $c_1 x^\lambda$ basis solution. Each real root of multiplicty two, generates $c_1 x^\lambda$ and $c_2 x^\lambda \ln(x)$ basis solutions. Each real root of multiplicty three, generates $c_1 x^\lambda$ and $c_2 x^\lambda \ln(x)$ and $c_3 x^\lambda \ln(x)^2$ basis solutions, and so on. Each complex root $\alpha \pm i\beta$ of multiplicity one generates $x^\alpha(c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity two generates $\ln(x)\, x^\alpha(c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity three generates $\ln(x)^2\, x^\alpha(c_1 \cos(\beta \ln(x)) + c_2$ basis solutions. And so on. Using the above show that the solution is

$$y = \frac{c_1}{x} + c_2 x + c_3 x^2$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = \frac{1}{x}$$
$$y_2 = x$$
$$y_3 = x^2$$

Summary
The solution(s) found are the following

$$y = \frac{c_1}{x} + c_2 x + c_3 x^2 \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x} + c_2 x + c_3 x^2$$

Verified OK.

6.14.1 Maple step by step solution

Let's solve

$x^3 y''' + x^2 y'' - 2xy' + 2y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

- Isolate 3rd derivative

  $y''' = -\frac{2y}{x^3} - \frac{y''x - 2y'}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''' + \frac{y''}{x} - \frac{2y'}{x^2} + \frac{2y}{x^3} = 0$

- Multiply by denominators of the ODE

  $x^3 y''' + x^2 y'' - 2xy' + 2y = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt} y(t)\right) t'(x)$

  - Compute derivative

    $y' = \frac{\frac{d}{dt} y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2} y(t)\right) t'(x)^2 + t''(x) \left(\frac{d}{dt} y(t)\right)$

  - Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2} y(t)}{x^2} - \frac{\frac{d}{dt} y(t)}{x^2}$

  - Calculate the 3rd derivative of y with respect to x , using the chain rule

    $y''' = \left(\frac{d^3}{dt^3} y(t)\right) t'(x)^3 + 3t'(x)\, t''(x) \left(\frac{d^2}{dt^2} y(t)\right) + t'''(x) \left(\frac{d}{dt} y(t)\right)$

- Compute derivative

  $y''' = \frac{\frac{d^3}{dt^3}y(t)}{x^3} - \frac{3\left(\frac{d^2}{dt^2}y(t)\right)}{x^3} + \frac{2\left(\frac{d}{dt}y(t)\right)}{x^3}$

  Substitute the change of variables back into the ODE

  $x^3\left(\frac{\frac{d^3}{dt^3}y(t)}{x^3} - \frac{3\left(\frac{d^2}{dt^2}y(t)\right)}{x^3} + \frac{2\left(\frac{d}{dt}y(t)\right)}{x^3}\right) + x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) - 2\frac{d}{dt}y(t) + 2y(t) = 0$

- Simplify

  $\frac{d^3}{dt^3}y(t) - 2\frac{d^2}{dt^2}y(t) - \frac{d}{dt}y(t) + 2y(t) = 0$

☐ Convert linear ODE into a system of first order ODEs

- Define new variable $y_1(t)$

  $y_1(t) = y(t)$

- Define new variable $y_2(t)$

  $y_2(t) = \frac{d}{dt}y(t)$

- Define new variable $y_3(t)$

  $y_3(t) = \frac{d^2}{dt^2}y(t)$

- Isolate for $\frac{d}{dt}y_3(t)$ using original ODE

  $\frac{d}{dt}y_3(t) = 2y_3(t) + y_2(t) - 2y_1(t)$

  Convert linear ODE into a system of first order ODEs

  $\left[y_2(t) = \frac{d}{dt}y_1(t), y_3(t) = \frac{d}{dt}y_2(t), \frac{d}{dt}y_3(t) = 2y_3(t) + y_2(t) - 2y_1(t)\right]$

- Define vector

  $\vec{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$

- System to solve

  $\frac{d}{dt}\vec{y}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 2 \end{bmatrix} \cdot \vec{y}(t)$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 2 \end{bmatrix}$$

- Rewrite the system as
  $\frac{d}{dt}\vec{y}(t) = A \cdot \vec{y}(t)$
- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A
  $$\left[\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]\right]$$
- Consider eigenpair
  $$\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right]$$
- Solution to homogeneous system from eigenpair
  $$\vec{y}_1 = \mathrm{e}^{-t} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$
- Consider eigenpair
  $$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$
- Solution to homogeneous system from eigenpair
  $$\vec{y}_2 = \mathrm{e}^{t} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$
- Consider eigenpair

$$\left[2, \left[\begin{array}{c} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_3 = \mathrm{e}^{2t} \cdot \left[\begin{array}{c} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{array}\right]$$

- General solution to the system of ODEs

$$\overrightarrow{y} = c_1 \overrightarrow{y}_1 + c_2 \overrightarrow{y}_2 + c_3 \overrightarrow{y}_3$$

- Substitute solutions into the general solution

$$\overrightarrow{y} = c_1 \mathrm{e}^{-t} \cdot \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right] + c_2 \mathrm{e}^{t} \cdot \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right] + c_3 \mathrm{e}^{2t} \cdot \left[\begin{array}{c} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y(t) = c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{t} + \frac{c_3 \mathrm{e}^{2t}}{4}$$

- Change variables back using $t = \ln(x)$

$$y = \frac{c_1}{x} + c_2 x + \frac{c_3 x^2}{4}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve(x^3*diff(y(x),x$3)+x^2*diff(y(x),x$2)-2*x*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_3 x^3 + c_2 x^2 + c_1}{x}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 22

```
DSolve[x^3*y'''[x]+x^2*y''[x]-2*x*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_3 x^2 + c_2 x + \frac{c_1}{x}$$

6.15 problem Problem 37

Internal problem ID [2739]
Internal file name [`OUTPUT/2231_Sunday_June_05_2022_02_55_39_AM_56463461/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.1, General Theory for Linear Differential Equations. page 502
Problem number: Problem 37.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_ODE_non_constant_coefficients_of_type_Euler"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_y]]
```

$$\boxed{x^3y''' + 3x^2y'' - 6xy' = 0}$$

This is Euler ODE of higher order. Let $y = x^\lambda$. Hence

$$\begin{aligned}
y' &= \lambda\, x^{\lambda-1} \\
y'' &= \lambda(\lambda-1)\, x^{\lambda-2} \\
y''' &= \lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3}
\end{aligned}$$

Substituting these back into

$$x^3y''' + 3x^2y'' - 6xy' = 0$$

gives

$$-6x\lambda\, x^{\lambda-1} + 3x^2\lambda(\lambda-1)\, x^{\lambda-2} + x^3\lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} = 0$$

Which simplifies to

$$-6\lambda\, x^\lambda + 3\lambda(\lambda-1)\, x^\lambda + \lambda(\lambda-1)\,(\lambda-2)\, x^\lambda = 0$$

And since $x^{\lambda} \neq 0$ then dividing through by x^{λ}, the above becomes

$$-6\lambda + 3\lambda(\lambda - 1) + \lambda(\lambda - 1)(\lambda - 2) = 0$$

Simplifying gives the characteristic equation as

$$\lambda^3 - 7\lambda = 0$$

Solving the above gives the following roots

$$\lambda_1 = 0$$
$$\lambda_2 = \sqrt{7}$$
$$\lambda_3 = -\sqrt{7}$$

This table summarises the result

root	multiplicity	type of root
0	1	real root
$\sqrt{7}$	1	real root
$-\sqrt{7}$	1	real root

The solution is generated by going over the above table. For each real root λ of multiplicity one generates a $c_1 x^{\lambda}$ basis solution. Each real root of multiplicty two, generates $c_1 x^{\lambda}$ and $c_2 x^{\lambda} \ln(x)$ basis solutions. Each real root of multiplicty three, generates $c_1 x^{\lambda}$ and $c_2 x^{\lambda} \ln(x)$ and $c_3 x^{\lambda} \ln(x)^2$ basis solutions, and so on. Each complex root $\alpha \pm i\beta$ of multiplicity one generates $x^{\alpha}(c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity two generates $\ln(x) x^{\alpha}(c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity three generates $\ln(x)^2 x^{\alpha}(c_1 \cos(\beta \ln(x)) + c_2$ basis solutions. And so on. Using the above show that the solution is

$$y = c_1 + c_2 x^{\sqrt{7}} + c_3 x^{-\sqrt{7}}$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = 1$$
$$y_2 = x^{\sqrt{7}}$$
$$y_3 = x^{-\sqrt{7}}$$

Summary
The solution(s) found are the following

$$y = c_1 + c_2 x^{\sqrt{7}} + c_3 x^{-\sqrt{7}} \tag{1}$$

Verification of solutions

$$y = c_1 + c_2 x^{\sqrt{7}} + c_3 x^{-\sqrt{7}}$$

Verified OK.

6.15.1 Maple step by step solution

Let's solve

$x^3 y''' + 3x^2 y'' - 6xy' = 0$

- Highest derivative means the order of the ODE is 3

y'''

- Isolate 3rd derivative

$y''' = -\frac{3(y''x-2y')}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y''' + \frac{3y''}{x} - \frac{6y'}{x^2} = 0$

- Multiply by denominators of the ODE

$y'''x^2 + 3y''x - 6y' = 0$

- Make a change of variables

$t = \ln(x)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

  $y' = \left(\frac{d}{dt}y(t)\right) t'(x)$

  - Compute derivative

  $y' = \frac{\frac{d}{dt}y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

  $y'' = \left(\frac{d^2}{dt^2}y(t)\right) t'(x)^2 + t''(x)\left(\frac{d}{dt}y(t)\right)$

  - Compute derivative

  $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  - Calculate the 3rd derivative of y with respect to x , using the chain rule

  $y''' = \left(\frac{d^3}{dt^3}y(t)\right) t'(x)^3 + 3t'(x)\, t''(x)\left(\frac{d^2}{dt^2}y(t)\right) + t'''(x)\left(\frac{d}{dt}y(t)\right)$

- Compute derivative

  $y''' = \frac{\frac{d^3}{dt^3}y(t)}{x^3} - \frac{3\left(\frac{d^2}{dt^2}y(t)\right)}{x^3} + \frac{2\left(\frac{d}{dt}y(t)\right)}{x^3}$

  Substitute the change of variables back into the ODE

  $\left(\frac{\frac{d^3}{dt^3}y(t)}{x^3} - \frac{3\left(\frac{d^2}{dt^2}y(t)\right)}{x^3} + \frac{2\left(\frac{d}{dt}y(t)\right)}{x^3}\right)x^2 + 3\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right)x - \frac{6\left(\frac{d}{dt}y(t)\right)}{x} = 0$

- Simplify

  $\frac{\frac{d^3}{dt^3}y(t) - 7\frac{d}{dt}y(t)}{x} = 0$

- Isolate 3rd derivative

  $\frac{d^3}{dt^3}y(t) = 7\frac{d}{dt}y(t)$

- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^3}{dt^3}y(t) - 7\frac{d}{dt}y(t) = 0$

- ☐ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(t)$

    $y_1(t) = y(t)$

  - Define new variable $y_2(t)$

    $y_2(t) = \frac{d}{dt}y(t)$

  - Define new variable $y_3(t)$

    $y_3(t) = \frac{d^2}{dt^2}y(t)$

  - Isolate for $\frac{d}{dt}y_3(t)$ using original ODE

    $\frac{d}{dt}y_3(t) = 7y_2(t)$

    Convert linear ODE into a system of first order ODEs

    $\left[y_2(t) = \frac{d}{dt}y_1(t), y_3(t) = \frac{d}{dt}y_2(t), \frac{d}{dt}y_3(t) = 7y_2(t)\right]$

- Define vector

  $\vec{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$

- System to solve

$$\frac{d}{dt}\vec{y}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 7 & 0 \end{bmatrix} \cdot \vec{y}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 7 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\frac{d}{dt}\vec{y}(t) = A \cdot \vec{y}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right], \left[\sqrt{7}, \begin{bmatrix} \frac{1}{7} \\ \frac{\sqrt{7}}{7} \\ 1 \end{bmatrix}\right], \left[-\sqrt{7}, \begin{bmatrix} \frac{1}{7} \\ -\frac{\sqrt{7}}{7} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\left[\sqrt{7}, \begin{bmatrix} \frac{1}{7} \\ \frac{\sqrt{7}}{7} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_2 = e^{\sqrt{7}\,t} \cdot \begin{bmatrix} \frac{1}{7} \\ \frac{\sqrt{7}}{7} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[-\sqrt{7}, \begin{bmatrix} \frac{1}{7} \\ -\frac{\sqrt{7}}{7} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_3 = e^{-\sqrt{7}\,t} \cdot \begin{bmatrix} \frac{1}{7} \\ -\frac{\sqrt{7}}{7} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\overrightarrow{y} = c_1 \overrightarrow{y}_1 + c_2 \overrightarrow{y}_2 + c_3 \overrightarrow{y}_3$$

- Substitute solutions into the general solution

$$\overrightarrow{y} = c_2 e^{\sqrt{7}\,t} \cdot \begin{bmatrix} \frac{1}{7} \\ \frac{\sqrt{7}}{7} \\ 1 \end{bmatrix} + c_3 e^{-\sqrt{7}\,t} \cdot \begin{bmatrix} \frac{1}{7} \\ -\frac{\sqrt{7}}{7} \\ 1 \end{bmatrix} + \begin{bmatrix} c_1 \\ 0 \\ 0 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y(t) = \frac{c_2 e^{\sqrt{7}\,t}}{7} + \frac{c_3 e^{-\sqrt{7}\,t}}{7} + c_1$$

- Change variables back using $t = \ln(x)$

$$y = \frac{c_2 e^{\sqrt{7}\,\ln(x)}}{7} + \frac{c_3 e^{-\sqrt{7}\,\ln(x)}}{7} + c_1$$

- Simplify

$$y = \frac{c_2 x^{\sqrt{7}}}{7} + \frac{c_3 x^{-\sqrt{7}}}{7} + c_1$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
dsolve(x^3*diff(y(x),x$3)+3*x^2*diff(y(x),x$2)-6*x*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = c_1 + c_2 x^{\sqrt{7}} + c_3 x^{-\sqrt{7}}$$

✓ Solution by Mathematica

Time used: 0.056 (sec). Leaf size: 41

```
DSolve[x^3*y'''[x]+3*x^2*y''[x]-6*x*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{c_1 x^{-\sqrt{7}}}{\sqrt{7}} + \frac{c_2 x^{\sqrt{7}}}{\sqrt{7}} + c_3$$

6.16 problem Problem 38

Internal problem ID [2740]
Internal file name [`OUTPUT/2232_Sunday_June_05_2022_02_55_40_AM_37374709/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.1, General Theory for Linear Differential Equations. page 502
Problem number: Problem 38.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + y' - 6y = 18\,\mathrm{e}^{5x}}$$

6.16.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 1, C = -6, f(x) = 18\,\mathrm{e}^{5x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' - 6y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = -6$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + \lambda e^{\lambda x} - 6 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda - 6 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = -6$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-1}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{1^2 - (4)\,(1)\,(-6)}\\ &= -\frac{1}{2} \pm \frac{5}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{1}{2} + \frac{5}{2}\\ \lambda_2 &= -\frac{1}{2} - \frac{5}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2\\ \lambda_2 &= -3\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(2)x} + c_2 e^{(-3)x}\end{aligned}$$

Or

$$y = e^{2x} c_1 + c_2 e^{-3x}$$

Therefore the homogeneous solution y_h is

$$y_h = e^{2x} c_1 + c_2 e^{-3x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$18\,\mathrm{e}^{5x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{5x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\mathrm{e}^{-3x}, \mathrm{e}^{2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \mathrm{e}^{5x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$24A_1 \mathrm{e}^{5x} = 18\,\mathrm{e}^{5x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{3}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{3\,\mathrm{e}^{5x}}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\mathrm{e}^{2x} c_1 + c_2 \mathrm{e}^{-3x}\right) + \left(\frac{3\,\mathrm{e}^{5x}}{4}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{2x} c_1 + c_2 \mathrm{e}^{-3x} + \frac{3\,\mathrm{e}^{5x}}{4} \tag{1}$$

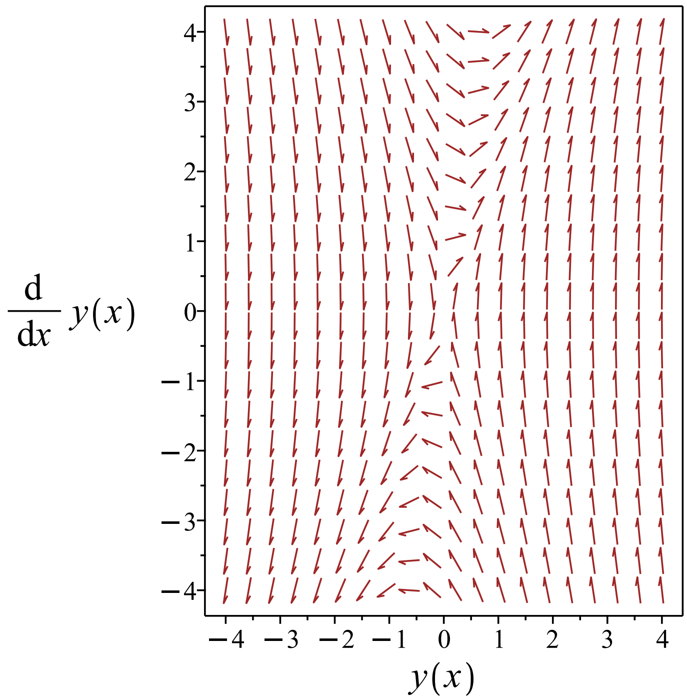


Figure 222: Slope field plot

Verification of solutions

$$y = e^{2x} c_1 + c_2 e^{-3x} + \frac{3\,e^{5x}}{4}$$

Verified OK.

6.16.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y' - 6y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= -6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{25}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 25 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{25z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 194: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = \frac{25}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-\frac{5x}{2}}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dx} \\
&= z_1 e^{-\frac{x}{2}} \\
&= z_1\left(\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-3x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{\mathrm{e}^{5x}}{5}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-3x}\right) + c_2\left(\mathrm{e}^{-3x}\left(\frac{\mathrm{e}^{5x}}{5}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + y' - 6y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1\mathrm{e}^{-3x} + \frac{c_2\mathrm{e}^{2x}}{5}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
18\,\mathrm{e}^{5x}
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\mathrm{e}^{5x}\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{\frac{\mathrm{e}^{2x}}{5}, \mathrm{e}^{-3x}\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1\mathrm{e}^{5x}
$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$24A_1 e^{5x} = 18\,e^{5x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{3}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{3\,e^{5x}}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-3x} + \frac{c_2 e^{2x}}{5}\right) + \left(\frac{3\,e^{5x}}{4}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-3x} + \frac{c_2 e^{2x}}{5} + \frac{3\,e^{5x}}{4} \tag{1}$$

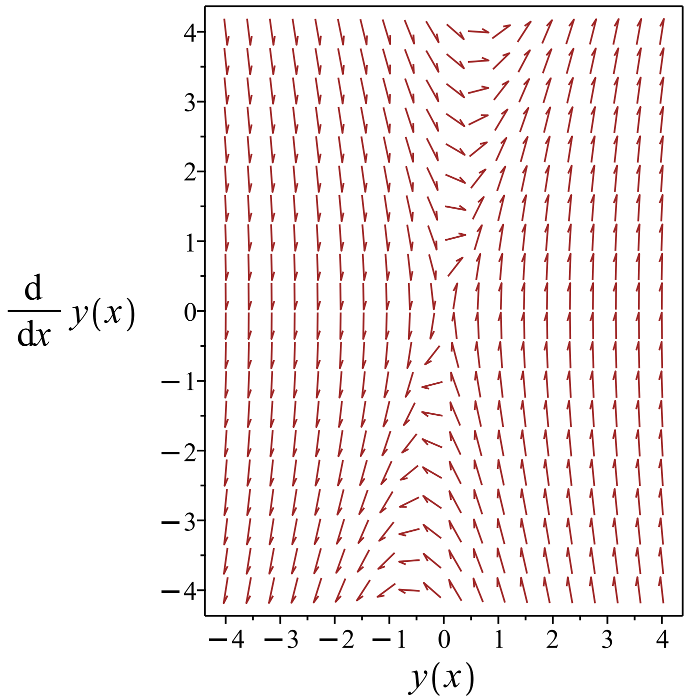


Figure 223: Slope field plot

Verification of solutions

$$y = c_1 e^{-3x} + \frac{c_2 e^{2x}}{5} + \frac{3\,e^{5x}}{4}$$

Verified OK.

6.16.3 Maple step by step solution

Let's solve

$y'' + y' - 6y = 18\,e^{5x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + r - 6 = 0$

- Factor the characteristic polynomial

  $(r+3)(r-2) = 0$

- Roots of the characteristic polynomial

$r = (-3, 2)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-3x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-3x} + c_2 e^{2x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x),y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x),y_2(x))} dx\right), f(x) = 18\,e^{5x}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-3x} & e^{2x} \\ -3\,e^{-3x} & 2\,e^{2x} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 5\,e^{-x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = \frac{18\left(e^{5x}\left(\int e^{3x} dx\right) - \left(\int e^{8x} dx\right)\right) e^{-3x}}{5}$

  - Compute integrals

    $y_p(x) = \frac{3\,e^{5x}}{4}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-3x} + c_2 e^{2x} + \frac{3\,e^{5x}}{4}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(diff(y(x),x$2)+diff(y(x),x)-6*y(x)=18*exp(5*x),y(x), singsol=all)
```

$$y(x) = \frac{(3\,\mathrm{e}^{8x} + 4c_1\mathrm{e}^{5x} + 4c_2)\,\mathrm{e}^{-3x}}{4}$$

✓ Solution by Mathematica

Time used: 0.017 (sec). Leaf size: 31

```
DSolve[y''[x]+y'[x]-6*y[x]==18*Exp[5*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{3e^{5x}}{4} + c_1e^{-3x} + c_2e^{2x}$$

6.17 problem Problem 39

Internal problem ID [2741]
Internal file name [`OUTPUT/2233_Sunday_June_05_2022_02_55_43_AM_21368893/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.1, General Theory for Linear Differential Equations. page 502
Problem number: Problem 39.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + y' - 2y = 4x^2 + 5}$$

6.17.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 1, C = -2, f(x) = 4x^2 + 5$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' - 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = -2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + \lambda \, \mathrm{e}^{\lambda x} - 2 \, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda - 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = -2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{1^2 - (4)(1)(-2)} \\ &= -\frac{1}{2} \pm \frac{3}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{1}{2} + \frac{3}{2} \\ \lambda_2 &= -\frac{1}{2} - \frac{3}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1 \\ \lambda_2 &= -2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(1)x} + c_2 e^{(-2)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-2x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^x, e^{-2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_3x^2 + A_2x + A_1$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_3x^2 - 2A_2x + 2xA_3 - 2A_1 + A_2 + 2A_3 = 4x^2 + 5$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{11}{2}, A_2 = -2, A_3 = -2\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -2x^2 - 2x - \frac{11}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^x + c_2 e^{-2x}\right) + \left(-2x^2 - 2x - \frac{11}{2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^x + c_2 e^{-2x} - 2x^2 - 2x - \frac{11}{2} \tag{1}$$

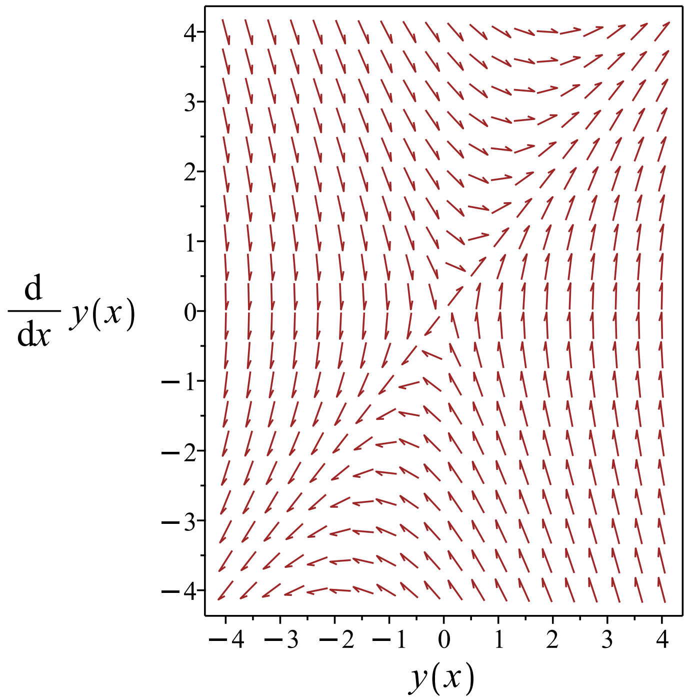


Figure 224: Slope field plot

Verification of solutions

$$y = c_1 e^{x} + c_2 e^{-2x} - 2x^2 - 2x - \frac{11}{2}$$

Verified OK.

6.17.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = 9$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \frac{9z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 196: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{9}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{3x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dx} \\
&= z_1 e^{-\frac{x}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{\mathrm{e}^{3x}}{3}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-2x}\right) + c_2 \left(\mathrm{e}^{-2x} \left(\frac{\mathrm{e}^{3x}}{3}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + y' - 2y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-2x} + \frac{c_2 \mathrm{e}^{x}}{3}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
x^2 + 1
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{1, x, x^2\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{\frac{\mathrm{e}^{x}}{3}, \mathrm{e}^{-2x}\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_3 x^2 + A_2 x + A_1
$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_3x^2 - 2A_2x + 2xA_3 - 2A_1 + A_2 + 2A_3 = 4x^2 + 5$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{11}{2}, A_2 = -2, A_3 = -2\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -2x^2 - 2x - \frac{11}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-2x} + \frac{c_2 e^x}{3}\right) + \left(-2x^2 - 2x - \frac{11}{2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-2x} + \frac{c_2 e^x}{3} - 2x^2 - 2x - \frac{11}{2} \tag{1}$$

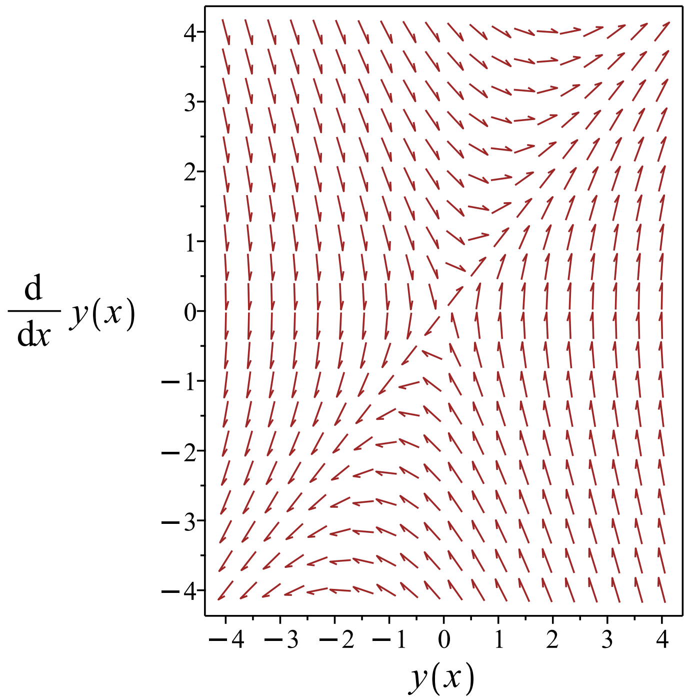


Figure 225: Slope field plot

Verification of solutions

$$y = c_1 e^{-2x} + \frac{c_2 e^{x}}{3} - 2x^2 - 2x - \frac{11}{2}$$

Verified OK.

6.17.3 Maple step by step solution

Let's solve

$y'' + y' - 2y = 4x^2 + 5$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + r - 2 = 0$

- Factor the characteristic polynomial

  $(r + 2)(r - 1) = 0$

- Roots of the characteristic polynomial

$r = (-2, 1)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-2x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 4x^2 + 5\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-2x} & \mathrm{e}^{x} \\ -2\,\mathrm{e}^{-2x} & \mathrm{e}^{x} \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = 3\,\mathrm{e}^{-x}$

  ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = \frac{\left(\mathrm{e}^{3x}\left(\int \mathrm{e}^{-x}(4x^2+5) dx\right) - \left(\int \mathrm{e}^{2x}(4x^2+5) dx\right)\right)\mathrm{e}^{-2x}}{3}$

  ◦ Compute integrals

  $y_p(x) = -2x^2 - 2x - \frac{11}{2}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{x} - 2x^2 - 2x - \frac{11}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 38

```
dsolve(diff(y(x),x$2)+diff(y(x),x)-2*y(x)=4*x^2+5,y(x), singsol=all)
```

$$y(x) = \frac{(-4x^2 - 4x - 11)\,\mathrm{e}^{-2x}\mathrm{e}^{2x}}{2} + \left(c_1\mathrm{e}^{3x} + c_2\right)\mathrm{e}^{-2x}$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 31

```
DSolve[y''[x]+y'[x]-2*y[x]==4*x^2+5,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -2x^2 - 2x + c_1e^{-2x} + c_2e^x - \frac{11}{2}$$

6.18 problem Problem 40

Internal problem ID [2742]
Internal file name [`OUTPUT/2234_Sunday_June_05_2022_02_55_45_AM_88425653/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.1, General Theory for Linear Differential Equations. page 502
Problem number: Problem 40.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _with_linear_symmetries]]
```

$$\boxed{y''' + 2y'' - y' - 2y = 4\,e^{2x}}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' + 2y'' - y' - 2y = 0$$

The characteristic equation is

$$\lambda^3 + 2\lambda^2 - \lambda - 2 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = -2$$
$$\lambda_3 = -1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{x}$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-x}$$
$$y_2 = e^{-2x}$$
$$y_3 = e^{x}$$

Now the particular solution to the given ODE is found

$$y''' + 2y'' - y' - 2y = 4\,e^{2x}$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$4\,e^{2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{x}, e^{-2x}, e^{-x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{2x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$12A_1 e^{2x} = 4\,e^{2x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^{2x}}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{x}) + \left(\frac{e^{2x}}{3}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{x} + \frac{e^{2x}}{3} \tag{1}$$

Verification of solutions

$$y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{x} + \frac{e^{2x}}{3}$$

Verified OK.

6.18.1 Maple step by step solution

Let's solve

$y''' + 2y'' - y' - 2y = 4\,e^{2x}$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$
  - Isolate for $y_3'(x)$ using original ODE

$y_3'(x) = 4\,e^{2x} - 2y_3(x) + y_2(x) + 2y_1(x)$

Convert linear ODE into a system of first order ODEs

$[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = 4\,e^{2x} - 2y_3(x) + y_2(x) + 2y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & -2 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 4\,e^{2x} \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ 4\,e^{2x} \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & -2 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right], \left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-2, \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = e^{-2x} \cdot \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3 + \vec{y}_p(x)$

□ Fundamental matrix

- ○ Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} \frac{e^{-2x}}{4} & e^{-x} & e^{x} \\ -\frac{e^{-2x}}{2} & -e^{-x} & e^{x} \\ e^{-2x} & e^{-x} & e^{x} \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

  $\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$
- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} \frac{e^{-2x}}{4} & e^{-x} & e^{x} \\ -\frac{e^{-2x}}{2} & -e^{-x} & e^{x} \\ e^{-2x} & e^{-x} & e^{x} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} \frac{1}{4} & 1 & 1 \\ -\frac{1}{2} & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} \frac{(e^{3x}+3e^{x}-1)e^{-2x}}{3} & -\frac{e^{-x}}{2} + \frac{e^{x}}{2} & \frac{(e^{3x}-3e^{x}+2)e^{-2x}}{6} \\ \frac{(e^{3x}-3e^{x}+2)e^{-2x}}{3} & \frac{e^{x}}{2} + \frac{e^{-x}}{2} & \frac{(e^{3x}+3e^{x}-4)e^{-2x}}{6} \\ \frac{(e^{3x}+3e^{x}-4)e^{-2x}}{3} & -\frac{e^{-x}}{2} + \frac{e^{x}}{2} & \frac{(e^{3x}-3e^{x}+8)e^{-2x}}{6} \end{bmatrix}$$

□ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

  $\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$
- Take the derivative of the particular solution

  $\vec{y}_p'(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$
- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$
- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$
- Cancel like terms

  $\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$
- Multiply by the inverse of the fundamental matrix

$$\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$$

- Integrate to solve for $\vec{v}(x)$

$$\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$$

- Plug $\vec{v}(x)$ into the equation for the particular solution

$$\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \left[\begin{array}{c} \frac{\left(e^{4x}-2\,e^{3x}+2\,e^{x}-1\right)e^{-2x}}{3} \\ -\frac{2\left(-e^{4x}+e^{3x}+e^{x}-1\right)e^{-2x}}{3} \\ \frac{2\left(2\,e^{4x}-e^{3x}+e^{x}-2\right)e^{-2x}}{3} \end{array}\right]$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3 + \left[\begin{array}{c} \frac{\left(e^{4x}-2\,e^{3x}+2\,e^{x}-1\right)e^{-2x}}{3} \\ -\frac{2\left(-e^{4x}+e^{3x}+e^{x}-1\right)e^{-2x}}{3} \\ \frac{2\left(2\,e^{4x}-e^{3x}+e^{x}-2\right)e^{-2x}}{3} \end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y = \frac{\left(4\,e^{4x}+12\,e^{3x}c_3-8\,e^{3x}+12 c_2 e^{x}+8\,e^{x}+3 c_1-4\right)e^{-2x}}{12}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 30

```
dsolve(diff(y(x),x$3)+2*diff(y(x),x$2)-diff(y(x),x)-2*y(x)=4*exp(2*x),y(x), singsol=all)
```

$$y(x) = \frac{(e^{4x} + 3c_1 e^{3x} + 3c_3 e^{x} + 3c_2)\, e^{-2x}}{3}$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 37

```
DSolve[y'''[x]+2*y''[x]-y'[x]-2*y[x]==4*Exp[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{2x}}{3} + c_1 e^{-2x} + c_2 e^{-x} + c_3 e^{x}$$

6.19 problem Problem 41

Internal problem ID [2743]
Internal file name [`OUTPUT/2235_Sunday_June_05_2022_02_55_47_AM_2481686/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.1, General Theory for Linear Differential Equations. page 502
Problem number: Problem 41.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _with_linear_symmetries]]
```

$$\boxed{y''' + y'' - 10y' + 8y = 24\,\mathrm{e}^{-3x}}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' + y'' - 10y' + 8y = 0$$

The characteristic equation is

$$\lambda^3 + \lambda^2 - 10\lambda + 8 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = 2$$
$$\lambda_3 = -4$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^x + c_2 e^{2x} + e^{-4x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^x$$
$$y_2 = e^{2x}$$
$$y_3 = e^{-4x}$$

Now the particular solution to the given ODE is found

$$y''' + y'' - 10y' + 8y = 24\,e^{-3x}$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$24\,e^{-3x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-3x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^x, e^{-4x}, e^{2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{-3x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$20A_1 e^{-3x} = 24\,e^{-3x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{6}{5}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{6\,e^{-3x}}{5}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^x + c_2 e^{2x} + e^{-4x} c_3\right) + \left(\frac{6\,e^{-3x}}{5}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^x + c_2 e^{2x} + e^{-4x} c_3 + \frac{6\,e^{-3x}}{5} \tag{1}$$

Verification of solutions

$$y = c_1 e^x + c_2 e^{2x} + e^{-4x} c_3 + \frac{6\,e^{-3x}}{5}$$

Verified OK.

6.19.1 Maple step by step solution

Let's solve

$y''' + y'' - 10y' + 8y = 24\,e^{-3x}$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$
  - Isolate for $y_3'(x)$ using original ODE

$y_3'(x) = 24\,\mathrm{e}^{-3x} - y_3(x) + 10y_2(x) - 8y_1(x)$

Convert linear ODE into a system of first order ODEs

$[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = 24\,\mathrm{e}^{-3x} - y_3(x) + 10y_2(x) - 8y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -8 & 10 & -1 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 24\,\mathrm{e}^{-3x} \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ 24\,\mathrm{e}^{-3x} \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -8 & 10 & -1 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-4, \begin{bmatrix} \frac{1}{16} \\ -\frac{1}{4} \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-4, \begin{bmatrix} \frac{1}{16} \\ -\frac{1}{4} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = e^{-4x} \cdot \begin{bmatrix} \frac{1}{16} \\ -\frac{1}{4} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{2x} \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3 + \vec{y}_p(x)$$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} \frac{e^{-4x}}{16} & e^{x} & \frac{e^{2x}}{4} \\ -\frac{e^{-4x}}{4} & e^{x} & \frac{e^{2x}}{2} \\ e^{-4x} & e^{x} & e^{2x} \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

  $\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} \frac{e^{-4x}}{16} & e^{x} & \frac{e^{2x}}{4} \\ -\frac{e^{-4x}}{4} & e^{x} & \frac{e^{2x}}{2} \\ e^{-4x} & e^{x} & e^{2x} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} \frac{1}{16} & 1 & \frac{1}{4} \\ -\frac{1}{4} & 1 & \frac{1}{2} \\ 1 & 1 & 1 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} -\frac{(10\,e^{6x}-24\,e^{5x}-1)e^{-4x}}{15} & \frac{(5\,e^{6x}-4\,e^{5x}-1)e^{-4x}}{10} & \frac{(5\,e^{6x}-6\,e^{5x}+1)e^{-4x}}{30} \\ -\frac{4(5\,e^{6x}-6\,e^{5x}+1)e^{-4x}}{15} & \frac{(5\,e^{6x}-2\,e^{5x}+2)e^{-4x}}{5} & \frac{(5\,e^{6x}-3\,e^{5x}-2)e^{-4x}}{15} \\ -\frac{8(5\,e^{6x}-3\,e^{5x}-2)e^{-4x}}{15} & -\frac{2(-5\,e^{6x}+e^{5x}+4)e^{-4x}}{5} & \frac{(10\,e^{6x}-3\,e^{5x}+8)e^{-4x}}{15} \end{bmatrix}$$

□ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

  $\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$

- Take the derivative of the particular solution

  $\vec{y}_p'(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- Cancel like terms

  $\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$

- Multiply by the inverse of the fundamental matrix

$$\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$$

- Integrate to solve for $\vec{v}(x)$

$$\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$$

- Plug $\vec{v}(x)$ into the equation for the particular solution

$$\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \left[\begin{array}{c} \frac{2\left(2\,\mathrm{e}^{6x}-3\,\mathrm{e}^{5x}+3\,\mathrm{e}^{x}-2\right)\mathrm{e}^{-4x}}{5} \\ \frac{2\left(4\,\mathrm{e}^{6x}-3\,\mathrm{e}^{5x}-9\,\mathrm{e}^{x}+8\right)\mathrm{e}^{-4x}}{5} \\ \frac{2\left(8\,\mathrm{e}^{6x}-3\,\mathrm{e}^{5x}+27\,\mathrm{e}^{x}-32\right)\mathrm{e}^{-4x}}{5} \end{array}\right]$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3 + \left[\begin{array}{c} \frac{2\left(2\,\mathrm{e}^{6x}-3\,\mathrm{e}^{5x}+3\,\mathrm{e}^{x}-2\right)\mathrm{e}^{-4x}}{5} \\ \frac{2\left(4\,\mathrm{e}^{6x}-3\,\mathrm{e}^{5x}-9\,\mathrm{e}^{x}+8\right)\mathrm{e}^{-4x}}{5} \\ \frac{2\left(8\,\mathrm{e}^{6x}-3\,\mathrm{e}^{5x}+27\,\mathrm{e}^{x}-32\right)\mathrm{e}^{-4x}}{5} \end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y = \frac{\left(20c_3\mathrm{e}^{6x}+64\,\mathrm{e}^{6x}+80c_2\mathrm{e}^{5x}-96\,\mathrm{e}^{5x}+96\,\mathrm{e}^{x}+5c_1-64\right)\mathrm{e}^{-4x}}{80}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 32

```
dsolve(diff(y(x),x$3)+diff(y(x),x$2)-10*diff(y(x),x)+8*y(x)=24*exp(-3*x),y(x), singsol=all)
```

$$y(x) = \frac{(5c_3 e^{6x} + 5c_1 e^{5x} + 6 e^{x} + 5c_2) e^{-4x}}{5}$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 37

```
DSolve[y'''[x]+y''[x]-10*y'[x]+8*y[x]==24*Exp[-3*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{6e^{-3x}}{5} + c_1 e^{-4x} + c_2 e^{x} + c_3 e^{2x}$$

6.20 problem Problem 42

Internal problem ID [2744]
Internal file name [`OUTPUT/2236_Sunday_June_05_2022_02_55_49_AM_28455852/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.1, General Theory for Linear Differential Equations. page 502
Problem number: Problem 42.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_y]]
```

$$\boxed{y''' + 5y'' + 6y' = 6\,\mathrm{e}^{-x}}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' + 5y'' + 6y' = 0$$

The characteristic equation is

$$\lambda^3 + 5\lambda^2 + 6\lambda = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = -2$$
$$\lambda_3 = -3$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 + c_2 e^{-2x} + e^{-3x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= 1 \\
y_2 &= e^{-2x} \\
y_3 &= e^{-3x}
\end{aligned}
$$

Now the particular solution to the given ODE is found

$$y''' + 5y'' + 6y' = 6\,e^{-x}$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$6\,e^{-x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^{-3x}, e^{-2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{-x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1 e^{-x} = 6\,e^{-x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -3]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -3\,\mathrm{e}^{-x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 + c_2\mathrm{e}^{-2x} + \mathrm{e}^{-3x}c_3\right) + \left(-3\,\mathrm{e}^{-x}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 + c_2\mathrm{e}^{-2x} + \mathrm{e}^{-3x}c_3 - 3\,\mathrm{e}^{-x} \tag{1}$$

Verification of solutions

$$y = c_1 + c_2\mathrm{e}^{-2x} + \mathrm{e}^{-3x}c_3 - 3\,\mathrm{e}^{-x}$$

Verified OK.

6.20.1 Maple step by step solution

Let's solve

$y''' + 5y'' + 6y' = 6\,\mathrm{e}^{-x}$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$
  - Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = 6\,\mathrm{e}^{-x} - 5y_3(x) - 6y_2(x)$

  Convert linear ODE into a system of first order ODEs

$$[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = 6\,\mathrm{e}^{-x} - 5y_3(x) - 6y_2(x)]$$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -6 & -5 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 6\,\mathrm{e}^{-x} \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ 6\,\mathrm{e}^{-x} \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -6 & -5 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-3, \begin{bmatrix} \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix}\right], \left[-2, \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right], \left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-3, \begin{bmatrix} \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = \mathrm{e}^{-3x} \cdot \left[\begin{array}{c} \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[-2, \left[\begin{array}{c} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_2 = \mathrm{e}^{-2x} \cdot \left[\begin{array}{c} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[0, \left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_3 = \left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\overrightarrow{y}_p($

$$\overrightarrow{y}(x) = c_1 \overrightarrow{y}_1 + c_2 \overrightarrow{y}_2 + c_3 \overrightarrow{y}_3 + \overrightarrow{y}_p(x)$$

☐ Fundamental matrix

  ◦ Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \left[\begin{array}{ccc} \frac{\mathrm{e}^{-3x}}{9} & \frac{\mathrm{e}^{-2x}}{4} & 1 \\ -\frac{\mathrm{e}^{-3x}}{3} & -\frac{\mathrm{e}^{-2x}}{2} & 0 \\ \mathrm{e}^{-3x} & \mathrm{e}^{-2x} & 0 \end{array}\right]$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t
  $\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$
- Substitute the value of $\phi(x)$ and $\phi(0)$
  $$\Phi(x) = \begin{bmatrix} \frac{e^{-3x}}{9} & \frac{e^{-2x}}{4} & 1 \\ -\frac{e^{-3x}}{3} & -\frac{e^{-2x}}{2} & 0 \\ e^{-3x} & e^{-2x} & 0 \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} \frac{1}{9} & \frac{1}{4} & 1 \\ -\frac{1}{3} & -\frac{1}{2} & 0 \\ 1 & 1 & 0 \end{bmatrix}}$$
- Evaluate and simplify to get the fundamental matrix
  $$\Phi(x) = \begin{bmatrix} 1 & \frac{2e^{-3x}}{3} - \frac{3e^{-2x}}{2} + \frac{5}{6} & \frac{e^{-3x}}{3} - \frac{e^{-2x}}{2} + \frac{1}{6} \\ 0 & -2e^{-3x} + 3e^{-2x} & -e^{-3x} + e^{-2x} \\ 0 & 6e^{-3x} - 6e^{-2x} & 3e^{-3x} - 2e^{-2x} \end{bmatrix}$$

□ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$
  $\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$
- Take the derivative of the particular solution
  $\vec{y}_p'(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$
- Substitute particular solution and its derivative into the system of ODEs
  $\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$
- The fundamental matrix has columns that are solutions to the homogeneous system so its der
  $A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$
- Cancel like terms
  $\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$
- Multiply by the inverse of the fundamental matrix
  $\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$
- Integrate to solve for $\vec{v}(x)$
  $\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$
- Plug $\vec{v}(x)$ into the equation for the particular solution

$$\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \left[\begin{array}{c} -3\,\mathrm{e}^{-x}+1+3\,\mathrm{e}^{-2x}-\mathrm{e}^{-3x} \\ 3\,\mathrm{e}^{-x}-6\,\mathrm{e}^{-2x}+3\,\mathrm{e}^{-3x} \\ -3\,\mathrm{e}^{-x}+12\,\mathrm{e}^{-2x}-9\,\mathrm{e}^{-3x} \end{array}\right]$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3 + \left[\begin{array}{c} -3\,\mathrm{e}^{-x}+1+3\,\mathrm{e}^{-2x}-\mathrm{e}^{-3x} \\ 3\,\mathrm{e}^{-x}-6\,\mathrm{e}^{-2x}+3\,\mathrm{e}^{-3x} \\ -3\,\mathrm{e}^{-x}+12\,\mathrm{e}^{-2x}-9\,\mathrm{e}^{-3x} \end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y = 1 + \frac{(-9+c_1)\,\mathrm{e}^{-3x}}{9} + \frac{(12+c_2)\,\mathrm{e}^{-2x}}{4} + c_3 - 3\,\mathrm{e}^{-x}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
-> Calling odsolve with the ODE`, diff(diff(_b(_a), _a), _a) = -5*(diff(_b(_a), _a))-6*_b(_a
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
   trying a double symmetry of the form [xi=0, eta=F(x)]
   -> Try solving first the homogeneous part of the ODE
      checking if the LODE has constant coefficients
      <- constant coefficients successful
   <- solving first the homogeneous part of the ODE successful
<- differential order: 3; linear nonhomogeneous with symmetry [0,1] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve(diff(y(x),x$3)+5*diff(y(x),x$2)+6*diff(y(x),x)=6*exp(-x),y(x), singsol=all)
```

$$y(x) = -\frac{c_1 e^{-3x}}{3} - \frac{e^{-2x} c_2}{2} - 3\,e^{-x} + c_3$$

✓ Solution by Mathematica

Time used: 0.05 (sec). Leaf size: 37

```
DSolve[y'''[x]+5*y''[x]+6*y'[x]==6*Exp[-x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -3e^{-x} - \frac{1}{3} c_1 e^{-3x} - \frac{1}{2} c_2 e^{-2x} + c_3$$

7 Chapter 8, Linear differential equations of order n. Section 8.3, The Method of Undetermined Coefficients. page 525

7.1 problem Problem 25

Internal problem ID [2745]
Internal file name [`OUTPUT/2237_Sunday_June_05_2022_02_55_51_AM_25015968/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.3, The Method of Undetermined Coefficients. page 525
Problem number: Problem 25.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + y = 6\,\mathrm{e}^{x}}$$

7.1.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = 6\,\mathrm{e}^{x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i \\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i \\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$6\,\mathrm{e}^x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \mathrm{e}^x$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 \mathrm{e}^x = 6\,\mathrm{e}^x$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 3]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 3\,\mathrm{e}^x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) + (3\,\mathrm{e}^x) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(x) + c_2 \sin(x) + 3\,e^x \tag{1}$$

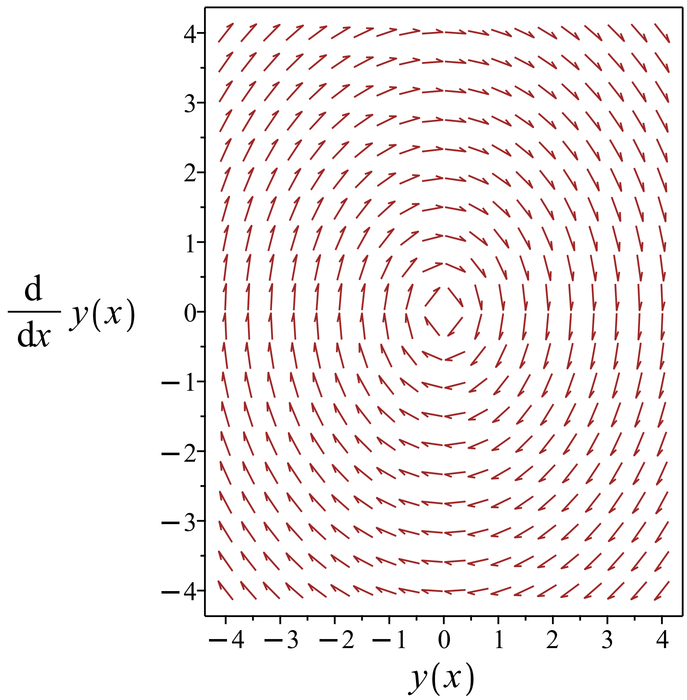


Figure 226: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x) + 3\,e^x$$

Verified OK.

7.1.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 201: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \cos(x)
\end{aligned}
$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \cos(x) \int \frac{1}{\cos(x)^2}\, dx \\ &= \cos(x)(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(x)) + c_2(\cos(x)(\tan(x))) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$6\,\mathrm{e}^{x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^x$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 e^x = 6\,e^x$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 3]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 3\,e^x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) + (3\,e^x) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(x) + c_2 \sin(x) + 3\,e^x \tag{1}$$

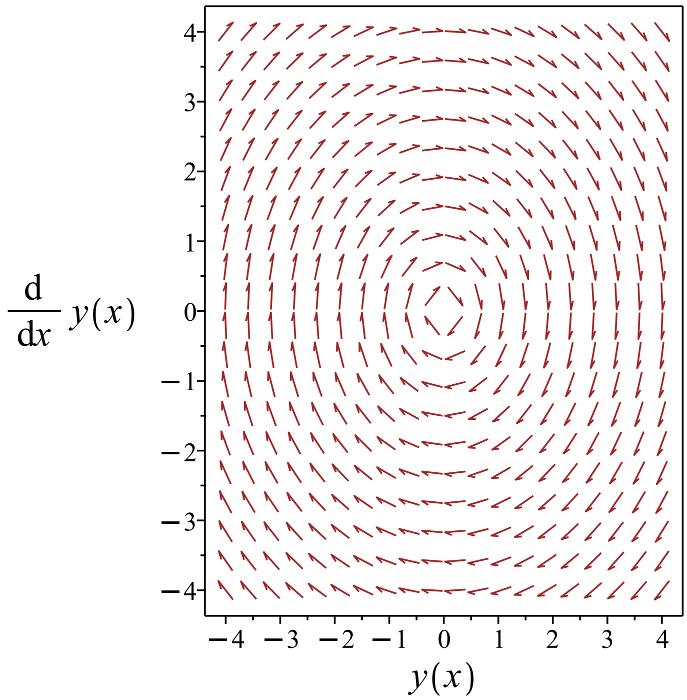


Figure 227: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x) + 3\,e^{x}$$

Verified OK.

7.1.3 Maple step by step solution

Let's solve

$y'' + y = 6\,e^{x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

$r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$
- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 6\,\mathrm{e}^{x}\right]$
- Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \left[\begin{array}{cc} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{array}\right]$
- Compute Wronskian

  $W(y_1(x), y_2(x)) = 1$
- Substitute functions into equation for $y_p(x)$

  $y_p(x) = -6\cos(x)\left(\int \mathrm{e}^{x}\sin(x)\,dx\right) + 6\sin(x)\left(\int \cos(x)\,\mathrm{e}^{x}dx\right)$
- Compute integrals

  $y_p(x) = 3\,\mathrm{e}^{x}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + 3\,\mathrm{e}^{x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 17

```
dsolve(diff(y(x),x$2)+y(x)=6*exp(x),y(x), singsol=all)
```

$$y(x) = \sin(x) c_2 + \cos(x) c_1 + 3 e^x$$

✓ Solution by Mathematica

Time used: 0.016 (sec). Leaf size: 21

```
DSolve[y''[x]+y[x]==6*Exp[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 3e^x + c_1 \cos(x) + c_2 \sin(x)$$

7.2 problem Problem 26

Internal problem ID [2746]
Internal file name [`OUTPUT/2238_Sunday_June_05_2022_02_55_54_AM_77301884/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.3, The Method of Undetermined Coefficients. page 525
Problem number: Problem 26.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 4y' + 4y = 5x\,\mathrm{e}^{-2x}}$$

7.2.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 4, C = 4, f(x) = 5x\,\mathrm{e}^{-2x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 4, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 4\lambda\,\mathrm{e}^{\lambda x} + 4\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4\lambda + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 4, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-4}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{(4)^2 - (4)\,(1)\,(4)}\\ &= -2\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 2$. Therefore the solution is

$$y = c_1\mathrm{e}^{-2x} + c_2 x\,\mathrm{e}^{-2x} \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1\mathrm{e}^{-2x} + c_2 x\,\mathrm{e}^{-2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$5x\,\mathrm{e}^{-2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\,\mathrm{e}^{-2x}, \mathrm{e}^{-2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\,\mathrm{e}^{-2x}, \mathrm{e}^{-2x}\}$$

Since e^{-2x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,\mathrm{e}^{-2x}, \mathrm{e}^{-2x}x^2\}]$$

Since $x\,\mathrm{e}^{-2x}$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^3\mathrm{e}^{-2x}, \mathrm{e}^{-2x}x^2\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1x^3\mathrm{e}^{-2x} + A_2\mathrm{e}^{-2x}x^2$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$6A_1x\,\mathrm{e}^{-2x} + 2A_2\mathrm{e}^{-2x} = 5x\,\mathrm{e}^{-2x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{5}{6}, A_2 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{5x^3\mathrm{e}^{-2x}}{6}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{-2x} + c_2x\,\mathrm{e}^{-2x}\right) + \left(\frac{5x^3\mathrm{e}^{-2x}}{6}\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{-2x}(c_2x + c_1) + \frac{5x^3\mathrm{e}^{-2x}}{6}$$

Summary
The solution(s) found are the following

$$y = e^{-2x}(c_2x + c_1) + \frac{5x^3 e^{-2x}}{6} \tag{1}$$

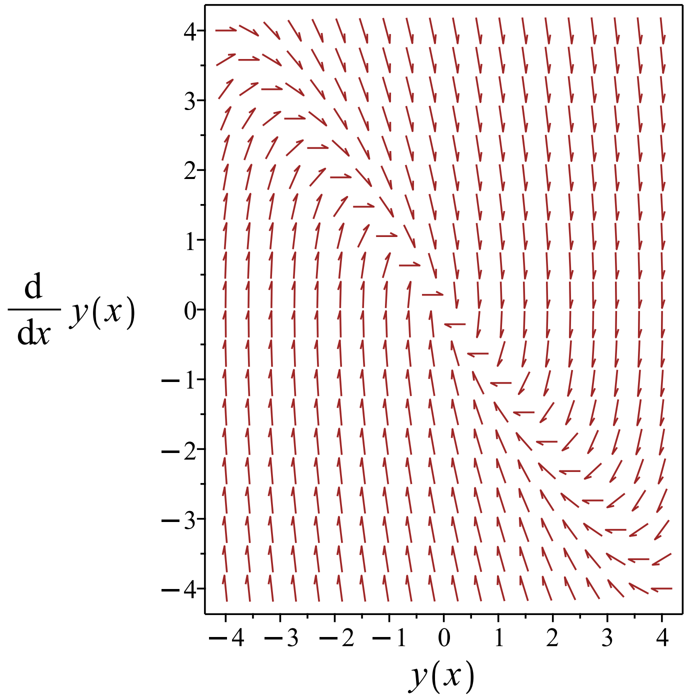


Figure 228: Slope field plot

Verification of solutions

$$y = e^{-2x}(c_2x + c_1) + \frac{5x^3 e^{-2x}}{6}$$

Verified OK.

7.2.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\,y' + \frac{\left(p(x)^2 + p'(x)\right)y}{2} = f(x)$$

Where $p(x) = 4$. Therefore, there is an integrating factor given by

$$
\begin{aligned}
M(x) &= e^{\frac{1}{2}\int p\,dx} \\
&= e^{\int 4\,dx} \\
&= \mathrm{e}^{2x}
\end{aligned}
$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$
\begin{aligned}
(M(x)y)'' &= 5\,\mathrm{e}^{2x} x\,\mathrm{e}^{-2x} \\
\left(\mathrm{e}^{2x} y\right)'' &= 5\,\mathrm{e}^{2x} x\,\mathrm{e}^{-2x}
\end{aligned}
$$

Integrating once gives

$$
\left(\mathrm{e}^{2x} y\right)' = \frac{5x^2}{2} + c_1
$$

Integrating again gives

$$
\left(\mathrm{e}^{2x} y\right) = \frac{5}{6}x^3 + c_1 x + c_2
$$

Hence the solution is

$$
y = \frac{\frac{5}{6}x^3 + c_1 x + c_2}{\mathrm{e}^{2x}}
$$

Or

$$
y = \frac{5x^3 \mathrm{e}^{-2x}}{6} + c_1 x\,\mathrm{e}^{-2x} + c_2 \mathrm{e}^{-2x}
$$

Summary
The solution(s) found are the following

$$
y = \frac{5x^3 \mathrm{e}^{-2x}}{6} + c_1 x\,\mathrm{e}^{-2x} + c_2 \mathrm{e}^{-2x} \tag{1}
$$

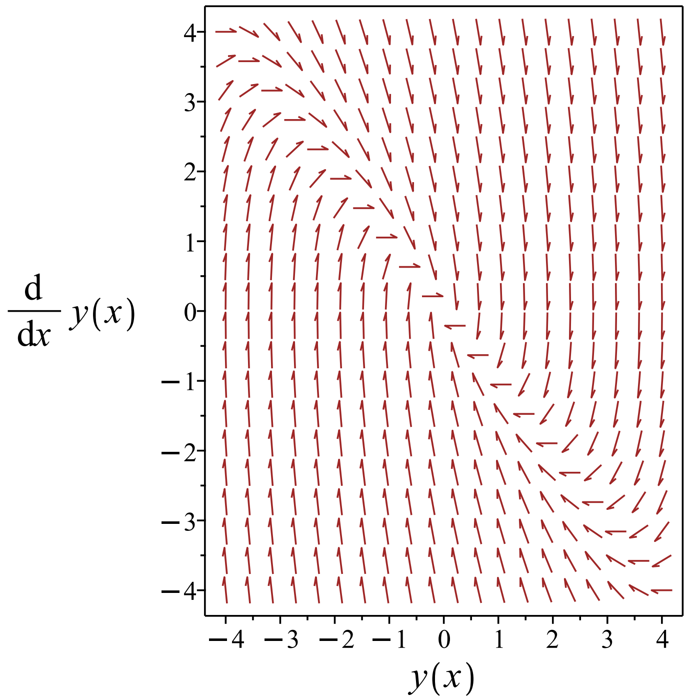


Figure 229: Slope field plot

Verification of solutions

$$y = \frac{5x^3 e^{-2x}}{6} + c_1 x\, e^{-2x} + c_2 e^{-2x}$$

Verified OK.

7.2.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 203: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4}{1}\,dx} \\
&= z_1 e^{-2x} \\
&= z_1\left(\mathrm{e}^{-2x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-4x}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-2x}\right) + c_2\left(\mathrm{e}^{-2x}(x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 4y' + 4y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-2x} + c_2 x\,\mathrm{e}^{-2x}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
5x\,\mathrm{e}^{-2x}
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{x\,\mathrm{e}^{-2x}, \mathrm{e}^{-2x}\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{x\,\mathrm{e}^{-2x}, \mathrm{e}^{-2x}\}
$$

Since e^{-2x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$
[\{x\,\mathrm{e}^{-2x}, \mathrm{e}^{-2x}x^2\}]
$$

Since $x\,\mathrm{e}^{-2x}$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$
[\{x^3\mathrm{e}^{-2x}, \mathrm{e}^{-2x}x^2\}]
$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^3 e^{-2x} + A_2 e^{-2x} x^2$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$6A_1 x\, e^{-2x} + 2A_2 e^{-2x} = 5x\, e^{-2x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{5}{6}, A_2 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{5x^3 e^{-2x}}{6}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-2x} + c_2 x\, e^{-2x}\right) + \left(\frac{5x^3 e^{-2x}}{6}\right) \end{aligned}$$

Which simplifies to

$$y = e^{-2x}(c_2 x + c_1) + \frac{5x^3 e^{-2x}}{6}$$

Summary

The solution(s) found are the following

$$y = e^{-2x}(c_2 x + c_1) + \frac{5x^3 e^{-2x}}{6} \tag{1}$$

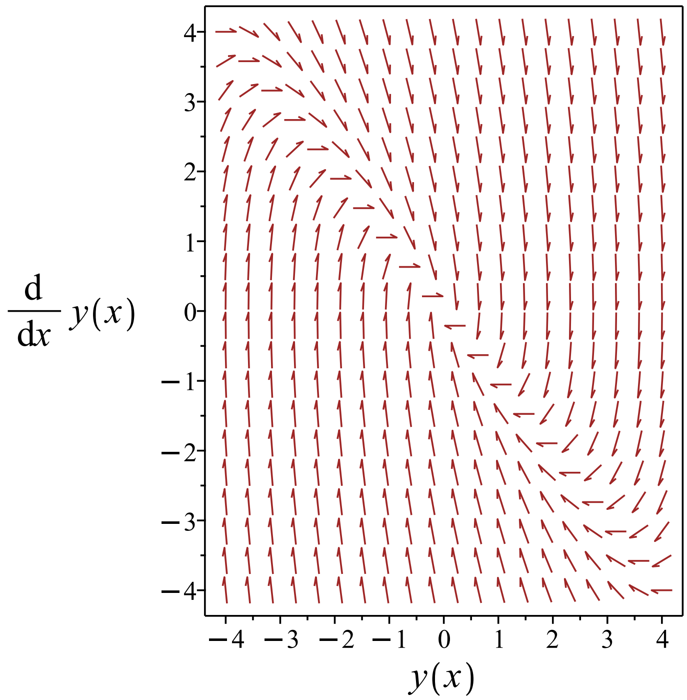


Figure 230: Slope field plot

Verification of solutions

$$y = e^{-2x}(c_2x + c_1) + \frac{5x^3 e^{-2x}}{6}$$

Verified OK.

7.2.4 Maple step by step solution

Let's solve

$y'' + 4y' + 4y = 5x\,e^{-2x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4r + 4 = 0$

- Factor the characteristic polynomial

  $(r + 2)^2 = 0$

- Root of the characteristic polynomial

$r = -2$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-2x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,\mathrm{e}^{-2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-2x} + c_2 x\,\mathrm{e}^{-2x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 5x\,\mathrm{e}^{-2x}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-2x} & x\,\mathrm{e}^{-2x} \\ -2\,\mathrm{e}^{-2x} & \mathrm{e}^{-2x} - 2x\,\mathrm{e}^{-2x} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = \mathrm{e}^{-4x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -5\,\mathrm{e}^{-2x}\left(\int x^2 dx - \left(\int x dx\right) x\right)$

  - Compute integrals

    $y_p(x) = \frac{5x^3 \mathrm{e}^{-2x}}{6}$

- Substitute particular solution into general solution to ODE

  $y = c_2 x\,\mathrm{e}^{-2x} + c_1 \mathrm{e}^{-2x} + \frac{5x^3 \mathrm{e}^{-2x}}{6}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(diff(y(x),x$2)+4*diff(y(x),x)+4*y(x)=5*x*exp(-2*x),y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{-2x}\left(c_2 + c_1 x + \frac{5}{6}x^3\right)$$

✓ Solution by Mathematica

Time used: 0.03 (sec). Leaf size: 29

```
DSolve[y''[x]+4*y'[x]+4*y[x]==5*x*Exp[-2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{6}e^{-2x}\left(5x^3 + 6c_2 x + 6c_1\right)$$

7.3 problem Problem 27

Internal problem ID [2747]
Internal file name [`OUTPUT/2239_Sunday_June_05_2022_02_55_56_AM_6096335/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.3, The Method of Undetermined Coefficients. page 525
Problem number: Problem 27.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 4y = 8\sin(2x)}$$

7.3.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 4, f(x) = 8\sin(2x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 4\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 4$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(4)} \\
&= \pm 2i
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= +2i \\
\lambda_2 &= -2i
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= 2i \\
\lambda_2 &= -2i
\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(2x) + c_2 \sin(2x))$$

Or

$$y = c_1 \cos(2x) + c_2 \sin(2x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(2x) + c_2 \sin(2x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$8 \sin(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(2x), \sin(2x)\}$$

Since $\cos(2x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x \cos(2x), x \sin(2x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x \cos(2x) + A_2 x \sin(2x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-4A_1 \sin(2x) + 4A_2 \cos(2x) = 8 \sin(2x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -2, A_2 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -2x \cos(2x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(2x) + c_2 \sin(2x)) + (-2x \cos(2x)) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \cos(2x) + c_2 \sin(2x) - 2x \cos(2x) \tag{1}$$

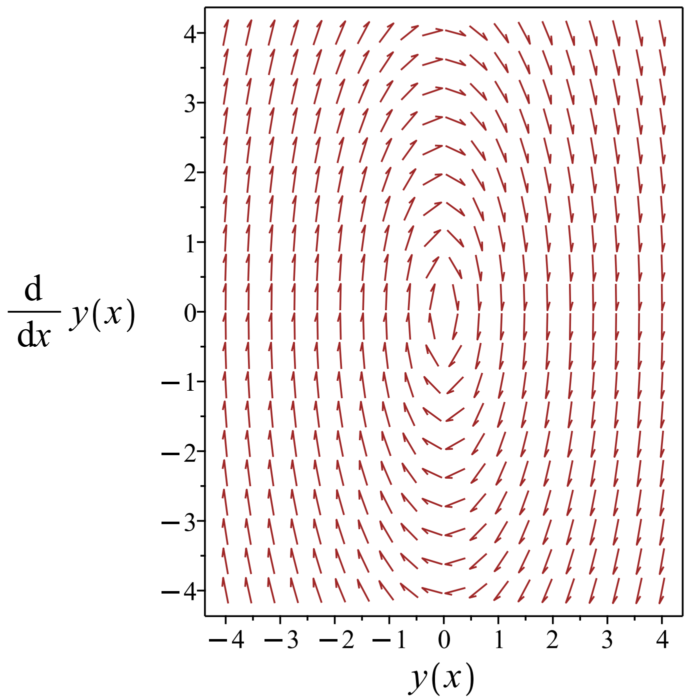


Figure 231: Slope field plot

Verification of solutions

$$y = c_1 \cos(2x) + c_2 \sin(2x) - 2x \cos(2x)$$

Verified OK.

7.3.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 205: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(2x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(2x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(2x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= \cos(2x) \int \frac{1}{\cos(2x)^2}\, dx \\
&= \cos(2x) \left(\frac{\tan(2x)}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\cos(2x)) + c_2\left(\cos(2x)\left(\frac{\tan(2x)}{2}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$8\sin(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sin(2x)}{2}, \cos(2x)\right\}$$

Since $\cos(2x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\cos(2x), x\sin(2x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\cos(2x) + A_2 x\sin(2x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-4A_1\sin(2x) + 4A_2\cos(2x) = 8\sin(2x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -2, A_2 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -2x\cos(2x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\cos(2x) + \frac{c_2\sin(2x)}{2}\right) + (-2x\cos(2x)) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2} - 2x \cos(2x) \tag{1}$$

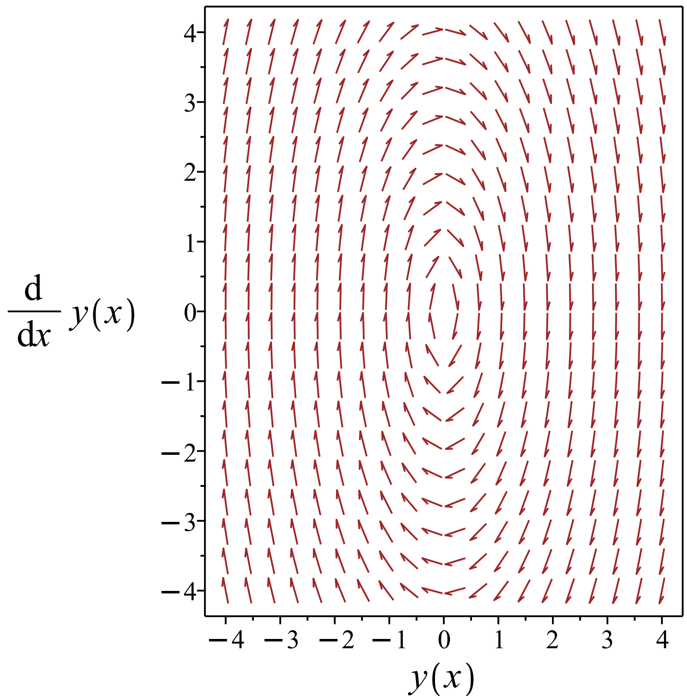


Figure 232: Slope field plot

Verification of solutions

$$y = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2} - 2x \cos(2x)$$

Verified OK.

7.3.3 Maple step by step solution

Let's solve

$y'' + 4y = 8\sin(2x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2 + 4 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (-2\,\mathrm{I}, 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(2x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(2x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(2x) + c_2 \sin(2x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 8 \sin(2x)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(2x) & \sin(2x) \\ -2\sin(2x) & 2\cos(2x) \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 2$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -2\cos(2x) \left(\int (1 - \cos(4x))\, dx\right) + 2\sin(2x) \left(\int \sin(4x)\, dx\right)$

  - Compute integrals

    $y_p(x) = \frac{\sin(2x)}{2} - 2x\cos(2x)$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(2x) + c_2 \sin(2x) + \frac{\sin(2x)}{2} - 2x\cos(2x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 21

```
dsolve(diff(y(x),x$2)+4*y(x)=8*sin(2*x),y(x), singsol=all)
```

$$y(x) = (-2x + c_1)\cos(2x) + \sin(2x)\, c_2$$

✓ Solution by Mathematica

Time used: 0.084 (sec). Leaf size: 29

```
DSolve[y''[x]+4*y[x]==8*Sin[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sin(x)\cos(x) + (-2x + c_1)\cos(2x) + c_2\sin(2x)$$

7.4 problem Problem 28

Internal problem ID [2748]
Internal file name [`OUTPUT/2240_Sunday_June_05_2022_02_55_58_AM_81023087/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.3, The Method of Undetermined Coefficients. page 525
Problem number: Problem 28.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - y' - 2y = 5\,\mathrm{e}^{2x}}$$

7.4.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -1, C = -2, f(x) = 5\,\mathrm{e}^{2x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - y' - 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -1, C = -2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - \lambda \,\mathrm{e}^{\lambda x} - 2 \,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - \lambda - 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -1, C = -2$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{1}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-1^2 - (4)\,(1)\,(-2)} \\
&= \frac{1}{2} \pm \frac{3}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= \frac{1}{2} + \frac{3}{2} \\
\lambda_2 &= \frac{1}{2} - \frac{3}{2}
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= 2 \\
\lambda_2 &= -1
\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}
y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\
y &= c_1 e^{(2)x} + c_2 e^{(-1)x}
\end{aligned}$$

Or

$$y = \mathrm{e}^{2x} c_1 + c_2 \mathrm{e}^{-x}$$

Therefore the homogeneous solution y_h is

$$y_h = \mathrm{e}^{2x} c_1 + c_2 \mathrm{e}^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$5\,\mathrm{e}^{2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\mathrm{e}^{-x}, \mathrm{e}^{2x}\}$$

Since e^{2x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,\mathrm{e}^{2x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\,\mathrm{e}^{2x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$3A_1\mathrm{e}^{2x} = 5\,\mathrm{e}^{2x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{5}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{5x\,\mathrm{e}^{2x}}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\mathrm{e}^{2x}c_1 + c_2\mathrm{e}^{-x}\right) + \left(\frac{5x\,\mathrm{e}^{2x}}{3}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = e^{2x}c_1 + c_2e^{-x} + \frac{5x\,e^{2x}}{3} \tag{1}$$

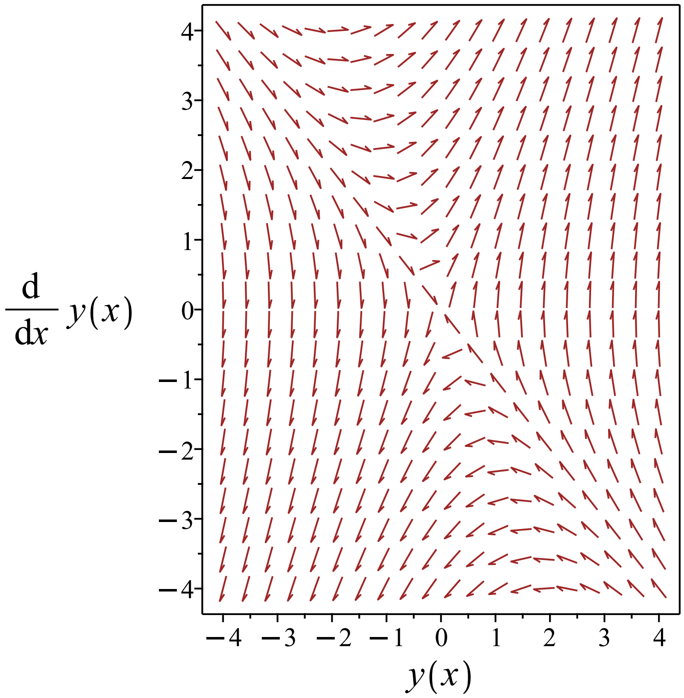


Figure 233: Slope field plot

Verification of solutions

$$y = e^{2x}c_1 + c_2e^{-x} + \frac{5x\,e^{2x}}{3}$$

Verified OK.

7.4.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= -1 \\
C &= -2
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A} \, dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{9}{4} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 9 \\
t &= 4
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \frac{9z(x)}{4} \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x) \, e^{-\int \frac{B}{2A} \, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 207: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{9}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{3x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-1}{1}\, dx} \\ &= z_1 e^{\frac{x}{2}} \\ &= z_1\left(\mathrm{e}^{\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{x}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{\mathrm{e}^{3x}}{3}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-x}\right) + c_2 \left(\mathrm{e}^{-x} \left(\frac{\mathrm{e}^{3x}}{3}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - y' - 2y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{2x}}{3}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^{-x}$$

$$y_2 = \frac{e^{2x}}{3}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{-x} & \frac{e^{2x}}{3} \\ \frac{d}{dx}(e^{-x}) & \frac{d}{dx}\left(\frac{e^{2x}}{3}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-x} & \frac{e^{2x}}{3} \\ -e^{-x} & \frac{2\,e^{2x}}{3} \end{vmatrix}$$

Therefore

$$W = \left(e^{-x}\right)\left(\frac{2\,e^{2x}}{3}\right) - \left(\frac{e^{2x}}{3}\right)\left(-e^{-x}\right)$$

Which simplifies to

$$W = e^{-x} e^{2x}$$

Which simplifies to

$$W = e^{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{5\,e^{4x}}{3}}{e^{x}}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{5\,e^{3x}}{3} dx$$

Hence

$$u_1 = -\frac{5\,e^{3x}}{9}$$

And Eq. (3) becomes

$$u_2 = \int \frac{5\,e^{-x} e^{2x}}{e^{x}}\, dx$$

Which simplifies to

$$u_2 = \int 5 dx$$

Hence

$$u_2 = 5x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{5\,e^{-x} e^{3x}}{9} + \frac{5x\,e^{2x}}{3}$$

Which simplifies to

$$y_p(x) = \frac{5\,e^{2x}(3x-1)}{9}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + \frac{c_2 e^{2x}}{3}\right) + \left(\frac{5\,e^{2x}(3x-1)}{9}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} + \frac{c_2 e^{2x}}{3} + \frac{5\,e^{2x}(3x-1)}{9} \tag{1}$$

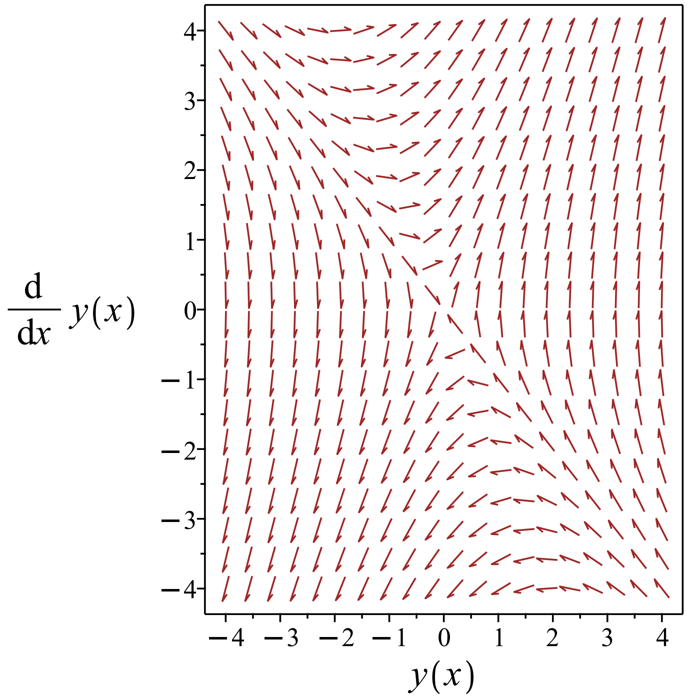


Figure 234: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} + \frac{c_2 e^{2x}}{3} + \frac{5\,e^{2x}(3x-1)}{9}$$

Verified OK.

7.4.3 Maple step by step solution

Let's solve

$y'' - y' - 2y = 5\,e^{2x}$

- Highest derivative means the order of the ODE is 2

y''

- Characteristic polynomial of homogeneous ODE

$r^2 - r - 2 = 0$

- Factor the characteristic polynomial

  $(r+1)(r-2) = 0$

- Roots of the characteristic polynomial

  $r = (-1, 2)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{2x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 5\,\mathrm{e}^{2x}\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-x} & \mathrm{e}^{2x} \\ -\mathrm{e}^{-x} & 2\,\mathrm{e}^{2x} \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = 3\,\mathrm{e}^{x}$

  ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\frac{5\,\mathrm{e}^{-x}\left(\int \mathrm{e}^{3x}dx\right)}{3} + \frac{5\,\mathrm{e}^{2x}\left(\int 1dx\right)}{3}$

  ◦ Compute integrals

  $y_p(x) = \frac{5\,\mathrm{e}^{2x}(3x-1)}{9}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{2x} + \frac{5\,\mathrm{e}^{2x}(3x-1)}{9}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 24

```
dsolve(diff(y(x),x$2)-diff(y(x),x)-2*y(x)=5*exp(2*x),y(x), singsol=all)
```

$$y(x) = \frac{(5x + 3c_1)\,\mathrm{e}^{2x}}{3} + \mathrm{e}^{-x} c_2$$

✓ Solution by Mathematica

Time used: 0.026 (sec). Leaf size: 31

```
DSolve[y''[x]-y'[x]-2*y[x]==5*Exp[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{-x} + e^{2x}\left(\frac{5x}{3} - \frac{5}{9} + c_2\right)$$

7.5 problem Problem 29

Internal problem ID [2749]
Internal file name [`OUTPUT/2241_Sunday_June_05_2022_02_56_01_AM_55212245/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.3, The Method of Undetermined Coefficients. page 525
Problem number: Problem 29.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 2y' + 5y = 3\sin(2x)}$$

7.5.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 2, C = 5, f(x) = 3\sin(2x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2y' + 5y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 2, C = 5$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 2\lambda\, e^{\lambda x} + 5\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2\lambda + 5 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = 5$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{2^2 - (4)\,(1)\,(5)}\\ &= -1 \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -1 + 2i\\ \lambda_2 &= -1 - 2i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -1 + 2i\\ \lambda_2 &= -1 - 2i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -1$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-x}(c_1 \cos(2x) + c_2 \sin(2x))$$

Therefore the homogeneous solution y_h is

$$y_h = e^{-x}(c_1 \cos(2x) + c_2 \sin(2x))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$3\sin(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-x}\cos(2x), e^{-x}\sin(2x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1\cos(2x) + A_2\sin(2x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1\cos(2x) + A_2\sin(2x) - 4A_1\sin(2x) + 4A_2\cos(2x) = 3\sin(2x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{12}{17}, A_2 = \frac{3}{17}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{12\cos(2x)}{17} + \frac{3\sin(2x)}{17}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{-x}(c_1\cos(2x) + c_2\sin(2x))\right) + \left(-\frac{12\cos(2x)}{17} + \frac{3\sin(2x)}{17}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^{-x}(c_1 \cos(2x) + c_2 \sin(2x)) - \frac{12 \cos(2x)}{17} + \frac{3 \sin(2x)}{17} \tag{1}$$

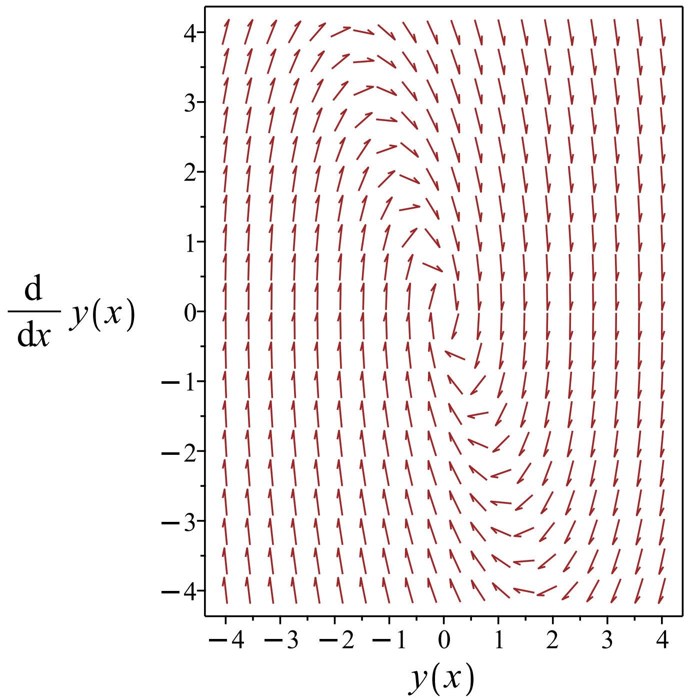


Figure 235: Slope field plot

Verification of solutions

$$y = e^{-x}(c_1 \cos(2x) + c_2 \sin(2x)) - \frac{12 \cos(2x)}{17} + \frac{3 \sin(2x)}{17}$$

Verified OK.

7.5.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y' + 5y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= 2 \\
C &= 5
\end{aligned}
\tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-4}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -4 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = -4z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 209: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(2x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{2}{1} \, dx} \\
&= z_1 e^{-x} \\
&= z_1 \left(\mathrm{e}^{-x}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x} \cos(2x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-2x}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{\tan(2x)}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-x} \cos(2x)\right) + c_2 \left(\mathrm{e}^{-x} \cos(2x) \left(\frac{\tan(2x)}{2}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2y' + 5y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \cos(2x)\, \mathrm{e}^{-x} c_1 + \frac{\sin(2x)\, \mathrm{e}^{-x} c_2}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$3 \sin(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^{-x}\cos(2x), \frac{e^{-x}\sin(2x)}{2}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1\cos(2x) + A_2\sin(2x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1\cos(2x) + A_2\sin(2x) - 4A_1\sin(2x) + 4A_2\cos(2x) = 3\sin(2x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{12}{17}, A_2 = \frac{3}{17}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{12\cos(2x)}{17} + \frac{3\sin(2x)}{17}$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p \\
&= \left(\cos(2x)\,e^{-x}c_1 + \frac{\sin(2x)\,e^{-x}c_2}{2}\right) + \left(-\frac{12\cos(2x)}{17} + \frac{3\sin(2x)}{17}\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \cos(2x)\,e^{-x}c_1 + \frac{\sin(2x)\,e^{-x}c_2}{2} - \frac{12\cos(2x)}{17} + \frac{3\sin(2x)}{17} \tag{1}$$

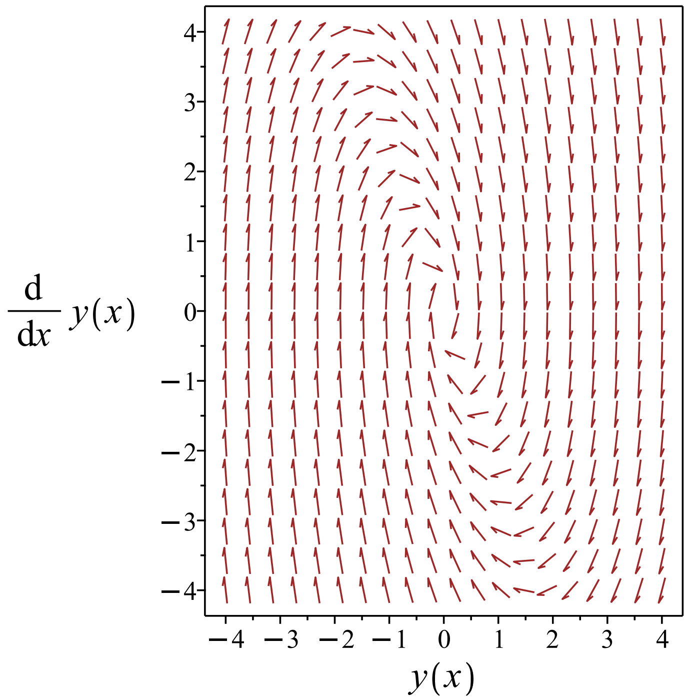


Figure 236: Slope field plot

<u>Verification of solutions</u>

$$y = \cos(2x)\,e^{-x}c_1 + \frac{\sin(2x)\,e^{-x}c_2}{2} - \frac{12\cos(2x)}{17} + \frac{3\sin(2x)}{17}$$

Verified OK.

7.5.3 Maple step by step solution

Let's solve

$y'' + 2y' + 5y = 3\sin(2x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 2r + 5 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-2)\pm(\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

$r = (-1-2\mathrm{I}, -1+2\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-x} \cos(2x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{-x} \sin(2x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = \cos(2x)\,\mathrm{e}^{-x} c_1 + \sin(2x)\,\mathrm{e}^{-x} c_2 + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 3\sin(2x)\right]$

  - Wronskian of solutions of the homogeneous equation

    $$W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-x}\cos(2x) & \mathrm{e}^{-x}\sin(2x) \\ -\mathrm{e}^{-x}\cos(2x) - 2\,\mathrm{e}^{-x}\sin(2x) & -\mathrm{e}^{-x}\sin(2x) + 2\,\mathrm{e}^{-x}\cos(2x) \end{bmatrix}$$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 2\,\mathrm{e}^{-2x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = \frac{3\,\mathrm{e}^{-x}\left(\sin(2x)\left(\int \sin(4x)\mathrm{e}^{x} dx\right) - 2\cos(2x)\left(\int \mathrm{e}^{x}\sin(2x)^2 dx\right)\right)}{4}$

  - Compute integrals

    $y_p(x) = -\frac{12\cos(2x)}{17} + \frac{3\sin(2x)}{17}$

- Substitute particular solution into general solution to ODE

  $y = \sin(2x)\,\mathrm{e}^{-x} c_2 + \cos(2x)\,\mathrm{e}^{-x} c_1 + \frac{3\sin(2x)}{17} - \frac{12\cos(2x)}{17}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 36

```
dsolve(diff(y(x),x$2)+2*diff(y(x),x)+5*y(x)=3*sin(2*x),y(x), singsol=all)
```

$$y(x) = \frac{\sin(2x)\left(17\,\mathrm{e}^{-x}c_2+3\right)}{17}+\cos(2x)\,\mathrm{e}^{-x}c_1-\frac{12\cos(2x)}{17}$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 45

```
DSolve[y''[x]+2*y'[x]+5*y[x]==3*Sin[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{17}e^{-x}((-12e^x+17c_2)\cos(2x)+(3e^x+17c_1)\sin(2x))$$

7.6 problem Problem 30

Internal problem ID [2750]
Internal file name [`OUTPUT/2242_Sunday_June_05_2022_02_56_03_AM_97144766/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.3, The Method of Undetermined Coefficients. page 525
Problem number: Problem 30.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _with_linear_symmetries]]
```

$$\boxed{y''' + 2y'' - 5y' - 6y = 4x^2}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' + 2y'' - 5y' - 6y = 0$$

The characteristic equation is

$$\lambda^3 + 2\lambda^2 - 5\lambda - 6 = 0$$

The roots of the above equation are

$$\lambda_1 = 2$$
$$\lambda_2 = -3$$
$$\lambda_3 = -1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + c_2 e^{-3x} + c_3 e^{2x}$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-x}$$
$$y_2 = e^{-3x}$$
$$y_3 = e^{2x}$$

Now the particular solution to the given ODE is found

$$y''' + 2y'' - 5y' - 6y = 4x^2$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-3x}, e^{-x}, e^{2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_3 x^2 + A_2 x + A_1$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-6A_3x^2 - 6A_2x - 10xA_3 - 6A_1 - 5A_2 + 4A_3 = 4x^2$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{37}{27}, A_2 = \frac{10}{9}, A_3 = -\frac{2}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{2}{3}x^2 + \frac{10}{9}x - \frac{37}{27}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 e^{-x} + c_2 e^{-3x} + c_3 e^{2x}) + \left(-\frac{2}{3}x^2 + \frac{10}{9}x - \frac{37}{27}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} + c_2 e^{-3x} + c_3 e^{2x} - \frac{2x^2}{3} + \frac{10x}{9} - \frac{37}{27} \tag{1}$$

Verification of solutions

$$y = c_1 e^{-x} + c_2 e^{-3x} + c_3 e^{2x} - \frac{2x^2}{3} + \frac{10x}{9} - \frac{37}{27}$$

Verified OK.

7.6.1 Maple step by step solution

Let's solve

$y''' + 2y'' - 5y' - 6y = 4x^2$

- Highest derivative means the order of the ODE is 3

  y'''

□ Convert linear ODE into a system of first order ODEs

  - Define new variable $y_1(x)$

    $y_1(x) = y$

  - Define new variable $y_2(x)$

    $y_2(x) = y'$

  - Define new variable $y_3(x)$

    $y_3(x) = y''$

  - Isolate for $y_3'(x)$ using original ODE

$y_3'(x) = 4x^2 - 2y_3(x) + 5y_2(x) + 6y_1(x)$

Convert linear ODE into a system of first order ODEs

$[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = 4x^2 - 2y_3(x) + 5y_2(x) + 6y_1(x)]$

- Define vector

$$\overrightarrow{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\overrightarrow{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & 5 & -2 \end{bmatrix} \cdot \overrightarrow{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 4x^2 \end{bmatrix}$$

- Define the forcing function

$$\overrightarrow{f}(x) = \begin{bmatrix} 0 \\ 0 \\ 4x^2 \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & 5 & -2 \end{bmatrix}$$

- Rewrite the system as

$\overrightarrow{y}'(x) = A \cdot \overrightarrow{y}(x) + \overrightarrow{f}$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-3, \begin{bmatrix} \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix}\right], \left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-3, \left[\begin{array}{c} \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = e^{-3x} \cdot \left[\begin{array}{c} \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[-1, \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_2 = e^{-x} \cdot \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[2, \left[\begin{array}{c} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_3 = e^{2x} \cdot \left[\begin{array}{c} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{array}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\overrightarrow{y}_p($

$$\overrightarrow{y}(x) = c_1\overrightarrow{y}_1 + c_2\overrightarrow{y}_2 + c_3\overrightarrow{y}_3 + \overrightarrow{y}_p(x)$$

☐ Fundamental matrix

  - Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} \frac{e^{-3x}}{9} & e^{-x} & \frac{e^{2x}}{4} \\ -\frac{e^{-3x}}{3} & -e^{-x} & \frac{e^{2x}}{2} \\ e^{-3x} & e^{-x} & e^{2x} \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

  $\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} \frac{e^{-3x}}{9} & e^{-x} & \frac{e^{2x}}{4} \\ -\frac{e^{-3x}}{3} & -e^{-x} & \frac{e^{2x}}{2} \\ e^{-3x} & e^{-x} & e^{2x} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} \frac{1}{9} & 1 & \frac{1}{4} \\ -\frac{1}{3} & -1 & \frac{1}{2} \\ 1 & 1 & 1 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} \frac{(e^{5x}+5\,e^{2x}-1)e^{-3x}}{5} & \frac{(8\,e^{5x}-5\,e^{2x}-3)e^{-3x}}{30} & \frac{(2\,e^{5x}-5\,e^{2x}+3)e^{-3x}}{30} \\ \frac{(2\,e^{5x}-5\,e^{2x}+3)e^{-3x}}{5} & \frac{(16\,e^{5x}+5\,e^{2x}+9)e^{-3x}}{30} & \frac{(4\,e^{5x}+5\,e^{2x}-9)e^{-3x}}{30} \\ \frac{(4\,e^{5x}+5\,e^{2x}-9)e^{-3x}}{5} & \frac{(32\,e^{5x}-5\,e^{2x}-27)e^{-3x}}{30} & \frac{(8\,e^{5x}-5\,e^{2x}+27)e^{-3x}}{30} \end{bmatrix}$$

□ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

  $\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$

- Take the derivative of the particular solution

  $\vec{y}_p'(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- Cancel like terms

  $\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$

- Multiply by the inverse of the fundamental matrix

$$\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$$

- Integrate to solve for $\vec{v}(x)$

$$\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$$

- Plug $\vec{v}(x)$ into the equation for the particular solution

$$\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \left[\begin{array}{c} \frac{e^{-3x}\left((-18x^2+30x-37)e^{3x}+36\,e^{2x}+\frac{9\,e^{5x}}{5}-\frac{4}{5}\right)}{27} \\ -\frac{2(-3\,e^{5x}+30x\,e^{3x}-25\,e^{3x}+30\,e^{2x}-2)e^{-3x}}{45} \\ \frac{4(e^{5x}-5\,e^{3x}+5\,e^{2x}-1)e^{-3x}}{15} \end{array}\right]$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3 + \left[\begin{array}{c} \frac{e^{-3x}\left((-18x^2+30x-37)e^{3x}+36\,e^{2x}+\frac{9\,e^{5x}}{5}-\frac{4}{5}\right)}{27} \\ -\frac{2(-3\,e^{5x}+30x\,e^{3x}-25\,e^{3x}+30\,e^{2x}-2)e^{-3x}}{45} \\ \frac{4(e^{5x}-5\,e^{3x}+5\,e^{2x}-1)e^{-3x}}{15} \end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y = \frac{e^{-3x}\left((-18x^2+30x-37)e^{3x}+27c_2e^{2x}+\frac{27c_3e^{5x}}{4}+3c_1+36\,e^{2x}+\frac{9\,e^{5x}}{5}-\frac{4}{5}\right)}{27}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 44

```
dsolve(diff(y(x),x$3)+2*diff(y(x),x$2)-5*diff(y(x),x)-6*y(x)=4*x^2,y(x), singsol=all)
```

$$y(x) = \frac{(-18x^2 + 30x - 37) e^{-3x} e^{3x}}{27} + (c_2 e^{2x} + c_3 e^{5x} + c_1) e^{-3x}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 45

```
DSolve[y'''[x]+2*y''[x]-5*y'[x]-6*y[x]==4*x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{2x^2}{3} + \frac{10x}{9} + c_1 e^{-3x} + c_2 e^{-x} + c_3 e^{2x} - \frac{37}{27}$$

7.7 problem Problem 31

Internal problem ID [2751]
Internal file name [`OUTPUT/2243_Sunday_June_05_2022_02_56_05_AM_62395484/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.3, The Method of Undetermined Coefficients. page 525
Problem number: Problem 31.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _with_linear_symmetries]]
```

$$\boxed{y''' - y'' + y' - y = 9\,\mathrm{e}^{-x}}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - y'' + y' - y = 0$$

The characteristic equation is

$$\lambda^3 - \lambda^2 + \lambda - 1 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = i$$
$$\lambda_3 = -i$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^x + e^{ix} c_2 + e^{-ix} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^x$$
$$y_2 = e^{ix}$$
$$y_3 = e^{-ix}$$

Now the particular solution to the given ODE is found

$$y''' - y'' + y' - y = 9\,e^{-x}$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$9\,e^{-x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^x, e^{ix}, e^{-ix}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{-x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-4A_1 e^{-x} = 9\,e^{-x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{9}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{9\,\mathrm{e}^{-x}}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{x} + \mathrm{e}^{ix} c_2 + \mathrm{e}^{-ix} c_3\right) + \left(-\frac{9\,\mathrm{e}^{-x}}{4}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \mathrm{e}^{x} + \mathrm{e}^{ix} c_2 + \mathrm{e}^{-ix} c_3 - \frac{9\,\mathrm{e}^{-x}}{4} \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{x} + \mathrm{e}^{ix} c_2 + \mathrm{e}^{-ix} c_3 - \frac{9\,\mathrm{e}^{-x}}{4}$$

Verified OK.

7.7.1 Maple step by step solution

Let's solve

$y''' - y'' + y' - y = 9\,\mathrm{e}^{-x}$

- Highest derivative means the order of the ODE is 3

  y'''

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$

  ◦ Isolate for $y_3'(x)$ using original ODE

$y_3'(x) = 9\,\mathrm{e}^{-x} + y_3(x) - y_2(x) + y_1(x)$

Convert linear ODE into a system of first order ODEs

$[y_2(x) = y_1'(x)\,, y_3(x) = y_2'(x)\,, y_3'(x) = 9\,\mathrm{e}^{-x} + y_3(x) - y_2(x) + y_1(x)]$

- Define vector

$$\vec{y}(x) = \left[\begin{array}{c} y_1(x) \\ y_2(x) \\ y_3(x) \end{array}\right]$$

- System to solve

$$\vec{y}'(x) = \left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 1 \end{array}\right] \cdot \vec{y}(x) + \left[\begin{array}{c} 0 \\ 0 \\ 9\,\mathrm{e}^{-x} \end{array}\right]$$

- Define the forcing function

$$\vec{f}(x) = \left[\begin{array}{c} 0 \\ 0 \\ 9\,\mathrm{e}^{-x} \end{array}\right]$$

- Define the coefficient matrix

$$A = \left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 1 \end{array}\right]$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1, \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]\right], \left[-\mathrm{I}, \left[\begin{array}{c} -1 \\ \mathrm{I} \\ 1 \end{array}\right]\right], \left[\mathrm{I}, \left[\begin{array}{c} -1 \\ -\mathrm{I} \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = e^x \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-I, \begin{bmatrix} -1 \\ I \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{-Ix} \cdot \begin{bmatrix} -1 \\ I \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$(\cos(x) - I\sin(x)) \cdot \begin{bmatrix} -1 \\ I \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} -\cos(x) + I\sin(x) \\ I(\cos(x) - I\sin(x)) \\ \cos(x) - I\sin(x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_2(x) = \begin{bmatrix} -\cos(x) \\ \sin(x) \\ \cos(x) \end{bmatrix}, \vec{y}_3(x) = \begin{bmatrix} \sin(x) \\ \cos(x) \\ -\sin(x) \end{bmatrix}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$\overrightarrow{y}(x) = c_1\overrightarrow{y}_1 + c_2\overrightarrow{y}_2(x) + c_3\overrightarrow{y}_3(x) + \overrightarrow{y}_p(x)$

- ☐ Fundamental matrix
  - Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

    $$\phi(x) = \begin{bmatrix} e^x & -\cos(x) & \sin(x) \\ e^x & \sin(x) & \cos(x) \\ e^x & \cos(x) & -\sin(x) \end{bmatrix}$$

  - The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

    $\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

  - Substitute the value of $\phi(x)$ and $\phi(0)$

    $$\Phi(x) = \begin{bmatrix} e^x & -\cos(x) & \sin(x) \\ e^x & \sin(x) & \cos(x) \\ e^x & \cos(x) & -\sin(x) \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}}$$

  - Evaluate and simplify to get the fundamental matrix

    $$\Phi(x) = \begin{bmatrix} \frac{e^x}{2} + \frac{\cos(x)}{2} - \frac{\sin(x)}{2} & \sin(x) & \frac{e^x}{2} - \frac{\cos(x)}{2} - \frac{\sin(x)}{2} \\ \frac{e^x}{2} - \frac{\cos(x)}{2} - \frac{\sin(x)}{2} & \cos(x) & \frac{e^x}{2} + \frac{\sin(x)}{2} - \frac{\cos(x)}{2} \\ \frac{e^x}{2} + \frac{\sin(x)}{2} - \frac{\cos(x)}{2} & -\sin(x) & \frac{e^x}{2} + \frac{\cos(x)}{2} + \frac{\sin(x)}{2} \end{bmatrix}$$

- ☐ Find a particular solution of the system of ODEs using variation of parameters
  - Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(x)$ and solve for $\overrightarrow{v}(x)$

    $\overrightarrow{y}_p(x) = \Phi(x) \cdot \overrightarrow{v}(x)$

  - Take the derivative of the particular solution

    $\overrightarrow{y}'_p(x) = \Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x)$

  - Substitute particular solution and its derivative into the system of ODEs

    $\Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$

  - The fundamental matrix has columns that are solutions to the homogeneous system so its der

    $A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$

  - Cancel like terms

$\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$

- Integrate to solve for $\vec{v}(x)$

  $\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$

- Plug $\vec{v}(x)$ into the equation for the particular solution

  $\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \left[\begin{array}{c} \frac{9\,\mathrm{e}^{x}}{4}-\frac{9\sin(x)}{2}-\frac{9\,\mathrm{e}^{-x}}{4} \\ \frac{9\,\mathrm{e}^{x}}{4}-\frac{9\cos(x)}{2}+\frac{9\,\mathrm{e}^{-x}}{4} \\ \frac{9\,\mathrm{e}^{x}}{4}+\frac{9\sin(x)}{2}-\frac{9\,\mathrm{e}^{-x}}{4} \end{array}\right]$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + \left[\begin{array}{c} \frac{9\,\mathrm{e}^{x}}{4}-\frac{9\sin(x)}{2}-\frac{9\,\mathrm{e}^{-x}}{4} \\ \frac{9\,\mathrm{e}^{x}}{4}-\frac{9\cos(x)}{2}+\frac{9\,\mathrm{e}^{-x}}{4} \\ \frac{9\,\mathrm{e}^{x}}{4}+\frac{9\sin(x)}{2}-\frac{9\,\mathrm{e}^{-x}}{4} \end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y = c_1\mathrm{e}^{x} + \frac{9\,\mathrm{e}^{x}}{4} - \frac{9\sin(x)}{2} - \frac{9\,\mathrm{e}^{-x}}{4} + c_3\sin(x) - c_2\cos(x)$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(y(x),x$3)-diff(y(x),x$2)+diff(y(x),x)-y(x)=9*exp(-x),y(x), singsol=all)
```

$$y(x) = -\frac{9\,\mathrm{e}^{-x}}{4} + \cos(x)\,c_1 + c_2\mathrm{e}^{x} + c_3\sin(x)$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 31

```
DSolve[y'''[x]-y''[x]+y'[x]-y[x]==9*Exp[-x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{9e^{-x}}{4} + c_3e^x + c_1\cos(x) + c_2\sin(x)$$

7.8 problem Problem 32

Internal problem ID [2752]
Internal file name [`OUTPUT/2244_Sunday_June_05_2022_02_56_07_AM_96434338/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.3, The Method of Undetermined Coefficients. page 525
Problem number: Problem 32.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

`[[_3rd_order, _linear, _nonhomogeneous]]`

$$\boxed{y''' + 3y'' + 3y' + y = 2\,e^{-x} + 3\,e^{2x}}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' + 3y'' + 3y' + y = 0$$

The characteristic equation is

$$\lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0$$

The roots of the above equation are

$$\lambda_1 = -1$$
$$\lambda_2 = -1$$
$$\lambda_3 = -1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + c_2 x\, e^{-x} + x^2 e^{-x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-x}$$
$$y_2 = x\, e^{-x}$$
$$y_3 = e^{-x} x^2$$

Now the particular solution to the given ODE is found

$$y''' + 3y'' + 3y' + y = 2\, e^{-x} + 3\, e^{2x}$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2\, e^{-x} + 3\, e^{2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-x}\}, \{e^{2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\, e^{-x}, e^{-x} x^2, e^{-x}\}$$

Since e^{-x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\, e^{-x}\}, \{e^{2x}\}]$$

Since $x\, e^{-x}$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{e^{-x} x^2\}, \{e^{2x}\}]$$

Since $e^{-x} x^2$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^3 e^{-x}\}, \{e^{2x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^3 e^{-x} + A_2 e^{2x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$6A_1\mathrm{e}^{-x} + 27A_2\mathrm{e}^{2x} = 2\,\mathrm{e}^{-x} + 3\,\mathrm{e}^{2x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{3}, A_2 = \frac{1}{9}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x^3\mathrm{e}^{-x}}{3} + \frac{\mathrm{e}^{2x}}{9}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{-x} + c_2x\,\mathrm{e}^{-x} + x^2\mathrm{e}^{-x}c_3\right) + \left(\frac{x^3\mathrm{e}^{-x}}{3} + \frac{\mathrm{e}^{2x}}{9}\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{-x}\left(c_3x^2 + c_2x + c_1\right) + \frac{x^3\mathrm{e}^{-x}}{3} + \frac{\mathrm{e}^{2x}}{9}$$

<u>Summary</u>

The solution(s) found are the following

$$y = \mathrm{e}^{-x}\left(c_3x^2 + c_2x + c_1\right) + \frac{x^3\mathrm{e}^{-x}}{3} + \frac{\mathrm{e}^{2x}}{9} \tag{1}$$

<u>Verification of solutions</u>

$$y = \mathrm{e}^{-x}\left(c_3x^2 + c_2x + c_1\right) + \frac{x^3\mathrm{e}^{-x}}{3} + \frac{\mathrm{e}^{2x}}{9}$$

Verified OK.

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 36

```
dsolve(diff(y(x),x$3)+3*diff(y(x),x$2)+3*diff(y(x),x)+y(x)=2*exp(-x)+3*exp(2*x),y(x), singso
```

$$y(x) = \frac{(9c_3x^2 + 3x^3 + 9c_2x + 9c_1)\,\mathrm{e}^{-x}}{9} + \frac{\mathrm{e}^{2x}}{9}$$

✓ Solution by Mathematica

Time used: 0.066 (sec). Leaf size: 41

```
DSolve[y'''[x]+3*y''[x]+3*y'[x]+y[x]==2*Exp[-x]+3*Exp[2*x],y[x],x,IncludeSingularSolutions -
```

$$y(x) \to \frac{1}{9}e^{-x}\left(3x^3 + 9c_3x^2 + e^{3x} + 9c_2x + 9c_1\right)$$

7.9 problem Problem 33

Internal problem ID [2753]
Internal file name [`OUTPUT/2245_Sunday_June_05_2022_02_56_10_AM_15687103/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.3, The Method of Undetermined Coefficients. page 525
Problem number: Problem 33.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 9y = 5\cos(2x)}$$

With initial conditions

$$[y(0) = 2, y'(0) = 3]$$

7.9.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= 9 \\ F &= 5\cos(2x) \end{aligned}$$

Hence the ode is

$$y'' + 9y = 5\cos(2x)$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 9$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = 5\cos(2x)$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

7.9.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 9, f(x) = 5\cos(2x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 9y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 9$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 9\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 9$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(9)}\\ &= \pm 3i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +3i\\ \lambda_2 &= -3i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 3i\\ \lambda_2 &= -3i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 3$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(3x) + c_2 \sin(3x))$$

Or

$$y = c_1 \cos(3x) + c_2 \sin(3x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(3x) + c_2 \sin(3x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$5\cos(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(3x), \sin(3x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(2x) + A_2 \sin(2x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$5A_1 \cos(2x) + 5A_2 \sin(2x) = 5\cos(2x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1, A_2 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \cos(2x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(3x) + c_2 \sin(3x)) + (\cos(2x)) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \cos(3x) + c_2 \sin(3x) + \cos(2x) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 0$ in the above gives

$$2 = c_1 + 1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -3c_1 \sin(3x) + 3c_2 \cos(3x) - 2\sin(2x)$$

substituting $y' = 3$ and $x = 0$ in the above gives

$$3 = 3c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

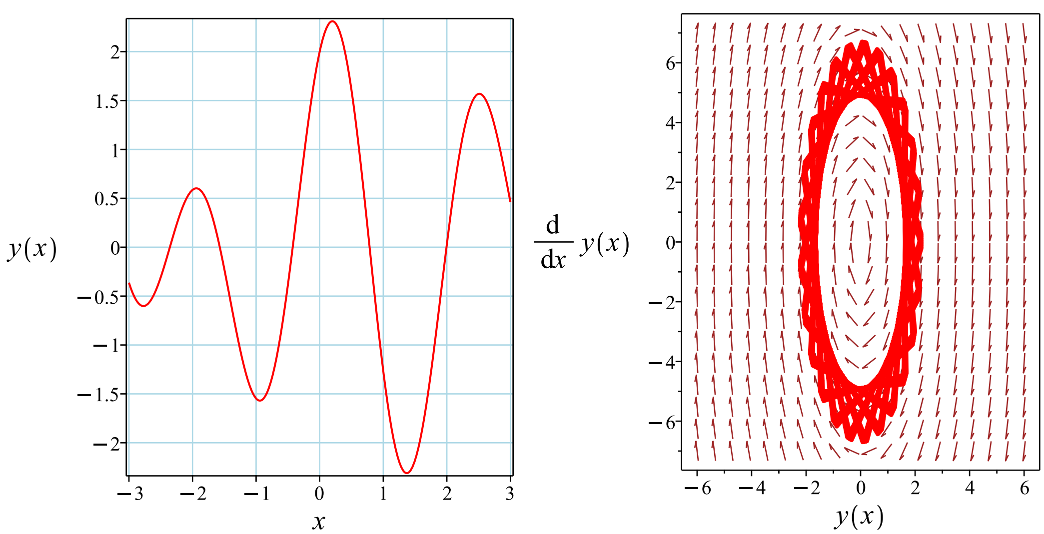
$$c_1 = 1$$
$$c_2 = 1$$

Substituting these values back in above solution results in

$$y = \cos(2x) + \cos(3x) + \sin(3x)$$

Summary
The solution(s) found are the following

$$y = \cos(2x) + \cos(3x) + \sin(3x) \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \cos(2x) + \cos(3x) + \sin(3x)$$

Verified OK.

7.9.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 9y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-9}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -9 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -9z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 213: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -9$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(3x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(3x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(3x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \cos(3x) \int \frac{1}{\cos(3x)^2}\, dx \\ &= \cos(3x) \left(\frac{\tan(3x)}{3}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(3x)) + c_2\left(\cos(3x) \left(\frac{\tan(3x)}{3}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 9y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$5\cos(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sin(3x)}{3}, \cos(3x)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(2x) + A_2 \sin(2x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$5A_1 \cos(2x) + 5A_2 \sin(2x) = 5\cos(2x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1, A_2 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \cos(2x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3}\right) + (\cos(2x)) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3} + \cos(2x) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 0$ in the above gives

$$2 = c_1 + 1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -3c_1 \sin(3x) + c_2 \cos(3x) - 2\sin(2x)$$

substituting $y' = 3$ and $x = 0$ in the above gives

$$3 = c_2 \tag{2A}$$

Equations $\{1A, 2A\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 1 \\ c_2 &= 3 \end{aligned}$$

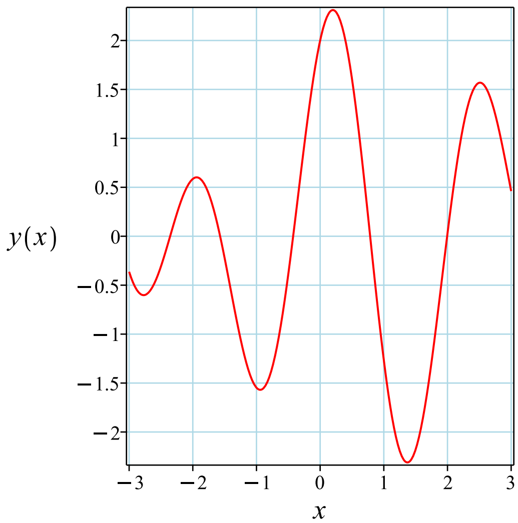
Substituting these values back in above solution results in

$$y = \cos(2x) + \cos(3x) + \sin(3x)$$

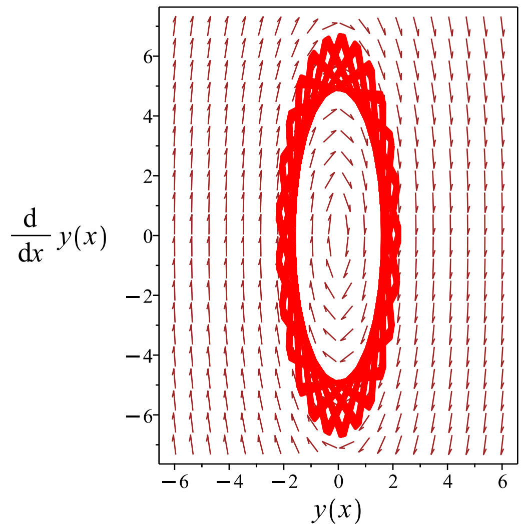
Summary

The solution(s) found are the following

$$y = \cos(2x) + \cos(3x) + \sin(3x) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \cos(2x) + \cos(3x) + \sin(3x)$$

Verified OK.

7.9.4 Maple step by step solution

Let's solve

$\left[y'' + 9y = 5\cos(2x), y(0) = 2, y'\Big|_{\{x=0\}} = 3\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 9 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-36})}{2}$

- Roots of the characteristic polynomial

  $r = (-3\,\mathrm{I}, 3\,\mathrm{I})$

- 1st solution of the homogeneous ODE

$y_1(x) = \cos(3x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(3x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(3x) + c_2 \sin(3x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 5\cos(2x)\right]$

- Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(3x) & \sin(3x) \\ -3\sin(3x) & 3\cos(3x) \end{bmatrix}$

- Compute Wronskian

  $W(y_1(x), y_2(x)) = 3$

- Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\frac{5\cos(3x)\left(\int(\sin(5x)+\sin(x))dx\right)}{6} + \frac{5\sin(3x)\left(\int(\cos(x)+\cos(5x))dx\right)}{6}$

- Compute integrals

  $y_p(x) = \cos(2x)$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(3x) + c_2 \sin(3x) + \cos(2x)$

□ Check validity of solution $y = c_1 \cos(3x) + c_2 \sin(3x) + \cos(2x)$

- Use initial condition $y(0) = 2$

  $2 = c_1 + 1$

- Compute derivative of the solution

  $y' = -3c_1 \sin(3x) + 3c_2 \cos(3x) - 2\sin(2x)$

- Use the initial condition $y'\Big|_{\{x=0\}} = 3$

  $3 = 3c_2$

- Solve for c_1 and c_2

  $$\{c_1 = 1, c_2 = 1\}$$

- Substitute constant values into general solution and simplify

  $$y = \cos(2x) + \cos(3x) + \sin(3x)$$

- Solution to the IVP

  $$y = \cos(2x) + \cos(3x) + \sin(3x)$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 17

```
dsolve([diff(y(x),x$2)+9*y(x)=5*cos(2*x),y(0) = 2, D(y)(0) = 3],y(x), singsol=all)
```

$$y(x) = \sin(3x) + \cos(3x) + \cos(2x)$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 18

```
DSolve[{y''[x]+9*y[x]==5*Cos[2*x],{y[0]==2,y'[0]==3}},y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to \sin(3x) + \cos(2x) + \cos(3x)$$

7.10 problem Problem 34

Internal problem ID [2754]
Internal file name [`OUTPUT/2246_Sunday_June_05_2022_02_56_13_AM_57390033/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.3, The Method of Undetermined Coefficients. page 525
Problem number: Problem 34.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' - y = 9x\,\mathrm{e}^{2x}}$$

With initial conditions

$$[y(0) = 0, y'(0) = 7]$$

7.10.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= -1 \\ F &= 9x\,\mathrm{e}^{2x} \end{aligned}$$

Hence the ode is

$$y'' - y = 9x\,e^{2x}$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = -1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = 9x\,e^{2x}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

7.10.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = -1, f(x) = 9x\,e^{2x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-1)} \\ &= \pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +1 \\ \lambda_2 &= -1\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1 \\ \lambda_2 &= -1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(1)x} + c_2 e^{(-1)x}\end{aligned}$$

Or

$$y = c_1 e^{x} + c_2 e^{-x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{x} + c_2 e^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$9x\, e^{2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\, e^{2x}, e^{2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{x}, e^{-x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 x\, e^{2x} + A_2 e^{2x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4A_1 e^{2x} + 3A_1 x\, e^{2x} + 3A_2 e^{2x} = 9x\, e^{2x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 3, A_2 = -4]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 3x\, e^{2x} - 4\, e^{2x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{x} + c_2 e^{-x}\right) + \left(3x\, e^{2x} - 4\, e^{2x}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{x} + c_2 e^{-x} + 3x\, e^{2x} - 4\, e^{2x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_1 + c_2 - 4 \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1 e^{x} - c_2 e^{-x} - 5\, e^{2x} + 6x\, e^{2x}$$

substituting $y' = 7$ and $x = 0$ in the above gives

$$7 = c_1 - c_2 - 5 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 8$$
$$c_2 = -4$$

Substituting these values back in above solution results in

$$y = 3x\,e^{2x} + 8\,e^{x} - 4\,e^{2x} - 4\,e^{-x}$$

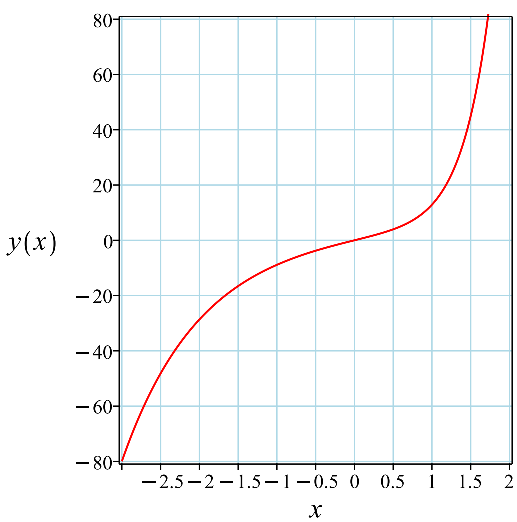
Which simplifies to

$$y = (3x - 4)\,e^{2x} + 8\,e^{x} - 4\,e^{-x}$$

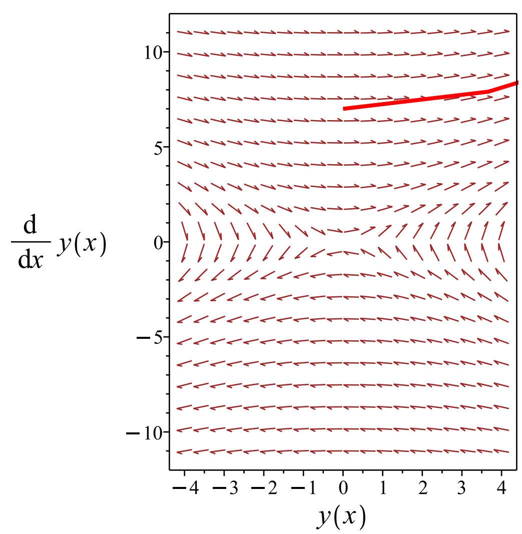
Summary

The solution(s) found are the following

$$y = (3x - 4)\,e^{2x} + 8\,e^{x} - 4\,e^{-x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = (3x - 4)\,e^{2x} + 8\,e^{x} - 4\,e^{-x}$$

Verified OK.

7.10.3 Solving using Kovacic algorithm

Writing the ode as

$$
\begin{align*}
y'' - y &= 0 \tag{1}\\
Ay'' + By' + Cy &= 0 \tag{2}
\end{align*}
$$

Comparing (1) and (2) shows that

$$
\begin{align*}
A &= 1 \\
B &= 0 \tag{3}\\
C &= -1
\end{align*}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = ye^{\int \frac{B}{2A}\,dx}
$$

Then (2) becomes

$$
z''(x) = rz(x) \tag{4}
$$

Where r is given by

$$
\begin{align*}
r &= \frac{s}{t} \tag{5}\\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{align*}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{1}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{align*}
s &= 1 \\
t &= 1
\end{align*}
$$

Therefore eq. (4) becomes

$$
z''(x) = z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\,dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 215: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-x}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= e^{-x}
\end{aligned}
$$

Which simplifies to

$$
y_1 = e^{-x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= e^{-x} \int \frac{1}{e^{-2x}}\, dx \\
&= e^{-x} \left(\frac{e^{2x}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(e^{-x}\right) + c_2 \left(e^{-x} \left(\frac{e^{2x}}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{-x} + \frac{c_2 e^{x}}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$9x\,e^{2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\,e^{2x}, e^{2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{e^{x}}{2}, e^{-x}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 x\,e^{2x} + A_2 e^{2x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4A_1 e^{2x} + 3A_1 x\,e^{2x} + 3A_2 e^{2x} = 9x\,e^{2x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 3, A_2 = -4]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 3x\,e^{2x} - 4\,e^{2x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + \frac{c_2 e^{x}}{2}\right) + \left(3x\,e^{2x} - 4\,e^{2x}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{-x} + \frac{c_2 e^x}{2} + 3x\,e^{2x} - 4\,e^{2x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_1 + \frac{c_2}{2} - 4 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -c_1 e^{-x} + \frac{c_2 e^x}{2} - 5\,e^{2x} + 6x\,e^{2x}$$

substituting $y' = 7$ and $x = 0$ in the above gives

$$7 = -c_1 + \frac{c_2}{2} - 5 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -4$$
$$c_2 = 16$$

Substituting these values back in above solution results in

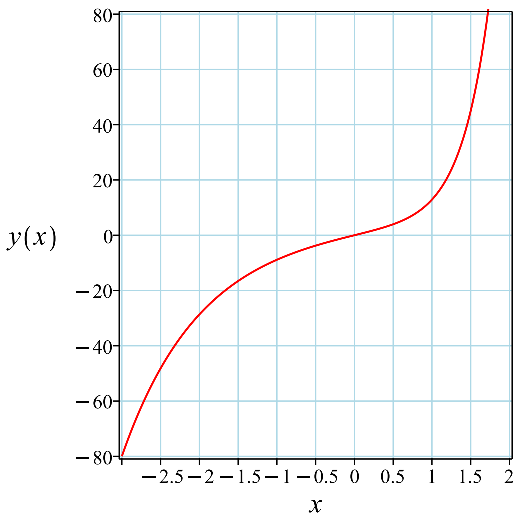
$$y = 3x\,e^{2x} + 8\,e^x - 4\,e^{2x} - 4\,e^{-x}$$

Which simplifies to

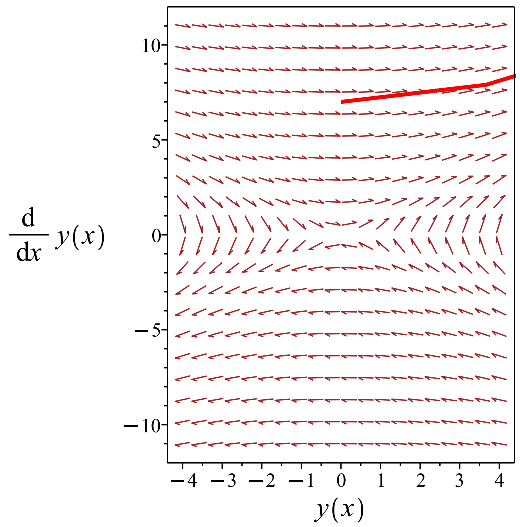
$$y = (3x - 4)\,e^{2x} + 8\,e^x - 4\,e^{-x}$$

<u>Summary</u>
The solution(s) found are the following

$$y = (3x - 4)\,e^{2x} + 8\,e^x - 4\,e^{-x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = (3x-4)\,\mathrm{e}^{2x} + 8\,\mathrm{e}^{x} - 4\,\mathrm{e}^{-x}$$

Verified OK.

7.10.4 Maple step by step solution

Let's solve

$$\left[y'' - y = 9x\,\mathrm{e}^{2x}, y(0) = 0, y'\Big|_{\{x=0\}} = 7\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 1 = 0$

- Factor the characteristic polynomial

  $(r-1)(r+1) = 0$

- Roots of the characteristic polynomial

  $r = (-1, 1)$

- 1st solution of the homogeneous ODE

$y_1(x) = \mathrm{e}^{-x}$

- 2nd solution of the homogeneous ODE

$y_2(x) = \mathrm{e}^{x}$

- General solution of the ODE

$y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

$y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 9x\,\mathrm{e}^{2x}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-x} & \mathrm{e}^{x} \\ -\mathrm{e}^{-x} & \mathrm{e}^{x} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 2$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\frac{9\,\mathrm{e}^{-x}\left(\int x\,\mathrm{e}^{3x}dx\right)}{2} + \frac{9\,\mathrm{e}^{x}\left(\int x\,\mathrm{e}^{x}dx\right)}{2}$

  - Compute integrals

    $y_p(x) = (3x-4)\,\mathrm{e}^{2x}$

- Substitute particular solution into general solution to ODE

$y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + (3x-4)\,\mathrm{e}^{2x}$

☐ Check validity of solution $y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + (3x-4)\,\mathrm{e}^{2x}$

  - Use initial condition $y(0) = 0$

    $0 = c_1 + c_2 - 4$

  - Compute derivative of the solution

    $y' = -c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + 3\,\mathrm{e}^{2x} + 2(3x-4)\,\mathrm{e}^{2x}$

  - Use the initial condition $y'\Big|_{\{x=0\}} = 7$

    $7 = -c_1 + c_2 - 5$

- Solve for c_1 and c_2

  $\{c_1 = -4, c_2 = 8\}$

- Substitute constant values into general solution and simplify

  $y = (3x-4)\,\mathrm{e}^{2x} + 8\,\mathrm{e}^{x} - 4\,\mathrm{e}^{-x}$

- Solution to the IVP

  $y = (3x-4)\,\mathrm{e}^{2x} + 8\,\mathrm{e}^{x} - 4\,\mathrm{e}^{-x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 25

```
dsolve([diff(y(x),x$2)-y(x)=9*x*exp(2*x),y(0) = 0, D(y)(0) = 7],y(x), singsol=all)
```

$$y(x) = -4\,\mathrm{e}^{-x} + 8\,\mathrm{e}^{x} + (3x-4)\,\mathrm{e}^{2x}$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 29

```
DSolve[{y''[x]-y[x]==9*x*Exp[2*x],{y[0]==0,y'[0]==7}},y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to e^{2x}(3x-4) - 4e^{-x} + 8e^{x}$$

7.11 problem Problem 35

Internal problem ID [2755]
Internal file name [`OUTPUT/2247_Sunday_June_05_2022_02_56_15_AM_46099453/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.3, The Method of Undetermined Coefficients. page 525
Problem number: Problem 35.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' + y' - 2y = -10\sin(x)}$$

With initial conditions

$$[y(0) = 2, y'(0) = 1]$$

7.11.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 1 \\ q(x) &= -2 \\ F &= -10\sin(x) \end{aligned}$$

Hence the ode is

$$y'' + y' - 2y = -10\sin(x)$$

The domain of $p(x) = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = -2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = -10\sin(x)$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

7.11.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 1, C = -2, f(x) = -10\sin(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' - 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = -2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + \lambda e^{\lambda x} - 2e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda - 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = -2$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{-1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{1^2 - (4)(1)(-2)} \\
&= -\frac{1}{2} \pm \frac{3}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= -\frac{1}{2} + \frac{3}{2} \\
\lambda_2 &= -\frac{1}{2} - \frac{3}{2}
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= 1 \\
\lambda_2 &= -2
\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}
y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\
y &= c_1 e^{(1)x} + c_2 e^{(-2)x}
\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-2x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$-10 \sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^x, e^{-2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x) + A_2 \sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-3A_1 \cos(x) - 3A_2 \sin(x) - A_1 \sin(x) + A_2 \cos(x) = -10 \sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1, A_2 = 3]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \cos(x) + 3\sin(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^x + c_2 e^{-2x}\right) + (\cos(x) + 3\sin(x)) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^x + c_2 e^{-2x} + \cos(x) + 3\sin(x) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 0$ in the above gives

$$2 = c_1 + c_2 + 1 \tag{1A}$$

Taking derivative of the solution gives

$$
y' = c_1 e^{x} - 2c_2 e^{-2x} - \sin(x) + 3\cos(x)
$$

substituting $y' = 1$ and $x = 0$ in the above gives

$$
1 = c_1 - 2c_2 + 3 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 0 \\
c_2 &= 1
\end{aligned}
$$

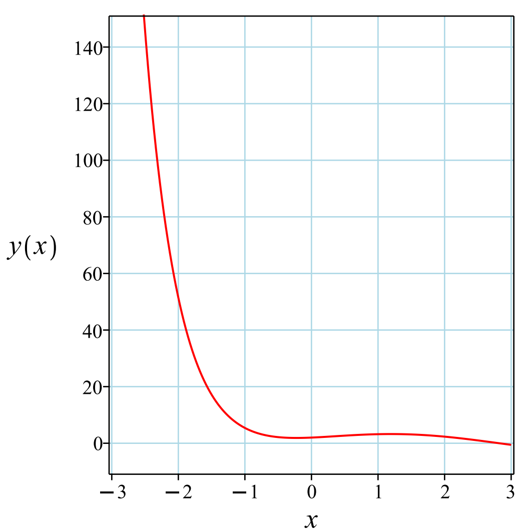
Substituting these values back in above solution results in

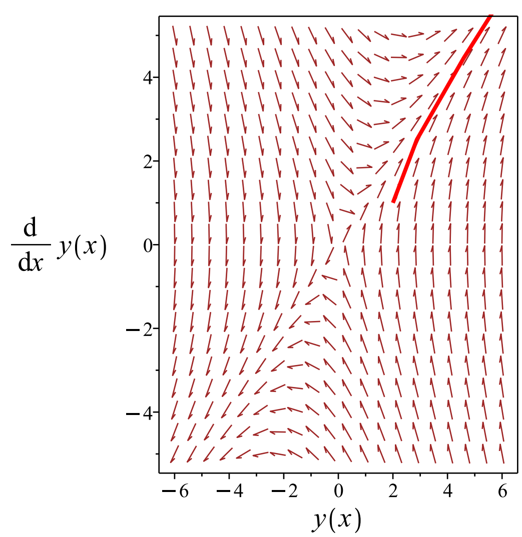
$$
y = 3\sin(x) + \cos(x) + e^{-2x}
$$

Summary

The solution(s) found are the following

$$
y = 3\sin(x) + \cos(x) + e^{-2x} \tag{1}
$$





(a) Solution plot

(b) Slope field plot

Verification of solutions

$$
y = 3\sin(x) + \cos(x) + e^{-2x}
$$

Verified OK.

7.11.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 9 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{9z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 217: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{9}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{3x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dx}
\end{aligned}
$$

$$
\begin{aligned}
&= z_1 e^{-\frac{x}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-2x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\, dx}}{\left(y_1\right)^2}\, dx \\
&= y_1 \int \frac{e^{-x}}{\left(y_1\right)^2}\, dx \\
&= y_1 \left(\frac{\mathrm{e}^{3x}}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-2x}\right) + c_2 \left(\mathrm{e}^{-2x} \left(\frac{\mathrm{e}^{3x}}{3}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + y' - 2y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-2x} + \frac{c_2 \mathrm{e}^{x}}{3}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$-10\sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{e^{x}}{3}, e^{-2x}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1\cos(x) + A_2\sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-3A_1\cos(x) - 3A_2\sin(x) - A_1\sin(x) + A_2\cos(x) = -10\sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1, A_2 = 3]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \cos(x) + 3\sin(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-2x} + \frac{c_2 e^{x}}{3}\right) + (\cos(x) + 3\sin(x)) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{-2x} + \frac{c_2 e^{x}}{3} + \cos(x) + 3\sin(x) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 0$ in the above gives

$$2 = c_1 + \frac{c_2}{3} + 1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -2c_1 e^{-2x} + \frac{c_2 e^{x}}{3} - \sin(x) + 3\cos(x)$$

substituting $y' = 1$ and $x = 0$ in the above gives

$$1 = -2c_1 + \frac{c_2}{3} + 3 \tag{2A}$$

Equations $\{1A,2A\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 1$$
$$c_2 = 0$$

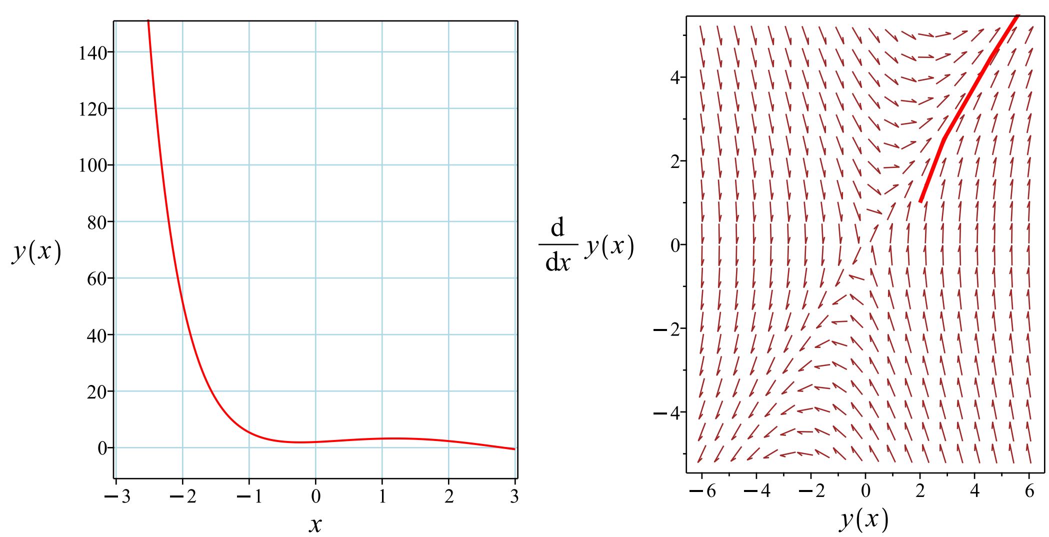
Substituting these values back in above solution results in

$$y = 3\sin(x) + \cos(x) + e^{-2x}$$

Summary

The solution(s) found are the following

$$y = 3\sin(x) + \cos(x) + e^{-2x} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = 3\sin(x) + \cos(x) + e^{-2x}$$

Verified OK.

7.11.4 Maple step by step solution

Let's solve

$$\left[y'' + y' - 2y = -10\sin(x),\, y(0) = 2,\, y'\Big|_{\{x=0\}} = 1\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + r - 2 = 0$

- Factor the characteristic polynomial

  $(r+2)(r-1) = 0$

- Roots of the characteristic polynomial

  $r = (-2, 1)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-2x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-2x} + c_2 e^{x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right),\, f(x) = -10\sin(x)\right]$

  ◦ Wronskian of solutions of the homogeneous equation

$$W(y_1(x)\,,y_2(x)) = \begin{bmatrix} \mathrm{e}^{-2x} & \mathrm{e}^{x} \\ -2\,\mathrm{e}^{-2x} & \mathrm{e}^{x} \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(x)\,,y_2(x)) = 3\,\mathrm{e}^{-x}$

- Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\frac{10\left(\mathrm{e}^{3x}\left(\int \mathrm{e}^{-x}\sin(x)dx\right)-\left(\int \sin(x)\mathrm{e}^{2x}dx\right)\right)\mathrm{e}^{-2x}}{3}$

- Compute integrals

  $y_p(x) = \cos(x) + 3\sin(x)$

• Substitute particular solution into general solution to ODE

  $y = c_1\mathrm{e}^{-2x} + c_2\mathrm{e}^{x} + \cos(x) + 3\sin(x)$

□ Check validity of solution $y = c_1\mathrm{e}^{-2x} + c_2\mathrm{e}^{x} + \cos(x) + 3\sin(x)$

- Use initial condition $y(0) = 2$

  $2 = c_1 + c_2 + 1$

- Compute derivative of the solution

  $y' = -2c_1\mathrm{e}^{-2x} + c_2\mathrm{e}^{x} - \sin(x) + 3\cos(x)$

- Use the initial condition $y'\Big|_{\{x=0\}} = 1$

  $1 = -2c_1 + c_2 + 3$

- Solve for c_1 and c_2

  $\{c_1 = 1, c_2 = 0\}$

- Substitute constant values into general solution and simplify

  $y = 3\sin(x) + \cos(x) + \mathrm{e}^{-2x}$

• Solution to the IVP

  $y = 3\sin(x) + \cos(x) + \mathrm{e}^{-2x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 15

```
dsolve([diff(y(x),x$2)+diff(y(x),x)-2*y(x)=-10*sin(x),y(0) = 2, D(y)(0) = 1],y(x), singsol=a
```

$$y(x) = e^{-2x} + \cos(x) + 3\sin(x)$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 17

```
DSolve[{y''[x]+y'[x]-2*y[x]==-10*Sin[x],{y[0]==2,y'[0]==1}},y[x],x,IncludeSingularSolutions
```

$$y(x) \to e^{-2x} + 3\sin(x) + \cos(x)$$

7.12 problem Problem 36

Internal problem ID [2756]
Internal file name [`OUTPUT/2248_Sunday_June_05_2022_02_56_18_AM_76438375/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.3, The Method of Undetermined Coefficients. page 525
Problem number: Problem 36.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y' - 2y = 4\cos(x) - 2\sin(x)}$$

With initial conditions

$$[y(0) = -1, y'(0) = 4]$$

7.12.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 1 \\ q(x) &= -2 \\ F &= 4\cos(x) - 2\sin(x) \end{aligned}$$

Hence the ode is

$$y'' + y' - 2y = 4\cos(x) - 2\sin(x)$$

The domain of $p(x) = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = -2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = 4\cos(x) - 2\sin(x)$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

7.12.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 1, C = -2, f(x) = 4\cos(x) - 2\sin(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' - 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = -2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + \lambda e^{\lambda x} - 2e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda - 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = -2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{1^2 - (4)(1)(-2)}\\ &= -\frac{1}{2} \pm \frac{3}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{1}{2} + \frac{3}{2}\\ \lambda_2 &= -\frac{1}{2} - \frac{3}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1\\ \lambda_2 &= -2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(1)x} + c_2 e^{(-2)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-2x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$4\cos(x) - 2\sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{x}, e^{-2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x) + A_2 \sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-3A_1 \cos(x) - 3A_2 \sin(x) - A_1 \sin(x) + A_2 \cos(x) = 4\cos(x) - 2\sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -1, A_2 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\cos(x) + \sin(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{x} + c_2 e^{-2x}\right) + (-\cos(x) + \sin(x)) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{x} + c_2 e^{-2x} - \cos(x) + \sin(x) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -1$ and $x = 0$ in the above gives

$$-1 = c_1 + c_2 - 1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1 e^x - 2c_2 e^{-2x} + \sin(x) + \cos(x)$$

substituting $y' = 4$ and $x = 0$ in the above gives

$$4 = c_1 - 2c_2 + 1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

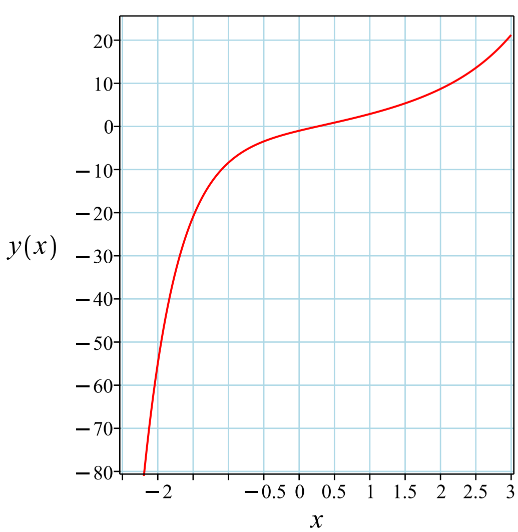
$$c_1 = 1$$
$$c_2 = -1$$

Substituting these values back in above solution results in

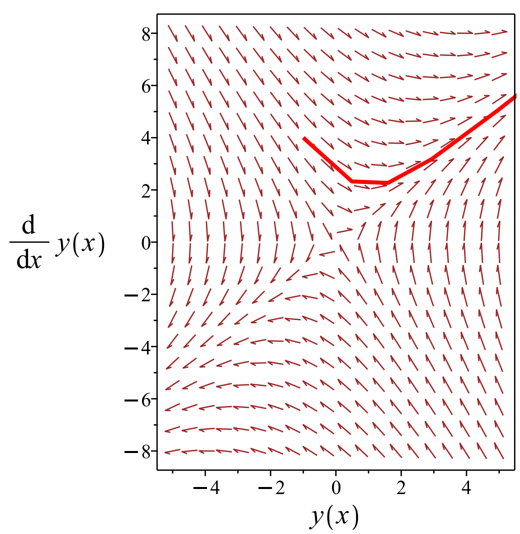
$$y = \sin(x) - \cos(x) + e^x - e^{-2x}$$

Summary
The solution(s) found are the following

$$y = \sin(x) - \cos(x) + e^x - e^{-2x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \sin(x) - \cos(x) + e^x - e^{-2x}$$

Verified OK.

7.12.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 9 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{9z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 219: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = \frac{9}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-\frac{3x}{2}}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dx}
\end{aligned}
$$

$$= z_1 e^{-\frac{x}{2}}$$
$$= z_1 \left(e^{-\frac{x}{2}}\right)$$

Which simplifies to

$$y_1 = e^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{1}{1}\, dx}}{(y_1)^2}\, dx$$
$$= y_1 \int \frac{e^{-x}}{(y_1)^2}\, dx$$
$$= y_1 \left(\frac{e^{3x}}{3}\right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1 \left(e^{-2x}\right) + c_2 \left(e^{-2x} \left(\frac{e^{3x}}{3}\right)\right)$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' - 2y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{-2x} + \frac{c_2 e^{x}}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$4\cos(x) - 2\sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{e^{x}}{3}, e^{-2x}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1\cos(x) + A_2\sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-3A_1\cos(x) - 3A_2\sin(x) - A_1\sin(x) + A_2\cos(x) = 4\cos(x) - 2\sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -1, A_2 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\cos(x) + \sin(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-2x} + \frac{c_2 e^{x}}{3}\right) + (-\cos(x) + \sin(x)) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{-2x} + \frac{c_2 e^{x}}{3} - \cos(x) + \sin(x) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -1$ and $x = 0$ in the above gives

$$-1 = c_1 + \frac{c_2}{3} - 1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -2c_1 e^{-2x} + \frac{c_2 e^{x}}{3} + \sin(x) + \cos(x)$$

substituting $y' = 4$ and $x = 0$ in the above gives

$$4 = -2c_1 + \frac{c_2}{3} + 1 \tag{2A}$$

Equations $\{1A, 2A\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -1$$
$$c_2 = 3$$

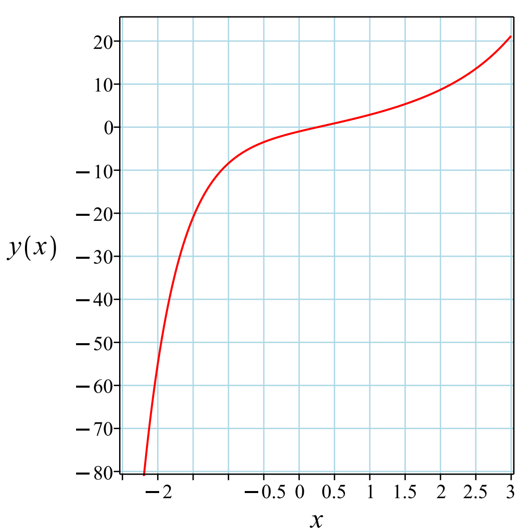
Substituting these values back in above solution results in

$$y = \sin(x) - \cos(x) + e^{x} - e^{-2x}$$

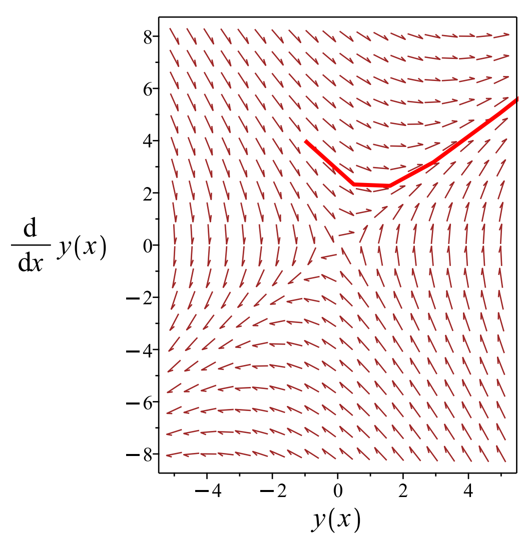
Summary

The solution(s) found are the following

$$y = \sin(x) - \cos(x) + e^{x} - e^{-2x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \sin(x) - \cos(x) + e^{x} - e^{-2x}$$

Verified OK.

7.12.4 Maple step by step solution

Let's solve

$$\left[y'' + y' - 2y = 4\cos(x) - 2\sin(x), y(0) = -1, y'\Big|_{\{x=0\}} = 4\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + r - 2 = 0$

- Factor the characteristic polynomial

  $(r+2)(r-1) = 0$

- Roots of the characteristic polynomial

  $r = (-2, 1)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-2x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-2x} + c_2 e^{x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 4\cos(x) - 2\sin(x)\right]$

  - Wronskian of solutions of the homogeneous equation

$$W(y_1(x)\,, y_2(x)) = \begin{bmatrix} e^{-2x} & e^{x} \\ -2\,e^{-2x} & e^{x} \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(x)\,, y_2(x)) = 3\,e^{-x}$

- Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\frac{2\left(e^{3x}\left(\int(-2\cos(x)+\sin(x))e^{-x}dx\right)-\left(\int(-2\cos(x)+\sin(x))e^{2x}dx\right)\right)e^{-2x}}{3}$

- Compute integrals

  $y_p(x) = -\cos(x) + \sin(x)$

- Substitute particular solution into general solution to ODE

  $y = c_1e^{-2x} + c_2e^{x} - \cos(x) + \sin(x)$

☐ Check validity of solution $y = c_1e^{-2x} + c_2e^{x} - \cos(x) + \sin(x)$

- Use initial condition $y(0) = -1$

  $-1 = c_1 + c_2 - 1$

- Compute derivative of the solution

  $y' = -2c_1e^{-2x} + c_2e^{x} + \sin(x) + \cos(x)$

- Use the initial condition $y'\Big|_{\{x=0\}} = 4$

  $4 = -2c_1 + c_2 + 1$

- Solve for c_1 and c_2

  $\{c_1 = -1, c_2 = 1\}$

- Substitute constant values into general solution and simplify

  $y = -((-\sin(x) + \cos(x))\,e^{2x} - e^{3x} + 1)\,e^{-2x}$

- Solution to the IVP

  $y = -((-\sin(x) + \cos(x))\,e^{2x} - e^{3x} + 1)\,e^{-2x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 19

```
dsolve([diff(y(x),x$2)+diff(y(x),x)-2*y(x)=4*cos(x)-2*sin(x),y(0) = -1, D(y)(0) = 4],y(x), s
```

$$y(x) = -\mathrm{e}^{-2x}\left((-\sin(x) + \cos(x))\,\mathrm{e}^{2x} - \mathrm{e}^{3x} + 1\right)$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 22

```
DSolve[{y''[x]+y'[x]-2*y[x]==4*Cos[x]-2*Sin[x],{y[0]==-1,y'[0]==4}},y[x],x,IncludeSingularSo
```

$$y(x) \to -e^{-2x} + e^{x} + \sin(x) - \cos(x)$$

7.13 problem Problem 38

Internal problem ID [2757]
Internal file name [`OUTPUT/2249_Sunday_June_05_2022_02_56_20_AM_24828777/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.3, The Method of Undetermined Coefficients. page 525
Problem number: Problem 38.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' + \omega^2 y = \frac{F_0 \cos(\omega t)}{m}}$$

With initial conditions

$$[y(0) = 1, y'(0) = 0]$$

7.13.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= \omega^2 \\ F &= \frac{F_0 \cos(\omega t)}{m} \end{aligned}$$

Hence the ode is

$$y'' + \omega^2 y = \frac{F_0 \cos(\omega t)}{m}$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = \omega^2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \frac{F_0 \cos(\omega t)}{m}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

7.13.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = 0, C = \omega^2, f(t) = \frac{F_0 \cos(\omega t)}{m}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + \omega^2 y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 0, C = \omega^2$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t} + \omega^2 \mathrm{e}^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + \omega^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = \omega^2$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(\omega^2)} \\
&= \pm\sqrt{-\omega^2}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= +\sqrt{-\omega^2} \\
\lambda_2 &= -\sqrt{-\omega^2}
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= \sqrt{-\omega^2} \\
\lambda_2 &= -\sqrt{-\omega^2}
\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}
y &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \\
y &= c_1 e^{\left(\sqrt{-\omega^2}\right)t} + c_2 e^{\left(-\sqrt{-\omega^2}\right)t}
\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{\sqrt{-\omega^2}\, t} + c_2 \mathrm{e}^{-\sqrt{-\omega^2}\, t}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{\sqrt{-\omega^2}\, t} + c_2 \mathrm{e}^{-\sqrt{-\omega^2}\, t}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$y_p(t) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \mathrm{e}^{\sqrt{-\omega^2}\, t}$$
$$y_2 = \mathrm{e}^{-\sqrt{-\omega^2}\, t}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(t)}{a W(t)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(t)}{a W(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{\sqrt{-\omega^2}\, t} & \mathrm{e}^{-\sqrt{-\omega^2}\, t} \\ \frac{d}{dt}\left(\mathrm{e}^{\sqrt{-\omega^2}\, t}\right) & \frac{d}{dt}\left(\mathrm{e}^{-\sqrt{-\omega^2}\, t}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{\sqrt{-\omega^2}\, t} & \mathrm{e}^{-\sqrt{-\omega^2}\, t} \\ \sqrt{-\omega^2}\, \mathrm{e}^{\sqrt{-\omega^2}\, t} & -\sqrt{-\omega^2}\, \mathrm{e}^{-\sqrt{-\omega^2}\, t} \end{vmatrix}$$

Therefore

$$W = \left(\mathrm{e}^{\sqrt{-\omega^2}\, t}\right)\left(-\sqrt{-\omega^2}\, \mathrm{e}^{-\sqrt{-\omega^2}\, t}\right) - \left(\mathrm{e}^{-\sqrt{-\omega^2}\, t}\right)\left(\sqrt{-\omega^2}\, \mathrm{e}^{\sqrt{-\omega^2}\, t}\right)$$

Which simplifies to

$$W = -2\, \mathrm{e}^{\sqrt{-\omega^2}\, t} \sqrt{-\omega^2}\, \mathrm{e}^{-\sqrt{-\omega^2}\, t}$$

Which simplifies to

$$W = -2\sqrt{-\omega^2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{e^{-\sqrt{-\omega^2}\, t} F_0 \cos(\omega t)}{m}}{-2\sqrt{-\omega^2}}\, dt$$

Which simplifies to

$$u_1 = -\int -\frac{e^{-\sqrt{-\omega^2}\, t} F_0 \cos(\omega t)}{2m\sqrt{-\omega^2}} dt$$

Hence

$$u_1 = -\frac{-\frac{F_0 e^{-\sqrt{-\omega^2}\, t}}{4\omega m} - \frac{F_0 t\, e^{-\sqrt{-\omega^2}\, t} \tan\left(\frac{\omega t}{2}\right)}{2m} + \frac{F_0 e^{-\sqrt{-\omega^2}\, t} \tan\left(\frac{\omega t}{2}\right)^2}{4\omega m} + \frac{F_0 \sqrt{-\omega^2}\, t\, e^{-\sqrt{-\omega^2}\, t}}{4m\omega} - \frac{F_0 \sqrt{-\omega^2}\, t\, e^{-\sqrt{-\omega^2}\, t} \tan\left(\frac{\omega t}{2}\right)^2}{4m\omega}}{\omega \left(1 + \tan\left(\frac{\omega t}{2}\right)^2\right)}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{e^{\sqrt{-\omega^2}\, t} F_0 \cos(\omega t)}{m}}{-2\sqrt{-\omega^2}}\, dt$$

Which simplifies to

$$u_2 = \int -\frac{e^{\sqrt{-\omega^2}\, t} F_0 \cos(\omega t)}{2m\sqrt{-\omega^2}} dt$$

Hence

$$u_2 = \frac{\frac{F_0 e^{\sqrt{-\omega^2}\, t}}{4\omega m} + \frac{F_0 t\, e^{\sqrt{-\omega^2}\, t} \tan\left(\frac{\omega t}{2}\right)}{2m} - \frac{F_0 e^{\sqrt{-\omega^2}\, t} \tan\left(\frac{\omega t}{2}\right)^2}{4\omega m} + \frac{F_0 \sqrt{-\omega^2}\, t\, e^{\sqrt{-\omega^2}\, t}}{4m\omega} - \frac{F_0 \sqrt{-\omega^2}\, t\, e^{\sqrt{-\omega^2}\, t} \tan\left(\frac{\omega t}{2}\right)^2}{4m\omega}}{\omega \left(1 + \tan\left(\frac{\omega t}{2}\right)^2\right)}$$

Which simplifies to

$$u_1 = \frac{F_0 e^{-\sqrt{-\omega^2}\, t} \left(-\sqrt{-\omega^2}\, t \cos(\omega t) + t\omega \sin(\omega t) + \cos(\omega t)\right)}{4m\, \omega^2}$$

$$u_2 = \frac{F_0 e^{\sqrt{-\omega^2}\, t} \left(\sqrt{-\omega^2}\, t \cos(\omega t) + t\omega \sin(\omega t) + \cos(\omega t)\right)}{4\omega^2 m}$$

Therefore the particular solution, from equation (1) is

$$
\begin{aligned}
y_p(t) = &\frac{F_0 \mathrm{e}^{-\sqrt{-\omega^2}\, t} \left(-\sqrt{-\omega^2}\, t \cos(\omega t) + t\omega \sin(\omega t) + \cos(\omega t)\right) \mathrm{e}^{\sqrt{-\omega^2}\, t}}{4m\,\omega^2} \\
&+ \frac{F_0 \mathrm{e}^{\sqrt{-\omega^2}\, t} \left(\sqrt{-\omega^2}\, t \cos(\omega t) + t\omega \sin(\omega t) + \cos(\omega t)\right) \mathrm{e}^{-\sqrt{-\omega^2}\, t}}{4\omega^2 m}
\end{aligned}
$$

Which simplifies to

$$
y_p(t) = \frac{F_0(t\omega \sin(\omega t) + \cos(\omega t))}{2\omega^2 m}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \mathrm{e}^{\sqrt{-\omega^2}\, t} + c_2 \mathrm{e}^{-\sqrt{-\omega^2}\, t}\right) + \left(\frac{F_0(t\omega \sin(\omega t) + \cos(\omega t))}{2\omega^2 m}\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \mathrm{e}^{\sqrt{-\omega^2}\, t} + c_2 \mathrm{e}^{-\sqrt{-\omega^2}\, t} + \frac{F_0(t\omega \sin(\omega t) + \cos(\omega t))}{2\omega^2 m} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $t = 0$ in the above gives

$$
1 = \frac{2m(c_1 + c_2)\,\omega^2 + F_0}{2\omega^2 m} \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = c_1 \sqrt{-\omega^2}\, \mathrm{e}^{\sqrt{-\omega^2}\, t} - c_2 \sqrt{-\omega^2}\, \mathrm{e}^{-\sqrt{-\omega^2}\, t} + \frac{F_0 t \cos(\omega t)}{2m}
$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$
0 = \sqrt{-\omega^2}\,(c_1 - c_2) \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= -\frac{-2\omega^2 m + F_0}{4\omega^2 m} \\
c_2 &= -\frac{-2\omega^2 m + F_0}{4\omega^2 m}
\end{aligned}
$$

Substituting these values back in above solution results in

$$y = \frac{2F_0 \sin(\omega t)\,\omega t + 2\,e^{\sqrt{-\omega^2}\,t}\omega^2 m + 2\,e^{-\sqrt{-\omega^2}\,t} m\,\omega^2 + 2F_0 \cos(\omega t) - F_0 e^{\sqrt{-\omega^2}\,t} - F_0 e^{-\sqrt{-\omega^2}\,t}}{4\omega^2 m}$$

Which simplifies to

$$y = \frac{(2\omega^2 m - F_0)\,e^{-\sqrt{-\omega^2}\,t} + (2\omega^2 m - F_0)\,e^{\sqrt{-\omega^2}\,t} + 2F_0(t\omega \sin(\omega t) + \cos(\omega t))}{4m\,\omega^2}$$

Summary

The solution(s) found are the following

$$y = \frac{(2\omega^2 m - F_0)\,e^{-\sqrt{-\omega^2}\,t} + (2\omega^2 m - F_0)\,e^{\sqrt{-\omega^2}\,t} + 2F_0(t\omega \sin(\omega t) + \cos(\omega t))}{4m\,\omega^2} \tag{1}$$

Verification of solutions

$$y = \frac{(2\omega^2 m - F_0)\,e^{-\sqrt{-\omega^2}\,t} + (2\omega^2 m - F_0)\,e^{\sqrt{-\omega^2}\,t} + 2F_0(t\omega \sin(\omega t) + \cos(\omega t))}{4m\,\omega^2}$$

Verified OK.

7.13.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + \omega^2 y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= \omega^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-\omega^2}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -\omega^2 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(-\omega^2\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 221: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -\omega^2$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = \mathrm{e}^{\sqrt{-\omega^2}\, t}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \mathrm{e}^{\sqrt{-\omega^2}\, t}
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{\sqrt{-\omega^2}\, t}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt
$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dt \\
&= \mathrm{e}^{\sqrt{-\omega^2}\, t} \int \frac{1}{\mathrm{e}^{2\sqrt{-\omega^2}\, t}}\, dt \\
&= \mathrm{e}^{\sqrt{-\omega^2}\, t} \left(\frac{\sqrt{-\omega^2}\, \mathrm{e}^{-2\sqrt{-\omega^2}\, t}}{2\omega^2} \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{\sqrt{-\omega^2}\, t}\right) + c_2 \left(\mathrm{e}^{\sqrt{-\omega^2}\, t} \left(\frac{\sqrt{-\omega^2}\, \mathrm{e}^{-2\sqrt{-\omega^2}\, t}}{2\omega^2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$
y'' + \omega^2 y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{\sqrt{-\omega^2}\, t} + \frac{c_2 \sqrt{-\omega^2}\, \mathrm{e}^{-\sqrt{-\omega^2}\, t}}{2\omega^2}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$
y_p(t) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \mathrm{e}^{\sqrt{-\omega^2}\, t} \\
y_2 &= \frac{\sqrt{-\omega^2}\, \mathrm{e}^{-\sqrt{-\omega^2}\, t}}{2\omega^2}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(t)}{a W(t)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(t)}{a W(t)} \tag{3}
$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{\sqrt{-\omega^2}\, t} & \frac{\sqrt{-\omega^2}\, e^{-\sqrt{-\omega^2}\, t}}{2\omega^2} \\ \frac{d}{dt}\left(e^{\sqrt{-\omega^2}\, t}\right) & \frac{d}{dt}\left(\frac{\sqrt{-\omega^2}\, e^{-\sqrt{-\omega^2}\, t}}{2\omega^2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{\sqrt{-\omega^2}\, t} & \frac{\sqrt{-\omega^2}\, e^{-\sqrt{-\omega^2}\, t}}{2\omega^2} \\ \sqrt{-\omega^2}\, e^{\sqrt{-\omega^2}\, t} & \frac{e^{-\sqrt{-\omega^2}\, t}}{2} \end{vmatrix}$$

Therefore

$$W = \left(e^{\sqrt{-\omega^2}\, t}\right)\left(\frac{e^{-\sqrt{-\omega^2}\, t}}{2}\right) - \left(\frac{\sqrt{-\omega^2}\, e^{-\sqrt{-\omega^2}\, t}}{2\omega^2}\right)\left(\sqrt{-\omega^2}\, e^{\sqrt{-\omega^2}\, t}\right)$$

Which simplifies to

$$W = e^{\sqrt{-\omega^2}\, t} e^{-\sqrt{-\omega^2}\, t}$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\sqrt{-\omega^2}\, e^{-\sqrt{-\omega^2}\, t} F_0 \cos(\omega t)}{2\omega^2 m}}{1}\, dt$$

Which simplifies to

$$u_1 = -\int \frac{\sqrt{-\omega^2}\, e^{-\sqrt{-\omega^2}\, t} F_0 \cos(\omega t)}{2\omega^2 m} dt$$

Hence

$$u_1 = -\frac{-\frac{F_0 e^{-\sqrt{-\omega^2}\, t}}{4\omega m} - \frac{F_0 t\, e^{-\sqrt{-\omega^2}\, t} \tan\left(\frac{\omega t}{2}\right)}{2m} + \frac{F_0 e^{-\sqrt{-\omega^2}\, t} \tan\left(\frac{\omega t}{2}\right)^2}{4\omega m} + \frac{F_0 \sqrt{-\omega^2}\, t\, e^{-\sqrt{-\omega^2}\, t}}{4m\omega} - \frac{F_0 \sqrt{-\omega^2}\, t\, e^{-\sqrt{-\omega^2}\, t} \tan\left(\frac{\omega t}{2}\right)^2}{4m\omega}}{\omega\left(1 + \tan\left(\frac{\omega t}{2}\right)^2\right)}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{e^{\sqrt{-\omega^2}\, t} F_0 \cos(\omega t)}{m}}{1} dt$$

Which simplifies to

$$u_2 = \int \frac{e^{\sqrt{-\omega^2}\, t} F_0 \cos(\omega t)}{m} dt$$

Hence

$$u_2 = \frac{\frac{F_0 t\, e^{\sqrt{-\omega^2}\, t}}{2m} - \frac{F_0 t\, e^{\sqrt{-\omega^2}\, t} \tan\left(\frac{\omega t}{2}\right)^2}{2m} - \frac{\sqrt{-\omega^2}\, F_0 e^{\sqrt{-\omega^2}\, t}}{2\omega^2 m} + \frac{\sqrt{-\omega^2}\, F_0 e^{\sqrt{-\omega^2}\, t} \tan\left(\frac{\omega t}{2}\right)^2}{2\omega^2 m} - \frac{F_0 \sqrt{-\omega^2}\, t\, e^{\sqrt{-\omega^2}\, t} \tan\left(\frac{\omega t}{2}\right)}{m\omega}}{1 + \tan\left(\frac{\omega t}{2}\right)^2}$$

Which simplifies to

$$u_1 = \frac{F_0 e^{-\sqrt{-\omega^2}\, t} \left(-\sqrt{-\omega^2}\, t \cos(\omega t) + t\omega \sin(\omega t) + \cos(\omega t)\right)}{4m\, \omega^2}$$

$$u_2 = \frac{F_0 \left(\left(-t\omega \sin(\omega t) - \cos(\omega t)\right) \sqrt{-\omega^2} + t\, \omega^2 \cos(\omega t)\right) e^{\sqrt{-\omega^2}\, t}}{2m\, \omega^2}$$

Therefore the particular solution, from equation (1) is

$$y_p(t) = \frac{F_0 e^{-\sqrt{-\omega^2}\, t} \left(-\sqrt{-\omega^2}\, t \cos(\omega t) + t\omega \sin(\omega t) + \cos(\omega t)\right) e^{\sqrt{-\omega^2}\, t}}{4m\, \omega^2} + \frac{F_0 \left(\left(-t\omega \sin(\omega t) - \cos(\omega t)\right) \sqrt{-\omega^2} + t\, \omega^2 \cos(\omega t)\right) e^{\sqrt{-\omega^2}\, t} \sqrt{-\omega^2}\, e^{-\sqrt{-\omega^2}\, t}}{4m\, \omega^4}$$

Which simplifies to

$$y_p(t) = \frac{F_0 (t\omega \sin(\omega t) + \cos(\omega t))}{2\omega^2 m}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{\sqrt{-\omega^2}\, t} + \frac{c_2 \sqrt{-\omega^2}\, e^{-\sqrt{-\omega^2}\, t}}{2\omega^2}\right) + \left(\frac{F_0 (t\omega \sin(\omega t) + \cos(\omega t))}{2\omega^2 m}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \mathrm{e}^{\sqrt{-\omega^2}\, t} + \frac{c_2 \sqrt{-\omega^2}\, \mathrm{e}^{-\sqrt{-\omega^2}\, t}}{2\omega^2} + \frac{F_0 (t\omega \sin(\omega t) + \cos(\omega t))}{2\omega^2 m} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $t = 0$ in the above gives

$$1 = \frac{2c_1\omega^2 m + \sqrt{-\omega^2}\, c_2 m + F_0}{2\omega^2 m} \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1 \sqrt{-\omega^2}\, \mathrm{e}^{\sqrt{-\omega^2}\, t} + \frac{c_2 \mathrm{e}^{-\sqrt{-\omega^2}\, t}}{2} + \frac{F_0 t \cos(\omega t)}{2m}$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = \sqrt{-\omega^2}\, c_1 + \frac{c_2}{2} \tag{2A}$$

Equations $\{1A, 2A\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -\frac{-2\omega^2 m + F_0}{4\omega^2 m}$$
$$c_2 = \frac{\sqrt{-\omega^2}\, (-2\omega^2 m + F_0)}{2\omega^2 m}$$

Substituting these values back in above solution results in

$$y = \frac{2F_0 \sin(\omega t)\, \omega t + 2\, \mathrm{e}^{\sqrt{-\omega^2}\, t}\omega^2 m + 2\, \mathrm{e}^{-\sqrt{-\omega^2}\, t} m\, \omega^2 + 2F_0 \cos(\omega t) - F_0 \mathrm{e}^{\sqrt{-\omega^2}\, t} - F_0 \mathrm{e}^{-\sqrt{-\omega^2}\, t}}{4\omega^2 m}$$

Which simplifies to

$$y = \frac{(2\omega^2 m - F_0)\, \mathrm{e}^{-\sqrt{-\omega^2}\, t} + (2\omega^2 m - F_0)\, \mathrm{e}^{\sqrt{-\omega^2}\, t} + 2F_0 (t\omega \sin(\omega t) + \cos(\omega t))}{4m\, \omega^2}$$

Summary

The solution(s) found are the following

$$y = \frac{(2\omega^2 m - F_0)\, \mathrm{e}^{-\sqrt{-\omega^2}\, t} + (2\omega^2 m - F_0)\, \mathrm{e}^{\sqrt{-\omega^2}\, t} + 2F_0 (t\omega \sin(\omega t) + \cos(\omega t))}{4m\, \omega^2} \tag{1}$$

Verification of solutions

$$y = \frac{(2\omega^2 m - F_0)\, \mathrm{e}^{-\sqrt{-\omega^2}\, t} + (2\omega^2 m - F_0)\, \mathrm{e}^{\sqrt{-\omega^2}\, t} + 2F_0 (t\omega \sin(\omega t) + \cos(\omega t))}{4m\, \omega^2}$$

Verified OK.

7.13.4 Maple step by step solution

Let's solve

$\left[y''+\omega^2 y=\frac{F_0\cos(\omega t)}{m}, y(0)=1, y'\Big|_{\{t=0\}}=0\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $\omega^2+r^2=0$

- Use quadratic formula to solve for r

  $r=\frac{0\pm\left(\sqrt{-4\omega^2}\right)}{2}$

- Roots of the characteristic polynomial

  $r=\left(\sqrt{-\omega^2}, -\sqrt{-\omega^2}\right)$

- 1st solution of the homogeneous ODE

  $y_1(t)=\mathrm{e}^{\sqrt{-\omega^2}\,t}$

- 2nd solution of the homogeneous ODE

  $y_2(t)=\mathrm{e}^{-\sqrt{-\omega^2}\,t}$

- General solution of the ODE

  $y=c_1y_1(t)+c_2y_2(t)+y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y=c_1\mathrm{e}^{\sqrt{-\omega^2}\,t}+c_2\mathrm{e}^{-\sqrt{-\omega^2}\,t}+y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t)=-y_1(t)\left(\int\frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right)+y_2(t)\left(\int\frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t)=\frac{F_0\cos(\omega t)}{m}\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(t), y_2(t))=\begin{bmatrix}\mathrm{e}^{\sqrt{-\omega^2}\,t} & \mathrm{e}^{-\sqrt{-\omega^2}\,t}\\ \sqrt{-\omega^2}\,\mathrm{e}^{\sqrt{-\omega^2}\,t} & -\sqrt{-\omega^2}\,\mathrm{e}^{-\sqrt{-\omega^2}\,t}\end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(t), y_2(t))=-2\sqrt{-\omega^2}$

- Substitute functions into equation for $y_p(t)$

  $y_p(t) = \frac{F_0\left(\mathrm{e}^{\sqrt{-\omega^2}\,t}\left(\int \mathrm{e}^{-\sqrt{-\omega^2}\,t}\cos(\omega t)dt\right)-\mathrm{e}^{-\sqrt{-\omega^2}\,t}\left(\int \mathrm{e}^{\sqrt{-\omega^2}\,t}\cos(\omega t)dt\right)\right)}{2m\sqrt{-\omega^2}}$

- Compute integrals

  $y_p(t) = \frac{F_0\sin(\omega t)t}{2\omega m}$

- Substitute particular solution into general solution to ODE

  $y = c_1\mathrm{e}^{\sqrt{-\omega^2}\,t} + c_2\mathrm{e}^{-\sqrt{-\omega^2}\,t} + \frac{F_0\sin(\omega t)t}{2\omega m}$

□ Check validity of solution $y = c_1\mathrm{e}^{\sqrt{-\omega^2}\,t} + c_2\mathrm{e}^{-\sqrt{-\omega^2}\,t} + \frac{F_0\sin(\omega t)t}{2\omega m}$

- Use initial condition $y(0) = 1$

  $1 = c_1 + c_2$

- Compute derivative of the solution

  $y' = c_1\sqrt{-\omega^2}\,\mathrm{e}^{\sqrt{-\omega^2}\,t} - c_2\sqrt{-\omega^2}\,\mathrm{e}^{-\sqrt{-\omega^2}\,t} + \frac{F_0 t\cos(\omega t)}{2m} + \frac{F_0\sin(\omega t)}{2\omega m}$

- Use the initial condition $y'\Big|_{\{t=0\}} = 0$

  $0 = \sqrt{-\omega^2}\,c_1 - \sqrt{-\omega^2}\,c_2$

- Solve for c_1 and c_2

  $\left\{c_1 = \frac{1}{2}, c_2 = \frac{1}{2}\right\}$

- Substitute constant values into general solution and simplify

  $y = \frac{t\sin(\omega t)F_0+\mathrm{e}^{\sqrt{-\omega^2}\,t}m\omega+\mathrm{e}^{-\sqrt{-\omega^2}\,t}m\omega}{2m\omega}$

- Solution to the IVP

  $y = \frac{t\sin(\omega t)F_0+\mathrm{e}^{\sqrt{-\omega^2}\,t}m\omega+\mathrm{e}^{-\sqrt{-\omega^2}\,t}m\omega}{2m\omega}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 23

```
dsolve([diff(y(t),t$2)+omega^2*y(t)=F__0/m*cos(omega*t),y(0) = 1, D(y)(0) = 0],y(t), singsol
```

$$y(t) = \cos(\omega t) + \frac{F_0 \sin(\omega t)\, t}{2\omega m}$$

✓ Solution by Mathematica

Time used: 0.082 (sec). Leaf size: 26

```
DSolve[{y''[t]+\[Omega]^2*y[t]==F0/m*Cos[\[Omega]*t],{y[0]==1,y'[0]==0}},y[t],t,IncludeSingu
```

$$y(t) \to \frac{\text{F0}t\sin(t\omega)}{2m\omega} + \cos(t\omega)$$

7.14 problem Problem 39

Internal problem ID [2758]
Internal file name [`OUTPUT/2250_Sunday_June_05_2022_02_56_23_AM_65732325/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.3, The Method of Undetermined Coefficients. page 525
Problem number: Problem 39.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 4y' + 6y = 7\,\mathrm{e}^{2x}}$$

7.14.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -4, C = 6, f(x) = 7\,\mathrm{e}^{2x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 4y' + 6y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -4, C = 6$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 4\lambda\, e^{\lambda x} + 6\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 4\lambda + 6 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -4, C = 6$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{4}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-4^2 - (4)\,(1)\,(6)} \\ &= 2 \pm i\sqrt{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= 2 + i\sqrt{2} \\ \lambda_2 &= 2 - i\sqrt{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2 + i\sqrt{2} \\ \lambda_2 &= 2 - i\sqrt{2}\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 2$ and $\beta = \sqrt{2}$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{2x}\left(c_1 \cos\left(\sqrt{2}\, x\right) + c_2 \sin\left(\sqrt{2}\, x\right)\right)$$

Therefore the homogeneous solution y_h is

$$y_h = e^{2x}\left(c_1 \cos\left(\sqrt{2}\, x\right) + c_2 \sin\left(\sqrt{2}\, x\right)\right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$7\,\mathrm{e}^{2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\mathrm{e}^{2x}\cos\left(\sqrt{2}\,x\right), \mathrm{e}^{2x}\sin\left(\sqrt{2}\,x\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1\mathrm{e}^{2x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1\mathrm{e}^{2x} = 7\,\mathrm{e}^{2x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{7}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{7\,\mathrm{e}^{2x}}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\mathrm{e}^{2x}\left(c_1\cos\left(\sqrt{2}\,x\right) + c_2\sin\left(\sqrt{2}\,x\right)\right)\right) + \left(\frac{7\,\mathrm{e}^{2x}}{2}\right) \end{aligned}$$

<u>Summary</u>

The solution(s) found are the following

$$y = \mathrm{e}^{2x}\left(c_1\cos\left(\sqrt{2}\,x\right) + c_2\sin\left(\sqrt{2}\,x\right)\right) + \frac{7\,\mathrm{e}^{2x}}{2} \tag{1}$$

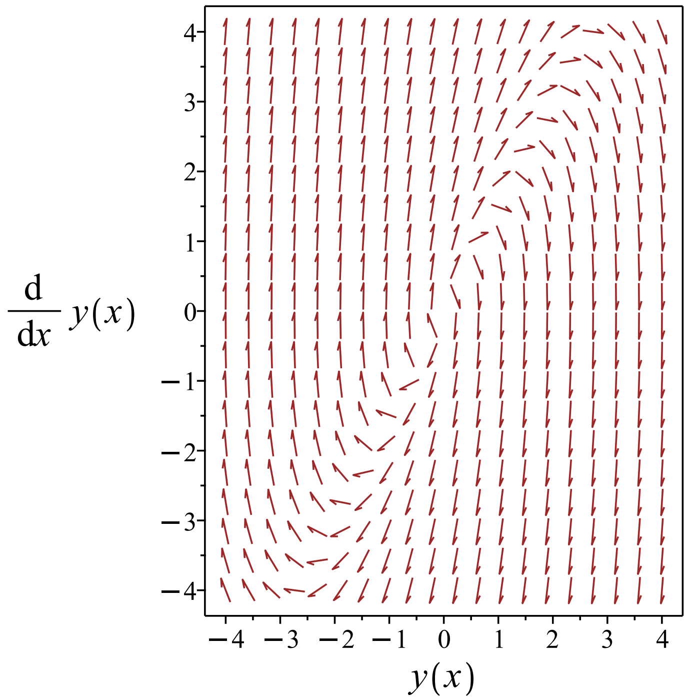


Figure 245: Slope field plot

Verification of solutions

$$y = e^{2x}\left(c_1 \cos\left(\sqrt{2}\, x\right) + c_2 \sin\left(\sqrt{2}\, x\right)\right) + \frac{7\, e^{2x}}{2}$$

Verified OK.

7.14.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 4y' + 6y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -4 \\ C &= 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-2}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -2 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -2z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 223: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -2$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos\left(\sqrt{2}\, x\right)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4}{1}\,dx} \\
&= z_1 e^{2x} \\
&= z_1\left(\mathrm{e}^{2x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{2x}\cos\left(\sqrt{2}\, x\right)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{4x}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{\sqrt{2}\tan\left(\sqrt{2}\, x\right)}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{2x} \cos\left(\sqrt{2}\, x\right)\right) + c_2 \left(\mathrm{e}^{2x} \cos\left(\sqrt{2}\, x\right) \left(\frac{\sqrt{2}\, \tan\left(\sqrt{2}\, x\right)}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 4y' + 6y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = \mathrm{e}^{2x} \cos\left(\sqrt{2}\, x\right) c_1 + \frac{c_2 \sin\left(\sqrt{2}\, x\right) \mathrm{e}^{2x} \sqrt{2}}{2}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
7\,\mathrm{e}^{2x}
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\mathrm{e}^{2x}\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{\mathrm{e}^{2x} \cos\left(\sqrt{2}\, x\right), \frac{\sin\left(\sqrt{2}\, x\right) \mathrm{e}^{2x} \sqrt{2}}{2}\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 \mathrm{e}^{2x}
$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1\mathrm{e}^{2x} = 7\,\mathrm{e}^{2x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{7}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{7\,\mathrm{e}^{2x}}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\mathrm{e}^{2x} \cos\left(\sqrt{2}\,x\right) c_1 + \frac{c_2 \sin\left(\sqrt{2}\,x\right) \mathrm{e}^{2x} \sqrt{2}}{2}\right) + \left(\frac{7\,\mathrm{e}^{2x}}{2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{2x} \cos\left(\sqrt{2}\,x\right) c_1 + \frac{c_2 \sin\left(\sqrt{2}\,x\right) \mathrm{e}^{2x} \sqrt{2}}{2} + \frac{7\,\mathrm{e}^{2x}}{2} \tag{1}$$

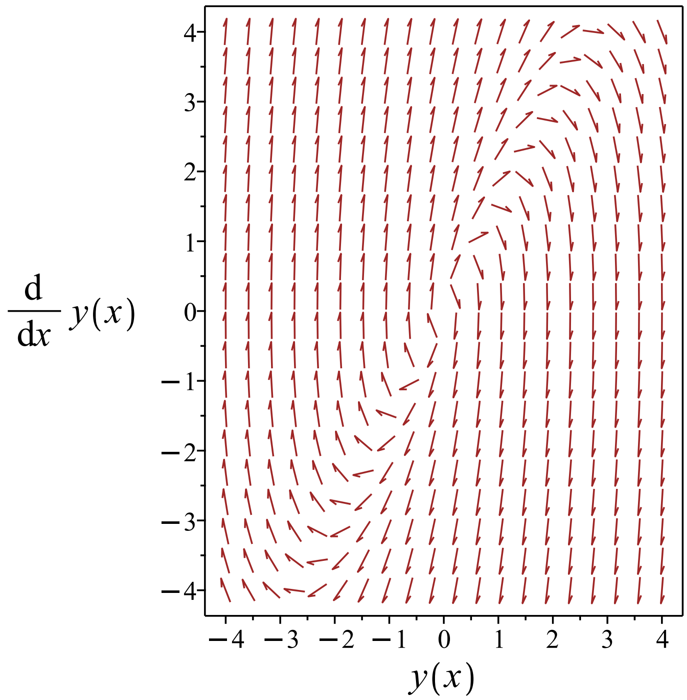


Figure 246: Slope field plot

Verification of solutions

$$y = e^{2x} \cos\left(\sqrt{2}\, x\right) c_1 + \frac{c_2 \sin\left(\sqrt{2}\, x\right) e^{2x} \sqrt{2}}{2} + \frac{7\, e^{2x}}{2}$$

Verified OK.

7.14.3 Maple step by step solution

Let's solve

$y'' - 4y' + 6y = 7\, e^{2x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 4r + 6 = 0$

- Use quadratic formula to solve for r

  $r = \frac{4 \pm \left(\sqrt{-8}\right)}{2}$

- Roots of the characteristic polynomial

$r = (2 - \mathrm{I}\sqrt{2}, 2 + \mathrm{I}\sqrt{2})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{2x} \cos\left(\sqrt{2}\, x\right)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{2x} \sin\left(\sqrt{2}\, x\right)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = \mathrm{e}^{2x} \cos\left(\sqrt{2}\, x\right) c_1 + \mathrm{e}^{2x} \sin\left(\sqrt{2}\, x\right) c_2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 7\, \mathrm{e}^{2x}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \left[\begin{array}{cc} \mathrm{e}^{2x} \cos\left(\sqrt{2}\, x\right) & \mathrm{e}^{2x} \sin\left(\sqrt{2}\, x\right) \\ 2\, \mathrm{e}^{2x} \cos\left(\sqrt{2}\, x\right) - \sin\left(\sqrt{2}\, x\right) \mathrm{e}^{2x} \sqrt{2} & 2\, \mathrm{e}^{2x} \sin\left(\sqrt{2}\, x\right) + \mathrm{e}^{2x} \sqrt{2}\, \cos\left(\sqrt{2}\, x\right) \end{array}\right]$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = \sqrt{2}\, \mathrm{e}^{4x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\frac{7\, \mathrm{e}^{2x} \sqrt{2}\left(\cos\left(\sqrt{2}\, x\right)\left(\int \sin\left(\sqrt{2}\, x\right) dx\right) - \sin\left(\sqrt{2}\, x\right)\left(\int \cos\left(\sqrt{2}\, x\right) dx\right)\right)}{2}$

  - Compute integrals

    $y_p(x) = \frac{7\, \mathrm{e}^{2x}}{2}$

- Substitute particular solution into general solution to ODE

  $y = \mathrm{e}^{2x} \cos\left(\sqrt{2}\, x\right) c_1 + \mathrm{e}^{2x} \sin\left(\sqrt{2}\, x\right) c_2 + \frac{7\, \mathrm{e}^{2x}}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 27

```
dsolve(diff(y(x),x$2)-4*diff(y(x),x)+6*y(x)=7*exp(2*x),y(x), singsol=all)
```

$$y(x) = e^{2x}\left(\frac{7}{2} + c_2 \sin\left(\sqrt{2}\, x\right) + \cos\left(\sqrt{2}\, x\right) c_1\right)$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 40

```
DSolve[y''[x]-4*y'[x]+6*y[x]==7*Exp[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}e^{2x}\left(2c_2 \cos\left(\sqrt{2}x\right) + 2c_1 \sin\left(\sqrt{2}x\right) + 7\right)$$

7.15 problem Problem 40

Internal problem ID [2759]
Internal file name [`OUTPUT/2251_Sunday_June_05_2022_02_56_26_AM_96941610/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.3, The Method of Undetermined Coefficients. page 525
Problem number: Problem 40.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y''' + y'' + y' + y = 4x\,\mathrm{e}^{x}}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' + y'' + y' + y = 0$$

The characteristic equation is

$$\lambda^3 + \lambda^2 + \lambda + 1 = 0$$

The roots of the above equation are

$$\lambda_1 = -1$$
$$\lambda_2 = i$$
$$\lambda_3 = -i$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-x} + \mathrm{e}^{ix} c_2 + \mathrm{e}^{-ix} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = \mathrm{e}^{-x}$$
$$y_2 = \mathrm{e}^{ix}$$
$$y_3 = \mathrm{e}^{-ix}$$

Now the particular solution to the given ODE is found

$$y''' + y'' + y' + y = 4x\,\mathrm{e}^{x}$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$4x\,\mathrm{e}^{x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\,\mathrm{e}^{x}, \mathrm{e}^{x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\mathrm{e}^{ix}, \mathrm{e}^{-x}, \mathrm{e}^{-ix}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 x\,\mathrm{e}^{x} + A_2 \mathrm{e}^{x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$6A_1 \mathrm{e}^{x} + 4A_1 x\,\mathrm{e}^{x} + 4A_2 \mathrm{e}^{x} = 4x\,\mathrm{e}^{x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 1, A_2 = -\frac{3}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x\,\mathrm{e}^{x} - \frac{3\,\mathrm{e}^{x}}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-x} + \mathrm{e}^{ix} c_2 + \mathrm{e}^{-ix} c_3\right) + \left(x\,\mathrm{e}^{x} - \frac{3\,\mathrm{e}^{x}}{2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-x} + \mathrm{e}^{ix} c_2 + \mathrm{e}^{-ix} c_3 + x\,\mathrm{e}^{x} - \frac{3\,\mathrm{e}^{x}}{2} \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{-x} + \mathrm{e}^{ix} c_2 + \mathrm{e}^{-ix} c_3 + x\,\mathrm{e}^{x} - \frac{3\,\mathrm{e}^{x}}{2}$$

Verified OK.

7.15.1 Maple step by step solution

Let's solve

$y''' + y'' + y' + y = 4x\,\mathrm{e}^{x}$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Convert linear ODE into a system of first order ODEs
  - ∘ Define new variable $y_1(x)$

    $y_1(x) = y$
  - ∘ Define new variable $y_2(x)$

    $y_2(x) = y'$
  - ∘ Define new variable $y_3(x)$

    $y_3(x) = y''$
  - ∘ Isolate for $y_3'(x)$ using original ODE

$y_3'(x) = 4x\,\mathrm{e}^x - y_3(x) - y_2(x) - y_1(x)$

Convert linear ODE into a system of first order ODEs

$[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = 4x\,\mathrm{e}^x - y_3(x) - y_2(x) - y_1(x)]$

- Define vector

$$\overrightarrow{y}(x) = \left[\begin{array}{c} y_1(x) \\ y_2(x) \\ y_3(x) \end{array}\right]$$

- System to solve

$$\overrightarrow{y}'(x) = \left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{array}\right] \cdot \overrightarrow{y}(x) + \left[\begin{array}{c} 0 \\ 0 \\ 4x\,\mathrm{e}^x \end{array}\right]$$

- Define the forcing function

$$\overrightarrow{f}(x) = \left[\begin{array}{c} 0 \\ 0 \\ 4x\,\mathrm{e}^x \end{array}\right]$$

- Define the coefficient matrix

$$A = \left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{array}\right]$$

- Rewrite the system as

$$\overrightarrow{y}'(x) = A \cdot \overrightarrow{y}(x) + \overrightarrow{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]\right], \left[-\mathrm{I}, \left[\begin{array}{c} -1 \\ \mathrm{I} \\ 1 \end{array}\right]\right], \left[\mathrm{I}, \left[\begin{array}{c} -1 \\ -\mathrm{I} \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[-1, \left[\begin{array}{c}1 \\ -1 \\ 1\end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_{1} = \mathrm{e}^{-x} \cdot \left[\begin{array}{c}1 \\ -1 \\ 1\end{array}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\mathrm{I}, \left[\begin{array}{c}-1 \\ \mathrm{I} \\ 1\end{array}\right]\right]$$

- Solution from eigenpair

$$\mathrm{e}^{-\mathrm{I} x} \cdot \left[\begin{array}{c}-1 \\ \mathrm{I} \\ 1\end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$(\cos(x) - \mathrm{I}\sin(x)) \cdot \left[\begin{array}{c}-1 \\ \mathrm{I} \\ 1\end{array}\right]$$

- Simplify expression

$$\left[\begin{array}{c}-\cos(x) + \mathrm{I}\sin(x) \\ \mathrm{I}(\cos(x) - \mathrm{I}\sin(x)) \\ \cos(x) - \mathrm{I}\sin(x)\end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\overrightarrow{y}_{2}(x) = \left[\begin{array}{c}-\cos(x) \\ \sin(x) \\ \cos(x)\end{array}\right], \overrightarrow{y}_{3}(x) = \left[\begin{array}{c}\sin(x) \\ \cos(x) \\ -\sin(x)\end{array}\right]\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\overrightarrow{y}_{p}($

$\overrightarrow{y}(x) = c_1\overrightarrow{y}_1 + c_2\overrightarrow{y}_2(x) + c_3\overrightarrow{y}_3(x) + \overrightarrow{y}_p(x)$

- ☐ Fundamental matrix
  - Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

    $$\phi(x) = \begin{bmatrix} e^{-x} & -\cos(x) & \sin(x) \\ -e^{-x} & \sin(x) & \cos(x) \\ e^{-x} & \cos(x) & -\sin(x) \end{bmatrix}$$
  - The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

    $\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$
  - Substitute the value of $\phi(x)$ and $\phi(0)$

    $$\Phi(x) = \begin{bmatrix} e^{-x} & -\cos(x) & \sin(x) \\ -e^{-x} & \sin(x) & \cos(x) \\ e^{-x} & \cos(x) & -\sin(x) \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}}$$
  - Evaluate and simplify to get the fundamental matrix

    $$\Phi(x) = \begin{bmatrix} \frac{e^{-x}}{2} + \frac{\cos(x)}{2} + \frac{\sin(x)}{2} & \sin(x) & \frac{e^{-x}}{2} - \frac{\cos(x)}{2} + \frac{\sin(x)}{2} \\ -\frac{e^{-x}}{2} - \frac{\sin(x)}{2} + \frac{\cos(x)}{2} & \cos(x) & -\frac{e^{-x}}{2} + \frac{\sin(x)}{2} + \frac{\cos(x)}{2} \\ \frac{e^{-x}}{2} - \frac{\cos(x)}{2} - \frac{\sin(x)}{2} & -\sin(x) & \frac{e^{-x}}{2} + \frac{\cos(x)}{2} - \frac{\sin(x)}{2} \end{bmatrix}$$
- ☐ Find a particular solution of the system of ODEs using variation of parameters
  - Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(x)$ and solve for $\overrightarrow{v}(x)$

    $\overrightarrow{y}_p(x) = \Phi(x) \cdot \overrightarrow{v}(x)$
  - Take the derivative of the particular solution

    $\overrightarrow{y}_p'(x) = \Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x)$
  - Substitute particular solution and its derivative into the system of ODEs

    $\Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$
  - The fundamental matrix has columns that are solutions to the homogeneous system so its der

    $A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$
  - Cancel like terms

$$\Phi(x)\cdot\vec{v}'(x)=\vec{f}(x)$$

- Multiply by the inverse of the fundamental matrix

$$\vec{v}'(x)=\frac{1}{\Phi(x)}\cdot\vec{f}(x)$$

- Integrate to solve for $\vec{v}(x)$

$$\vec{v}(x)=\int_0^x \frac{1}{\Phi(s)}\cdot\vec{f}(s)\,ds$$

- Plug $\vec{v}(x)$ into the equation for the particular solution

$$\vec{y}_p(x)=\Phi(x)\cdot\left(\int_0^x \frac{1}{\Phi(s)}\cdot\vec{f}(s)\,ds\right)$$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x)=\begin{bmatrix} x\,\mathrm{e}^{x}-\frac{3\,\mathrm{e}^{x}}{2}+\frac{\mathrm{e}^{-x}}{2}+\cos(x)+\sin(x) \\ x\,\mathrm{e}^{x}-\frac{\mathrm{e}^{x}}{2}-\frac{\mathrm{e}^{-x}}{2}+\cos(x)-\sin(x) \\ x\,\mathrm{e}^{x}+\frac{\mathrm{e}^{x}}{2}+\frac{\mathrm{e}^{-x}}{2}-\cos(x)-\sin(x) \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{y}(x)=c_1\vec{y}_1+c_2\vec{y}_2(x)+c_3\vec{y}_3(x)+\begin{bmatrix} x\,\mathrm{e}^{x}-\frac{3\,\mathrm{e}^{x}}{2}+\frac{\mathrm{e}^{-x}}{2}+\cos(x)+\sin(x) \\ x\,\mathrm{e}^{x}-\frac{\mathrm{e}^{x}}{2}-\frac{\mathrm{e}^{-x}}{2}+\cos(x)-\sin(x) \\ x\,\mathrm{e}^{x}+\frac{\mathrm{e}^{x}}{2}+\frac{\mathrm{e}^{-x}}{2}-\cos(x)-\sin(x) \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y=c_1\mathrm{e}^{-x}+x\,\mathrm{e}^{x}-\frac{3\,\mathrm{e}^{x}}{2}+\frac{\mathrm{e}^{-x}}{2}+\cos(x)+\sin(x)+c_3\sin(x)-c_2\cos(x)$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 27

```
dsolve(diff(y(x),x$3)+diff(y(x),x$2)+diff(y(x),x)+y(x)=4*x*exp(x),y(x), singsol=all)
```

$$y(x) = c_3 \mathrm{e}^{-x} + \cos(x) c_1 + x \,\mathrm{e}^{x} + \sin(x) c_2 - \frac{3\,\mathrm{e}^{x}}{2}$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 36

```
DSolve[y'''[x]+y''[x]+y'[x]+y[x]==4*x*Exp[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^x x - \frac{3e^x}{2} + c_3 e^{-x} + c_1 \cos(x) + c_2 \sin(x)$$

7.16 problem Problem 41

Internal problem ID [2760]
Internal file name [`OUTPUT/2252_Sunday_June_05_2022_02_56_28_AM_61127683/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.3, The Method of Undetermined Coefficients. page 525
Problem number: Problem 41.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_y]]
```

$$\boxed{y'''' + 104y''' + 2740y'' = 5\,\mathrm{e}^{-2x}\cos(3x)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' + 104y''' + 2740y'' = 0$$

The characteristic equation is

$$\lambda^4 + 104\lambda^3 + 2740\lambda^2 = 0$$

The roots of the above equation are

$$\begin{aligned}
\lambda_1 &= 0 \\
\lambda_2 &= 0 \\
\lambda_3 &= -52 + 6i \\
\lambda_4 &= -52 - 6i
\end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = c_2 x + c_1 + \mathrm{e}^{(-52+6i)x} c_3 + \mathrm{e}^{(-52-6i)x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned} y_1 &= 1 \\ y_2 &= x \\ y_3 &= \mathrm{e}^{(-52+6i)x} \\ y_4 &= \mathrm{e}^{(-52-6i)x} \end{aligned}$$

Now the particular solution to the given ODE is found

$$y'''' + 104y''' + 2740y'' = 5\,\mathrm{e}^{-2x} \cos(3x)$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$5\,\mathrm{e}^{-2x} \cos(3x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{-2x} \cos(3x), \mathrm{e}^{-2x} \sin(3x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, x, \mathrm{e}^{(-52-6i)x}, \mathrm{e}^{(-52+6i)x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \mathrm{e}^{-2x} \cos(3x) + A_2 \mathrm{e}^{-2x} \sin(3x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned} &-9035A_1\mathrm{e}^{-2x} \cos(3x) + 31824A_1\mathrm{e}^{-2x} \sin(3x) \\ &- 9035A_2\mathrm{e}^{-2x} \sin(3x) - 31824A_2\mathrm{e}^{-2x} \cos(3x) = 5\,\mathrm{e}^{-2x} \cos(3x) \end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{3475}{84184477}, A_2 = -\frac{12240}{84184477}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{3475\,\mathrm{e}^{-2x}\cos(3x)}{84184477} - \frac{12240\,\mathrm{e}^{-2x}\sin(3x)}{84184477}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_2 x + c_1 + \mathrm{e}^{(-52+6i)x} c_3 + \mathrm{e}^{(-52-6i)x} c_4\right) + \left(-\frac{3475\,\mathrm{e}^{-2x}\cos(3x)}{84184477} - \frac{12240\,\mathrm{e}^{-2x}\sin(3x)}{84184477}\right) \end{aligned}$$

<u>Summary</u>

The solution(s) found are the following

$$y = c_2 x + c_1 + \mathrm{e}^{(-52+6i)x} c_3 + \mathrm{e}^{(-52-6i)x} c_4 - \frac{3475\,\mathrm{e}^{-2x}\cos(3x)}{84184477} - \frac{12240\,\mathrm{e}^{-2x}\sin(3x)}{84184477} \tag{1}$$

<u>Verification of solutions</u>

$$y = c_2 x + c_1 + \mathrm{e}^{(-52+6i)x} c_3 + \mathrm{e}^{(-52-6i)x} c_4 - \frac{3475\,\mathrm{e}^{-2x}\cos(3x)}{84184477} - \frac{12240\,\mathrm{e}^{-2x}\sin(3x)}{84184477}$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
-> Calling odsolve with the ODE`, diff(diff(_b(_a), _a), _a) = 5*exp(-2*_a)*cos(3*_a)-104*(d
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
   trying a double symmetry of the form [xi=0, eta=F(x)]
   -> Try solving first the homogeneous part of the ODE
      checking if the LODE has constant coefficients
      <- constant coefficients successful
   <- solving first the homogeneous part of the ODE successful
<- differential order: 4; linear nonhomogeneous with symmetry [0,1] successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 58

```
dsolve(diff(y(x),x$4)+104*diff(y(x),x$3)+2740*diff(y(x),x$2)=5*exp(-2*x)*cos(3*x),y(x), sing
```

$$y(x) = \frac{\left((667c_1 + 156c_2)\cos(6x) - 156\left(c_1 - \frac{667c_2}{156}\right)\sin(6x)\right)e^{-52x}}{1876900} + \frac{5(-695\cos(3x) - 2448\sin(3x))e^{-2x}}{84184477} + c_3x + c_4$$

✓ Solution by Mathematica

Time used: 4.755 (sec). Leaf size: 82

```
DSolve[y''''[x]+104*y'''[x]+2740*y''[x]==5*Exp[-2*x]*Cos[3*x],y[x],x,IncludeSingularSolution
```

$$y(x) \to -\frac{12240e^{-2x}\sin(3x)}{84184477} - \frac{3475e^{-2x}\cos(3x)}{84184477} + c_4x + \frac{(156c_1 + 667c_2)e^{-52x}\cos(6x)}{1876900} + \frac{(667c_1 - 156c_2)e^{-52x}\sin(6x)}{1876900} + c_3$$

7.17 problem Problem 46

Internal problem ID [2761]
Internal file name [`OUTPUT/2253_Sunday_June_05_2022_02_56_30_AM_69860173/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.3, The Method of Undetermined Coefficients. page 525
Problem number: Problem 46.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 2y' - 3y = \sin(x)^2}$$

7.17.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 2, C = -3, f(x) = \sin(x)^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2y' - 3y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 2, C = -3$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 2\lambda \,\mathrm{e}^{\lambda x} - 3\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2\lambda - 3 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = -3$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{2^2 - (4)\,(1)\,(-3)}\\ &= -1 \pm 2\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -1 + 2\\ \lambda_2 &= -1 - 2\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1\\ \lambda_2 &= -3\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(1)x} + c_2 e^{(-3)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-3x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-3x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}, \{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{x}, e^{-3x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 + A_2 \cos(2x) + A_3 \sin(2x)$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-7A_2 \cos(2x) - 7A_3 \sin(2x) - 4A_2 \sin(2x) + 4A_3 \cos(2x) - 3A_1 = \sin(x)^2$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{6}, A_2 = \frac{7}{130}, A_3 = -\frac{2}{65}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{1}{6} + \frac{7\cos(2x)}{130} - \frac{2\sin(2x)}{65}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{x} + c_2 e^{-3x}\right) + \left(-\frac{1}{6} + \frac{7\cos(2x)}{130} - \frac{2\sin(2x)}{65}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{x} + c_2 e^{-3x} - \frac{1}{6} + \frac{7\cos(2x)}{130} - \frac{2\sin(2x)}{65} \tag{1}$$

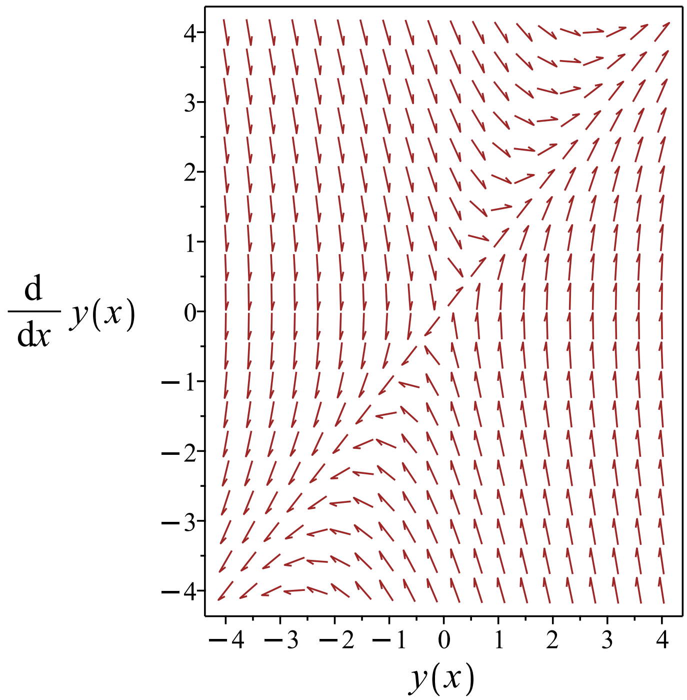


Figure 247: Slope field plot

Verification of solutions

$$y = c_1 e^{x} + c_2 e^{-3x} - \frac{1}{6} + \frac{7\cos(2x)}{130} - \frac{2\sin(2x)}{65}$$

Verified OK.

7.17.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y' - 3y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= 2 \\
C &= -3
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = ye^{\int \frac{B}{2A}\,dx}
$$

Then (2) becomes

$$
z''(x) = rz(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{4}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 4 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = 4z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\,dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 226: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s)\\
&= 0 - 0\\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-2x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\
&= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dx}\\
&= z_1 e^{-x}\\
&= z_1\left(\mathrm{e}^{-x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = e^{-3x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-2x}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{e^{4x}}{4}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(e^{-3x}\right) + c_2 \left(e^{-3x}\left(\frac{e^{4x}}{4}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2y' - 3y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{-3x} + \frac{c_2 e^{x}}{4}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{1\}, \{\cos(2x), \sin(2x)\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{\frac{e^{x}}{4}, e^{-3x}\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 + A_2 \cos(2x) + A_3 \sin(2x)
$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
-7A_2 \cos(2x) - 7A_3 \sin(2x) - 4A_2 \sin(2x) + 4A_3 \cos(2x) - 3A_1 = \sin(x)^2
$$

Solving for the unknowns by comparing coefficients results in

$$
\left[A_1 = -\frac{1}{6}, A_2 = \frac{7}{130}, A_3 = -\frac{2}{65}\right]
$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$
y_p = -\frac{1}{6} + \frac{7\cos(2x)}{130} - \frac{2\sin(2x)}{65}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 e^{-3x} + \frac{c_2 e^{x}}{4}\right) + \left(-\frac{1}{6} + \frac{7\cos(2x)}{130} - \frac{2\sin(2x)}{65}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 e^{-3x} + \frac{c_2 e^{x}}{4} - \frac{1}{6} + \frac{7\cos(2x)}{130} - \frac{2\sin(2x)}{65} \tag{1}
$$

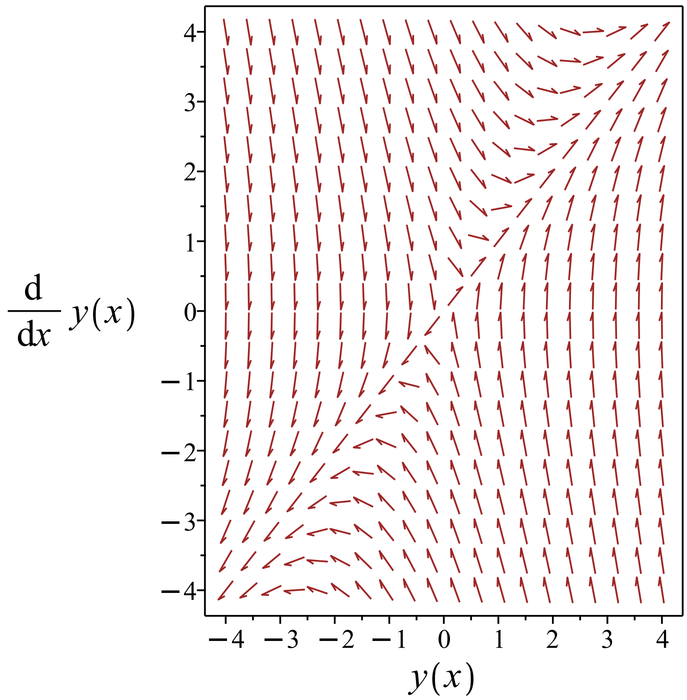


Figure 248: Slope field plot

<u>Verification of solutions</u>

$$y = c_1 e^{-3x} + \frac{c_2 e^{x}}{4} - \frac{1}{6} + \frac{7\cos(2x)}{130} - \frac{2\sin(2x)}{65}$$

Verified OK.

7.17.3 Maple step by step solution

Let's solve

$y'' + 2y' - 3y = \sin(x)^2$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 2r - 3 = 0$

- Factor the characteristic polynomial

  $(r+3)(r-1) = 0$

- Roots of the characteristic polynomial

$r = (-3, 1)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-3x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-3x} + c_2 e^{x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \sin(x)^2\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-3x} & e^{x} \\ -3\,e^{-3x} & e^{x} \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = 4\,e^{-2x}$

  ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = \frac{\left(e^{4x}\left(\int \sin(x)^2 e^{-x} dx\right) - \left(\int e^{3x} \sin(x)^2 dx\right)\right) e^{-3x}}{4}$

  ◦ Compute integrals

  $y_p(x) = -\frac{1}{6} + \frac{7\cos(2x)}{130} - \frac{2\sin(2x)}{65}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-3x} + c_2 e^{x} - \frac{1}{6} + \frac{7\cos(2x)}{130} - \frac{2\sin(2x)}{65}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 36

```
dsolve(diff(y(x),x$2)+2*diff(y(x),x)-3*y(x)=sin(x)^2,y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{-3x}\left(\left(-\frac{1}{6}-\frac{2\sin(2x)}{65}+\frac{7\cos(2x)}{130}\right)\mathrm{e}^{3x}+\mathrm{e}^{4x}c_1+c_2\right)$$

✓ Solution by Mathematica

Time used: 0.098 (sec). Leaf size: 39

```
DSolve[y''[x]+2*y'[x]-3*y[x]==Sin[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{2}{65}\sin(2x)+\frac{7}{130}\cos(2x)+c_1e^{-3x}+c_2e^x-\frac{1}{6}$$

7.18 problem Problem 47

Internal problem ID [2762]
Internal file name [`OUTPUT/2254_Sunday_June_05_2022_02_56_32_AM_68604457/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.3, The Method of Undetermined Coefficients. page 525
Problem number: Problem 47.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 6y = \sin(x)^2 \cos(x)^2}$$

7.18.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 6, f(x) = \sin(x)^2 \cos(x)^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 6y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 6$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 6\,e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 6 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 6$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(6)} \\ &= \pm i\sqrt{6}\end{aligned}$$

Hence

$$\lambda_1 = +i\sqrt{6}$$
$$\lambda_2 = -i\sqrt{6}$$

Which simplifies to

$$\lambda_1 = i\sqrt{6}$$
$$\lambda_2 = -i\sqrt{6}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = \sqrt{6}$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0\left(c_1 \cos\left(\sqrt{6}\,x\right) + c_2 \sin\left(\sqrt{6}\,x\right)\right)$$

Or

$$y = c_1 \cos\left(\sqrt{6}\, x\right) + c_2 \sin\left(\sqrt{6}\, x\right)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos\left(\sqrt{6}\, x\right) + c_2 \sin\left(\sqrt{6}\, x\right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)^2 \cos(x)^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}, \{\cos(4x), \sin(4x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\cos\left(\sqrt{6}\, x\right), \sin\left(\sqrt{6}\, x\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 + A_2 \cos(4x) + A_3 \sin(4x)$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-10A_2 \cos(4x) - 10A_3 \sin(4x) + 6A_1 = \sin(x)^2 \cos(x)^2$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{48}, A_2 = \frac{1}{80}, A_3 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{1}{48} + \frac{\cos(4x)}{80}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \cos\left(\sqrt{6}\, x\right) + c_2 \sin\left(\sqrt{6}\, x\right)\right) + \left(\frac{1}{48} + \frac{\cos(4x)}{80}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \cos\left(\sqrt{6}\, x\right) + c_2 \sin\left(\sqrt{6}\, x\right) + \frac{1}{48} + \frac{\cos(4x)}{80} \tag{1}
$$

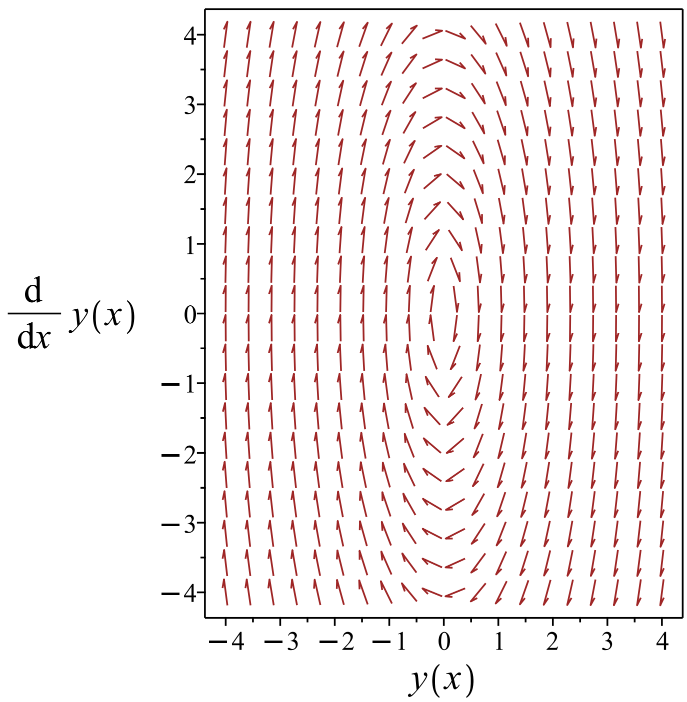


Figure 249: Slope field plot

Verification of solutions

$$
y = c_1 \cos\left(\sqrt{6}\, x\right) + c_2 \sin\left(\sqrt{6}\, x\right) + \frac{1}{48} + \frac{\cos(4x)}{80}
$$

Verified OK.

7.18.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 6y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-6}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -6 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -6z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 228: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -6$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos\left(\sqrt{6}\, x\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \cos\left(\sqrt{6}\, x\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \cos\left(\sqrt{6}\, x\right)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= \cos\left(\sqrt{6}\, x\right) \int \frac{1}{\cos\left(\sqrt{6}\, x\right)^2}\, dx \\
&= \cos\left(\sqrt{6}\, x\right) \left(\frac{\sqrt{6}\, \tan\left(\sqrt{6}\, x\right)}{6}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\cos\left(\sqrt{6}\, x\right)\right) + c_2 \left(\cos\left(\sqrt{6}\, x\right) \left(\frac{\sqrt{6}\, \tan\left(\sqrt{6}\, x\right)}{6}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 6y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos\left(\sqrt{6}\, x\right) + \frac{c_2\sqrt{6}\, \sin\left(\sqrt{6}\, x\right)}{6}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)^2 \cos(x)^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}, \{\cos(4x), \sin(4x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sqrt{6}\, \sin\left(\sqrt{6}\, x\right)}{6}, \cos\left(\sqrt{6}\, x\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 + A_2 \cos(4x) + A_3 \sin(4x)$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-10A_2 \cos(4x) - 10A_3 \sin(4x) + 6A_1 = \sin(x)^2 \cos(x)^2$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{48}, A_2 = \frac{1}{80}, A_3 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{1}{48} + \frac{\cos(4x)}{80}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \cos\left(\sqrt{6}\, x\right) + \frac{c_2 \sqrt{6}\, \sin\left(\sqrt{6}\, x\right)}{6}\right) + \left(\frac{1}{48} + \frac{\cos(4x)}{80}\right)
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = c_1 \cos\left(\sqrt{6}\, x\right) + \frac{c_2 \sqrt{6}\, \sin\left(\sqrt{6}\, x\right)}{6} + \frac{1}{48} + \frac{\cos(4x)}{80} \tag{1}
$$

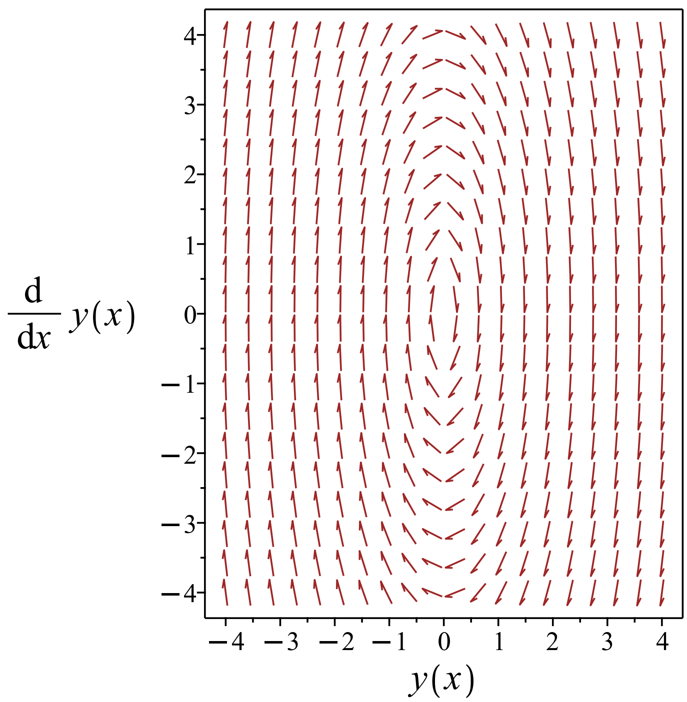


Figure 250: Slope field plot

<u>Verification of solutions</u>

$$
y = c_1 \cos\left(\sqrt{6}\, x\right) + \frac{c_2 \sqrt{6}\, \sin\left(\sqrt{6}\, x\right)}{6} + \frac{1}{48} + \frac{\cos(4x)}{80}
$$

Verified OK.

7.18.3 Maple step by step solution

Let's solve

$y'' + 6y = \sin(x)^2 \cos(x)^2$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 6 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-24})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\mathrm{I}\sqrt{6}, \mathrm{I}\sqrt{6}\right)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos\left(\sqrt{6}\, x\right)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin\left(\sqrt{6}\, x\right)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos\left(\sqrt{6}\, x\right) + c_2 \sin\left(\sqrt{6}\, x\right) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \sin(x)^2 \cos(x)^2\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos\left(\sqrt{6}\, x\right) & \sin\left(\sqrt{6}\, x\right) \\ -\sqrt{6}\sin\left(\sqrt{6}\, x\right) & \sqrt{6}\cos\left(\sqrt{6}\, x\right) \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = \sqrt{6}$

- ○ Substitute functions into equation for $y_p(x)$

$$y_p(x) = \frac{\sqrt{6}\left(\cos\left(\sqrt{6}\,x\right)\left(\int(-1+\cos(4x))\sin\left(\sqrt{6}\,x\right)dx\right)-\sin\left(\sqrt{6}\,x\right)\left(\int(-1+\cos(4x))\cos\left(\sqrt{6}\,x\right)dx\right)\right)}{48}$$

- Compute integrals

  $y_p(x) = \frac{1}{48} + \frac{\cos(4x)}{80}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos\left(\sqrt{6}\,x\right) + c_2 \sin\left(\sqrt{6}\,x\right) + \frac{1}{48} + \frac{\cos(4x)}{80}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 28

```
dsolve(diff(y(x),x$2)+6*y(x)=sin(x)^2*cos(x)^2,y(x), singsol=all)
```

$$y(x) = \sin\left(\sqrt{6}\,x\right)c_2 + \cos\left(\sqrt{6}\,x\right)c_1 + \frac{1}{48} + \frac{\cos(4x)}{80}$$

✓ Solution by Mathematica

Time used: 0.756 (sec). Leaf size: 39

```
DSolve[y''[x]+6*y[x]==Sin[x]^2*Cos[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{80}\cos(4x) + c_1 \cos\left(\sqrt{6}x\right) + c_2 \sin\left(\sqrt{6}x\right) + \frac{1}{48}$$

8 Chapter 8, Linear differential equations of order n. Section 8.4, Complex-Valued Trial Solutions. page 529

8.1 problem Problem 1

Internal problem ID [2763]
Internal file name [`OUTPUT/2255_Sunday_June_05_2022_02_56_35_AM_9892585/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.4, Complex-Valued Trial Solutions. page 529
Problem number: Problem 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 16y = 20\cos(4x)}$$

8.1.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = -16, f(x) = 20\cos(4x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 16y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -16$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 16\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 16 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -16$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-16)}\\ &= \pm 4\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +4\\ \lambda_2 &= -4\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 4\\ \lambda_2 &= -4\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(4)x} + c_2 e^{(-4)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{4x} + c_2 \mathrm{e}^{-4x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{4x} + c_2 \mathrm{e}^{-4x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$20\cos(4x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(4x), \sin(4x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-4x}, e^{4x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1\cos(4x) + A_2\sin(4x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-32A_1\cos(4x) - 32A_2\sin(4x) = 20\cos(4x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{5}{8}, A_2 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{5\cos(4x)}{8}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{4x} + c_2 e^{-4x}\right) + \left(-\frac{5\cos(4x)}{8}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{4x} + c_2 e^{-4x} - \frac{5\cos(4x)}{8} \tag{1}$$

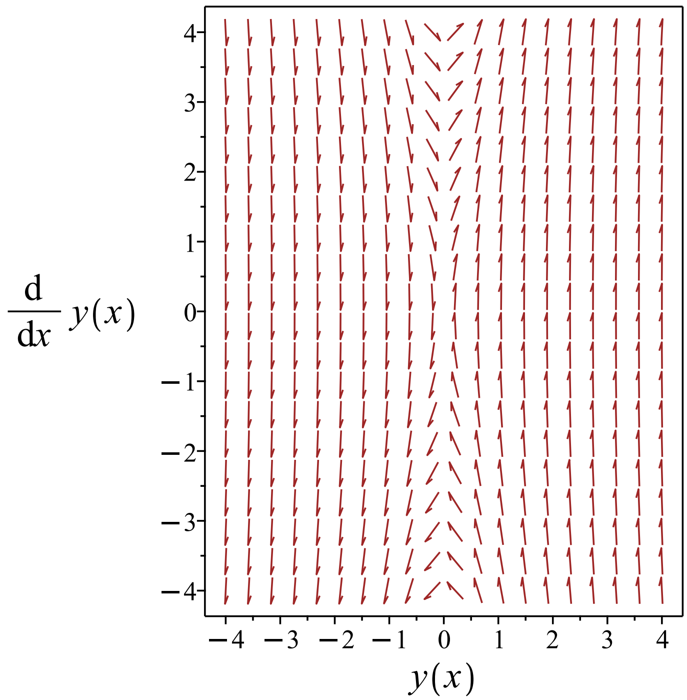


Figure 251: Slope field plot

Verification of solutions

$$y = c_1 e^{4x} + c_2 e^{-4x} - \frac{5\cos(4x)}{8}$$

Verified OK.

8.1.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 16y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -16 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{16}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 16 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 16 z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 230: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 16$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-4x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \mathrm{e}^{-4x} \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-4x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= \mathrm{e}^{-4x} \int \frac{1}{\mathrm{e}^{-8x}}\, dx \\
&= \mathrm{e}^{-4x} \left(\frac{\mathrm{e}^{8x}}{8}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-4x}\right) + c_2 \left(\mathrm{e}^{-4x} \left(\frac{\mathrm{e}^{8x}}{8}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 16y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-4x} + \frac{\mathrm{e}^{4x} c_2}{8}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$20\cos(4x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(4x), \sin(4x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{e^{4x}}{8}, e^{-4x}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1\cos(4x) + A_2\sin(4x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-32A_1\cos(4x) - 32A_2\sin(4x) = 20\cos(4x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{5}{8}, A_2 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{5\cos(4x)}{8}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-4x} + \frac{e^{4x}c_2}{8}\right) + \left(-\frac{5\cos(4x)}{8}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-4x} + \frac{e^{4x} c_2}{8} - \frac{5\cos(4x)}{8} \tag{1}$$

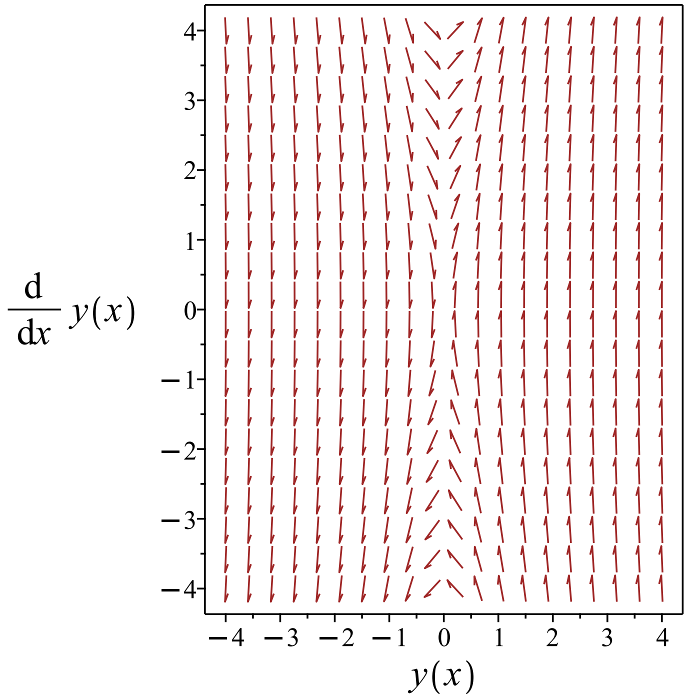


Figure 252: Slope field plot

Verification of solutions

$$y = c_1 e^{-4x} + \frac{e^{4x} c_2}{8} - \frac{5\cos(4x)}{8}$$

Verified OK.

8.1.3 Maple step by step solution

Let's solve

$y'' - 16y = 20\cos(4x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2 - 16 = 0$

- Factor the characteristic polynomial

  $(r-4)(r+4) = 0$

- Roots of the characteristic polynomial

  $r = (-4, 4)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-4x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{4x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-4x} + e^{4x} c_2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 20\cos(4x)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-4x} & e^{4x} \\ -4\,e^{-4x} & 4\,e^{4x} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 8$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\frac{5\,e^{-4x}\left(\int e^{4x}\cos(4x)dx\right)}{2} + \frac{5\,e^{4x}\left(\int e^{-4x}\cos(4x)dx\right)}{2}$

  - Compute integrals

    $y_p(x) = -\frac{5\cos(4x)}{8}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-4x} + e^{4x} c_2 - \frac{5\cos(4x)}{8}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(y(x),x$2)-16*y(x)=20*cos(4*x),y(x), singsol=all)
```

$$y(x) = c_2 e^{4x} + e^{-4x} c_1 - \frac{5\cos(4x)}{8}$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 30

```
DSolve[y''[x]-16*y[x]==20*Cos[4*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{5}{8}\cos(4x) + c_1 e^{4x} + c_2 e^{-4x}$$

8.2 problem Problem 2

Internal problem ID [2764]
Internal file name [`OUTPUT/2256_Sunday_June_05_2022_02_56_37_AM_1048561/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.4, Complex-Valued Trial Solutions. page 529
Problem number: Problem 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' + 2y' + y = 50 \sin(3x)}$$

8.2.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 2, C = 1, f(x) = 50 \sin(3x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2y' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 2, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 2\lambda\,\mathrm{e}^{\lambda x} + \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(2)^2 - (4)(1)(1)} \\ &= -1\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 1$. Therefore the solution is

$$y = c_1 \mathrm{e}^{-x} + c_2 x\,\mathrm{e}^{-x} \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{-x} + c_2 x\,\mathrm{e}^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$50 \sin(3x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(3x), \sin(3x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\,\mathrm{e}^{-x}, \mathrm{e}^{-x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(3x) + A_2 \sin(3x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-8A_1 \cos(3x) - 8A_2 \sin(3x) - 6A_1 \sin(3x) + 6A_2 \cos(3x) = 50 \sin(3x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -3, A_2 = -4]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -3\cos(3x) - 4\sin(3x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-x} + c_2 x\,\mathrm{e}^{-x}\right) + \left(-3\cos(3x) - 4\sin(3x)\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{-x}(c_2 x + c_1) - 3\cos(3x) - 4\sin(3x)$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-x}(c_2 x + c_1) - 3\cos(3x) - 4\sin(3x) \tag{1}$$

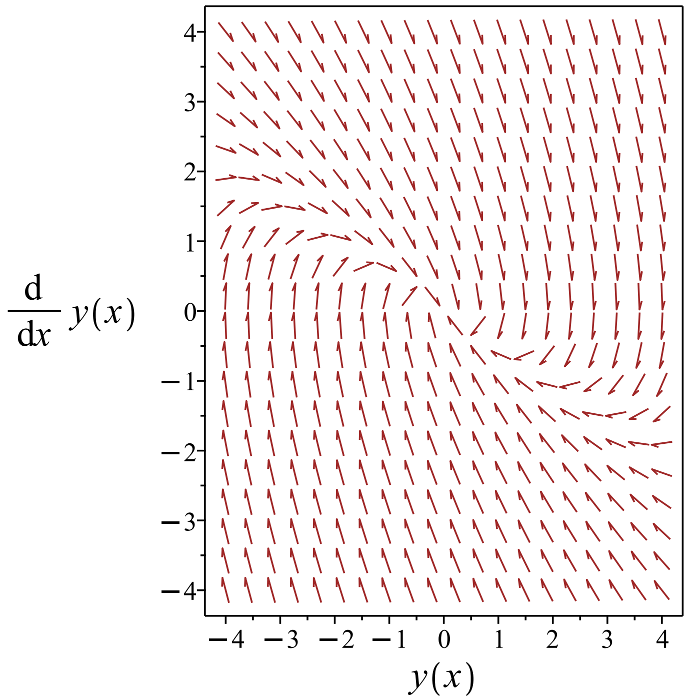


Figure 253: Slope field plot

Verification of solutions

$$y = e^{-x}(c_2x + c_1) - 3\cos(3x) - 4\sin(3x)$$

Verified OK.

8.2.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\,y' + \frac{(p(x)^2 + p'(x))\,y}{2} = f(x)$$

Where $p(x) = 2$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int 2\,dx} \\ &= e^{x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$
\begin{aligned}
(M(x)y)'' &= 50\,\mathrm{e}^{x} \sin(3x) \\
(\mathrm{e}^{x} y)'' &= 50\,\mathrm{e}^{x} \sin(3x)
\end{aligned}
$$

Integrating once gives

$$
(\mathrm{e}^{x} y)' = 5\,\mathrm{e}^{x}(-3\cos(3x) + \sin(3x)) + c_1
$$

Integrating again gives

$$
(\mathrm{e}^{x} y) = -4\,\mathrm{e}^{x} \sin(3x) - 3\,\mathrm{e}^{x} \cos(3x) + c_1 x + c_2
$$

Hence the solution is

$$
y = \frac{-4\,\mathrm{e}^{x} \sin(3x) - 3\,\mathrm{e}^{x} \cos(3x) + c_1 x + c_2}{\mathrm{e}^{x}}
$$

Or

$$
y = -12\cos(x)^{3} - 16\cos(x)^{2}\sin(x) + c_1 x\,\mathrm{e}^{-x} + 9\cos(x) + 4\sin(x) + c_2 \mathrm{e}^{-x}
$$

Summary

The solution(s) found are the following

$$
y = -12\cos(x)^{3} - 16\cos(x)^{2}\sin(x) + c_1 x\,\mathrm{e}^{-x} + 9\cos(x) + 4\sin(x) + c_2 \mathrm{e}^{-x} \tag{1}
$$

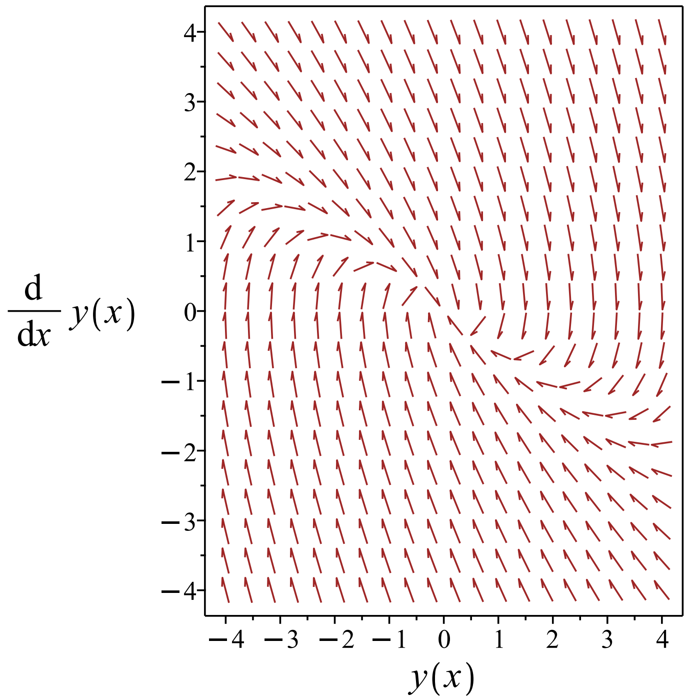


Figure 254: Slope field plot

<u>Verification of solutions</u>

$$y = -12\cos(x)^3 - 16\cos(x)^2\sin(x) + c_1 x\,\mathrm{e}^{-x} + 9\cos(x) + 4\sin(x) + c_2\mathrm{e}^{-x}$$

Verified OK.

8.2.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 232: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dx} \\
&= z_1 e^{-x} \\
&= z_1\left(\mathrm{e}^{-x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-2x}}{\left(y_1\right)^2}\,dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-x}\right) + c_2 \left(\mathrm{e}^{-x}(x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 2y' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-x} + c_2 x\, \mathrm{e}^{-x}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
50 \sin(3x)
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\cos(3x), \sin(3x)\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{x\, \mathrm{e}^{-x}, \mathrm{e}^{-x}\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 \cos(3x) + A_2 \sin(3x)
$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
-8A_1 \cos(3x) - 8A_2 \sin(3x) - 6A_1 \sin(3x) + 6A_2 \cos(3x) = 50 \sin(3x)
$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -3, A_2 = -4]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -3\cos(3x) - 4\sin(3x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \mathrm{e}^{-x} + c_2 x\,\mathrm{e}^{-x}) + (-3\cos(3x) - 4\sin(3x)) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{-x}(c_2 x + c_1) - 3\cos(3x) - 4\sin(3x)$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-x}(c_2 x + c_1) - 3\cos(3x) - 4\sin(3x) \tag{1}$$

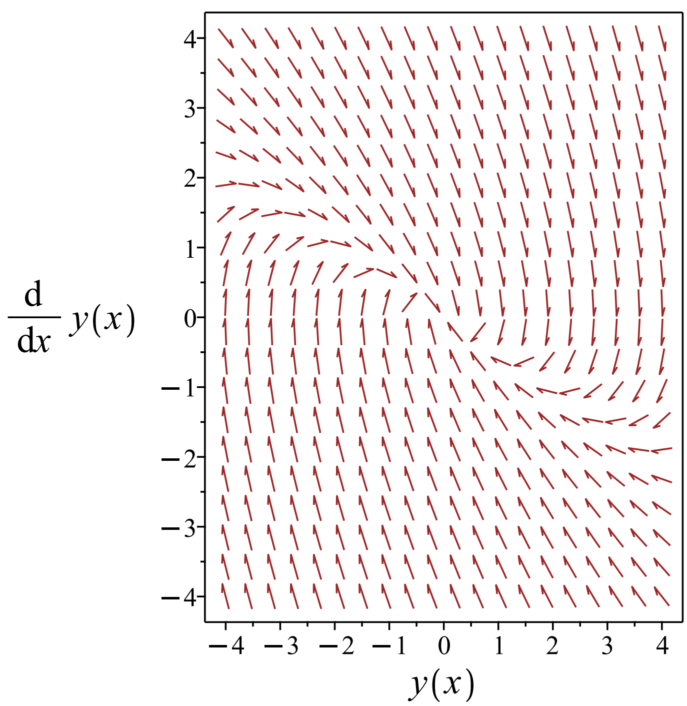


Figure 255: Slope field plot

Verification of solutions

$$y = e^{-x}(c_2 x + c_1) - 3\cos(3x) - 4\sin(3x)$$

Verified OK.

8.2.4 Maple step by step solution

Let's solve

$y'' + 2y' + y = 50\sin(3x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 2r + 1 = 0$

- Factor the characteristic polynomial

  $(r+1)^2 = 0$

- Root of the characteristic polynomial

  $r = -1$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,e^{-x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-x} + c_2 x\,e^{-x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 50\sin(3x)\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-x} & x\,e^{-x} \\ -e^{-x} & e^{-x} - x\,e^{-x} \end{bmatrix}$

- Compute Wronskian

  $W(y_1(x), y_2(x)) = \mathrm{e}^{-2x}$

- Substitute functions into equation for $y_p(x)$

  $y_p(x) = 50\,\mathrm{e}^{-x}\left(-\left(\int x\sin(3x)\,\mathrm{e}^{x}dx\right) + x\left(\int \mathrm{e}^{x}\sin(3x)\,dx\right)\right)$

- Compute integrals

  $y_p(x) = -3\cos(3x) - 4\sin(3x)$

- Substitute particular solution into general solution to ODE

  $y = c_2 x\,\mathrm{e}^{-x} + c_1\mathrm{e}^{-x} - 4\sin(3x) - 3\cos(3x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 27

```
dsolve(diff(y(x),x$2)+2*diff(y(x),x)+y(x)=50*sin(3*x),y(x), singsol=all)
```

$$y(x) = (c_1 x + c_2)\,\mathrm{e}^{-x} - 3\cos(3x) - 4\sin(3x)$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 34

```
DSolve[y''[x]+2*y'[x]+y[x]==50*Sin[3*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -3\cos(3x) + e^{-x}\left(-4e^{x}\sin(3x) + c_2 x + c_1\right)$$

8.3 problem Problem 3

Internal problem ID [2765]
Internal file name [`OUTPUT/2257_Sunday_June_05_2022_02_56_39_AM_14082930/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.4, Complex-Valued Trial Solutions. page 529
Problem number: Problem 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - y = 10\cos(x)\,e^{2x}}$$

8.3.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = -1, f(x) = 10\cos(x)\,e^{2x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-1)} \\ &= \pm 1\end{aligned}$$

Hence

$$\lambda_1 = +1$$
$$\lambda_2 = -1$$

Which simplifies to

$$\lambda_1 = 1$$
$$\lambda_2 = -1$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(1)x} + c_2 e^{(-1)x}$$

Or

$$y = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$10 \cos(x) e^{2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x) e^{2x}, \sin(x) e^{2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{x}, e^{-x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x) e^{2x} + A_2 \sin(x) e^{2x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 \cos(x) e^{2x} - 4A_1 \sin(x) e^{2x} + 2A_2 \sin(x) e^{2x} + 4A_2 \cos(x) e^{2x} = 10 \cos(x) e^{2x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1, A_2 = 2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \cos(x) e^{2x} + 2 \sin(x) e^{2x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{x} + c_2 e^{-x}\right) + \left(\cos(x) e^{2x} + 2 \sin(x) e^{2x}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{x} + c_2 e^{-x} + \cos(x) e^{2x} + 2 \sin(x) e^{2x} \tag{1}$$

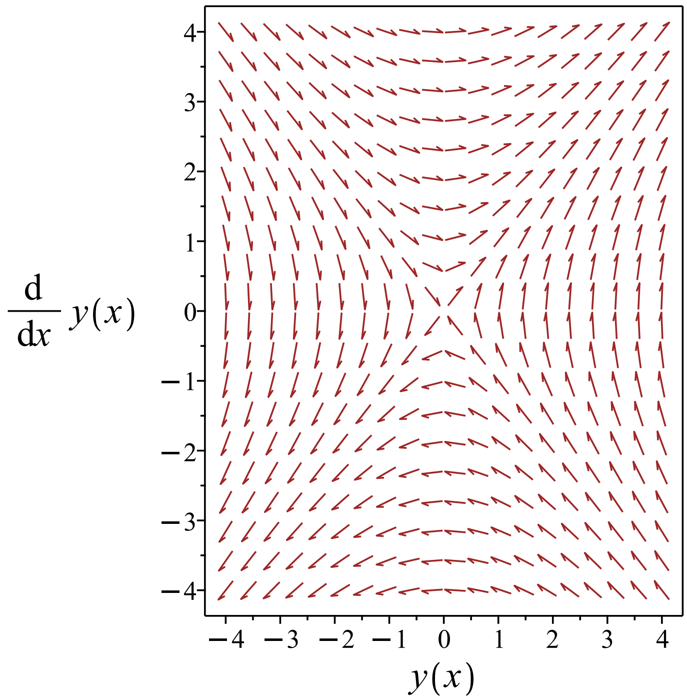


Figure 256: Slope field plot

Verification of solutions

$$y = c_1 e^x + c_2 e^{-x} + \cos(x) e^{2x} + 2\sin(x) e^{2x}$$

Verified OK.

8.3.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 234: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \mathrm{e}^{-x} \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \mathrm{e}^{-x} \int \frac{1}{\mathrm{e}^{-2x}}\,dx \\ &= \mathrm{e}^{-x}\left(\frac{\mathrm{e}^{2x}}{2}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-x}\right) + c_2 \left(\mathrm{e}^{-x} \left(\frac{\mathrm{e}^{2x}}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{x}}{2}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
10 \cos (x) \mathrm{e}^{2x}
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\cos (x) \mathrm{e}^{2x}, \sin (x) \mathrm{e}^{2x}\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{\frac{\mathrm{e}^{x}}{2}, \mathrm{e}^{-x}\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 \cos (x) \mathrm{e}^{2x} + A_2 \sin (x) \mathrm{e}^{2x}
$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 \cos(x) e^{2x} - 4A_1 \sin(x) e^{2x} + 2A_2 \sin(x) e^{2x} + 4A_2 \cos(x) e^{2x} = 10 \cos(x) e^{2x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1, A_2 = 2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \cos(x) e^{2x} + 2 \sin(x) e^{2x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + \frac{c_2 e^{x}}{2}\right) + \left(\cos(x) e^{2x} + 2 \sin(x) e^{2x}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-x} + \frac{c_2 e^{x}}{2} + \cos(x) e^{2x} + 2 \sin(x) e^{2x} \tag{1}$$

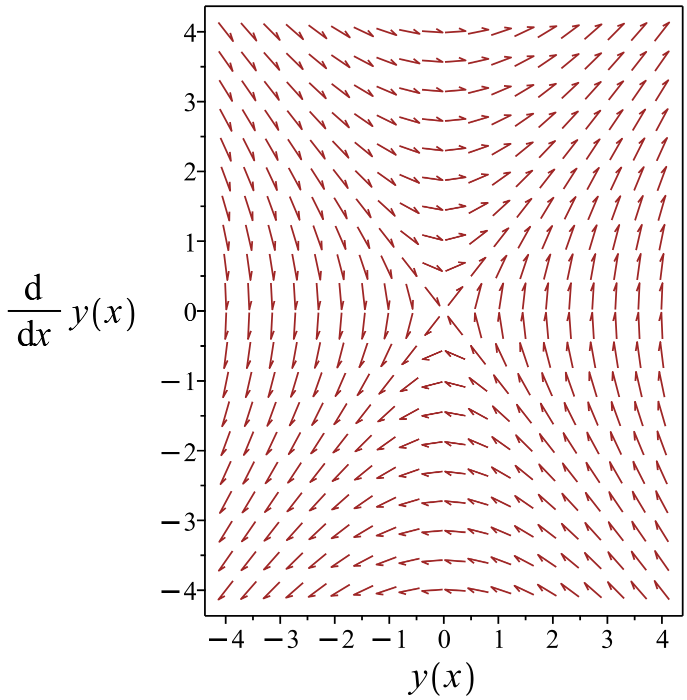


Figure 257: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} + \frac{c_2 e^{x}}{2} + \cos(x) e^{2x} + 2\sin(x) e^{2x}$$

Verified OK.

8.3.3 Maple step by step solution

Let's solve

$y'' - y = 10\cos(x) e^{2x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 1 = 0$

- Factor the characteristic polynomial

  $(r-1)(r+1) = 0$

- Roots of the characteristic polynomial

$r = (-1, 1)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + y_p(x)$

- ☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 10 \cos(x) \mathrm{e}^{2x}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-x} & \mathrm{e}^{x} \\ -\mathrm{e}^{-x} & \mathrm{e}^{x} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 2$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -5\,\mathrm{e}^{-x} \left(\int \cos(x) \mathrm{e}^{3x} dx\right) + 5\,\mathrm{e}^{x} \left(\int \cos(x) \mathrm{e}^{x} dx\right)$

  - Compute integrals

    $y_p(x) = (\cos(x) + 2\sin(x)) \mathrm{e}^{2x}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + (\cos(x) + 2\sin(x)) \mathrm{e}^{2x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(diff(y(x),x$2)-y(x)=10*exp(2*x)*cos(x),y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{-x}c_2 + \mathrm{e}^{x}c_1 + \mathrm{e}^{2x}(2\sin(x) + \cos(x))$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 33

```
DSolve[y''[x]-y[x]==10*Exp[2*x]*Cos[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1e^x + c_2e^{-x} + e^{2x}(2\sin(x) + \cos(x))$$

8.4 problem Problem 4

Internal problem ID [2766]
Internal file name [`OUTPUT/2258_Sunday_June_05_2022_02_56_41_AM_45151139/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.4, Complex-Valued Trial Solutions. page 529
Problem number: Problem 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 4y' + 4y = 169 \sin(3x)}$$

8.4.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 4, C = 4, f(x) = 169 \sin(3x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 4, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 4\lambda e^{\lambda x} + 4 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4\lambda + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 4, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-4}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(4)^2 - (4)(1)(4)} \\ &= -2\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 2$. Therefore the solution is

$$y = c_1 e^{-2x} + c_2 x e^{-2x} \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{-2x} + c_2 x e^{-2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$169 \sin(3x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(3x), \sin(3x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x e^{-2x}, e^{-2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(3x) + A_2 \sin(3x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-5A_1 \cos(3x) - 5A_2 \sin(3x) - 12A_1 \sin(3x) + 12A_2 \cos(3x) = 169 \sin(3x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -12, A_2 = -5]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -12 \cos(3x) - 5 \sin(3x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-2x} + c_2 x \,\mathrm{e}^{-2x}\right) + (-12 \cos(3x) - 5 \sin(3x)) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{-2x}(c_2 x + c_1) - 12 \cos(3x) - 5 \sin(3x)$$

Summary
The solution(s) found are the following

$$y = \mathrm{e}^{-2x}(c_2 x + c_1) - 12 \cos(3x) - 5 \sin(3x) \tag{1}$$

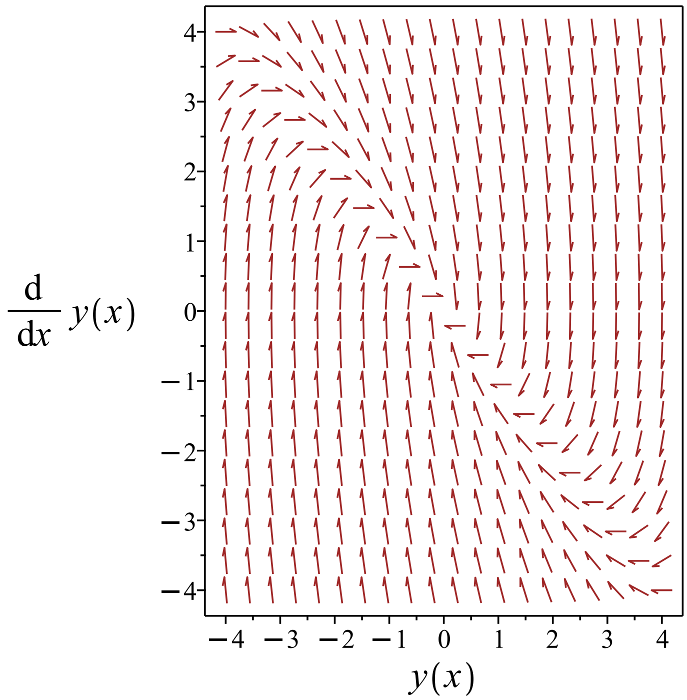


Figure 258: Slope field plot

Verification of solutions

$$y = e^{-2x}(c_2 x + c_1) - 12\cos(3x) - 5\sin(3x)$$

Verified OK.

8.4.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\,y' + \frac{\left(p(x)^2 + p'(x)\right)y}{2} = f(x)$$

Where $p(x) = 4$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int 4\,dx} \\ &= e^{2x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$
\begin{aligned}
(M(x)y)'' &= 169\,\mathrm{e}^{2x}\sin(3x) \\
\left(\mathrm{e}^{2x}y\right)'' &= 169\,\mathrm{e}^{2x}\sin(3x)
\end{aligned}
$$

Integrating once gives

$$
\left(\mathrm{e}^{2x}y\right)' = 13\,\mathrm{e}^{2x}(-3\cos(3x) + 2\sin(3x)) + c_1
$$

Integrating again gives

$$
\left(\mathrm{e}^{2x}y\right) = \mathrm{e}^{2x}(-12\cos(3x) - 5\sin(3x)) + c_1x + c_2
$$

Hence the solution is

$$
y = \frac{\mathrm{e}^{2x}(-12\cos(3x) - 5\sin(3x)) + c_1x + c_2}{\mathrm{e}^{2x}}
$$

Or

$$
y = -48\cos(x)^3 - 20\cos(x)^2\sin(x) + 36\cos(x) + 5\sin(x) + c_1x\,\mathrm{e}^{-2x} + c_2\mathrm{e}^{-2x}
$$

Summary

The solution(s) found are the following

$$
y = -48\cos(x)^3 - 20\cos(x)^2\sin(x) + 36\cos(x) + 5\sin(x) + c_1x\,\mathrm{e}^{-2x} + c_2\mathrm{e}^{-2x} \tag{1}
$$

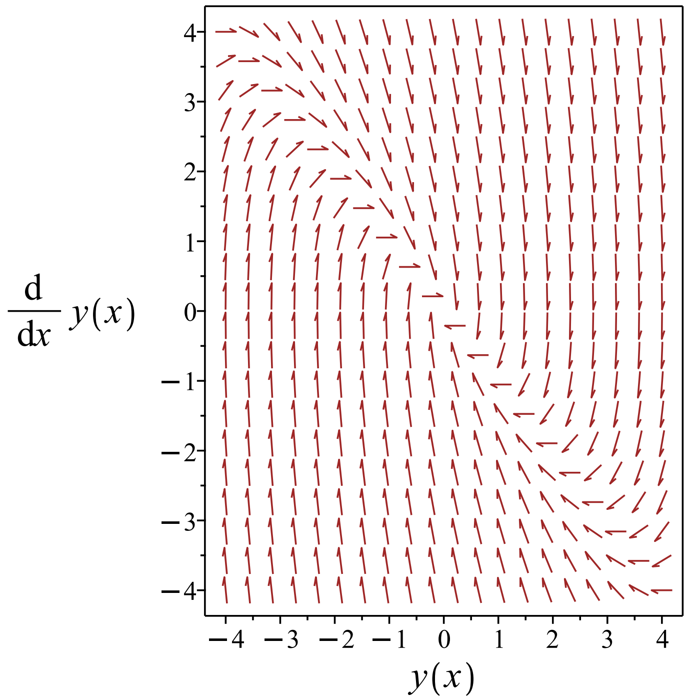


Figure 259: Slope field plot

Verification of solutions

$$y = -48\cos(x)^3 - 20\cos(x)^2\sin(x) + 36\cos(x) + 5\sin(x) + c_1 x\,e^{-2x} + c_2 e^{-2x}$$

Verified OK.

8.4.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 236: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4}{1}\,dx} \\
&= z_1 e^{-2x} \\
&= z_1 \left(\mathrm{e}^{-2x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-2x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-4x}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-2x}\right) + c_2 \left(\mathrm{e}^{-2x}(x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 4y' + 4y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-2x} + c_2 x \,\mathrm{e}^{-2x}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
169 \sin(3x)
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\cos(3x), \sin(3x)\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{x\,\mathrm{e}^{-2x}, \mathrm{e}^{-2x}\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 \cos(3x) + A_2 \sin(3x)
$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
-5A_1 \cos(3x) - 5A_2 \sin(3x) - 12A_1 \sin(3x) + 12A_2 \cos(3x) = 169 \sin(3x)
$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -12, A_2 = -5]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -12\cos(3x) - 5\sin(3x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-2x} + c_2 x\,\mathrm{e}^{-2x}\right) + \left(-12\cos(3x) - 5\sin(3x)\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{-2x}(c_2 x + c_1) - 12\cos(3x) - 5\sin(3x)$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-2x}(c_2 x + c_1) - 12\cos(3x) - 5\sin(3x) \tag{1}$$

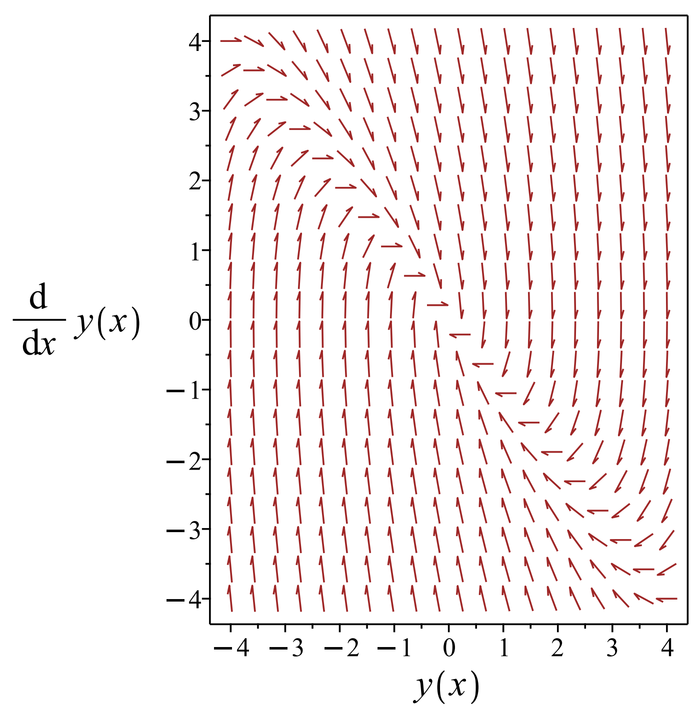


Figure 260: Slope field plot

Verification of solutions

$$y = e^{-2x}(c_2x + c_1) - 12\cos(3x) - 5\sin(3x)$$

Verified OK.

8.4.4 Maple step by step solution

Let's solve

$y'' + 4y' + 4y = 169\sin(3x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4r + 4 = 0$

- Factor the characteristic polynomial

  $(r+2)^2 = 0$

- Root of the characteristic polynomial

  $r = -2$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-2x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,e^{-2x}$

- General solution of the ODE

  $y = c_1y_1(x) + c_2y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1e^{-2x} + c_2x\,e^{-2x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 169\sin(3x)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-2x} & x\,e^{-2x} \\ -2\,e^{-2x} & e^{-2x} - 2x\,e^{-2x} \end{bmatrix}$

- Compute Wronskian

  $W(y_1(x), y_2(x)) = e^{-4x}$

- Substitute functions into equation for $y_p(x)$

  $y_p(x) = 169\,e^{-2x}\left(-\left(\int x\sin(3x)\,e^{2x}dx\right) + x\left(\int e^{2x}\sin(3x)\,dx\right)\right)$

- Compute integrals

  $y_p(x) = -12\cos(3x) - 5\sin(3x)$

- Substitute particular solution into general solution to ODE

  $y = c_2 x\,e^{-2x} + c_1 e^{-2x} - 5\sin(3x) - 12\cos(3x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 27

```
dsolve(diff(y(x),x$2)+4*diff(y(x),x)+4*y(x)=169*sin(3*x),y(x), singsol=all)
```

$$y(x) = (c_1 x + c_2)\,e^{-2x} - 12\cos(3x) - 5\sin(3x)$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 36

```
DSolve[y''[x]+4*y'[x]+4*y[x]==169*Sin[3*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -12\cos(3x) + e^{-2x}\left(-5e^{2x}\sin(3x) + c_2 x + c_1\right)$$

8.5 problem Problem 5

Internal problem ID [2767]
Internal file name [`OUTPUT/2259_Sunday_June_05_2022_02_56_44_AM_92419375/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.4, Complex-Valued Trial Solutions. page 529
Problem number: Problem 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - y' - 2y = 40 \sin(x)^2}$$

8.5.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -1, C = -2, f(x) = 40 \sin(x)^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - y' - 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -1, C = -2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - \lambda e^{\lambda x} - 2 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - \lambda - 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -1, C = -2$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-1^2 - (4)(1)(-2)} \\
&= \frac{1}{2} \pm \frac{3}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= \frac{1}{2} + \frac{3}{2} \\
\lambda_2 &= \frac{1}{2} - \frac{3}{2}
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= 2 \\
\lambda_2 &= -1
\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}
y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\
y &= c_1 e^{(2)x} + c_2 e^{(-1)x}
\end{aligned}$$

Or

$$y = e^{2x} c_1 + c_2 e^{-x}$$

Therefore the homogeneous solution y_h is

$$y_h = e^{2x}c_1 + c_2 e^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$40 \sin(x)^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}, \{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-x}, e^{2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 + A_2 \cos(2x) + A_3 \sin(2x)$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-6A_2 \cos(2x) - 6A_3 \sin(2x) + 2A_2 \sin(2x) - 2A_3 \cos(2x) - 2A_1 = 40 \sin(x)^2$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -10, A_2 = 3, A_3 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -10 + 3\cos(2x) + \sin(2x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{2x}c_1 + c_2 e^{-x}\right) + \left(-10 + 3\cos(2x) + \sin(2x)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^{2x}c_1 + c_2e^{-x} - 10 + 3\cos(2x) + \sin(2x) \tag{1}$$

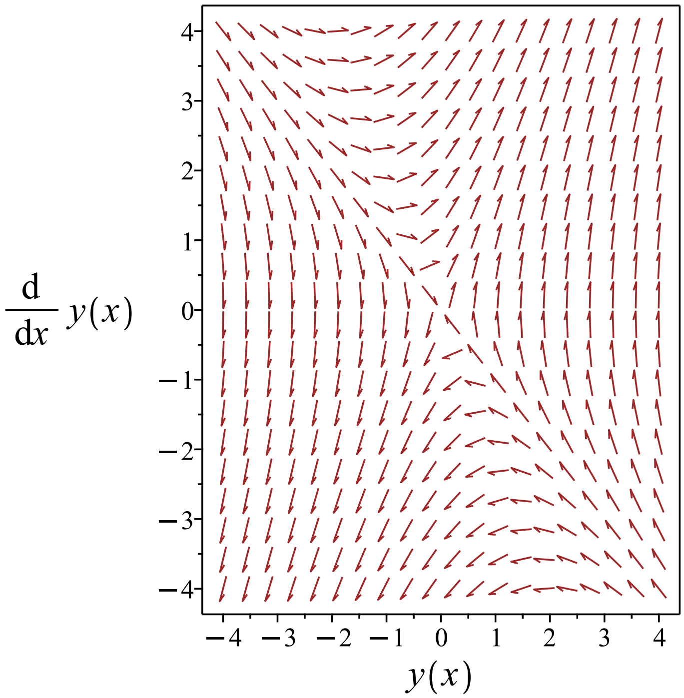


Figure 261: Slope field plot

Verification of solutions

$$y = e^{2x}c_1 + c_2e^{-x} - 10 + 3\cos(2x) + \sin(2x)$$

Verified OK.

8.5.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -1 \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 9 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{9z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 238: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{9}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{3x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-1}{1}\,dx} \\
&= z_1 e^{\frac{x}{2}} \\
&= z_1 \left(\mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = e^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{x}}{(y_1)^2}\, dx \\ &= y_1 \left(\frac{e^{3x}}{3}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(e^{-x}\right) + c_2 \left(e^{-x} \left(\frac{e^{3x}}{3}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - y' - 2y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{-x} + \frac{c_2 e^{2x}}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$40 \sin(x)^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}, \{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{e^{2x}}{3}, e^{-x}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 + A_2 \cos(2x) + A_3 \sin(2x)$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-6A_2 \cos(2x) - 6A_3 \sin(2x) + 2A_2 \sin(2x) - 2A_3 \cos(2x) - 2A_1 = 40 \sin(x)^2$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -10, A_2 = 3, A_3 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -10 + 3\cos(2x) + \sin(2x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + \frac{c_2 e^{2x}}{3}\right) + (-10 + 3\cos(2x) + \sin(2x)) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} + \frac{c_2 e^{2x}}{3} - 10 + 3\cos(2x) + \sin(2x) \tag{1}$$

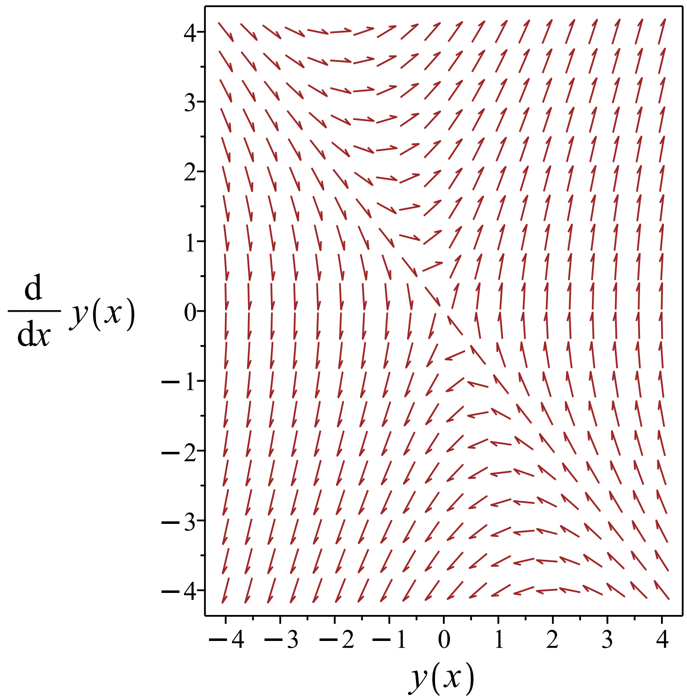


Figure 262: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} + \frac{c_2 e^{2x}}{3} - 10 + 3\cos(2x) + \sin(2x)$$

Verified OK.

8.5.3 Maple step by step solution

Let's solve

$y'' - y' - 2y = 40\sin(x)^2$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - r - 2 = 0$

- Factor the characteristic polynomial

  $(r+1)(r-2) = 0$

- Roots of the characteristic polynomial

$r = (-1, 2)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-x}$
- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{2x}$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{2x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 40 \sin(x)^2\right]$
- Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-x} & \mathrm{e}^{2x} \\ -\mathrm{e}^{-x} & 2\,\mathrm{e}^{2x} \end{bmatrix}$
- Compute Wronskian

  $W(y_1(x), y_2(x)) = 3\,\mathrm{e}^{x}$
- Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\frac{40\,\mathrm{e}^{-x} \left(\int \mathrm{e}^{x} \sin(x)^2 dx\right)}{3} + \frac{40\,\mathrm{e}^{2x} \left(\int \mathrm{e}^{-2x} \sin(x)^2 dx\right)}{3}$
- Compute integrals

  $y_p(x) = -10 + 3\cos(2x) + \sin(2x)$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{2x} - 10 + 3\cos(2x) + \sin(2x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 28

```
dsolve(diff(y(x),x$2)-diff(y(x),x)-2*y(x)=40*sin(x)^2,y(x), singsol=all)
```

$$y(x) = e^{-x}c_2 + c_1e^{2x} - 10 + \sin(2x) + 3\cos(2x)$$

✓ Solution by Mathematica

Time used: 0.091 (sec). Leaf size: 33

```
DSolve[y''[x]-y'[x]-2*y[x]==40*Sin[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sin(2x) + 3\cos(2x) + c_1e^{-x} + c_2e^{2x} - 10$$

8.6 problem Problem 6

Internal problem ID [2768]
Internal file name [`OUTPUT/2260_Sunday_June_05_2022_02_56_46_AM_81534391/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.4, Complex-Valued Trial Solutions. page 529
Problem number: Problem 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y = 3\,\mathrm{e}^{x} \cos(2x)}$$

8.6.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = 3\,\mathrm{e}^{x} \cos(2x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)}\\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i\\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i\\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$3\,\mathrm{e}^x \cos(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^x \cos(2x), \mathrm{e}^x \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \mathrm{e}^x \cos(2x) + A_2 \mathrm{e}^x \sin(2x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1\mathrm{e}^x \cos(2x) - 4A_1\mathrm{e}^x \sin(2x) - 2A_2\mathrm{e}^x \sin(2x) + 4A_2\mathrm{e}^x \cos(2x) = 3\,\mathrm{e}^x \cos(2x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{3}{10}, A_2 = \frac{3}{5}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{3\,\mathrm{e}^x \cos(2x)}{10} + \frac{3\,\mathrm{e}^x \sin(2x)}{5}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) + \left(-\frac{3\,\mathrm{e}^x \cos(2x)}{10} + \frac{3\,\mathrm{e}^x \sin(2x)}{5}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \cos(x) + c_2 \sin(x) - \frac{3\,\mathrm{e}^{x} \cos(2x)}{10} + \frac{3\,\mathrm{e}^{x} \sin(2x)}{5} \tag{1}$$

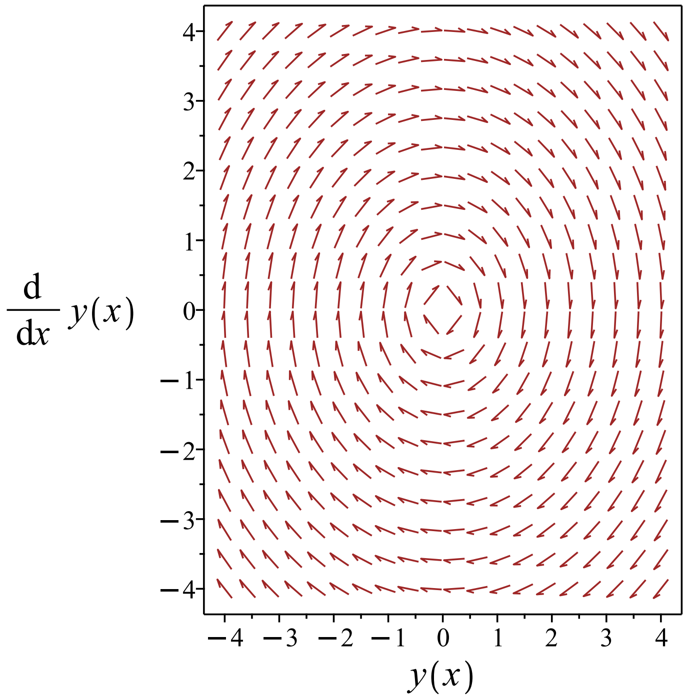


Figure 263: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x) - \frac{3\,\mathrm{e}^{x} \cos(2x)}{10} + \frac{3\,\mathrm{e}^{x} \sin(2x)}{5}$$

Verified OK.

8.6.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= 0 \\
C &= 1
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = ye^{\int \frac{B}{2A}\,dx}
$$

Then (2) becomes

$$
z''(x) = rz(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-1}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -1 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = -z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\,dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 240: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \cos(x)
\end{aligned}
$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \cos(x) \int \frac{1}{\cos(x)^2}\, dx \\ &= \cos(x)\,(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(x)) + c_2(\cos(x)\,(\tan(x))) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$3\,\mathrm{e}^{x} \cos(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^x \cos(2x), e^x \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^x \cos(2x) + A_2 e^x \sin(2x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1 e^x \cos(2x) - 4A_1 e^x \sin(2x) - 2A_2 e^x \sin(2x) + 4A_2 e^x \cos(2x) = 3\,e^x \cos(2x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{3}{10}, A_2 = \frac{3}{5}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{3\,e^x \cos(2x)}{10} + \frac{3\,e^x \sin(2x)}{5}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) + \left(-\frac{3\,e^x \cos(2x)}{10} + \frac{3\,e^x \sin(2x)}{5}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(x) + c_2 \sin(x) - \frac{3\,e^x \cos(2x)}{10} + \frac{3\,e^x \sin(2x)}{5} \tag{1}$$

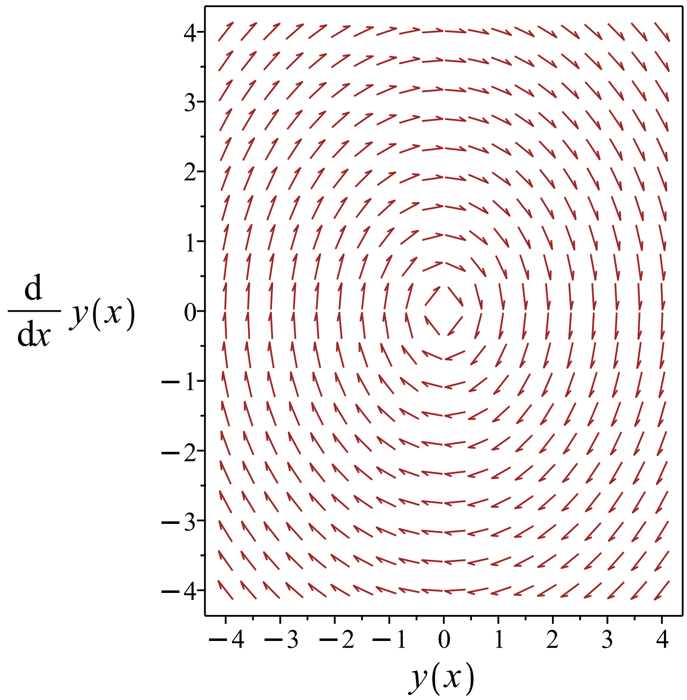


Figure 264: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x) - \frac{3\,e^x \cos(2x)}{10} + \frac{3\,e^x \sin(2x)}{5}$$

Verified OK.

8.6.3 Maple step by step solution

Let's solve

$y'' + y = 3\,e^x \cos(2x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

$r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$
- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 3\,\mathrm{e}^{x}\cos(2x)\right]$
- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$
- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = 1$
- ○ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -3\cos(x)\left(\int \sin(x)\,\mathrm{e}^{x}\cos(2x)\,dx\right) + 3\sin(x)\left(\int \cos(x)\,\mathrm{e}^{x}\cos(2x)\,dx\right)$
- ○ Compute integrals

  $y_p(x) = -\frac{3\,\mathrm{e}^{x}(\cos(2x) - 2\sin(2x))}{10}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(x) + c_2 \sin(x) - \frac{3\,\mathrm{e}^{x}(\cos(2x) - 2\sin(2x))}{10}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 29

```
dsolve(diff(y(x),x$2)+y(x)=3*exp(x)*cos(2*x),y(x), singsol=all)
```

$$y(x) = \cos(x) c_1 + \frac{3 e^x \sin(2x)}{5} - \frac{3 e^x \cos(2x)}{10} + \sin(x) c_2$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 34

```
DSolve[y''[x]+y[x]==3*Exp[x]*Cos[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{3}{10} e^x (\cos(2x) - 2\sin(2x)) + c_1 \cos(x) + c_2 \sin(x)$$

8.7 problem Problem 7

Internal problem ID [2769]
Internal file name [`OUTPUT/2261_Sunday_June_05_2022_02_56_48_AM_82603819/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.4, Complex-Valued Trial Solutions. page 529
Problem number: Problem 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 2y' + 2y = 2\,\mathrm{e}^{-x} \sin(x)}$$

8.7.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 2, C = 2, f(x) = 2\,\mathrm{e}^{-x} \sin(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2y' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 2, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 2\lambda\, e^{\lambda x} + 2\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{2^2 - (4)\,(1)\,(2)}\\ &= -1 \pm i\end{aligned}$$

Hence

$$\lambda_1 = -1 + i$$
$$\lambda_2 = -1 - i$$

Which simplifies to

$$\lambda_1 = -1 + i$$
$$\lambda_2 = -1 - i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -1$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-x}(c_1 \cos(x) + c_2 \sin(x))$$

Therefore the homogeneous solution y_h is

$$y_h = \mathrm{e}^{-x}(c_1 \cos(x) + c_2 \sin(x))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2\,\mathrm{e}^{-x}\sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{-x}\cos(x), \mathrm{e}^{-x}\sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\mathrm{e}^{-x}\cos(x), \mathrm{e}^{-x}\sin(x)\}$$

Since $\mathrm{e}^{-x}\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,\mathrm{e}^{-x}\cos(x), x\,\mathrm{e}^{-x}\sin(x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\,\mathrm{e}^{-x}\cos(x) + A_2 x\,\mathrm{e}^{-x}\sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1\mathrm{e}^{-x}\sin(x) + 2A_2\mathrm{e}^{-x}\cos(x) = 2\,\mathrm{e}^{-x}\sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -1, A_2 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -x\,\mathrm{e}^{-x}\cos(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\mathrm{e}^{-x}(c_1\cos(x) + c_2\sin(x))\right) + \left(-x\,\mathrm{e}^{-x}\cos(x)\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = e^{-x}(c_1 \cos(x) + c_2 \sin(x)) - x e^{-x} \cos(x) \tag{1}$$

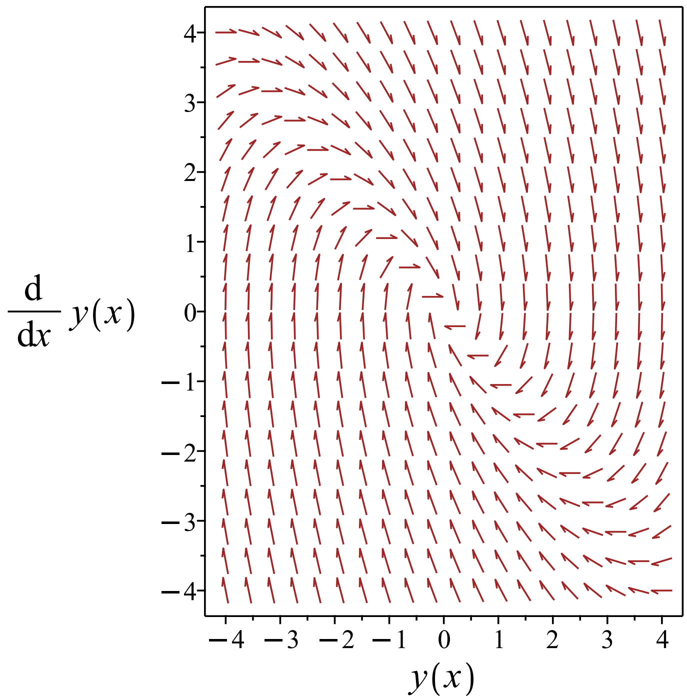


Figure 265: Slope field plot

Verification of solutions

$$y = e^{-x}(c_1 \cos(x) + c_2 \sin(x)) - x e^{-x} \cos(x)$$

Verified OK.

8.7.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 242: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 0\\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dx}\\ &= z_1 e^{-x}\\ &= z_1\left(\mathrm{e}^{-x}\right)\end{aligned}$$

Which simplifies to

$$y_1 = e^{-x} \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2x}}{(y_1)^2}\,dx \\
&= y_1(\tan(x))
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(e^{-x} \cos(x)\right) + c_2\left(e^{-x} \cos(x) (\tan(x))\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2y' + 2y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = e^{-x} \cos(x)\, c_1 + e^{-x} \sin(x)\, c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2\, e^{-x} \sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-x}\cos(x), e^{-x}\sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-x}\cos(x), e^{-x}\sin(x)\}$$

Since $e^{-x}\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,e^{-x}\cos(x), x\,e^{-x}\sin(x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\,e^{-x}\cos(x) + A_2 x\,e^{-x}\sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1 e^{-x}\sin(x) + 2A_2 e^{-x}\cos(x) = 2\,e^{-x}\sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -1, A_2 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -x\,e^{-x}\cos(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{-x}\cos(x)\,c_1 + e^{-x}\sin(x)\,c_2\right) + \left(-x\,e^{-x}\cos(x)\right) \end{aligned}$$

Which simplifies to

$$y = e^{-x}(c_1\cos(x) + c_2\sin(x)) - x\,e^{-x}\cos(x)$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-x}(c_1 \cos(x) + c_2 \sin(x)) - x\,\mathrm{e}^{-x} \cos(x) \tag{1}$$

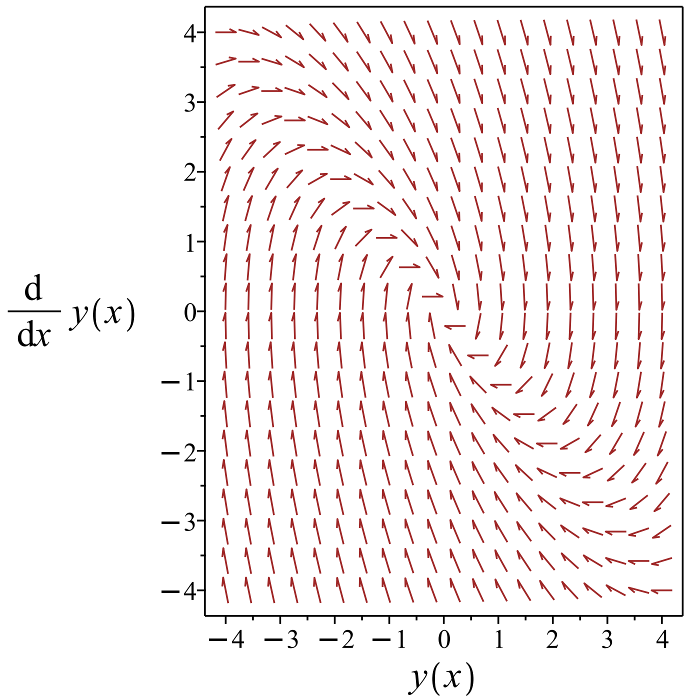


Figure 266: Slope field plot

Verification of solutions

$$y = \mathrm{e}^{-x}(c_1 \cos(x) + c_2 \sin(x)) - x\,\mathrm{e}^{-x} \cos(x)$$

Verified OK.

8.7.3 Maple step by step solution

Let's solve

$y'' + 2y' + 2y = 2\,\mathrm{e}^{-x} \sin(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 2r + 2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-2)\pm(\sqrt{-4})}{2}$
- Roots of the characteristic polynomial

  $r = (-1 - \mathrm{I}, -1 + \mathrm{I})$
- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-x} \cos(x)$
- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{-x} \sin(x)$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE

  $y = \mathrm{e}^{-x} \cos(x) c_1 + \mathrm{e}^{-x} \sin(x) c_2 + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 2\,\mathrm{e}^{-x} \sin(x)\right]$
- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-x} \cos(x) & \mathrm{e}^{-x} \sin(x) \\ -\mathrm{e}^{-x} \cos(x) - \mathrm{e}^{-x} \sin(x) & -\mathrm{e}^{-x} \sin(x) + \mathrm{e}^{-x} \cos(x) \end{bmatrix}$
- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = \mathrm{e}^{-2x}$
- ○ Substitute functions into equation for $y_p(x)$

  $y_p(x) = \mathrm{e}^{-x}\left(-2\cos(x)\left(\int \sin(x)^2 dx\right) + \sin(x)\left(\int \sin(2x) dx\right)\right)$
- ○ Compute integrals

  $y_p(x) = \frac{(\sin(x) - 2\cos(x) x)\mathrm{e}^{-x}}{2}$

- Substitute particular solution into general solution to ODE

  $y = \mathrm{e}^{-x} \cos(x) c_1 + \mathrm{e}^{-x} \sin(x) c_2 + \frac{(\sin(x) - 2\cos(x) x)\mathrm{e}^{-x}}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 25

```
dsolve(diff(y(x),x$2)+2*diff(y(x),x)+2*y(x)=2*exp(-x)*sin(x),y(x), singsol=all)
```

$$y(x) = e^{-x}(\sin(x) c_2 + \cos(x) c_1 - x\cos(x) + \sin(x))$$

✓ Solution by Mathematica

Time used: 0.049 (sec). Leaf size: 34

```
DSolve[y''[x]+2*y'[x]+2*y[x]==2*Exp[-x]*Sin[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}e^{-x}(\sin(x) - 2x\cos(x) + 2c_2\cos(x) + 2c_1\sin(x))$$

8.8 problem Problem 8

Internal problem ID [2770]
Internal file name [`OUTPUT/2262_Sunday_June_05_2022_02_56_50_AM_15779565/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.4, Complex-Valued Trial Solutions. page 529
Problem number: Problem 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 4y = 100\,e^{x} x \sin(x)}$$

8.8.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = -4, f(x) = 100\,e^{x} x \sin(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 4\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(-4)}\\ &= \pm 2\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +2\\ \lambda_2 &= -2\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2\\ \lambda_2 &= -2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(2)x} + c_2 e^{(-2)x}\end{aligned}$$

Or

$$y = \mathrm{e}^{2x} c_1 + c_2 \mathrm{e}^{-2x}$$

Therefore the homogeneous solution y_h is

$$y_h = \mathrm{e}^{2x} c_1 + c_2 \mathrm{e}^{-2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$100\, e^{x} x \sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x)\, e^{x}, e^{x} \sin(x), e^{x} x \cos(x), e^{x} x \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-2x}, e^{2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x)\, e^{x} + A_2 e^{x} \sin(x) + A_3 e^{x} x \cos(x) + A_4 e^{x} x \sin(x)$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&-2A_1 \sin(x)\, e^{x} + 2A_2 e^{x} \cos(x) + 2A_3 e^{x} \cos(x) - 2A_3 e^{x} x \sin(x) - 2A_3 e^{x} \sin(x) \\ &+ 2A_4 e^{x} \sin(x) + 2A_4 e^{x} x \cos(x) + 2A_4 e^{x} \cos(x) - 4A_1 \cos(x)\, e^{x} \\ &- 4A_2 e^{x} \sin(x) - 4A_3 e^{x} x \cos(x) - 4A_4 e^{x} x \sin(x) = 100\, e^{x} x \sin(x)\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -14, A_2 = 2, A_3 = -10, A_4 = -20]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -14 \cos(x)\, e^{x} + 2\, e^{x} \sin(x) - 10\, e^{x} x \cos(x) - 20\, e^{x} x \sin(x)$$

Therefore the general solution is

$$\begin{aligned}y &= y_h + y_p \\ &= \left(e^{2x} c_1 + c_2 e^{-2x}\right) + \left(-14 \cos(x)\, e^{x} + 2\, e^{x} \sin(x) - 10\, e^{x} x \cos(x) - 20\, e^{x} x \sin(x)\right)\end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^{2x} c_1 + c_2 e^{-2x} - 14\cos(x) e^x + 2 e^x \sin(x) - 10 e^x x\cos(x) - 20 e^x x\sin(x) \tag{1}$$

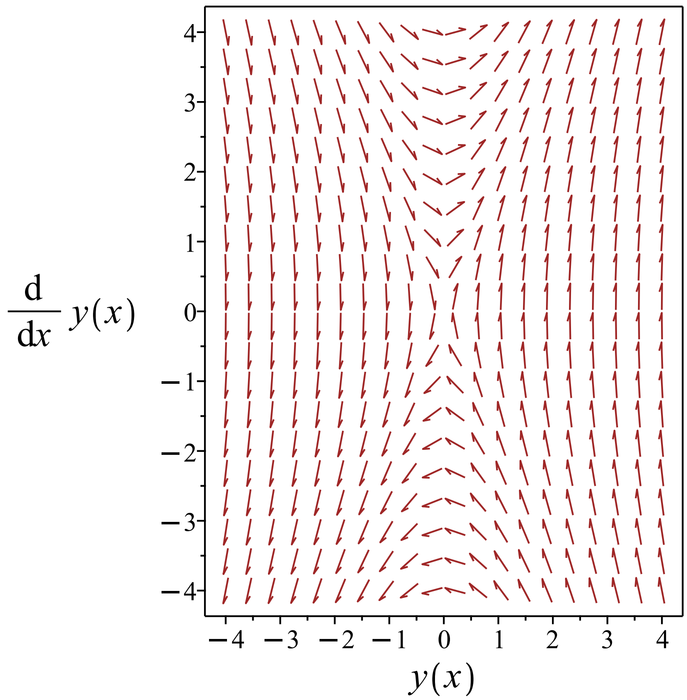


Figure 267: Slope field plot

Verification of solutions

$$y = e^{2x} c_1 + c_2 e^{-2x} - 14\cos(x) e^x + 2 e^x \sin(x) - 10 e^x x\cos(x) - 20 e^x x\sin(x)$$

Verified OK.

8.8.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 4y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 244: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-2x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \mathrm{e}^{-2x} \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \mathrm{e}^{-2x} \int \frac{1}{\mathrm{e}^{-4x}}\, dx \\ &= \mathrm{e}^{-2x} \left(\frac{\mathrm{e}^{4x}}{4}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-2x}\right) + c_2 \left(\mathrm{e}^{-2x} \left(\frac{\mathrm{e}^{4x}}{4}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 4y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-2x} + \frac{c_2 \mathrm{e}^{2x}}{4}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$100\,\mathrm{e}^{x} x \sin\left(x\right)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos\left(x\right)\mathrm{e}^{x}, \mathrm{e}^{x}\sin\left(x\right), \mathrm{e}^{x} x \cos\left(x\right), \mathrm{e}^{x} x \sin\left(x\right)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\mathrm{e}^{2x}}{4}, \mathrm{e}^{-2x}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos\left(x\right)\mathrm{e}^{x} + A_2 \mathrm{e}^{x}\sin\left(x\right) + A_3 \mathrm{e}^{x} x \cos\left(x\right) + A_4 \mathrm{e}^{x} x \sin\left(x\right)$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&-2A_1 \sin\left(x\right)\mathrm{e}^{x} + 2A_2 \mathrm{e}^{x}\cos\left(x\right) + 2A_3 \mathrm{e}^{x}\cos\left(x\right) - 2A_3 \mathrm{e}^{x} x \sin\left(x\right) - 2A_3 \mathrm{e}^{x}\sin\left(x\right)\\ &+ 2A_4 \mathrm{e}^{x}\sin\left(x\right) + 2A_4 \mathrm{e}^{x} x \cos\left(x\right) + 2A_4 \mathrm{e}^{x}\cos\left(x\right) - 4A_1 \cos\left(x\right)\mathrm{e}^{x}\\ &- 4A_2 \mathrm{e}^{x}\sin\left(x\right) - 4A_3 \mathrm{e}^{x} x \cos\left(x\right) - 4A_4 \mathrm{e}^{x} x \sin\left(x\right) = 100\,\mathrm{e}^{x} x \sin\left(x\right)\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -14, A_2 = 2, A_3 = -10, A_4 = -20]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -14\cos\left(x\right)\mathrm{e}^{x} + 2\,\mathrm{e}^{x}\sin\left(x\right) - 10\,\mathrm{e}^{x} x \cos\left(x\right) - 20\,\mathrm{e}^{x} x \sin\left(x\right)$$

Therefore the general solution is

$$\begin{aligned}y &= y_h + y_p\\ &= \left(c_1 \mathrm{e}^{-2x} + \frac{c_2 \mathrm{e}^{2x}}{4}\right) + \left(-14\cos\left(x\right)\mathrm{e}^{x} + 2\,\mathrm{e}^{x}\sin\left(x\right) - 10\,\mathrm{e}^{x} x \cos\left(x\right) - 20\,\mathrm{e}^{x} x \sin\left(x\right)\right)\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-2x} + \frac{c_2 e^{2x}}{4} - 14\cos(x)\,e^{x} + 2\,e^{x}\sin(x) - 10\,e^{x} x\cos(x) - 20\,e^{x} x\sin(x) \quad (1)$$

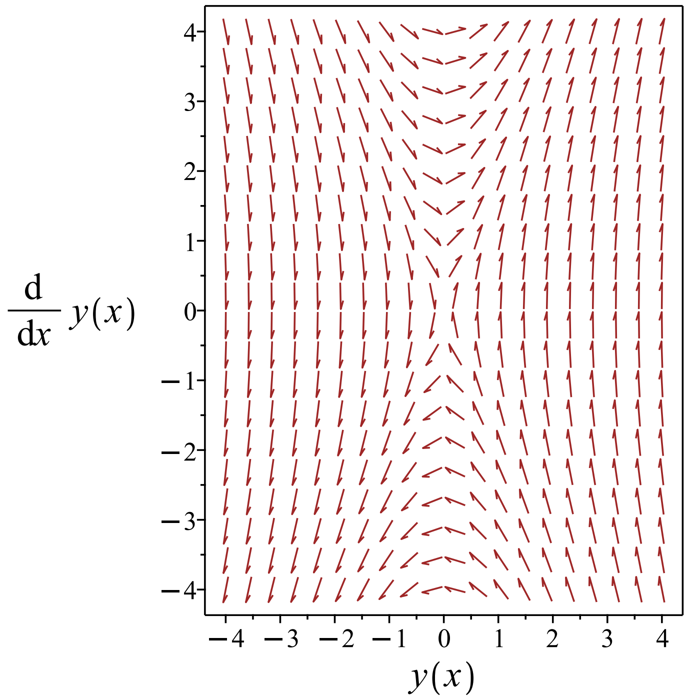


Figure 268: Slope field plot

Verification of solutions

$$y = c_1 e^{-2x} + \frac{c_2 e^{2x}}{4} - 14\cos(x)\,e^{x} + 2\,e^{x}\sin(x) - 10\,e^{x} x\cos(x) - 20\,e^{x} x\sin(x)$$

Verified OK.

8.8.3 Maple step by step solution

Let's solve

$y'' - 4y = 100\,e^{x} x\sin(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2 - 4 = 0$

- Factor the characteristic polynomial

  $(r-2)(r+2) = 0$

- Roots of the characteristic polynomial

  $r = (-2, 2)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-2x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{2x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 100\,\mathrm{e}^x x \sin(x)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-2x} & \mathrm{e}^{2x} \\ -2\,\mathrm{e}^{-2x} & 2\,\mathrm{e}^{2x} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 4$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -25\,\mathrm{e}^{-2x}\left(\int \mathrm{e}^{3x} \sin(x)\, x dx\right) + 25\,\mathrm{e}^{2x}\left(\int x\,\mathrm{e}^{-x} \sin(x)\, dx\right)$

  - Compute integrals

    $y_p(x) = -2\,\mathrm{e}^x (10x \sin(x) + 5\cos(x)\, x - \sin(x) + 7\cos(x))$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{2x} - 2\,\mathrm{e}^x (10x \sin(x) + 5\cos(x)\, x - \sin(x) + 7\cos(x))$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 51

```
dsolve(diff(y(x),x$2)-4*y(x)=100*x*exp(x)*sin(x),y(x), singsol=all)
```

$$y(x) = \left(-10x\cos\left(x\right)\mathrm{e}^{3x} + \mathrm{e}^{4x}c_1 - 20x\sin\left(x\right)\mathrm{e}^{3x} - 14\cos\left(x\right)\mathrm{e}^{3x} + 2\,\mathrm{e}^{3x}\sin\left(x\right) + c_2\right)\mathrm{e}^{-2x}$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 44

```
DSolve[y''[x]-4*y[x]==100*x*Exp[x]*Sin[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1e^{2x} + c_2e^{-2x} - 2e^x((10x-1)\sin(x) + (5x+7)\cos(x))$$

8.9 problem Problem 9

Internal problem ID [2771]
Internal file name [`OUTPUT/2263_Sunday_June_05_2022_02_56_53_AM_69436000/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.4, Complex-Valued Trial Solutions. page 529
Problem number: Problem 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 2y' + 5y = 4\,\mathrm{e}^{-x} \cos(2x)}$$

8.9.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 2, C = 5, f(x) = 4\,\mathrm{e}^{-x} \cos(2x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2y' + 5y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 2, C = 5$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 2\lambda \, e^{\lambda x} + 5 \, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2\lambda + 5 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = 5$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{2^2 - (4)\,(1)\,(5)} \\ &= -1 \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -1 + 2i \\ \lambda_2 &= -1 - 2i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -1 + 2i \\ \lambda_2 &= -1 - 2i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -1$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-x}(c_1 \cos(2x) + c_2 \sin(2x))$$

Therefore the homogeneous solution y_h is

$$y_h = \mathrm{e}^{-x}(c_1 \cos(2x) + c_2 \sin(2x))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$4\,e^{-x}\cos(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-x}\cos(2x)\,, e^{-x}\sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-x}\cos(2x)\,, e^{-x}\sin(2x)\}$$

Since $e^{-x}\cos(2x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,e^{-x}\cos(2x)\,, x\,e^{-x}\sin(2x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\,e^{-x}\cos(2x) + A_2 x\,e^{-x}\sin(2x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-4A_1 e^{-x}\sin(2x) + 4A_2 e^{-x}\cos(2x) = 4\,e^{-x}\cos(2x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x\,e^{-x}\sin(2x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{-x}(c_1\cos(2x) + c_2\sin(2x))\right) + \left(x\,e^{-x}\sin(2x)\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = e^{-x}(c_1 \cos(2x) + c_2 \sin(2x)) + x e^{-x} \sin(2x) \tag{1}$$

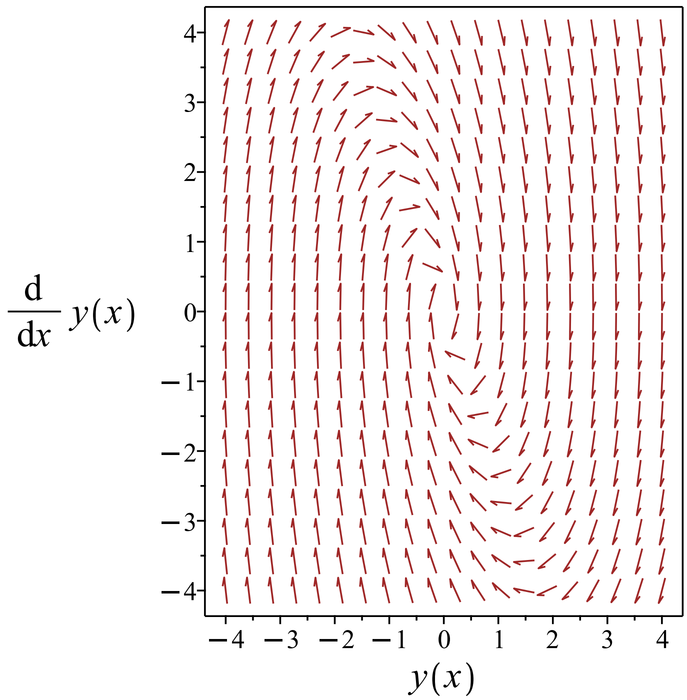


Figure 269: Slope field plot

Verification of solutions

$$y = e^{-x}(c_1 \cos(2x) + c_2 \sin(2x)) + x e^{-x} \sin(2x)$$

Verified OK.

8.9.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y' + 5y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= 5 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 246: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(2x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dx} \\
&= z_1 e^{-x} \\
&= z_1 \left(\mathrm{e}^{-x}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x} \cos(2x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-2x}}{(y_1)^2}\, dx \\ &= y_1 \left(\frac{\tan(2x)}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-x} \cos(2x)\right) + c_2 \left(\mathrm{e}^{-x} \cos(2x) \left(\frac{\tan(2x)}{2}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2y' + 5y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \cos(2x)\, \mathrm{e}^{-x} c_1 + \frac{\sin(2x)\, \mathrm{e}^{-x} c_2}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$4\, \mathrm{e}^{-x} \cos(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-x} \cos (2x), e^{-x} \sin (2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^{-x} \cos (2x), \frac{e^{-x} \sin (2x)}{2}\right\}$$

Since $e^{-x} \cos (2x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,e^{-x} \cos (2x), x\,e^{-x} \sin (2x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\,e^{-x} \cos (2x) + A_2 x\,e^{-x} \sin (2x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-4A_1 e^{-x} \sin (2x) + 4A_2 e^{-x} \cos (2x) = 4\,e^{-x} \cos (2x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x\,e^{-x} \sin (2x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\cos (2x)\,e^{-x} c_1 + \frac{\sin (2x)\,e^{-x} c_2}{2}\right) + \left(x\,e^{-x} \sin (2x)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \cos (2x)\,e^{-x} c_1 + \frac{\sin (2x)\,e^{-x} c_2}{2} + x\,e^{-x} \sin (2x) \tag{1}$$

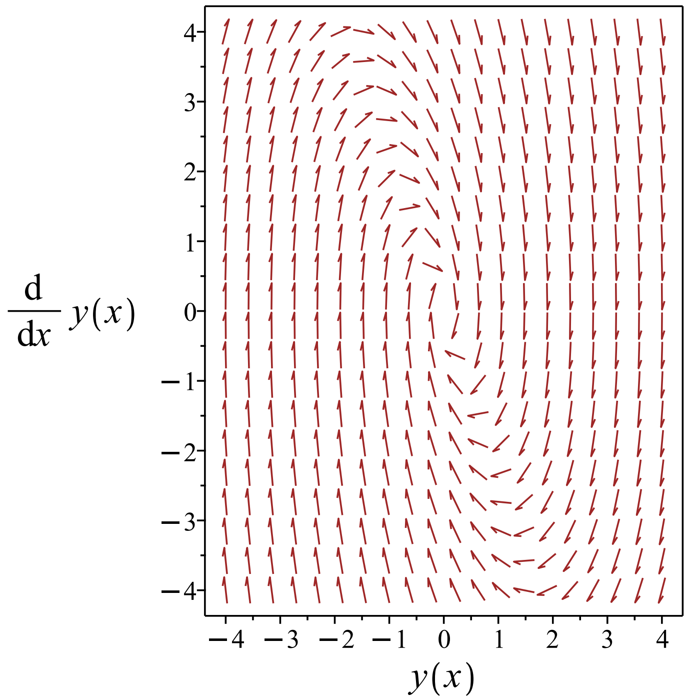


Figure 270: Slope field plot

Verification of solutions

$$y = \cos(2x)\,e^{-x}c_1 + \frac{\sin(2x)\,e^{-x}c_2}{2} + x\,e^{-x}\sin(2x)$$

Verified OK.

8.9.3 Maple step by step solution

Let's solve

$y'' + 2y' + 5y = 4\,e^{-x}\cos(2x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 2r + 5 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-2)\pm(\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

$r = (-1 - 2\mathrm{I}, -1 + 2\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-x} \cos(2x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{-x} \sin(2x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = \cos(2x)\, \mathrm{e}^{-x} c_1 + \sin(2x)\, \mathrm{e}^{-x} c_2 + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 4\,\mathrm{e}^{-x} \cos(2x)\right]$

- ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-x} \cos(2x) & \mathrm{e}^{-x} \sin(2x) \\ -\mathrm{e}^{-x} \cos(2x) - 2\,\mathrm{e}^{-x} \sin(2x) & -\mathrm{e}^{-x} \sin(2x) + 2\,\mathrm{e}^{-x} \cos(2x) \end{bmatrix}$

- ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = 2\,\mathrm{e}^{-2x}$

- ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\mathrm{e}^{-x}\left(\cos(2x)\left(\int \sin(4x)\, dx\right) - 2\sin(2x)\left(\int \cos(2x)^2\, dx\right)\right)$

- ◦ Compute integrals

  $y_p(x) = \frac{(4x\sin(2x) + \cos(2x))\mathrm{e}^{-x}}{4}$

- Substitute particular solution into general solution to ODE

  $y = \cos(2x)\, \mathrm{e}^{-x} c_1 + \sin(2x)\, \mathrm{e}^{-x} c_2 + \frac{(4x\sin(2x) + \cos(2x))\mathrm{e}^{-x}}{4}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve(diff(y(x),x$2)+2*diff(y(x),x)+5*y(x)=4*exp(-x)*cos(2*x),y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{-x}\left(\cos(2x)\left(\frac{1}{2}+c_1\right)+\sin(2x)\,(c_2+x)\right)$$

✓ Solution by Mathematica

Time used: 0.059 (sec). Leaf size: 36

```
DSolve[y''[x]+2*y'[x]+5*y[x]==4*Exp[-x]*Cos[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4}e^{-x}((1+4c_2)\cos(2x)+4(x+c_1)\sin(2x))$$

8.10 problem Problem 10

Internal problem ID [2772]
Internal file name [`OUTPUT/2264_Sunday_June_05_2022_02_56_56_AM_72898369/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.4, Complex-Valued Trial Solutions. page 529
Problem number: Problem 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 2y' + 10y = 24\,\mathrm{e}^{x} \cos(3x)}$$

8.10.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -2, C = 10, f(x) = 24\,\mathrm{e}^{x} \cos(3x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 2y' + 10y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = 10$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 2\lambda \,\mathrm{e}^{\lambda x} + 10\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda + 10 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = 10$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-2^2 - (4)\,(1)\,(10)}\\ &= 1 \pm 3i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= 1 + 3i\\ \lambda_2 &= 1 - 3i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1 + 3i\\ \lambda_2 &= 1 - 3i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 1$ and $\beta = 3$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{x}(c_1 \cos(3x) + c_2 \sin(3x))$$

Therefore the homogeneous solution y_h is

$$y_h = \mathrm{e}^{x}(c_1 \cos(3x) + c_2 \sin(3x))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$24\,e^{x}\cos(3x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{x}\cos(3x), e^{x}\sin(3x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{x}\cos(3x), e^{x}\sin(3x)\}$$

Since $e^{x}\cos(3x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,e^{x}\cos(3x), x\sin(3x)\,e^{x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\,e^{x}\cos(3x) + A_2 x\sin(3x)\,e^{x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-6A_1 e^{x}\sin(3x) + 6A_2\cos(3x)\,e^{x} = 24\,e^{x}\cos(3x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 4]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 4x\sin(3x)\,e^{x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (e^{x}(c_1\cos(3x) + c_2\sin(3x))) + (4x\sin(3x)\,e^{x}) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^x(c_1 \cos(3x) + c_2 \sin(3x)) + 4x \sin(3x) e^x \tag{1}$$

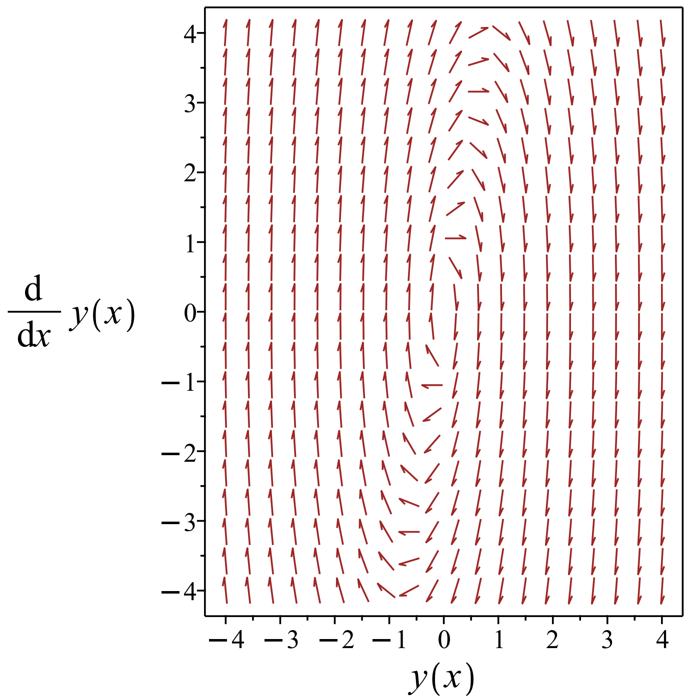


Figure 271: Slope field plot

Verification of solutions

$$y = e^x(c_1 \cos(3x) + c_2 \sin(3x)) + 4x \sin(3x) e^x$$

Verified OK.

8.10.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' + 10y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \\ C &= 10 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-9}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -9 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -9z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 248: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -9$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(3x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2}{1}\,dx} \\
&= z_1 e^{x} \\
&= z_1\left(\mathrm{e}^{x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = e^x \cos(3x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2x}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{\tan(3x)}{3}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 (e^x \cos(3x)) + c_2 \left(e^x \cos(3x) \left(\frac{\tan(3x)}{3}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 2y' + 10y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = e^x \cos(3x)\, c_1 + \frac{e^x \sin(3x)\, c_2}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$24\, e^x \cos(3x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^x \cos(3x), e^x \sin(3x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^x \cos(3x), \frac{e^x \sin(3x)}{3}\right\}$$

Since $e^x \cos(3x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,e^x \cos(3x), x \sin(3x)\,e^x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\,e^x \cos(3x) + A_2 x \sin(3x)\,e^x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-6A_1 e^x \sin(3x) + 6A_2 \cos(3x)\,e^x = 24\,e^x \cos(3x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 4]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 4x \sin(3x)\,e^x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^x \cos(3x)\,c_1 + \frac{e^x \sin(3x)\,c_2}{3}\right) + (4x \sin(3x)\,e^x) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^x \cos(3x)\,c_1 + \frac{e^x \sin(3x)\,c_2}{3} + 4x \sin(3x)\,e^x \tag{1}$$

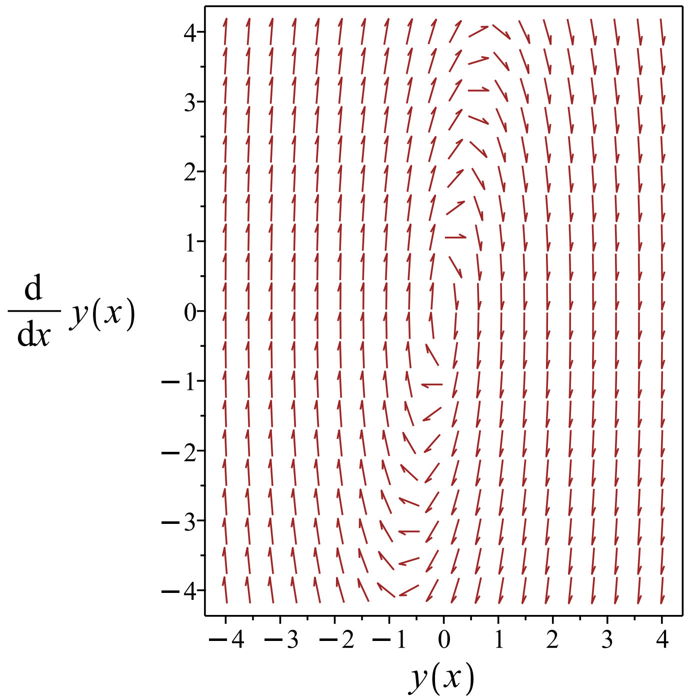


Figure 272: Slope field plot

Verification of solutions

$$y = e^{x} \cos(3x) c_1 + \frac{e^{x} \sin(3x) c_2}{3} + 4x \sin(3x) e^{x}$$

Verified OK.

8.10.3 Maple step by step solution

Let's solve

$y'' - 2y' + 10y = 24 e^{x} \cos(3x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 2r + 10 = 0$

- Use quadratic formula to solve for r

  $r = \frac{2 \pm (\sqrt{-36})}{2}$

- Roots of the characteristic polynomial

$r = (1 - 3\,\mathrm{I}, 1 + 3\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^x \cos(3x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^x \sin(3x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = \mathrm{e}^x \cos(3x)\, c_1 + \mathrm{e}^x \sin(3x)\, c_2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 24\,\mathrm{e}^x \cos(3x)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^x \cos(3x) & \mathrm{e}^x \sin(3x) \\ \mathrm{e}^x \cos(3x) - 3\,\mathrm{e}^x \sin(3x) & \mathrm{e}^x \sin(3x) + 3\,\mathrm{e}^x \cos(3x) \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 3\,\mathrm{e}^{2x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -4\,\mathrm{e}^x \left(\cos(3x)\left(\int \sin(6x)\, dx\right) - 2\sin(3x)\left(\int \cos(3x)^2\, dx\right)\right)$

  - Compute integrals

    $y_p(x) = \frac{2(6x\sin(3x) + \cos(3x))\mathrm{e}^x}{3}$

- Substitute particular solution into general solution to ODE

  $y = \mathrm{e}^x \cos(3x)\, c_1 + \mathrm{e}^x \sin(3x)\, c_2 + \frac{2(6x\sin(3x) + \cos(3x))\mathrm{e}^x}{3}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 31

```
dsolve(diff(y(x),x$2)-2*diff(y(x),x)+10*y(x)=24*exp(x)*cos(3*x),y(x), singsol=all)
```

$$y(x) = \frac{e^{x}(3c_1 + 4)\cos(3x)}{3} + 4\,e^{x}\sin(3x)\left(x + \frac{c_2}{4}\right)$$

✓ Solution by Mathematica

Time used: 0.055 (sec). Leaf size: 36

```
DSolve[y''[x]-2*y'[x]+10*y[x]==24*Exp[x]*Cos[3*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{3}e^{x}((2 + 3c_2)\cos(3x) + 3(4x + c_1)\sin(3x))$$

8.11 problem Problem 11

Internal problem ID [2773]
Internal file name [`OUTPUT/2265_Sunday_June_05_2022_02_56_59_AM_98263515/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.4, Complex-Valued Trial Solutions. page 529
Problem number: Problem 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 16y = 34\,e^{x} + 16\cos(4x) - 8\sin(4x)}$$

8.11.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 16, f(x) = 34\,e^{x} + 16\cos(4x) - 8\sin(4x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 16y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 16$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 16\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 16 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 16$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(16)}\\ &= \pm 4i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +4i\\ \lambda_2 &= -4i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 4i\\ \lambda_2 &= -4i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 4$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(4x) + c_2 \sin(4x))$$

Or

$$y = c_1 \cos(4x) + c_2 \sin(4x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(4x) + c_2 \sin(4x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$34\,\mathrm{e}^{x} + 16\cos(4x) - 8\sin(4x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{x}\}, \{\cos(4x), \sin(4x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(4x), \sin(4x)\}$$

Since $\cos(4x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{\mathrm{e}^{x}\}, \{x\cos(4x), x\sin(4x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1\mathrm{e}^{x} + A_2 x\cos(4x) + A_3 x\sin(4x)$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$17A_1\mathrm{e}^{x} - 8A_2\sin(4x) + 8A_3\cos(4x) = 34\,\mathrm{e}^{x} + 16\cos(4x) - 8\sin(4x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 2, A_2 = 1, A_3 = 2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2\,\mathrm{e}^{x} + x\cos(4x) + 2x\sin(4x)$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 \cos(4x) + c_2 \sin(4x)) + (2\,\mathrm{e}^{x} + x \cos(4x) + 2x \sin(4x))
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \cos(4x) + c_2 \sin(4x) + 2\,\mathrm{e}^{x} + x \cos(4x) + 2x \sin(4x) \tag{1}
$$

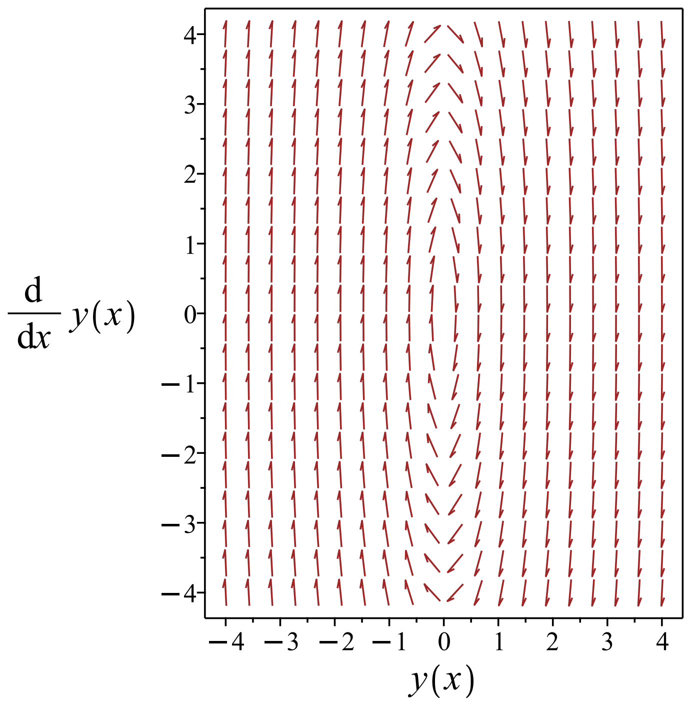


Figure 273: Slope field plot

Verification of solutions

$$
y = c_1 \cos(4x) + c_2 \sin(4x) + 2\,\mathrm{e}^{x} + x \cos(4x) + 2x \sin(4x)
$$

Verified OK.

8.11.2 Solving using Kovacic algorithm

Writing the ode as

$$
\begin{align}
y'' + 16y &= 0 \tag{1} \\
Ay'' + By' + Cy &= 0 \tag{2}
\end{align}
$$

Comparing (1) and (2) shows that

$$
\begin{align}
A &= 1 \\
B &= 0 \tag{3} \\
C &= 16
\end{align}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = ye^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$
z''(x) = rz(x) \tag{4}
$$

Where r is given by

$$
\begin{align}
r &= \frac{s}{t} \tag{5} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{align}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-16}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{align}
s &= -16 \\
t &= 1
\end{align}
$$

Therefore eq. (4) becomes

$$
z''(x) = -16z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 250: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -16$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(4x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \cos(4x)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \cos(4x)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= \cos(4x) \int \frac{1}{\cos(4x)^2}\, dx \\
&= \cos(4x) \left(\frac{\tan(4x)}{4}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\cos(4x)) + c_2\left(\cos(4x) \left(\frac{\tan(4x)}{4}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 16y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(4x) + \frac{c_2 \sin(4x)}{4}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$34\,e^x + 16\cos(4x) - 8\sin(4x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^x\}, \{\cos(4x), \sin(4x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sin(4x)}{4}, \cos(4x)\right\}$$

Since $\cos(4x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{e^x\}, \{x\cos(4x), x\sin(4x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 e^x + A_2 x\cos(4x) + A_3 x\sin(4x)$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$17A_1 e^x - 8A_2\sin(4x) + 8A_3\cos(4x) = 34\,e^x + 16\cos(4x) - 8\sin(4x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 2, A_2 = 1, A_3 = 2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2\,e^x + x\cos(4x) + 2x\sin(4x)$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \cos(4x) + \frac{c_2 \sin(4x)}{4}\right) + \left(2\,\mathrm{e}^{x} + x \cos(4x) + 2x \sin(4x)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1 \cos(4x) + \frac{c_2 \sin(4x)}{4} + 2\,\mathrm{e}^{x} + x \cos(4x) + 2x \sin(4x) \tag{1}
$$

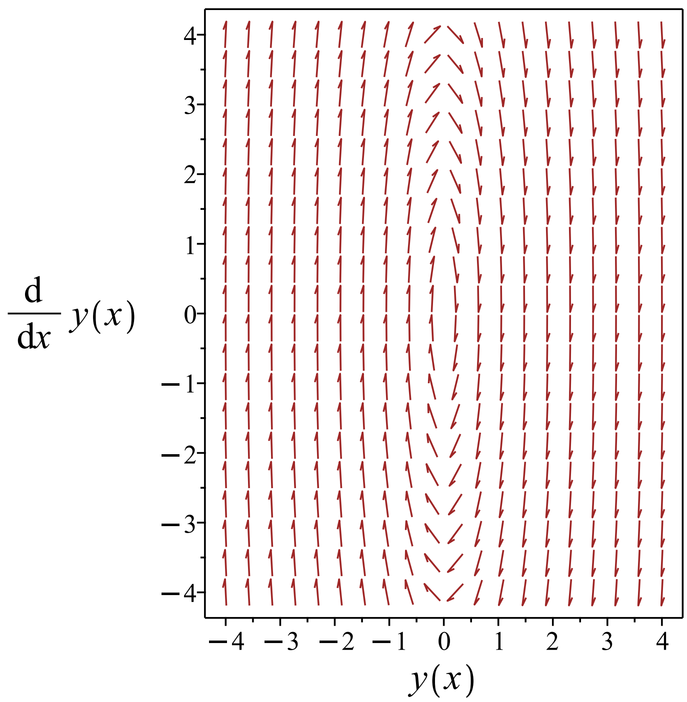


Figure 274: Slope field plot

Verification of solutions

$$
y = c_1 \cos(4x) + \frac{c_2 \sin(4x)}{4} + 2\,\mathrm{e}^{x} + x \cos(4x) + 2x \sin(4x)
$$

Verified OK.

8.11.3 Maple step by step solution

Let's solve

$y'' + 16y = 34\,\mathrm{e}^{x} + 16\cos(4x) - 8\sin(4x)$

- Highest derivative means the order of the ODE is 2

  y''
- Characteristic polynomial of homogeneous ODE

  $r^2 + 16 = 0$
- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-64})}{2}$
- Roots of the characteristic polynomial

  $r = (-4\,\mathrm{I}, 4\,\mathrm{I})$
- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(4x)$
- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(4x)$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(4x) + c_2 \sin(4x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 34\,\mathrm{e}^{x} + 16\cos(4x) - 8\right.$
  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(4x) & \sin(4x) \\ -4\sin(4x) & 4\cos(4x) \end{bmatrix}$
  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 4$
  - Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\frac{\cos(4x)\left(\int \sin(4x)(17\,\mathrm{e}^x+8\cos(4x)-4\sin(4x))dx\right)}{2} + \frac{\sin(4x)\left(\int \cos(4x)(17\,\mathrm{e}^x+8\cos(4x)-4\sin(4x))dx\right)}{2}$$

- Compute integrals

$$y_p(x) = 2\,\mathrm{e}^x + 2x\sin(4x) + x\cos(4x) - \frac{\sin(4x)}{4}$$

- Substitute particular solution into general solution to ODE

$$y = c_1\cos(4x) + c_2\sin(4x) + 2\,\mathrm{e}^x + 2x\sin(4x) + x\cos(4x) - \frac{\sin(4x)}{4}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 31

```
dsolve(diff(y(x),x$2)+16*y(x)=34*exp(x)+16*cos(4*x)-8*sin(4*x),y(x), singsol=all)
```

$$y(x) = \frac{(4c_2 + 8x - 1)\sin(4x)}{4} + (c_1 + x)\cos(4x) + 2\,\mathrm{e}^x$$

✓ Solution by Mathematica

Time used: 0.623 (sec). Leaf size: 37

```
DSolve[y''[x]+16*y[x]==34*Exp[x]+16*Cos[4*x]-8*Sin[4*x],y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to 2e^x + \left(x + \frac{1}{4} + c_1\right)\cos(4x) + \left(2x - \frac{1}{8} + c_2\right)\sin(4x)$$

9 Chapter 8, Linear differential equations of order n. Section 8.7, The Variation of Parameters Method. page 556

9.1 problem Problem 1

Internal problem ID [2774]
Internal file name [`OUTPUT/2266_Sunday_June_05_2022_02_57_01_AM_15360658/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.7, The Variation of Parameters Method. page 556
Problem number: Problem 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 6y' + 9y = 4\,\mathrm{e}^{3x} \ln(x)}$$

9.1.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -6, C = 9, f(x) = 4\,\mathrm{e}^{3x} \ln(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 6y' + 9y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -6, C = 9$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 6\lambda\, e^{\lambda x} + 9\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 6\lambda + 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -6, C = 9$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{6}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(-6)^2 - (4)(1)(9)}\\ &= 3\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -3$. Therefore the solution is

$$y = c_1 e^{3x} + c_2 x\, e^{3x} \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{3x} + c_2 x\, e^{3x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}y_1 &= e^{3x}\\ y_2 &= x\, e^{3x}\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{3x} & x\,e^{3x} \\ \frac{d}{dx}(e^{3x}) & \frac{d}{dx}(x\,e^{3x}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{3x} & x\,e^{3x} \\ 3\,e^{3x} & e^{3x} + 3x\,e^{3x} \end{vmatrix}$$

Therefore

$$W = \left(e^{3x}\right)\left(e^{3x} + 3x\,e^{3x}\right) - \left(x\,e^{3x}\right)\left(3\,e^{3x}\right)$$

Which simplifies to

$$W = e^{6x}$$

Which simplifies to

$$W = e^{6x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{4x\,e^{6x}\ln(x)}{e^{6x}}\,dx$$

Which simplifies to

$$u_1 = -\int 4\ln(x)\,x dx$$

Hence

$$u_1 = -2\ln(x)\,x^2 + x^2$$

And Eq. (3) becomes

$$u_2 = \int \frac{4\,\mathrm{e}^{6x} \ln(x)}{\mathrm{e}^{6x}}\, dx$$

Which simplifies to

$$u_2 = \int 4\ln(x)\, dx$$

Hence

$$u_2 = 4\ln(x)\, x - 4x$$

Which simplifies to

$$u_1 = x^2(1 - 2\ln(x))$$
$$u_2 = 4\ln(x)\, x - 4x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = x^2(1 - 2\ln(x))\, \mathrm{e}^{3x} + (4\ln(x)\, x - 4x)\, x\, \mathrm{e}^{3x}$$

Which simplifies to

$$y_p(x) = x^2\mathrm{e}^{3x}(2\ln(x) - 3)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{3x} + c_2 x\, \mathrm{e}^{3x}\right) + \left(x^2\mathrm{e}^{3x}(2\ln(x) - 3)\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{3x}(c_2 x + c_1) + x^2\mathrm{e}^{3x}(2\ln(x) - 3)$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{3x}(c_2 x + c_1) + x^2\mathrm{e}^{3x}(2\ln(x) - 3) \tag{1}$$

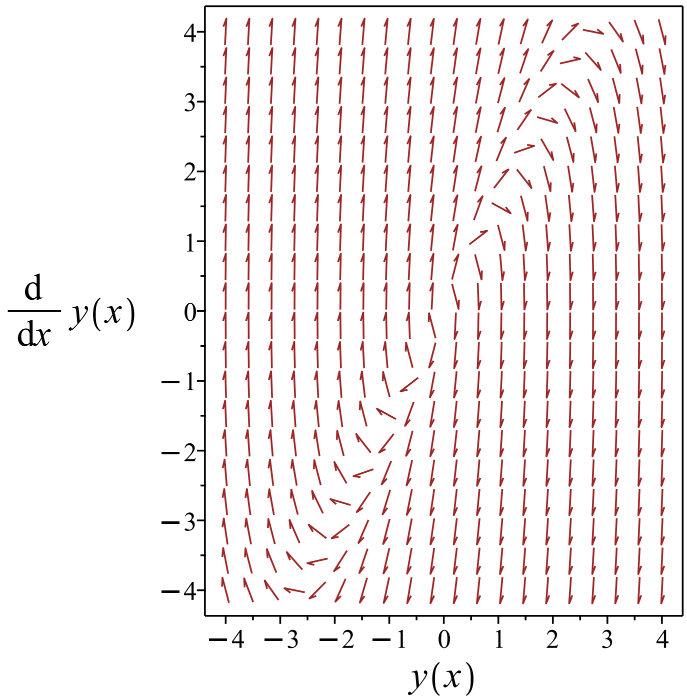


Figure 275: Slope field plot

Verification of solutions

$$y = e^{3x}(c_2x + c_1) + x^2e^{3x}(2\ln(x) - 3)$$

Verified OK.

9.1.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\,y' + \frac{\left(p(x)^2 + p'(x)\right)y}{2} = f(x)$$

Where $p(x) = -6$. Therefore, there is an integrating factor given by

$$\begin{aligned}M(x) &= e^{\frac{1}{2}\int p\,dx}\\ &= e^{\int -6\,dx}\\ &= e^{-3x}\end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$
\begin{aligned}
(M(x)y)'' &= 4\,\mathrm{e}^{-3x}\mathrm{e}^{3x}\ln(x) \\
\left(\mathrm{e}^{-3x}y\right)'' &= 4\,\mathrm{e}^{-3x}\mathrm{e}^{3x}\ln(x)
\end{aligned}
$$

Integrating once gives

$$
\left(\mathrm{e}^{-3x}y\right)' = 4\ln(x)\,x - 4x + c_1
$$

Integrating again gives

$$
\left(\mathrm{e}^{-3x}y\right) = x(2\ln(x)\,x + c_1 - 3x) + c_2
$$

Hence the solution is

$$
y = \frac{x(2\ln(x)\,x + c_1 - 3x) + c_2}{\mathrm{e}^{-3x}}
$$

Or

$$
y = 2x^2\mathrm{e}^{3x}\ln(x) + c_1 x\,\mathrm{e}^{3x} - 3x^2\mathrm{e}^{3x} + c_2\mathrm{e}^{3x}
$$

Summary

The solution(s) found are the following

$$
y = 2x^2\mathrm{e}^{3x}\ln(x) + c_1 x\,\mathrm{e}^{3x} - 3x^2\mathrm{e}^{3x} + c_2\mathrm{e}^{3x} \tag{1}
$$

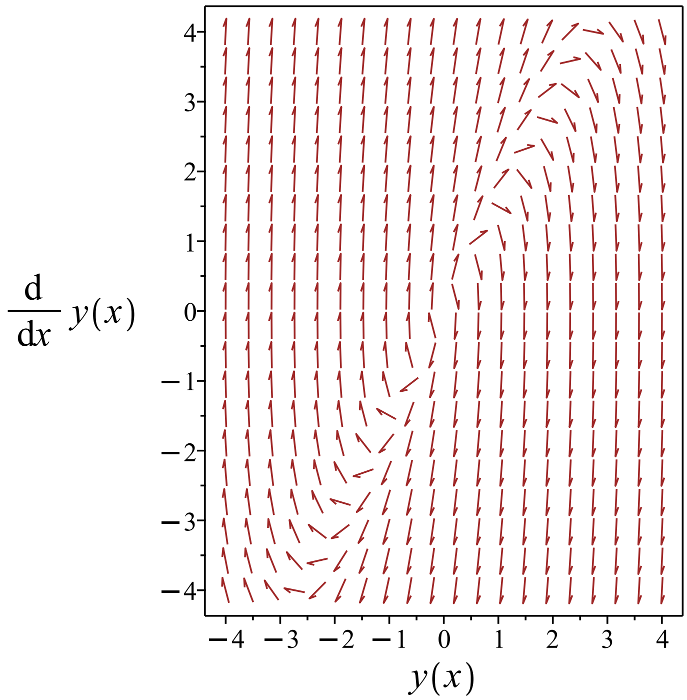


Figure 276: Slope field plot

Verification of solutions

$$y = 2x^2 e^{3x} \ln(x) + c_1 x\, e^{3x} - 3x^2 e^{3x} + c_2 e^{3x}$$

Verified OK.

9.1.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 6y' + 9y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -6 \\ C &= 9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 252: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-6}{1}\,dx} \\
&= z_1 e^{3x} \\
&= z_1\left(\mathrm{e}^{3x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{3x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-6}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{6x}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{3x}\right) + c_2 \left(\mathrm{e}^{3x}(x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 6y' + 9y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{3x} + c_2 x \,\mathrm{e}^{3x}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \mathrm{e}^{3x} \\
y_2 &= x \,\mathrm{e}^{3x}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{3x} & x\,e^{3x} \\ \frac{d}{dx}\left(e^{3x}\right) & \frac{d}{dx}\left(x\,e^{3x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{3x} & x\,e^{3x} \\ 3\,e^{3x} & e^{3x} + 3x\,e^{3x} \end{vmatrix}$$

Therefore

$$W = \left(e^{3x}\right)\left(e^{3x} + 3x\,e^{3x}\right) - \left(x\,e^{3x}\right)\left(3\,e^{3x}\right)$$

Which simplifies to

$$W = e^{6x}$$

Which simplifies to

$$W = e^{6x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{4x\,e^{6x}\ln(x)}{e^{6x}}\,dx$$

Which simplifies to

$$u_1 = -\int 4\ln(x)\,x\,dx$$

Hence

$$u_1 = -2\ln(x)\,x^2 + x^2$$

And Eq. (3) becomes

$$u_2 = \int \frac{4\,e^{6x}\ln(x)}{e^{6x}}\,dx$$

Which simplifies to

$$u_2 = \int 4 \ln (x)\, dx$$

Hence

$$u_2 = 4 \ln (x)\, x - 4x$$

Which simplifies to

$$\begin{aligned} u_1 &= x^2 (1 - 2 \ln (x)) \\ u_2 &= 4 \ln (x)\, x - 4x \end{aligned}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = x^2 (1 - 2 \ln (x))\, \mathrm{e}^{3x} + (4 \ln (x)\, x - 4x)\, x\, \mathrm{e}^{3x}$$

Which simplifies to

$$y_p(x) = x^2 \mathrm{e}^{3x} (2 \ln (x) - 3)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{3x} + c_2 x\, \mathrm{e}^{3x}\right) + \left(x^2 \mathrm{e}^{3x} (2 \ln (x) - 3)\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{3x} (c_2 x + c_1) + x^2 \mathrm{e}^{3x} (2 \ln (x) - 3)$$

<u>Summary</u>
The solution(s) found are the following

$$y = \mathrm{e}^{3x} (c_2 x + c_1) + x^2 \mathrm{e}^{3x} (2 \ln (x) - 3) \tag{1}$$

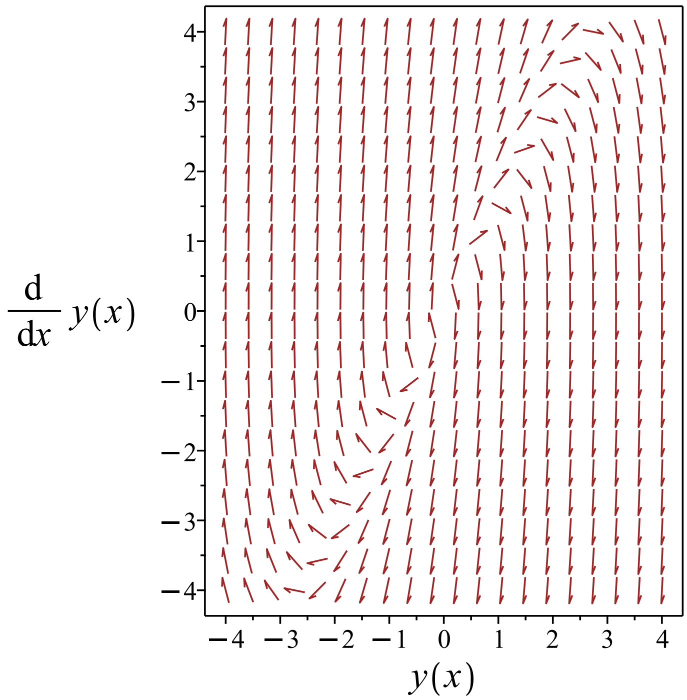


Figure 277: Slope field plot

Verification of solutions

$$y = e^{3x}(c_2x + c_1) + x^2e^{3x}(2\ln(x) - 3)$$

Verified OK.

9.1.4 Maple step by step solution

Let's solve

$y'' - 6y' + 9y = 4\,e^{3x}\ln(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 6r + 9 = 0$

- Factor the characteristic polynomial

  $(r-3)^2 = 0$

- Root of the characteristic polynomial

$r = 3$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{3x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\, e^{3x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{3x} + c_2 x\, e^{3x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 4\, e^{3x} \ln(x)\right]$

- Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{3x} & x\, e^{3x} \\ 3\, e^{3x} & e^{3x} + 3x\, e^{3x} \end{bmatrix}$

- Compute Wronskian

  $W(y_1(x), y_2(x)) = e^{6x}$

- Substitute functions into equation for $y_p(x)$

  $y_p(x) = -4\, e^{3x}\left(\int \ln(x)\, x dx - \left(\int \ln(x)\, dx\right) x\right)$

- Compute integrals

  $y_p(x) = x^2 e^{3x}(2\ln(x) - 3)$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{3x} + c_2 x\, e^{3x} + x^2 e^{3x}(2\ln(x) - 3)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve(diff(y(x),x$2)-6*diff(y(x),x)+9*y(x)=4*exp(3*x)*ln(x),y(x), singsol=all)
```

$$y(x) = e^{3x}\left(2\ln\left(x\right)x^2 + c_1 x - 3x^2 + c_2\right)$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 30

```
DSolve[y''[x]-6*y'[x]+9*y[x]==4*Exp[3*x]*Log[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{3x}\left(-3x^2 + 2x^2 \log(x) + c_2 x + c_1\right)$$

9.2 problem Problem 2

Internal problem ID [2775]
Internal file name [`OUTPUT/2267_Sunday_June_05_2022_02_57_03_AM_63963978/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.7, The Variation of Parameters Method. page 556
Problem number: Problem 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' + 4y' + 4y = \frac{e^{-2x}}{x^2}}$$

9.2.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 4, C = 4, f(x) = \frac{e^{-2x}}{x^2}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 4, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 4\lambda \,\mathrm{e}^{\lambda x} + 4 \,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4\lambda + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 4, C = 4$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{-4}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{(4)^2 - (4)\,(1)\,(4)} \\
&= -2
\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 2$. Therefore the solution is

$$y = c_1 \mathrm{e}^{-2x} + c_2 x \,\mathrm{e}^{-2x} \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{-2x} + c_2 x \,\mathrm{e}^{-2x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}
y_1 &= \mathrm{e}^{-2x} \\
y_2 &= x \,\mathrm{e}^{-2x}
\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{-2x} & x\,e^{-2x} \\ \frac{d}{dx}\left(e^{-2x}\right) & \frac{d}{dx}\left(x\,e^{-2x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-2x} & x\,e^{-2x} \\ -2\,e^{-2x} & e^{-2x} - 2x\,e^{-2x} \end{vmatrix}$$

Therefore

$$W = \left(e^{-2x}\right)\left(e^{-2x} - 2x\,e^{-2x}\right) - \left(x\,e^{-2x}\right)\left(-2\,e^{-2x}\right)$$

Which simplifies to

$$W = e^{-4x}$$

Which simplifies to

$$W = e^{-4x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{e^{-4x}}{x}}{e^{-4x}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{1}{x} dx$$

Hence

$$u_1 = -\ln(x)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{e^{-4x}}{x^2}}{e^{-4x}} dx$$

Which simplifies to

$$u_2 = \int \frac{1}{x^2} dx$$

Hence

$$u_2 = -\frac{1}{x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\ln(x)\, e^{-2x} - e^{-2x}$$

Which simplifies to

$$y_p(x) = e^{-2x}(-1 - \ln(x))$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-2x} + c_2 x\, e^{-2x}\right) + \left(e^{-2x}(-1 - \ln(x))\right) \end{aligned}$$

Which simplifies to

$$y = e^{-2x}(c_2 x + c_1) + e^{-2x}(-1 - \ln(x))$$

<u>Summary</u>
The solution(s) found are the following

$$y = e^{-2x}(c_2 x + c_1) + e^{-2x}(-1 - \ln(x)) \tag{1}$$

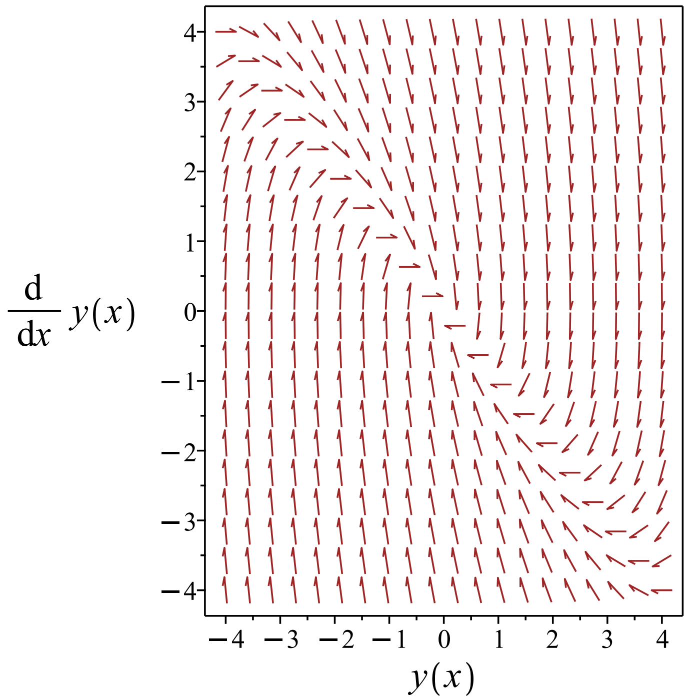


Figure 278: Slope field plot

Verification of solutions

$$y = \mathrm{e}^{-2x}(c_2 x + c_1) + \mathrm{e}^{-2x}(-1 - \ln(x))$$

Verified OK.

9.2.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{(p(x)^2 + p'(x))\, y}{2} = f(x)$$

Where $p(x) = 4$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int 4\,dx} \\ &= \mathrm{e}^{2x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$
\begin{aligned}
(M(x)y)'' &= \frac{e^{2x}e^{-2x}}{x^2} \\
\left(e^{2x}y\right)'' &= \frac{e^{2x}e^{-2x}}{x^2}
\end{aligned}
$$

Integrating once gives

$$
\left(e^{2x}y\right)' = -\frac{1}{x} + c_1
$$

Integrating again gives

$$
\left(e^{2x}y\right) = c_1x - \ln(x) + c_2
$$

Hence the solution is

$$
y = \frac{c_1x - \ln(x) + c_2}{e^{2x}}
$$

Or

$$
y = c_1x\,e^{-2x} + c_2e^{-2x} - \ln(x)\,e^{-2x}
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = c_1x\,e^{-2x} + c_2e^{-2x} - \ln(x)\,e^{-2x} \tag{1}
$$

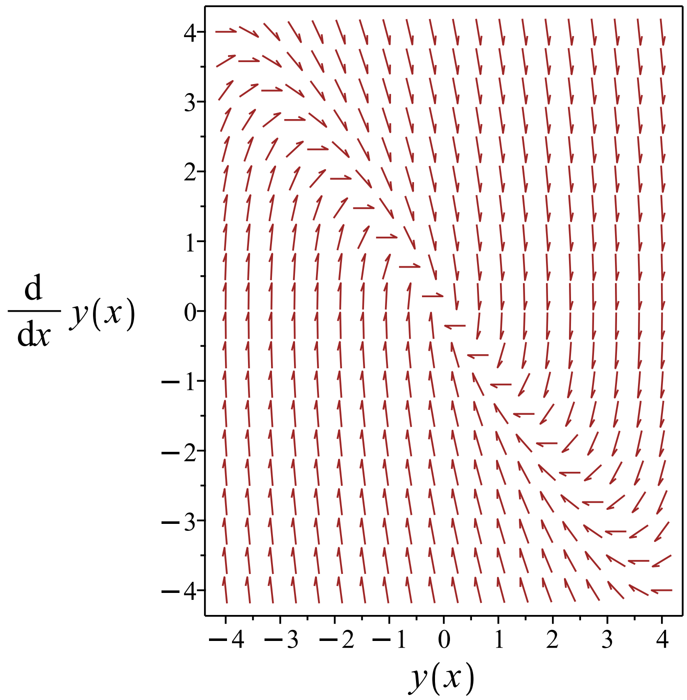


Figure 279: Slope field plot

Verification of solutions

$$y = c_1 x \,\mathrm{e}^{-2x} + c_2 \mathrm{e}^{-2x} - \ln(x)\,\mathrm{e}^{-2x}$$

Verified OK.

9.2.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 254: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - -\infty\\ &= \infty\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{4}{1}\,dx}\\ &= z_1 e^{-2x}\\ &= z_1\left(\mathrm{e}^{-2x}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{4}{1}\,dx}}{(y_1)^2}\,dx\\ &= y_1 \int \frac{e^{-4x}}{(y_1)^2}\,dx\\ &= y_1(x)\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-2x}\right) + c_2\left(\mathrm{e}^{-2x}(x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 4y' + 4y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-2x} + c_2 x\,\mathrm{e}^{-2x}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \mathrm{e}^{-2x} \\
y_2 &= x\,\mathrm{e}^{-2x}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{-2x} & x\,\mathrm{e}^{-2x} \\ \frac{d}{dx}\left(\mathrm{e}^{-2x}\right) & \frac{d}{dx}\left(x\,\mathrm{e}^{-2x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{-2x} & x\,\mathrm{e}^{-2x} \\ -2\,\mathrm{e}^{-2x} & \mathrm{e}^{-2x} - 2x\,\mathrm{e}^{-2x} \end{vmatrix}$$

Therefore

$$W = \left(\mathrm{e}^{-2x}\right)\left(\mathrm{e}^{-2x} - 2x\,\mathrm{e}^{-2x}\right) - \left(x\,\mathrm{e}^{-2x}\right)\left(-2\,\mathrm{e}^{-2x}\right)$$

Which simplifies to

$$W = \mathrm{e}^{-4x}$$

Which simplifies to

$$W = \mathrm{e}^{-4x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\mathrm{e}^{-4x}}{x}}{\mathrm{e}^{-4x}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{1}{x}dx$$

Hence

$$u_1 = -\ln(x)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{\mathrm{e}^{-4x}}{x^2}}{\mathrm{e}^{-4x}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{1}{x^2}dx$$

Hence

$$u_2 = -\frac{1}{x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\ln(x)\, \mathrm{e}^{-2x} - \mathrm{e}^{-2x}$$

Which simplifies to

$$y_p(x) = \mathrm{e}^{-2x}(-1 - \ln(x))$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-2x} + c_2 x\, \mathrm{e}^{-2x}\right) + \left(\mathrm{e}^{-2x}(-1 - \ln(x))\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{-2x}(c_2 x + c_1) + \mathrm{e}^{-2x}(-1 - \ln(x))$$

Summary
The solution(s) found are the following

$$y = \mathrm{e}^{-2x}(c_2 x + c_1) + \mathrm{e}^{-2x}(-1 - \ln(x)) \tag{1}$$

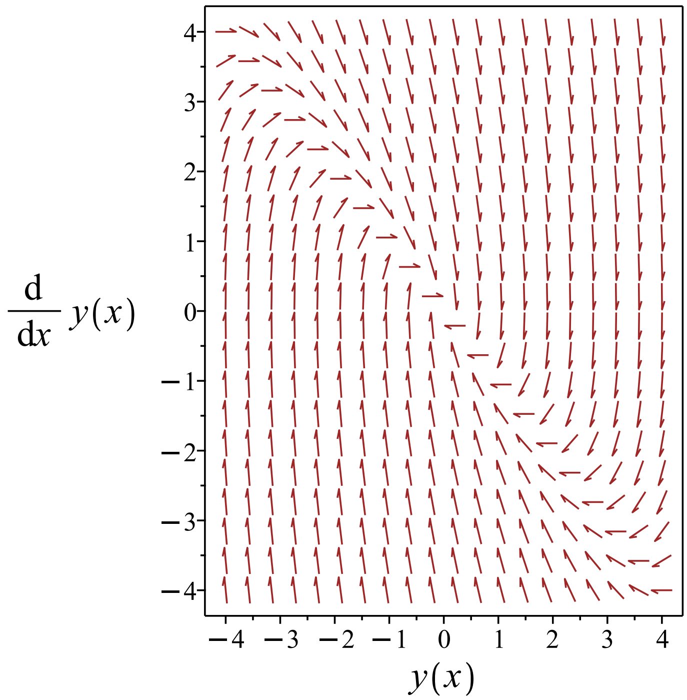


Figure 280: Slope field plot

Verification of solutions

$$y = e^{-2x}(c_2 x + c_1) + e^{-2x}(-1 - \ln(x))$$

Verified OK.

9.2.4 Maple step by step solution

Let's solve

$y'' + 4y' + 4y = \frac{e^{-2x}}{x^2}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4r + 4 = 0$

- Factor the characteristic polynomial

  $(r+2)^2 = 0$

- Root of the characteristic polynomial

$r = -2$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-2x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,\mathrm{e}^{-2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-2x} + c_2 x\,\mathrm{e}^{-2x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = \frac{\mathrm{e}^{-2x}}{x^2}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-2x} & x\,\mathrm{e}^{-2x} \\ -2\,\mathrm{e}^{-2x} & \mathrm{e}^{-2x} - 2x\,\mathrm{e}^{-2x} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = \mathrm{e}^{-4x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = \mathrm{e}^{-2x}\left(-\left(\int \frac{1}{x}dx\right) + \left(\int \frac{1}{x^2}dx\right)x\right)$

  - Compute integrals

    $y_p(x) = -\mathrm{e}^{-2x}(\ln(x) + 1)$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-2x} + c_2 x\,\mathrm{e}^{-2x} - \mathrm{e}^{-2x}(\ln(x) + 1)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 19

```
dsolve(diff(y(x),x$2)+4*diff(y(x),x)+4*y(x)=x^(-2)*exp(-2*x),y(x), singsol=all)
```

$$y(x) = e^{-2x}(-1 + c_1 x - \ln(x) + c_2)$$

✓ Solution by Mathematica
Time used: 0.029 (sec). Leaf size: 23

```
DSolve[y''[x]+4*y'[x]+4*y[x]==x^(-2)*Exp[-2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-2x}(-\log(x) + c_2 x - 1 + c_1)$$

9.3 problem Problem 3

Internal problem ID [2776]
Internal file name [`OUTPUT/2268_Sunday_June_05_2022_02_57_05_AM_70335767/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.7, The Variation of Parameters Method. page 556
Problem number: Problem 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 9y = 18 \sec(3x)^3}$$

9.3.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 9, f(x) = 18 \sec(3x)^3$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 9y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 9$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 9 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 9$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(9)} \\ &= \pm 3i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +3i \\ \lambda_2 &= -3i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 3i \\ \lambda_2 &= -3i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 3$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(3x) + c_2 \sin(3x))$$

Or

$$y = c_1 \cos(3x) + c_2 \sin(3x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(3x) + c_2 \sin(3x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= \cos(3x) \\ y_2 &= \sin(3x) \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(3x) & \sin(3x) \\ \frac{d}{dx}(\cos(3x)) & \frac{d}{dx}(\sin(3x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(3x) & \sin(3x) \\ -3\sin(3x) & 3\cos(3x) \end{vmatrix}$$

Therefore

$$W = (\cos (3x)) (3 \cos (3x)) - (\sin (3x)) (-3 \sin (3x))$$

Which simplifies to

$$W = 3 \cos (3x)^2 + 3 \sin (3x)^2$$

Which simplifies to

$$W = 3$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{18 \sin (3x) \sec (3x)^3}{3} dx$$

Which simplifies to

$$u_1 = -\int 6 \tan (3x) \sec (3x)^2 dx$$

Hence

$$u_1 = -\sec (3x)^2$$

And Eq. (3) becomes

$$u_2 = \int \frac{18 \cos (3x) \sec (3x)^3}{3} dx$$

Which simplifies to

$$u_2 = \int 6 \sec (3x)^2 dx$$

Hence

$$u_2 = 2 \tan (3x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\sec (3x)^2 \cos (3x) + 2 \tan (3x) \sin (3x)$$

Which simplifies to

$$y_p(x) = -2 \cos (3x) + \sec (3x)$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 \cos(3x) + c_2 \sin(3x)) + (-2\cos(3x) + \sec(3x))
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \cos(3x) + c_2 \sin(3x) - 2\cos(3x) + \sec(3x) \tag{1}
$$

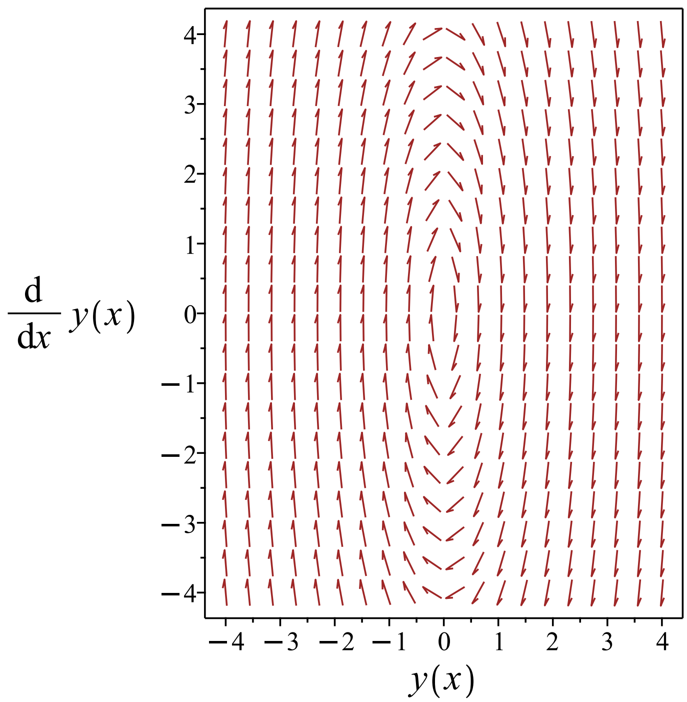


Figure 281: Slope field plot

Verification of solutions

$$
y = c_1 \cos(3x) + c_2 \sin(3x) - 2\cos(3x) + \sec(3x)
$$

Verified OK.

9.3.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 9y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-9}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -9 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -9z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 256: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -9$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(3x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}
$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}
y_1 &= z_1 \\
&= \cos(3x)
\end{aligned}$$

Which simplifies to

$$y_1 = \cos(3x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\
&= \cos(3x) \int \frac{1}{\cos(3x)^2}\,dx \\
&= \cos(3x)\left(\frac{\tan(3x)}{3}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\cos(3x)) + c_2\left(\cos(3x)\left(\frac{\tan(3x)}{3}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 9y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(3x)$$
$$y_2 = \frac{\sin(3x)}{3}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(3x) & \frac{\sin(3x)}{3} \\ \frac{d}{dx}(\cos(3x)) & \frac{d}{dx}\left(\frac{\sin(3x)}{3}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(3x) & \frac{\sin(3x)}{3} \\ -3\sin(3x) & \cos(3x) \end{vmatrix}$$

Therefore

$$W = (\cos(3x))(\cos(3x)) - \left(\frac{\sin(3x)}{3}\right)(-3\sin(3x))$$

Which simplifies to

$$W = \cos(3x)^2 + \sin(3x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{6\sin(3x)\sec(3x)^3}{1}\,dx$$

Which simplifies to

$$u_1 = -\int 6\tan(3x)\sec(3x)^2\,dx$$

Hence

$$u_1 = -\sec(3x)^2$$

And Eq. (3) becomes

$$u_2 = \int \frac{18\cos(3x)\sec(3x)^3}{1}\,dx$$

Which simplifies to

$$u_2 = \int 18\sec(3x)^2\,dx$$

Hence

$$u_2 = 6\tan(3x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\sec(3x)^2\cos(3x) + 2\tan(3x)\sin(3x)$$

Which simplifies to

$$y_p(x) = -2\cos(3x) + \sec(3x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3}\right) + (-2\cos(3x) + \sec(3x)) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3} - 2\cos(3x) + \sec(3x) \tag{1}$$

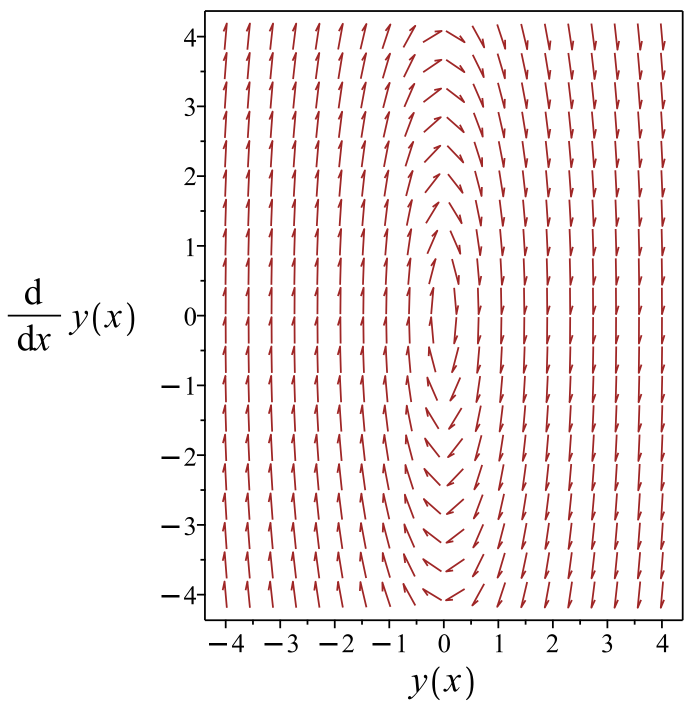


Figure 282: Slope field plot

Verification of solutions

$$y = c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3} - 2\cos(3x) + \sec(3x)$$

Verified OK.

9.3.3 Maple step by step solution

Let's solve

$y'' + 9y = 18 \sec(3x)^3$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 9 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-36})}{2}$

- Roots of the characteristic polynomial

  $r = (-3\,\mathrm{I}, 3\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(3x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(3x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(3x) + c_2 \sin(3x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ∘ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 18 \sec(3x)^3\right]$

- ∘ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(3x) & \sin(3x) \\ -3\sin(3x) & 3\cos(3x) \end{bmatrix}$

- ∘ Compute Wronskian

  $W(y_1(x), y_2(x)) = 3$

- ∘ Substitute functions into equation for $y_p(x)$

$$y_p(x) = -6\cos(3x)\left(\int \tan(3x)\sec(3x)^2\, dx\right) + 6\sin(3x)\left(\int \sec(3x)^2\, dx\right)$$

- Compute integrals

$$y_p(x) = -2\cos(3x) + \sec(3x)$$

- Substitute particular solution into general solution to ODE

$$y = c_1 \cos(3x) + c_2 \sin(3x) - 2\cos(3x) + \sec(3x)$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(y(x),x$2)+9*y(x)=18*sec(3*x)^3,y(x), singsol=all)
```

$$y(x) = (c_1 - 2)\cos(3x) + \sin(3x)\, c_2 + \sec(3x)$$

✓ Solution by Mathematica
Time used: 0.139 (sec). Leaf size: 32

```
DSolve[y''[x]+9*y[x]==18*Sec[3*x]^3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}\sec(3x)((-2 + c_1)\cos(6x) + c_2\sin(6x) + c_1)$$

9.4 problem Problem 4

Internal problem ID [2777]
Internal file name [`OUTPUT/2269_Sunday_June_05_2022_02_57_08_AM_51508744/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.7, The Variation of Parameters Method. page 556
Problem number: Problem 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 6y' + 9y = \frac{2\,e^{-3x}}{x^2+1}}$$

9.4.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 6, C = 9, f(x) = \frac{2\,e^{-3x}}{x^2+1}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$.

y_h is the solution to

$$y'' + 6y' + 9y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 6, C = 9$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 6\lambda\,\mathrm{e}^{\lambda x} + 9\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 6\lambda + 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 6, C = 9$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-6}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{(6)^2 - (4)\,(1)\,(9)} \\ &= -3\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 3$. Therefore the solution is

$$y = c_1 \mathrm{e}^{-3x} + c_2 x\,\mathrm{e}^{-3x} \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{-3x} + c_2 x\,\mathrm{e}^{-3x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^{-3x}$$
$$y_2 = x\,e^{-3x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{-3x} & x\,e^{-3x} \\ \frac{d}{dx}\left(e^{-3x}\right) & \frac{d}{dx}\left(x\,e^{-3x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-3x} & x\,e^{-3x} \\ -3\,e^{-3x} & e^{-3x} - 3x\,e^{-3x} \end{vmatrix}$$

Therefore

$$W = \left(e^{-3x}\right)\left(e^{-3x} - 3x\,e^{-3x}\right) - \left(x\,e^{-3x}\right)\left(-3\,e^{-3x}\right)$$

Which simplifies to

$$W = e^{-6x}$$

Which simplifies to

$$W = e^{-6x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{2x\,e^{-6x}}{x^2+1}}{e^{-6x}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{2x}{x^2+1} dx$$

Hence

$$u_1 = -\ln\left(x^2+1\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{2\,\mathrm{e}^{-6x}}{x^2+1}}{\mathrm{e}^{-6x}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{2}{x^2+1} dx$$

Hence

$$u_2 = 2\arctan\left(x\right)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\ln\left(x^2+1\right)\mathrm{e}^{-3x} + 2\arctan\left(x\right) x\,\mathrm{e}^{-3x}$$

Which simplifies to

$$y_p(x) = \mathrm{e}^{-3x}\left(-\ln\left(x^2+1\right) + 2\arctan\left(x\right) x\right)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-3x} + c_2 x\,\mathrm{e}^{-3x}\right) + \left(\mathrm{e}^{-3x}\left(-\ln\left(x^2+1\right) + 2\arctan\left(x\right) x\right)\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{-3x}(c_2 x + c_1) + \mathrm{e}^{-3x}\left(-\ln\left(x^2+1\right) + 2\arctan\left(x\right) x\right)$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-3x}(c_2 x + c_1) + \mathrm{e}^{-3x}\left(-\ln\left(x^2+1\right) + 2\arctan\left(x\right) x\right) \tag{1}$$

Verification of solutions

$$y = \mathrm{e}^{-3x}(c_2x + c_1) + \mathrm{e}^{-3x}\left(-\ln\left(x^2+1\right) + 2\arctan\left(x\right)x\right)$$

Verified OK.

9.4.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\,y' + \frac{\left(p(x)^2 + p'(x)\right)y}{2} = f(x)$$

Where $p(x) = 6$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int 6\,dx} \\ &= \mathrm{e}^{3x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)y)'' &= \frac{2\,\mathrm{e}^{3x}\mathrm{e}^{-3x}}{x^2+1} \\ \left(\mathrm{e}^{3x}y\right)'' &= \frac{2\,\mathrm{e}^{3x}\mathrm{e}^{-3x}}{x^2+1} \end{aligned}$$

Integrating once gives

$$\left(\mathrm{e}^{3x}y\right)' = 2\arctan\left(x\right) + c_1$$

Integrating again gives

$$\left(\mathrm{e}^{3x}y\right) = c_1x + 2\arctan\left(x\right)x - \ln\left(x^2+1\right) + c_2$$

Hence the solution is

$$y = \frac{c_1x + 2\arctan\left(x\right)x - \ln\left(x^2+1\right) + c_2}{\mathrm{e}^{3x}}$$

Or

$$y = c_1x\,\mathrm{e}^{-3x} + 2\arctan\left(x\right)x\,\mathrm{e}^{-3x} + c_2\mathrm{e}^{-3x} - \ln\left(x^2+1\right)\mathrm{e}^{-3x}$$

Summary
The solution(s) found are the following

$$y = c_1 x\,\mathrm{e}^{-3x} + 2\arctan(x)\,x\,\mathrm{e}^{-3x} + c_2\mathrm{e}^{-3x} - \ln\left(x^2+1\right)\mathrm{e}^{-3x} \tag{1}$$

Verification of solutions

$$y = c_1 x\,\mathrm{e}^{-3x} + 2\arctan(x)\,x\,\mathrm{e}^{-3x} + c_2\mathrm{e}^{-3x} - \ln\left(x^2+1\right)\mathrm{e}^{-3x}$$

Verified OK.

9.4.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 6y' + 9y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 6 \\ C &= 9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 258: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{6}{1}\,dx} \\
&= z_1 e^{-3x} \\
&= z_1 \left(\mathrm{e}^{-3x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-3x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{6}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-6x}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-3x}\right) + c_2\left(\mathrm{e}^{-3x}(x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 6y' + 9y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-3x} + c_2 x\, \mathrm{e}^{-3x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= \mathrm{e}^{-3x} \\ y_2 &= x\, \mathrm{e}^{-3x} \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{-3x} & x\, \mathrm{e}^{-3x} \\ \frac{d}{dx}\left(\mathrm{e}^{-3x}\right) & \frac{d}{dx}\left(x\, \mathrm{e}^{-3x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{-3x} & x\, \mathrm{e}^{-3x} \\ -3\, \mathrm{e}^{-3x} & \mathrm{e}^{-3x} - 3x\, \mathrm{e}^{-3x} \end{vmatrix}$$

Therefore

$$W = \left(\mathrm{e}^{-3x}\right)\left(\mathrm{e}^{-3x} - 3x\, \mathrm{e}^{-3x}\right) - \left(x\, \mathrm{e}^{-3x}\right)\left(-3\, \mathrm{e}^{-3x}\right)$$

Which simplifies to

$$W = e^{-6x}$$

Which simplifies to

$$W = e^{-6x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{2x\,e^{-6x}}{x^2+1}}{e^{-6x}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{2x}{x^2+1}dx$$

Hence

$$u_1 = -\ln\left(x^2+1\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{2\,e^{-6x}}{x^2+1}}{e^{-6x}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{2}{x^2+1}dx$$

Hence

$$u_2 = 2\arctan(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\ln\left(x^2+1\right)e^{-3x} + 2\arctan(x)\,x\,e^{-3x}$$

Which simplifies to

$$y_p(x) = e^{-3x}\left(-\ln\left(x^2+1\right) + 2\arctan(x)\,x\right)$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \mathrm{e}^{-3x} + c_2 x\,\mathrm{e}^{-3x}\right) + \left(\mathrm{e}^{-3x}\left(-\ln\left(x^2+1\right) + 2\arctan\left(x\right)x\right)\right)
\end{aligned}
$$

Which simplifies to

$$
y = \mathrm{e}^{-3x}(c_2 x + c_1) + \mathrm{e}^{-3x}\left(-\ln\left(x^2+1\right) + 2\arctan\left(x\right)x\right)
$$

Summary

The solution(s) found are the following

$$
y = \mathrm{e}^{-3x}(c_2 x + c_1) + \mathrm{e}^{-3x}\left(-\ln\left(x^2+1\right) + 2\arctan\left(x\right)x\right) \tag{1}
$$

Verification of solutions

$$
y = \mathrm{e}^{-3x}(c_2 x + c_1) + \mathrm{e}^{-3x}\left(-\ln\left(x^2+1\right) + 2\arctan\left(x\right)x\right)
$$

Verified OK.

9.4.4 Maple step by step solution

Let's solve

$y'' + 6y' + 9y = \frac{2\,\mathrm{e}^{-3x}}{x^2+1}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 6r + 9 = 0$

- Factor the characteristic polynomial

  $(r+3)^2 = 0$

- Root of the characteristic polynomial

  $r = -3$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-3x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,\mathrm{e}^{-3x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-3x} + c_2 x\,\mathrm{e}^{-3x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = \frac{2\,\mathrm{e}^{-3x}}{x^2+1}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-3x} & x\,\mathrm{e}^{-3x} \\ -3\,\mathrm{e}^{-3x} & \mathrm{e}^{-3x} - 3x\,\mathrm{e}^{-3x} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = \mathrm{e}^{-6x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -2\,\mathrm{e}^{-3x}\left(\int \frac{x}{x^2+1}dx - \left(\int \frac{1}{x^2+1}dx\right)x\right)$

  - Compute integrals

    $y_p(x) = \mathrm{e}^{-3x}(-\ln(x^2+1) + 2\arctan(x)\,x)$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-3x} + c_2 x\,\mathrm{e}^{-3x} + \mathrm{e}^{-3x}(-\ln(x^2+1) + 2\arctan(x)\,x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(diff(y(x),x$2)+6*diff(y(x),x)+9*y(x)=2*exp(-3*x)/(x^2+1),y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{-3x}\left(c_2 + c_1 x - \ln\left(x^2+1\right) + 2x \arctan\left(x\right)\right)$$

✓ Solution by Mathematica

Time used: 0.043 (sec). Leaf size: 31

```
DSolve[y''[x]+6*y'[x]+9*y[x]==2*Exp[-3*x]/(x^2+1),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-3x}\left(2x \arctan(x) - \log\left(x^2+1\right) + c_2 x + c_1\right)$$

9.5 problem Problem 5

Internal problem ID [2778]
Internal file name [`OUTPUT/2270_Sunday_June_05_2022_02_57_09_AM_77182596/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.7, The Variation of Parameters Method. page 556
Problem number: Problem 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 4y = \frac{8}{e^{2x} + 1}}$$

9.5.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = -4, f(x) = \frac{8}{e^{2x}+1}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 4\,e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(-4)}\\ &= \pm 2\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +2\\ \lambda_2 &= -2\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2\\ \lambda_2 &= -2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(2)x} + c_2 e^{(-2)x}\end{aligned}$$

Or

$$y = e^{2x} c_1 + c_2 e^{-2x}$$

Therefore the homogeneous solution y_h is

$$y_h = e^{2x} c_1 + c_2 e^{-2x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^{2x}$$
$$y_2 = e^{-2x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{2x} & e^{-2x} \\ \frac{d}{dx}\left(e^{2x}\right) & \frac{d}{dx}\left(e^{-2x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{2x} & e^{-2x} \\ 2\,e^{2x} & -2\,e^{-2x} \end{vmatrix}$$

Therefore

$$W = \left(e^{2x}\right)\left(-2\,e^{-2x}\right) - \left(e^{-2x}\right)\left(2\,e^{2x}\right)$$

Which simplifies to

$$W = -4\,e^{-2x} e^{2x}$$

Which simplifies to

$$W = -4$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{8\,\mathrm{e}^{-2x}}{\mathrm{e}^{2x}+1}}{-4}\,dx$$

Which simplifies to

$$u_1 = -\int -\frac{2\,\mathrm{e}^{-2x}}{\mathrm{e}^{2x}+1}dx$$

Hence

$$u_1 = \ln\left(\mathrm{e}^{2x}+1\right) - \mathrm{e}^{-2x} - 2\ln\left(\mathrm{e}^{x}\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{8\,\mathrm{e}^{2x}}{\mathrm{e}^{2x}+1}}{-4}\,dx$$

Which simplifies to

$$u_2 = \int -\frac{2\,\mathrm{e}^{2x}}{\mathrm{e}^{2x}+1}dx$$

Hence

$$u_2 = -\ln\left(\mathrm{e}^{2x}+1\right)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(\ln\left(\mathrm{e}^{2x}+1\right) - \mathrm{e}^{-2x} - 2\ln\left(\mathrm{e}^{x}\right)\right)\mathrm{e}^{2x} - \ln\left(\mathrm{e}^{2x}+1\right)\mathrm{e}^{-2x}$$

Which simplifies to

$$y_p(x) = \left(-\mathrm{e}^{-2x}+\mathrm{e}^{2x}\right)\ln\left(\mathrm{e}^{2x}+1\right) - 2\,\mathrm{e}^{2x}\ln\left(\mathrm{e}^{x}\right) - 1$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\mathrm{e}^{2x}c_1 + c_2\mathrm{e}^{-2x}\right) + \left(\left(-\mathrm{e}^{-2x}+\mathrm{e}^{2x}\right)\ln\left(\mathrm{e}^{2x}+1\right) - 2\,\mathrm{e}^{2x}\ln\left(\mathrm{e}^{x}\right) - 1\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^{2x}c_1 + c_2 e^{-2x} + \left(-e^{-2x} + e^{2x}\right)\ln\left(e^{2x} + 1\right) - 2\,e^{2x}\ln\left(e^x\right) - 1 \tag{1}$$

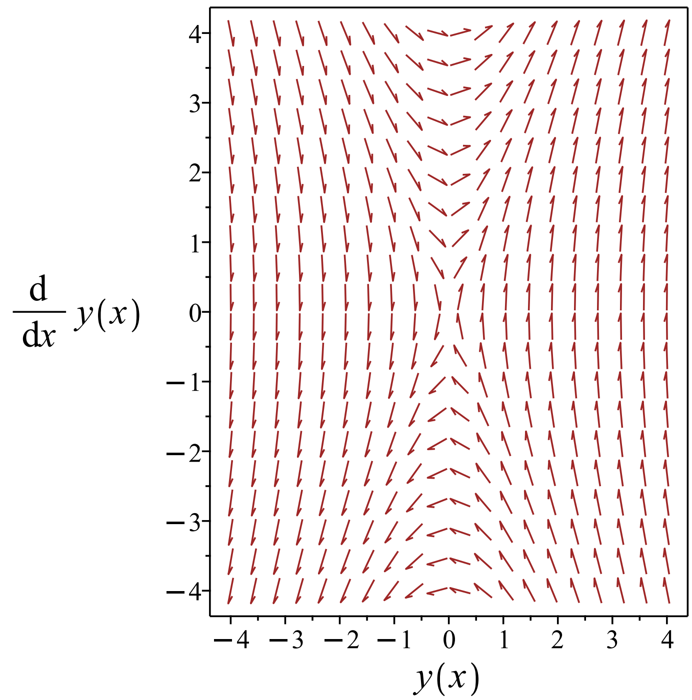


Figure 283: Slope field plot

Verification of solutions

$$y = e^{2x}c_1 + c_2 e^{-2x} + \left(-e^{-2x} + e^{2x}\right)\ln\left(e^{2x} + 1\right) - 2\,e^{2x}\ln\left(e^x\right) - 1$$

Verified OK.

9.5.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 260: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-2x}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \mathrm{e}^{-2x}
\end{aligned}
$$

Which simplifies to

$$y_1 = e^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\
&= e^{-2x} \int \frac{1}{e^{-4x}}\,dx \\
&= e^{-2x}\left(\frac{e^{4x}}{4}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(e^{-2x}\right) + c_2\left(e^{-2x}\left(\frac{e^{4x}}{4}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 4y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{-2x} + \frac{c_2 e^{2x}}{4}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^{-2x}$$
$$y_2 = \frac{e^{2x}}{4}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{-2x} & \frac{e^{2x}}{4} \\ \frac{d}{dx}\left(e^{-2x}\right) & \frac{d}{dx}\left(\frac{e^{2x}}{4}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-2x} & \frac{e^{2x}}{4} \\ -2\,e^{-2x} & \frac{e^{2x}}{2} \end{vmatrix}$$

Therefore

$$W = \left(e^{-2x}\right)\left(\frac{e^{2x}}{2}\right) - \left(\frac{e^{2x}}{4}\right)\left(-2\,e^{-2x}\right)$$

Which simplifies to

$$W = e^{-2x} e^{2x}$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{2\,\mathrm{e}^{2x}}{\mathrm{e}^{2x}+1}}{1}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{2\,\mathrm{e}^{2x}}{\mathrm{e}^{2x}+1}dx$$

Hence

$$u_1 = -\ln\left(\mathrm{e}^{2x}+1\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{8\,\mathrm{e}^{-2x}}{\mathrm{e}^{2x}+1}}{1}\,dx$$

Which simplifies to

$$u_2 = \int \frac{8\,\mathrm{e}^{-2x}}{\mathrm{e}^{2x}+1}dx$$

Hence

$$u_2 = 4\ln\left(\mathrm{e}^{2x}+1\right) - 4\,\mathrm{e}^{-2x} - 8\ln\left(\mathrm{e}^{x}\right)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\ln\left(\mathrm{e}^{2x}+1\right)\mathrm{e}^{-2x} + \frac{\left(4\ln\left(\mathrm{e}^{2x}+1\right) - 4\,\mathrm{e}^{-2x} - 8\ln\left(\mathrm{e}^{x}\right)\right)\mathrm{e}^{2x}}{4}$$

Which simplifies to

$$y_p(x) = \left(-\mathrm{e}^{-2x} + \mathrm{e}^{2x}\right)\ln\left(\mathrm{e}^{2x}+1\right) - 2\,\mathrm{e}^{2x}\ln\left(\mathrm{e}^{x}\right) - 1$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{-2x} + \frac{c_2\mathrm{e}^{2x}}{4}\right) + \left(\left(-\mathrm{e}^{-2x} + \mathrm{e}^{2x}\right)\ln\left(\mathrm{e}^{2x}+1\right) - 2\,\mathrm{e}^{2x}\ln\left(\mathrm{e}^{x}\right) - 1\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-2x} + \frac{c_2 \mathrm{e}^{2x}}{4} + \left(-\mathrm{e}^{-2x} + \mathrm{e}^{2x}\right) \ln\left(\mathrm{e}^{2x} + 1\right) - 2\,\mathrm{e}^{2x} \ln\left(\mathrm{e}^{x}\right) - 1 \tag{1}$$

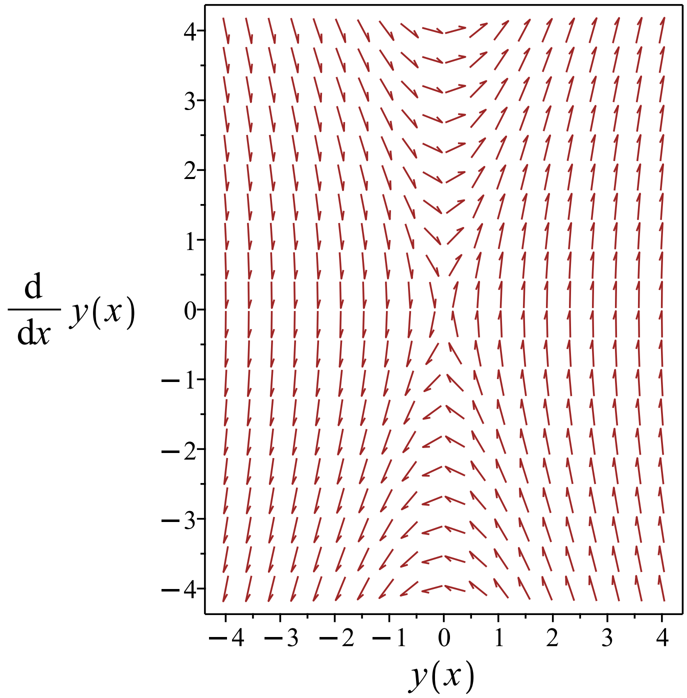


Figure 284: Slope field plot

Verification of solutions

$$y = c_1 \mathrm{e}^{-2x} + \frac{c_2 \mathrm{e}^{2x}}{4} + \left(-\mathrm{e}^{-2x} + \mathrm{e}^{2x}\right) \ln\left(\mathrm{e}^{2x} + 1\right) - 2\,\mathrm{e}^{2x} \ln\left(\mathrm{e}^{x}\right) - 1$$

Verified OK.

9.5.3 Maple step by step solution

Let's solve

$y'' - 4y = \frac{8}{\mathrm{e}^{2x}+1}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2 - 4 = 0$

- Factor the characteristic polynomial

  $(r-2)(r+2) = 0$

- Roots of the characteristic polynomial

  $r = (-2, 2)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-2x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{2x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = \frac{8}{\mathrm{e}^{2x}+1}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-2x} & \mathrm{e}^{2x} \\ -2\,\mathrm{e}^{-2x} & 2\,\mathrm{e}^{2x} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 4$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -2\,\mathrm{e}^{-2x}\left(\int \frac{\mathrm{e}^{2x}}{\mathrm{e}^{2x}+1}dx\right) + 2\,\mathrm{e}^{2x}\left(\int \frac{\mathrm{e}^{-2x}}{\mathrm{e}^{2x}+1}dx\right)$

  - Compute integrals

    $y_p(x) = (-\mathrm{e}^{-2x} + \mathrm{e}^{2x})\ln(\mathrm{e}^{2x}+1) - 2\,\mathrm{e}^{2x}\ln(\mathrm{e}^{x}) - 1$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{2x} + (-\mathrm{e}^{-2x} + \mathrm{e}^{2x})\ln(\mathrm{e}^{2x}+1) - 2\,\mathrm{e}^{2x}\ln(\mathrm{e}^{x}) - 1$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 43

```
dsolve(diff(y(x),x$2)-4*y(x)=8/(exp(2*x)+1),y(x), singsol=all)
```

$$y(x) = \left(-\mathrm{e}^{-2x} + \mathrm{e}^{2x}\right) \ln\left(\mathrm{e}^{2x} + 1\right) + \left(c_1 - 2\ln\left(\mathrm{e}^{x}\right)\right) \mathrm{e}^{2x} + \mathrm{e}^{-2x} c_2 - 1$$

✓ Solution by Mathematica

Time used: 0.086 (sec). Leaf size: 56

```
DSolve[y''[x]-4*y[x]==8/(Exp[2*x]+1),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-2x}\left(2e^{4x} \operatorname{arctanh}\left(2e^{2x} + 1\right) - e^{2x} - \log\left(e^{2x} + 1\right) + c_1 e^{4x} + c_2\right)$$

9.6 problem Problem 6

Internal problem ID [2779]
Internal file name [`OUTPUT/2271_Sunday_June_05_2022_02_57_11_AM_47497370/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.7, The Variation of Parameters Method. page 556
Problem number: Problem 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 4y' + 5y = e^{2x} \tan(x)}$$

9.6.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -4, C = 5, f(x) = e^{2x} \tan(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 4y' + 5y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -4, C = 5$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 4\lambda\, e^{\lambda x} + 5\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 4\lambda + 5 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -4, C = 5$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{4}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-4^2 - (4)\,(1)\,(5)} \\
&= 2 \pm i
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= 2 + i \\
\lambda_2 &= 2 - i
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= 2 + i \\
\lambda_2 &= 2 - i
\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 2$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{2x}(c_1 \cos(x) + c_2 \sin(x))$$

Therefore the homogeneous solution y_h is

$$y_h = e^{2x}(c_1 \cos(x) + c_2 \sin(x))$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x)\,\mathrm{e}^{2x}$$
$$y_2 = \sin(x)\,\mathrm{e}^{2x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x)\,\mathrm{e}^{2x} & \sin(x)\,\mathrm{e}^{2x} \\ \frac{d}{dx}\left(\cos(x)\,\mathrm{e}^{2x}\right) & \frac{d}{dx}\left(\sin(x)\,\mathrm{e}^{2x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x)\,\mathrm{e}^{2x} & \sin(x)\,\mathrm{e}^{2x} \\ -\sin(x)\,\mathrm{e}^{2x} + 2\cos(x)\,\mathrm{e}^{2x} & \cos(x)\,\mathrm{e}^{2x} + 2\sin(x)\,\mathrm{e}^{2x} \end{vmatrix}$$

Therefore

$$W = \left(\cos(x)\,\mathrm{e}^{2x}\right)\left(\cos(x)\,\mathrm{e}^{2x} + 2\sin(x)\,\mathrm{e}^{2x}\right) - \left(\sin(x)\,\mathrm{e}^{2x}\right)\left(-\sin(x)\,\mathrm{e}^{2x} + 2\cos(x)\,\mathrm{e}^{2x}\right)$$

Which simplifies to

$$W = \mathrm{e}^{4x}\cos(x)^2 + \sin(x)^2\,\mathrm{e}^{4x}$$

Which simplifies to

$$W = e^{4x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(x)\,e^{4x}\tan(x)}{e^{4x}}\,dx$$

Which simplifies to

$$u_1 = -\int \tan(x)\sin(x)\,dx$$

Hence

$$u_1 = \sin(x) - \ln(\sec(x) + \tan(x))$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(x)\,e^{4x}\tan(x)}{e^{4x}}\,dx$$

Which simplifies to

$$u_2 = \int \sin(x)\,dx$$

Hence

$$u_2 = -\cos(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = (\sin(x) - \ln(\sec(x) + \tan(x)))\cos(x)\,e^{2x} - \sin(x)\cos(x)\,e^{2x}$$

Which simplifies to

$$y_p(x) = -\cos(x)\,e^{2x}\ln(\sec(x) + \tan(x))$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{2x}(c_1\cos(x) + c_2\sin(x))\right) + \left(-\cos(x)\,e^{2x}\ln(\sec(x) + \tan(x))\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^{2x}(c_1 \cos(x) + c_2 \sin(x)) - \cos(x) e^{2x} \ln(\sec(x) + \tan(x)) \tag{1}$$

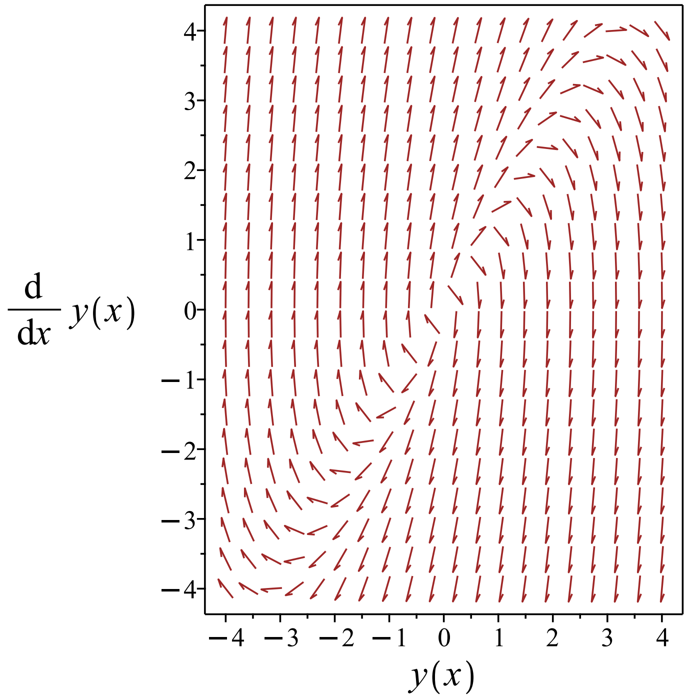


Figure 285: Slope field plot

Verification of solutions

$$y = e^{2x}(c_1 \cos(x) + c_2 \sin(x)) - \cos(x) e^{2x} \ln(\sec(x) + \tan(x))$$

Verified OK.

9.6.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 4y' + 5y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -4 \\ C &= 5 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 262: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4}{1}\,dx} \\
&= z_1 e^{2x} \\
&= z_1\left(\mathrm{e}^{2x}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \cos(x)\,e^{2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{4x}}{(y_1)^2}\,dx \\
&= y_1(\tan(x))
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\cos(x)\,e^{2x}\right) + c_2\left(\cos(x)\,e^{2x}(\tan(x))\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 4y' + 5y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \cos(x)\,e^{2x}c_1 + \sin(x)\,e^{2x}c_2$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x)\, e^{2x}$$
$$y_2 = \sin(x)\, e^{2x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x)\, e^{2x} & \sin(x)\, e^{2x} \\ \frac{d}{dx}\left(\cos(x)\, e^{2x}\right) & \frac{d}{dx}\left(\sin(x)\, e^{2x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x)\, e^{2x} & \sin(x)\, e^{2x} \\ -\sin(x)\, e^{2x} + 2\cos(x)\, e^{2x} & \cos(x)\, e^{2x} + 2\sin(x)\, e^{2x} \end{vmatrix}$$

Therefore

$$W = \left(\cos(x)\, e^{2x}\right)\left(\cos(x)\, e^{2x} + 2\sin(x)\, e^{2x}\right) - \left(\sin(x)\, e^{2x}\right)\left(-\sin(x)\, e^{2x} + 2\cos(x)\, e^{2x}\right)$$

Which simplifies to

$$W = e^{4x}\cos(x)^2 + \sin(x)^2 e^{4x}$$

Which simplifies to

$$W = e^{4x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(x) e^{4x} \tan(x)}{e^{4x}} dx$$

Which simplifies to

$$u_1 = -\int \tan(x) \sin(x) dx$$

Hence

$$u_1 = \sin(x) - \ln(\sec(x) + \tan(x))$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(x) e^{4x} \tan(x)}{e^{4x}} dx$$

Which simplifies to

$$u_2 = \int \sin(x) dx$$

Hence

$$u_2 = -\cos(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = (\sin(x) - \ln(\sec(x) + \tan(x))) \cos(x) e^{2x} - \sin(x) \cos(x) e^{2x}$$

Which simplifies to

$$y_p(x) = -\cos(x) e^{2x} \ln(\sec(x) + \tan(x))$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\cos(x) e^{2x} c_1 + \sin(x) e^{2x} c_2\right) + \left(-\cos(x) e^{2x} \ln(\sec(x) + \tan(x))\right) \end{aligned}$$

Which simplifies to

$$y = e^{2x}(c_1 \cos(x) + c_2 \sin(x)) - \cos(x) e^{2x} \ln(\sec(x) + \tan(x))$$

Summary

The solution(s) found are the following

$$y = e^{2x}(c_1 \cos(x) + c_2 \sin(x)) - \cos(x) e^{2x} \ln(\sec(x) + \tan(x)) \tag{1}$$

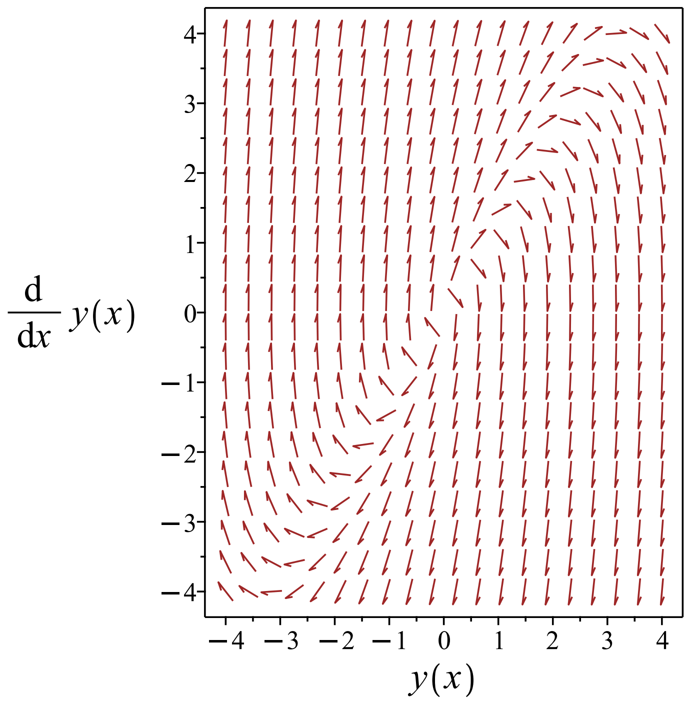


Figure 286: Slope field plot

Verification of solutions

$$y = e^{2x}(c_1 \cos(x) + c_2 \sin(x)) - \cos(x) e^{2x} \ln(\sec(x) + \tan(x))$$

Verified OK.

9.6.3 Maple step by step solution

Let's solve

$y'' - 4y' + 5y = e^{2x} \tan(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 4r + 5 = 0$

- Use quadratic formula to solve for r

  $r = \frac{4 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (2 - \mathrm{I}, 2 + \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)\, e^{2x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)\, e^{2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = \cos(x)\, e^{2x} c_1 + \sin(x)\, e^{2x} c_2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = e^{2x} \tan(x)\right]$

- ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x)\, e^{2x} & \sin(x)\, e^{2x} \\ -\sin(x)\, e^{2x} + 2\cos(x)\, e^{2x} & \cos(x)\, e^{2x} + 2\sin(x)\, e^{2x} \end{bmatrix}$

- ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = e^{4x}$

- ◦ Substitute functions into equation for $y_p(x)$

$$y_p(x) = -e^{2x}\left(\cos(x)\left(\int \tan(x)\sin(x)\,dx\right) - \sin(x)\left(\int \sin(x)\,dx\right)\right)$$

- Compute integrals

$$y_p(x) = -\cos(x)\,e^{2x}\ln(\sec(x) + \tan(x))$$

- Substitute particular solution into general solution to ODE

$$y = -\cos(x)\,e^{2x}\ln(\sec(x) + \tan(x)) + \cos(x)\,e^{2x}c_1 + \sin(x)\,e^{2x}c_2$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 28

```
dsolve(diff(y(x),x$2)-4*diff(y(x),x)+5*y(x)=exp(2*x)*tan(x),y(x), singsol=all)
```

$$y(x) = e^{2x}(\sin(x)\,c_2 + \cos(x)\,c_1 - \cos(x)\ln(\sec(x) + \tan(x)))$$

✓ Solution by Mathematica

Time used: 0.063 (sec). Leaf size: 29

```
DSolve[y''[x]-4*y'[x]+5*y[x]==Exp[2*x]*Tan[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{2x}(\cos(x)(-\operatorname{arctanh}(\sin(x))) + c_2\cos(x) + c_1\sin(x))$$

9.7 problem Problem 7

Internal problem ID [2780]
Internal file name [`OUTPUT/2272_Sunday_June_05_2022_02_57_14_AM_32443035/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.7, The Variation of Parameters Method. page 556
Problem number: Problem 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 9y = \frac{36}{4 - \cos(3x)^2}}$$

9.7.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 9, f(x) = -\frac{72}{-7+\cos(6x)}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 9y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 9$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 9\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 9$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(9)}\\ &= \pm 3i\end{aligned}$$

Hence

$$\lambda_1 = +3i$$
$$\lambda_2 = -3i$$

Which simplifies to

$$\lambda_1 = 3i$$
$$\lambda_2 = -3i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 3$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(3x) + c_2 \sin(3x))$$

Or

$$y = c_1 \cos(3x) + c_2 \sin(3x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(3x) + c_2 \sin(3x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(3x)$$
$$y_2 = \sin(3x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(3x) & \sin(3x) \\ \frac{d}{dx}(\cos(3x)) & \frac{d}{dx}(\sin(3x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(3x) & \sin(3x) \\ -3\sin(3x) & 3\cos(3x) \end{vmatrix}$$

Therefore

$$W = (\cos(3x))(3\cos(3x)) - (\sin(3x))(-3\sin(3x))$$

Which simplifies to

$$W = 3\cos(3x)^2 + 3\sin(3x)^2$$

Which simplifies to

$$W = 3$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-\frac{72\sin(3x)}{-7+\cos(6x)}}{3}\, dx$$

Which simplifies to

$$u_1 = -\int -\frac{12\sin(3x)}{\cos(3x)^2 - 4} dx$$

Hence

$$u_1 = \ln(\cos(3x) + 2) - \ln(\cos(3x) - 2)$$

And Eq. (3) becomes

$$u_2 = \int \frac{-\frac{72\cos(3x)}{-7+\cos(6x)}}{3}\, dx$$

Which simplifies to

$$u_2 = \int -\frac{12\cos(3x)}{\cos(3x)^2 - 4} dx$$

Hence

$$u_2 = \frac{4\sqrt{3}\,\arctan\left(\frac{\sqrt{3}\,\sin(3x)}{3}\right)}{3}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = (\ln(\cos(3x) + 2) - \ln(\cos(3x) - 2))\cos(3x) + \frac{4\sqrt{3}\,\arctan\left(\frac{\sqrt{3}\,\sin(3x)}{3}\right)\sin(3x)}{3}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 \cos(3x) + c_2 \sin(3x)) \\
&\quad + \left((\ln(\cos(3x)+2) - \ln(\cos(3x)-2)) \cos(3x) + \frac{4\sqrt{3}\, \arctan\left(\frac{\sqrt{3}\, \sin(3x)}{3}\right) \sin(3x)}{3} \right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 \cos(3x) + c_2 \sin(3x) + (\ln(\cos(3x)+2) - \ln(\cos(3x)-2)) \cos(3x) \\
&\quad + \frac{4\sqrt{3}\, \arctan\left(\frac{\sqrt{3}\, \sin(3x)}{3}\right) \sin(3x)}{3}
\end{aligned}
\tag{1}
$$

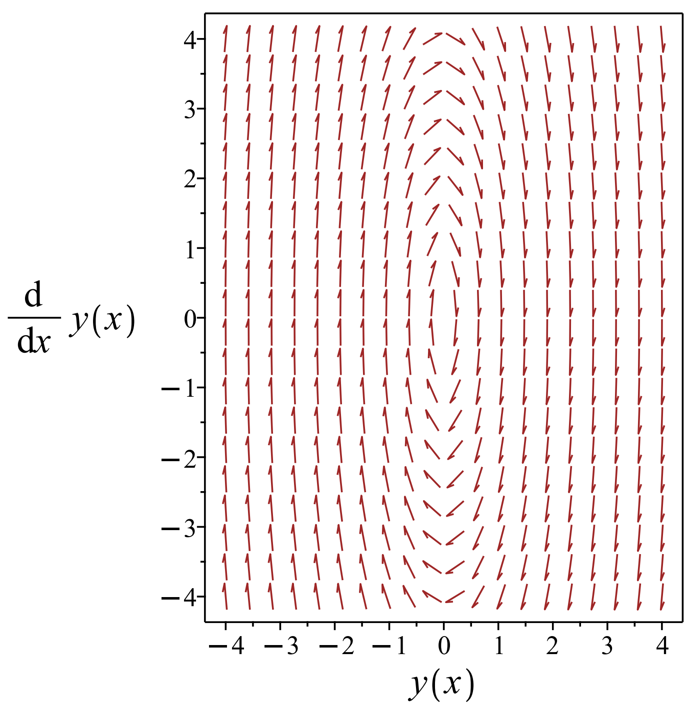


Figure 287: Slope field plot

Verification of solutions

$$y = c_1 \cos(3x) + c_2 \sin(3x) + (\ln(\cos(3x) + 2) - \ln(\cos(3x) - 2)) \cos(3x) + \frac{4\sqrt{3} \arctan\left(\frac{\sqrt{3}\sin(3x)}{3}\right) \sin(3x)}{3}$$

Verified OK.

9.7.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 9y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-9}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -9 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -9z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 264: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -9$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(3x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(3x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(3x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \cos(3x) \int \frac{1}{\cos(3x)^2}\,dx \\ &= \cos(3x) \left(\frac{\tan(3x)}{3}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(3x)) + c_2\left(\cos(3x)\left(\frac{\tan(3x)}{3}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 9y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(3x)$$
$$y_2 = \frac{\sin(3x)}{3}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(3x) & \frac{\sin(3x)}{3} \\ \frac{d}{dx}(\cos(3x)) & \frac{d}{dx}\left(\frac{\sin(3x)}{3}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(3x) & \frac{\sin(3x)}{3} \\ -3\sin(3x) & \cos(3x) \end{vmatrix}$$

Therefore

$$W = (\cos(3x))(\cos(3x)) - \left(\frac{\sin(3x)}{3}\right)(-3\sin(3x))$$

Which simplifies to

$$W = \cos(3x)^2 + \sin(3x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-\frac{24\sin(3x)}{-7+\cos(6x)}}{1} dx$$

Which simplifies to

$$u_1 = -\int -\frac{12\sin(3x)}{\cos(3x)^2 - 4} dx$$

Hence

$$u_1 = \ln(\cos(3x) + 2) - \ln(\cos(3x) - 2)$$

And Eq. (3) becomes

$$u_2 = \int \frac{-\frac{72\cos(3x)}{-7+\cos(6x)}}{1} dx$$

Which simplifies to

$$u_2 = \int -\frac{36\cos(3x)}{\cos(3x)^2 - 4} dx$$

Hence

$$u_2 = 4\sqrt{3}\,\arctan\left(\frac{\sqrt{3}\,\sin(3x)}{3}\right)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = (\ln(\cos(3x)+2) - \ln(\cos(3x)-2))\cos(3x) + \frac{4\sqrt{3}\arctan\left(\frac{\sqrt{3}\sin(3x)}{3}\right)\sin(3x)}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\cos(3x) + \frac{c_2\sin(3x)}{3}\right) \\ &\quad + \left((\ln(\cos(3x)+2) - \ln(\cos(3x)-2))\cos(3x) + \frac{4\sqrt{3}\arctan\left(\frac{\sqrt{3}\sin(3x)}{3}\right)\sin(3x)}{3}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = {} & c_1\cos(3x) + \frac{c_2\sin(3x)}{3} + (\ln(\cos(3x)+2) - \ln(\cos(3x)-2))\cos(3x) \\ & + \frac{4\sqrt{3}\arctan\left(\frac{\sqrt{3}\sin(3x)}{3}\right)\sin(3x)}{3} \end{aligned} \tag{1}$$

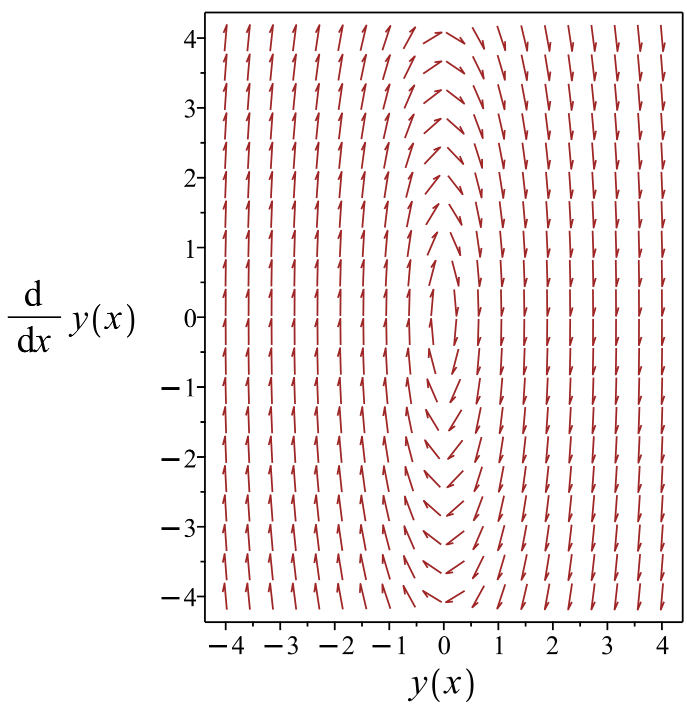


Figure 288: Slope field plot

Verification of solutions

$$y = c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3} + (\ln(\cos(3x) + 2) - \ln(\cos(3x) - 2)) \cos(3x) + \frac{4\sqrt{3}\, \arctan\left(\frac{\sqrt{3}\, \sin(3x)}{3}\right) \sin(3x)}{3}$$

Verified OK.

9.7.3 Maple step by step solution

Let's solve

$y'' + 9y = -\frac{72}{-7+\cos(6x)}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 9 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-36})}{2}$

- Roots of the characteristic polynomial

  $r = (-3\,\mathrm{I}, 3\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(3x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(3x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(3x) + c_2 \sin(3x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = -\frac{72}{-7+\cos(6x)}\right]$

  ◦ Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(3x) & \sin(3x) \\ -3\sin(3x) & 3\cos(3x) \end{bmatrix}$$

- Compute Wronskian

  $$W(y_1(x), y_2(x)) = 3$$

- Substitute functions into equation for $y_p(x)$

  $$y_p(x) = 12\cos(3x)\left(\int \frac{\sin(3x)}{\cos(3x)^2-4}dx\right) - 12\sin(3x)\left(\int \frac{\cos(3x)}{\cos(3x)^2-4}dx\right)$$

- Compute integrals

  $$y_p(x) = \frac{4\sqrt{3}\arctan\left(\frac{\sqrt{3}\sin(3x)}{3}\right)\sin(3x)}{3} - \cos(3x)\left(-\ln(\cos(3x)+2) + \ln(\cos(3x)-2)\right)$$

- Substitute particular solution into general solution to ODE

  $$y = c_1\cos(3x) + c_2\sin(3x) + \frac{4\sqrt{3}\arctan\left(\frac{\sqrt{3}\sin(3x)}{3}\right)\sin(3x)}{3} - \cos(3x)\left(-\ln(\cos(3x)+2) + \ln(\cos(3$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 61

```
dsolve(diff(y(x),x$2)+9*y(x)=36/(4-cos(3*x)^2),y(x), singsol=all)
```

$$y(x) = \frac{4\sqrt{3}\arctan\left(\frac{\sqrt{3}\sin(3x)}{3}\right)\sin(3x)}{3} + \sin(3x)c_2 + \cos(3x)\ln(\cos(3x)+2) - \cos(3x)\ln(\cos(3x)-2) + \cos(3x)c_1$$

✓ Solution by Mathematica

Time used: 0.227 (sec). Leaf size: 61

```
DSolve[y''[x]+9*y[x]==36/(4-Cos[3*x]^2),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{4\sin(3x)\arctan\left(\frac{\sin(3x)}{\sqrt{3}}\right)}{\sqrt{3}} + c_2\sin(3x) + \cos(3x)(-\log(2-\cos(3x)) + \log(\cos(3x)+2) + c_1)$$

9.8 problem Problem 8

Internal problem ID [2781]
Internal file name [`OUTPUT/2273_Sunday_June_05_2022_02_57_17_AM_45743087/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.7, The Variation of Parameters Method. page 556
Problem number: Problem 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 10y' + 25y = \frac{2\,e^{5x}}{x^2+4}}$$

9.8.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -10, C = 25, f(x) = \frac{2\,e^{5x}}{x^2+4}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$.

y_h is the solution to

$$y'' - 10y' + 25y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -10, C = 25$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 10\lambda\, e^{\lambda x} + 25\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 10\lambda + 25 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -10, C = 25$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{10}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(-10)^2 - (4)(1)(25)}\\ &= 5\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -5$. Therefore the solution is

$$y = c_1 e^{5x} + c_2 x\, e^{5x} \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{5x} + c_2 x\, e^{5x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \mathrm{e}^{5x} \\
y_2 &= x\,\mathrm{e}^{5x}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$
W = \begin{vmatrix} \mathrm{e}^{5x} & x\,\mathrm{e}^{5x} \\ \frac{d}{dx}\left(\mathrm{e}^{5x}\right) & \frac{d}{dx}\left(x\,\mathrm{e}^{5x}\right) \end{vmatrix}
$$

Which gives

$$
W = \begin{vmatrix} \mathrm{e}^{5x} & x\,\mathrm{e}^{5x} \\ 5\,\mathrm{e}^{5x} & \mathrm{e}^{5x} + 5x\,\mathrm{e}^{5x} \end{vmatrix}
$$

Therefore

$$
W = \left(\mathrm{e}^{5x}\right)\left(\mathrm{e}^{5x} + 5x\,\mathrm{e}^{5x}\right) - \left(x\,\mathrm{e}^{5x}\right)\left(5\,\mathrm{e}^{5x}\right)
$$

Which simplifies to

$$
W = \mathrm{e}^{10x}
$$

Which simplifies to

$$
W = \mathrm{e}^{10x}
$$

Therefore Eq. (2) becomes

$$
u_1 = -\int \frac{\frac{2x\,\mathrm{e}^{10x}}{x^2+4}}{\mathrm{e}^{10x}}\,dx
$$

Which simplifies to

$$u_1 = -\int \frac{2x}{x^2+4} dx$$

Hence

$$u_1 = -\ln\left(x^2+4\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{2\,\mathrm{e}^{10x}}{x^2+4}}{\mathrm{e}^{10x}}\, dx$$

Which simplifies to

$$u_2 = \int \frac{2}{x^2+4} dx$$

Hence

$$u_2 = \arctan\left(\frac{x}{2}\right)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\ln\left(x^2+4\right)\mathrm{e}^{5x} + \arctan\left(\frac{x}{2}\right) x\,\mathrm{e}^{5x}$$

Which simplifies to

$$y_p(x) = \mathrm{e}^{5x}\left(-\ln\left(x^2+4\right) + x \arctan\left(\frac{x}{2}\right)\right)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{5x} + c_2 x\,\mathrm{e}^{5x}\right) + \left(\mathrm{e}^{5x}\left(-\ln\left(x^2+4\right) + x \arctan\left(\frac{x}{2}\right)\right)\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{5x}(c_2 x + c_1) + \mathrm{e}^{5x}\left(-\ln\left(x^2+4\right) + x \arctan\left(\frac{x}{2}\right)\right)$$

Summary
The solution(s) found are the following

$$y = e^{5x}(c_2x + c_1) + e^{5x}\left(-\ln\left(x^2 + 4\right) + x\arctan\left(\frac{x}{2}\right)\right) \tag{1}$$

Verification of solutions

$$y = e^{5x}(c_2x + c_1) + e^{5x}\left(-\ln\left(x^2 + 4\right) + x\arctan\left(\frac{x}{2}\right)\right)$$

Verified OK.

9.8.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\,y' + \frac{\left(p(x)^2 + p'(x)\right)y}{2} = f(x)$$

Where $p(x) = -10$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int -10\,dx} \\ &= e^{-5x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)y)'' &= \frac{2\,e^{-5x}e^{5x}}{x^2 + 4} \\ \left(e^{-5x}y\right)'' &= \frac{2\,e^{-5x}e^{5x}}{x^2 + 4} \end{aligned}$$

Integrating once gives

$$\left(e^{-5x}y\right)' = \arctan\left(\frac{x}{2}\right) + c_1$$

Integrating again gives

$$\left(e^{-5x}y\right) = c_1x + x\arctan\left(\frac{x}{2}\right) + 2\ln(2) - \ln\left(x^2 + 4\right) + c_2$$

Hence the solution is

$$y = \frac{c_1x + x\arctan\left(\frac{x}{2}\right) + 2\ln(2) - \ln\left(x^2 + 4\right) + c_2}{e^{-5x}}$$

Or

$$y = c_1 x\,\mathrm{e}^{5x} + \arctan\left(\frac{x}{2}\right) x\,\mathrm{e}^{5x} + c_2\mathrm{e}^{5x} - \ln\left(x^2+4\right)\mathrm{e}^{5x} + 2\ln\left(2\right)\mathrm{e}^{5x}$$

Summary
The solution(s) found are the following

$$y = c_1 x\,\mathrm{e}^{5x} + \arctan\left(\frac{x}{2}\right) x\,\mathrm{e}^{5x} + c_2\mathrm{e}^{5x} - \ln\left(x^2+4\right)\mathrm{e}^{5x} + 2\ln\left(2\right)\mathrm{e}^{5x} \tag{1}$$

Verification of solutions

$$y = c_1 x\,\mathrm{e}^{5x} + \arctan\left(\frac{x}{2}\right) x\,\mathrm{e}^{5x} + c_2\mathrm{e}^{5x} - \ln\left(x^2+4\right)\mathrm{e}^{5x} + 2\ln\left(2\right)\mathrm{e}^{5x}$$

Verified OK.

9.8.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 10y' + 25y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -10 \\ C &= 25 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 0 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = 0 \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 266: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-10}{1}\,dx} \\
&= z_1 e^{5x} \\
&= z_1 \left(\mathrm{e}^{5x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{5x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-10}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{10x}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{5x}\right) + c_2\left(\mathrm{e}^{5x}(x)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 10y' + 25y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{5x} + c_2 x \,\mathrm{e}^{5x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= \mathrm{e}^{5x} \\ y_2 &= x\,\mathrm{e}^{5x} \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{5x} & x\,\mathrm{e}^{5x} \\ \frac{d}{dx}(\mathrm{e}^{5x}) & \frac{d}{dx}(x\,\mathrm{e}^{5x}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{5x} & x\, e^{5x} \\ 5\, e^{5x} & e^{5x} + 5x\, e^{5x} \end{vmatrix}$$

Therefore

$$W = \left(e^{5x}\right) \left(e^{5x} + 5x\, e^{5x}\right) - \left(x\, e^{5x}\right) \left(5\, e^{5x}\right)$$

Which simplifies to

$$W = e^{10x}$$

Which simplifies to

$$W = e^{10x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{2x\, e^{10x}}{x^2+4}}{e^{10x}}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{2x}{x^2 + 4} dx$$

Hence

$$u_1 = -\ln\left(x^2 + 4\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{2\, e^{10x}}{x^2+4}}{e^{10x}}\, dx$$

Which simplifies to

$$u_2 = \int \frac{2}{x^2 + 4} dx$$

Hence

$$u_2 = \arctan\left(\frac{x}{2}\right)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\ln\left(x^2+4\right) e^{5x} + \arctan\left(\frac{x}{2}\right) x\, e^{5x}$$

Which simplifies to

$$y_p(x) = e^{5x}\left(-\ln\left(x^2+4\right) + x \arctan\left(\frac{x}{2}\right)\right)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{5x} + c_2 x\, e^{5x}\right) + \left(e^{5x}\left(-\ln\left(x^2+4\right) + x \arctan\left(\frac{x}{2}\right)\right)\right) \end{aligned}$$

Which simplifies to

$$y = e^{5x}(c_2 x + c_1) + e^{5x}\left(-\ln\left(x^2+4\right) + x \arctan\left(\frac{x}{2}\right)\right)$$

Summary

The solution(s) found are the following

$$y = e^{5x}(c_2 x + c_1) + e^{5x}\left(-\ln\left(x^2+4\right) + x \arctan\left(\frac{x}{2}\right)\right) \tag{1}$$

Verification of solutions

$$y = e^{5x}(c_2 x + c_1) + e^{5x}\left(-\ln\left(x^2+4\right) + x \arctan\left(\frac{x}{2}\right)\right)$$

Verified OK.

9.8.4 Maple step by step solution

Let's solve

$y'' - 10y' + 25y = \frac{2\,e^{5x}}{x^2+4}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 10r + 25 = 0$

- Factor the characteristic polynomial

  $(r-5)^2 = 0$

- Root of the characteristic polynomial

  $r = 5$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{5x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,\mathrm{e}^{5x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{5x} + c_2 x\,\mathrm{e}^{5x} + y_p(x)$

- ☐ Find a particular solution $y_p(x)$ of the ODE
  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = \frac{2\,\mathrm{e}^{5x}}{x^2+4}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{5x} & x\,\mathrm{e}^{5x} \\ 5\,\mathrm{e}^{5x} & \mathrm{e}^{5x} + 5x\,\mathrm{e}^{5x} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = \mathrm{e}^{10x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -2\,\mathrm{e}^{5x}\left(\int \frac{x}{x^2+4}dx - \left(\int \frac{1}{x^2+4}dx\right)x\right)$

  - Compute integrals

    $y_p(x) = \mathrm{e}^{5x}\left(-\ln(x^2+4) + x\arctan\left(\frac{x}{2}\right)\right)$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{5x} + c_2 x\,\mathrm{e}^{5x} + \mathrm{e}^{5x}\left(-\ln(x^2+4) + x\arctan\left(\frac{x}{2}\right)\right)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 28

```
dsolve(diff(y(x),x$2)-10*diff(y(x),x)+25*y(x)=2*exp(5*x)/(4+x^2),y(x), singsol=all)
```

$$y(x) = e^{5x}\left(c_2 + c_1x - \ln\left(x^2+4\right) + x\arctan\left(\frac{x}{2}\right)\right)$$

✓ Solution by Mathematica

Time used: 0.039 (sec). Leaf size: 34

```
DSolve[y''[x]-10*y'[x]+25*y[x]==2*Exp[5*x]/(4+x^2),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{5x}\left(x\arctan\left(\frac{x}{2}\right) - \log\left(x^2+4\right) + c_2x + c_1\right)$$

9.9 problem Problem 9

Internal problem ID [2782]
Internal file name [`OUTPUT/2274_Sunday_June_05_2022_02_57_19_AM_28200098/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.7, The Variation of Parameters Method. page 556
Problem number: Problem 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 6y' + 13y = 4\,e^{3x}\sec(2x)^2}$$

9.9.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -6, C = 13, f(x) = 4\,e^{3x}\sec(2x)^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 6y' + 13y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -6, C = 13$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 6\lambda \,\mathrm{e}^{\lambda x} + 13 \,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 6\lambda + 13 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -6, C = 13$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{6}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-6^2 - (4)\,(1)\,(13)}\\ &= 3 \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= 3 + 2i\\ \lambda_2 &= 3 - 2i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 3 + 2i\\ \lambda_2 &= 3 - 2i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 3$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{3x}(c_1 \cos(2x) + c_2 \sin(2x))$$

Therefore the homogeneous solution y_h is

$$y_h = e^{3x}(c_1 \cos(2x) + c_2 \sin(2x))$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(2x)\, e^{3x}$$
$$y_2 = \sin(2x)\, e^{3x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(2x)\, e^{3x} & \sin(2x)\, e^{3x} \\ \frac{d}{dx}(\cos(2x)\, e^{3x}) & \frac{d}{dx}(\sin(2x)\, e^{3x}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(2x)\, e^{3x} & \sin(2x)\, e^{3x} \\ -2\sin(2x)\, e^{3x} + 3\cos(2x)\, e^{3x} & 2\cos(2x)\, e^{3x} + 3\sin(2x)\, e^{3x} \end{vmatrix}$$

Therefore

$$
\begin{aligned}
W = &\left(\cos(2x)\,\mathrm{e}^{3x}\right)\left(2\cos(2x)\,\mathrm{e}^{3x} + 3\sin(2x)\,\mathrm{e}^{3x}\right) \\
&- \left(\sin(2x)\,\mathrm{e}^{3x}\right)\left(-2\sin(2x)\,\mathrm{e}^{3x} + 3\cos(2x)\,\mathrm{e}^{3x}\right)
\end{aligned}
$$

Which simplifies to

$$
W = 2\sin(2x)^2\,\mathrm{e}^{6x} + 2\,\mathrm{e}^{6x}\cos(2x)^2
$$

Which simplifies to

$$
W = 2\,\mathrm{e}^{6x}
$$

Therefore Eq. (2) becomes

$$
u_1 = -\int \frac{4\sin(2x)\,\mathrm{e}^{6x}\sec(2x)^2}{2\,\mathrm{e}^{6x}}\,dx
$$

Which simplifies to

$$
u_1 = -\int 2\tan(2x)\sec(2x)\,dx
$$

Hence

$$
u_1 = -\sec(2x)
$$

And Eq. (3) becomes

$$
u_2 = \int \frac{4\cos(2x)\,\mathrm{e}^{6x}\sec(2x)^2}{2\,\mathrm{e}^{6x}}\,dx
$$

Which simplifies to

$$
u_2 = \int 2\sec(2x)\,dx
$$

Hence

$$
u_2 = \ln(\sec(2x) + \tan(2x))
$$

Therefore the particular solution, from equation (1) is

$$
y_p(x) = -\sec(2x)\cos(2x)\,\mathrm{e}^{3x} + \ln(\sec(2x) + \tan(2x))\sin(2x)\,\mathrm{e}^{3x}
$$

Which simplifies to

$$
y_p(x) = e^{3x}(-1 + \ln(\sec(2x) + \tan(2x))\sin(2x))
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(e^{3x}(c_1 \cos(2x) + c_2 \sin(2x))\right) + \left(e^{3x}(-1 + \ln(\sec(2x) + \tan(2x))\sin(2x))\right)
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = e^{3x}(c_1 \cos(2x) + c_2 \sin(2x)) + e^{3x}(-1 + \ln(\sec(2x) + \tan(2x))\sin(2x)) \quad (1)
$$

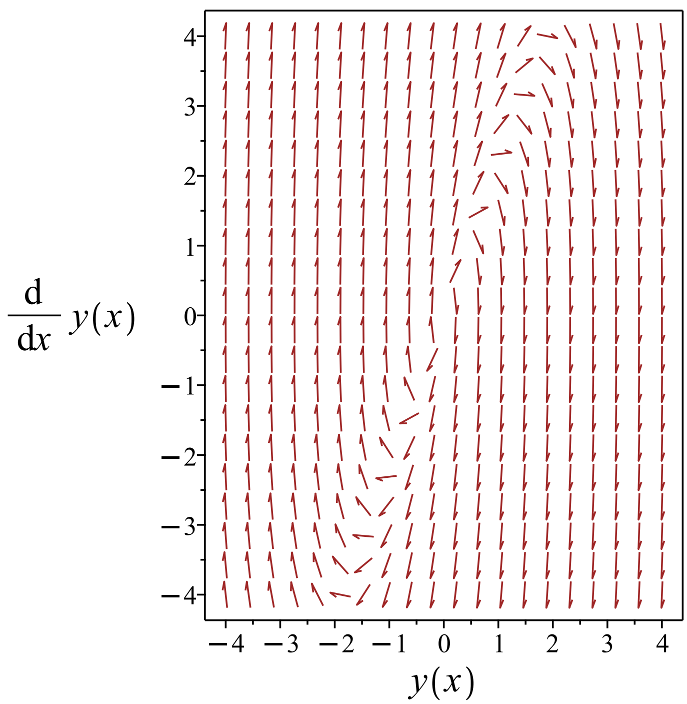


Figure 289: Slope field plot

<u>Verification of solutions</u>

$$
y = e^{3x}(c_1 \cos(2x) + c_2 \sin(2x)) + e^{3x}(-1 + \ln(\sec(2x) + \tan(2x))\sin(2x))
$$

Verified OK.

9.9.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 6y' + 13y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -6 \\ C &= 13 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 268: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(2x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-6}{1}\,dx}
\end{aligned}
$$

$$
\begin{aligned}
&= z_1 e^{3x} \\
&= z_1 \left(\mathrm{e}^{3x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \cos(2x)\,\mathrm{e}^{3x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-6}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{6x}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{\tan(2x)}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\cos(2x)\,\mathrm{e}^{3x}\right) + c_2 \left(\cos(2x)\,\mathrm{e}^{3x} \left(\frac{\tan(2x)}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 6y' + 13y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = \mathrm{e}^{3x} \cos(2x)\, c_1 + \frac{\sin(2x)\,\mathrm{e}^{3x} c_2}{2}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(2x)\,\mathrm{e}^{3x}$$
$$y_2 = \frac{\sin(2x)\,\mathrm{e}^{3x}}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(2x)\,\mathrm{e}^{3x} & \frac{\sin(2x)\mathrm{e}^{3x}}{2} \\ \frac{d}{dx}\left(\cos(2x)\,\mathrm{e}^{3x}\right) & \frac{d}{dx}\left(\frac{\sin(2x)\mathrm{e}^{3x}}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(2x)\,\mathrm{e}^{3x} & \frac{\sin(2x)\mathrm{e}^{3x}}{2} \\ -2\sin(2x)\,\mathrm{e}^{3x} + 3\cos(2x)\,\mathrm{e}^{3x} & \cos(2x)\,\mathrm{e}^{3x} + \frac{3\sin(2x)\mathrm{e}^{3x}}{2} \end{vmatrix}$$

Therefore

$$W = \left(\cos(2x)\,\mathrm{e}^{3x}\right)\left(\cos(2x)\,\mathrm{e}^{3x} + \frac{3\sin(2x)\,\mathrm{e}^{3x}}{2}\right) - \left(\frac{\sin(2x)\,\mathrm{e}^{3x}}{2}\right)\left(-2\sin(2x)\,\mathrm{e}^{3x} + 3\cos(2x)\,\mathrm{e}^{3x}\right)$$

Which simplifies to

$$W = \sin(2x)^2 e^{6x} + e^{6x} \cos(2x)^2$$

Which simplifies to

$$W = e^{6x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{2 \sin(2x) e^{6x} \sec(2x)^2}{e^{6x}} dx$$

Which simplifies to

$$u_1 = -\int 2 \tan(2x) \sec(2x) dx$$

Hence

$$u_1 = -\sec(2x)$$

And Eq. (3) becomes

$$u_2 = \int \frac{4 \cos(2x) e^{6x} \sec(2x)^2}{e^{6x}} dx$$

Which simplifies to

$$u_2 = \int 4 \sec(2x) dx$$

Hence

$$u_2 = 2 \ln(\sec(2x) + \tan(2x))$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\sec(2x) \cos(2x) e^{3x} + \ln(\sec(2x) + \tan(2x)) \sin(2x) e^{3x}$$

Which simplifies to

$$y_p(x) = e^{3x}(-1 + \ln(\sec(2x) + \tan(2x)) \sin(2x))$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(e^{3x} \cos(2x) c_1 + \frac{\sin(2x) e^{3x} c_2}{2}\right) + \left(e^{3x}(-1 + \ln(\sec(2x) + \tan(2x)) \sin(2x))\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = e^{3x} \cos(2x) c_1 + \frac{\sin(2x) e^{3x} c_2}{2} + e^{3x}(-1 + \ln(\sec(2x) + \tan(2x)) \sin(2x)) \quad (1)
$$

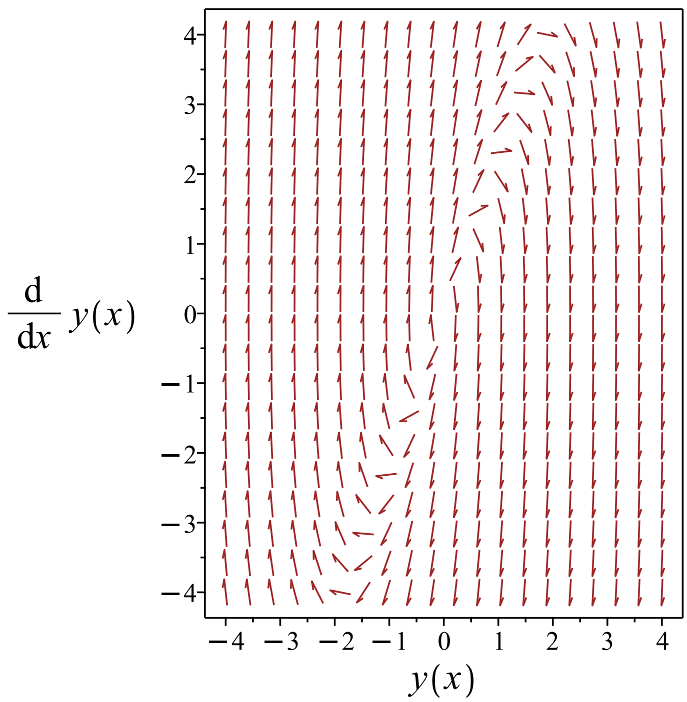


Figure 290: Slope field plot

Verification of solutions

$$
y = e^{3x} \cos(2x) c_1 + \frac{\sin(2x) e^{3x} c_2}{2} + e^{3x}(-1 + \ln(\sec(2x) + \tan(2x)) \sin(2x))
$$

Verified OK.

9.9.3 Maple step by step solution

Let's solve

$y'' - 6y' + 13y = 4\,\mathrm{e}^{3x} \sec(2x)^2$

- Highest derivative means the order of the ODE is 2

y''

- Characteristic polynomial of homogeneous ODE

$r^2 - 6r + 13 = 0$

- Use quadratic formula to solve for r

$r = \frac{6 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

$r = (3 - 2\,\mathrm{I}, 3 + 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

$y_1(x) = \cos(2x)\,\mathrm{e}^{3x}$

- 2nd solution of the homogeneous ODE

$y_2(x) = \sin(2x)\,\mathrm{e}^{3x}$

- General solution of the ODE

$y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

$y = \mathrm{e}^{3x} \cos(2x)\, c_1 + \sin(2x)\,\mathrm{e}^{3x} c_2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 4\,\mathrm{e}^{3x} \sec(2x)^2\right]$

  - Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(2x)\,\mathrm{e}^{3x} & \sin(2x)\,\mathrm{e}^{3x} \\ -2\sin(2x)\,\mathrm{e}^{3x} + 3\cos(2x)\,\mathrm{e}^{3x} & 2\cos(2x)\,\mathrm{e}^{3x} + 3\sin(2x)\,\mathrm{e}^{3x} \end{bmatrix}$

  - Compute Wronskian

  $W(y_1(x), y_2(x)) = 2\,\mathrm{e}^{6x}$

  - Substitute functions into equation for $y_p(x)$

$$y_p(x) = 2\,e^{3x}\left(-\cos(2x)\left(\int \tan(2x)\sec(2x)\,dx\right) + \sin(2x)\left(\int \sec(2x)\,dx\right)\right)$$

- Compute integrals

$$y_p(x) = e^{3x}(-1 + \ln(\sec(2x) + \tan(2x))\sin(2x))$$

- Substitute particular solution into general solution to ODE

$$y = e^{3x}\cos(2x)\,c_1 + \sin(2x)\,e^{3x}c_2 + e^{3x}(-1 + \ln(\sec(2x) + \tan(2x))\sin(2x))$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 38

```
dsolve(diff(y(x),x$2)-6*diff(y(x),x)+13*y(x)=4*exp(3*x)*sec(2*x)^2,y(x), singsol=all)
```

$$y(x) = e^{3x}(\sin(2x)\,c_2 + c_1\cos(2x) - 1 + \sin(2x)\ln(\sec(2x) + \tan(2x)))$$

✓ Solution by Mathematica

Time used: 0.122 (sec). Leaf size: 37

```
DSolve[y''[x]-6*y'[x]+13*y[x]==4*Exp[3*x]*Sec[2*x]^2,y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to e^{3x}\left(c_2\cos(2x) + \sin(2x)\coth^{-1}(\sin(2x)) + c_1\sin(2x) - 1\right)$$

9.10 problem Problem 10

Internal problem ID [2783]
Internal file name [`OUTPUT/2275_Sunday_June_05_2022_02_57_21_AM_64945049/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.7, The Variation of Parameters Method. page 556
Problem number: Problem 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' + y = \sec(x) + 4\,\mathrm{e}^{x}}$$

9.10.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = \sec(x) + 4\,\mathrm{e}^{x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\
&= \pm i
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= +i \\
\lambda_2 &= -i
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= i \\
\lambda_2 &= -i
\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x)$$
$$y_2 = \sin(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ \frac{d}{dx}(\cos(x)) & \frac{d}{dx}(\sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

Therefore

$$W = (\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))$$

Which simplifies to

$$W = \cos(x)^2 + \sin(x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(x)(\sec(x) + 4\,\mathrm{e}^x)}{1}\,dx$$

Which simplifies to

$$u_1 = -\int (4\,\mathrm{e}^x \sin(x) + \tan(x))\,dx$$

Hence

$$u_1 = 2\cos(x)\,\mathrm{e}^x - 2\,\mathrm{e}^x \sin(x) + \ln(\cos(x))$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(x)(\sec(x) + 4\,\mathrm{e}^x)}{1}\,dx$$

Which simplifies to

$$u_2 = \int (4\cos(x)\,\mathrm{e}^x + 1)\,dx$$

Hence

$$u_2 = x + 2\cos(x)\,\mathrm{e}^x + 2\,\mathrm{e}^x \sin(x)$$

Which simplifies to

$$u_1 = (2\cos(x) - 2\sin(x))\,\mathrm{e}^x + \ln(\cos(x))$$
$$u_2 = x + (2\cos(x) + 2\sin(x))\,\mathrm{e}^x$$

Therefore the particular solution, from equation (1) is

$$\begin{aligned} y_p(x) = {} & ((2\cos(x) - 2\sin(x))\,\mathrm{e}^x + \ln(\cos(x)))\cos(x) \\ & + (x + (2\cos(x) + 2\sin(x))\,\mathrm{e}^x)\sin(x) \end{aligned}$$

Which simplifies to

$$y_p(x) = \cos(x)\ln(\cos(x)) + x\sin(x) + 2\,\mathrm{e}^{x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1\cos(x) + c_2\sin(x)) + (\cos(x)\ln(\cos(x)) + x\sin(x) + 2\,\mathrm{e}^{x}) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1\cos(x) + c_2\sin(x) + \cos(x)\ln(\cos(x)) + x\sin(x) + 2\,\mathrm{e}^{x} \tag{1}$$

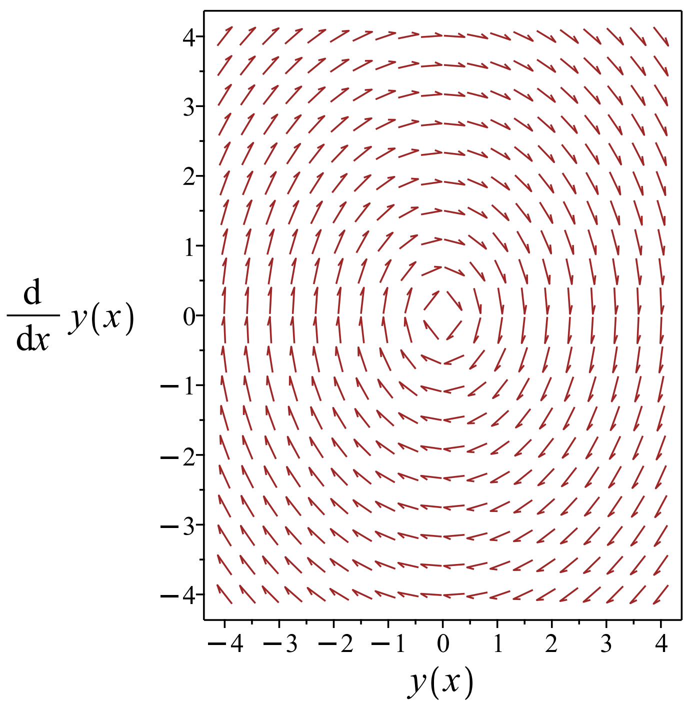


Figure 291: Slope field plot

Verification of solutions

$$y = c_1\cos(x) + c_2\sin(x) + \cos(x)\ln(\cos(x)) + x\sin(x) + 2\,\mathrm{e}^{x}$$

Verified OK.

9.10.2 Solving using Kovacic algorithm

Writing the ode as

$$
\begin{aligned}
y'' + y &= 0 && (1)\\
Ay'' + By' + Cy &= 0 && (2)
\end{aligned}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1\\
B &= 0 \qquad (3)\\
C &= 1
\end{aligned}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = ye^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$
z''(x) = rz(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} && (5)\\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-1}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -1\\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = -z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 270: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \cos(x)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \cos(x)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= \cos(x) \int \frac{1}{\cos(x)^2}\, dx \\
&= \cos(x)\,(\tan(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\cos(x)) + c_2(\cos(x)\,(\tan(x)))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \cos(x) + c_2 \sin(x)
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x)$$
$$y_2 = \sin(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ \frac{d}{dx}(\cos(x)) & \frac{d}{dx}(\sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

Therefore

$$W = (\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))$$

Which simplifies to

$$W = \cos(x)^2 + \sin(x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(x)(\sec(x) + 4\,\mathrm{e}^x)}{1}\,dx$$

Which simplifies to

$$u_1 = -\int (4\,\mathrm{e}^x \sin(x) + \tan(x))\,dx$$

Hence

$$u_1 = 2\cos(x)\,\mathrm{e}^x - 2\,\mathrm{e}^x \sin(x) + \ln(\cos(x))$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(x)(\sec(x) + 4\,\mathrm{e}^x)}{1}\,dx$$

Which simplifies to

$$u_2 = \int (4\cos(x)\,\mathrm{e}^x + 1)\,dx$$

Hence

$$u_2 = x + 2\cos(x)\,\mathrm{e}^x + 2\,\mathrm{e}^x \sin(x)$$

Which simplifies to

$$\begin{aligned}
u_1 &= (2\cos(x) - 2\sin(x))\,\mathrm{e}^x + \ln(\cos(x)) \\
u_2 &= x + (2\cos(x) + 2\sin(x))\,\mathrm{e}^x
\end{aligned}$$

Therefore the particular solution, from equation (1) is

$$\begin{aligned}
y_p(x) = {} & ((2\cos(x) - 2\sin(x))\,\mathrm{e}^x + \ln(\cos(x)))\cos(x) \\
& + (x + (2\cos(x) + 2\sin(x))\,\mathrm{e}^x)\sin(x)
\end{aligned}$$

Which simplifies to

$$y_p(x) = \cos(x)\ln(\cos(x)) + x\sin(x) + 2\,\mathrm{e}^x$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 \cos(x) + c_2 \sin(x)) + (\cos(x) \ln(\cos(x)) + x \sin(x) + 2 e^{x})
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \cos(x) + c_2 \sin(x) + \cos(x) \ln(\cos(x)) + x \sin(x) + 2 e^{x} \tag{1}
$$

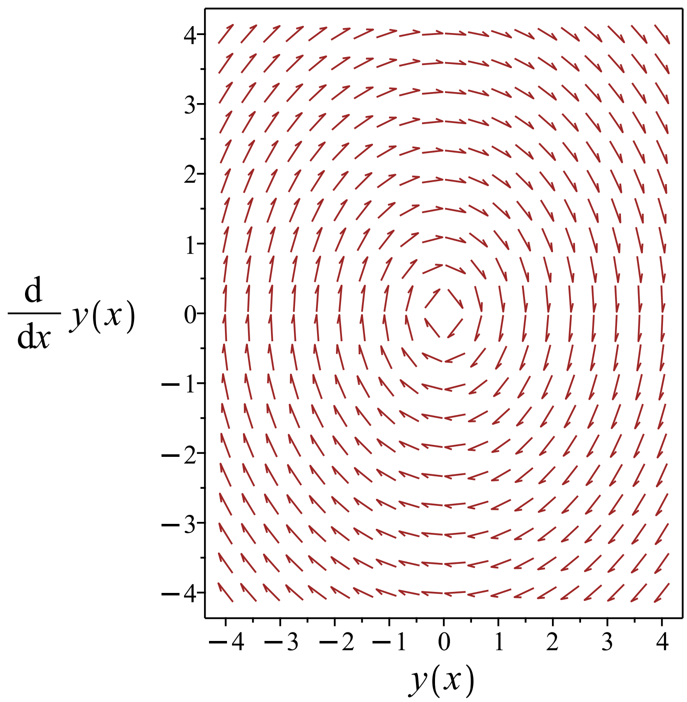


Figure 292: Slope field plot

Verification of solutions

$$
y = c_1 \cos(x) + c_2 \sin(x) + \cos(x) \ln(\cos(x)) + x \sin(x) + 2 e^{x}
$$

Verified OK.

9.10.3 Maple step by step solution

Let's solve

$y'' + y = \sec(x) + 4\,\mathrm{e}^{x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ∘ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \sec(x) + 4\,\mathrm{e}^{x}\right]$

- ∘ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$

- ∘ Compute Wronskian

  $W(y_1(x), y_2(x)) = 1$

- ∘ Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\cos(x)\left(\int (4\,e^x \sin(x) + \tan(x))\, dx\right) + \sin(x)\left(\int (4\cos(x)\,e^x + 1)\, dx\right)$$

- Compute integrals

$$y_p(x) = \cos(x)\ln(\cos(x)) + x\sin(x) + 2\,e^x$$

- Substitute particular solution into general solution to ODE

$$y = c_1\cos(x) + c_2\sin(x) + \cos(x)\ln(\cos(x)) + x\sin(x) + 2\,e^x$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 25

```
dsolve(diff(y(x),x$2)+y(x)=sec(x)+4*exp(x),y(x), singsol=all)
```

$$y(x) = \cos(x)\ln(\cos(x)) + \cos(x)\,c_1 + \sin(x)\,(c_2 + x) + 2\,e^x$$

✓ Solution by Mathematica

Time used: 0.065 (sec). Leaf size: 91

```
DSolve[y''[x]+y[x]==4*Exp[x]*Sec[x],y[x],x,IncludeSingularSolutions -> True]
```

$$\begin{aligned} y(x) \to & -4ie^x \operatorname{Hypergeometric2F1}\left(-\frac{i}{2}, 1, 1-\frac{i}{2}, -e^{2ix}\right)\cos(x) \\ & + \left(\frac{8}{5}+\frac{4i}{5}\right)e^{(1+2i)x}\operatorname{Hypergeometric2F1}\left(1, 1-\frac{i}{2}, 2-\frac{i}{2}, -e^{2ix}\right)\cos(x) \\ & + 4e^x\sin(x) + c_1\cos(x) + c_2\sin(x) \end{aligned}$$

9.11 problem Problem 11

Internal problem ID [2784]
Internal file name [`OUTPUT/2276_Sunday_June_05_2022_02_57_23_AM_99059270/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.7, The Variation of Parameters Method. page 556
Problem number: Problem 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y = \csc(x) + 2x^2 + 5x + 1}$$

9.11.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = \csc(x) + 2x^2 + 5x + 1$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\ &= \pm i \end{aligned}$$

Hence

$$\begin{aligned} \lambda_1 &= +i \\ \lambda_2 &= -i \end{aligned}$$

Which simplifies to

$$\begin{aligned} \lambda_1 &= i \\ \lambda_2 &= -i \end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x)$$
$$y_2 = \sin(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ \frac{d}{dx}(\cos(x)) & \frac{d}{dx}(\sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

Therefore

$$W = (\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))$$

Which simplifies to

$$W = \cos(x)^2 + \sin(x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(x)\left(\csc(x) + 2x^2 + 5x + 1\right)}{1} dx$$

Which simplifies to

$$u_1 = -\int \left(1 + \left(2x^2 + 5x + 1\right)\sin(x)\right) dx$$

Hence

$$u_1 = -x + 2x^2\cos(x) - 3\cos(x) - 4x\sin(x) - 5\sin(x) + 5\cos(x)x$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(x)\left(\csc(x) + 2x^2 + 5x + 1\right)}{1} dx$$

Which simplifies to

$$u_2 = \int \cos(x)\left(\csc(x) + 2x^2 + 5x + 1\right) dx$$

Hence

$$u_2 = 2x^2\sin(x) - 3\sin(x) + 4\cos(x)x + 5\cos(x) + 5x\sin(x) + \ln(\sin(x))$$

Which simplifies to

$$\begin{aligned} u_1 &= \left(2x^2 + 5x - 3\right)\cos(x) + (-4x - 5)\sin(x) - x \\ u_2 &= \ln(\sin(x)) + \left(2x^2 + 5x - 3\right)\sin(x) + (4x + 5)\cos(x) \end{aligned}$$

Therefore the particular solution, from equation (1) is

$$
\begin{aligned}
y_p(x) = &\left((2x^2+5x-3)\cos(x)+(-4x-5)\sin(x)-x\right)\cos(x) \\
&+\left(\ln(\sin(x))+\left(2x^2+5x-3\right)\sin(x)+(4x+5)\cos(x)\right)\sin(x)
\end{aligned}
$$

Which simplifies to

$$
y_p(x) = -\cos(x)\,x+\sin(x)\ln(\sin(x))+2x^2+5x-3
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1\cos(x)+c_2\sin(x))+\left(-\cos(x)\,x+\sin(x)\ln(\sin(x))+2x^2+5x-3\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1\cos(x)+c_2\sin(x)-\cos(x)\,x+\sin(x)\ln(\sin(x))+2x^2+5x-3 \tag{1}
$$

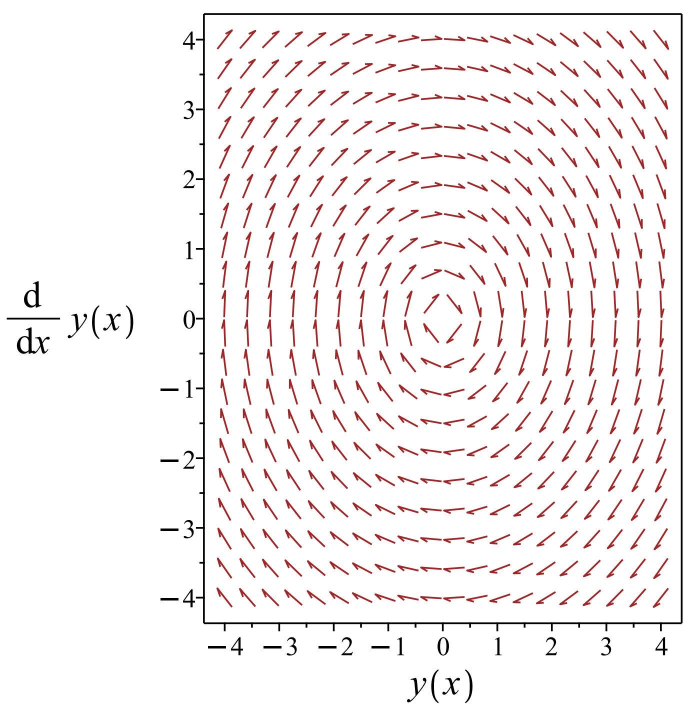


Figure 293: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x) - \cos(x) x + \sin(x) \ln(\sin(x)) + 2x^2 + 5x - 3$$

Verified OK.

9.11.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 272: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \cos(x) \int \frac{1}{\cos(x)^2}\,dx \\ &= \cos(x)(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(x)) + c_2(\cos(x)(\tan(x))) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x)$$
$$y_2 = \sin(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ \frac{d}{dx}(\cos(x)) & \frac{d}{dx}(\sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

Therefore

$$W = (\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))$$

Which simplifies to

$$W = \cos(x)^2 + \sin(x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(x)(\csc(x) + 2x^2 + 5x + 1)}{1}\, dx$$

Which simplifies to

$$u_1 = -\int \left(1 + \left(2x^2 + 5x + 1\right)\sin(x)\right) dx$$

Hence

$$u_1 = -x + 2x^2\cos(x) - 3\cos(x) - 4x\sin(x) - 5\sin(x) + 5\cos(x)\, x$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(x)(\csc(x) + 2x^2 + 5x + 1)}{1}\, dx$$

Which simplifies to

$$u_2 = \int \cos(x)\left(\csc(x) + 2x^2 + 5x + 1\right) dx$$

Hence

$$u_2 = 2x^2\sin(x) - 3\sin(x) + 4\cos(x)\, x + 5\cos(x) + 5x\sin(x) + \ln(\sin(x))$$

Which simplifies to

$$u_1 = \left(2x^2 + 5x - 3\right)\cos(x) + (-4x - 5)\sin(x) - x$$
$$u_2 = \ln(\sin(x)) + \left(2x^2 + 5x - 3\right)\sin(x) + (4x + 5)\cos(x)$$

Therefore the particular solution, from equation (1) is

$$\begin{aligned} y_p(x) = &\left((2x^2+5x-3)\cos(x)+(-4x-5)\sin(x)-x\right)\cos(x) \\ &+\left(\ln(\sin(x))+(2x^2+5x-3)\sin(x)+(4x+5)\cos(x)\right)\sin(x) \end{aligned}$$

Which simplifies to

$$y_p(x) = -\cos(x)\,x+\sin(x)\ln(\sin(x))+2x^2+5x-3$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1\cos(x)+c_2\sin(x))+\left(-\cos(x)\,x+\sin(x)\ln(\sin(x))+2x^2+5x-3\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\cos(x)+c_2\sin(x)-\cos(x)\,x+\sin(x)\ln(\sin(x))+2x^2+5x-3 \tag{1}$$

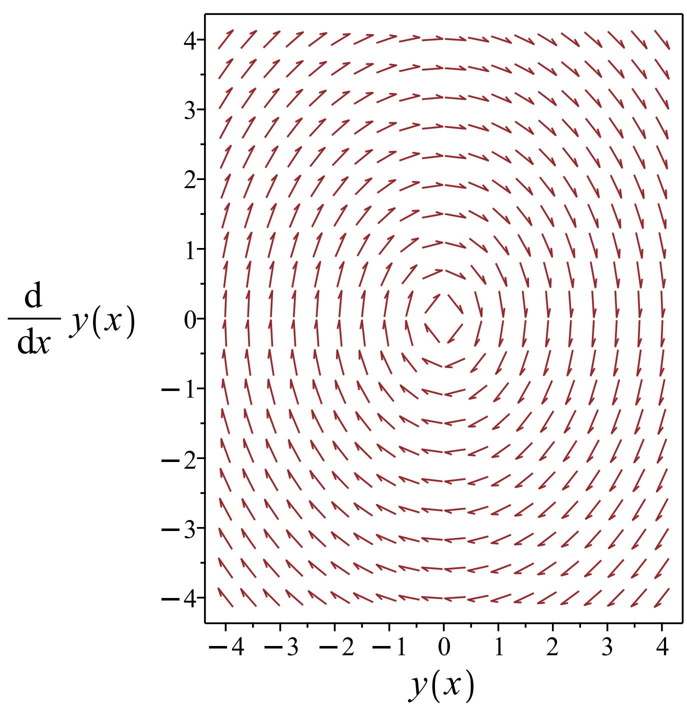


Figure 294: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x) - \cos(x)\, x + \sin(x) \ln(\sin(x)) + 2x^2 + 5x - 3$$

Verified OK.

9.11.3 Maple step by step solution

Let's solve

$y'' + y = \csc(x) + 2x^2 + 5x + 1$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \csc(x) + 2x^2 + 5x + 1\right.$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$

- Compute Wronskian

  $W(y_1(x), y_2(x)) = 1$
- Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\cos(x)\left(\int (1 + (2x^2 + 5x + 1)\sin(x))\, dx\right) + \sin(x)\left(\int \cos(x)(\csc(x) + 2x^2 + 5x + 1)\right.$
- Compute integrals

  $y_p(x) = -\cos(x)\, x + \sin(x)\ln(\sin(x)) + 2x^2 + 5x - 3$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(x) + c_2 \sin(x) - \cos(x)\, x + \sin(x)\ln(\sin(x)) + 2x^2 + 5x - 3$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 32

```
dsolve(diff(y(x),x$2)+y(x)=csc(x)+2*x^2+5*x+1,y(x), singsol=all)
```

$$y(x) = \sin(x)\ln(\sin(x)) + (c_1 - x)\cos(x) + 2x^2 + \sin(x)\, c_2 + 5x - 3$$

✓ Solution by Mathematica

Time used: 0.208 (sec). Leaf size: 33

```
DSolve[y''[x]+y[x]==Csc[x]+2*x^2+5*x+1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 2x^2 + 5x + (-x + c_1)\cos(x) + \sin(x)(\log(\sin(x)) + c_2) - 3$$

9.12 problem Problem 12

Internal problem ID [2785]
Internal file name [`OUTPUT/2277_Sunday_June_05_2022_02_57_26_AM_63859560/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.7, The Variation of Parameters Method. page 556
Problem number: Problem 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' - y = 2\tanh(x)}$$

9.12.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = -1, f(x) = 2\tanh(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-1)}\\ &= \pm 1\end{aligned}$$

Hence

$$\lambda_1 = +1$$
$$\lambda_2 = -1$$

Which simplifies to

$$\lambda_1 = 1$$
$$\lambda_2 = -1$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(1)x} + c_2 e^{(-1)x}$$

Or

$$y = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \mathrm{e}^x$$
$$y_2 = \mathrm{e}^{-x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^x & \mathrm{e}^{-x} \\ \frac{d}{dx}(\mathrm{e}^x) & \frac{d}{dx}(\mathrm{e}^{-x}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^x & \mathrm{e}^{-x} \\ \mathrm{e}^x & -\mathrm{e}^{-x} \end{vmatrix}$$

Therefore

$$W = (\mathrm{e}^x)(-\mathrm{e}^{-x}) - (\mathrm{e}^{-x})(\mathrm{e}^x)$$

Which simplifies to

$$W = -2\,\mathrm{e}^{-x}\mathrm{e}^x$$

Which simplifies to

$$W = -2$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{2\,\mathrm{e}^{-x} \tanh(x)}{-2}\, dx$$

Which simplifies to

$$u_1 = -\int -\mathrm{e}^{-x} \tanh(x)\, dx$$

Hence

$$u_1 = -\sinh(x) + 2\arctan(\mathrm{e}^{x}) + \cosh(x)$$

And Eq. (3) becomes

$$u_2 = \int \frac{2\,\mathrm{e}^{x} \tanh(x)}{-2}\, dx$$

Which simplifies to

$$u_2 = \int -\mathrm{e}^{x} \tanh(x)\, dx$$

Hence

$$u_2 = -\sinh(x) + 2\arctan(\mathrm{e}^{x}) - \cosh(x)$$

Which simplifies to

$$u_1 = \mathrm{e}^{-x} + 2\arctan(\mathrm{e}^{x})$$
$$u_2 = -\mathrm{e}^{x} + 2\arctan(\mathrm{e}^{x})$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(\mathrm{e}^{-x} + 2\arctan(\mathrm{e}^{x})\right)\mathrm{e}^{x} + \left(-\mathrm{e}^{x} + 2\arctan(\mathrm{e}^{x})\right)\mathrm{e}^{-x}$$

Which simplifies to

$$y_p(x) = 2\arctan(\mathrm{e}^{x})\left(\mathrm{e}^{x} + \mathrm{e}^{-x}\right)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-x}\right) + \left(2\arctan(\mathrm{e}^{x})\left(\mathrm{e}^{x} + \mathrm{e}^{-x}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^x + c_2 e^{-x} + 2 \arctan(e^x)(e^x + e^{-x}) \tag{1}$$

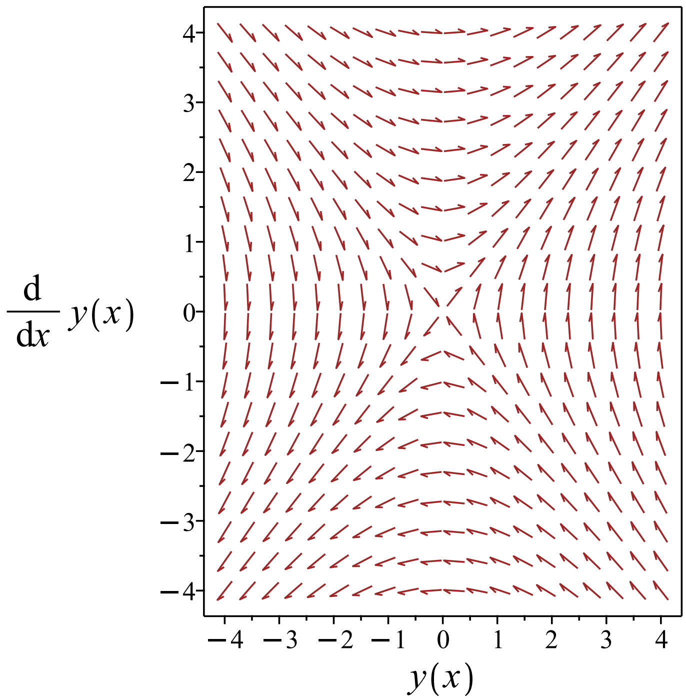


Figure 295: Slope field plot

Verification of solutions

$$y = c_1 e^x + c_2 e^{-x} + 2 \arctan(e^x)(e^x + e^{-x})$$

Verified OK.

9.12.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 274: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \mathrm{e}^{-x} \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \mathrm{e}^{-x} \int \frac{1}{\mathrm{e}^{-2x}}\, dx \\ &= \mathrm{e}^{-x} \left(\frac{\mathrm{e}^{2x}}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-x}\right) + c_2 \left(\mathrm{e}^{-x} \left(\frac{\mathrm{e}^{2x}}{2}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{x}}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}
y_1 &= \mathrm{e}^{-x} \\
y_2 &= \frac{\mathrm{e}^{x}}{2}
\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{-x} & \frac{\mathrm{e}^{x}}{2} \\ \frac{d}{dx}\left(\mathrm{e}^{-x}\right) & \frac{d}{dx}\left(\frac{\mathrm{e}^{x}}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{-x} & \frac{\mathrm{e}^{x}}{2} \\ -\mathrm{e}^{-x} & \frac{\mathrm{e}^{x}}{2} \end{vmatrix}$$

Therefore

$$W = \left(\mathrm{e}^{-x}\right)\left(\frac{\mathrm{e}^{x}}{2}\right) - \left(\frac{\mathrm{e}^{x}}{2}\right)\left(-\mathrm{e}^{-x}\right)$$

Which simplifies to

$$W = \mathrm{e}^{-x}\mathrm{e}^{x}$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{e^x \tanh(x)}{1} dx$$

Which simplifies to

$$u_1 = -\int e^x \tanh(x) dx$$

Hence

$$u_1 = -\sinh(x) + 2\arctan(e^x) - \cosh(x)$$

And Eq. (3) becomes

$$u_2 = \int \frac{2\,e^{-x} \tanh(x)}{1} dx$$

Which simplifies to

$$u_2 = \int 2\,e^{-x} \tanh(x) dx$$

Hence

$$u_2 = -2\sinh(x) + 4\arctan(e^x) + 2\cosh(x)$$

Which simplifies to

$$u_1 = -e^x + 2\arctan(e^x)$$
$$u_2 = 2\,e^{-x} + 4\arctan(e^x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = (-e^x + 2\arctan(e^x))\,e^{-x} + \frac{(2\,e^{-x} + 4\arctan(e^x))\,e^x}{2}$$

Which simplifies to

$$y_p(x) = 2\arctan(e^x)\left(e^x + e^{-x}\right)$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{x}}{2}\right) + \left(2 \arctan\left(\mathrm{e}^{x}\right) \left(\mathrm{e}^{x} + \mathrm{e}^{-x}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{x}}{2} + 2 \arctan\left(\mathrm{e}^{x}\right) \left(\mathrm{e}^{x} + \mathrm{e}^{-x}\right) \tag{1}
$$

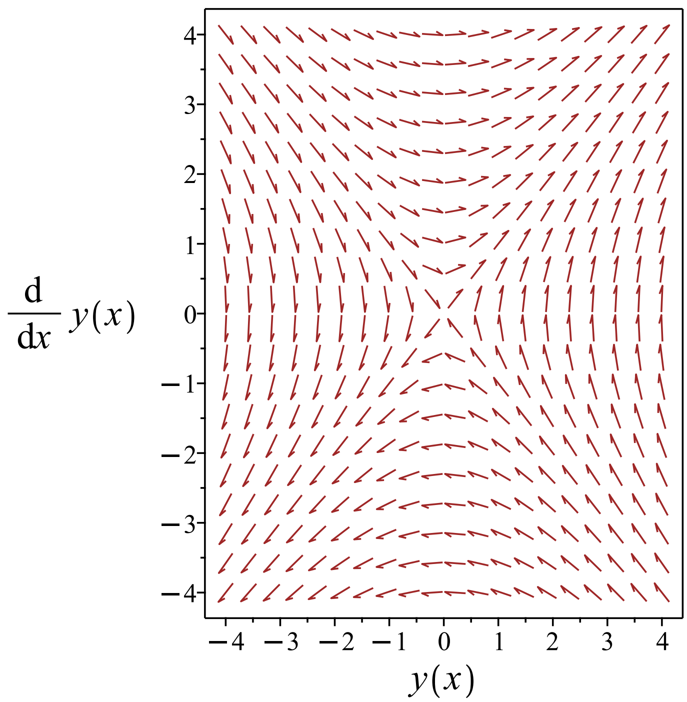


Figure 296: Slope field plot

Verification of solutions

$$
y = c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{x}}{2} + 2 \arctan\left(\mathrm{e}^{x}\right) \left(\mathrm{e}^{x} + \mathrm{e}^{-x}\right)
$$

Verified OK.

9.12.3 Maple step by step solution

Let's solve

$y'' - y = 2\tanh(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 1 = 0$

- Factor the characteristic polynomial

  $(r-1)(r+1) = 0$

- Roots of the characteristic polynomial

  $r = (-1, 1)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-x} + c_2 e^{x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ∘ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 2\tanh(x)\right]$

- ∘ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-x} & e^{x} \\ -e^{-x} & e^{x} \end{bmatrix}$

- ∘ Compute Wronskian

  $W(y_1(x), y_2(x)) = 2$

- ∘ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -e^{-x}\left(\int e^{x}\tanh(x)\, dx\right) + e^{x}\left(\int e^{-x}\tanh(x)\, dx\right)$

- Compute integrals

  $$y_p(x) = 2\arctan(e^x)(e^x + e^{-x})$$

- Substitute particular solution into general solution to ODE

  $$y = c_1 e^{-x} + c_2 e^x + 2\arctan(e^x)(e^x + e^{-x})$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(diff(y(x),x$2)-y(x)=2*tanh(x),y(x), singsol=all)
```

$$y(x) = (c_2 + 2\arctan(e^x))e^{-x} + e^x(c_1 + 2\arctan(e^x))$$

✓ Solution by Mathematica

Time used: 0.06 (sec). Leaf size: 35

```
DSolve[y''[x]-y[x]==2*Tanh[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x}\left(2\left(e^{2x} + 1\right)\arctan(e^x) + c_1 e^{2x} + c_2\right)$$

9.13 problem Problem 13

Internal problem ID [2786]
Internal file name [`OUTPUT/2278_Sunday_June_05_2022_02_57_28_AM_35811178/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.7, The Variation of Parameters Method. page 556
Problem number: Problem 13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 2my' + m^2 y = \frac{e^{mx}}{x^2+1}}$$

9.13.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -2m, C = m^2, f(x) = \frac{e^{mx}}{x^2+1}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 2my' + m^2 y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2m, C = m^2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 2m\lambda\, e^{\lambda x} + m^2 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2m\lambda + m^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2m, C = m^2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2m}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(-2m)^2 - (4)(1)(m^2)}\\ &= m\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -m$. Therefore the solution is

$$y = c_1 e^{mx} + c_2 x\, e^{mx} \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{mx} + c_2 x\, e^{mx}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}y_1 &= e^{mx}\\ y_2 &= x\, e^{mx}\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{mx} & x\,e^{mx} \\ \frac{d}{dx}(e^{mx}) & \frac{d}{dx}(x\,e^{mx}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{mx} & x\,e^{mx} \\ m\,e^{mx} & e^{mx} + xm\,e^{mx} \end{vmatrix}$$

Therefore

$$W = (e^{mx})\,(e^{mx} + xm\,e^{mx}) - (x\,e^{mx})\,(m\,e^{mx})$$

Which simplifies to

$$W = e^{2mx}$$

Which simplifies to

$$W = e^{2mx}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{x\,e^{2mx}}{x^2+1}}{e^{2mx}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{x}{x^2+1} dx$$

Hence

$$u_1 = -\frac{\ln(x^2+1)}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{e^{2mx}}{x^2+1}}{e^{2mx}} dx$$

Which simplifies to

$$u_2 = \int \frac{1}{x^2+1} dx$$

Hence

$$u_2 = \arctan(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{\ln(x^2+1)\, e^{mx}}{2} + \arctan(x)\, x\, e^{mx}$$

Which simplifies to

$$y_p(x) = e^{mx}\left(-\frac{\ln(x^2+1)}{2} + \arctan(x)\, x\right)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 e^{mx} + c_2 x\, e^{mx}) + \left(e^{mx}\left(-\frac{\ln(x^2+1)}{2} + \arctan(x)\, x\right)\right) \end{aligned}$$

Which simplifies to

$$y = e^{mx}(c_2 x + c_1) + e^{mx}\left(-\frac{\ln(x^2+1)}{2} + \arctan(x)\, x\right)$$

Summary

The solution(s) found are the following

$$y = e^{mx}(c_2 x + c_1) + e^{mx}\left(-\frac{\ln(x^2+1)}{2} + \arctan(x)\, x\right) \tag{1}$$

Verification of solutions

$$y = e^{mx}(c_2 x + c_1) + e^{mx}\left(-\frac{\ln(x^2+1)}{2} + \arctan(x)\, x\right)$$

Verified OK.

9.13.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = -2m$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\, dx} \\ &= e^{\int -2m\, dx} \\ &= \mathrm{e}^{-mx} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)y)'' &= \frac{\mathrm{e}^{-mx}\mathrm{e}^{mx}}{x^2+1} \\ \left(\mathrm{e}^{-mx}y\right)'' &= \frac{\mathrm{e}^{-mx}\mathrm{e}^{mx}}{x^2+1} \end{aligned}$$

Integrating once gives

$$\left(\mathrm{e}^{-mx}y\right)' = \arctan(x) + c_1$$

Integrating again gives

$$\left(\mathrm{e}^{-mx}y\right) = c_1 x + \arctan(x)\, x - \frac{\ln(x^2+1)}{2} + c_2$$

Hence the solution is

$$y = \frac{c_1 x + \arctan(x)\, x - \frac{\ln(x^2+1)}{2} + c_2}{\mathrm{e}^{-mx}}$$

Or

$$y = c_1 x\, \mathrm{e}^{mx} + x \arctan(x)\, \mathrm{e}^{mx} + c_2 \mathrm{e}^{mx} - \frac{\mathrm{e}^{mx}\ln(x^2+1)}{2}$$

Summary

The solution(s) found are the following

$$y = c_1 x\, \mathrm{e}^{mx} + x \arctan(x)\, \mathrm{e}^{mx} + c_2 \mathrm{e}^{mx} - \frac{\mathrm{e}^{mx}\ln(x^2+1)}{2} \tag{1}$$

Verification of solutions

$$y = c_1 x \, \mathrm{e}^{mx} + x \arctan(x) \, \mathrm{e}^{mx} + c_2 \mathrm{e}^{mx} - \frac{\mathrm{e}^{mx} \ln(x^2+1)}{2}$$

Verified OK.

9.13.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2my' + m^2 y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2m \\ C &= m^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 276: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2m}{1}\,dx} \\
&= z_1 e^{mx} \\
&= z_1 (\mathrm{e}^{mx})
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{mx}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2m}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2mx}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^{mx}) + c_2(\mathrm{e}^{mx}(x))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 2my' + m^2 y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{mx} + c_2 x\, e^{mx}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= e^{mx} \\ y_2 &= x\, e^{mx} \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{mx} & x\, e^{mx} \\ \frac{d}{dx}(e^{mx}) & \frac{d}{dx}(x\, e^{mx}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{mx} & x\, e^{mx} \\ m\, e^{mx} & e^{mx} + xm\, e^{mx} \end{vmatrix}$$

Therefore

$$W = (e^{mx})(e^{mx} + xm\, e^{mx}) - (x\, e^{mx})(m\, e^{mx})$$

Which simplifies to

$$W = e^{2mx}$$

Which simplifies to

$$W = e^{2mx}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{x\,e^{2mx}}{x^2+1}}{e^{2mx}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{x}{x^2+1}dx$$

Hence

$$u_1 = -\frac{\ln\left(x^2+1\right)}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{e^{2mx}}{x^2+1}}{e^{2mx}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{1}{x^2+1}dx$$

Hence

$$u_2 = \arctan\left(x\right)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{e^{mx}\ln\left(x^2+1\right)}{2} + x\arctan\left(x\right)e^{mx}$$

Which simplifies to

$$y_p(x) = e^{mx}\left(-\frac{\ln\left(x^2+1\right)}{2} + \arctan\left(x\right)x\right)$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 e^{mx} + c_2 x\, e^{mx}) + \left(e^{mx}\left(-\frac{\ln(x^2+1)}{2} + \arctan(x)\, x\right)\right)
\end{aligned}
$$

Which simplifies to

$$
y = e^{mx}(c_2 x + c_1) + e^{mx}\left(-\frac{\ln(x^2+1)}{2} + \arctan(x)\, x\right)
$$

Summary
The solution(s) found are the following

$$
y = e^{mx}(c_2 x + c_1) + e^{mx}\left(-\frac{\ln(x^2+1)}{2} + \arctan(x)\, x\right) \tag{1}
$$

Verification of solutions

$$
y = e^{mx}(c_2 x + c_1) + e^{mx}\left(-\frac{\ln(x^2+1)}{2} + \arctan(x)\, x\right)
$$

Verified OK.

9.13.4 Maple step by step solution

Let's solve

$y'' - 2my' + m^2 y = \frac{e^{mx}}{x^2+1}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $m^2 - 2mr + r^2 = 0$

- Factor the characteristic polynomial

  $(m-r)^2 = 0$

- Root of the characteristic polynomial

  $r = m$

- 1st solution of the homogeneous ODE

$y_1(x) = e^{mx}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,e^{mx}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{mx} + c_2 x\,e^{mx} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \frac{e^{mx}}{x^2+1}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{mx} & x\,e^{mx} \\ m\,e^{mx} & e^{mx} + xm\,e^{mx} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = e^{2mx}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = e^{mx}\left(-\left(\int \frac{x}{x^2+1} dx\right) + \left(\int \frac{1}{x^2+1} dx\right) x\right)$

  - Compute integrals

    $y_p(x) = -\frac{e^{mx}(-2\arctan(x)x + \ln(x^2+1))}{2}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{mx} + c_2 x\,e^{mx} - \frac{e^{mx}(-2\arctan(x)x + \ln(x^2+1))}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 26

```
dsolve(diff(y(x),x$2)-2*m*diff(y(x),x)+m^2*y(x)=exp(m*x)/(1+x^2),y(x), singsol=all)
```

$$y(x) = e^{mx}\left(c_2 + c_1 x - \frac{\ln(x^2+1)}{2} + x \arctan(x)\right)$$

✓ Solution by Mathematica

Time used: 0.034 (sec). Leaf size: 37

```
DSolve[y''[x]-2*m*y'[x]+m^2*y[x]==Exp[m*x]/(1+x^2),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2} e^{mx} \left(2x \arctan(x) - \log\left(x^2+1\right) + 2(c_2 x + c_1)\right)$$

9.14 problem Problem 13

Internal problem ID [2787]
Internal file name [`OUTPUT/2279_Sunday_June_05_2022_02_57_30_AM_52127018/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.7, The Variation of Parameters Method. page 556
Problem number: Problem 13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 2y' + y = \frac{4\,\mathrm{e}^{x} \ln(x)}{x^{3}}}$$

9.14.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -2, C = 1, f(x) = \frac{4\,\mathrm{e}^{x} \ln(x)}{x^{3}}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 2y' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 2\lambda\, e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(-2)^2 - (4)(1)(1)}\\ &= 1\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -1$. Therefore the solution is

$$y = c_1 e^{x} + c_2 x\, e^{x} \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{x} + c_2 x\, e^{x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}y_1 &= e^{x}\\ y_2 &= x\, e^{x}\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^x & x\,e^x \\ \frac{d}{dx}(e^x) & \frac{d}{dx}(x\,e^x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^x & x\,e^x \\ e^x & x\,e^x + e^x \end{vmatrix}$$

Therefore

$$W = (e^x)(x\,e^x + e^x) - (x\,e^x)(e^x)$$

Which simplifies to

$$W = e^{2x}$$

Which simplifies to

$$W = e^{2x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{4\,e^{2x}\ln(x)}{x^2}}{e^{2x}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{4\ln(x)}{x^2}dx$$

Hence

$$u_1 = \frac{4\ln(x)}{x} + \frac{4}{x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{4\,\mathrm{e}^{2x}\ln(x)}{x^3}}{\mathrm{e}^{2x}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{4\ln(x)}{x^3}dx$$

Hence

$$u_2 = -\frac{2\ln(x)}{x^2} - \frac{1}{x^2}$$

Which simplifies to

$$u_1 = \frac{4\ln(x) + 4}{x}$$
$$u_2 = \frac{-1 - 2\ln(x)}{x^2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{(4\ln(x) + 4)\,\mathrm{e}^x}{x} + \frac{(-1 - 2\ln(x))\,\mathrm{e}^x}{x}$$

Which simplifies to

$$y_p(x) = \frac{2\,\mathrm{e}^x \ln(x) + 3\,\mathrm{e}^x}{x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1\mathrm{e}^x + c_2 x\,\mathrm{e}^x) + \left(\frac{2\,\mathrm{e}^x \ln(x) + 3\,\mathrm{e}^x}{x}\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^x(c_2 x + c_1) + \frac{2\,\mathrm{e}^x \ln(x) + 3\,\mathrm{e}^x}{x}$$

Summary

The solution(s) found are the following

$$y = e^{x}(c_2 x + c_1) + \frac{2\,e^{x}\ln(x) + 3\,e^{x}}{x} \tag{1}$$

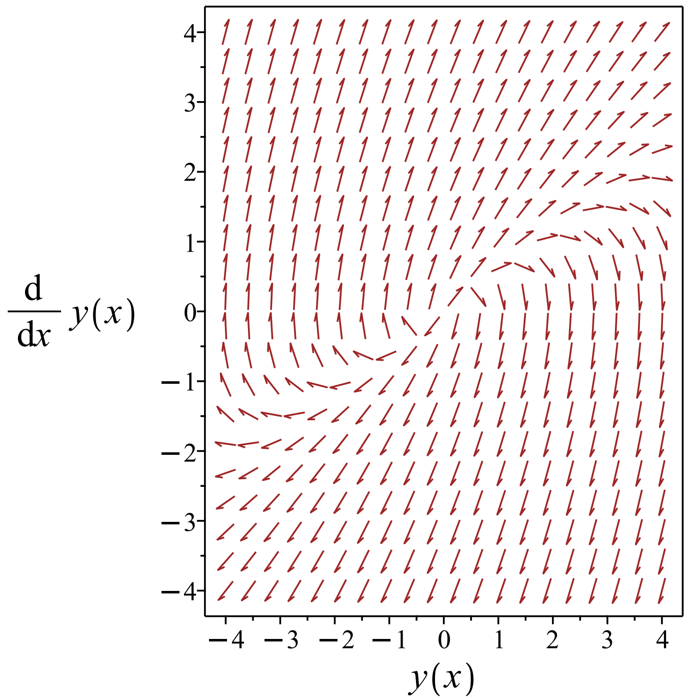


Figure 297: Slope field plot

Verification of solutions

$$y = e^{x}(c_2 x + c_1) + \frac{2\,e^{x}\ln(x) + 3\,e^{x}}{x}$$

Verified OK.

9.14.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\,y' + \frac{\left(p(x)^2 + p'(x)\right)y}{2} = f(x)$$

Where $p(x) = -2$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int -2\,dx} \\ &= \mathrm{e}^{-x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)y)'' &= \frac{4\,\mathrm{e}^{-x}\mathrm{e}^{x}\ln(x)}{x^3} \\ \left(\mathrm{e}^{-x}y\right)'' &= \frac{4\,\mathrm{e}^{-x}\mathrm{e}^{x}\ln(x)}{x^3} \end{aligned}$$

Integrating once gives

$$\left(\mathrm{e}^{-x}y\right)' = \frac{-1-2\ln(x)}{x^2} + c_1$$

Integrating again gives

$$\left(\mathrm{e}^{-x}y\right) = \frac{c_1x^2 + 2\ln(x) + 3}{x} + c_2$$

Hence the solution is

$$y = \frac{\frac{c_1x^2+2\ln(x)+3}{x} + c_2}{\mathrm{e}^{-x}}$$

Or

$$y = c_1x\,\mathrm{e}^{x} + c_2\mathrm{e}^{x} + \frac{2\,\mathrm{e}^{x}\ln(x)}{x} + \frac{3\,\mathrm{e}^{x}}{x}$$

Summary

The solution(s) found are the following

$$y = c_1x\,\mathrm{e}^{x} + c_2\mathrm{e}^{x} + \frac{2\,\mathrm{e}^{x}\ln(x)}{x} + \frac{3\,\mathrm{e}^{x}}{x} \tag{1}$$

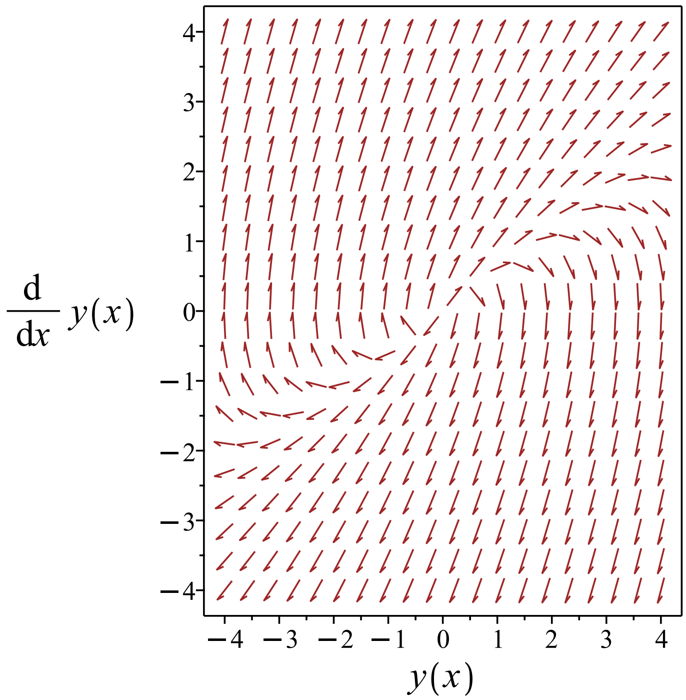


Figure 298: Slope field plot

Verification of solutions

$$y = c_1 x\,\mathrm{e}^{x} + c_2\mathrm{e}^{x} + \frac{2\,\mathrm{e}^{x}\ln(x)}{x} + \frac{3\,\mathrm{e}^{x}}{x}$$

Verified OK.

9.14.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 278: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2}{1}\,dx} \\
&= z_1 e^{x} \\
&= z_1(\mathrm{e}^{x})
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2x}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^x) + c_2(\mathrm{e}^x(x))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 2y' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^x + c_2 x\,\mathrm{e}^x
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \mathrm{e}^x \\
y_2 &= x\,\mathrm{e}^x
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^x & x\,e^x \\ \frac{d}{dx}(e^x) & \frac{d}{dx}(x\,e^x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^x & x\,e^x \\ e^x & x\,e^x + e^x \end{vmatrix}$$

Therefore

$$W = (e^x)(x\,e^x + e^x) - (x\,e^x)(e^x)$$

Which simplifies to

$$W = e^{2x}$$

Which simplifies to

$$W = e^{2x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{4\,e^{2x}\ln(x)}{x^2}}{e^{2x}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{4\ln(x)}{x^2}dx$$

Hence

$$u_1 = \frac{4\ln(x)}{x} + \frac{4}{x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{4\,e^{2x}\ln(x)}{x^3}}{e^{2x}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{4 \ln (x)}{x^3} dx$$

Hence

$$u_2 = -\frac{2 \ln (x)}{x^2} - \frac{1}{x^2}$$

Which simplifies to

$$u_1 = \frac{4 \ln (x) + 4}{x}$$

$$u_2 = \frac{-1 - 2 \ln (x)}{x^2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{(4 \ln (x) + 4) \, \mathrm{e}^x}{x} + \frac{(-1 - 2 \ln (x)) \, \mathrm{e}^x}{x}$$

Which simplifies to

$$y_p(x) = \frac{2 \, \mathrm{e}^x \ln (x) + 3 \, \mathrm{e}^x}{x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \mathrm{e}^x + c_2 x \, \mathrm{e}^x) + \left(\frac{2 \, \mathrm{e}^x \ln (x) + 3 \, \mathrm{e}^x}{x} \right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^x (c_2 x + c_1) + \frac{2 \, \mathrm{e}^x \ln (x) + 3 \, \mathrm{e}^x}{x}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^x (c_2 x + c_1) + \frac{2 \, \mathrm{e}^x \ln (x) + 3 \, \mathrm{e}^x}{x} \tag{1}$$

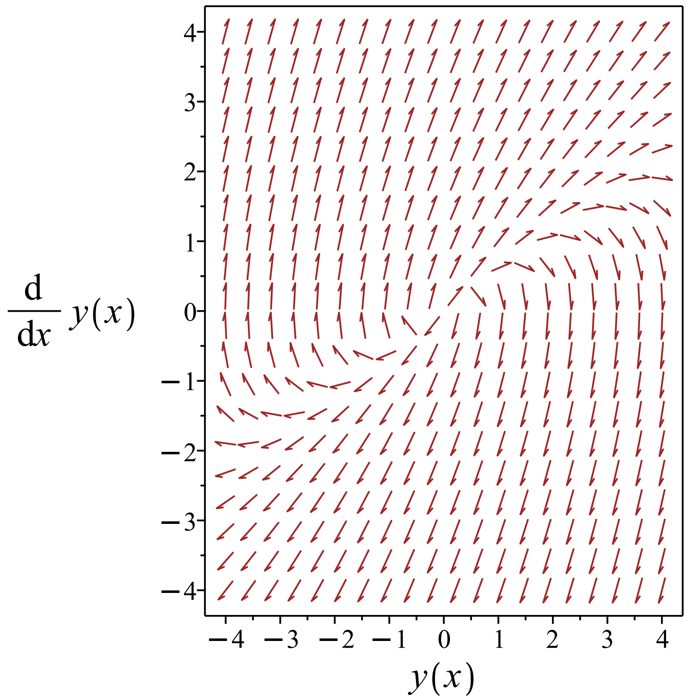


Figure 299: Slope field plot

Verification of solutions

$$y = e^x(c_2x + c_1) + \frac{2\,e^x \ln(x) + 3\,e^x}{x}$$

Verified OK.

9.14.4 Maple step by step solution

Let's solve

$y'' - 2y' + y = \frac{4\,e^x \ln(x)}{x^3}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 2r + 1 = 0$

- Factor the characteristic polynomial

  $(r-1)^2 = 0$

- Root of the characteristic polynomial

$r = 1$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^x$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,\mathrm{e}^x$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^x + c_2 x\,\mathrm{e}^x + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \frac{4\,\mathrm{e}^x \ln(x)}{x^3}\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^x & x\,\mathrm{e}^x \\ \mathrm{e}^x & x\,\mathrm{e}^x + \mathrm{e}^x \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = \mathrm{e}^{2x}$

- ○ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -4\,\mathrm{e}^x\left(\int \frac{\ln(x)}{x^2} dx - \left(\int \frac{\ln(x)}{x^3} dx\right) x\right)$

- ○ Compute integrals

  $y_p(x) = \frac{2\,\mathrm{e}^x \ln(x) + 3\,\mathrm{e}^x}{x}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^x + c_2 x\,\mathrm{e}^x + \frac{2\,\mathrm{e}^x \ln(x) + 3\,\mathrm{e}^x}{x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
dsolve(diff(y(x),x$2)-2*diff(y(x),x)+y(x)=4*exp(x)*x^(-3)*ln(x),y(x), singsol=all)
```

$$y(x) = \frac{e^{x}(c_1x^2 + c_2x + 2\ln(x) + 3)}{x}$$

✓ Solution by Mathematica

Time used: 0.036 (sec). Leaf size: 28

```
DSolve[y''[x]-2*y'[x]+y[x]==4*Exp[x]*x^(-3)*Log[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{x}(c_2x^2 + 2\log(x) + c_1x + 3)}{x}$$

9.15 problem Problem 15

Internal problem ID [2788]
Internal file name [`OUTPUT/2280_Sunday_June_05_2022_02_57_32_AM_20489786/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.7, The Variation of Parameters Method. page 556
Problem number: Problem 15.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 2y' + y = \frac{e^{-x}}{\sqrt{-x^2+4}}}$$

9.15.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 2, C = 1, f(x) = \frac{e^{-x}}{\sqrt{-x^2+4}}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$.

y_h is the solution to

$$y'' + 2y' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 2, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 2\lambda\, \mathrm{e}^{\lambda x} + \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = 1$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{-2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(2)^2 - (4)(1)(1)} \\
&= -1
\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 1$. Therefore the solution is

$$y = c_1 \mathrm{e}^{-x} + c_2 x\, \mathrm{e}^{-x} \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{-x} + c_2 x\, \mathrm{e}^{-x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}
y_1 &= \mathrm{e}^{-x} \\
y_2 &= x\, \mathrm{e}^{-x}
\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{-x} & x\,\mathrm{e}^{-x} \\ \frac{d}{dx}(\mathrm{e}^{-x}) & \frac{d}{dx}(x\,\mathrm{e}^{-x}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{-x} & x\,\mathrm{e}^{-x} \\ -\mathrm{e}^{-x} & \mathrm{e}^{-x} - x\,\mathrm{e}^{-x} \end{vmatrix}$$

Therefore

$$W = (\mathrm{e}^{-x})(\mathrm{e}^{-x} - x\,\mathrm{e}^{-x}) - (x\,\mathrm{e}^{-x})(-\mathrm{e}^{-x})$$

Which simplifies to

$$W = \mathrm{e}^{-2x}$$

Which simplifies to

$$W = \mathrm{e}^{-2x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{x\,\mathrm{e}^{-2x}}{\sqrt{-x^2+4}}}{\mathrm{e}^{-2x}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{x}{\sqrt{-x^2+4}}dx$$

Hence

$$u_1 = -\frac{(x-2)(x+2)}{\sqrt{-x^2+4}}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{e^{-2x}}{\sqrt{-x^2+4}}}{e^{-2x}}\, dx$$

Which simplifies to

$$u_2 = \int \frac{1}{\sqrt{-x^2+4}} dx$$

Hence

$$u_2 = \arcsin\left(\frac{x}{2}\right)$$

Which simplifies to

$$u_1 = \sqrt{-x^2+4}$$
$$u_2 = \arcsin\left(\frac{x}{2}\right)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \sqrt{-x^2+4}\, e^{-x} + \arcsin\left(\frac{x}{2}\right) x\, e^{-x}$$

Which simplifies to

$$y_p(x) = e^{-x}\left(\sqrt{-x^2+4} + x \arcsin\left(\frac{x}{2}\right)\right)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + c_2 x\, e^{-x}\right) + \left(e^{-x}\left(\sqrt{-x^2+4} + x \arcsin\left(\frac{x}{2}\right)\right)\right) \end{aligned}$$

Which simplifies to

$$y = e^{-x}(c_2 x + c_1) + e^{-x}\left(\sqrt{-x^2+4} + x \arcsin\left(\frac{x}{2}\right)\right)$$

Summary

The solution(s) found are the following

$$y = e^{-x}(c_2 x + c_1) + e^{-x}\left(\sqrt{-x^2+4} + x \arcsin\left(\frac{x}{2}\right)\right) \tag{1}$$

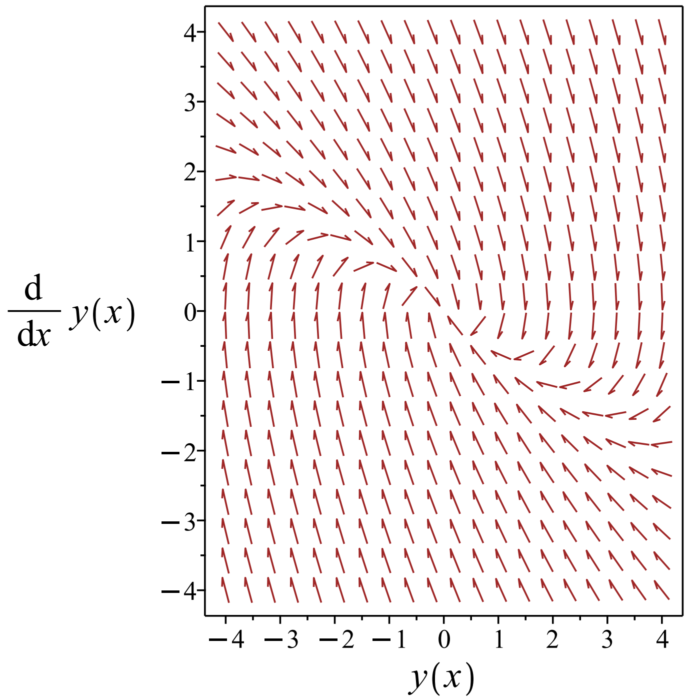


Figure 300: Slope field plot

Verification of solutions

$$y = e^{-x}(c_2 x + c_1) + e^{-x}\left(\sqrt{-x^2+4} + x \arcsin\left(\frac{x}{2}\right)\right)$$

Verified OK.

9.15.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = 2$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int 2\,dx} \\ &= e^{x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$
\begin{aligned}
(M(x)y)'' &= \frac{e^x e^{-x}}{\sqrt{-x^2+4}} \\
(e^x y)'' &= \frac{e^x e^{-x}}{\sqrt{-x^2+4}}
\end{aligned}
$$

Integrating once gives

$$
(e^x y)' = \arcsin\left(\frac{x}{2}\right) + c_1
$$

Integrating again gives

$$
(e^x y) = c_1 x + x \arcsin\left(\frac{x}{2}\right) + \sqrt{-x^2+4} + c_2
$$

Hence the solution is

$$
y = \frac{c_1 x + x \arcsin\left(\frac{x}{2}\right) + \sqrt{-x^2+4} + c_2}{e^x}
$$

Or

$$
y = c_1 x\, e^{-x} + \arcsin\left(\frac{x}{2}\right) x\, e^{-x} + c_2 e^{-x} + \sqrt{-x^2+4}\, e^{-x}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x\, e^{-x} + \arcsin\left(\frac{x}{2}\right) x\, e^{-x} + c_2 e^{-x} + \sqrt{-x^2+4}\, e^{-x} \tag{1}
$$

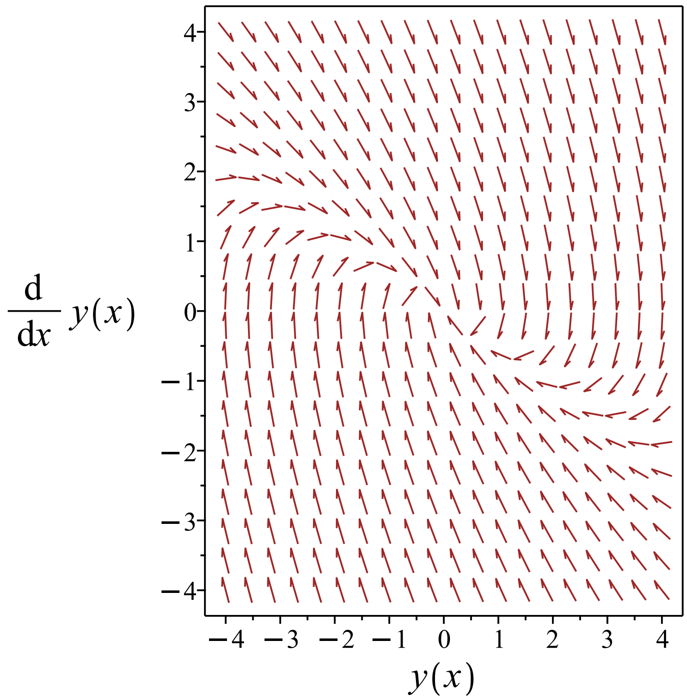


Figure 301: Slope field plot

Verification of solutions

$$y = c_1 x\, e^{-x} + \arcsin\left(\frac{x}{2}\right) x\, e^{-x} + c_2 e^{-x} + \sqrt{-x^2+4}\, e^{-x}$$

Verified OK.

9.15.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y' + y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 280: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dx} \\
&= z_1 e^{-x} \\
&= z_1 \left(\mathrm{e}^{-x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2x}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(e^{-x}\right) + c_2 \left(e^{-x}(x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 2y' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 e^{-x} + c_2 x \, e^{-x}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= e^{-x} \\
y_2 &= x \, e^{-x}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{-x} & x\,\mathrm{e}^{-x} \\ \frac{d}{dx}(\mathrm{e}^{-x}) & \frac{d}{dx}(x\,\mathrm{e}^{-x}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{-x} & x\,\mathrm{e}^{-x} \\ -\mathrm{e}^{-x} & \mathrm{e}^{-x} - x\,\mathrm{e}^{-x} \end{vmatrix}$$

Therefore

$$W = \left(\mathrm{e}^{-x}\right)\left(\mathrm{e}^{-x} - x\,\mathrm{e}^{-x}\right) - \left(x\,\mathrm{e}^{-x}\right)\left(-\mathrm{e}^{-x}\right)$$

Which simplifies to

$$W = \mathrm{e}^{-2x}$$

Which simplifies to

$$W = \mathrm{e}^{-2x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{x\,\mathrm{e}^{-2x}}{\sqrt{-x^2+4}}}{\mathrm{e}^{-2x}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{x}{\sqrt{-x^2+4}}dx$$

Hence

$$u_1 = -\frac{(x-2)(x+2)}{\sqrt{-x^2+4}}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{\mathrm{e}^{-2x}}{\sqrt{-x^2+4}}}{\mathrm{e}^{-2x}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{1}{\sqrt{-x^2+4}}dx$$

Hence

$$u_2 = \arcsin\left(\frac{x}{2}\right)$$

Which simplifies to

$$u_1 = \sqrt{-x^2+4}$$
$$u_2 = \arcsin\left(\frac{x}{2}\right)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \sqrt{-x^2+4}\,\mathrm{e}^{-x} + \arcsin\left(\frac{x}{2}\right) x\,\mathrm{e}^{-x}$$

Which simplifies to

$$y_p(x) = \mathrm{e}^{-x}\left(\sqrt{-x^2+4} + x \arcsin\left(\frac{x}{2}\right)\right)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-x} + c_2 x\,\mathrm{e}^{-x}\right) + \left(\mathrm{e}^{-x}\left(\sqrt{-x^2+4} + x \arcsin\left(\frac{x}{2}\right)\right)\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{-x}(c_2 x + c_1) + \mathrm{e}^{-x}\left(\sqrt{-x^2+4} + x \arcsin\left(\frac{x}{2}\right)\right)$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-x}(c_2 x + c_1) + \mathrm{e}^{-x}\left(\sqrt{-x^2+4} + x \arcsin\left(\frac{x}{2}\right)\right) \tag{1}$$

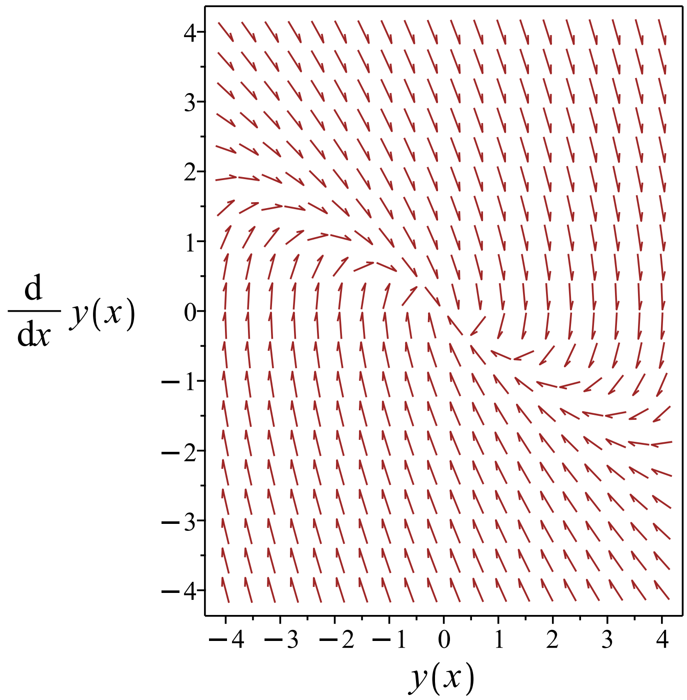


Figure 302: Slope field plot

Verification of solutions

$$y = e^{-x}(c_2 x + c_1) + e^{-x}\left(\sqrt{-x^2+4} + x \arcsin\left(\frac{x}{2}\right)\right)$$

Verified OK.

9.15.4 Maple step by step solution

Let's solve

$y'' + 2y' + y = \frac{e^{-x}}{\sqrt{-x^2+4}}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 2r + 1 = 0$

- Factor the characteristic polynomial

  $(r+1)^2 = 0$

- Root of the characteristic polynomial

$r = -1$

- 1st solution of the homogeneous ODE

$$y_1(x) = \mathrm{e}^{-x}$$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

$$y_2(x) = x\,\mathrm{e}^{-x}$$

- General solution of the ODE

$$y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$$

- Substitute in solutions of the homogeneous ODE

$$y = c_1 \mathrm{e}^{-x} + c_2 x\,\mathrm{e}^{-x} + y_p(x)$$

□ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

$$\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = \frac{\mathrm{e}^{-x}}{\sqrt{-x^2+4}}\right]$$

- ○ Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \left[\begin{array}{cc} \mathrm{e}^{-x} & x\,\mathrm{e}^{-x} \\ -\mathrm{e}^{-x} & \mathrm{e}^{-x} - x\,\mathrm{e}^{-x} \end{array}\right]$$

- ○ Compute Wronskian

$$W(y_1(x), y_2(x)) = \mathrm{e}^{-2x}$$

- ○ Substitute functions into equation for $y_p(x)$

$$y_p(x) = \mathrm{e}^{-x}\left(-\left(\int \frac{x}{\sqrt{-x^2+4}}dx\right) + \left(\int \frac{1}{\sqrt{-x^2+4}}dx\right)x\right)$$

- ○ Compute integrals

$$y_p(x) = -\frac{\mathrm{e}^{-x}\left(-x \arcsin\left(\frac{x}{2}\right)\sqrt{-x^2+4}+x^2-4\right)}{\sqrt{-x^2+4}}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 \mathrm{e}^{-x} + c_2 x\,\mathrm{e}^{-x} - \frac{\mathrm{e}^{-x}\left(-x \arcsin\left(\frac{x}{2}\right)\sqrt{-x^2+4}+x^2-4\right)}{\sqrt{-x^2+4}}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 49

```
dsolve(diff(y(x),x$2)+2*diff(y(x),x)+y(x)=exp(-x)/sqrt(4-x^2),y(x), singsol=all)
```

$$y(x) = -\frac{\mathrm{e}^{-x}\left(\left(-c_1 x - \arcsin\left(\frac{x}{2}\right) x - c_2\right)\sqrt{-x^2+4} + x^2 - 4\right)}{\sqrt{-x^2+4}}$$

✓ Solution by Mathematica

Time used: 0.085 (sec). Leaf size: 50

```
DSolve[y''[x]+2*y'[x]+y[x]==Exp[-x]/Sqrt[4-x^2],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x}\left(-2x\arctan\left(\frac{\sqrt{4-x^2}}{x+2}\right) + \sqrt{4-x^2} + c_2 x + c_1\right)$$

9.16 problem Problem 16

Internal problem ID [2789]
Internal file name [`OUTPUT/2281_Sunday_June_05_2022_02_57_34_AM_49762758/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.7, The Variation of Parameters Method. page 556
Problem number: Problem 16.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 2y' + 17y = \frac{64\,\mathrm{e}^{-x}}{3 + \sin(4x)^2}}$$

9.16.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 2, C = 17, f(x) = -\frac{128\,\mathrm{e}^{-x}}{-7+\cos(8x)}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2y' + 17y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 2, C = 17$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 2\lambda\, e^{\lambda x} + 17\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2\lambda + 17 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = 17$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{2^2 - (4)\,(1)\,(17)}\\ &= -1 \pm 4i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -1 + 4i\\ \lambda_2 &= -1 - 4i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -1 + 4i\\ \lambda_2 &= -1 - 4i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -1$ and $\beta = 4$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-x}(c_1 \cos(4x) + c_2 \sin(4x))$$

Therefore the homogeneous solution y_h is

$$y_h = \mathrm{e}^{-x}(c_1 \cos(4x) + c_2 \sin(4x))$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \mathrm{e}^{-x} \cos(4x)$$
$$y_2 = \mathrm{e}^{-x} \sin(4x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{-x} \cos(4x) & \mathrm{e}^{-x} \sin(4x) \\ \frac{d}{dx}(\mathrm{e}^{-x} \cos(4x)) & \frac{d}{dx}(\mathrm{e}^{-x} \sin(4x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{-x} \cos(4x) & \mathrm{e}^{-x} \sin(4x) \\ -\mathrm{e}^{-x} \cos(4x) - 4\,\mathrm{e}^{-x} \sin(4x) & -\mathrm{e}^{-x} \sin(4x) + 4\,\mathrm{e}^{-x} \cos(4x) \end{vmatrix}$$

Therefore

$$W = \left(e^{-x} \cos(4x)\right) \left(-e^{-x} \sin(4x) + 4\,e^{-x} \cos(4x)\right) - \left(e^{-x} \sin(4x)\right) \left(-e^{-x} \cos(4x) - 4\,e^{-x} \sin(4x)\right)$$

Which simplifies to

$$W = 4\,e^{-2x} \sin(4x)^2 + 4\,e^{-2x} \cos(4x)^2$$

Which simplifies to

$$W = 4\,e^{-2x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-\frac{128\,e^{-2x} \sin(4x)}{-7+\cos(8x)}}{4\,e^{-2x}}\,dx$$

Which simplifies to

$$u_1 = -\int -\frac{16 \sin(4x)}{\cos(4x)^2 - 4}dx$$

Hence

$$u_1 = \ln(\cos(4x) + 2) - \ln(\cos(4x) - 2)$$

And Eq. (3) becomes

$$u_2 = \int \frac{-\frac{128\,e^{-2x} \cos(4x)}{-7+\cos(8x)}}{4\,e^{-2x}}\,dx$$

Which simplifies to

$$u_2 = \int -\frac{16 \cos(4x)}{\cos(4x)^2 - 4}dx$$

Hence

$$u_2 = \frac{4\sqrt{3}\, \arctan\left(\frac{\sqrt{3}\, \sin(4x)}{3}\right)}{3}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = (\ln(\cos(4x)+2) - \ln(\cos(4x)-2))\,\mathrm{e}^{-x}\cos(4x) + \frac{4\sqrt{3}\,\arctan\left(\frac{\sqrt{3}\,\sin(4x)}{3}\right)\mathrm{e}^{-x}\sin(4x)}{3}$$

Which simplifies to

$$y_p(x) = \frac{4\left(\sqrt{3}\,\arctan\left(\frac{\sqrt{3}\,\sin(4x)}{3}\right)\sin(4x) - \frac{3\cos(4x)(-\ln(\cos(4x)+2)+\ln(\cos(4x)-2))}{4}\right)\mathrm{e}^{-x}}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\mathrm{e}^{-x}(c_1\cos(4x)+c_2\sin(4x))\right) \\ &\quad + \left(\frac{4\left(\sqrt{3}\,\arctan\left(\frac{\sqrt{3}\,\sin(4x)}{3}\right)\sin(4x) - \frac{3\cos(4x)(-\ln(\cos(4x)+2)+\ln(\cos(4x)-2))}{4}\right)\mathrm{e}^{-x}}{3}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$\begin{aligned} y &= \mathrm{e}^{-x}(c_1\cos(4x)+c_2\sin(4x)) \\ &\quad + \frac{4\left(\sqrt{3}\,\arctan\left(\frac{\sqrt{3}\,\sin(4x)}{3}\right)\sin(4x) - \frac{3\cos(4x)(-\ln(\cos(4x)+2)+\ln(\cos(4x)-2))}{4}\right)\mathrm{e}^{-x}}{3} \end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned} y &= \mathrm{e}^{-x}(c_1\cos(4x)+c_2\sin(4x)) \\ &\quad + \frac{4\left(\sqrt{3}\,\arctan\left(\frac{\sqrt{3}\,\sin(4x)}{3}\right)\sin(4x) - \frac{3\cos(4x)(-\ln(\cos(4x)+2)+\ln(\cos(4x)-2))}{4}\right)\mathrm{e}^{-x}}{3} \end{aligned}$$

Verified OK.

9.16.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y' + 17y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= 17 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-16}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -16 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -16z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 282: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -16$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(4x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\, dx} \end{aligned}$$

$$
\begin{aligned}
&= z_1 e^{-x} \\
&= z_1 \left(\mathrm{e}^{-x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-x} \cos(4x)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-2x}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{\tan(4x)}{4}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-x} \cos(4x)\right) + c_2 \left(\mathrm{e}^{-x} \cos(4x) \left(\frac{\tan(4x)}{4}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$
y'' + 2y' + 17y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = \mathrm{e}^{-x} \cos(4x)\, c_1 + \frac{\mathrm{e}^{-x} \sin(4x)\, c_2}{4}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \mathrm{e}^{-x} \cos(4x)$$
$$y_2 = \frac{\mathrm{e}^{-x} \sin(4x)}{4}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{-x} \cos(4x) & \frac{\mathrm{e}^{-x} \sin(4x)}{4} \\ \frac{d}{dx}\left(\mathrm{e}^{-x} \cos(4x)\right) & \frac{d}{dx}\left(\frac{\mathrm{e}^{-x} \sin(4x)}{4}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{-x} \cos(4x) & \frac{\mathrm{e}^{-x} \sin(4x)}{4} \\ -\mathrm{e}^{-x} \cos(4x) - 4\,\mathrm{e}^{-x} \sin(4x) & -\frac{\mathrm{e}^{-x} \sin(4x)}{4} + \mathrm{e}^{-x} \cos(4x) \end{vmatrix}$$

Therefore

$$W = \left(\mathrm{e}^{-x} \cos(4x)\right)\left(-\frac{\mathrm{e}^{-x} \sin(4x)}{4} + \mathrm{e}^{-x} \cos(4x)\right) - \left(\frac{\mathrm{e}^{-x} \sin(4x)}{4}\right)\left(-\mathrm{e}^{-x} \cos(4x) - 4\,\mathrm{e}^{-x} \sin(4x)\right)$$

Which simplifies to

$$W = \mathrm{e}^{-2x} \sin(4x)^2 + \mathrm{e}^{-2x} \cos(4x)^2$$

Which simplifies to

$$W = \mathrm{e}^{-2x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-\frac{32\,\mathrm{e}^{-2x}\sin(4x)}{-7+\cos(8x)}}{\mathrm{e}^{-2x}}\,dx$$

Which simplifies to

$$u_1 = -\int -\frac{16\sin(4x)}{\cos(4x)^2 - 4}dx$$

Hence

$$u_1 = \ln(\cos(4x) + 2) - \ln(\cos(4x) - 2)$$

And Eq. (3) becomes

$$u_2 = \int \frac{-\frac{128\,\mathrm{e}^{-2x}\cos(4x)}{-7+\cos(8x)}}{\mathrm{e}^{-2x}}\,dx$$

Which simplifies to

$$u_2 = \int -\frac{64\cos(4x)}{\cos(4x)^2 - 4}dx$$

Hence

$$u_2 = \frac{16\sqrt{3}\,\arctan\left(\frac{\sqrt{3}\,\sin(4x)}{3}\right)}{3}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = (\ln(\cos(4x) + 2) - \ln(\cos(4x) - 2))\,\mathrm{e}^{-x}\cos(4x) + \frac{4\sqrt{3}\,\arctan\left(\frac{\sqrt{3}\,\sin(4x)}{3}\right)\mathrm{e}^{-x}\sin(4x)}{3}$$

Which simplifies to

$$y_p(x) = \frac{4\left(\sqrt{3}\, \arctan\left(\frac{\sqrt{3}\, \sin(4x)}{3}\right) \sin\left(4x\right) - \frac{3\cos(4x)(-\ln(\cos(4x)+2)+\ln(\cos(4x)-2))}{4}\right) \mathrm{e}^{-x}}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\mathrm{e}^{-x} \cos\left(4x\right) c_1 + \frac{\mathrm{e}^{-x} \sin\left(4x\right) c_2}{4}\right) \\ &\quad + \left(\frac{4\left(\sqrt{3}\, \arctan\left(\frac{\sqrt{3}\, \sin(4x)}{3}\right) \sin\left(4x\right) - \frac{3\cos(4x)(-\ln(\cos(4x)+2)+\ln(\cos(4x)-2))}{4}\right) \mathrm{e}^{-x}}{3}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$\begin{aligned} y &= \mathrm{e}^{-x} \cos\left(4x\right) c_1 + \frac{\mathrm{e}^{-x} \sin\left(4x\right) c_2}{4} \\ &\quad + \frac{4\left(\sqrt{3}\, \arctan\left(\frac{\sqrt{3}\, \sin(4x)}{3}\right) \sin\left(4x\right) - \frac{3\cos(4x)(-\ln(\cos(4x)+2)+\ln(\cos(4x)-2))}{4}\right) \mathrm{e}^{-x}}{3} \end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned} y &= \mathrm{e}^{-x} \cos\left(4x\right) c_1 + \frac{\mathrm{e}^{-x} \sin\left(4x\right) c_2}{4} \\ &\quad + \frac{4\left(\sqrt{3}\, \arctan\left(\frac{\sqrt{3}\, \sin(4x)}{3}\right) \sin\left(4x\right) - \frac{3\cos(4x)(-\ln(\cos(4x)+2)+\ln(\cos(4x)-2))}{4}\right) \mathrm{e}^{-x}}{3} \end{aligned}$$

Verified OK.

9.16.3 Maple step by step solution

Let's solve

$y'' + 2y' + 17y = -\frac{128\,\mathrm{e}^{-x}}{-7+\cos(8x)}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2 + 2r + 17 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-2)\pm(\sqrt{-64})}{2}$

- Roots of the characteristic polynomial

  $r = (-1-4\,\mathrm{I}, -1+4\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-x}\cos(4x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{-x}\sin(4x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = \mathrm{e}^{-x}\cos(4x)\,c_1 + \mathrm{e}^{-x}\sin(4x)\,c_2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = -\frac{128\,\mathrm{e}^{-x}}{-7+\cos(8x)}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-x}\cos(4x) & \mathrm{e}^{-x}\sin(4x) \\ -\mathrm{e}^{-x}\cos(4x) - 4\,\mathrm{e}^{-x}\sin(4x) & -\mathrm{e}^{-x}\sin(4x) + 4\,\mathrm{e}^{-x}\cos(4x) \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 4\,\mathrm{e}^{-2x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -16\,\mathrm{e}^{-x}\left(\sin(4x)\left(\int \frac{\cos(4x)}{\cos(4x)^2-4}dx\right) - \cos(4x)\left(\int \frac{\sin(4x)}{\cos(4x)^2-4}dx\right)\right)$

  - Compute integrals

    $y_p(x) = \frac{4\left(\sqrt{3}\arctan\left(\frac{\sqrt{3}\sin(4x)}{3}\right)\sin(4x) - \frac{3\cos(4x)(-\ln(\cos(4x)+2)+\ln(\cos(4x)-2))}{4}\right)\mathrm{e}^{-x}}{3}$

- Substitute particular solution into general solution to ODE

  $y = \mathrm{e}^{-x}\cos(4x)\,c_1 + \mathrm{e}^{-x}\sin(4x)\,c_2 + \frac{4\left(\sqrt{3}\arctan\left(\frac{\sqrt{3}\sin(4x)}{3}\right)\sin(4x) - \frac{3\cos(4x)(-\ln(\cos(4x)+2)+\ln(\cos(4x)-2))}{4}\right)\mathrm{e}^{-x}}{3}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 70

```
dsolve(diff(y(x),x$2)+2*diff(y(x),x)+17*y(x)=64*exp(-x)/(3+sin(4*x)^2),y(x), singsol=all)
```

$$y(x) = \frac{e^{-x}\left(4\sin(4x)\sqrt{3}\arctan\left(\frac{\sqrt{3}\sin(4x)}{3}\right)+3\ln(\cos(4x)+2)\cos(4x)-3\ln(\cos(4x)-2)\cos(4x)+3\cos(\right.}{3}$$

✓ Solution by Mathematica

Time used: 0.238 (sec). Leaf size: 72

```
DSolve[y''[x]+2*y'[x]+17*y[x]==64*Exp[-x]/(3+Sin[4*x]^2),y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to \frac{1}{3}e^{-x}\left(4\sqrt{3}\sin(4x)\arctan\left(\frac{\sin(4x)}{\sqrt{3}}\right)+3c_1\sin(4x)\right.$$
$$\left.+3\cos(4x)(-\log(2-\cos(4x))+\log(\cos(4x)+2)+c_2)\right)$$

9.17 problem Problem 17

Internal problem ID [2790]
Internal file name [`OUTPUT/2282_Sunday_June_05_2022_02_57_37_AM_58118268/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.7, The Variation of Parameters Method. page 556
Problem number: Problem 17.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' + 4y' + 4y = \frac{4\,\mathrm{e}^{-2x}}{x^2+1} + 2x^2 - 1}$$

9.17.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 4, C = 4, f(x) = \frac{2x^4-1+x^2+4\,\mathrm{e}^{-2x}}{x^2+1}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$.

y_h is the solution to

$$y'' + 4y' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 4, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 4\lambda \, \mathrm{e}^{\lambda x} + 4 \, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4\lambda + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 4, C = 4$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{-4}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(4)^2 - (4)(1)(4)} \\
&= -2
\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 2$. Therefore the solution is

$$y = c_1 \mathrm{e}^{-2x} + c_2 x \, \mathrm{e}^{-2x} \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{-2x} + c_2 x \, \mathrm{e}^{-2x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \mathrm{e}^{-2x}$$
$$y_2 = x\,\mathrm{e}^{-2x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{-2x} & x\,\mathrm{e}^{-2x} \\ \frac{d}{dx}\left(\mathrm{e}^{-2x}\right) & \frac{d}{dx}\left(x\,\mathrm{e}^{-2x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{-2x} & x\,\mathrm{e}^{-2x} \\ -2\,\mathrm{e}^{-2x} & \mathrm{e}^{-2x} - 2x\,\mathrm{e}^{-2x} \end{vmatrix}$$

Therefore

$$W = \left(\mathrm{e}^{-2x}\right)\left(\mathrm{e}^{-2x} - 2x\,\mathrm{e}^{-2x}\right) - \left(x\,\mathrm{e}^{-2x}\right)\left(-2\,\mathrm{e}^{-2x}\right)$$

Which simplifies to

$$W = \mathrm{e}^{-4x}$$

Which simplifies to

$$W = \mathrm{e}^{-4x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{x\,\mathrm{e}^{-2x}\left(2x^4 - 1 + x^2 + 4\,\mathrm{e}^{-2x}\right)}{x^2+1}}{\mathrm{e}^{-4x}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{(2x^5 + x^3 - x)\,\mathrm{e}^{2x} + 4x}{x^2 + 1} dx$$

Hence

$$u_1 = -2\ln\left(4x^2 + 4\right) - \frac{(2x - 1)\,\mathrm{e}^{2x}}{4} - \frac{\mathrm{e}^{2x}(8x^3 - 12x^2 + 4x - 2)}{8}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{\mathrm{e}^{-2x}(2x^4 - 1 + x^2 + 4\,\mathrm{e}^{-2x})}{x^2 + 1}}{\mathrm{e}^{-4x}} dx$$

Which simplifies to

$$u_2 = \int \frac{4 + (2x^4 + x^2 - 1)\,\mathrm{e}^{2x}}{x^2 + 1} dx$$

Hence

$$u_2 = 4\arctan(x) + \frac{\mathrm{e}^{2x}}{2} + \frac{\mathrm{e}^{2x}(4x^2 - 4x - 2)}{4}$$

Which simplifies to

$$\begin{aligned} u_1 &= -2\ln\left(x^2 + 1\right) + \frac{(-2x^3 + 3x^2 - 2x + 1)\,\mathrm{e}^{2x}}{2} - 4\ln(2) \\ u_2 &= \left(x^2 - x\right)\mathrm{e}^{2x} + 4\arctan(x) \end{aligned}$$

Therefore the particular solution, from equation (1) is

$$\begin{aligned} y_p(x) = {} & \left(-2\ln\left(x^2 + 1\right) + \frac{(-2x^3 + 3x^2 - 2x + 1)\,\mathrm{e}^{2x}}{2} - 4\ln(2)\right)\mathrm{e}^{-2x} \\ & + \left(\left(x^2 - x\right)\mathrm{e}^{2x} + 4\arctan(x)\right) x\,\mathrm{e}^{-2x} \end{aligned}$$

Which simplifies to

$$y_p(x) = \frac{(x - 1)^2}{2} - 2\,\mathrm{e}^{-2x}\left(-2\arctan(x)\,x + \ln\left(x^2 + 1\right) + 2\ln(2)\right)$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \mathrm{e}^{-2x} + c_2 x\, \mathrm{e}^{-2x}\right) + \left(\frac{(x-1)^2}{2} - 2\,\mathrm{e}^{-2x}\left(-2\arctan(x)\, x + \ln\left(x^2+1\right) + 2\ln(2)\right)\right)
\end{aligned}
$$

Which simplifies to

$$
y = \mathrm{e}^{-2x}(c_2 x + c_1) + \frac{(x-1)^2}{2} - 2\,\mathrm{e}^{-2x}\left(-2\arctan(x)\, x + \ln\left(x^2+1\right) + 2\ln(2)\right)
$$

Summary

The solution(s) found are the following

$$
y = \mathrm{e}^{-2x}(c_2 x + c_1) + \frac{(x-1)^2}{2} - 2\,\mathrm{e}^{-2x}\left(-2\arctan(x)\, x + \ln\left(x^2+1\right) + 2\ln(2)\right) \quad (1)
$$

Verification of solutions

$$
y = \mathrm{e}^{-2x}(c_2 x + c_1) + \frac{(x-1)^2}{2} - 2\,\mathrm{e}^{-2x}\left(-2\arctan(x)\, x + \ln\left(x^2+1\right) + 2\ln(2)\right)
$$

Verified OK.

9.17.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$
y'' + p(x)\, y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)
$$

Where $p(x) = 4$. Therefore, there is an integrating factor given by

$$
\begin{aligned}
M(x) &= e^{\frac{1}{2}\int p\, dx} \\
&= e^{\int 4\, dx} \\
&= \mathrm{e}^{2x}
\end{aligned}
$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$
\begin{aligned}
(M(x)y)'' &= \frac{\mathrm{e}^{2x}\left(2x^4 - 1 + x^2 + 4\,\mathrm{e}^{-2x}\right)}{x^2+1} \\
\left(\mathrm{e}^{2x} y\right)'' &= \frac{\mathrm{e}^{2x}\left(2x^4 - 1 + x^2 + 4\,\mathrm{e}^{-2x}\right)}{x^2+1}
\end{aligned}
$$

Integrating once gives

$$\left(\mathrm{e}^{2x}y\right)' = \left(x^2 - x\right)\mathrm{e}^{2x} + 4\arctan\left(x\right) + c_1$$

Integrating again gives

$$\left(\mathrm{e}^{2x}y\right) = -2\ln\left(x^2+1\right) + \frac{\left(x^2-2x+1\right)\mathrm{e}^{2x}}{2} + c_1x + 4\arctan\left(x\right)x + c_2$$

Hence the solution is

$$y = \frac{-2\ln\left(x^2+1\right) + \frac{\left(x^2-2x+1\right)\mathrm{e}^{2x}}{2} + c_1x + 4\arctan\left(x\right)x + c_2}{\mathrm{e}^{2x}}$$

Or

$$y = \frac{x^2}{2} - x + c_1x\,\mathrm{e}^{-2x} + 4\arctan\left(x\right)x\,\mathrm{e}^{-2x} + c_2\mathrm{e}^{-2x} - 2\,\mathrm{e}^{-2x}\ln\left(x^2+1\right) + \frac{1}{2}$$

Summary
The solution(s) found are the following

$$y = \frac{x^2}{2} - x + c_1x\,\mathrm{e}^{-2x} + 4\arctan\left(x\right)x\,\mathrm{e}^{-2x} + c_2\mathrm{e}^{-2x} - 2\,\mathrm{e}^{-2x}\ln\left(x^2+1\right) + \frac{1}{2} \tag{1}$$

Verification of solutions

$$y = \frac{x^2}{2} - x + c_1x\,\mathrm{e}^{-2x} + 4\arctan\left(x\right)x\,\mathrm{e}^{-2x} + c_2\mathrm{e}^{-2x} - 2\,\mathrm{e}^{-2x}\ln\left(x^2+1\right) + \frac{1}{2}$$

Verified OK.

9.17.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 284: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4}{1}\,dx} \\
&= z_1 e^{-2x} \\
&= z_1 \left(\mathrm{e}^{-2x}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{4}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-4x}}{(y_1)^2}\, dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{-2x}\right) + c_2\left(\mathrm{e}^{-2x}(x)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y' + 4y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1\mathrm{e}^{-2x} + c_2 x\, \mathrm{e}^{-2x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= \mathrm{e}^{-2x} \\ y_2 &= x\,\mathrm{e}^{-2x} \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{-2x} & x\,\mathrm{e}^{-2x} \\ \frac{d}{dx}\left(\mathrm{e}^{-2x}\right) & \frac{d}{dx}\left(x\,\mathrm{e}^{-2x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{-2x} & x\,\mathrm{e}^{-2x} \\ -2\,\mathrm{e}^{-2x} & \mathrm{e}^{-2x} - 2x\,\mathrm{e}^{-2x} \end{vmatrix}$$

Therefore

$$W = \left(\mathrm{e}^{-2x}\right)\left(\mathrm{e}^{-2x} - 2x\,\mathrm{e}^{-2x}\right) - \left(x\,\mathrm{e}^{-2x}\right)\left(-2\,\mathrm{e}^{-2x}\right)$$

Which simplifies to

$$W = \mathrm{e}^{-4x}$$

Which simplifies to

$$W = \mathrm{e}^{-4x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{x\,\mathrm{e}^{-2x}\left(2x^4-1+x^2+4\,\mathrm{e}^{-2x}\right)}{x^2+1}}{\mathrm{e}^{-4x}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{\left(2x^5+x^3-x\right)\mathrm{e}^{2x}+4x}{x^2+1}dx$$

Hence

$$u_1 = -2\ln\left(4x^2+4\right) - \frac{(2x-1)\,\mathrm{e}^{2x}}{4} - \frac{\mathrm{e}^{2x}\left(8x^3-12x^2+4x-2\right)}{8}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{\mathrm{e}^{-2x}\left(2x^4-1+x^2+4\,\mathrm{e}^{-2x}\right)}{x^2+1}}{\mathrm{e}^{-4x}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{4+\left(2x^4+x^2-1\right)\mathrm{e}^{2x}}{x^2+1}dx$$

Hence

$$u_2 = 4\arctan\left(x\right) + \frac{\mathrm{e}^{2x}}{2} + \frac{\mathrm{e}^{2x}\left(4x^2-4x-2\right)}{4}$$

Which simplifies to

$$\begin{aligned}
u_1 &= -2\ln\left(x^2+1\right) + \frac{\left(-2x^3+3x^2-2x+1\right)\mathrm{e}^{2x}}{2} - 4\ln\left(2\right) \\
u_2 &= \left(x^2-x\right)\mathrm{e}^{2x} + 4\arctan\left(x\right)
\end{aligned}$$

Therefore the particular solution, from equation (1) is

$$\begin{aligned}
y_p(x) = {} & \left(-2\ln\left(x^2+1\right) + \frac{\left(-2x^3+3x^2-2x+1\right)\mathrm{e}^{2x}}{2} - 4\ln\left(2\right)\right)\mathrm{e}^{-2x} \\
& + \left(\left(x^2-x\right)\mathrm{e}^{2x} + 4\arctan\left(x\right)\right)x\,\mathrm{e}^{-2x}
\end{aligned}$$

Which simplifies to

$$y_p(x) = \frac{(x-1)^2}{2} - 2\,\mathrm{e}^{-2x}\left(-2\arctan\left(x\right)x + \ln\left(x^2+1\right) + 2\ln\left(2\right)\right)$$

Therefore the general solution is

$$\begin{aligned}y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-2x} + c_2 x\,\mathrm{e}^{-2x}\right) + \left(\frac{(x-1)^2}{2} - 2\,\mathrm{e}^{-2x}\left(-2\arctan\left(x\right)x + \ln\left(x^2+1\right) + 2\ln\left(2\right)\right)\right)\end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{-2x}(c_2 x + c_1) + \frac{(x-1)^2}{2} - 2\,\mathrm{e}^{-2x}\left(-2\arctan\left(x\right)x + \ln\left(x^2+1\right) + 2\ln\left(2\right)\right)$$

Summary
The solution(s) found are the following

$$y = \mathrm{e}^{-2x}(c_2 x + c_1) + \frac{(x-1)^2}{2} - 2\,\mathrm{e}^{-2x}\left(-2\arctan\left(x\right)x + \ln\left(x^2+1\right) + 2\ln\left(2\right)\right) \tag{1}$$

Verification of solutions

$$y = \mathrm{e}^{-2x}(c_2 x + c_1) + \frac{(x-1)^2}{2} - 2\,\mathrm{e}^{-2x}\left(-2\arctan\left(x\right)x + \ln\left(x^2+1\right) + 2\ln\left(2\right)\right)$$

Verified OK.

9.17.4 Maple step by step solution

Let's solve

$y'' + 4y' + 4y = \frac{2x^4 - 1 + x^2 + 4\,\mathrm{e}^{-2x}}{x^2+1}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4r + 4 = 0$

- Factor the characteristic polynomial

$(r+2)^2 = 0$

- Root of the characteristic polynomial

  $r = -2$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-2x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,\mathrm{e}^{-2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-2x} + c_2 x\,\mathrm{e}^{-2x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = \frac{2x^4-1+x^2+4\,\mathrm{e}^{-2x}}{x^2+1}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-2x} & x\,\mathrm{e}^{-2x} \\ -2\,\mathrm{e}^{-2x} & \mathrm{e}^{-2x} - 2x\,\mathrm{e}^{-2x} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = \mathrm{e}^{-4x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = \mathrm{e}^{-2x}\left(-\left(\int \frac{(2x^5+x^3-x)\mathrm{e}^{2x}+4x}{x^2+1}dx\right) + \left(\int \frac{4+(2x^4+x^2-1)\mathrm{e}^{2x}}{x^2+1}dx\right)x\right)$

  - Compute integrals

    $y_p(x) = \frac{(x-1)^2}{2} - 2\,\mathrm{e}^{-2x}(-2\arctan(x)\,x + \ln(x^2+1) + 2\ln(2))$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-2x} + c_2 x\,\mathrm{e}^{-2x} + \frac{(x-1)^2}{2} - 2\,\mathrm{e}^{-2x}(-2\arctan(x)\,x + \ln(x^2+1) + 2\ln(2))$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 35

```
dsolve(diff(y(x),x$2)+4*diff(y(x),x)+4*y(x)=4*exp(-2*x)/(1+x^2)+2*x^2-1,y(x), singsol=all)
```

$$y(x) = \frac{(x-1)^2}{2} + e^{-2x}\left(c_1 x + 4x \arctan(x) + c_2 - 2\ln\left(x^2+1\right)\right)$$

✓ Solution by Mathematica

Time used: 0.58 (sec). Leaf size: 59

```
DSolve[y''[x]+4*y'[x]+4*y[x]==4*Exp[-2*x]/(1+x^2)+2*x^2-1,y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to \frac{1}{2}e^{-2x}\left(8x\arctan(x) + e^{2x}x^2 - 4\log\left(x^2+1\right) - 2e^{2x}x + e^{2x} + 2c_2 x + 2c_1\right)$$

9.18 problem Problem 18

Internal problem ID [2791]
Internal file name [`OUTPUT/2283_Sunday_June_05_2022_02_57_39_AM_27040637/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.7, The Variation of Parameters Method. page 556
Problem number: Problem 18.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 4y' + 4y = 15\ln(x)\,\mathrm{e}^{-2x} + 25\cos(x)}$$

9.18.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 4, C = 4, f(x) = 15\ln(x)\,\mathrm{e}^{-2x} + 25\cos(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 4, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 4\lambda \,\mathrm{e}^{\lambda x} + 4\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4\lambda + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 4, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-4}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{(4)^2 - (4)\,(1)\,(4)}\\ &= -2\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 2$. Therefore the solution is

$$y = c_1 \mathrm{e}^{-2x} + c_2 x\,\mathrm{e}^{-2x} \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{-2x} + c_2 x\,\mathrm{e}^{-2x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}y_1 &= \mathrm{e}^{-2x}\\ y_2 &= x\,\mathrm{e}^{-2x}\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{-2x} & x\,\mathrm{e}^{-2x} \\ \frac{d}{dx}\left(\mathrm{e}^{-2x}\right) & \frac{d}{dx}\left(x\,\mathrm{e}^{-2x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{-2x} & x\,\mathrm{e}^{-2x} \\ -2\,\mathrm{e}^{-2x} & \mathrm{e}^{-2x} - 2x\,\mathrm{e}^{-2x} \end{vmatrix}$$

Therefore

$$W = \left(\mathrm{e}^{-2x}\right)\left(\mathrm{e}^{-2x} - 2x\,\mathrm{e}^{-2x}\right) - \left(x\,\mathrm{e}^{-2x}\right)\left(-2\,\mathrm{e}^{-2x}\right)$$

Which simplifies to

$$W = \mathrm{e}^{-4x}$$

Which simplifies to

$$W = \mathrm{e}^{-4x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x\,\mathrm{e}^{-2x}\left(15\ln(x)\,\mathrm{e}^{-2x} + 25\cos(x)\right)}{\mathrm{e}^{-4x}}\,dx$$

Which simplifies to

$$u_1 = -\int 5x\left(5\cos(x)\,\mathrm{e}^{2x} + 3\ln(x)\right)dx$$

Hence

$$u_1 = -\frac{15\ln(x)\,x^2}{2} + \frac{15x^2}{4} - 25\left(\frac{2x}{5} - \frac{3}{25}\right) e^{2x}\cos(x) + 25\left(-\frac{x}{5} + \frac{4}{25}\right) e^{2x}\sin(x)$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{-2x}(15\ln(x)\,e^{-2x} + 25\cos(x))}{e^{-4x}}\,dx$$

Which simplifies to

$$u_2 = \int \left(25\cos(x)\,e^{2x} + 15\ln(x)\right) dx$$

Hence

$$u_2 = 10\cos(x)\,e^{2x} + 5\sin(x)\,e^{2x} + 15\ln(x)\,x - 15x$$

Which simplifies to

$$u_1 = ((-10x+3)\cos(x) + (4-5x)\sin(x))\,e^{2x} - \frac{15\left(\ln(x) - \frac{1}{2}\right)x^2}{2}$$
$$u_2 = 5\,e^{2x}(\sin(x) + 2\cos(x)) + 15(\ln(x) - 1)\,x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(((-10x+3)\cos(x) + (4-5x)\sin(x))\,e^{2x} - \frac{15\left(\ln(x) - \frac{1}{2}\right)x^2}{2}\right) e^{-2x} + \left(5\,e^{2x}(\sin(x) + 2\cos(x)) + 15(\ln(x) - 1)\,x\right) x\,e^{-2x}$$

Which simplifies to

$$y_p(x) = \frac{15x^2\left(\ln(x) - \frac{3}{2}\right)e^{-2x}}{2} + 3\cos(x) + 4\sin(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-2x} + c_2 x\,e^{-2x}\right) + \left(\frac{15x^2\left(\ln(x) - \frac{3}{2}\right)e^{-2x}}{2} + 3\cos(x) + 4\sin(x)\right) \end{aligned}$$

Which simplifies to

$$y = e^{-2x}(c_2 x + c_1) + \frac{15x^2 \left(\ln(x) - \frac{3}{2}\right) e^{-2x}}{2} + 3\cos(x) + 4\sin(x)$$

Summary

The solution(s) found are the following

$$y = e^{-2x}(c_2 x + c_1) + \frac{15x^2 \left(\ln(x) - \frac{3}{2}\right) e^{-2x}}{2} + 3\cos(x) + 4\sin(x) \tag{1}$$

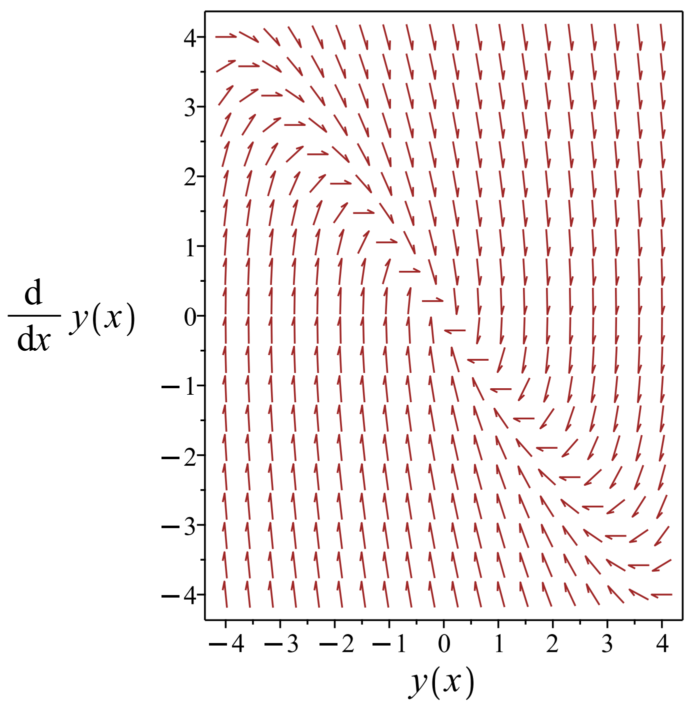


Figure 303: Slope field plot

Verification of solutions

$$y = e^{-2x}(c_2 x + c_1) + \frac{15x^2 \left(\ln(x) - \frac{3}{2}\right) e^{-2x}}{2} + 3\cos(x) + 4\sin(x)$$

Verified OK.

9.18.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = 4$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\, dx} \\ &= e^{\int 4\, dx} \\ &= \mathrm{e}^{2x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)y)'' &= \left(15 \ln(x)\, \mathrm{e}^{-2x} + 25 \cos(x)\right) \mathrm{e}^{2x} \\ \left(\mathrm{e}^{2x} y\right)'' &= \left(15 \ln(x)\, \mathrm{e}^{-2x} + 25 \cos(x)\right) \mathrm{e}^{2x} \end{aligned}$$

Integrating once gives

$$\left(\mathrm{e}^{2x} y\right)' = 5\, \mathrm{e}^{2x} (\sin(x) + 2\cos(x)) + 15(\ln(x) - 1)\, x + c_1$$

Integrating again gives

$$\left(\mathrm{e}^{2x} y\right) = (3\cos(x) + 4\sin(x))\, \mathrm{e}^{2x} + c_1 x + \frac{15 \ln(x)\, x^2}{2} - \frac{45x^2}{4} + c_2$$

Hence the solution is

$$y = \frac{(3\cos(x) + 4\sin(x))\, \mathrm{e}^{2x} + c_1 x + \frac{15\ln(x) x^2}{2} - \frac{45x^2}{4} + c_2}{\mathrm{e}^{2x}}$$

Or

$$y = \frac{15\, \mathrm{e}^{-2x} \ln(x)\, x^2}{2} + 3\cos(x) + 4\sin(x) + c_1 x\, \mathrm{e}^{-2x} - \frac{45\, \mathrm{e}^{-2x} x^2}{4} + c_2 \mathrm{e}^{-2x}$$

Summary

The solution(s) found are the following

$$y = \frac{15\, \mathrm{e}^{-2x} \ln(x)\, x^2}{2} + 3\cos(x) + 4\sin(x) + c_1 x\, \mathrm{e}^{-2x} - \frac{45\, \mathrm{e}^{-2x} x^2}{4} + c_2 \mathrm{e}^{-2x} \tag{1}$$

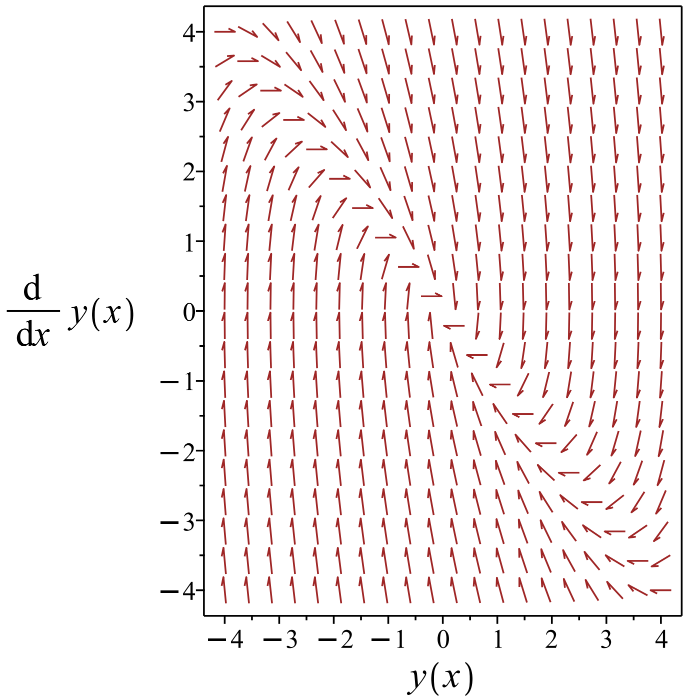


Figure 304: Slope field plot

Verification of solutions

$$y = \frac{15\,e^{-2x}\ln(x)\,x^2}{2} + 3\cos(x) + 4\sin(x) + c_1 x\,e^{-2x} - \frac{45\,e^{-2x}x^2}{4} + c_2 e^{-2x}$$

Verified OK.

9.18.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 286: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{4}{1}\,dx} \\ &= z_1 e^{-2x} \\ &= z_1\left(\mathrm{e}^{-2x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{4}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-4x}}{(y_1)^2}\,dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-2x}\right) + c_2 \left(\mathrm{e}^{-2x}(x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 4y' + 4y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-2x} + c_2 x \,\mathrm{e}^{-2x}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \mathrm{e}^{-2x} \\
y_2 &= x \,\mathrm{e}^{-2x}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{-2x} & x\,\mathrm{e}^{-2x} \\ \frac{d}{dx}\left(\mathrm{e}^{-2x}\right) & \frac{d}{dx}\left(x\,\mathrm{e}^{-2x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{-2x} & x\,\mathrm{e}^{-2x} \\ -2\,\mathrm{e}^{-2x} & \mathrm{e}^{-2x} - 2x\,\mathrm{e}^{-2x} \end{vmatrix}$$

Therefore

$$W = \left(\mathrm{e}^{-2x}\right)\left(\mathrm{e}^{-2x} - 2x\,\mathrm{e}^{-2x}\right) - \left(x\,\mathrm{e}^{-2x}\right)\left(-2\,\mathrm{e}^{-2x}\right)$$

Which simplifies to

$$W = \mathrm{e}^{-4x}$$

Which simplifies to

$$W = \mathrm{e}^{-4x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x\,\mathrm{e}^{-2x}\left(15\ln(x)\,\mathrm{e}^{-2x} + 25\cos(x)\right)}{\mathrm{e}^{-4x}}\,dx$$

Which simplifies to

$$u_1 = -\int 5x\left(5\cos(x)\,\mathrm{e}^{2x} + 3\ln(x)\right)dx$$

Hence

$$u_1 = -\frac{15\ln(x)\,x^2}{2} + \frac{15x^2}{4} - 25\left(\frac{2x}{5} - \frac{3}{25}\right)\mathrm{e}^{2x}\cos(x) + 25\left(-\frac{x}{5} + \frac{4}{25}\right)\mathrm{e}^{2x}\sin(x)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\mathrm{e}^{-2x}\left(15\ln(x)\,\mathrm{e}^{-2x} + 25\cos(x)\right)}{\mathrm{e}^{-4x}}\,dx$$

Which simplifies to

$$u_2 = \int \left(25\cos(x)\,\mathrm{e}^{2x} + 15\ln(x)\right) dx$$

Hence

$$u_2 = 10\cos(x)\,\mathrm{e}^{2x} + 5\sin(x)\,\mathrm{e}^{2x} + 15\ln(x)\,x - 15x$$

Which simplifies to

$$u_1 = ((-10x+3)\cos(x) + (4-5x)\sin(x))\,\mathrm{e}^{2x} - \frac{15\left(\ln(x) - \frac{1}{2}\right)x^2}{2}$$

$$u_2 = 5\,\mathrm{e}^{2x}(\sin(x) + 2\cos(x)) + 15(\ln(x) - 1)\,x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(((-10x+3)\cos(x) + (4-5x)\sin(x))\,\mathrm{e}^{2x} - \frac{15\left(\ln(x) - \frac{1}{2}\right)x^2}{2}\right)\mathrm{e}^{-2x} + \left(5\,\mathrm{e}^{2x}(\sin(x) + 2\cos(x)) + 15(\ln(x) - 1)\,x\right)x\,\mathrm{e}^{-2x}$$

Which simplifies to

$$y_p(x) = \frac{15x^2\left(\ln(x) - \frac{3}{2}\right)\mathrm{e}^{-2x}}{2} + 3\cos(x) + 4\sin(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-2x} + c_2 x\,\mathrm{e}^{-2x}\right) + \left(\frac{15x^2\left(\ln(x) - \frac{3}{2}\right)\mathrm{e}^{-2x}}{2} + 3\cos(x) + 4\sin(x)\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{-2x}(c_2 x + c_1) + \frac{15x^2\left(\ln(x) - \frac{3}{2}\right)\mathrm{e}^{-2x}}{2} + 3\cos(x) + 4\sin(x)$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-2x}(c_2 x + c_1) + \frac{15x^2\left(\ln(x) - \frac{3}{2}\right)\mathrm{e}^{-2x}}{2} + 3\cos(x) + 4\sin(x) \tag{1}$$

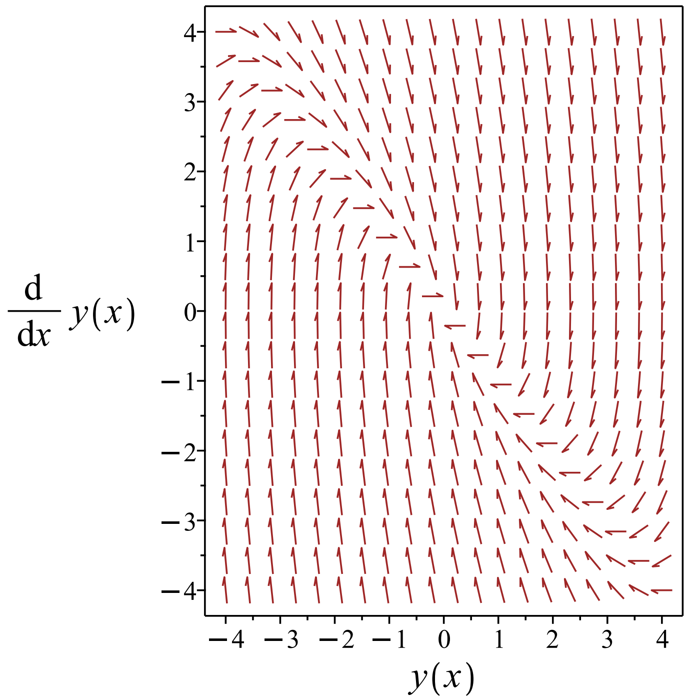


Figure 305: Slope field plot

Verification of solutions

$$y = e^{-2x}(c_2x + c_1) + \frac{15x^2\left(\ln(x) - \frac{3}{2}\right)e^{-2x}}{2} + 3\cos(x) + 4\sin(x)$$

Verified OK.

9.18.4 Maple step by step solution

Let's solve

$y'' + 4y' + 4y = 15\ln(x)\,e^{-2x} + 25\cos(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4r + 4 = 0$

- Factor the characteristic polynomial

  $(r+2)^2 = 0$

- Root of the characteristic polynomial

$r = -2$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-2x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,\mathrm{e}^{-2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-2x} + c_2 x\,\mathrm{e}^{-2x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x),y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x),y_2(x))} dx\right), f(x) = 15\ln(x)\,\mathrm{e}^{-2x} + 25\cos(\right.$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \left[\begin{array}{cc} \mathrm{e}^{-2x} & x\,\mathrm{e}^{-2x} \\ -2\,\mathrm{e}^{-2x} & \mathrm{e}^{-2x} - 2x\,\mathrm{e}^{-2x} \end{array}\right]$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = \mathrm{e}^{-4x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = 5\,\mathrm{e}^{-2x}\left(-\left(\int x(5\cos(x)\,\mathrm{e}^{2x} + 3\ln(x))\,dx\right) + x\left(\int (5\cos(x)\,\mathrm{e}^{2x} + 3\ln(x))\,dx\right)\right)$

  - Compute integrals

    $y_p(x) = \frac{15x^2\left(\ln(x) - \frac{3}{2}\right)\mathrm{e}^{-2x}}{2} + 3\cos(x) + 4\sin(x)$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-2x} + c_2 x\,\mathrm{e}^{-2x} + \frac{15x^2\left(\ln(x) - \frac{3}{2}\right)\mathrm{e}^{-2x}}{2} + 3\cos(x) + 4\sin(x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 39

```
dsolve(diff(y(x),x$2)+4*diff(y(x),x)+4*y(x)=15*exp(-2*x)*ln(x)+25*cos(x),y(x), singsol=all)
```

$$y(x) = \frac{(30 \ln (x) x^2 - 45x^2 + 4c_1 x + 4c_2) e^{-2x}}{4} + 3\cos(x) + 4\sin(x)$$

✓ Solution by Mathematica

Time used: 0.211 (sec). Leaf size: 54

```
DSolve[y''[x]+4*y'[x]+4*y[x]==15*Exp[-2*x]*Log[x]+25*Cos[x],y[x],x,IncludeSingularSolutions
```

$$y(x) \to \frac{1}{4} e^{-2x} \left(-45x^2 + 30x^2 \log(x) + 16e^{2x} \sin(x) + 12e^{2x} \cos(x) + 4c_2 x + 4c_1\right)$$

9.19 problem Problem 19

Internal problem ID [2792]
Internal file name [`OUTPUT/2284_Sunday_June_05_2022_02_57_41_AM_83086885/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.7, The Variation of Parameters Method. page 556
Problem number: Problem 19.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

`[[_3rd_order, _linear, _nonhomogeneous]]`

$$\boxed{y''' - 3y'' + 3y' - y = \frac{2\,\mathrm{e}^{x}}{x^{2}}}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - 3y'' + 3y' - y = 0$$

The characteristic equation is

$$\lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = 1$$
$$\lambda_3 = 1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^x + c_2 x\, \mathrm{e}^x + x^2 \mathrm{e}^x c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned} y_1 &= \mathrm{e}^x \\ y_2 &= x\, \mathrm{e}^x \\ y_3 &= x^2 \mathrm{e}^x \end{aligned}$$

Now the particular solution to the given ODE is found

$$y''' - 3y'' + 3y' - y = \frac{2\, \mathrm{e}^x}{x^2}$$

Let the particular solution be

$$y_p = U_1 y_1 + U_2 y_2 + U_3 y_3$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$U_i = (-1)^{n-i} \int \frac{F(x) W_i(x)}{a W(x)}\, dx$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$W = \begin{bmatrix} \mathrm{e}^x & x\, \mathrm{e}^x & x^2 \mathrm{e}^x \\ \mathrm{e}^x & \mathrm{e}^x (x+1) & \mathrm{e}^x x(x+2) \\ \mathrm{e}^x & \mathrm{e}^x (x+2) & \mathrm{e}^x (x^2+4x+2) \end{bmatrix}$$

$$|W| = 2\, \mathrm{e}^{3x}$$

The determinant simplifies to

$$|W| = 2\,\mathrm{e}^{3x}$$

Now we determine W_i for each U_i.

$$\begin{aligned}
W_1(x) &= \det\begin{bmatrix} x\,\mathrm{e}^x & x^2\mathrm{e}^x \\ \mathrm{e}^x(x+1) & \mathrm{e}^x x(x+2) \end{bmatrix} \\
&= x^2\mathrm{e}^{2x}
\end{aligned}$$

$$\begin{aligned}
W_2(x) &= \det\begin{bmatrix} \mathrm{e}^x & x^2\mathrm{e}^x \\ \mathrm{e}^x & \mathrm{e}^x x(x+2) \end{bmatrix} \\
&= 2x\,\mathrm{e}^{2x}
\end{aligned}$$

$$\begin{aligned}
W_3(x) &= \det\begin{bmatrix} \mathrm{e}^x & x\,\mathrm{e}^x \\ \mathrm{e}^x & \mathrm{e}^x(x+1) \end{bmatrix} \\
&= \mathrm{e}^{2x}
\end{aligned}$$

Now we are ready to evaluate each $U_i(x)$.

$$\begin{aligned}
U_1 &= (-1)^{3-1}\int \frac{F(x)W_1(x)}{aW(x)}\,dx \\
&= (-1)^2\int \frac{\left(\frac{2\,\mathrm{e}^x}{x^2}\right)\left(x^2\mathrm{e}^{2x}\right)}{(1)\left(2\,\mathrm{e}^{3x}\right)}\,dx \\
&= \int \frac{2\,\mathrm{e}^x\mathrm{e}^{2x}}{2\,\mathrm{e}^{3x}}\,dx \\
&= \int (1)\,dx \\
&= x
\end{aligned}$$

$$\begin{aligned}
U_2 &= (-1)^{3-2}\int \frac{F(x)W_2(x)}{aW(x)}\,dx \\
&= (-1)^1\int \frac{\left(\frac{2\,\mathrm{e}^x}{x^2}\right)\left(2x\,\mathrm{e}^{2x}\right)}{(1)\left(2\,\mathrm{e}^{3x}\right)}\,dx \\
&= -\int \frac{\frac{4\,\mathrm{e}^x\mathrm{e}^{2x}}{x}}{2\,\mathrm{e}^{3x}}\,dx \\
&= -\int \left(\frac{2}{x}\right)dx \\
&= -2\ln(x)
\end{aligned}$$

$$
\begin{aligned}
U_3 &= (-1)^{3-3} \int \frac{F(x)W_3(x)}{aW(x)}\,dx \\
&= (-1)^0 \int \frac{\left(\frac{2\,e^{x}}{x^2}\right)\left(e^{2x}\right)}{(1)\left(2\,e^{3x}\right)}\,dx \\
&= \int \frac{\frac{2\,e^{x} e^{2x}}{x^2}}{2\,e^{3x}}\,dx \\
&= \int \left(\frac{1}{x^2}\right) dx \\
&= -\frac{1}{x}
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3
$$

Hence

$$
\begin{aligned}
y_p = {} & (x)\left(e^{x}\right) \\
& + \left(-2\ln(x)\right)\left(x\,e^{x}\right) \\
& + \left(-\frac{1}{x}\right)\left(x^2 e^{x}\right)
\end{aligned}
$$

Therefore the particular solution is

$$
y_p = -2\ln(x)\,x\,e^{x}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 e^{x} + c_2 x\,e^{x} + x^2 e^{x} c_3\right) + \left(-2\ln(x)\,x\,e^{x}\right)
\end{aligned}
$$

Which simplifies to

$$
y = e^{x}\left(c_3 x^2 + c_2 x + c_1\right) - 2\ln(x)\,x\,e^{x}
$$

Summary

The solution(s) found are the following

$$
y = e^{x}\left(c_3 x^2 + c_2 x + c_1\right) - 2\ln(x)\,x\,e^{x} \tag{1}
$$

Verification of solutions

$$
y = e^{x}\left(c_3 x^2 + c_2 x + c_1\right) - 2\ln(x)\,x\,e^{x}
$$

Verified OK.

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
dsolve(diff(y(x),x$3)-3*diff(y(x),x$2)+3*diff(y(x),x)-y(x)=2*x^(-2)*exp(x),y(x), singsol=all
```

$$y(x) = e^{x}\left(-2x\ln\left(x\right)+c_1+c_2x+c_3x^2\right)$$

✓ Solution by Mathematica

Time used: 0.393 (sec). Leaf size: 627

```
DSolve[y'''[x]-6*y''[x]+3*y'[x]-y[x]==2*x^(-2)*Exp[x],y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to \frac{2i(\text{Root}[\#1^3-6\#1^2+3\#1-1\&,1]-\text{Root}[\#1^3-6\#1^2+3\#1-1\&,2])\exp\left(x\text{Root}[\#1^3-6\#1^2\right.}{}$$
$$+\frac{2i(\text{Root}[\#1^3-6\#1^2+3\#1-1\&,2]-\text{Root}[\#1^3-6\#1^2+3\#1-1\&,3])\exp\left(x\text{Root}[\#1^3-6\#1\right.}{}$$
$$-\frac{2i(\text{Root}[\#1^3-6\#1^2+3\#1-1\&,1]-\text{Root}[\#1^3-6\#1^2+3\#1-1\&,3])\exp\left(x\text{Root}[\#1^3-6\#1\right.}{}$$
$$+c_2\exp\left(x\text{Root}\left[\#1^3-6\#1^2+3\#1-1\&,2\right]\right)$$
$$+c_3\exp\left(x\text{Root}\left[\#1^3-6\#1^2+3\#1-1\&,3\right]\right)$$
$$+c_1\exp\left(x\text{Root}\left[\#1^3-6\#1^2+3\#1-1\&,1\right]\right)$$

9.20 problem Problem 20

Internal problem ID [2793]
Internal file name [`OUTPUT/2285_Sunday_June_05_2022_02_57_43_AM_95749743/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.7, The Variation of Parameters Method. page 556
Problem number: Problem 20.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

`[[_3rd_order, _linear, _nonhomogeneous]]`

$$\boxed{y''' - 6y'' + 12y' - 8y = 36\,\mathrm{e}^{2x} \ln(x)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - 6y'' + 12y' - 8y = 0$$

The characteristic equation is

$$\lambda^3 - 6\lambda^2 + 12\lambda - 8 = 0$$

The roots of the above equation are

$$\lambda_1 = 2$$
$$\lambda_2 = 2$$
$$\lambda_3 = 2$$

Therefore the homogeneous solution is

$$y_h(x) = \mathrm{e}^{2x} c_1 + x\,\mathrm{e}^{2x} c_2 + x^2 \mathrm{e}^{2x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned}
y_1 &= \mathrm{e}^{2x} \\
y_2 &= x\,\mathrm{e}^{2x} \\
y_3 &= x^2 \mathrm{e}^{2x}
\end{aligned}$$

Now the particular solution to the given ODE is found

$$y''' - 6y'' + 12y' - 8y = 36\,\mathrm{e}^{2x} \ln(x)$$

Let the particular solution be

$$y_p = U_1 y_1 + U_2 y_2 + U_3 y_3$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$U_i = (-1)^{n-i} \int \frac{F(x) W_i(x)}{a W(x)}\, dx$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$\begin{aligned}
W &= \begin{bmatrix} \mathrm{e}^{2x} & x\,\mathrm{e}^{2x} & x^2 \mathrm{e}^{2x} \\ 2\,\mathrm{e}^{2x} & \mathrm{e}^{2x}(1+2x) & 2\,\mathrm{e}^{2x} x(x+1) \\ 4\,\mathrm{e}^{2x} & 4\,\mathrm{e}^{2x}(x+1) & (4x^2+8x+2)\,\mathrm{e}^{2x} \end{bmatrix} \\
|W| &= 2\,\mathrm{e}^{6x}
\end{aligned}$$

The determinant simplifies to

$$|W| = 2\,\mathrm{e}^{6x}$$

Now we determine W_i for each U_i.

$$\begin{aligned}
W_1(x) &= \det \begin{bmatrix} x\,\mathrm{e}^{2x} & x^2\mathrm{e}^{2x} \\ \mathrm{e}^{2x}(1+2x) & 2\,\mathrm{e}^{2x}x(x+1) \end{bmatrix} \\
&= x^2\mathrm{e}^{4x}
\end{aligned}$$

$$\begin{aligned}
W_2(x) &= \det \begin{bmatrix} \mathrm{e}^{2x} & x^2\mathrm{e}^{2x} \\ 2\,\mathrm{e}^{2x} & 2\,\mathrm{e}^{2x}x(x+1) \end{bmatrix} \\
&= 2x\,\mathrm{e}^{4x}
\end{aligned}$$

$$\begin{aligned}
W_3(x) &= \det \begin{bmatrix} \mathrm{e}^{2x} & x\,\mathrm{e}^{2x} \\ 2\,\mathrm{e}^{2x} & \mathrm{e}^{2x}(1+2x) \end{bmatrix} \\
&= \mathrm{e}^{4x}
\end{aligned}$$

Now we are ready to evaluate each $U_i(x)$.

$$\begin{aligned}
U_1 &= (-1)^{3-1} \int \frac{F(x)W_1(x)}{aW(x)}\,dx \\
&= (-1)^2 \int \frac{\left(36\,\mathrm{e}^{2x}\ln(x)\right)\left(x^2\mathrm{e}^{4x}\right)}{(1)\left(2\,\mathrm{e}^{6x}\right)}\,dx \\
&= \int \frac{36\,\mathrm{e}^{2x}\ln(x)\,x^2\mathrm{e}^{4x}}{2\,\mathrm{e}^{6x}}\,dx \\
&= \int \left(18\ln(x)\,x^2\right)\,dx \\
&= 6x^3\ln(x) - 2x^3
\end{aligned}$$

$$\begin{aligned}
U_2 &= (-1)^{3-2} \int \frac{F(x)W_2(x)}{aW(x)}\,dx \\
&= (-1)^1 \int \frac{\left(36\,\mathrm{e}^{2x}\ln(x)\right)\left(2x\,\mathrm{e}^{4x}\right)}{(1)\left(2\,\mathrm{e}^{6x}\right)}\,dx \\
&= -\int \frac{72\,\mathrm{e}^{2x}\ln(x)\,x\,\mathrm{e}^{4x}}{2\,\mathrm{e}^{6x}}\,dx \\
&= -\int \left(36\ln(x)\,x\right)\,dx \\
&= -18\ln(x)\,x^2 + 9x^2 \\
&= -18\ln(x)\,x^2 + 9x^2
\end{aligned}$$

$$
\begin{aligned}
U_3 &= (-1)^{3-3} \int \frac{F(x)W_3(x)}{aW(x)}\,dx \\
&= (-1)^0 \int \frac{(36\,\mathrm{e}^{2x}\ln(x))\,(\mathrm{e}^{4x})}{(1)\,(2\,\mathrm{e}^{6x})}\,dx \\
&= \int \frac{36\,\mathrm{e}^{2x}\ln(x)\,\mathrm{e}^{4x}}{2\,\mathrm{e}^{6x}}\,dx \\
&= \int (18\ln(x))\,dx \\
&= 18\ln(x)\,x - 18x
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1y_1 + U_2y_2 + U_3y_3
$$

Hence

$$
\begin{aligned}
y_p = &\left(6x^3\ln(x) - 2x^3\right)\left(\mathrm{e}^{2x}\right) \\
&+ \left(-18\ln(x)\,x^2 + 9x^2\right)\left(x\,\mathrm{e}^{2x}\right) \\
&+ (18\ln(x)\,x - 18x)\left(x^2\mathrm{e}^{2x}\right)
\end{aligned}
$$

Therefore the particular solution is

$$
y_p = x^3\mathrm{e}^{2x}(-11 + 6\ln(x))
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\mathrm{e}^{2x}c_1 + x\,\mathrm{e}^{2x}c_2 + x^2\mathrm{e}^{2x}c_3\right) + \left(x^3\mathrm{e}^{2x}(-11 + 6\ln(x))\right)
\end{aligned}
$$

Which simplifies to

$$
y = \mathrm{e}^{2x}\left(c_3x^2 + c_2x + c_1\right) + x^3\mathrm{e}^{2x}(-11 + 6\ln(x))
$$

Summary

The solution(s) found are the following

$$
y = \mathrm{e}^{2x}\left(c_3x^2 + c_2x + c_1\right) + x^3\mathrm{e}^{2x}(-11 + 6\ln(x)) \tag{1}
$$

Verification of solutions

$$
y = \mathrm{e}^{2x}\left(c_3x^2 + c_2x + c_1\right) + x^3\mathrm{e}^{2x}(-11 + 6\ln(x))
$$

Verified OK.

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 31

```
dsolve(diff(y(x),x$3)-6*diff(y(x),x$2)+12*diff(y(x),x)-8*y(x)=36*exp(2*x)*ln(x),y(x), singso
```

$$y(x) = \mathrm{e}^{2x}\left(6x^3 \ln\left(x\right) - 11x^3 + c_1 + c_2x + c_3x^2\right)$$

✓ Solution by Mathematica

Time used: 0.009 (sec). Leaf size: 36

```
DSolve[y'''[x]-6*y''[x]+12*y'[x]-8*y[x]==36*Exp[2*x]*Log[x],y[x],x,IncludeSingularSolutions
```

$$y(x) \to e^{2x}\left(-11x^3 + 6x^3 \log(x) + c_3x^2 + c_2x + c_1\right)$$

9.21 problem Problem 21

Internal problem ID [2794]
Internal file name [`OUTPUT/2286_Sunday_June_05_2022_02_57_46_AM_89412987/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.7, The Variation of Parameters Method. page 556
Problem number: Problem 21.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

`[[_3rd_order, _linear, _nonhomogeneous]]`

$$\boxed{y''' + 3y'' + 3y' + y = \frac{2\,\mathrm{e}^{-x}}{x^2 + 1}}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' + 3y'' + 3y' + y = 0$$

The characteristic equation is

$$\lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0$$

The roots of the above equation are

$$\lambda_1 = -1$$
$$\lambda_2 = -1$$
$$\lambda_3 = -1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + c_2 x\, e^{-x} + x^2 e^{-x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-x}$$
$$y_2 = x\, e^{-x}$$
$$y_3 = e^{-x} x^2$$

Now the particular solution to the given ODE is found

$$y''' + 3y'' + 3y' + y = \frac{2\, e^{-x}}{x^2 + 1}$$

Let the particular solution be

$$y_p = U_1 y_1 + U_2 y_2 + U_3 y_3$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$U_i = (-1)^{n-i} \int \frac{F(x) W_i(x)}{a W(x)}\, dx$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$W = \begin{bmatrix} e^{-x} & x\, e^{-x} & e^{-x} x^2 \\ -e^{-x} & e^{-x}(1-x) & -e^{-x} x(x-2) \\ e^{-x} & e^{-x}(x-2) & e^{-x}(x^2 - 4x + 2) \end{bmatrix}$$
$$|W| = 2\, e^{-3x}$$

The determinant simplifies to

$$|W| = 2\,\mathrm{e}^{-3x}$$

Now we determine W_i for each U_i.

$$\begin{aligned}
W_1(x) &= \det\begin{bmatrix} x\,\mathrm{e}^{-x} & \mathrm{e}^{-x}x^2 \\ \mathrm{e}^{-x}(1-x) & -\mathrm{e}^{-x}x(x-2) \end{bmatrix} \\
&= \mathrm{e}^{-2x}x^2
\end{aligned}$$

$$\begin{aligned}
W_2(x) &= \det\begin{bmatrix} \mathrm{e}^{-x} & \mathrm{e}^{-x}x^2 \\ -\mathrm{e}^{-x} & -\mathrm{e}^{-x}x(x-2) \end{bmatrix} \\
&= 2x\,\mathrm{e}^{-2x}
\end{aligned}$$

$$\begin{aligned}
W_3(x) &= \det\begin{bmatrix} \mathrm{e}^{-x} & x\,\mathrm{e}^{-x} \\ -\mathrm{e}^{-x} & \mathrm{e}^{-x}(1-x) \end{bmatrix} \\
&= \mathrm{e}^{-2x}
\end{aligned}$$

Now we are ready to evaluate each $U_i(x)$.

$$\begin{aligned}
U_1 &= (-1)^{3-1}\int \frac{F(x)W_1(x)}{aW(x)}\,dx \\
&= (-1)^2\int \frac{\left(\frac{2\,\mathrm{e}^{-x}}{x^2+1}\right)\left(\mathrm{e}^{-2x}x^2\right)}{(1)\left(2\,\mathrm{e}^{-3x}\right)}\,dx \\
&= \int \frac{\frac{2\,\mathrm{e}^{-x}\mathrm{e}^{-2x}x^2}{x^2+1}}{2\,\mathrm{e}^{-3x}}\,dx \\
&= \int \left(\frac{x^2}{x^2+1}\right)dx \\
&= x - \arctan(x)
\end{aligned}$$

$$
\begin{aligned}
U_2 &= (-1)^{3-2} \int \frac{F(x)W_2(x)}{aW(x)}\, dx \\
&= (-1)^1 \int \frac{\left(\frac{2\,\mathrm{e}^{-x}}{x^2+1}\right) \left(2x\,\mathrm{e}^{-2x}\right)}{(1)\left(2\,\mathrm{e}^{-3x}\right)}\, dx \\
&= -\int \frac{\frac{4\,\mathrm{e}^{-x} x\,\mathrm{e}^{-2x}}{x^2+1}}{2\,\mathrm{e}^{-3x}}\, dx \\
&= -\int \left(\frac{2x}{x^2+1}\right) dx \\
&= -\ln\left(x^2+1\right)
\end{aligned}
$$

$$
\begin{aligned}
U_3 &= (-1)^{3-3} \int \frac{F(x)W_3(x)}{aW(x)}\, dx \\
&= (-1)^0 \int \frac{\left(\frac{2\,\mathrm{e}^{-x}}{x^2+1}\right) \left(\mathrm{e}^{-2x}\right)}{(1)\left(2\,\mathrm{e}^{-3x}\right)}\, dx \\
&= \int \frac{\frac{2\,\mathrm{e}^{-x}\mathrm{e}^{-2x}}{x^2+1}}{2\,\mathrm{e}^{-3x}}\, dx \\
&= \int \left(\frac{1}{x^2+1}\right) dx \\
&= \arctan(x)
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3
$$

Hence

$$
\begin{aligned}
y_p = &(x - \arctan(x))\left(\mathrm{e}^{-x}\right) \\
&+ \left(-\ln\left(x^2+1\right)\right)\left(x\,\mathrm{e}^{-x}\right) \\
&+ (\arctan(x))\left(\mathrm{e}^{-x} x^2\right)
\end{aligned}
$$

Therefore the particular solution is

$$
y_p = \mathrm{e}^{-x}\left(x - \arctan(x) - \ln\left(x^2+1\right)x + \arctan(x)\,x^2\right)
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \mathrm{e}^{-x} + c_2 x\,\mathrm{e}^{-x} + x^2 \mathrm{e}^{-x} c_3\right) + \left(\mathrm{e}^{-x}\left(x - \arctan(x) - \ln\left(x^2+1\right)x + \arctan(x)\,x^2\right)\right)
\end{aligned}
$$

Which simplifies to

$$y = e^{-x}(c_3x^2 + c_2x + c_1) + e^{-x}(x - \arctan(x) - \ln(x^2+1)x + \arctan(x)x^2)$$

Summary

The solution(s) found are the following

$$y = e^{-x}(c_3x^2 + c_2x + c_1) + e^{-x}(x - \arctan(x) - \ln(x^2+1)x + \arctan(x)x^2) \quad (1)$$

Verification of solutions

$$y = e^{-x}(c_3x^2 + c_2x + c_1) + e^{-x}(x - \arctan(x) - \ln(x^2+1)x + \arctan(x)x^2)$$

Verified OK.

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 39

```
dsolve(diff(y(x),x$3)+3*diff(y(x),x$2)+3*diff(y(x),x)+y(x)=2*exp(-x)/(1+x^2),y(x), singsol=a
```

$$y(x) = e^{-x}(x^2\arctan(x) - x\ln(x^2+1) - \arctan(x) + x + c_1 + c_2x + c_3x^2)$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 42

```
DSolve[y'''[x]+3*y''[x]+3*y'[x]+y[x]==2*Exp[-x]/(1+x^2),y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to e^{-x}((x^2-1)\arctan(x) - x\log(x^2+1) + c_3x^2 + x + c_2x + c_1)$$

9.22 problem Problem 22

Internal problem ID [2795]
Internal file name [`OUTPUT/2287_Sunday_June_05_2022_02_57_49_AM_20383296/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.7, The Variation of Parameters Method. page 556
Problem number: Problem 22.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_y]]
```

$$\boxed{y''' - 6y'' + 9y' = 12\,\mathrm{e}^{3x}}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - 6y'' + 9y' = 0$$

The characteristic equation is

$$\lambda^3 - 6\lambda^2 + 9\lambda = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 3$$
$$\lambda_3 = 3$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 + c_2 e^{3x} + x\, e^{3x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = 1$$
$$y_2 = e^{3x}$$
$$y_3 = x\, e^{3x}$$

Now the particular solution to the given ODE is found

$$y''' - 6y'' + 9y' = 12\, e^{3x}$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$12\, e^{3x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{3x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, x\, e^{3x}, e^{3x}\}$$

Since e^{3x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\, e^{3x}\}]$$

Since $x\, e^{3x}$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2 e^{3x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^2 e^{3x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$6A_1 e^{3x} = 12\, e^{3x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2x^2 e^{3x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 + c_2 e^{3x} + x\, e^{3x} c_3\right) + \left(2x^2 e^{3x}\right) \end{aligned}$$

Which simplifies to

$$y = (c_3 x + c_2)\, e^{3x} + c_1 + 2x^2 e^{3x}$$

Summary

The solution(s) found are the following

$$y = (c_3 x + c_2)\, e^{3x} + c_1 + 2x^2 e^{3x} \tag{1}$$

Verification of solutions

$$y = (c_3 x + c_2)\, e^{3x} + c_1 + 2x^2 e^{3x}$$

Verified OK.

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
-> Calling odsolve with the ODE`, diff(diff(_b(_a), _a), _a) = 6*(diff(_b(_a), _a))-9*_b(_a)
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
   trying a double symmetry of the form [xi=0, eta=F(x)]
   -> Try solving first the homogeneous part of the ODE
      checking if the LODE has constant coefficients
      <- constant coefficients successful
   <- solving first the homogeneous part of the ODE successful
<- differential order: 3; linear nonhomogeneous with symmetry [0,1] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 31

```
dsolve(diff(y(x),x$3)-6*diff(y(x),x$2)+9*diff(y(x),x)=12*exp(3*x),y(x), singsol=all)
```

$$y(x) = \frac{(4 + 18x^2 + 3(-4 + c_1)\, x - c_1 + 3c_2)\, e^{3x}}{9} + c_3$$

✓ Solution by Mathematica

Time used: 0.06 (sec). Leaf size: 39

```
DSolve[y'''[x]-6*y''[x]+9*y'[x]==12*Exp[3*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{9} e^{3x} \left(18x^2 + 3(-4 + c_2)x + 4 + 3c_1 - c_2\right) + c_3$$

9.23 problem Problem 23

Internal problem ID [2796]
Internal file name [`OUTPUT/2288_Sunday_June_05_2022_02_57_51_AM_50500525/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.7, The Variation of Parameters Method. page 556
Problem number: Problem 23.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' - 9y = F(x)}$$

9.23.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = -9, f(x) = F(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 9y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -9$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 9\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -9$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(-9)} \\
&= \pm 3
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= +3 \\
\lambda_2 &= -3
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= 3 \\
\lambda_2 &= -3
\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}
y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\
y &= c_1 e^{(3)x} + c_2 e^{(-3)x}
\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{3x} + c_2 \mathrm{e}^{-3x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{3x} + c_2 \mathrm{e}^{-3x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^{3x}$$
$$y_2 = e^{-3x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{3x} & e^{-3x} \\ \frac{d}{dx}(e^{3x}) & \frac{d}{dx}(e^{-3x}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{3x} & e^{-3x} \\ 3\,e^{3x} & -3\,e^{-3x} \end{vmatrix}$$

Therefore

$$W = (e^{3x})\left(-3\,e^{-3x}\right) - \left(e^{-3x}\right)\left(3\,e^{3x}\right)$$

Which simplifies to

$$W = -6\,e^{3x} e^{-3x}$$

Which simplifies to

$$W = -6$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{e^{-3x} F(x)}{-6} \, dx$$

Which simplifies to

$$u_1 = -\int -\frac{e^{-3x} F(x)}{6} dx$$

Hence

$$u_1 = -\left(\int_0^x -\frac{e^{-3\alpha} F(\alpha)}{6} d\alpha\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{3x} F(x)}{-6} \, dx$$

Which simplifies to

$$u_2 = \int -\frac{e^{3x} F(x)}{6} dx$$

Hence

$$u_2 = \int_0^x -\frac{e^{3\alpha} F(\alpha)}{6} d\alpha$$

Which simplifies to

$$u_1 = \frac{\left(\int_0^x e^{-3\alpha} F(\alpha) \, d\alpha\right)}{6}$$
$$u_2 = -\frac{\left(\int_0^x e^{3\alpha} F(\alpha) \, d\alpha\right)}{6}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\left(\int_0^x e^{-3\alpha} F(\alpha) \, d\alpha\right) e^{3x}}{6} - \frac{\left(\int_0^x e^{3\alpha} F(\alpha) \, d\alpha\right) e^{-3x}}{6}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{3x} + c_2 e^{-3x}\right) + \left(\frac{\left(\int_0^x e^{-3\alpha} F(\alpha) \, d\alpha\right) e^{3x}}{6} - \frac{\left(\int_0^x e^{3\alpha} F(\alpha) \, d\alpha\right) e^{-3x}}{6}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$
y = c_1 e^{3x} + c_2 e^{-3x} + \frac{\left(\int_0^x e^{-3\alpha} F(\alpha) \, d\alpha\right) e^{3x}}{6} - \frac{\left(\int_0^x e^{3\alpha} F(\alpha) \, d\alpha\right) e^{-3x}}{6} \tag{1}
$$

Verification of solutions

$$
y = c_1 e^{3x} + c_2 e^{-3x} + \frac{\left(\int_0^x e^{-3\alpha} F(\alpha) \, d\alpha\right) e^{3x}}{6} - \frac{\left(\int_0^x e^{3\alpha} F(\alpha) \, d\alpha\right) e^{-3x}}{6}
$$

Verified OK.

9.23.2 Solving using Kovacic algorithm

Writing the ode as

$$
\begin{align*}
y'' - 9y &= 0 \tag{1} \\
Ay'' + By' + Cy &= 0 \tag{2}
\end{align*}
$$

Comparing (1) and (2) shows that

$$
\begin{align*}
A &= 1 \\
B &= 0 \tag{3} \\
C &= -9
\end{align*}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A} \, dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \tag{4}
$$

Where r is given by

$$
\begin{align*}
r &= \frac{s}{t} \tag{5} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{align*}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{9}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$s = 9$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 9z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 288: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 9$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-3x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \mathrm{e}^{-3x} \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-3x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \mathrm{e}^{-3x} \int \frac{1}{\mathrm{e}^{-6x}}\,dx \\ &= \mathrm{e}^{-3x} \left(\frac{\mathrm{e}^{6x}}{6}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-3x}\right) + c_2 \left(\mathrm{e}^{-3x} \left(\frac{\mathrm{e}^{6x}}{6}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 9y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-3x} + \frac{c_2 \mathrm{e}^{3x}}{6}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \mathrm{e}^{-3x}$$
$$y_2 = \frac{\mathrm{e}^{3x}}{6}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{-3x} & \frac{\mathrm{e}^{3x}}{6} \\ \frac{d}{dx}\left(\mathrm{e}^{-3x}\right) & \frac{d}{dx}\left(\frac{\mathrm{e}^{3x}}{6}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{-3x} & \frac{\mathrm{e}^{3x}}{6} \\ -3\,\mathrm{e}^{-3x} & \frac{\mathrm{e}^{3x}}{2} \end{vmatrix}$$

Therefore

$$W = \left(\mathrm{e}^{-3x}\right)\left(\frac{\mathrm{e}^{3x}}{2}\right) - \left(\frac{\mathrm{e}^{3x}}{6}\right)\left(-3\,\mathrm{e}^{-3x}\right)$$

Which simplifies to

$$W = \mathrm{e}^{3x}\mathrm{e}^{-3x}$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\mathrm{e}^{3x}F(x)}{6}}{1}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{\mathrm{e}^{3x}F(x)}{6}dx$$

Hence

$$u_1 = -\left(\int_0^x \frac{\mathrm{e}^{3\alpha}F(\alpha)}{6}d\alpha\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\mathrm{e}^{-3x}F(x)}{1}\,dx$$

Which simplifies to

$$u_2 = \int \mathrm{e}^{-3x}F(x)\,dx$$

Hence

$$u_2 = \int_0^x \mathrm{e}^{-3\alpha}F(\alpha)\,d\alpha$$

Which simplifies to

$$u_1 = -\frac{\left(\int_0^x \mathrm{e}^{3\alpha} F(\alpha)\, d\alpha\right)}{6}$$

$$u_2 = \int_0^x \mathrm{e}^{-3\alpha} F(\alpha)\, d\alpha$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{\left(\int_0^x \mathrm{e}^{3\alpha} F(\alpha)\, d\alpha\right) \mathrm{e}^{-3x}}{6} + \frac{\left(\int_0^x \mathrm{e}^{-3\alpha} F(\alpha)\, d\alpha\right) \mathrm{e}^{3x}}{6}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-3x} + \frac{c_2 \mathrm{e}^{3x}}{6}\right) + \left(-\frac{\left(\int_0^x \mathrm{e}^{3\alpha} F(\alpha)\, d\alpha\right) \mathrm{e}^{-3x}}{6} + \frac{\left(\int_0^x \mathrm{e}^{-3\alpha} F(\alpha)\, d\alpha\right) \mathrm{e}^{3x}}{6}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-3x} + \frac{c_2 \mathrm{e}^{3x}}{6} - \frac{\left(\int_0^x \mathrm{e}^{3\alpha} F(\alpha)\, d\alpha\right) \mathrm{e}^{-3x}}{6} + \frac{\left(\int_0^x \mathrm{e}^{-3\alpha} F(\alpha)\, d\alpha\right) \mathrm{e}^{3x}}{6} \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{-3x} + \frac{c_2 \mathrm{e}^{3x}}{6} - \frac{\left(\int_0^x \mathrm{e}^{3\alpha} F(\alpha)\, d\alpha\right) \mathrm{e}^{-3x}}{6} + \frac{\left(\int_0^x \mathrm{e}^{-3\alpha} F(\alpha)\, d\alpha\right) \mathrm{e}^{3x}}{6}$$

Verified OK.

9.23.3 Maple step by step solution

Let's solve

$y'' - 9y = F(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 9 = 0$

- Factor the characteristic polynomial

  $(r-3)(r+3)=0$
- Roots of the characteristic polynomial

  $r=(-3,3)$
- 1st solution of the homogeneous ODE

  $y_1(x)=\mathrm{e}^{-3x}$
- 2nd solution of the homogeneous ODE

  $y_2(x)=\mathrm{e}^{3x}$
- General solution of the ODE

  $y=c_1y_1(x)+c_2y_2(x)+y_p(x)$
- Substitute in solutions of the homogeneous ODE

  $y=c_1\mathrm{e}^{-3x}+c_2\mathrm{e}^{3x}+y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x)=-y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right)+y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x)=F(x)\right]$
  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x),y_2(x))=\begin{bmatrix} \mathrm{e}^{-3x} & \mathrm{e}^{3x} \\ -3\,\mathrm{e}^{-3x} & 3\,\mathrm{e}^{3x} \end{bmatrix}$
  - Compute Wronskian

    $W(y_1(x),y_2(x))=6$
  - Substitute functions into equation for $y_p(x)$

    $y_p(x)=-\frac{\mathrm{e}^{-3x}\left(\int \mathrm{e}^{3x}F(x)dx\right)}{6}+\frac{\mathrm{e}^{3x}\left(\int \mathrm{e}^{-3x}F(x)dx\right)}{6}$
  - Compute integrals

    $y_p(x)=-\frac{\mathrm{e}^{-3x}\left(\int \mathrm{e}^{3x}F(x)dx\right)}{6}+\frac{\mathrm{e}^{3x}\left(\int \mathrm{e}^{-3x}F(x)dx\right)}{6}$
- Substitute particular solution into general solution to ODE

  $y=c_1\mathrm{e}^{-3x}+c_2\mathrm{e}^{3x}-\frac{\mathrm{e}^{-3x}\left(\int \mathrm{e}^{3x}F(x)dx\right)}{6}+\frac{\mathrm{e}^{3x}\left(\int \mathrm{e}^{-3x}F(x)dx\right)}{6}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 47

```
dsolve(diff(y(x),x$2)-9*y(x)=F(x),y(x), singsol=all)
```

$$y(x) = c_2 e^{3x} + c_1 e^{-3x} + \frac{\left(\int e^{-3x} F(x)\, dx\right) e^{3x}}{6} - \frac{\left(\int e^{3x} F(x)\, dx\right) e^{-3x}}{6}$$

✓ Solution by Mathematica

Time used: 0.063 (sec). Leaf size: 66

```
DSolve[y''[x]-y[x]==F[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x}\left(e^{2x} \int_1^x \frac{1}{2} e^{-K[1]} F(K[1]) dK[1] + \int_1^x -\frac{1}{2} e^{K[2]} F(K[2]) dK[2] + c_1 e^{2x} + c_2\right)$$

9.24 problem Problem 24

Internal problem ID [2797]
Internal file name [`OUTPUT/2289_Sunday_June_05_2022_02_57_54_AM_36733998/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.7, The Variation of Parameters Method. page 556
Problem number: Problem 24.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 5y' + 4y = F(x)}$$

9.24.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 5, C = 4, f(x) = F(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 5y' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 5, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 5\lambda\, \mathrm{e}^{\lambda x} + 4\, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 5\lambda + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 5, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-5}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{5^2 - (4)\,(1)\,(4)}\\ &= -\frac{5}{2} \pm \frac{3}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{5}{2} + \frac{3}{2}\\ \lambda_2 &= -\frac{5}{2} - \frac{3}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -1\\ \lambda_2 &= -4\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(-1)x} + c_2 e^{(-4)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{-4x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{-4x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \mathrm{e}^{-x}$$
$$y_2 = \mathrm{e}^{-4x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{-x} & \mathrm{e}^{-4x} \\ \frac{d}{dx}(\mathrm{e}^{-x}) & \frac{d}{dx}(\mathrm{e}^{-4x}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{-x} & \mathrm{e}^{-4x} \\ -\mathrm{e}^{-x} & -4\,\mathrm{e}^{-4x} \end{vmatrix}$$

Therefore

$$W = \left(\mathrm{e}^{-x}\right)\left(-4\,\mathrm{e}^{-4x}\right) - \left(\mathrm{e}^{-4x}\right)\left(-\mathrm{e}^{-x}\right)$$

Which simplifies to

$$W = -3\,\mathrm{e}^{-x}\mathrm{e}^{-4x}$$

Which simplifies to

$$W = -3\,e^{-5x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{e^{-4x} F(x)}{-3\,e^{-5x}}\,dx$$

Which simplifies to

$$u_1 = -\int -\frac{F(x)\,e^{x}}{3}dx$$

Hence

$$u_1 = -\left(\int_0^x -\frac{F(\alpha)\,e^{\alpha}}{3}d\alpha\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{-x} F(x)}{-3\,e^{-5x}}\,dx$$

Which simplifies to

$$u_2 = \int -\frac{F(x)\,e^{4x}}{3}dx$$

Hence

$$u_2 = \int_0^x -\frac{F(\alpha)\,e^{4\alpha}}{3}d\alpha$$

Which simplifies to

$$u_1 = \frac{\left(\int_0^x F(\alpha)\,e^{\alpha}d\alpha\right)}{3}$$

$$u_2 = -\frac{\left(\int_0^x F(\alpha)\,e^{4\alpha}d\alpha\right)}{3}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\left(\int_0^x F(\alpha)\,e^{\alpha}d\alpha\right)e^{-x}}{3} - \frac{\left(\int_0^x F(\alpha)\,e^{4\alpha}d\alpha\right)e^{-4x}}{3}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{-4x}\right) + \left(\frac{\left(\int_0^x F(\alpha)\, \mathrm{e}^{\alpha} d\alpha\right) \mathrm{e}^{-x}}{3} - \frac{\left(\int_0^x F(\alpha)\, \mathrm{e}^{4\alpha} d\alpha\right) \mathrm{e}^{-4x}}{3}\right)
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{-4x} + \frac{\left(\int_0^x F(\alpha)\, \mathrm{e}^{\alpha} d\alpha\right) \mathrm{e}^{-x}}{3} - \frac{\left(\int_0^x F(\alpha)\, \mathrm{e}^{4\alpha} d\alpha\right) \mathrm{e}^{-4x}}{3} \tag{1}
$$

<u>Verification of solutions</u>

$$
y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{-4x} + \frac{\left(\int_0^x F(\alpha)\, \mathrm{e}^{\alpha} d\alpha\right) \mathrm{e}^{-x}}{3} - \frac{\left(\int_0^x F(\alpha)\, \mathrm{e}^{4\alpha} d\alpha\right) \mathrm{e}^{-4x}}{3}
$$

Verified OK.

9.24.2 Solving using Kovacic algorithm

Writing the ode as

$$
y'' + 5y' + 4y = 0 \tag{1}
$$
$$
Ay'' + By' + Cy = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= 5 \\
C &= 4
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 9 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{9z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 290: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{9}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{3x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{5}{1}\,dx} \\
&= z_1 e^{-\frac{5x}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{5x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-4x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{5}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-5x}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{\mathrm{e}^{3x}}{3}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-4x}\right) + c_2\left(\mathrm{e}^{-4x}\left(\frac{\mathrm{e}^{3x}}{3}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 5y' + 4y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1\mathrm{e}^{-4x} + \frac{c_2\mathrm{e}^{-x}}{3}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \mathrm{e}^{-4x} \\
y_2 &= \frac{\mathrm{e}^{-x}}{3}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{-4x} & \frac{e^{-x}}{3} \\ \frac{d}{dx}\left(e^{-4x}\right) & \frac{d}{dx}\left(\frac{e^{-x}}{3}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-4x} & \frac{e^{-x}}{3} \\ -4\,e^{-4x} & -\frac{e^{-x}}{3} \end{vmatrix}$$

Therefore

$$W = \left(e^{-4x}\right)\left(-\frac{e^{-x}}{3}\right) - \left(\frac{e^{-x}}{3}\right)\left(-4\,e^{-4x}\right)$$

Which simplifies to

$$W = e^{-x} e^{-4x}$$

Which simplifies to

$$W = e^{-5x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{e^{-x} F(x)}{3}}{e^{-5x}}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{F(x)\,e^{4x}}{3} dx$$

Hence

$$u_1 = -\left(\int_0^x \frac{F(\alpha)\,e^{4\alpha}}{3} d\alpha\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{-4x} F(x)}{e^{-5x}}\, dx$$

Which simplifies to

$$u_2 = \int F(x)\,\mathrm{e}^{x} dx$$

Hence

$$u_2 = \int_0^x F(\alpha)\,\mathrm{e}^{\alpha} d\alpha$$

Which simplifies to

$$u_1 = -\frac{\left(\int_0^x F(\alpha)\,\mathrm{e}^{4\alpha} d\alpha\right)}{3}$$
$$u_2 = \int_0^x F(\alpha)\,\mathrm{e}^{\alpha} d\alpha$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{\left(\int_0^x F(\alpha)\,\mathrm{e}^{4\alpha} d\alpha\right)\mathrm{e}^{-4x}}{3} + \frac{\left(\int_0^x F(\alpha)\,\mathrm{e}^{\alpha} d\alpha\right)\mathrm{e}^{-x}}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-4x} + \frac{c_2 \mathrm{e}^{-x}}{3}\right) + \left(-\frac{\left(\int_0^x F(\alpha)\,\mathrm{e}^{4\alpha} d\alpha\right)\mathrm{e}^{-4x}}{3} + \frac{\left(\int_0^x F(\alpha)\,\mathrm{e}^{\alpha} d\alpha\right)\mathrm{e}^{-x}}{3}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-4x} + \frac{c_2 \mathrm{e}^{-x}}{3} - \frac{\left(\int_0^x F(\alpha)\,\mathrm{e}^{4\alpha} d\alpha\right)\mathrm{e}^{-4x}}{3} + \frac{\left(\int_0^x F(\alpha)\,\mathrm{e}^{\alpha} d\alpha\right)\mathrm{e}^{-x}}{3} \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{-4x} + \frac{c_2 \mathrm{e}^{-x}}{3} - \frac{\left(\int_0^x F(\alpha)\,\mathrm{e}^{4\alpha} d\alpha\right)\mathrm{e}^{-4x}}{3} + \frac{\left(\int_0^x F(\alpha)\,\mathrm{e}^{\alpha} d\alpha\right)\mathrm{e}^{-x}}{3}$$

Verified OK.

9.24.3 Maple step by step solution

Let's solve

$y'' + 5y' + 4y = F(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 5r + 4 = 0$

- Factor the characteristic polynomial

  $(r + 4)(r + 1) = 0$

- Roots of the characteristic polynomial

  $r = (-4, -1)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-4x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{-x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-4x} + c_2 e^{-x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ∘ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = F(x)\right]$

- ∘ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-4x} & e^{-x} \\ -4 e^{-4x} & -e^{-x} \end{bmatrix}$

- ∘ Compute Wronskian

  $W(y_1(x), y_2(x)) = 3 e^{-5x}$

- ∘ Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\frac{e^{-4x} \left(\int F(x) e^{4x} dx\right)}{3} + \frac{e^{-x} \left(\int F(x) e^{x} dx\right)}{3}$$

- Compute integrals

$$y_p(x) = -\frac{e^{-4x} \left(\int F(x) e^{4x} dx\right)}{3} + \frac{e^{-x} \left(\int F(x) e^{x} dx\right)}{3}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 e^{-4x} + c_2 e^{-x} - \frac{e^{-4x} \left(\int F(x) e^{4x} dx\right)}{3} + \frac{e^{-x} \left(\int F(x) e^{x} dx\right)}{3}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 45

```
dsolve(diff(y(x),x$2)+5*diff(y(x),x)+4*y(x)=F(x),y(x), singsol=all)
```

$$y(x) = e^{-x} c_2 + e^{-4x} c_1 + \frac{\left(\int e^{x} F(x)\, dx\right) e^{-x}}{3} - \frac{\left(\int F(x)\, e^{4x} dx\right) e^{-4x}}{3}$$

✓ Solution by Mathematica

Time used: 0.06 (sec). Leaf size: 66

```
DSolve[y''[x]+5*y'[x]+4*y[x]==F[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-4x} \left(\int_1^x -\frac{1}{3} e^{4K[1]} F(K[1]) dK[1] + e^{3x} \int_1^x \frac{1}{3} e^{K[2]} F(K[2]) dK[2] + c_2 e^{3x} + c_1 \right)$$

9.25 problem Problem 25

Internal problem ID [2798]
Internal file name [`OUTPUT/2290_Sunday_June_05_2022_02_57_57_AM_20886169/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.7, The Variation of Parameters Method. page 556
Problem number: Problem 25.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y' - 2y = F(x)}$$

9.25.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 1, C = -2, f(x) = F(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' - 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = -2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + \lambda e^{\lambda x} - 2 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda - 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = -2$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{-1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{1^2 - (4)(1)(-2)} \\
&= -\frac{1}{2} \pm \frac{3}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= -\frac{1}{2} + \frac{3}{2} \\
\lambda_2 &= -\frac{1}{2} - \frac{3}{2}
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= 1 \\
\lambda_2 &= -2
\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}
y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\
y &= c_1 e^{(1)x} + c_2 e^{(-2)x}
\end{aligned}$$

Or

$$y = c_1 e^{x} + c_2 e^{-2x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{x} + c_2 e^{-2x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^x$$
$$y_2 = e^{-2x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^x & e^{-2x} \\ \frac{d}{dx}(e^x) & \frac{d}{dx}(e^{-2x}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^x & e^{-2x} \\ e^x & -2\,e^{-2x} \end{vmatrix}$$

Therefore

$$W = (e^x)\left(-2\,e^{-2x}\right) - \left(e^{-2x}\right)(e^x)$$

Which simplifies to

$$W = -3\,e^{-2x} e^x$$

Which simplifies to

$$W = -3\,\mathrm{e}^{-x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\mathrm{e}^{-2x} F(x)}{-3\,\mathrm{e}^{-x}}\,dx$$

Which simplifies to

$$u_1 = -\int -\frac{\mathrm{e}^{-x} F(x)}{3} dx$$

Hence

$$u_1 = -\left(\int_0^x -\frac{\mathrm{e}^{-\alpha} F(\alpha)}{3} d\alpha\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{F(x)\,\mathrm{e}^{x}}{-3\,\mathrm{e}^{-x}}\,dx$$

Which simplifies to

$$u_2 = \int -\frac{F(x)\,\mathrm{e}^{2x}}{3} dx$$

Hence

$$u_2 = \int_0^x -\frac{F(\alpha)\,\mathrm{e}^{2\alpha}}{3} d\alpha$$

Which simplifies to

$$u_1 = \frac{\left(\int_0^x \mathrm{e}^{-\alpha} F(\alpha)\,d\alpha\right)}{3}$$

$$u_2 = -\frac{\left(\int_0^x F(\alpha)\,\mathrm{e}^{2\alpha} d\alpha\right)}{3}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\left(\int_0^x \mathrm{e}^{-\alpha} F(\alpha)\,d\alpha\right)\mathrm{e}^{x}}{3} - \frac{\left(\int_0^x F(\alpha)\,\mathrm{e}^{2\alpha} d\alpha\right)\mathrm{e}^{-2x}}{3}$$

Which simplifies to

$$y_p(x) = -\frac{\left(-\left(\int_0^x \mathrm{e}^{-\alpha} F(\alpha)\, d\alpha\right) \mathrm{e}^{3x} + \int_0^x F(\alpha)\, \mathrm{e}^{2\alpha} d\alpha\right) \mathrm{e}^{-2x}}{3}$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-2x}\right) + \left(-\frac{\left(-\left(\int_0^x \mathrm{e}^{-\alpha} F(\alpha)\, d\alpha\right) \mathrm{e}^{3x} + \int_0^x F(\alpha)\, \mathrm{e}^{2\alpha} d\alpha\right) \mathrm{e}^{-2x}}{3}\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-2x} - \frac{\left(-\left(\int_0^x \mathrm{e}^{-\alpha} F(\alpha)\, d\alpha\right) \mathrm{e}^{3x} + \int_0^x F(\alpha)\, \mathrm{e}^{2\alpha} d\alpha\right) \mathrm{e}^{-2x}}{3} \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-2x} - \frac{\left(-\left(\int_0^x \mathrm{e}^{-\alpha} F(\alpha)\, d\alpha\right) \mathrm{e}^{3x} + \int_0^x F(\alpha)\, \mathrm{e}^{2\alpha} d\alpha\right) \mathrm{e}^{-2x}}{3}$$

Verified OK.

9.25.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}
A &= 1 \\
B &= 1 \\
C &= -2
\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 9 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{9z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 292: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{9}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{3x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dx} \\ &= z_1 e^{-\frac{x}{2}} \\ &= z_1 \left(\mathrm{e}^{-\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-x}}{(y_1)^2}\,dx \\ &= y_1 \left(\frac{\mathrm{e}^{3x}}{3}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-2x}\right) + c_2\left(\mathrm{e}^{-2x}\left(\frac{\mathrm{e}^{3x}}{3}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$
y'' + y' - 2y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-2x} + \frac{c_2 \mathrm{e}^{x}}{3}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \mathrm{e}^{-2x} \\
y_2 &= \frac{\mathrm{e}^{x}}{3}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{-2x} & \frac{e^{x}}{3} \\ \frac{d}{dx}(e^{-2x}) & \frac{d}{dx}\left(\frac{e^{x}}{3}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-2x} & \frac{e^{x}}{3} \\ -2\,e^{-2x} & \frac{e^{x}}{3} \end{vmatrix}$$

Therefore

$$W = \left(e^{-2x}\right)\left(\frac{e^{x}}{3}\right) - \left(\frac{e^{x}}{3}\right)\left(-2\,e^{-2x}\right)$$

Which simplifies to

$$W = e^{-2x} e^{x}$$

Which simplifies to

$$W = e^{-x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{F(x) e^{x}}{3}}{e^{-x}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{F(x)\,e^{2x}}{3} dx$$

Hence

$$u_1 = -\left(\int_0^x \frac{F(\alpha)\,e^{2\alpha}}{3} d\alpha\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{-2x} F(x)}{e^{-x}}\,dx$$

Which simplifies to

$$u_2 = \int e^{-x} F(x)\,dx$$

Hence

$$u_2 = \int_0^x e^{-\alpha} F(\alpha)\,d\alpha$$

Which simplifies to

$$u_1 = -\frac{\left(\int_0^x F(\alpha)\,e^{2\alpha} d\alpha\right)}{3}$$

$$u_2 = \int_0^x e^{-\alpha} F(\alpha)\,d\alpha$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{\left(\int_0^x F(\alpha)\,e^{2\alpha} d\alpha\right) e^{-2x}}{3} + \frac{\left(\int_0^x e^{-\alpha} F(\alpha)\,d\alpha\right) e^{x}}{3}$$

Which simplifies to

$$y_p(x) = -\frac{\left(-\left(\int_0^x e^{-\alpha} F(\alpha)\,d\alpha\right) e^{3x} + \int_0^x F(\alpha)\,e^{2\alpha} d\alpha\right) e^{-2x}}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-2x} + \frac{c_2 e^{x}}{3}\right) + \left(-\frac{\left(-\left(\int_0^x e^{-\alpha} F(\alpha)\,d\alpha\right) e^{3x} + \int_0^x F(\alpha)\,e^{2\alpha} d\alpha\right) e^{-2x}}{3}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-2x} + \frac{c_2 e^{x}}{3} - \frac{\left(-\left(\int_0^x e^{-\alpha} F(\alpha)\,d\alpha\right) e^{3x} + \int_0^x F(\alpha)\,e^{2\alpha} d\alpha\right) e^{-2x}}{3} \tag{1}$$

Verification of solutions

$$y = c_1 e^{-2x} + \frac{c_2 e^{x}}{3} - \frac{\left(-\left(\int_0^x e^{-\alpha} F(\alpha)\,d\alpha\right) e^{3x} + \int_0^x F(\alpha)\,e^{2\alpha} d\alpha\right) e^{-2x}}{3}$$

Verified OK.

9.25.3 Maple step by step solution

Let's solve

$y'' + y' - 2y = F(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + r - 2 = 0$

- Factor the characteristic polynomial

  $(r+2)(r-1) = 0$

- Roots of the characteristic polynomial

  $r = (-2, 1)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-2x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-2x} + c_2 e^{x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = F(x)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-2x} & e^{x} \\ -2\,e^{-2x} & e^{x} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 3\,e^{-x}$

  - Substitute functions into equation for $y_p(x)$

$$y_p(x) = \frac{\left(e^{3x}\left(\int e^{-x}F(x)dx\right) - \left(\int F(x)e^{2x}dx\right)\right)e^{-2x}}{3}$$

- Compute integrals

$$y_p(x) = \frac{\left(e^{3x}\left(\int e^{-x}F(x)dx\right) - \left(\int F(x)e^{2x}dx\right)\right)e^{-2x}}{3}$$

- Substitute particular solution into general solution to ODE

$$y = c_1e^{-2x} + c_2e^x + \frac{\left(e^{3x}\left(\int e^{-x}F(x)dx\right) - \left(\int F(x)e^{2x}dx\right)\right)e^{-2x}}{3}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 46

```
dsolve(diff(y(x),x$2)+diff(y(x),x)-2*y(x)=F(x),y(x), singsol=all)
```

$$y(x) = \frac{\left(\left(\int e^{-x}F(x)\,dx\right)e^{3x} + 3c_1e^{3x} - \left(\int F(x)\,e^{2x}dx\right) + 3c_2\right)e^{-2x}}{3}$$

✓ Solution by Mathematica

Time used: 0.053 (sec). Leaf size: 68

```
DSolve[y''[x]+y'[x]-2*y[x]==F[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-2x}\left(\int_1^x -\frac{1}{3}e^{2K[1]}F(K[1])dK[1] + e^{3x}\int_1^x \frac{1}{3}e^{-K[2]}F(K[2])dK[2] + c_2e^{3x} + c_1\right)$$

9.26 problem Problem 26

Internal problem ID [2799]
Internal file name [`OUTPUT/2291_Sunday_June_05_2022_02_57_59_AM_46227829/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.7, The Variation of Parameters Method. page 556
Problem number: Problem 26.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 4y' - 12y = F(x)}$$

9.26.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 4, C = -12, f(x) = F(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y' - 12y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 4, C = -12$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 4\lambda \, e^{\lambda x} - 12 \, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4\lambda - 12 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 4, C = -12$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{-4}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{4^2 - (4)\,(1)\,(-12)} \\
&= -2 \pm 4
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= -2 + 4 \\
\lambda_2 &= -2 - 4
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= 2 \\
\lambda_2 &= -6
\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}
y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\
y &= c_1 e^{(2)x} + c_2 e^{(-6)x}
\end{aligned}$$

Or

$$y = e^{2x} c_1 + c_2 e^{-6x}$$

Therefore the homogeneous solution y_h is

$$y_h = e^{2x} c_1 + c_2 e^{-6x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= \mathrm{e}^{2x} \\ y_2 &= \mathrm{e}^{-6x} \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{2x} & \mathrm{e}^{-6x} \\ \frac{d}{dx}(\mathrm{e}^{2x}) & \frac{d}{dx}(\mathrm{e}^{-6x}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{2x} & \mathrm{e}^{-6x} \\ 2\,\mathrm{e}^{2x} & -6\,\mathrm{e}^{-6x} \end{vmatrix}$$

Therefore

$$W = (\mathrm{e}^{2x})\left(-6\,\mathrm{e}^{-6x}\right) - \left(\mathrm{e}^{-6x}\right)\left(2\,\mathrm{e}^{2x}\right)$$

Which simplifies to

$$W = -8\,\mathrm{e}^{2x}\mathrm{e}^{-6x}$$

Which simplifies to

$$W = -8\,\mathrm{e}^{-4x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{e^{-6x} F(x)}{-8\,e^{-4x}}\,dx$$

Which simplifies to

$$u_1 = -\int -\frac{e^{-2x} F(x)}{8} dx$$

Hence

$$u_1 = -\left(\int_0^x -\frac{e^{-2\alpha} F(\alpha)}{8} d\alpha\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{F(x)\,e^{2x}}{-8\,e^{-4x}}\,dx$$

Which simplifies to

$$u_2 = \int -\frac{F(x)\,e^{6x}}{8} dx$$

Hence

$$u_2 = \int_0^x -\frac{F(\alpha)\,e^{6\alpha}}{8} d\alpha$$

Which simplifies to

$$u_1 = \frac{\left(\int_0^x e^{-2\alpha} F(\alpha)\,d\alpha\right)}{8}$$

$$u_2 = -\frac{\left(\int_0^x F(\alpha)\,e^{6\alpha} d\alpha\right)}{8}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\left(\int_0^x e^{-2\alpha} F(\alpha)\,d\alpha\right) e^{2x}}{8} - \frac{\left(\int_0^x F(\alpha)\,e^{6\alpha} d\alpha\right) e^{-6x}}{8}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{2x} c_1 + c_2 e^{-6x}\right) + \left(\frac{\left(\int_0^x e^{-2\alpha} F(\alpha)\,d\alpha\right) e^{2x}}{8} - \frac{\left(\int_0^x F(\alpha)\,e^{6\alpha} d\alpha\right) e^{-6x}}{8}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = e^{2x}c_1 + c_2e^{-6x} + \frac{\left(\int_0^x e^{-2\alpha}F(\alpha)\,d\alpha\right)e^{2x}}{8} - \frac{\left(\int_0^x F(\alpha)\,e^{6\alpha}d\alpha\right)e^{-6x}}{8} \tag{1}$$

Verification of solutions

$$y = e^{2x}c_1 + c_2e^{-6x} + \frac{\left(\int_0^x e^{-2\alpha}F(\alpha)\,d\alpha\right)e^{2x}}{8} - \frac{\left(\int_0^x F(\alpha)\,e^{6\alpha}d\alpha\right)e^{-6x}}{8}$$

Verified OK.

9.26.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y' - 12y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4 \\ C &= -12 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{16}{1} \tag{6}$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 16 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = 16z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 294: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 16$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-4x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4}{1}\,dx} \\
&= z_1 e^{-2x} \\
&= z_1 \left(\mathrm{e}^{-2x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-6x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{\mathrm{e}^{-4x}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{\mathrm{e}^{8x}}{8}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-6x}\right) + c_2 \left(\mathrm{e}^{-6x} \left(\frac{\mathrm{e}^{8x}}{8}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$y''+4y'-12y=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-6x} + \frac{c_2 \mathrm{e}^{2x}}{8}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \mathrm{e}^{-6x}$$

$$y_2 = \frac{\mathrm{e}^{2x}}{8}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{-6x} & \frac{\mathrm{e}^{2x}}{8} \\ \frac{d}{dx}\left(\mathrm{e}^{-6x}\right) & \frac{d}{dx}\left(\frac{\mathrm{e}^{2x}}{8}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-6x} & \frac{e^{2x}}{8} \\ -6\,e^{-6x} & \frac{e^{2x}}{4} \end{vmatrix}$$

Therefore

$$W = \left(e^{-6x}\right)\left(\frac{e^{2x}}{4}\right) - \left(\frac{e^{2x}}{8}\right)\left(-6\,e^{-6x}\right)$$

Which simplifies to

$$W = e^{2x}e^{-6x}$$

Which simplifies to

$$W = e^{-4x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{F(x)e^{2x}}{8}}{e^{-4x}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{F(x)\,e^{6x}}{8}dx$$

Hence

$$u_1 = -\left(\int_0^x \frac{F(\alpha)\,e^{6\alpha}}{8}d\alpha\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{-6x}F(x)}{e^{-4x}}\,dx$$

Which simplifies to

$$u_2 = \int e^{-2x}F(x)\,dx$$

Hence

$$u_2 = \int_0^x e^{-2\alpha}F(\alpha)\,d\alpha$$

Which simplifies to

$$u_1 = -\frac{\left(\int_0^x F(\alpha)\,\mathrm{e}^{6\alpha}d\alpha\right)}{8}$$
$$u_2 = \int_0^x \mathrm{e}^{-2\alpha}F(\alpha)\,d\alpha$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{\left(\int_0^x F(\alpha)\,\mathrm{e}^{6\alpha}d\alpha\right)\mathrm{e}^{-6x}}{8} + \frac{\left(\int_0^x \mathrm{e}^{-2\alpha}F(\alpha)\,d\alpha\right)\mathrm{e}^{2x}}{8}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{-6x} + \frac{c_2\mathrm{e}^{2x}}{8}\right) + \left(-\frac{\left(\int_0^x F(\alpha)\,\mathrm{e}^{6\alpha}d\alpha\right)\mathrm{e}^{-6x}}{8} + \frac{\left(\int_0^x \mathrm{e}^{-2\alpha}F(\alpha)\,d\alpha\right)\mathrm{e}^{2x}}{8}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\mathrm{e}^{-6x} + \frac{c_2\mathrm{e}^{2x}}{8} - \frac{\left(\int_0^x F(\alpha)\,\mathrm{e}^{6\alpha}d\alpha\right)\mathrm{e}^{-6x}}{8} + \frac{\left(\int_0^x \mathrm{e}^{-2\alpha}F(\alpha)\,d\alpha\right)\mathrm{e}^{2x}}{8} \tag{1}$$

Verification of solutions

$$y = c_1\mathrm{e}^{-6x} + \frac{c_2\mathrm{e}^{2x}}{8} - \frac{\left(\int_0^x F(\alpha)\,\mathrm{e}^{6\alpha}d\alpha\right)\mathrm{e}^{-6x}}{8} + \frac{\left(\int_0^x \mathrm{e}^{-2\alpha}F(\alpha)\,d\alpha\right)\mathrm{e}^{2x}}{8}$$

Verified OK.

9.26.3 Maple step by step solution

Let's solve

$y'' + 4y' - 12y = F(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4r - 12 = 0$

- Factor the characteristic polynomial

  $(r+6)(r-2) = 0$

- Roots of the characteristic polynomial

  $r = (-6, 2)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-6x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-6x} + c_2 e^{2x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = F(x)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-6x} & e^{2x} \\ -6\,e^{-6x} & 2\,e^{2x} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 8\,e^{-4x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = \frac{\left(e^{8x}\left(\int e^{-2x}F(x)dx\right) - \left(\int F(x)e^{6x}dx\right)\right)e^{-6x}}{8}$

  - Compute integrals

    $y_p(x) = \frac{\left(e^{8x}\left(\int e^{-2x}F(x)dx\right) - \left(\int F(x)e^{6x}dx\right)\right)e^{-6x}}{8}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-6x} + c_2 e^{2x} + \frac{\left(e^{8x}\left(\int e^{-2x}F(x)dx\right) - \left(\int F(x)e^{6x}dx\right)\right)e^{-6x}}{8}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 45

```
dsolve(diff(y(x),x$2)+4*diff(y(x),x)-12*y(x)=F(x),y(x), singsol=all)
```

$$y(x) = -\frac{\left(-\left(\int F(x)\,\mathrm{e}^{-2x}dx\right)\mathrm{e}^{8x} - 8c_1\mathrm{e}^{8x} + \int F(x)\,\mathrm{e}^{6x}dx - 8c_2\right)\mathrm{e}^{-6x}}{8}$$

✓ Solution by Mathematica

Time used: 0.077 (sec). Leaf size: 68

```
DSolve[y''[x]+4*y'[x]-12*y[x]==F[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-6x}\left(\int_1^x -\frac{1}{8}e^{6K[1]}F(K[1])dK[1] + e^{8x}\int_1^x \frac{1}{8}e^{-2K[2]}F(K[2])dK[2] + c_2e^{8x} + c_1\right)$$

9.27 problem Problem 27

Internal problem ID [2800]
Internal file name [`OUTPUT/2292_Sunday_June_05_2022_02_58_03_AM_48418941/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.7, The Variation of Parameters Method. page 556
Problem number: Problem 27.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 4y' + 4y = 5x\,\mathrm{e}^{2x}}$$

With initial conditions

$$[y(0) = 1, y'(0) = 0]$$

9.27.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -4 \\ q(x) &= 4 \\ F &= 5x\,\mathrm{e}^{2x} \end{aligned}$$

Hence the ode is

$$y'' - 4y' + 4y = 5x\,\mathrm{e}^{2x}$$

The domain of $p(x) = -4$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 4$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = 5x\,\mathrm{e}^{2x}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

9.27.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -4, C = 4, f(x) = 5x\,\mathrm{e}^{2x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 4y' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -4, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 4\lambda\, e^{\lambda x} + 4\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 4\lambda + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -4, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{4}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{(-4)^2 - (4)\,(1)\,(4)}\\ &= 2\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -2$. Therefore the solution is

$$y = c_1 e^{2x} + c_2 x\, e^{2x} \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = e^{2x} c_1 + x\, e^{2x} c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$5x\, e^{2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\, e^{2x}, e^{2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\, e^{2x}, e^{2x}\}$$

Since e^{2x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,e^{2x}, x^2e^{2x}\}]$$

Since $x\,e^{2x}$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2e^{2x}, x^3e^{2x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1x^2e^{2x} + A_2x^3e^{2x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1e^{2x} + 6A_2x\,e^{2x} = 5x\,e^{2x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{5}{6}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{5x^3e^{2x}}{6}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{2x}c_1 + x\,e^{2x}c_2\right) + \left(\frac{5x^3e^{2x}}{6}\right) \end{aligned}$$

Which simplifies to

$$y = e^{2x}(c_2x + c_1) + \frac{5x^3e^{2x}}{6}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = e^{2x}(c_2 x + c_1) + \frac{5x^3 e^{2x}}{6} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$1 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 2\,e^{2x}(c_2 x + c_1) + c_2 e^{2x} + \frac{5x^2 e^{2x}}{2} + \frac{5x^3 e^{2x}}{3}$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$0 = 2c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 1 \\ c_2 &= -2 \end{aligned}$$

Substituting these values back in above solution results in

$$y = -2x\,e^{2x} + e^{2x} + \frac{5x^3 e^{2x}}{6}$$

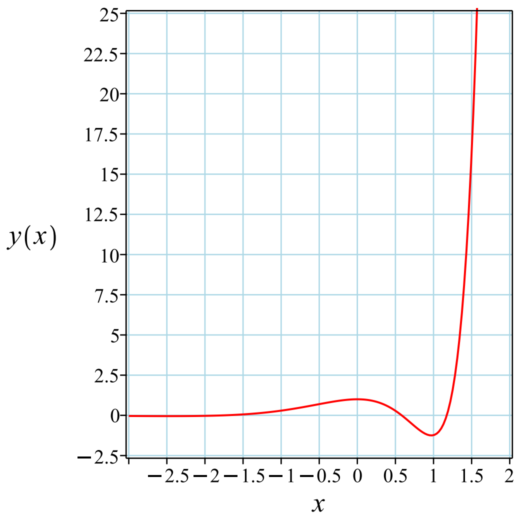
Which simplifies to

$$y = e^{2x}\left(1 - 2x + \frac{5}{6}x^3\right)$$

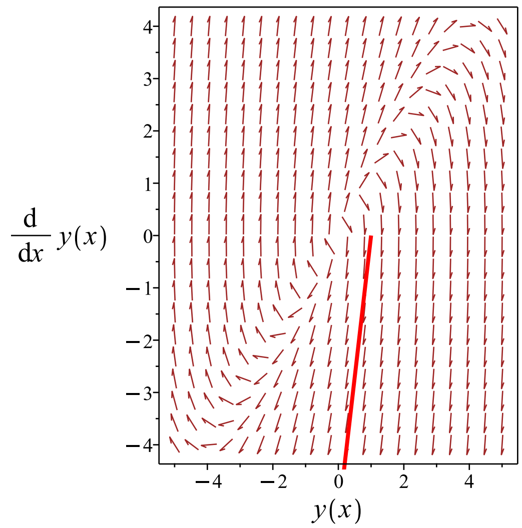
Summary

The solution(s) found are the following

$$y = e^{2x}\left(1 - 2x + \frac{5}{6}x^3\right) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = e^{2x}\left(1 - 2x + \frac{5}{6}x^3\right)$$

Verified OK.

9.27.3 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = -4$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\, dx} \\ &= e^{\int -4\, dx} \\ &= e^{-2x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)y)'' &= 5\, e^{2x} x\, e^{-2x} \\ \left(e^{-2x} y\right)'' &= 5\, e^{2x} x\, e^{-2x} \end{aligned}$$

Integrating once gives

$$\left(\mathrm{e}^{-2x} y\right)' = \frac{5x^2}{2} + c_1$$

Integrating again gives

$$\left(\mathrm{e}^{-2x} y\right) = \frac{5}{6}x^3 + c_1 x + c_2$$

Hence the solution is

$$y = \frac{\frac{5}{6}x^3 + c_1 x + c_2}{\mathrm{e}^{-2x}}$$

Or

$$y = \frac{5x^3 \mathrm{e}^{2x}}{6} + c_1 x\, \mathrm{e}^{2x} + c_2 \mathrm{e}^{2x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{5x^3 \mathrm{e}^{2x}}{6} + c_1 x\, \mathrm{e}^{2x} + c_2 \mathrm{e}^{2x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$1 = c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{5x^2 \mathrm{e}^{2x}}{2} + \frac{5x^3 \mathrm{e}^{2x}}{3} + \mathrm{e}^{2x} c_1 + 2c_1 x\, \mathrm{e}^{2x} + 2c_2 \mathrm{e}^{2x}$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$0 = c_1 + 2c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -2$$
$$c_2 = 1$$

Substituting these values back in above solution results in

$$y = -2x\, \mathrm{e}^{2x} + \mathrm{e}^{2x} + \frac{5x^3 \mathrm{e}^{2x}}{6}$$

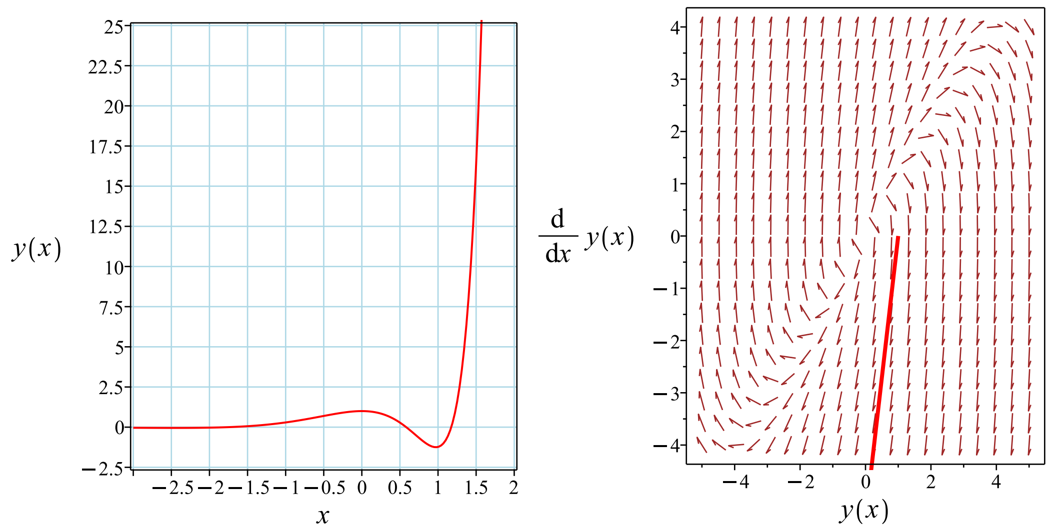
Which simplifies to

$$y = \mathrm{e}^{2x}\left(1 - 2x + \frac{5}{6}x^3\right)$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{2x}\left(1 - 2x + \frac{5}{6}x^3\right) \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \mathrm{e}^{2x}\left(1 - 2x + \frac{5}{6}x^3\right)$$

Verified OK.

9.27.4 Solving using Kovacic algorithm

Writing the ode as

$$
\begin{align*}
y'' - 4y' + 4y &= 0 \tag{1}\\
Ay'' + By' + Cy &= 0 \tag{2}
\end{align*}
$$

Comparing (1) and (2) shows that

$$
\begin{align*}
A &= 1 \\
B &= -4 \tag{3}\\
C &= 4
\end{align*}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = ye^{\int \frac{B}{2A}\,dx}
$$

Then (2) becomes

$$
z''(x) = rz(x) \tag{4}
$$

Where r is given by

$$
\begin{align*}
r &= \frac{s}{t} \tag{5}\\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{align*}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{0}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{align*}
s &= 0\\
t &= 1
\end{align*}
$$

Therefore eq. (4) becomes

$$
z''(x) = 0 \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\,dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 296: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4}{1}\,dx}
\end{aligned}
$$

$$
\begin{aligned}
&= z_1 e^{2x} \\
&= z_1 \left(\mathrm{e}^{2x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{2x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{4x}}{(y_1)^2}\, dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{2x}\right) + c_2 \left(\mathrm{e}^{2x}(x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 4y' + 4y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = \mathrm{e}^{2x} c_1 + x\,\mathrm{e}^{2x} c_2
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$5x\,e^{2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\,e^{2x}, e^{2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\,e^{2x}, e^{2x}\}$$

Since e^{2x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,e^{2x}, x^2e^{2x}\}]$$

Since $x\,e^{2x}$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2e^{2x}, x^3e^{2x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1x^2e^{2x} + A_2x^3e^{2x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1e^{2x} + 6A_2x\,e^{2x} = 5x\,e^{2x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{5}{6}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{5x^3e^{2x}}{6}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\mathrm{e}^{2x} c_1 + x\,\mathrm{e}^{2x} c_2\right) + \left(\frac{5x^3 \mathrm{e}^{2x}}{6}\right)
\end{aligned}
$$

Which simplifies to

$$
y = \mathrm{e}^{2x} (c_2 x + c_1) + \frac{5x^3 \mathrm{e}^{2x}}{6}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = \mathrm{e}^{2x} (c_2 x + c_1) + \frac{5x^3 \mathrm{e}^{2x}}{6} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$
1 = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = 2\,\mathrm{e}^{2x} (c_2 x + c_1) + c_2 \mathrm{e}^{2x} + \frac{5x^2 \mathrm{e}^{2x}}{2} + \frac{5x^3 \mathrm{e}^{2x}}{3}
$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$
0 = 2c_1 + c_2 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 1 \\
c_2 &= -2
\end{aligned}
$$

Substituting these values back in above solution results in

$$
y = -2x\,\mathrm{e}^{2x} + \mathrm{e}^{2x} + \frac{5x^3 \mathrm{e}^{2x}}{6}
$$

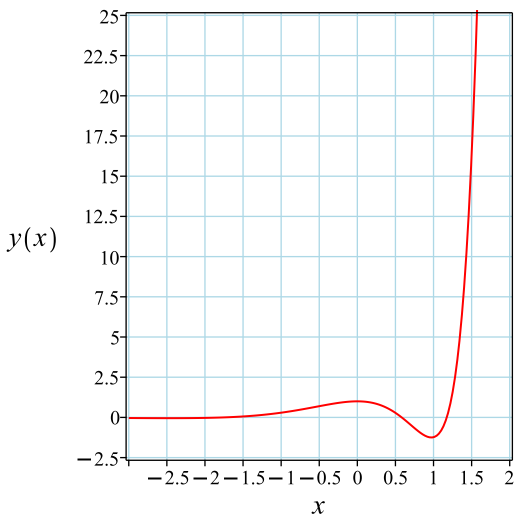
Which simplifies to

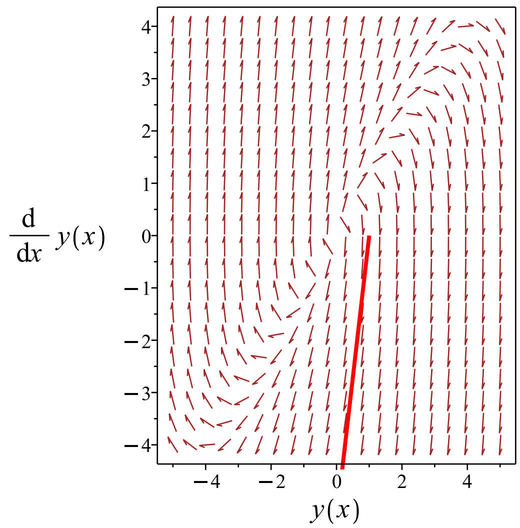
$$
y = \mathrm{e}^{2x} \left(1 - 2x + \frac{5}{6} x^3\right)
$$

Summary

The solution(s) found are the following

$$y = e^{2x}\left(1 - 2x + \frac{5}{6}x^3\right) \tag{1}$$





(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = e^{2x}\left(1 - 2x + \frac{5}{6}x^3\right)$$

Verified OK.

9.27.5 Maple step by step solution

Let's solve

$$\left[y'' - 4y' + 4y = 5x\,e^{2x}, y(0) = 1, y'\Big|_{\{x=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 4r + 4 = 0$

- Factor the characteristic polynomial

$(r-2)^2 = 0$

- Root of the characteristic polynomial

  $r = 2$
- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{2x}$
- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,\mathrm{e}^{2x}$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE

  $y = \mathrm{e}^{2x} c_1 + x\,\mathrm{e}^{2x} c_2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 5x\,\mathrm{e}^{2x}\right]$
- Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{2x} & x\,\mathrm{e}^{2x} \\ 2\,\mathrm{e}^{2x} & \mathrm{e}^{2x} + 2x\,\mathrm{e}^{2x} \end{bmatrix}$
- Compute Wronskian

  $W(y_1(x), y_2(x)) = \mathrm{e}^{4x}$
- Substitute functions into equation for $y_p(x)$

  $y_p(x) = -5\,\mathrm{e}^{2x}\left(\int x^2 dx - \left(\int x dx\right) x\right)$
- Compute integrals

  $y_p(x) = \frac{5x^3 \mathrm{e}^{2x}}{6}$

- Substitute particular solution into general solution to ODE

  $y = \mathrm{e}^{2x} c_1 + x\,\mathrm{e}^{2x} c_2 + \frac{5x^3 \mathrm{e}^{2x}}{6}$

□ Check validity of solution $y = \mathrm{e}^{2x} c_1 + x\mathrm{e}^{2x} c_2 + \frac{5x^3 \mathrm{e}^{2x}}{6}$

- Use initial condition $y(0) = 1$

  $1 = c_1$
- Compute derivative of the solution

$$y' = 2\,\mathrm{e}^{2x} c_1 + c_2 \mathrm{e}^{2x} + 2x\,\mathrm{e}^{2x} c_2 + \frac{5x^2 \mathrm{e}^{2x}}{2} + \frac{5x^3 \mathrm{e}^{2x}}{3}$$

- Use the initial condition $y'\Big|_{\{x=0\}} = 0$

  $$0 = 2c_1 + c_2$$

- Solve for c_1 and c_2

  $$\{c_1 = 1, c_2 = -2\}$$

- Substitute constant values into general solution and simplify

  $$y = \mathrm{e}^{2x}\left(1 - 2x + \frac{5}{6}x^3\right)$$

- Solution to the IVP

  $$y = \mathrm{e}^{2x}\left(1 - 2x + \frac{5}{6}x^3\right)$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 19

```
dsolve([diff(y(x),x$2)-4*diff(y(x),x)+4*y(x)=5*x*exp(2*x),y(0) = 1, D(y)(0) = 0],y(x), sings
```

$$y(x) = \frac{\mathrm{e}^{2x}(5x^3 - 12x + 6)}{6}$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 24

```
DSolve[{y''[x]-4*y'[x]+4*y[x]==5*x*Exp[2*x],{y[0]==1,y'[0]==0}},y[x],x,IncludeSingularSoluti
```

$$y(x) \to \frac{1}{6}e^{2x}\left(5x^3 - 12x + 6\right)$$

9.28 problem Problem 28

Internal problem ID [2801]
Internal file name [`OUTPUT/2293_Sunday_June_05_2022_02_58_06_AM_79223964/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.7, The Variation of Parameters Method. page 556
Problem number: Problem 28.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' + y = \sec(x)}$$

With initial conditions

$$[y(0) = 0, y'(0) = 1]$$

9.28.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= 1 \\ F &= \sec(x) \end{aligned}$$

Hence the ode is

$$y'' + y = \sec(x)$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = \sec(x)$ is

$$\left\{x < \frac{1}{2}\pi + \pi\_Z142 \vee \frac{1}{2}\pi + \pi\_Z142 < x\right\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

9.28.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = \sec(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i \\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i \\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x)$$
$$y_2 = \sin(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ \frac{d}{dx}(\cos(x)) & \frac{d}{dx}(\sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

Therefore

$$W = (\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))$$

Which simplifies to

$$W = \cos(x)^2 + \sin(x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sec(x)\sin(x)}{1}\,dx$$

Which simplifies to

$$u_1 = -\int \tan(x)\,dx$$

Hence

$$u_1 = \ln(\cos(x))$$

And Eq. (3) becomes

$$u_2 = \int \frac{\sec(x)\cos(x)}{1}\,dx$$

Which simplifies to

$$u_2 = \int 1dx$$

Hence

$$u_2 = x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \cos(x)\ln(\cos(x)) + x\sin(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1\cos(x) + c_2\sin(x)) + (\cos(x)\ln(\cos(x)) + x\sin(x)) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1\cos(x) + c_2\sin(x) + \cos(x)\ln(\cos(x)) + x\sin(x) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -c_1 \sin(x) + c_2 \cos(x) - \sin(x) \ln(\cos(x)) + \cos(x) x$$

substituting $y' = 1$ and $x = 0$ in the above gives

$$1 = c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

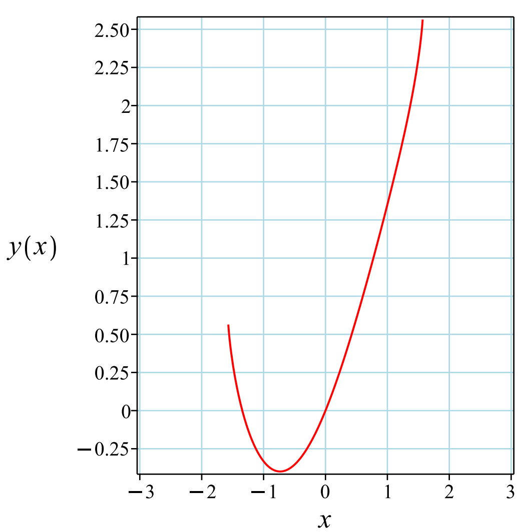
$$c_1 = 0$$
$$c_2 = 1$$

Substituting these values back in above solution results in

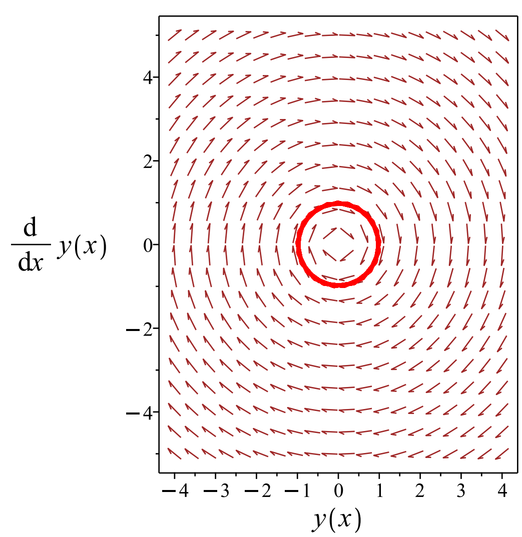
$$y = \sin(x) + \cos(x) \ln(\cos(x)) + x \sin(x)$$

<u>Summary</u>
The solution(s) found are the following

$$y = \sin(x) + \cos(x) \ln(\cos(x)) + x \sin(x) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \sin(x) + \cos(x)\ln(\cos(x)) + x\sin(x)$$

Verified OK.

9.28.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 298: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \cos(x) \int \frac{1}{\cos(x)^2}\,dx \\ &= \cos(x)(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(x)) + c_2(\cos(x)(\tan(x))) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x)$$
$$y_2 = \sin(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ \frac{d}{dx}(\cos(x)) & \frac{d}{dx}(\sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

Therefore

$$W = (\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))$$

Which simplifies to

$$W = \cos(x)^2 + \sin(x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sec(x)\sin(x)}{1}\,dx$$

Which simplifies to

$$u_1 = -\int \tan(x)\,dx$$

Hence

$$u_1 = \ln(\cos(x))$$

And Eq. (3) becomes

$$u_2 = \int \frac{\sec(x)\cos(x)}{1}\,dx$$

Which simplifies to

$$u_2 = \int 1 dx$$

Hence

$$u_2 = x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \cos(x)\ln(\cos(x)) + x\sin(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1\cos(x) + c_2\sin(x)) + (\cos(x)\ln(\cos(x)) + x\sin(x)) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \cos(x) + c_2 \sin(x) + \cos(x)\ln(\cos(x)) + x\sin(x) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -c_1 \sin(x) + c_2 \cos(x) - \sin(x)\ln(\cos(x)) + \cos(x)\, x$$

substituting $y' = 1$ and $x = 0$ in the above gives

$$1 = c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 0$$
$$c_2 = 1$$

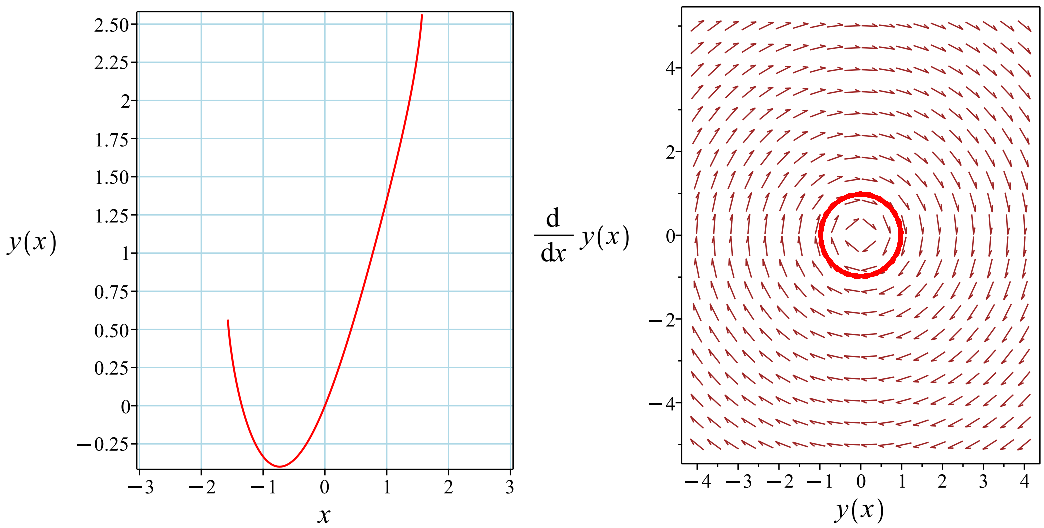
Substituting these values back in above solution results in

$$y = \sin(x) + \cos(x)\ln(\cos(x)) + x\sin(x)$$

Summary

The solution(s) found are the following

$$y = \sin(x) + \cos(x)\ln(\cos(x)) + x\sin(x) \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \sin(x) + \cos(x)\ln(\cos(x)) + x\sin(x)$$

Verified OK.

9.28.4 Maple step by step solution

Let's solve

$$\left[y'' + y = \sec(x), y(0) = 0, y'\Big|_{\{x=0\}} = 1\right]$$

- Highest derivative means the order of the ODE is 2
  y''
- Characteristic polynomial of homogeneous ODE
  $r^2 + 1 = 0$
- Use quadratic formula to solve for r
  $r = \frac{0 \pm (\sqrt{-4})}{2}$
- Roots of the characteristic polynomial
  $r = (-\mathrm{I}, \mathrm{I})$
- 1st solution of the homogeneous ODE

$y_1(x) = \cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = \sec(x)\right]$

- Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$

- Compute Wronskian

  $W(y_1(x), y_2(x)) = 1$

- Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\cos(x)\left(\int \tan(x)\, dx\right) + \sin(x)\left(\int 1 dx\right)$

- Compute integrals

  $y_p(x) = \cos(x)\ln(\cos(x)) + x\sin(x)$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + \cos(x)\ln(\cos(x)) + x\sin(x)$

□ Check validity of solution $y = c_1 \cos(x) + c_2 \sin(x) + \cos(x)\ln(\cos(x)) + x\sin(x)$

- Use initial condition $y(0) = 0$

  $0 = c_1$

- Compute derivative of the solution

  $y' = -c_1 \sin(x) + c_2 \cos(x) - \sin(x)\ln(\cos(x)) + \cos(x)\, x$

- Use the initial condition $y'\Big|_{\{x=0\}} = 1$

  $1 = c_2$

- Solve for c_1 and c_2

  $$\{c_1 = 0, c_2 = 1\}$$

- Substitute constant values into general solution and simplify

  $$y = \sin(x) + \cos(x)\ln(\cos(x)) + x\sin(x)$$

- Solution to the IVP

  $$y = \sin(x) + \cos(x)\ln(\cos(x)) + x\sin(x)$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 18

```
dsolve([diff(y(x),x$2)+y(x)=sec(x),y(0) = 0, D(y)(0) = 1],y(x), singsol=all)
```

$$y(x) = \sin(x) + x\sin(x) - \cos(x)\ln(\sec(x))$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 24

```
DSolve[{y''[x]-4*y'[x]+4*y[x]==5*x*Exp[2*x],{y[0]==1,y'[0]==0}},y[x],x,IncludeSingularSoluti
```

$$y(x) \to \frac{1}{6}e^{2x}\left(5x^3 - 12x + 6\right)$$

10 Chapter 8, Linear differential equations of order n. Section 8.8, A Differential Equation with Nonconstant Coefficients. page 567

10.1 problem Problem 14

Internal problem ID [2802]
Internal file name [`OUTPUT/2294_Sunday_June_05_2022_02_58_08_AM_51088327/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.8, A Differential Equation with Nonconstant Coefficients. page 567
Problem number: Problem 14.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "exact linear second order ode", "second_order_integrable_as_is", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_1", "second_order_change_of_variable_on_y_method_2", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _nonhomogeneous]]
```

$$\boxed{x^2y'' + 4xy' + 2y = 4\ln(x)}$$

10.1.1 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = 4x, C = 2, f(x) = 4\ln(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' + 4xy' + 2y = 0$$

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} + 4xrx^{r-1} + 2x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r + 4r\,x^r + 2x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) + 4r + 2 = 0$$

Or

$$r^2 + 3r + 2 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= -2 \\ r_2 &= -1 \end{aligned}$$

Since the roots are real and distinct, then the general solution is

$$y = c_1y_1 + c_2y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = \frac{c_1}{x^2} + \frac{c_2}{x}$$

Next, we find the particular solution to the ODE

$$x^2y'' + 4xy' + 2y = 4\ln(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x^2}$$
$$y_2 = \frac{1}{x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x^2} & \frac{1}{x} \\ \frac{d}{dx}\left(\frac{1}{x^2}\right) & \frac{d}{dx}\left(\frac{1}{x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x^2} & \frac{1}{x} \\ -\frac{2}{x^3} & -\frac{1}{x^2} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x^2}\right)\left(-\frac{1}{x^2}\right) - \left(\frac{1}{x}\right)\left(-\frac{2}{x^3}\right)$$

Which simplifies to

$$W = \frac{1}{x^4}$$

Which simplifies to

$$W = \frac{1}{x^4}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{4\ln(x)}{x}}{\frac{1}{x^2}}\,dx$$

Which simplifies to

$$u_1 = -\int 4\ln(x)\,x dx$$

Hence

$$u_1 = -2\ln(x)\,x^2 + x^2$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{4\ln(x)}{x^2}}{\frac{1}{x^2}}\,dx$$

Which simplifies to

$$u_2 = \int 4\ln(x)\,dx$$

Hence

$$u_2 = 4\ln(x)\,x - 4x$$

Which simplifies to

$$u_1 = x^2(1 - 2\ln(x))$$
$$u_2 = 4\ln(x)\,x - 4x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = 1 - 2\ln(x) + \frac{4\ln(x)\,x - 4x}{x}$$

Which simplifies to

$$y_p(x) = 2\ln(x) - 3$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= 2\ln(x) - 3 + \frac{c_1}{x^2} + \frac{c_2}{x}
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = 2\ln(x) - 3 + \frac{c_1}{x^2} + \frac{c_2}{x} \tag{1}
$$

<u>Verification of solutions</u>

$$
y = 2\ln(x) - 3 + \frac{c_1}{x^2} + \frac{c_2}{x}
$$

Verified OK.

10.1.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$
y'' + p(x)\,y' + \frac{\left(p(x)^2 + p'(x)\right)y}{2} = f(x)
$$

Where $p(x) = \frac{4}{x}$. Therefore, there is an integrating factor given by

$$
\begin{aligned}
M(x) &= e^{\frac{1}{2}\int p\,dx} \\
&= e^{\int \frac{4}{x}\,dx} \\
&= x^2
\end{aligned}
$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$
\begin{aligned}
(M(x)y)'' &= 4\ln(x) \\
\left(x^2y\right)'' &= 4\ln(x)
\end{aligned}
$$

Integrating once gives

$$
\left(x^2y\right)' = 4\ln(x)\,x - 4x + c_1
$$

Integrating again gives

$$
\left(x^2y\right) = x(2\ln(x)\,x + c_1 - 3x) + c_2
$$

Hence the solution is

$$y = \frac{x(2\ln(x)\,x + c_1 - 3x) + c_2}{x^2}$$

Or

$$y = 2\ln(x) + \frac{c_1}{x} + \frac{c_2}{x^2} - 3$$

Summary

The solution(s) found are the following

$$y = 2\ln(x) + \frac{c_1}{x} + \frac{c_2}{x^2} - 3 \tag{1}$$

Verification of solutions

$$y = 2\ln(x) + \frac{c_1}{x} + \frac{c_2}{x^2} - 3$$

Verified OK.

10.1.3 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' + 4xy' + 2y = 0$$

In normal form the ode

$$x^2y'' + 4xy' + 2y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$p(x) = \frac{4}{x}$$
$$q(x) = \frac{2}{x^2}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'\,(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'\,(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)} dx \\ &= \int \mathrm{e}^{-\left(\int \frac{4}{x}dx\right)} dx \\ &= \int e^{-4\ln(x)}\, dx \\ &= \int \frac{1}{x^4} dx \\ &= -\frac{1}{3x^3} \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned} q_1(\tau) &= \frac{q(x)}{\tau'\,(x)^2} \\ &= \frac{\frac{2}{x^2}}{\frac{1}{x^8}} \\ &= 2x^6 \end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2} y(\tau) + q_1 y(\tau) = 0$$

$$\frac{d^2}{d\tau^2} y(\tau) + 2x^6 y(\tau) = 0$$

But in terms of τ

$$2x^6 = \frac{2}{9\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{2y(\tau)}{9\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$9\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + 2y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$9\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 2\tau^r = 0$$

Simplifying gives

$$9r(r-1)\,\tau^r + 0\,\tau^r + 2\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$9r(r-1) + 0 + 2 = 0$$

Or

$$9r^2 - 9r + 2 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{3}$$
$$r_2 = \frac{2}{3}$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1y_1 + c_2y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = c_1\tau^{\frac{1}{3}} + c_2\tau^{\frac{2}{3}}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{c_1 3^{\frac{2}{3}}\left(-\frac{1}{x^3}\right)^{\frac{1}{3}}}{3} + \frac{c_2 3^{\frac{1}{3}}\left(-\frac{1}{x^3}\right)^{\frac{2}{3}}}{3}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{c_1 3^{\frac{2}{3}} \left(-\frac{1}{x^3}\right)^{\frac{1}{3}}}{3} + \frac{c_2 3^{\frac{1}{3}} \left(-\frac{1}{x^3}\right)^{\frac{2}{3}}}{3}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \left(-\frac{1}{x^3}\right)^{\frac{1}{3}}$$

$$y_2 = \left(-\frac{1}{x^3}\right)^{\frac{2}{3}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \left(-\frac{1}{x^3}\right)^{\frac{1}{3}} & \left(-\frac{1}{x^3}\right)^{\frac{2}{3}} \\ \frac{d}{dx}\left(\left(-\frac{1}{x^3}\right)^{\frac{1}{3}}\right) & \frac{d}{dx}\left(\left(-\frac{1}{x^3}\right)^{\frac{2}{3}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \left(-\frac{1}{x^3}\right)^{\frac{1}{3}} & \left(-\frac{1}{x^3}\right)^{\frac{2}{3}} \\ \frac{1}{\left(-\frac{1}{x^3}\right)^{\frac{2}{3}} x^4} & \frac{2}{\left(-\frac{1}{x^3}\right)^{\frac{1}{3}} x^4} \end{vmatrix}$$

Therefore

$$W=\left(\left(-\frac{1}{x^3}\right)^{\frac{1}{3}}\right)\left(\frac{2}{\left(-\frac{1}{x^3}\right)^{\frac{1}{3}}x^4}\right)-\left(\left(-\frac{1}{x^3}\right)^{\frac{2}{3}}\right)\left(\frac{1}{\left(-\frac{1}{x^3}\right)^{\frac{2}{3}}x^4}\right)$$

Which simplifies to

$$W=\frac{1}{x^4}$$

Which simplifies to

$$W=\frac{1}{x^4}$$

Therefore Eq. (2) becomes

$$u_1=-\int\frac{4\left(-\frac{1}{x^3}\right)^{\frac{2}{3}}\ln(x)}{\frac{1}{x^2}}dx$$

Which simplifies to

$$u_1=-\int 4\left(-\frac{1}{x^3}\right)^{\frac{2}{3}}\ln(x)\,x^2dx$$

Hence

$$u_1=-4\left(-\frac{1}{x^3}\right)^{\frac{2}{3}}x^3\ln(x)+4\left(-\frac{1}{x^3}\right)^{\frac{2}{3}}x^3$$

And Eq. (3) becomes

$$u_2=\int\frac{4\left(-\frac{1}{x^3}\right)^{\frac{1}{3}}\ln(x)}{\frac{1}{x^2}}dx$$

Which simplifies to

$$u_2=\int 4\left(-\frac{1}{x^3}\right)^{\frac{1}{3}}\ln(x)\,x^2dx$$

Hence

$$u_2=2\left(-\frac{1}{x^3}\right)^{\frac{1}{3}}x^3\ln(x)-\left(-\frac{1}{x^3}\right)^{\frac{1}{3}}x^3$$

Which simplifies to

$$u_1 = -4\left(-\frac{1}{x^3}\right)^{\frac{2}{3}} x^3(\ln(x) - 1)$$

$$u_2 = \left(-\frac{1}{x^3}\right)^{\frac{1}{3}} x^3(-1 + 2\ln(x))$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = 2\ln(x) - 3$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1 3^{\frac{2}{3}} \left(-\frac{1}{x^3}\right)^{\frac{1}{3}}}{3} + \frac{c_2 3^{\frac{1}{3}} \left(-\frac{1}{x^3}\right)^{\frac{2}{3}}}{3}\right) + (2\ln(x) - 3) \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{c_1 3^{\frac{2}{3}} \left(-\frac{1}{x^3}\right)^{\frac{1}{3}}}{3} + \frac{c_2 3^{\frac{1}{3}} \left(-\frac{1}{x^3}\right)^{\frac{2}{3}}}{3} + 2\ln(x) - 3 \tag{1}$$

<u>Verification of solutions</u>

$$y = \frac{c_1 3^{\frac{2}{3}} \left(-\frac{1}{x^3}\right)^{\frac{1}{3}}}{3} + \frac{c_2 3^{\frac{1}{3}} \left(-\frac{1}{x^3}\right)^{\frac{2}{3}}}{3} + 2\ln(x) - 3$$

Verified OK.

10.1.4 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = 4x, C = 2, f(x) = 4\ln(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. Solving for y_h from

$$x^2y''+4xy'+2y=0$$

In normal form the ode

$$x^2y''+4xy'+2y=0 \tag{1}$$

Becomes

$$y''+p(x)\,y'+q(x)\,y=0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{4}{x} \\ q(x) &= \frac{2}{x^2} \end{aligned}$$

Applying change of variables $\tau=g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau)+p_1\left(\frac{d}{d\tau}y(\tau)\right)+q_1y(\tau)=0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau)=\frac{\tau''(x)+p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau)=\frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1=c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{2}\,\sqrt{\frac{1}{x^2}}}{c} \\ \tau'' &= -\frac{\sqrt{2}}{c\sqrt{\frac{1}{x^2}}\,x^3} \end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$
\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{-\frac{\sqrt{2}}{c\sqrt{\frac{1}{x^2}}\,x^3} + \frac{4}{x}\frac{\sqrt{2}\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{2}\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\
&= \frac{3c\sqrt{2}}{2}
\end{aligned}
$$

Therefore ode (3) now becomes

$$
\begin{gathered}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) = 0 \\
\frac{d^2}{d\tau^2}y(\tau) + \frac{3c\sqrt{2}\left(\frac{d}{d\tau}y(\tau)\right)}{2} + c^2 y(\tau) = 0 \qquad (7)
\end{gathered}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
y(\tau) = \mathrm{e}^{-\frac{3\sqrt{2}\,c\tau}{4}}\left(c_1\cosh\left(\frac{\sqrt{2}\,c\tau}{4}\right) + ic_2\sinh\left(\frac{\sqrt{2}\,c\tau}{4}\right)\right)
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int \sqrt{2}\sqrt{\frac{1}{x^2}}dx}{c} \\
&= \frac{\sqrt{2}\sqrt{\frac{1}{x^2}}\,x\ln(x)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = \frac{c_1\cosh\left(\frac{\ln(x)}{2}\right) + ic_2\sinh\left(\frac{\ln(x)}{2}\right)}{x^{\frac{3}{2}}}
$$

Now the particular solution to this ODE is found

$$
x^2y'' + 4xy' + 2y = 4\ln(x)
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \left(-\frac{1}{x^3}\right)^{\frac{1}{3}}$$

$$y_2 = \left(-\frac{1}{x^3}\right)^{\frac{2}{3}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \left(-\frac{1}{x^3}\right)^{\frac{1}{3}} & \left(-\frac{1}{x^3}\right)^{\frac{2}{3}} \\ \frac{d}{dx}\left(\left(-\frac{1}{x^3}\right)^{\frac{1}{3}}\right) & \frac{d}{dx}\left(\left(-\frac{1}{x^3}\right)^{\frac{2}{3}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \left(-\frac{1}{x^3}\right)^{\frac{1}{3}} & \left(-\frac{1}{x^3}\right)^{\frac{2}{3}} \\ \frac{1}{\left(-\frac{1}{x^3}\right)^{\frac{2}{3}} x^4} & \frac{2}{\left(-\frac{1}{x^3}\right)^{\frac{1}{3}} x^4} \end{vmatrix}$$

Therefore

$$W = \left(\left(-\frac{1}{x^3}\right)^{\frac{1}{3}}\right)\left(\frac{2}{\left(-\frac{1}{x^3}\right)^{\frac{1}{3}} x^4}\right) - \left(\left(-\frac{1}{x^3}\right)^{\frac{2}{3}}\right)\left(\frac{1}{\left(-\frac{1}{x^3}\right)^{\frac{2}{3}} x^4}\right)$$

Which simplifies to

$$W = \frac{1}{x^4}$$

Which simplifies to

$$W = \frac{1}{x^4}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{4\left(-\frac{1}{x^3}\right)^{\frac{2}{3}} \ln(x)}{\frac{1}{x^2}} dx$$

Which simplifies to

$$u_1 = -\int 4\left(-\frac{1}{x^3}\right)^{\frac{2}{3}} \ln(x)\, x^2 dx$$

Hence

$$u_1 = -4\left(-\frac{1}{x^3}\right)^{\frac{2}{3}} x^3 \ln(x) + 4\left(-\frac{1}{x^3}\right)^{\frac{2}{3}} x^3$$

And Eq. (3) becomes

$$u_2 = \int \frac{4\left(-\frac{1}{x^3}\right)^{\frac{1}{3}} \ln(x)}{\frac{1}{x^2}} dx$$

Which simplifies to

$$u_2 = \int 4\left(-\frac{1}{x^3}\right)^{\frac{1}{3}} \ln(x)\, x^2 dx$$

Hence

$$u_2 = 2\left(-\frac{1}{x^3}\right)^{\frac{1}{3}} x^3 \ln(x) - \left(-\frac{1}{x^3}\right)^{\frac{1}{3}} x^3$$

Which simplifies to

$$u_1 = -4\left(-\frac{1}{x^3}\right)^{\frac{2}{3}} x^3(\ln(x) - 1)$$

$$u_2 = \left(-\frac{1}{x^3}\right)^{\frac{1}{3}} x^3(-1 + 2\ln(x))$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = 2\ln(x) - 3$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1 \cosh\left(\frac{\ln(x)}{2}\right) + ic_2 \sinh\left(\frac{\ln(x)}{2}\right)}{x^{\frac{3}{2}}}\right) + (2\ln(x) - 3) \\ &= 2\ln(x) - 3 + \frac{c_1 \cosh\left(\frac{\ln(x)}{2}\right) + ic_2 \sinh\left(\frac{\ln(x)}{2}\right)}{x^{\frac{3}{2}}} \end{aligned}$$

Which simplifies to

$$y = 2\ln(x) - 3 + \frac{c_1 \cosh\left(\frac{\ln(x)}{2}\right) + ic_2 \sinh\left(\frac{\ln(x)}{2}\right)}{x^{\frac{3}{2}}}$$

Summary

The solution(s) found are the following

$$y = 2\ln(x) - 3 + \frac{c_1 \cosh\left(\frac{\ln(x)}{2}\right) + ic_2 \sinh\left(\frac{\ln(x)}{2}\right)}{x^{\frac{3}{2}}} \tag{1}$$

Verification of solutions

$$y = 2\ln(x) - 3 + \frac{c_1 \cosh\left(\frac{\ln(x)}{2}\right) + ic_2 \sinh\left(\frac{\ln(x)}{2}\right)}{x^{\frac{3}{2}}}$$

Verified OK.

10.1.5 Solving as second order change of variable on y method 1 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' + 4xy' + 2y = 0$$

In normal form the given ode is written as

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{4}{x} \\ q(x) &= \frac{2}{x^2} \end{aligned}$$

Calculating the Liouville ode invariant Q given by

$$\begin{aligned} Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\ &= \frac{2}{x^2} - \frac{\left(\frac{4}{x}\right)'}{2} - \frac{\left(\frac{4}{x}\right)^2}{4} \\ &= \frac{2}{x^2} - \frac{\left(-\frac{4}{x^2}\right)}{2} - \frac{\left(\frac{16}{x^2}\right)}{4} \\ &= \frac{2}{x^2} - \left(-\frac{2}{x^2}\right) - \frac{4}{x^2} \\ &= 0 \end{aligned}$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$y = v(x)\, z(x) \tag{3}$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$\begin{aligned} z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)} \\ &= e^{-\int \frac{\frac{4}{x}}{2}} \\ &= \frac{1}{x^2} \end{aligned} \tag{5}$$

Hence (3) becomes

$$y = \frac{v(x)}{x^2} \tag{4}$$

Applying this change of variable to the original ode results in

$$v''(x) = 4 \ln(x)$$

Which is now solved for $v(x)$ Integrating once gives

$$v'(x) = 4\ln(x)\,x - 4x + c_1$$

Integrating again gives

$$v(x) = 2\ln(x)\,x^2 - 3x^2 + c_1x + c_2$$

Now that $v(x)$ is known, then

$$\begin{aligned} y &= v(x)\,z(x) \\ &= \left(-3x^2 + c_1x + 2\ln(x)\,x^2 + c_2\right)(z(x)) \end{aligned} \tag{7}$$

But from (5)

$$z(x) = \frac{1}{x^2}$$

Hence (7) becomes

$$y = \frac{-3x^2 + c_1x + 2\ln(x)\,x^2 + c_2}{x^2}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{-3x^2 + c_1x + 2\ln(x)\,x^2 + c_2}{x^2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \left(-\frac{1}{x^3}\right)^{\frac{1}{3}}$$

$$y_2 = \left(-\frac{1}{x^3}\right)^{\frac{2}{3}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \left(-\frac{1}{x^3}\right)^{\frac{1}{3}} & \left(-\frac{1}{x^3}\right)^{\frac{2}{3}} \\ \frac{d}{dx}\left(\left(-\frac{1}{x^3}\right)^{\frac{1}{3}}\right) & \frac{d}{dx}\left(\left(-\frac{1}{x^3}\right)^{\frac{2}{3}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \left(-\frac{1}{x^3}\right)^{\frac{1}{3}} & \left(-\frac{1}{x^3}\right)^{\frac{2}{3}} \\ \frac{1}{\left(-\frac{1}{x^3}\right)^{\frac{2}{3}} x^4} & \frac{2}{\left(-\frac{1}{x^3}\right)^{\frac{1}{3}} x^4} \end{vmatrix}$$

Therefore

$$W = \left(\left(-\frac{1}{x^3}\right)^{\frac{1}{3}}\right)\left(\frac{2}{\left(-\frac{1}{x^3}\right)^{\frac{1}{3}} x^4}\right) - \left(\left(-\frac{1}{x^3}\right)^{\frac{2}{3}}\right)\left(\frac{1}{\left(-\frac{1}{x^3}\right)^{\frac{2}{3}} x^4}\right)$$

Which simplifies to

$$W = \frac{1}{x^4}$$

Which simplifies to

$$W = \frac{1}{x^4}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{4\left(-\frac{1}{x^3}\right)^{\frac{2}{3}} \ln(x)}{\frac{1}{x^2}} \, dx$$

Which simplifies to

$$u_1 = -\int 4\left(-\frac{1}{x^3}\right)^{\frac{2}{3}} \ln(x)\, x^2 dx$$

Hence

$$u_1 = -4\left(-\frac{1}{x^3}\right)^{\frac{2}{3}} x^3 \ln(x) + 4\left(-\frac{1}{x^3}\right)^{\frac{2}{3}} x^3$$

And Eq. (3) becomes

$$u_2 = \int \frac{4\left(-\frac{1}{x^3}\right)^{\frac{1}{3}} \ln(x)}{\frac{1}{x^2}} dx$$

Which simplifies to

$$u_2 = \int 4\left(-\frac{1}{x^3}\right)^{\frac{1}{3}} \ln(x)\, x^2 dx$$

Hence

$$u_2 = 2\left(-\frac{1}{x^3}\right)^{\frac{1}{3}} x^3 \ln(x) - \left(-\frac{1}{x^3}\right)^{\frac{1}{3}} x^3$$

Which simplifies to

$$u_1 = -4\left(-\frac{1}{x^3}\right)^{\frac{2}{3}} x^3(\ln(x) - 1)$$

$$u_2 = \left(-\frac{1}{x^3}\right)^{\frac{1}{3}} x^3(-1 + 2\ln(x))$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = 2\ln(x) - 3$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{-3x^2 + c_1 x + 2\ln(x)\, x^2 + c_2}{x^2}\right) + (2\ln(x) - 3) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{-3x^2 + c_1x + 2\ln(x)\,x^2 + c_2}{x^2} + 2\ln(x) - 3 \tag{1}$$

Verification of solutions

$$y = \frac{-3x^2 + c_1x + 2\ln(x)\,x^2 + c_2}{x^2} + 2\ln(x) - 3$$

Verified OK.

10.1.6 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = 4x, C = 2, f(x) = 4\ln(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' + 4xy' + 2y = 0$$

In normal form the ode

$$x^2y'' + 4xy' + 2y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$p(x) = \frac{4}{x}$$
$$q(x) = \frac{2}{x^2}$$

Applying change of variables on the depndent variable $y = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right)v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right)v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2}+\frac{np}{x}+q=0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2}+\frac{4n}{x^2}+\frac{2}{x^2}=0 \tag{5}$$

Solving (5) for n gives

$$n=-1 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x)+\frac{2v'(x)}{x}&=0\\ v''(x)+\frac{2v'(x)}{x}&=0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x)=v'(x)$$

Then (7) becomes

$$u'(x)+\frac{2u(x)}{x}=0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u'&=F(x,u)\\ &=f(x)g(u)\\ &=-\frac{2u}{x} \end{aligned}$$

Where $f(x)=-\frac{2}{x}$ and $g(u)=u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du&=-\frac{2}{x}\,dx\\ \int\frac{1}{u}\,du&=\int-\frac{2}{x}\,dx\\ \ln(u)&=-2\ln(x)+c_1\\ u&=\mathrm{e}^{-2\ln(x)+c_1}\\ &=\frac{c_1}{x^2} \end{aligned}$$

Now that $u(x)$ is known, then

$$
\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\, dx + c_2 \\
&= -\frac{c_1}{x} + c_2
\end{aligned}
$$

Hence

$$
\begin{aligned}
y &= v(x)\, x^n \\
&= \frac{-\frac{c_1}{x} + c_2}{x} \\
&= \frac{c_2 x - c_1}{x^2}
\end{aligned}
$$

Now the particular solution to this ODE is found

$$
x^2 y'' + 4xy' + 2y = 4 \ln(x)
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \frac{1}{x^2} \\
y_2 &= \frac{1}{x}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x^2} & \frac{1}{x} \\ \frac{d}{dx}\left(\frac{1}{x^2}\right) & \frac{d}{dx}\left(\frac{1}{x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x^2} & \frac{1}{x} \\ -\frac{2}{x^3} & -\frac{1}{x^2} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x^2}\right)\left(-\frac{1}{x^2}\right) - \left(\frac{1}{x}\right)\left(-\frac{2}{x^3}\right)$$

Which simplifies to

$$W = \frac{1}{x^4}$$

Which simplifies to

$$W = \frac{1}{x^4}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{4\ln(x)}{x}}{\frac{1}{x^2}} dx$$

Which simplifies to

$$u_1 = -\int 4\ln(x)\, x dx$$

Hence

$$u_1 = -2\ln(x)\, x^2 + x^2$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{4\ln(x)}{x^2}}{\frac{1}{x^2}} dx$$

Which simplifies to

$$u_2 = \int 4 \ln(x)\, dx$$

Hence

$$u_2 = 4 \ln(x)\, x - 4x$$

Which simplifies to

$$u_1 = x^2(1 - 2 \ln(x))$$
$$u_2 = 4 \ln(x)\, x - 4x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = 1 - 2 \ln(x) + \frac{4 \ln(x)\, x - 4x}{x}$$

Which simplifies to

$$y_p(x) = 2 \ln(x) - 3$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{-\frac{c_1}{x} + c_2}{x}\right) + (2 \ln(x) - 3) \\ &= 2 \ln(x) - 3 + \frac{-\frac{c_1}{x} + c_2}{x} \end{aligned}$$

Which simplifies to

$$y = 2 \ln(x) - 3 + \frac{-\frac{c_1}{x} + c_2}{x}$$

Summary

The solution(s) found are the following

$$y = 2 \ln(x) - 3 + \frac{-\frac{c_1}{x} + c_2}{x} \tag{1}$$

Verification of solutions

$$y = 2 \ln(x) - 3 + \frac{-\frac{c_1}{x} + c_2}{x}$$

Verified OK.

10.1.7 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$
\begin{aligned}
\int (x^2y'' + 4xy' + 2y)\, dx &= \int 4\ln(x)\, dx \\
y'x^2 + 2yx &= 4\ln(x)\, x - 4x + c_1
\end{aligned}
$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
y' + p(x)y = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= \frac{2}{x} \\
q(x) &= \frac{4\ln(x)\, x - 4x + c_1}{x^2}
\end{aligned}
$$

Hence the ode is

$$
y' + \frac{2y}{x} = \frac{4\ln(x)\, x - 4x + c_1}{x^2}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int \frac{2}{x} dx} \\
&= x^2
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{4\ln(x)\, x - 4x + c_1}{x^2}\right) \\
\frac{d}{dx}(x^2 y) &= (x^2)\left(\frac{4\ln(x)\, x - 4x + c_1}{x^2}\right) \\
d(x^2 y) &= (4\ln(x)\, x - 4x + c_1)\, dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
x^2 y &= \int 4\ln(x)\, x - 4x + c_1\, dx \\
x^2 y &= -3x^2 + c_1 x + 2\ln(x)\, x^2 + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = x^2$ results in

$$
y = \frac{-3x^2 + c_1 x + 2\ln(x)\, x^2}{x^2} + \frac{c_2}{x^2}
$$

which simplifies to

$$y = \frac{-3x^2 + c_1 x + 2\ln(x)\, x^2 + c_2}{x^2}$$

Summary

The solution(s) found are the following

$$y = \frac{-3x^2 + c_1 x + 2\ln(x)\, x^2 + c_2}{x^2} \tag{1}$$

Verification of solutions

$$y = \frac{-3x^2 + c_1 x + 2\ln(x)\, x^2 + c_2}{x^2}$$

Verified OK.

10.1.8 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$x^2 y'' + 4xy' + 2y = 4\ln(x)$$

Integrating both sides of the ODE w.r.t x gives

$$\int (x^2 y'' + 4xy' + 2y)\, dx = \int 4\ln(x)\, dx$$
$$y' x^2 + 2yx = 4\ln(x)\, x - 4x + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{2}{x}$$
$$q(x) = \frac{4\ln(x)\, x - 4x + c_1}{x^2}$$

Hence the ode is

$$y' + \frac{2y}{x} = \frac{4\ln(x)\, x - 4x + c_1}{x^2}$$

The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int \frac{2}{x}dx} \\
&= x^2
\end{aligned}$$

The ode becomes

$$\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{4\ln(x)\,x - 4x + c_1}{x^2}\right) \\
\frac{d}{dx}\left(x^2 y\right) &= \left(x^2\right)\left(\frac{4\ln(x)\,x - 4x + c_1}{x^2}\right) \\
d\left(x^2 y\right) &= (4\ln(x)\,x - 4x + c_1)\, dx
\end{aligned}$$

Integrating gives

$$\begin{aligned}
x^2 y &= \int 4\ln(x)\,x - 4x + c_1\, dx \\
x^2 y &= -3x^2 + c_1 x + 2\ln(x)\,x^2 + c_2
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x^2$ results in

$$y = \frac{-3x^2 + c_1 x + 2\ln(x)\,x^2}{x^2} + \frac{c_2}{x^2}$$

which simplifies to

$$y = \frac{-3x^2 + c_1 x + 2\ln(x)\,x^2 + c_2}{x^2}$$

Summary

The solution(s) found are the following

$$y = \frac{-3x^2 + c_1 x + 2\ln(x)\,x^2 + c_2}{x^2} \tag{1}$$

Verification of solutions

$$y = \frac{-3x^2 + c_1 x + 2\ln(x)\,x^2 + c_2}{x^2}$$

Verified OK.

10.1.9 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' + 4xy' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 4x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 300: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{4x}{x^2} \, dx}
\end{aligned}
$$

$$= z_1 e^{-2\ln(x)}$$
$$= z_1 \left(\frac{1}{x^2}\right)$$

Which simplifies to

$$y_1 = \frac{1}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{4x}{x^2}\, dx}}{(y_1)^2}\, dx$$
$$= y_1 \int \frac{e^{-4\ln(x)}}{(y_1)^2}\, dx$$
$$= y_1(x)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1 \left(\frac{1}{x^2}\right) + c_2 \left(\frac{1}{x^2}(x)\right)$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' + 4xy' + 2y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \frac{c_1}{x^2} + \frac{c_2}{x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x^2}$$

$$y_2 = \frac{1}{x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x^2} & \frac{1}{x} \\ \frac{d}{dx}\left(\frac{1}{x^2}\right) & \frac{d}{dx}\left(\frac{1}{x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x^2} & \frac{1}{x} \\ -\frac{2}{x^3} & -\frac{1}{x^2} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x^2}\right)\left(-\frac{1}{x^2}\right) - \left(\frac{1}{x}\right)\left(-\frac{2}{x^3}\right)$$

Which simplifies to

$$W = \frac{1}{x^4}$$

Which simplifies to

$$W = \frac{1}{x^4}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{4\ln(x)}{x}}{\frac{1}{x^2}}\, dx$$

Which simplifies to

$$u_1 = -\int 4\ln(x)\, x dx$$

Hence

$$u_1 = -2\ln(x)\, x^2 + x^2$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{4\ln(x)}{x^2}}{\frac{1}{x^2}}\, dx$$

Which simplifies to

$$u_2 = \int 4\ln(x)\, dx$$

Hence

$$u_2 = 4\ln(x)\, x - 4x$$

Which simplifies to

$$u_1 = x^2(1 - 2\ln(x))$$
$$u_2 = 4\ln(x)\, x - 4x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = 1 - 2\ln(x) + \frac{4\ln(x)\, x - 4x}{x}$$

Which simplifies to

$$y_p(x) = 2\ln(x) - 3$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\frac{c_1}{x^2} + \frac{c_2}{x}\right) + (2\ln(x) - 3)
\end{aligned}
$$

Which simplifies to

$$
y = \frac{c_2 x + c_1}{x^2} + 2\ln(x) - 3
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_2 x + c_1}{x^2} + 2\ln(x) - 3 \tag{1}
$$

Verification of solutions

$$
y = \frac{c_2 x + c_1}{x^2} + 2\ln(x) - 3
$$

Verified OK.

10.1.10 Solving as exact linear second order ode ode

An ode of the form

$$
p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)
$$

is exact if

$$
p''(x) - q'(x) + r(x) = 0 \tag{1}
$$

For the given ode we have

$$
\begin{aligned}
p(x) &= x^2 \\
q(x) &= 4x \\
r(x) &= 2 \\
s(x) &= 4\ln(x)
\end{aligned}
$$

Hence

$$
\begin{aligned}
p''(x) &= 2 \\
q'(x) &= 4
\end{aligned}
$$

Therefore (1) becomes

$$2-(4)+(2)=0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$\left(p(x)\,y'+\left(q(x)-p'(x)\right)y\right)'=s(x)$$

Integrating gives

$$p(x)\,y'+\left(q(x)-p'(x)\right)y=\int s(x)\;dx$$

Substituting the above values for p,q,r,s gives

$$y'x^2+2yx=\int 4\ln\left(x\right)\,dx$$

We now have a first order ode to solve which is

$$y'x^2+2yx=4\ln\left(x\right)x-4x+c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y'+p(x)y=q(x)$$

Where here

$$\begin{aligned}p(x)&=\frac{2}{x}\\ q(x)&=\frac{4\ln\left(x\right)x-4x+c_1}{x^2}\end{aligned}$$

Hence the ode is

$$y'+\frac{2y}{x}=\frac{4\ln\left(x\right)x-4x+c_1}{x^2}$$

The integrating factor μ is

$$\begin{aligned}\mu&=\mathrm{e}^{\int\frac{2}{x}dx}\\ &=x^2\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y)&=(\mu)\left(\frac{4\ln\left(x\right)x-4x+c_1}{x^2}\right)\\ \frac{\mathrm{d}}{\mathrm{d}x}\left(x^2y\right)&=\left(x^2\right)\left(\frac{4\ln\left(x\right)x-4x+c_1}{x^2}\right)\\ \mathrm{d}\left(x^2y\right)&=\left(4\ln\left(x\right)x-4x+c_1\right)\,\mathrm{d}x\end{aligned}$$

Integrating gives

$$x^2y = \int 4\ln(x)\,x - 4x + c_1\,dx$$
$$x^2y = -3x^2 + c_1x + 2\ln(x)\,x^2 + c_2$$

Dividing both sides by the integrating factor $\mu = x^2$ results in

$$y = \frac{-3x^2 + c_1x + 2\ln(x)\,x^2}{x^2} + \frac{c_2}{x^2}$$

which simplifies to

$$y = \frac{-3x^2 + c_1x + 2\ln(x)\,x^2 + c_2}{x^2}$$

Summary

The solution(s) found are the following

$$y = \frac{-3x^2 + c_1x + 2\ln(x)\,x^2 + c_2}{x^2} \tag{1}$$

Verification of solutions

$$y = \frac{-3x^2 + c_1x + 2\ln(x)\,x^2 + c_2}{x^2}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve(x^2*diff(y(x),x$2)+4*x*diff(y(x),x)+2*y(x)=4*ln(x),y(x), singsol=all)
```

$$y(x) = 2\ln(x) + \frac{c_1}{x} - 3 + \frac{c_2}{x^2}$$

✓ Solution by Mathematica

Time used: 0.016 (sec). Leaf size: 23

```
DSolve[x^2*y''[x]+4*x*y'[x]+2*y[x]==4*Log[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_1}{x^2} + 2\log(x) + \frac{c_2}{x} - 3$$

10.2 problem Problem 15

Internal problem ID [2803]
Internal file name [`OUTPUT/2295_Sunday_June_05_2022_02_58_10_AM_46815860/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.8, A Differential Equation with Nonconstant Coefficients. page 567
Problem number: Problem 15.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "exact linear second order ode", "second_order_integrable_as_is", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_1", "second_order_change_of_variable_on_y_method_2", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _nonhomogeneous]]
```

$$\boxed{x^2y'' + 4xy' + 2y = \cos(x)}$$

10.2.1 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = 4x, C = 2, f(x) = \cos(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' + 4xy' + 2y = 0$$

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} + 4xrx^{r-1} + 2x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r + 4r\,x^r + 2x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) + 4r + 2 = 0$$

Or

$$r^2 + 3r + 2 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -2$$
$$r_2 = -1$$

Since the roots are real and distinct, then the general solution is

$$y = c_1y_1 + c_2y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = \frac{c_1}{x^2} + \frac{c_2}{x}$$

Next, we find the particular solution to the ODE

$$x^2y'' + 4xy' + 2y = \cos(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x^2}$$
$$y_2 = \frac{1}{x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x^2} & \frac{1}{x} \\ \frac{d}{dx}\left(\frac{1}{x^2}\right) & \frac{d}{dx}\left(\frac{1}{x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x^2} & \frac{1}{x} \\ -\frac{2}{x^3} & -\frac{1}{x^2} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x^2}\right)\left(-\frac{1}{x^2}\right) - \left(\frac{1}{x}\right)\left(-\frac{2}{x^3}\right)$$

Which simplifies to

$$W = \frac{1}{x^4}$$

Which simplifies to

$$
W = \frac{1}{x^4}
$$

Therefore Eq. (2) becomes

$$
u_1 = -\int \frac{\frac{\cos(x)}{x}}{\frac{1}{x^2}} dx
$$

Which simplifies to

$$
u_1 = -\int \cos(x) x dx
$$

Hence

$$
u_1 = -x \sin(x) - \cos(x)
$$

And Eq. (3) becomes

$$
u_2 = \int \frac{\frac{\cos(x)}{x^2}}{\frac{1}{x^2}} dx
$$

Which simplifies to

$$
u_2 = \int \cos(x) dx
$$

Hence

$$
u_2 = \sin(x)
$$

Therefore the particular solution, from equation (1) is

$$
y_p(x) = \frac{-x \sin(x) - \cos(x)}{x^2} + \frac{\sin(x)}{x}
$$

Which simplifies to

$$
y_p(x) = -\frac{\cos(x)}{x^2}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \frac{c_2 x - \cos(x) + c_1}{x^2}
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = \frac{c_2x - \cos(x) + c_1}{x^2} \tag{1}$$

Verification of solutions

$$y = \frac{c_2x - \cos(x) + c_1}{x^2}$$

Verified OK.

10.2.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = \frac{4}{x}$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\, dx} \\ &= e^{\int \frac{4}{x}\, dx} \\ &= x^2 \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)y)'' &= \cos(x) \\ \left(x^2 y\right)'' &= \cos(x) \end{aligned}$$

Integrating once gives

$$\left(x^2 y\right)' = \sin(x) + c_1$$

Integrating again gives

$$\left(x^2 y\right) = c_1 x - \cos(x) + c_2$$

Hence the solution is

$$y = \frac{c_1x - \cos(x) + c_2}{x^2}$$

Or

$$y = \frac{c_1}{x} + \frac{c_2}{x^2} - \frac{\cos(x)}{x^2}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1}{x} + \frac{c_2}{x^2} - \frac{\cos(x)}{x^2} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x} + \frac{c_2}{x^2} - \frac{\cos(x)}{x^2}$$

Verified OK.

10.2.3 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$x^2y'' + 4xy' + 2y = 0$$

In normal form the ode

$$x^2y'' + 4xy' + 2y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$p(x) = \frac{4}{x}$$
$$q(x) = \frac{2}{x^2}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$
$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\
&= \int \mathrm{e}^{-\left(\int \frac{4}{x}dx\right)}dx \\
&= \int e^{-4\ln(x)}\,dx \\
&= \int \frac{1}{x^4}dx \\
&= -\frac{1}{3x^3}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{2}{x^2}}{\frac{1}{x^8}} \\
&= 2x^6
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0$$
$$\frac{d^2}{d\tau^2}y(\tau) + 2x^6 y(\tau) = 0$$

But in terms of τ

$$2x^6 = \frac{2}{9\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{2y(\tau)}{9\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$9\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + 2y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$9\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 2\tau^r = 0$$

Simplifying gives

$$9r(r-1)\,\tau^r + 0\,\tau^r + 2\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$9r(r-1) + 0 + 2 = 0$$

Or

$$9r^2 - 9r + 2 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{3}$$
$$r_2 = \frac{2}{3}$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1y_1 + c_2y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = c_1\tau^{\frac{1}{3}} + c_2\tau^{\frac{2}{3}}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{c_1 3^{\frac{2}{3}}\left(-\frac{1}{x^3}\right)^{\frac{1}{3}}}{3} + \frac{c_2 3^{\frac{1}{3}}\left(-\frac{1}{x^3}\right)^{\frac{2}{3}}}{3}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{c_1 3^{\frac{2}{3}} \left(-\frac{1}{x^3}\right)^{\frac{1}{3}}}{3} + \frac{c_2 3^{\frac{1}{3}} \left(-\frac{1}{x^3}\right)^{\frac{2}{3}}}{3}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \left(-\frac{1}{x^3}\right)^{\frac{1}{3}}$$

$$y_2 = \left(-\frac{1}{x^3}\right)^{\frac{2}{3}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \left(-\frac{1}{x^3}\right)^{\frac{1}{3}} & \left(-\frac{1}{x^3}\right)^{\frac{2}{3}} \\ \frac{d}{dx}\left(\left(-\frac{1}{x^3}\right)^{\frac{1}{3}}\right) & \frac{d}{dx}\left(\left(-\frac{1}{x^3}\right)^{\frac{2}{3}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \left(-\frac{1}{x^3}\right)^{\frac{1}{3}} & \left(-\frac{1}{x^3}\right)^{\frac{2}{3}} \\ \frac{1}{\left(-\frac{1}{x^3}\right)^{\frac{2}{3}} x^4} & \frac{2}{\left(-\frac{1}{x^3}\right)^{\frac{1}{3}} x^4} \end{vmatrix}$$

Therefore

$$W = \left(\left(-\frac{1}{x^3}\right)^{\frac{1}{3}}\right)\left(\frac{2}{\left(-\frac{1}{x^3}\right)^{\frac{1}{3}} x^4}\right) - \left(\left(-\frac{1}{x^3}\right)^{\frac{2}{3}}\right)\left(\frac{1}{\left(-\frac{1}{x^3}\right)^{\frac{2}{3}} x^4}\right)$$

Which simplifies to

$$W = \frac{1}{x^4}$$

Which simplifies to

$$W = \frac{1}{x^4}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\left(-\frac{1}{x^3}\right)^{\frac{2}{3}} \cos(x)}{\frac{1}{x^2}} dx$$

Which simplifies to

$$u_1 = -\int \left(-\frac{1}{x^3}\right)^{\frac{2}{3}} \cos(x) x^2 dx$$

Hence

$$u_1 = -\left(-\frac{1}{x^3}\right)^{\frac{2}{3}} \sin(x) x^2$$

And Eq. (3) becomes

$$u_2 = \int \frac{\left(-\frac{1}{x^3}\right)^{\frac{1}{3}} \cos(x)}{\frac{1}{x^2}} dx$$

Which simplifies to

$$u_2 = \int \left(-\frac{1}{x^3}\right)^{\frac{1}{3}} \cos(x) x^2 dx$$

Hence

$$u_2 = \left(-\frac{1}{x^3}\right)^{\frac{1}{3}} \cos(x) x + \left(-\frac{1}{x^3}\right)^{\frac{1}{3}} \sin(x) x^2$$

Which simplifies to

$$u_1 = -\left(-\frac{1}{x^3}\right)^{\frac{2}{3}} \sin(x)\, x^2$$

$$u_2 = \left(-\frac{1}{x^3}\right)^{\frac{1}{3}} x(x\sin(x) + \cos(x))$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\sin(x)}{x} - \frac{x\sin(x) + \cos(x)}{x^2}$$

Which simplifies to

$$y_p(x) = -\frac{\cos(x)}{x^2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1 3^{\frac{2}{3}} \left(-\frac{1}{x^3}\right)^{\frac{1}{3}}}{3} + \frac{c_2 3^{\frac{1}{3}} \left(-\frac{1}{x^3}\right)^{\frac{2}{3}}}{3}\right) + \left(-\frac{\cos(x)}{x^2}\right) \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{c_1 3^{\frac{2}{3}} \left(-\frac{1}{x^3}\right)^{\frac{1}{3}}}{3} + \frac{c_2 3^{\frac{1}{3}} \left(-\frac{1}{x^3}\right)^{\frac{2}{3}}}{3} - \frac{\cos(x)}{x^2} \tag{1}$$

<u>Verification of solutions</u>

$$y = \frac{c_1 3^{\frac{2}{3}} \left(-\frac{1}{x^3}\right)^{\frac{1}{3}}}{3} + \frac{c_2 3^{\frac{1}{3}} \left(-\frac{1}{x^3}\right)^{\frac{2}{3}}}{3} - \frac{\cos(x)}{x^2}$$

Verified OK.

10.2.4 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = 4x, C = 2, f(x) = \cos(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' + 4xy' + 2y = 0$$

In normal form the ode

$$x^2y'' + 4xy' + 2y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{4}{x} \\ q(x) &= \frac{2}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{2}\,\sqrt{\frac{1}{x^2}}}{c} \end{aligned} \tag{6}$$

$$\tau'' = -\frac{\sqrt{2}}{c\sqrt{\frac{1}{x^2}}\,x^3}$$

Substituting the above into (4) results in

$$
\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{-\frac{\sqrt{2}}{c\sqrt{\frac{1}{x^2}}\,x^3} + \frac{4}{x}\frac{\sqrt{2}\,\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{2}\,\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\
&= \frac{3c\sqrt{2}}{2}
\end{aligned}
$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + \frac{3c\sqrt{2}\left(\frac{d}{d\tau}y(\tau)\right)}{2} + c^2 y(\tau) &= 0 \qquad (7)
\end{aligned}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
y(\tau) = \mathrm{e}^{-\frac{3\sqrt{2}\,c\tau}{4}}\left(c_1 \cosh\left(\frac{\sqrt{2}\,c\tau}{4}\right) + ic_2 \sinh\left(\frac{\sqrt{2}\,c\tau}{4}\right)\right)
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int \sqrt{2}\,\sqrt{\frac{1}{x^2}}dx}{c} \\
&= \frac{\sqrt{2}\,\sqrt{\frac{1}{x^2}}\,x\ln(x)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = \frac{c_1 \cosh\left(\frac{\ln(x)}{2}\right) + ic_2 \sinh\left(\frac{\ln(x)}{2}\right)}{x^{\frac{3}{2}}}
$$

Now the particular solution to this ODE is found

$$
x^2 y'' + 4xy' + 2y = \cos(x)
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \left(-\frac{1}{x^3}\right)^{\frac{1}{3}}$$

$$y_2 = \left(-\frac{1}{x^3}\right)^{\frac{2}{3}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \left(-\frac{1}{x^3}\right)^{\frac{1}{3}} & \left(-\frac{1}{x^3}\right)^{\frac{2}{3}} \\ \frac{d}{dx}\left(\left(-\frac{1}{x^3}\right)^{\frac{1}{3}}\right) & \frac{d}{dx}\left(\left(-\frac{1}{x^3}\right)^{\frac{2}{3}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \left(-\frac{1}{x^3}\right)^{\frac{1}{3}} & \left(-\frac{1}{x^3}\right)^{\frac{2}{3}} \\ \frac{1}{\left(-\frac{1}{x^3}\right)^{\frac{2}{3}} x^4} & \frac{2}{\left(-\frac{1}{x^3}\right)^{\frac{1}{3}} x^4} \end{vmatrix}$$

Therefore

$$W = \left(\left(-\frac{1}{x^3}\right)^{\frac{1}{3}}\right)\left(\frac{2}{\left(-\frac{1}{x^3}\right)^{\frac{1}{3}} x^4}\right) - \left(\left(-\frac{1}{x^3}\right)^{\frac{2}{3}}\right)\left(\frac{1}{\left(-\frac{1}{x^3}\right)^{\frac{2}{3}} x^4}\right)$$

Which simplifies to

$$W = \frac{1}{x^4}$$

Which simplifies to

$$W = \frac{1}{x^4}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\left(-\frac{1}{x^3}\right)^{\frac{2}{3}} \cos\left(x\right)}{\frac{1}{x^2}}dx$$

Which simplifies to

$$u_1 = -\int \left(-\frac{1}{x^3}\right)^{\frac{2}{3}} \cos\left(x\right) x^2 dx$$

Hence

$$u_1 = -\left(-\frac{1}{x^3}\right)^{\frac{2}{3}} \sin\left(x\right) x^2$$

And Eq. (3) becomes

$$u_2 = \int \frac{\left(-\frac{1}{x^3}\right)^{\frac{1}{3}} \cos\left(x\right)}{\frac{1}{x^2}}dx$$

Which simplifies to

$$u_2 = \int \left(-\frac{1}{x^3}\right)^{\frac{1}{3}} \cos\left(x\right) x^2 dx$$

Hence

$$u_2 = \left(-\frac{1}{x^3}\right)^{\frac{1}{3}} \cos\left(x\right) x + \left(-\frac{1}{x^3}\right)^{\frac{1}{3}} \sin\left(x\right) x^2$$

Which simplifies to

$$u_1 = -\left(-\frac{1}{x^3}\right)^{\frac{2}{3}} \sin\left(x\right) x^2$$

$$u_2 = \left(-\frac{1}{x^3}\right)^{\frac{1}{3}} x(x\sin\left(x\right) + \cos\left(x\right))$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\sin(x)}{x} - \frac{x \sin(x) + \cos(x)}{x^2}$$

Which simplifies to

$$y_p(x) = -\frac{\cos(x)}{x^2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1 \cosh\left(\frac{\ln(x)}{2}\right) + i c_2 \sinh\left(\frac{\ln(x)}{2}\right)}{x^{\frac{3}{2}}}\right) + \left(-\frac{\cos(x)}{x^2}\right) \\ &= -\frac{\cos(x)}{x^2} + \frac{c_1 \cosh\left(\frac{\ln(x)}{2}\right) + i c_2 \sinh\left(\frac{\ln(x)}{2}\right)}{x^{\frac{3}{2}}} \end{aligned}$$

Which simplifies to

$$y = -\frac{\cos(x)}{x^2} + \frac{c_1 \cosh\left(\frac{\ln(x)}{2}\right) + i c_2 \sinh\left(\frac{\ln(x)}{2}\right)}{x^{\frac{3}{2}}}$$

Summary

The solution(s) found are the following

$$y = -\frac{\cos(x)}{x^2} + \frac{c_1 \cosh\left(\frac{\ln(x)}{2}\right) + i c_2 \sinh\left(\frac{\ln(x)}{2}\right)}{x^{\frac{3}{2}}} \tag{1}$$

Verification of solutions

$$y = -\frac{\cos(x)}{x^2} + \frac{c_1 \cosh\left(\frac{\ln(x)}{2}\right) + i c_2 \sinh\left(\frac{\ln(x)}{2}\right)}{x^{\frac{3}{2}}}$$

Verified OK.

10.2.5 Solving as second order change of variable on y method 1 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' + 4xy' + 2y = 0$$

In normal form the given ode is written as

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned}p(x) &= \frac{4}{x}\\ q(x) &= \frac{2}{x^2}\end{aligned}$$

Calculating the Liouville ode invariant Q given by

$$\begin{aligned}Q &= q - \frac{p'}{2} - \frac{p^2}{4}\\ &= \frac{2}{x^2} - \frac{\left(\frac{4}{x}\right)'}{2} - \frac{\left(\frac{4}{x}\right)^2}{4}\\ &= \frac{2}{x^2} - \frac{\left(-\frac{4}{x^2}\right)}{2} - \frac{\left(\frac{16}{x^2}\right)}{4}\\ &= \frac{2}{x^2} - \left(-\frac{2}{x^2}\right) - \frac{4}{x^2}\\ &= 0\end{aligned}$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$y = v(x)\,z(x) \tag{3}$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$\begin{aligned}z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2}dx\right)}\\ &= e^{-\int \frac{\frac{4}{x}}{2}}\\ &= \frac{1}{x^2}\end{aligned} \tag{5}$$

Hence (3) becomes

$$y = \frac{v(x)}{x^2} \tag{4}$$

Applying this change of variable to the original ode results in

$$v''(x) = \cos(x)$$

Which is now solved for $v(x)$ Integrating once gives

$$v'(x) = \sin(x) + c_1$$

Integrating again gives

$$v(x) = -\cos(x) + c_1 x + c_2$$

Now that $v(x)$ is known, then

$$\begin{aligned} y &= v(x)\, z(x) \\ &= (c_1 x - \cos(x) + c_2)\,(z(x)) \end{aligned} \tag{7}$$

But from (5)

$$z(x) = \frac{1}{x^2}$$

Hence (7) becomes

$$y = \frac{c_1 x - \cos(x) + c_2}{x^2}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{c_1 x - \cos(x) + c_2}{x^2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \left(-\frac{1}{x^3}\right)^{\frac{1}{3}}$$

$$y_2 = \left(-\frac{1}{x^3}\right)^{\frac{2}{3}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \left(-\frac{1}{x^3}\right)^{\frac{1}{3}} & \left(-\frac{1}{x^3}\right)^{\frac{2}{3}} \\ \frac{d}{dx}\left(\left(-\frac{1}{x^3}\right)^{\frac{1}{3}}\right) & \frac{d}{dx}\left(\left(-\frac{1}{x^3}\right)^{\frac{2}{3}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \left(-\frac{1}{x^3}\right)^{\frac{1}{3}} & \left(-\frac{1}{x^3}\right)^{\frac{2}{3}} \\ \frac{1}{\left(-\frac{1}{x^3}\right)^{\frac{2}{3}} x^4} & \frac{2}{\left(-\frac{1}{x^3}\right)^{\frac{1}{3}} x^4} \end{vmatrix}$$

Therefore

$$W = \left(\left(-\frac{1}{x^3}\right)^{\frac{1}{3}}\right)\left(\frac{2}{\left(-\frac{1}{x^3}\right)^{\frac{1}{3}} x^4}\right) - \left(\left(-\frac{1}{x^3}\right)^{\frac{2}{3}}\right)\left(\frac{1}{\left(-\frac{1}{x^3}\right)^{\frac{2}{3}} x^4}\right)$$

Which simplifies to

$$W = \frac{1}{x^4}$$

Which simplifies to

$$W = \frac{1}{x^4}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\left(-\frac{1}{x^3}\right)^{\frac{2}{3}} \cos(x)}{\frac{1}{x^2}} dx$$

Which simplifies to

$$u_1 = -\int \left(-\frac{1}{x^3}\right)^{\frac{2}{3}} \cos(x)\, x^2 dx$$

Hence

$$u_1 = -\left(-\frac{1}{x^3}\right)^{\frac{2}{3}} \sin(x)\, x^2$$

And Eq. (3) becomes

$$u_2 = \int \frac{\left(-\frac{1}{x^3}\right)^{\frac{1}{3}} \cos(x)}{\frac{1}{x^2}} dx$$

Which simplifies to

$$u_2 = \int \left(-\frac{1}{x^3}\right)^{\frac{1}{3}} \cos(x)\, x^2 dx$$

Hence

$$u_2 = \left(-\frac{1}{x^3}\right)^{\frac{1}{3}} \cos(x)\, x + \left(-\frac{1}{x^3}\right)^{\frac{1}{3}} \sin(x)\, x^2$$

Which simplifies to

$$u_1 = -\left(-\frac{1}{x^3}\right)^{\frac{2}{3}} \sin(x)\, x^2$$

$$u_2 = \left(-\frac{1}{x^3}\right)^{\frac{1}{3}} x(x \sin(x) + \cos(x))$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\sin(x)}{x} - \frac{x\sin(x) + \cos(x)}{x^2}$$

Which simplifies to

$$y_p(x) = -\frac{\cos(x)}{x^2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1 x - \cos(x) + c_2}{x^2}\right) + \left(-\frac{\cos(x)}{x^2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 x - \cos(x) + c_2}{x^2} - \frac{\cos(x)}{x^2} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x - \cos(x) + c_2}{x^2} - \frac{\cos(x)}{x^2}$$

Verified OK.

10.2.6 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = 4x, C = 2, f(x) = \cos(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' + 4xy' + 2y = 0$$

In normal form the ode

$$x^2y'' + 4xy' + 2y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{4}{x} \\ q(x) &= \frac{2}{x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{4n}{x^2} + \frac{2}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = -1 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{2v'(x)}{x} &= 0 \\ v''(x) + \frac{2v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{2u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{2u}{x} \end{aligned}$$

Where $f(x) = -\frac{2}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= -\frac{2}{x}\,dx \\ \int \frac{1}{u}\,du &= \int -\frac{2}{x}\,dx \\ \ln(u) &= -2\ln(x) + c_1 \\ u &= \mathrm{e}^{-2\ln(x)+c_1} \\ &= \frac{c_1}{x^2} \end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\,dx + c_2 \\ &= -\frac{c_1}{x} + c_2 \end{aligned}$$

Hence

$$\begin{aligned} y &= v(x)\,x^n \\ &= \frac{-\frac{c_1}{x} + c_2}{x} \\ &= \frac{c_2 x - c_1}{x^2} \end{aligned}$$

Now the particular solution to this ODE is found

$$x^2y'' + 4xy' + 2y = \cos(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x^2}$$
$$y_2 = \frac{1}{x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x^2} & \frac{1}{x} \\ \frac{d}{dx}\left(\frac{1}{x^2}\right) & \frac{d}{dx}\left(\frac{1}{x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x^2} & \frac{1}{x} \\ -\frac{2}{x^3} & -\frac{1}{x^2} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x^2}\right)\left(-\frac{1}{x^2}\right) - \left(\frac{1}{x}\right)\left(-\frac{2}{x^3}\right)$$

Which simplifies to

$$W = \frac{1}{x^4}$$

Which simplifies to

$$W = \frac{1}{x^4}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\cos(x)}{x}}{\frac{1}{x^2}} \, dx$$

Which simplifies to

$$u_1 = -\int \cos(x) \, x dx$$

Hence

$$u_1 = -x \sin(x) - \cos(x)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{\cos(x)}{x^2}}{\frac{1}{x^2}} \, dx$$

Which simplifies to

$$u_2 = \int \cos(x) \, dx$$

Hence

$$u_2 = \sin(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{-x \sin(x) - \cos(x)}{x^2} + \frac{\sin(x)}{x}$$

Which simplifies to

$$y_p(x) = -\frac{\cos(x)}{x^2}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\frac{-\frac{c_1}{x} + c_2}{x}\right) + \left(-\frac{\cos(x)}{x^2}\right) \\
&= -\frac{\cos(x)}{x^2} + \frac{-\frac{c_1}{x} + c_2}{x}
\end{aligned}
$$

Which simplifies to

$$
y = \frac{-\cos(x) + c_2 x - c_1}{x^2}
$$

Summary

The solution(s) found are the following

$$
y = \frac{-\cos(x) + c_2 x - c_1}{x^2} \tag{1}
$$

Verification of solutions

$$
y = \frac{-\cos(x) + c_2 x - c_1}{x^2}
$$

Verified OK.

10.2.7 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$
\begin{aligned}
\int (x^2 y'' + 4xy' + 2y)\, dx &= \int \cos(x)\, dx \\
y' x^2 + 2yx &= \sin(x) + c_1
\end{aligned}
$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
y' + p(x)y = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= \frac{2}{x} \\
q(x) &= \frac{\sin(x) + c_1}{x^2}
\end{aligned}
$$

Hence the ode is

$$y' + \frac{2y}{x} = \frac{\sin(x) + c_1}{x^2}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{2}{x} dx} \\ &= x^2\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{\sin(x) + c_1}{x^2}\right) \\ \frac{d}{dx}(x^2 y) &= (x^2)\left(\frac{\sin(x) + c_1}{x^2}\right) \\ d(x^2 y) &= (\sin(x) + c_1)\, dx\end{aligned}$$

Integrating gives

$$\begin{aligned}x^2 y &= \int \sin(x) + c_1 \, dx \\ x^2 y &= c_1 x - \cos(x) + c_2\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x^2$ results in

$$y = \frac{c_1 x - \cos(x)}{x^2} + \frac{c_2}{x^2}$$

which simplifies to

$$y = \frac{c_1 x - \cos(x) + c_2}{x^2}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1 x - \cos(x) + c_2}{x^2} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x - \cos(x) + c_2}{x^2}$$

Verified OK.

10.2.8 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$x^2y'' + 4xy' + 2y = \cos(x)$$

Integrating both sides of the ODE w.r.t x gives

$$\int (x^2y'' + 4xy' + 2y)\, dx = \int \cos(x)\, dx$$
$$y'x^2 + 2yx = \sin(x) + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{2}{x}$$
$$q(x) = \frac{\sin(x) + c_1}{x^2}$$

Hence the ode is

$$y' + \frac{2y}{x} = \frac{\sin(x) + c_1}{x^2}$$

The integrating factor μ is

$$\mu = e^{\int \frac{2}{x} dx}$$
$$= x^2$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{\sin(x) + c_1}{x^2}\right)$$
$$\frac{d}{dx}(x^2y) = (x^2)\left(\frac{\sin(x) + c_1}{x^2}\right)$$
$$d(x^2y) = (\sin(x) + c_1)\, dx$$

Integrating gives

$$x^2y = \int \sin(x) + c_1\, dx$$
$$x^2y = c_1x - \cos(x) + c_2$$

Dividing both sides by the integrating factor $\mu = x^2$ results in

$$y = \frac{c_1 x - \cos(x)}{x^2} + \frac{c_2}{x^2}$$

which simplifies to

$$y = \frac{c_1 x - \cos(x) + c_2}{x^2}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 x - \cos(x) + c_2}{x^2} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x - \cos(x) + c_2}{x^2}$$

Verified OK.

10.2.9 Solving using Kovacic algorithm

Writing the ode as

$$x^2 y'' + 4xy' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 4x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 301: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4x}{x^2}\,dx} \\
&= z_1 e^{-2\ln(x)} \\
&= z_1\left(\frac{1}{x^2}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-4\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{x^2}\right) + c_2 \left(\frac{1}{x^2}(x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
x^2 y'' + 4xy' + 2y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = \frac{c_1}{x^2} + \frac{c_2}{x}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \frac{1}{x^2} \\
y_2 &= \frac{1}{x}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x^2} & \frac{1}{x} \\ \frac{d}{dx}\left(\frac{1}{x^2}\right) & \frac{d}{dx}\left(\frac{1}{x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x^2} & \frac{1}{x} \\ -\frac{2}{x^3} & -\frac{1}{x^2} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x^2}\right)\left(-\frac{1}{x^2}\right) - \left(\frac{1}{x}\right)\left(-\frac{2}{x^3}\right)$$

Which simplifies to

$$W = \frac{1}{x^4}$$

Which simplifies to

$$W = \frac{1}{x^4}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\cos(x)}{x}}{\frac{1}{x^2}} dx$$

Which simplifies to

$$u_1 = -\int \cos(x)\, x dx$$

Hence

$$u_1 = -x \sin(x) - \cos(x)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{\cos(x)}{x^2}}{\frac{1}{x^2}} dx$$

Which simplifies to

$$u_2 = \int \cos(x)\, dx$$

Hence

$$u_2 = \sin(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{-x\sin(x) - \cos(x)}{x^2} + \frac{\sin(x)}{x}$$

Which simplifies to

$$y_p(x) = -\frac{\cos(x)}{x^2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1}{x^2} + \frac{c_2}{x}\right) + \left(-\frac{\cos(x)}{x^2}\right) \end{aligned}$$

Which simplifies to

$$y = \frac{c_2 x + c_1}{x^2} - \frac{\cos(x)}{x^2}$$

Summary

The solution(s) found are the following

$$y = \frac{c_2 x + c_1}{x^2} - \frac{\cos(x)}{x^2} \tag{1}$$

Verification of solutions

$$y = \frac{c_2 x + c_1}{x^2} - \frac{\cos(x)}{x^2}$$

Verified OK.

10.2.10 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= x^2 \\ q(x) &= 4x \\ r(x) &= 2 \\ s(x) &= \cos(x) \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 2 \\ q'(x) &= 4 \end{aligned}$$

Therefore (1) becomes

$$2 - (4) + (2) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$\left(p(x)\, y' + (q(x) - p'(x))\, y\right)' = s(x)$$

Integrating gives

$$p(x)\, y' + (q(x) - p'(x))\, y = \int s(x)\, dx$$

Substituting the above values for p, q, r, s gives

$$y'x^2 + 2yx = \int \cos(x)\, dx$$

We now have a first order ode to solve which is

$$y'x^2 + 2yx = \sin(x) + c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{2}{x} \\ q(x) &= \frac{\sin(x) + c_1}{x^2} \end{aligned}$$

Hence the ode is

$$y' + \frac{2y}{x} = \frac{\sin(x) + c_1}{x^2}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int \frac{2}{x} dx} \\ &= x^2 \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{\sin(x) + c_1}{x^2}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}(x^2 y) &= (x^2)\left(\frac{\sin(x) + c_1}{x^2}\right) \\ \mathrm{d}(x^2 y) &= (\sin(x) + c_1)\, \mathrm{d}x \end{aligned}$$

Integrating gives

$$\begin{aligned} x^2 y &= \int \sin(x) + c_1 \,\mathrm{d}x \\ x^2 y &= c_1 x - \cos(x) + c_2 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = x^2$ results in

$$y = \frac{c_1 x - \cos(x)}{x^2} + \frac{c_2}{x^2}$$

which simplifies to

$$y = \frac{c_1 x - \cos(x) + c_2}{x^2}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 x - \cos(x) + c_2}{x^2} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x - \cos(x) + c_2}{x^2}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(x^2*diff(y(x),x$2)+4*x*diff(y(x),x)+2*y(x)=cos(x),y(x), singsol=all)
```

$$y(x) = \frac{c_2 + c_1 x - \cos(x)}{x^2}$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 20

```
DSolve[x^2*y''[x]+4*x*y'[x]+2*y[x]==Cos[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{-\cos(x) + c_2 x + c_1}{x^2}$$

10.3 problem Problem 16

Internal problem ID [2804]
Internal file name [`OUTPUT/2296_Sunday_June_05_2022_02_58_12_AM_49847731/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.8, A Differential Equation with Nonconstant Coefficients. page 567
Problem number: Problem 16.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + xy' + 9y = 9\ln(x)}$$

10.3.1 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = x, C = 9, f(x) = 9\ln(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$.

Solving for y_h from

$$x^2y'' + xy' + 9y = 0$$

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} + xrx^{r-1} + 9x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r + r\,x^r + 9x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) + r + 9 = 0$$

Or

$$r^2 + 9 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= -3i \\ r_2 &= 3i \end{aligned}$$

The roots are complex conjugate of each others. Let the roots be

$$\begin{aligned} r_1 &= \alpha + i\beta \\ r_2 &= \alpha - i\beta \end{aligned}$$

Where in this case $\alpha = 0$ and $\beta = -3$. Hence the solution becomes

$$\begin{aligned} y &= c_1x^{r_1} + c_2x^{r_2} \\ &= c_1x^{\alpha+i\beta} + c_2x^{\alpha-i\beta} \\ &= x^\alpha\left(c_1x^{i\beta} + c_2x^{-i\beta}\right) \\ &= x^\alpha\left(c_1e^{\ln\left(x^{i\beta}\right)} + c_2e^{\ln\left(x^{-i\beta}\right)}\right) \\ &= x^\alpha\left(c_1e^{i(\beta\ln x)} + c_2e^{-i(\beta\ln x)}\right) \end{aligned}$$

Using the values for $\alpha = 0, \beta = -3$, the above becomes

$$y = x^0\left(c_1e^{-3i\ln(x)} + c_2e^{3i\ln(x)}\right)$$

Using Euler relation, the expression $c_1 e^{iA} + c_2 e^{-iA}$ is transformed to $c_1 \cos A + c_1 \sin A$ where the constants are free to change. Applying this to the above result gives

$$y = c_1 \cos (3 \ln (x)) + c_2 \sin (3 \ln (x))$$

Next, we find the particular solution to the ODE

$$x^2 y'' + x y' + 9y = 9 \ln (x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= \cos (3 \ln (x)) \\ y_2 &= -\sin (3 \ln (x)) \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos (3 \ln (x)) & -\sin (3 \ln (x)) \\ \frac{d}{dx}(\cos (3 \ln (x))) & \frac{d}{dx}(-\sin (3 \ln (x))) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos (3 \ln (x)) & -\sin (3 \ln (x)) \\ -\frac{3 \sin(3 \ln(x))}{x} & -\frac{3 \cos(3 \ln(x))}{x} \end{vmatrix}$$

Therefore

$$W = (\cos(3\ln(x)))\left(-\frac{3\cos(3\ln(x))}{x}\right) - (-\sin(3\ln(x)))\left(-\frac{3\sin(3\ln(x))}{x}\right)$$

Which simplifies to

$$W = -\frac{3\left(\cos(3\ln(x))^2 + \sin(3\ln(x))^2\right)}{x}$$

Which simplifies to

$$W = -\frac{3}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-9\sin(3\ln(x))\ln(x)}{-3x}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{3\sin(3\ln(x))\ln(x)}{x}dx$$

Hence

$$u_1 = -\frac{\sin(3\ln(x))}{3} + \cos(3\ln(x))\ln(x)$$

And Eq. (3) becomes

$$u_2 = \int \frac{9\cos(3\ln(x))\ln(x)}{-3x}\,dx$$

Which simplifies to

$$u_2 = \int -\frac{3\cos(3\ln(x))\ln(x)}{x}dx$$

Hence

$$u_2 = -\frac{\cos(3\ln(x))}{3} - \sin(3\ln(x))\ln(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-\frac{\sin(3\ln(x))}{3} + \cos(3\ln(x))\ln(x)\right)\cos(3\ln(x)) - \left(-\frac{\cos(3\ln(x))}{3} - \sin(3\ln(x))\ln(x)\right)\sin(3\ln(x))$$

Which simplifies to

$$y_p(x) = \ln(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \ln(x) + c_1\cos(3\ln(x)) + c_2\sin(3\ln(x)) \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \ln(x) + c_1\cos(3\ln(x)) + c_2\sin(3\ln(x)) \tag{1}$$

<u>Verification of solutions</u>

$$y = \ln(x) + c_1\cos(3\ln(x)) + c_2\sin(3\ln(x))$$

Verified OK.

10.3.2 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' + xy' + 9y = 0$$

In normal form the ode

$$x^2y'' + xy' + 9y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = \frac{9}{x^2}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$
$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int e^{-\left(\int p(x)dx\right)}dx \\
&= \int e^{-\left(\int \frac{1}{x}dx\right)}dx \\
&= \int e^{-\ln(x)}\,dx \\
&= \int \frac{1}{x}dx \\
&= \ln(x)
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{9}{x^2}}{\frac{1}{x^2}} \\
&= 9
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0$$
$$\frac{d^2}{d\tau^2}y(\tau) + 9y(\tau) = 0$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = 9$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda\tau} + 9\,\mathrm{e}^{\lambda\tau} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$\lambda^2 + 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 9$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(9)}\\ &= \pm 3i\end{aligned}$$

Hence

$$\lambda_1 = +3i$$
$$\lambda_2 = -3i$$

Which simplifies to

$$\lambda_1 = 3i$$
$$\lambda_2 = -3i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 3$. Therefore the final solution, when using Euler relation, can be written as

$$y(\tau) = e^{\alpha\tau}(c_1 \cos(\beta\tau) + c_2 \sin(\beta\tau))$$

Which becomes

$$y(\tau) = e^0(c_1 \cos(3\tau) + c_2 \sin(3\tau))$$

Or

$$y(\tau) = c_1 \cos(3\tau) + c_2 \sin(3\tau)$$

The above solution is now transformed back to y using (6) which results in

$$y = c_1 \cos(3\ln(x)) + c_2 \sin(3\ln(x))$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(3\ln(x)) + c_2 \sin(3\ln(x))$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= 4\cos(\ln(x))^3 - 3\cos(\ln(x)) \\ y_2 &= 4\sin(\ln(x))\cos(\ln(x))^2 - \sin(\ln(x)) \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} 4\cos(\ln(x))^3 - 3\cos(\ln(x)) & 4\sin(\ln(x))\cos(\ln(x))^2 - \sin(\ln(x)) \\ \frac{d}{dx}\left(4\cos(\ln(x))^3 - 3\cos(\ln(x))\right) & \frac{d}{dx}\left(4\sin(\ln(x))\cos(\ln(x))^2 - \sin(\ln(x))\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} 4\cos(\ln(x))^3 - 3\cos(\ln(x)) & 4\sin(\ln(x))\cos(\ln(x))^2 - \sin(\ln(x)) \\ -\frac{12\cos(\ln(x))^2\sin(\ln(x))}{x} + \frac{3\sin(\ln(x))}{x} & \frac{4\cos(\ln(x))^3}{x} - \frac{8\sin(\ln(x))^2\cos(\ln(x))}{x} - \frac{\cos(\ln(x))}{x} \end{vmatrix}$$

Therefore

$$W = \left(4\cos(\ln(x))^3 - 3\cos(\ln(x))\right)\left(\frac{4\cos(\ln(x))^3}{x} - \frac{8\sin(\ln(x))^2\cos(\ln(x))}{x} - \frac{\cos(\ln(x))}{x}\right) - \left(4\sin(\ln(x))\cos(\ln(x))^2 - \sin(\ln(x))\right)\left(-\frac{12\cos(\ln(x))^2\sin(\ln(x))}{x} + \frac{3\sin(\ln(x))}{x}\right)$$

Which simplifies to

$$W = \frac{16\cos(\ln(x))^6 + 16\cos(\ln(x))^4\sin(\ln(x))^2 - 16\cos(\ln(x))^4 + 3\cos(\ln(x))^2 + 3\sin(\ln(x))^2}{x}$$

Which simplifies to

$$W = \frac{3}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{9\left(4\sin(\ln(x))\cos(\ln(x))^2 - \sin(\ln(x))\right)\ln(x)}{3x}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{3\sin(3\ln(x))\ln(x)}{x}\,dx$$

Hence

$$u_1 = -\frac{\sin(3\ln(x))}{3} + \cos(3\ln(x))\ln(x)$$

And Eq. (3) becomes

$$u_2 = \int \frac{9\left(4\cos(\ln(x))^3 - 3\cos(\ln(x))\right)\ln(x)}{3x}\,dx$$

Which simplifies to

$$u_2 = \int \frac{3\cos(3\ln(x))\ln(x)}{x}dx$$

Hence

$$u_2 = \frac{\cos(3\ln(x))}{3} + \sin(3\ln(x))\ln(x)$$

Therefore the particular solution, from equation (1) is

$$\begin{aligned} y_p(x) = &\left(-\frac{\sin(3\ln(x))}{3} + \cos(3\ln(x))\ln(x)\right)\left(4\cos(\ln(x))^3 - 3\cos(\ln(x))\right) \\ &+ \left(\frac{\cos(3\ln(x))}{3} + \sin(3\ln(x))\ln(x)\right)\left(4\sin(\ln(x))\cos(\ln(x))^2 - \sin(\ln(x))\right) \end{aligned}$$

Which simplifies to

$$y_p(x) = \ln(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1\cos(3\ln(x)) + c_2\sin(3\ln(x))) + (\ln(x)) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \ln(x) + c_1\cos(3\ln(x)) + c_2\sin(3\ln(x)) \tag{1}$$

Verification of solutions

$$y = \ln(x) + c_1\cos(3\ln(x)) + c_2\sin(3\ln(x))$$

Verified OK.

10.3.3 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = x, C = 9, f(x) = 9\ln(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' + xy' + 9y = 0$$

In normal form the ode

$$x^2y'' + xy' + 9y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= \frac{9}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{3\sqrt{\frac{1}{x^2}}}{c} \end{aligned} \tag{6}$$

$$\tau'' = -\frac{3}{c\sqrt{\frac{1}{x^2}}\,x^3}$$

Substituting the above into (4) results in

$$
\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{-\frac{3}{c\sqrt{\frac{1}{x^2}}\,x^3} + \frac{1}{x}\frac{3\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{3\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\
&= 0
\end{aligned}
$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2} y(\tau) + c^2 y(\tau) &= 0 \qquad (7)
\end{aligned}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int 3\sqrt{\frac{1}{x^2}}dx}{c} \\
&= \frac{3\sqrt{\frac{1}{x^2}}\,x \ln(x)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = c_1 \cos(3\ln(x)) + c_2 \sin(3\ln(x))
$$

Now the particular solution to this ODE is found

$$
x^2 y'' + x y' + 9y = 9\ln(x)
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= 4\cos(\ln(x))^3 - 3\cos(\ln(x)) \\ y_2 &= 4\sin(\ln(x))\cos(\ln(x))^2 - \sin(\ln(x)) \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} 4\cos(\ln(x))^3 - 3\cos(\ln(x)) & 4\sin(\ln(x))\cos(\ln(x))^2 - \sin(\ln(x)) \\ \frac{d}{dx}\left(4\cos(\ln(x))^3 - 3\cos(\ln(x))\right) & \frac{d}{dx}\left(4\sin(\ln(x))\cos(\ln(x))^2 - \sin(\ln(x))\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} 4\cos(\ln(x))^3 - 3\cos(\ln(x)) & 4\sin(\ln(x))\cos(\ln(x))^2 - \sin(\ln(x)) \\ -\frac{12\cos(\ln(x))^2\sin(\ln(x))}{x} + \frac{3\sin(\ln(x))}{x} & \frac{4\cos(\ln(x))^3}{x} - \frac{8\sin(\ln(x))^2\cos(\ln(x))}{x} - \frac{\cos(\ln(x))}{x} \end{vmatrix}$$

Therefore

$$\begin{aligned} W = \left(4\cos(\ln(x))^3 - 3\cos(\ln(x))\right)&\left(\frac{4\cos(\ln(x))^3}{x} - \frac{8\sin(\ln(x))^2\cos(\ln(x))}{x} \right. \\ &\left. - \frac{\cos(\ln(x))}{x}\right) - \left(4\sin(\ln(x))\cos(\ln(x))^2 \right. \\ &\left. - \sin(\ln(x))\right)\left(-\frac{12\cos(\ln(x))^2\sin(\ln(x))}{x} + \frac{3\sin(\ln(x))}{x}\right) \end{aligned}$$

Which simplifies to

$$W = \frac{16\cos(\ln(x))^6 + 16\cos(\ln(x))^4\sin(\ln(x))^2 - 16\cos(\ln(x))^4 + 3\cos(\ln(x))^2 + 3\sin(\ln(x))^2}{x}$$

Which simplifies to

$$W = \frac{3}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{9\left(4\sin(\ln(x))\cos(\ln(x))^2 - \sin(\ln(x))\right)\ln(x)}{3x}dx$$

Which simplifies to

$$u_1 = -\int \frac{3\sin(3\ln(x))\ln(x)}{x}dx$$

Hence

$$u_1 = -\frac{\sin(3\ln(x))}{3} + \cos(3\ln(x))\ln(x)$$

And Eq. (3) becomes

$$u_2 = \int \frac{9\left(4\cos(\ln(x))^3 - 3\cos(\ln(x))\right)\ln(x)}{3x}dx$$

Which simplifies to

$$u_2 = \int \frac{3\cos(3\ln(x))\ln(x)}{x}dx$$

Hence

$$u_2 = \frac{\cos(3\ln(x))}{3} + \sin(3\ln(x))\ln(x)$$

Therefore the particular solution, from equation (1) is

$$\begin{aligned} y_p(x) = &\left(-\frac{\sin(3\ln(x))}{3} + \cos(3\ln(x))\ln(x)\right)\left(4\cos(\ln(x))^3 - 3\cos(\ln(x))\right) \\ &+ \left(\frac{\cos(3\ln(x))}{3} + \sin(3\ln(x))\ln(x)\right)\left(4\sin(\ln(x))\cos(\ln(x))^2 - \sin(\ln(x))\right) \end{aligned}$$

Which simplifies to

$$y_p(x) = \ln(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(3\ln(x)) + c_2 \sin(3\ln(x))) + (\ln(x)) \\ &= \ln(x) + c_1 \cos(3\ln(x)) + c_2 \sin(3\ln(x)) \end{aligned}$$

Which simplifies to

$$y = \ln(x) + c_1 \cos(3\ln(x)) + c_2 \sin(3\ln(x))$$

Summary
The solution(s) found are the following

$$y = \ln(x) + c_1 \cos(3\ln(x)) + c_2 \sin(3\ln(x)) \tag{1}$$

Verification of solutions

$$y = \ln(x) + c_1 \cos(3\ln(x)) + c_2 \sin(3\ln(x))$$

Verified OK.

10.3.4 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = x, C = 9, f(x) = 9\ln(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' + xy' + 9y = 0$$

In normal form the ode

$$x^2y'' + xy' + 9y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned}p(x) &= \frac{1}{x}\\ q(x) &= \frac{9}{x^2}\end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{n}{x^2} + \frac{9}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 3i \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned}v''(x) + \left(\frac{6i}{x} + \frac{1}{x}\right) v'(x) &= 0\\ v''(x) + \frac{(1+6i)\,v'(x)}{x} &= 0\end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{(1+6i)\,u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= \frac{(-1-6i)\,u}{x}
\end{aligned}
$$

Where $f(x) = \frac{-1-6i}{x}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= \frac{-1-6i}{x}\,dx \\
\int \frac{1}{u}\,du &= \int \frac{-1-6i}{x}\,dx \\
\ln(u) &= (-1-6i)\ln(x) + c_1 \\
u &= \mathrm{e}^{(-1-6i)\ln(x)+c_1} \\
&= c_1 \mathrm{e}^{(-1-6i)\ln(x)}
\end{aligned}
$$

Which simplifies to

$$
u(x) = \frac{c_1 x^{-6i}}{x}
$$

Now that $u(x)$ is known, then

$$
\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\,dx + c_2 \\
&= \frac{i c_1 x^{-6i}}{6} + c_2
\end{aligned}
$$

Hence

$$
\begin{aligned}
y &= v(x)\,x^n \\
&= \left(\frac{i c_1 x^{-6i}}{6} + c_2\right) x^{3i} \\
&= x^{3i} c_2 + \frac{i x^{-3i} c_1}{6}
\end{aligned}
$$

Now the particular solution to this ODE is found

$$
x^2 y'' + x y' + 9y = 9\ln(x)
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x^{3i}$$
$$y_2 = x^{3i}x^{-6i}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^{3i} & x^{3i}x^{-6i} \\ \frac{d}{dx}(x^{3i}) & \frac{d}{dx}(x^{3i}x^{-6i}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^{3i} & x^{3i}x^{-6i} \\ \frac{3ix^{3i}}{x} & -\frac{3ix^{3i}x^{-6i}}{x} \end{vmatrix}$$

Therefore

$$W = \left(x^{3i}\right)\left(-\frac{3ix^{3i}x^{-6i}}{x}\right) - \left(x^{3i}x^{-6i}\right)\left(\frac{3ix^{3i}}{x}\right)$$

Which simplifies to

$$W = -\frac{6ix^{6i}x^{-6i}}{x}$$

Which simplifies to

$$W = -\frac{6i}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{9x^{3i}x^{-6i}\ln(x)}{-6ix}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{3ix^{-1-3i}\ln(x)}{2}dx$$

Hence

$$u_1 = \mathit{undefined}$$

And Eq. (3) becomes

$$u_2 = \int \frac{9x^{3i}\ln(x)}{-6ix}\,dx$$

Which simplifies to

$$u_2 = \int \frac{3ix^{-1+3i}\ln(x)}{2}dx$$

Hence

$$u_2 = \mathit{undefined}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \text{undefined } x^{3i} + \text{undefined } x^{3i}x^{-6i}$$

Which simplifies to

$$y_p(x) = \text{undefined } \left(x^{3i} + x^{-3i}\right)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\left(\frac{ic_1x^{-6i}}{6} + c_2\right)x^{3i}\right) + \left(\text{undefined } \left(x^{3i} + x^{-3i}\right)\right) \\ &= \text{undefined } \left(x^{3i} + x^{-3i}\right) + \left(\frac{ic_1x^{-6i}}{6} + c_2\right)x^{3i} \end{aligned}$$

Which simplifies to

$$y = \frac{(ic_1 + \text{undefined})\, x^{-3i}}{6} + (\text{undefined} + c_2)\, x^{3i}$$

Summary

The solution(s) found are the following

$$y = \frac{(ic_1 + \text{undefined})\, x^{-3i}}{6} + (\text{undefined} + c_2)\, x^{3i} \tag{1}$$

Verification of solutions

$$y = \frac{(ic_1 + \text{undefined})\, x^{-3i}}{6} + (\text{undefined} + c_2)\, x^{3i}$$

Verified OK.

10.3.5 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' + xy' + 9y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= 9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-37}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -37 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{37}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 302: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{37}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{37}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + 3i \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - 3i
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{37}{4x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{37}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + 3i \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - 3i
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{37}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}+3i$	$\frac{1}{2}-3i$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}+3i$	$\frac{1}{2}-3i$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2} - 3i$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= \frac{1}{2} - 3i - \left(\frac{1}{2} - 3i\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{\frac{1}{2} - 3i}{x} + (-)(0) \\ &= \frac{\frac{1}{2} - 3i}{x} \\ &= \frac{\frac{1}{2} - 3i}{x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{\frac{1}{2} - 3i}{x}\right)(0) + \left(\left(\frac{-\frac{1}{2} + 3i}{x^2}\right) + \left(\frac{\frac{1}{2} - 3i}{x}\right)^2 - \left(-\frac{37}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \frac{\frac{1}{2} - 3i}{x} dx} \\
&= x^{\frac{1}{2} - 3i}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x}{x^2} \, dx} \\
&= z_1 e^{-\frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x^{-3i}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2} \, dx \\
&= y_1 \left(-\frac{i x^{6i}}{6}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^{-3i}\right) + c_2 \left(x^{-3i} \left(-\frac{i x^{6i}}{6}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
x^2 y'' + xy' + 9y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = x^{-3i} c_1 - \frac{i c_2 x^{3i}}{6}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= x^{-3i} \\
y_2 &= -\frac{i x^{3i}}{6}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^{-3i} & -\frac{i x^{3i}}{6} \\ \frac{d}{dx}\left(x^{-3i}\right) & \frac{d}{dx}\left(-\frac{i x^{3i}}{6}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^{-3i} & -\frac{i x^{3i}}{6} \\ -\frac{3i x^{-3i}}{x} & \frac{x^{3i}}{2x} \end{vmatrix}$$

Therefore

$$W = \left(x^{-3i}\right)\left(\frac{x^{3i}}{2x}\right) - \left(-\frac{i x^{3i}}{6}\right)\left(-\frac{3i x^{-3i}}{x}\right)$$

Which simplifies to

$$W = \frac{x^{3i} x^{-3i}}{x}$$

Which simplifies to

$$W = \frac{1}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-\frac{3i x^{3i} \ln(x)}{2}}{x}\, dx$$

Which simplifies to

$$u_1 = -\int -\frac{3i x^{-1+3i} \ln(x)}{2} dx$$

Hence

$$u_1 = \mathit{undefined}$$

And Eq. (3) becomes

$$u_2 = \int \frac{9x^{-3i} \ln(x)}{x}\, dx$$

Which simplifies to

$$u_2 = \int 9x^{-1-3i} \ln(x) \, dx$$

Hence

$$u_2 = undefined$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \text{undefined } x^{-3i} - i \, \text{undefined } x^{3i}$$

Which simplifies to

$$y_p(x) = \left(i x^{3i} + x^{-3i}\right) \text{undefined}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(x^{-3i} c_1 - \frac{i c_2 x^{3i}}{6}\right) + \left(\left(i x^{3i} + x^{-3i}\right) \text{undefined}\right) \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = x^{-3i} c_1 - \frac{i c_2 x^{3i}}{6} + \left(i x^{3i} + x^{-3i}\right) \text{undefined} \tag{1}$$

<u>Verification of solutions</u>

$$y = x^{-3i} c_1 - \frac{i c_2 x^{3i}}{6} + \left(i x^{3i} + x^{-3i}\right) \text{undefined}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   <- LODE of Euler type successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+9*y(x)=9*ln(x),y(x), singsol=all)
```

$$y(x) = \sin(3\ln(x))\, c_2 + \cos(3\ln(x))\, c_1 + \ln(x)$$

✓ Solution by Mathematica

Time used: 0.155 (sec). Leaf size: 24

```
DSolve[x^2*y''[x]+x*y'[x]+9*y[x]==9*Log[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \log(x) + c_1 \cos(3\log(x)) + c_2 \sin(3\log(x))$$

10.4 problem Problem 17

Internal problem ID [2805]
Internal file name [`OUTPUT/2297_Sunday_June_05_2022_02_58_15_AM_15173724/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.8, A Differential Equation with Nonconstant Coefficients. page 567
Problem number: Problem 17.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{x^2y'' - xy' + 5y = 8x\ln(x)^2}$$

10.4.1 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -x, C = 5, f(x) = 8x\ln(x)^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$.

Solving for y_h from

$$x^2y'' - xy' + 5y = 0$$

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} - xrx^{r-1} + 5x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r - r\,x^r + 5x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) - r + 5 = 0$$

Or

$$r^2 - 2r + 5 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= 1 - 2i \\ r_2 &= 1 + 2i \end{aligned}$$

The roots are complex conjugate of each others. Let the roots be

$$\begin{aligned} r_1 &= \alpha + i\beta \\ r_2 &= \alpha - i\beta \end{aligned}$$

Where in this case $\alpha = 1$ and $\beta = -2$. Hence the solution becomes

$$\begin{aligned} y &= c_1x^{r_1} + c_2x^{r_2} \\ &= c_1x^{\alpha+i\beta} + c_2x^{\alpha-i\beta} \\ &= x^\alpha\left(c_1x^{i\beta} + c_2x^{-i\beta}\right) \\ &= x^\alpha\left(c_1e^{\ln\left(x^{i\beta}\right)} + c_2e^{\ln\left(x^{-i\beta}\right)}\right) \\ &= x^\alpha\left(c_1e^{i(\beta\ln x)} + c_2e^{-i(\beta\ln x)}\right) \end{aligned}$$

Using the values for $\alpha = 1, \beta = -2$, the above becomes

$$y = x^1\left(c_1e^{-2i\ln(x)} + c_2e^{2i\ln(x)}\right)$$

Using Euler relation, the expression $c_1 e^{iA} + c_2 e^{-iA}$ is transformed to $c_1 \cos A + c_1 \sin A$ where the constants are free to change. Applying this to the above result gives

$$y = x(c_1 \cos(2\ln(x)) + c_2 \sin(2\ln(x)))$$

Next, we find the particular solution to the ODE

$$x^2 y'' - xy' + 5y = 8x \ln(x)^2$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= x \cos(2\ln(x)) \\ y_2 &= -x \sin(2\ln(x)) \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x \cos(2\ln(x)) & -x \sin(2\ln(x)) \\ \frac{d}{dx}(x \cos(2\ln(x))) & \frac{d}{dx}(-x \sin(2\ln(x))) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x \cos(2\ln(x)) & -x \sin(2\ln(x)) \\ \cos(2\ln(x)) - 2\sin(2\ln(x)) & -\sin(2\ln(x)) - 2\cos(2\ln(x)) \end{vmatrix}$$

Therefore

$$\begin{aligned}W = &(x \cos(2\ln(x))) (-\sin(2\ln(x)) - 2\cos(2\ln(x))) \\ &- (-x\sin(2\ln(x))) (\cos(2\ln(x)) - 2\sin(2\ln(x)))\end{aligned}$$

Which simplifies to

$$W = -2x\cos(2\ln(x))^2 - 2x\sin(2\ln(x))^2$$

Which simplifies to

$$W = -2x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-8x^2\sin(2\ln(x))\ln(x)^2}{-2x^3}dx$$

Which simplifies to

$$u_1 = -\int \frac{4\sin(2\ln(x))\ln(x)^2}{x}dx$$

Hence

$$u_1 = 2\ln(x)^2\cos(2\ln(x)) - \cos(2\ln(x)) - 2\sin(2\ln(x))\ln(x)$$

And Eq. (3) becomes

$$u_2 = \int \frac{8x^2\cos(2\ln(x))\ln(x)^2}{-2x^3}dx$$

Which simplifies to

$$u_2 = \int -\frac{4\cos(2\ln(x))\ln(x)^2}{x}dx$$

Hence

$$u_2 = -2\sin(2\ln(x))\ln(x)^2 + \sin(2\ln(x)) - 2\ln(x)\cos(2\ln(x))$$

Therefore the particular solution, from equation (1) is

$$\begin{aligned}y_p(x) = &\left(2\ln(x)^2\cos(2\ln(x)) - \cos(2\ln(x)) - 2\sin(2\ln(x))\ln(x)\right)x\cos(2\ln(x)) \\ &- \left(-2\sin(2\ln(x))\ln(x)^2 + \sin(2\ln(x)) - 2\ln(x)\cos(2\ln(x))\right)x\sin(2\ln(x))\end{aligned}$$

Which simplifies to

$$y_p(x) = 2x\ln(x)^2 - x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= x(-1 + 2\ln(x)^2 + c_1\cos(2\ln(x)) + c_2\sin(2\ln(x))) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = x(-1 + 2\ln(x)^2 + c_1\cos(2\ln(x)) + c_2\sin(2\ln(x))) \tag{1}$$

Verification of solutions

$$y = x(-1 + 2\ln(x)^2 + c_1\cos(2\ln(x)) + c_2\sin(2\ln(x)))$$

Verified OK.

10.4.2 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' - xy' + 5y = 0$$

In normal form the ode

$$x^2y'' - xy' + 5y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= \frac{5}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\ &= \int \mathrm{e}^{-\left(\int -\frac{1}{x}dx\right)}dx \\ &= \int e^{\ln(x)}\,dx \\ &= \int x dx \\ &= \frac{x^2}{2}\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\ &= \frac{\frac{5}{x^2}}{x^2} \\ &= \frac{5}{x^4}\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}\frac{d^2}{d\tau^2}y(\tau) + q_1y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + \frac{5y(\tau)}{x^4} &= 0\end{aligned}$$

But in terms of τ

$$\frac{5}{x^4} = \frac{5}{4\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{5y(\tau)}{4\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$4\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + 5y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$4\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 5\tau^r = 0$$

Simplifying gives

$$4r(r-1)\,\tau^r + 0\,\tau^r + 5\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$4r(r-1) + 0 + 5 = 0$$

Or

$$4r^2 - 4r + 5 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{2} - i$$
$$r_2 = \frac{1}{2} + i$$

The roots are complex conjugate of each others. Let the roots be

$$r_1 = \alpha + i\beta$$
$$r_2 = \alpha - i\beta$$

Where in this case $\alpha = \frac{1}{2}$ and $\beta = -1$. Hence the solution becomes

$$\begin{aligned}
y(\tau) &= c_1\tau^{r_1} + c_2\tau^{r_2}\\
&= c_1\tau^{\alpha+i\beta} + c_2\tau^{\alpha-i\beta}\\
&= \tau^{\alpha}\left(c_1\tau^{i\beta} + c_2\tau^{-i\beta}\right)\\
&= \tau^{\alpha}\left(c_1e^{\ln(\tau^{i\beta})} + c_2e^{\ln(\tau^{-i\beta})}\right)\\
&= \tau^{\alpha}\left(c_1e^{i(\beta\ln\tau)} + c_2e^{-i(\beta\ln\tau)}\right)
\end{aligned}$$

Using the values for $\alpha = \frac{1}{2}, \beta = -1$, the above becomes

$$y(\tau) = \tau^{\frac{1}{2}}\left(c_1e^{-i\ln(\tau)} + c_2e^{i\ln(\tau)}\right)$$

Using Euler relation, the expression $c_1e^{iA} + c_2e^{-iA}$ is transformed to $c_1\cos A + c_1\sin A$ where the constants are free to change. Applying this to the above result gives

$$y(\tau) = \sqrt{\tau}(c_1\cos(\ln(\tau)) + c_2\sin(\ln(\tau)))$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{\sqrt{2}\,x(c_1\cos(-\ln(2) + 2\ln(x)) + c_2\sin(-\ln(2) + 2\ln(x)))}{2}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{\sqrt{2}\,x(c_1\cos(-\ln(2) + 2\ln(x)) + c_2\sin(-\ln(2) + 2\ln(x)))}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}
y_1 &= \sqrt{2}\,x\cos(\ln(2))\cos(\ln(x))^2 - \frac{\sqrt{2}\,x\cos(\ln(2))}{2}\\
&\quad + \sqrt{2}\,x\sin(\ln(2))\sin(\ln(x))\cos(\ln(x))
\end{aligned}$$

$$\begin{aligned}
y_2 &= -\sqrt{2}\,x\sin(\ln(2))\cos(\ln(x))^2 + \frac{\sqrt{2}\,x\sin(\ln(2))}{2}\\
&\quad + \sqrt{2}\,x\cos(\ln(2))\sin(\ln(x))\cos(\ln(x))
\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \sqrt{2}\, x \cos(\ln(2)) \cos(\ln(x))^2 - \frac{\sqrt{2}\, x \cos(\ln(2))}{2} + \sqrt{2}\, x \sin(\ln(2)) \sin(\ln(x)) \cos(\ln(x)) & -\sqrt{2} \\ \frac{d}{dx}\left(\sqrt{2}\, x \cos(\ln(2)) \cos(\ln(x))^2 - \frac{\sqrt{2}\, x \cos(\ln(2))}{2} + \sqrt{2}\, x \sin(\ln(2)) \sin(\ln(x)) \cos(\ln(x))\right) & \frac{d}{dx}\left(-\sqrt{2}\right. \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \sqrt{2}\, x \cos(\ln(2)) \cos(\ln(x))^2 - \frac{\sqrt{2}\, x \cos(\ln(2))}{2} + \sqrt{2}\, x \\ \sqrt{2} \cos(\ln(2)) \cos(\ln(x))^2 - 2\sqrt{2} \cos(\ln(2)) \cos(\ln(x)) \sin(\ln(x)) - \frac{\sqrt{2} \cos(\ln(2))}{2} + \sqrt{2} \sin(\ln(2)) \end{vmatrix}$$

Therefore

$$
\begin{aligned}
W = &\left(\sqrt{2}\, x \cos(\ln(2)) \cos(\ln(x))^2 - \frac{\sqrt{2}\, x \cos(\ln(2))}{2} \right. \\
&\left. + \sqrt{2}\, x \sin(\ln(2)) \sin(\ln(x)) \cos(\ln(x))\right) \left(-\sqrt{2}\, \sin(\ln(2)) \cos(\ln(x))^2 \right. \\
&+ 2\sqrt{2}\, \sin(\ln(2)) \sin(\ln(x)) \cos(\ln(x)) + \frac{\sqrt{2}\, \sin(\ln(2))}{2} \\
&+ \sqrt{2}\, \cos(\ln(2)) \cos(\ln(x)) \sin(\ln(x)) + \sqrt{2}\, \cos(\ln(2)) \cos(\ln(x))^2 \\
&\left. - \sqrt{2}\, \cos(\ln(2)) \sin(\ln(x))^2 \right) \\
&- \left(-\sqrt{2}\, x \sin(\ln(2)) \cos(\ln(x))^2 + \frac{\sqrt{2}\, x \sin(\ln(2))}{2} \right. \\
&\left. + \sqrt{2}\, x \cos(\ln(2)) \sin(\ln(x)) \cos(\ln(x))\right) \left(\sqrt{2}\, \cos(\ln(2)) \cos(\ln(x))^2 \right. \\
&- 2\sqrt{2}\, \cos(\ln(2)) \cos(\ln(x)) \sin(\ln(x)) - \frac{\sqrt{2}\, \cos(\ln(2))}{2} \\
&+ \sqrt{2}\, \sin(\ln(2)) \sin(\ln(x)) \cos(\ln(x)) + \sqrt{2}\, \sin(\ln(2)) \cos(\ln(x))^2 \\
&\left. - \sqrt{2}\, \sin(\ln(2)) \sin(\ln(x))^2 \right)
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
W = &\, 2x \cos(\ln(2))^2 \cos(\ln(x))^4 - x \cos(\ln(2))^2 \cos(\ln(x))^2 \\
&+ x \cos(\ln(2))^2 \sin(\ln(x))^2 + 2x \sin(\ln(2))^2 \cos(\ln(x))^4 \\
&- x \sin(\ln(2))^2 \cos(\ln(x))^2 + x \sin(\ln(2))^2 \sin(\ln(x))^2 \\
&+ 2x \cos(\ln(2))^2 \cos(\ln(x))^2 \sin(\ln(x))^2 + 2x \sin(\ln(2))^2 \sin(\ln(x))^2 \cos(\ln(x))^2
\end{aligned}
$$

Which simplifies to

$$W = x$$

Therefore Eq. (2) becomes

$$
u_1 = - \int \frac{8\left(-\sqrt{2}\, x \sin(\ln(2)) \cos(\ln(x))^2 + \frac{\sqrt{2}\, x \sin(\ln(2))}{2} + \sqrt{2}\, x \cos(\ln(2)) \sin(\ln(x)) \cos(\ln(x))\right) x \ln(x)}{x^3}
$$

Which simplifies to

$$u_1 = -\int -\frac{4(\sin(\ln(2))\cos(2\ln(x)) - \cos(\ln(2))\sin(2\ln(x)))\sqrt{2}\,\ln(x)^2}{x}dx$$

Hence

$$u_1 = 4\sqrt{2}\left(\frac{\sin(\ln(2))\left(4\sin(2\ln(x))\ln(x)^2 - 2\sin(2\ln(x)) + 4\ln(x)\cos(2\ln(x))\right)}{8} - \frac{\cos(\ln(2))\left(-4\ln(x)^2\cos(2\ln(x)) + 2\cos(2\ln(x)) + 4\sin(2\ln(x))\ln(x)\right)}{8}\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{8\left(\sqrt{2}\,x\cos(\ln(2))\cos(\ln(x))^2 - \frac{\sqrt{2}\,x\cos(\ln(2))}{2} + \sqrt{2}\,x\sin(\ln(2))\sin(\ln(x))\cos(\ln(x))\right)x\ln(x)^2}{x^3}\,dx$$

Which simplifies to

$$u_2 = \int \frac{4(\cos(\ln(2))\cos(2\ln(x)) + \sin(\ln(2))\sin(2\ln(x)))\sqrt{2}\,\ln(x)^2}{x}dx$$

Hence

$$u_2 = 4\sqrt{2}\left(\frac{\cos(\ln(2))\left(4\sin(2\ln(x))\ln(x)^2 - 2\sin(2\ln(x)) + 4\ln(x)\cos(2\ln(x))\right)}{8} + \frac{\sin(\ln(2))\left(-4\ln(x)^2\cos(2\ln(x)) + 2\cos(2\ln(x)) + 4\sin(2\ln(x))\ln(x)\right)}{8}\right)$$

Which simplifies to

$$u_1 = 2\sqrt{2}\left(\left(\cos(\ln(2))\ln(x)^2 + \ln(x)\sin(\ln(2)) - \frac{\cos(\ln(2))}{2}\right)\cos(2\ln(x)) + \sin(2\ln(x))\left(\ln(x)^2\sin(\ln(2)) - \ln(x)\cos(\ln(2)) - \frac{\sin(\ln(2))}{2}\right)\right)$$

$$u_2 = 2\left(\left(-\ln(x)^2\sin(\ln(2)) + \ln(x)\cos(\ln(2)) + \frac{\sin(\ln(2))}{2}\right)\cos(2\ln(x)) + \sin(2\ln(x))\left(\cos(\ln(2))\ln(x)^2 + \ln(x)\sin(\ln(2)) - \frac{\cos(\ln(2))}{2}\right)\right)\sqrt{2}$$

Therefore the particular solution, from equation (1) is

$$
\begin{aligned}
y_p(x) = 2\sqrt{2}\Bigg(\Bigg(&\cos(\ln(2))\ln(x)^2 + \ln(x)\sin(\ln(2)) - \frac{\cos(\ln(2))}{2}\Bigg)\cos(2\ln(x)) \\
&+ \sin(2\ln(x))\Bigg(\ln(x)^2\sin(\ln(2)) - \ln(x)\cos(\ln(2)) \\
&- \frac{\sin(\ln(2))}{2}\Bigg)\Bigg)\Bigg(\sqrt{2}\,x\cos(\ln(2))\cos(\ln(x))^2 - \frac{\sqrt{2}\,x\cos(\ln(2))}{2} \\
&+ \sqrt{2}\,x\sin(\ln(2))\sin(\ln(x))\cos(\ln(x))\Bigg) \\
&+ 2\Bigg(\Bigg(-\ln(x)^2\sin(\ln(2)) + \ln(x)\cos(\ln(2)) + \frac{\sin(\ln(2))}{2}\Bigg)\cos(2\ln(x)) \\
&+ \sin(2\ln(x))\Bigg(\cos(\ln(2))\ln(x)^2 + \ln(x)\sin(\ln(2)) \\
&- \frac{\cos(\ln(2))}{2}\Bigg)\Bigg)\sqrt{2}\Bigg(-\sqrt{2}\,x\sin(\ln(2))\cos(\ln(x))^2 + \frac{\sqrt{2}\,x\sin(\ln(2))}{2} \\
&+ \sqrt{2}\,x\cos(\ln(2))\sin(\ln(x))\cos(\ln(x))\Bigg)
\end{aligned}
$$

Which simplifies to

$$
y_p(x) = 2x\ln(x)^2 - x
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\frac{\sqrt{2}\,x(c_1\cos(-\ln(2)+2\ln(x)) + c_2\sin(-\ln(2)+2\ln(x)))}{2}\right) + (2x\ln(x)^2 - x)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{\sqrt{2}\,x(c_1\cos(-\ln(2)+2\ln(x)) + c_2\sin(-\ln(2)+2\ln(x)))}{2} + 2x\ln(x)^2 - x \quad (1)
$$

Verification of solutions

$$
y = \frac{\sqrt{2}\,x(c_1\cos(-\ln(2)+2\ln(x)) + c_2\sin(-\ln(2)+2\ln(x)))}{2} + 2x\ln(x)^2 - x
$$

Verified OK.

10.4.3 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -x, C = 5, f(x) = 8x\ln(x)^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - xy' + 5y = 0$$

In normal form the ode

$$x^2y'' - xy' + 5y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= \frac{5}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{5}\,\sqrt{\frac{1}{x^2}}}{c} \\ \tau'' &= -\frac{\sqrt{5}}{c\sqrt{\frac{1}{x^2}}\,x^3} \end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$
\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'\left(x\right)^2} \\
&= \frac{-\frac{\sqrt{5}}{c\sqrt{\frac{1}{x^2}}\,x^3} - \frac{1}{x}\frac{\sqrt{5}\,\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{5}\,\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\
&= -\frac{2c\sqrt{5}}{5}
\end{aligned}
$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) - \frac{2c\sqrt{5}\left(\frac{d}{d\tau}y(\tau)\right)}{5} + c^2 y(\tau) &= 0
\end{aligned} \tag{7}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
y(\tau) = \mathrm{e}^{\frac{\sqrt{5}\,c\tau}{5}}\left(c_1\cos\left(\frac{2\sqrt{5}\,c\tau}{5}\right) + c_2\sin\left(\frac{2\sqrt{5}\,c\tau}{5}\right)\right)
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int \sqrt{5}\,\sqrt{\frac{1}{x^2}}dx}{c} \\
&= \frac{\sqrt{5}\,\sqrt{\frac{1}{x^2}}\,x\ln\left(x\right)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = x(c_1\cos\left(2\ln\left(x\right)\right) + c_2\sin\left(2\ln\left(x\right)\right))
$$

Now the particular solution to this ODE is found

$$
x^2y'' - xy' + 5y = 8x\ln\left(x\right)^2
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= \sqrt{2}\, x \cos(\ln(2)) \cos(\ln(x))^2 - \frac{\sqrt{2}\, x \cos(\ln(2))}{2} \\ &\quad + \sqrt{2}\, x \sin(\ln(2)) \sin(\ln(x)) \cos(\ln(x)) \\ y_2 &= -\sqrt{2}\, x \sin(\ln(2)) \cos(\ln(x))^2 + \frac{\sqrt{2}\, x \sin(\ln(2))}{2} \\ &\quad + \sqrt{2}\, x \cos(\ln(2)) \sin(\ln(x)) \cos(\ln(x)) \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \sqrt{2}\, x \cos(\ln(2)) \cos(\ln(x))^2 - \frac{\sqrt{2}\, x \cos(\ln(2))}{2} + \sqrt{2}\, x \sin(\ln(2)) \sin(\ln(x)) \cos(\ln(x)) & -\sqrt{2} \\ \frac{d}{dx}\left(\sqrt{2}\, x \cos(\ln(2)) \cos(\ln(x))^2 - \frac{\sqrt{2}\, x \cos(\ln(2))}{2} + \sqrt{2}\, x \sin(\ln(2)) \sin(\ln(x)) \cos(\ln(x))\right) & \frac{d}{dx}\left(-\sqrt{2}\right. \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \sqrt{2}\, x \cos(\ln(2)) \cos(\ln(x))^2 - \frac{\sqrt{2}\, x \cos(\ln(2))}{2} + \sqrt{2}\, x \\ \sqrt{2}\, \cos(\ln(2)) \cos(\ln(x))^2 - 2\sqrt{2}\, \cos(\ln(2)) \cos(\ln(x)) \sin(\ln(x)) - \frac{\sqrt{2}\, \cos(\ln(2))}{2} + \sqrt{2}\, \sin(\ln(2)) \end{vmatrix}$$

Therefore

$$
\begin{aligned}
W = \Bigg(&\sqrt{2}\, x \cos(\ln(2)) \cos(\ln(x))^{2} - \frac{\sqrt{2}\, x \cos(\ln(2))}{2} \\
&+ \sqrt{2}\, x \sin(\ln(2)) \sin(\ln(x)) \cos(\ln(x))\Bigg) \Bigg(-\sqrt{2}\, \sin(\ln(2)) \cos(\ln(x))^{2} \\
&+ 2\sqrt{2}\, \sin(\ln(2)) \sin(\ln(x)) \cos(\ln(x)) + \frac{\sqrt{2}\, \sin(\ln(2))}{2} \\
&+ \sqrt{2}\, \cos(\ln(2)) \cos(\ln(x)) \sin(\ln(x)) + \sqrt{2}\, \cos(\ln(2)) \cos(\ln(x))^{2} \\
&- \sqrt{2}\, \cos(\ln(2)) \sin(\ln(x))^{2}\Bigg) \\
- \Bigg(&-\sqrt{2}\, x \sin(\ln(2)) \cos(\ln(x))^{2} + \frac{\sqrt{2}\, x \sin(\ln(2))}{2} \\
&+ \sqrt{2}\, x \cos(\ln(2)) \sin(\ln(x)) \cos(\ln(x))\Bigg) \Bigg(\sqrt{2}\, \cos(\ln(2)) \cos(\ln(x))^{2} \\
&- 2\sqrt{2}\, \cos(\ln(2)) \cos(\ln(x)) \sin(\ln(x)) - \frac{\sqrt{2}\, \cos(\ln(2))}{2} \\
&+ \sqrt{2}\, \sin(\ln(2)) \sin(\ln(x)) \cos(\ln(x)) + \sqrt{2}\, \sin(\ln(2)) \cos(\ln(x))^{2} \\
&- \sqrt{2}\, \sin(\ln(2)) \sin(\ln(x))^{2}\Bigg)
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
W = {} & 2x \cos(\ln(2))^{2} \cos(\ln(x))^{4} - x \cos(\ln(2))^{2} \cos(\ln(x))^{2} \\
&+ x \cos(\ln(2))^{2} \sin(\ln(x))^{2} + 2x \sin(\ln(2))^{2} \cos(\ln(x))^{4} \\
&- x \sin(\ln(2))^{2} \cos(\ln(x))^{2} + x \sin(\ln(2))^{2} \sin(\ln(x))^{2} \\
&+ 2x \cos(\ln(2))^{2} \cos(\ln(x))^{2} \sin(\ln(x))^{2} + 2x \sin(\ln(2))^{2} \sin(\ln(x))^{2} \cos(\ln(x))^{2}
\end{aligned}
$$

Which simplifies to

$$
W = x
$$

Therefore Eq. (2) becomes

$$
u_1 = - \int \frac{8\left(-\sqrt{2}\, x \sin(\ln(2)) \cos(\ln(x))^{2} + \frac{\sqrt{2}\, x \sin(\ln(2))}{2} + \sqrt{2}\, x \cos(\ln(2)) \sin(\ln(x)) \cos(\ln(x))\right) x \ln(x)^{2}}{x^{3}}
$$

Which simplifies to

$$u_1 = -\int -\frac{4(\sin(\ln(2))\cos(2\ln(x)) - \cos(\ln(2))\sin(2\ln(x)))\sqrt{2}\,\ln(x)^2}{x}dx$$

Hence

$$u_1 = 4\sqrt{2}\left(\frac{\sin(\ln(2))\left(4\sin(2\ln(x))\ln(x)^2 - 2\sin(2\ln(x)) + 4\ln(x)\cos(2\ln(x))\right)}{8} - \frac{\cos(\ln(2))\left(-4\ln(x)^2\cos(2\ln(x)) + 2\cos(2\ln(x)) + 4\sin(2\ln(x))\ln(x)\right)}{8}\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{8\left(\sqrt{2}\,x\cos(\ln(2))\cos(\ln(x))^2 - \frac{\sqrt{2}\,x\cos(\ln(2))}{2} + \sqrt{2}\,x\sin(\ln(2))\sin(\ln(x))\cos(\ln(x))\right)x\ln(x)^2}{x^3}dx$$

Which simplifies to

$$u_2 = \int \frac{4(\cos(\ln(2))\cos(2\ln(x)) + \sin(\ln(2))\sin(2\ln(x)))\sqrt{2}\,\ln(x)^2}{x}dx$$

Hence

$$u_2 = 4\sqrt{2}\left(\frac{\cos(\ln(2))\left(4\sin(2\ln(x))\ln(x)^2 - 2\sin(2\ln(x)) + 4\ln(x)\cos(2\ln(x))\right)}{8} + \frac{\sin(\ln(2))\left(-4\ln(x)^2\cos(2\ln(x)) + 2\cos(2\ln(x)) + 4\sin(2\ln(x))\ln(x)\right)}{8}\right)$$

Which simplifies to

$$u_1 = 2\sqrt{2}\left(\left(\cos(\ln(2))\ln(x)^2 + \ln(x)\sin(\ln(2)) - \frac{\cos(\ln(2))}{2}\right)\cos(2\ln(x)) + \sin(2\ln(x))\left(\ln(x)^2\sin(\ln(2)) - \ln(x)\cos(\ln(2)) - \frac{\sin(\ln(2))}{2}\right)\right)$$

$$u_2 = 2\left(\left(-\ln(x)^2\sin(\ln(2)) + \ln(x)\cos(\ln(2)) + \frac{\sin(\ln(2))}{2}\right)\cos(2\ln(x)) + \sin(2\ln(x))\left(\cos(\ln(2))\ln(x)^2 + \ln(x)\sin(\ln(2)) - \frac{\cos(\ln(2))}{2}\right)\right)\sqrt{2}$$

Therefore the particular solution, from equation (1) is

$$
\begin{aligned}
y_p(x) = 2\sqrt{2}\Bigg(\Bigg(\cos(\ln(2))\ln(x)^2 + \ln(x)\sin(\ln(2)) - \frac{\cos(\ln(2))}{2}\Bigg)\cos(2\ln(x)) \\
+ \sin(2\ln(x))\Bigg(\ln(x)^2\sin(\ln(2)) - \ln(x)\cos(\ln(2)) \\
- \frac{\sin(\ln(2))}{2}\Bigg)\Bigg)\Bigg(\sqrt{2}\,x\cos(\ln(2))\cos(\ln(x))^2 - \frac{\sqrt{2}\,x\cos(\ln(2))}{2} \\
+ \sqrt{2}\,x\sin(\ln(2))\sin(\ln(x))\cos(\ln(x))\Bigg) \\
+ 2\Bigg(\Bigg(-\ln(x)^2\sin(\ln(2)) + \ln(x)\cos(\ln(2)) + \frac{\sin(\ln(2))}{2}\Bigg)\cos(2\ln(x)) \\
+ \sin(2\ln(x))\Bigg(\cos(\ln(2))\ln(x)^2 + \ln(x)\sin(\ln(2)) \\
- \frac{\cos(\ln(2))}{2}\Bigg)\Bigg)\sqrt{2}\Bigg(-\sqrt{2}\,x\sin(\ln(2))\cos(\ln(x))^2 + \frac{\sqrt{2}\,x\sin(\ln(2))}{2} \\
+ \sqrt{2}\,x\cos(\ln(2))\sin(\ln(x))\cos(\ln(x))\Bigg)
\end{aligned}
$$

Which simplifies to

$$
y_p(x) = 2x\ln(x)^2 - x
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (x(c_1\cos(2\ln(x)) + c_2\sin(2\ln(x)))) + \left(2x\ln(x)^2 - x\right) \\
&= 2x\ln(x)^2 - x + x(c_1\cos(2\ln(x)) + c_2\sin(2\ln(x)))
\end{aligned}
$$

Which simplifies to

$$
y = x\left(-1 + 2\ln(x)^2 + c_1\cos(2\ln(x)) + c_2\sin(2\ln(x))\right)
$$

Summary

The solution(s) found are the following

$$
y = x\left(-1 + 2\ln(x)^2 + c_1\cos(2\ln(x)) + c_2\sin(2\ln(x))\right) \tag{1}
$$

Verification of solutions

$$y = x\left(-1 + 2\ln(x)^2 + c_1\cos(2\ln(x)) + c_2\sin(2\ln(x))\right)$$

Verified OK.

10.4.4 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -x, C = 5, f(x) = 8x\ln(x)^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - xy' + 5y = 0$$

In normal form the ode

$$x^2y'' - xy' + 5y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned}p(x) &= -\frac{1}{x}\\ q(x) &= \frac{5}{x^2}\end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right)v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right)v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{n}{x^2} + \frac{5}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 1 + 2i \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \left(\frac{2+4i}{x} - \frac{1}{x}\right) v'(x) &= 0 \\ v''(x) + \frac{(1+4i)\, v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{(1+4i)\, u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= \frac{(-1-4i)\, u}{x} \end{aligned}$$

Where $f(x) = \frac{-1-4i}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= \frac{-1-4i}{x}\, dx \\ \int \frac{1}{u}\, du &= \int \frac{-1-4i}{x}\, dx \\ \ln(u) &= (-1-4i)\ln(x) + c_1 \\ u &= e^{(-1-4i)\ln(x) + c_1} \\ &= c_1 e^{(-1-4i)\ln(x)} \end{aligned}$$

Which simplifies to

$$u(x) = \frac{c_1 x^{-4i}}{x}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\, dx + c_2 \\ &= \frac{i c_1 x^{-4i}}{4} + c_2 \end{aligned}$$

Hence

$$\begin{aligned} y &= v(x)\, x^n \\ &= \left(\frac{i c_1 x^{-4i}}{4} + c_2\right) x^{1+2i} \\ &= x^{1+2i} c_2 + \frac{i x^{1-2i} c_1}{4} \end{aligned}$$

Now the particular solution to this ODE is found

$$x^2 y'' - x y' + 5y = 8x \ln(x)^2$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= x^{2i} x \\ y_2 &= x^{2i} x\, x^{-4i} \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^{2i}x & x^{2i}x\,x^{-4i} \\ \frac{d}{dx}(x^{2i}x) & \frac{d}{dx}(x^{2i}x\,x^{-4i}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^{2i}x & x^{2i}x\,x^{-4i} \\ 2ix^{2i} + x^{2i} & -2ix^{2i}x^{-4i} + x^{2i}x^{-4i} \end{vmatrix}$$

Therefore

$$W = (x^{2i}x)\left(-2ix^{2i}x^{-4i} + x^{2i}x^{-4i}\right) - \left(x^{2i}x\,x^{-4i}\right)\left(2ix^{2i} + x^{2i}\right)$$

Which simplifies to

$$W = -4ix^{4i}x\,x^{-4i}$$

Which simplifies to

$$W = -4ix$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{8x^{2i}x^2x^{-4i}\ln(x)^2}{-4ix^3}\,dx$$

Which simplifies to

$$u_1 = -\int 2ix^{-1-2i}\ln(x)^2\,dx$$

Hence

$$u_1 = undefined$$

And Eq. (3) becomes

$$u_2 = \int \frac{8x^{2i}x^2 \ln(x)^2}{-4ix^3}\, dx$$

Which simplifies to

$$u_2 = \int 2ix^{-1+2i} \ln(x)^2\, dx$$

Hence

$$u_2 = undefined$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \text{undefined } x^{2i}x + \text{undefined } x^{2i}x\, x^{-4i}$$

Which simplifies to

$$y_p(x) = \text{undefined}\left(x^{1+2i} + x^{1-2i}\right)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\left(\frac{ic_1 x^{-4i}}{4} + c_2\right)x^{1+2i}\right) + \left(\text{undefined}\left(x^{1+2i} + x^{1-2i}\right)\right) \\ &= \text{undefined}\left(x^{1+2i} + x^{1-2i}\right) + \left(\frac{ic_1 x^{-4i}}{4} + c_2\right)x^{1+2i} \end{aligned}$$

Which simplifies to

$$y = \frac{(ic_1 + \text{undefined})\, x^{1-2i}}{4} + (\text{undefined} + c_2)\, x^{1+2i}$$

Summary

The solution(s) found are the following

$$y = \frac{(ic_1 + \text{undefined})\, x^{1-2i}}{4} + (\text{undefined} + c_2)\, x^{1+2i} \tag{1}$$

Verification of solutions

$$y = \frac{(ic_1 + \text{undefined})\, x^{1-2i}}{4} + (\text{undefined} + c_2)\, x^{1+2i}$$

Verified OK.

10.4.5 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' - xy' + 5y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -x \\ C &= 5 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-17}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -17 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{17}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 303: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{17}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{17}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + 2i \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - 2i
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{17}{4x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{17}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + 2i \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - 2i
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{17}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2} + 2i$	$\frac{1}{2} - 2i$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2} + 2i$	$\frac{1}{2} - 2i$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_{\infty}^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_{\infty}^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_{\infty}^{-} = \frac{1}{2} - 2i$ then

$$\begin{aligned} d &= \alpha_{\infty}^{-} - \left(\alpha_{c_1}^{-}\right) \\ &= \frac{1}{2} - 2i - \left(\frac{1}{2} - 2i\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_{\infty}$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (-)[\sqrt{r}]_{\infty} \\ &= \frac{\frac{1}{2} - 2i}{x} + (-)\,(0) \\ &= \frac{\frac{1}{2} - 2i}{x} \\ &= \frac{\frac{1}{2} - 2i}{x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{\frac{1}{2} - 2i}{x}\right)(0) + \left(\left(\frac{-\frac{1}{2} + 2i}{x^2}\right) + \left(\frac{\frac{1}{2} - 2i}{x}\right)^2 - \left(-\frac{17}{4x^2}\right) \right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \frac{\frac{1}{2}-2i}{x} dx} \\
&= x^{\frac{1}{2}-2i}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x}{x^2} \, dx} \\
&= z_1 e^{\frac{\ln(x)}{2}} \\
&= z_1\left(\sqrt{x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x^{1-2i}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{x^2} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{\ln(x)}}{(y_1)^2} \, dx \\
&= y_1\left(-\frac{i x^{4i}}{4}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^{1-2i}\right) + c_2\left(x^{1-2i}\left(-\frac{i x^{4i}}{4}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' - xy' + 5y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = x^{1-2i}c_1 - \frac{ic_2x^{1+2i}}{4}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x^{1-2i}$$

$$y_2 = -\frac{ix^{1+2i}}{4}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^{1-2i} & -\frac{ix^{1+2i}}{4} \\ \frac{d}{dx}\left(x^{1-2i}\right) & \frac{d}{dx}\left(-\frac{ix^{1+2i}}{4}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^{1-2i} & -\frac{i x^{1+2i}}{4} \\ \frac{(1-2i)x^{1-2i}}{x} & \frac{\left(\frac{1}{2}-\frac{i}{4}\right)x^{1+2i}}{x} \end{vmatrix}$$

Therefore

$$W = \left(x^{1-2i}\right)\left(\frac{\left(\frac{1}{2}-\frac{i}{4}\right)x^{1+2i}}{x}\right) - \left(-\frac{i x^{1+2i}}{4}\right)\left(\frac{(1-2i)\,x^{1-2i}}{x}\right)$$

Which simplifies to

$$W = \frac{x^{1-2i}x^{1+2i}}{x}$$

Which simplifies to

$$W = x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-2i x^{1+2i} x \ln(x)^2}{x^3}\,dx$$

Which simplifies to

$$u_1 = -\int -2i x^{-1+2i} \ln(x)^2\,dx$$

Hence

$$u_1 = \mathit{undefined}$$

And Eq. (3) becomes

$$u_2 = \int \frac{8x^{1-2i} x \ln(x)^2}{x^3}\,dx$$

Which simplifies to

$$u_2 = \int 8x^{-1-2i} \ln(x)^2\,dx$$

Hence

$$u_2 = \mathit{undefined}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \text{undefined } x^{1-2i} - i \,\text{undefined } x^{1+2i}$$

Which simplifies to

$$y_p(x) = \left(i x^{1+2i} + x^{1-2i}\right) \text{undefined}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(x^{1-2i} c_1 - \frac{i c_2 x^{1+2i}}{4}\right) + \left(\left(i x^{1+2i} + x^{1-2i}\right) \text{undefined}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = x^{1-2i} c_1 - \frac{i c_2 x^{1+2i}}{4} + \left(i x^{1+2i} + x^{1-2i}\right) \text{undefined} \tag{1}$$

Verification of solutions

$$y = x^{1-2i} c_1 - \frac{i c_2 x^{1+2i}}{4} + \left(i x^{1+2i} + x^{1-2i}\right) \text{undefined}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   <- LODE of Euler type successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 28

```
dsolve(x^2*diff(y(x),x$2)-x*diff(y(x),x)+5*y(x)=8*x*(ln(x))^2,y(x), singsol=all)
```

$$y(x) = x\left(-1 + \sin\left(2\ln\left(x\right)\right) c_2 + \cos\left(2\ln\left(x\right)\right) c_1 + 2\ln\left(x\right)^2\right)$$

✓ Solution by Mathematica

Time used: 0.154 (sec). Leaf size: 31

```
DSolve[x^2*y''[x]-x*y'[x]+5*y[x]==8*x*(Log[x])^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x\left(2\log^2(x) + c_2\cos(2\log(x)) + c_1\sin(2\log(x)) - 1\right)$$

10.5 problem Problem 18

Internal problem ID [2806]
Internal file name [`OUTPUT/2298_Sunday_June_05_2022_02_58_17_AM_89165808/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.8, A Differential Equation with Nonconstant Coefficients. page 567
Problem number: Problem 18.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_1", "second_order_change_of_variable_on_y_method_2", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{x^2y'' - 4xy' + 6y = x^4\sin(x)}$$

10.5.1 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -4x, C = 6, f(x) = x^4 \sin(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - 4xy' + 6y = 0$$

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} - 4xrx^{r-1} + 6x^r = 0$$

Simplifying gives

$$r(r-1)\, x^r - 4r\, x^r + 6x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) - 4r + 6 = 0$$

Or

$$r^2 - 5r + 6 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = 2$$
$$r_2 = 3$$

Since the roots are real and distinct, then the general solution is

$$y = c_1y_1 + c_2y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = c_2x^3 + c_1x^2$$

Next, we find the particular solution to the ODE

$$x^2y'' - 4xy' + 6y = x^4 \sin(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x^2$$
$$y_2 = x^3$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^2 & x^3 \\ \frac{d}{dx}(x^2) & \frac{d}{dx}(x^3) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix}$$

Therefore

$$W = (x^2)(3x^2) - (x^3)(2x)$$

Which simplifies to

$$W = x^4$$

Which simplifies to

$$W = x^4$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^7 \sin(x)}{x^6} \, dx$$

Which simplifies to

$$u_1 = -\int x \sin(x) \, dx$$

Hence

$$u_1 = -\sin(x) + \cos(x) x$$

And Eq. (3) becomes

$$u_2 = \int \frac{\sin(x) x^6}{x^6} \, dx$$

Which simplifies to

$$u_2 = \int \sin(x) \, dx$$

Hence

$$u_2 = -\cos(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = (-\sin(x) + \cos(x) x) x^2 - \cos(x) x^3$$

Which simplifies to

$$y_p(x) = -x^2 \sin(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= x^2(c_2 x - \sin(x) + c_1) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = x^2(c_2x - \sin(x) + c_1) \tag{1}$$

Verification of solutions

$$y = x^2(c_2x - \sin(x) + c_1)$$

Verified OK.

10.5.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\,y' + \frac{(p(x)^2 + p'(x))\,y}{2} = f(x)$$

Where $p(x) = -\frac{4}{x}$. Therefore, there is an integrating factor given by

$$\begin{aligned}
M(x) &= e^{\frac{1}{2}\int p\,dx} \\
&= e^{\int -\frac{4}{x}\,dx} \\
&= \frac{1}{x^2}
\end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned}
(M(x)y)'' &= \sin(x) \\
\left(\frac{y}{x^2}\right)'' &= \sin(x)
\end{aligned}$$

Integrating once gives

$$\left(\frac{y}{x^2}\right)' = -\cos(x) + c_1$$

Integrating again gives

$$\left(\frac{y}{x^2}\right) = c_1x - \sin(x) + c_2$$

Hence the solution is

$$y = \frac{c_1x - \sin(x) + c_2}{\frac{1}{x^2}}$$

Or

$$y = c_1x^3 + c_2x^2 - x^2\sin(x)$$

Summary
The solution(s) found are the following

$$y = c_1 x^3 + c_2 x^2 - x^2 \sin(x) \tag{1}$$

Verification of solutions

$$y = c_1 x^3 + c_2 x^2 - x^2 \sin(x)$$

Verified OK.

10.5.3 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$x^2 y'' - 4xy' + 6y = 0$$

In normal form the ode

$$x^2 y'' - 4xy' + 6y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{4}{x} \\ q(x) &= \frac{6}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\
&= \int \mathrm{e}^{-\left(\int -\frac{4}{x}dx\right)}dx \\
&= \int e^{4\ln(x)}\,dx \\
&= \int x^4 dx \\
&= \frac{x^5}{5}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{6}{x^2}}{x^8} \\
&= \frac{6}{x^{10}}
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + \frac{6y(\tau)}{x^{10}} &= 0
\end{aligned}$$

But in terms of τ

$$\frac{6}{x^{10}} = \frac{6}{25\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{6y(\tau)}{25\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$25\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + 6y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$25\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 6\tau^r = 0$$

Simplifying gives

$$25r(r-1)\,\tau^r + 0\,\tau^r + 6\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$25r(r-1) + 0 + 6 = 0$$

Or

$$25r^2 - 25r + 6 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= \frac{2}{5} \\ r_2 &= \frac{3}{5} \end{aligned}$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1 y_1 + c_2 y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = c_1\tau^{\frac{2}{5}} + c_2\tau^{\frac{3}{5}}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{c_1 5^{\frac{3}{5}} (x^5)^{\frac{2}{5}}}{5} + \frac{c_2 5^{\frac{2}{5}} (x^5)^{\frac{3}{5}}}{5}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{c_1 5^{\frac{3}{5}} (x^5)^{\frac{2}{5}}}{5} + \frac{c_2 5^{\frac{2}{5}} (x^5)^{\frac{3}{5}}}{5}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \left(x^5\right)^{\frac{2}{5}}$$

$$y_2 = \left(x^5\right)^{\frac{3}{5}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \left(x^5\right)^{\frac{2}{5}} & \left(x^5\right)^{\frac{3}{5}} \\ \frac{d}{dx}\left(\left(x^5\right)^{\frac{2}{5}}\right) & \frac{d}{dx}\left(\left(x^5\right)^{\frac{3}{5}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \left(x^5\right)^{\frac{2}{5}} & \left(x^5\right)^{\frac{3}{5}} \\ \frac{2x^4}{\left(x^5\right)^{\frac{3}{5}}} & \frac{3x^4}{\left(x^5\right)^{\frac{2}{5}}} \end{vmatrix}$$

Therefore

$$W = \left(\left(x^5\right)^{\frac{2}{5}}\right)\left(\frac{3x^4}{\left(x^5\right)^{\frac{2}{5}}}\right) - \left(\left(x^5\right)^{\frac{3}{5}}\right)\left(\frac{2x^4}{\left(x^5\right)^{\frac{3}{5}}}\right)$$

Which simplifies to

$$W = x^4$$

Which simplifies to

$$W = x^4$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\left(x^5\right)^{\frac{3}{5}} x^4 \sin\left(x\right)}{x^6}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{\left(x^5\right)^{\frac{3}{5}} \sin\left(x\right)}{x^2} dx$$

Hence

$$u_1 = \frac{\left(x^5\right)^{\frac{3}{5}} \cos\left(x\right)}{x^2} - \frac{\left(x^5\right)^{\frac{3}{5}} \sin\left(x\right)}{x^3}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\left(x^5\right)^{\frac{2}{5}} x^4 \sin\left(x\right)}{x^6}\, dx$$

Which simplifies to

$$u_2 = \int \frac{\left(x^5\right)^{\frac{2}{5}} \sin\left(x\right)}{x^2} dx$$

Hence

$$u_2 = -\frac{\left(x^5\right)^{\frac{2}{5}} \cos\left(x\right)}{x^2}$$

Which simplifies to

$$u_1 = \frac{\left(-\sin\left(x\right) + \cos\left(x\right) x\right) \left(x^5\right)^{\frac{3}{5}}}{x^3}$$

$$u_2 = -\frac{\left(x^5\right)^{\frac{2}{5}} \cos\left(x\right)}{x^2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-\sin\left(x\right) + \cos\left(x\right) x\right) x^2 - \cos\left(x\right) x^3$$

Which simplifies to

$$y_p(x) = -x^2 \sin\left(x\right)$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\frac{c_1 5^{\frac{3}{5}} (x^5)^{\frac{2}{5}}}{5} + \frac{c_2 5^{\frac{2}{5}} (x^5)^{\frac{3}{5}}}{5}\right) + \left(-x^2 \sin(x)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 5^{\frac{3}{5}} (x^5)^{\frac{2}{5}}}{5} + \frac{c_2 5^{\frac{2}{5}} (x^5)^{\frac{3}{5}}}{5} - x^2 \sin(x) \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 5^{\frac{3}{5}} (x^5)^{\frac{2}{5}}}{5} + \frac{c_2 5^{\frac{2}{5}} (x^5)^{\frac{3}{5}}}{5} - x^2 \sin(x)
$$

Verified OK.

10.5.4 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$
Ay''(x) + By'(x) + Cy(x) = f(x)
$$

Where $A = x^2, B = -4x, C = 6, f(x) = x^4 \sin(x)$. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$
x^2y'' - 4xy' + 6y = 0
$$

In normal form the ode

$$
x^2y'' - 4xy' + 6y = 0 \tag{1}
$$

Becomes

$$
y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}
$$

Where

$$
\begin{aligned}
p(x) &= -\frac{4}{x} \\
q(x) &= \frac{6}{x^2}
\end{aligned}
$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}\tau' &= \frac{1}{c}\sqrt{q}\\ &= \frac{\sqrt{6}\,\sqrt{\frac{1}{x^2}}}{c}\\ \tau'' &= -\frac{\sqrt{6}}{c\sqrt{\frac{1}{x^2}}\,x^3}\end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned}p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2}\\ &= \frac{-\frac{\sqrt{6}}{c\sqrt{\frac{1}{x^2}}\,x^3} - \frac{4}{x}\frac{\sqrt{6}\,\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{6}\,\sqrt{\frac{1}{x^2}}}{c}\right)^2}\\ &= -\frac{5c\sqrt{6}}{6}\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0\\ \frac{d^2}{d\tau^2}y(\tau) - \frac{5c\sqrt{6}\left(\frac{d}{d\tau}y(\tau)\right)}{6} + c^2 y(\tau) &= 0\end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = \mathrm{e}^{\frac{5\sqrt{6}\,c\tau}{12}}\left(c_1\cosh\left(\frac{\sqrt{6}\,c\tau}{12}\right) + ic_2\sinh\left(\frac{\sqrt{6}\,c\tau}{12}\right)\right)$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int \sqrt{6}\,\sqrt{\frac{1}{x^2}}dx}{c} \\
&= \frac{\sqrt{6}\,\sqrt{\frac{1}{x^2}}\,x\ln(x)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = x^{\frac{5}{2}}\left(c_1\cosh\left(\frac{\ln(x)}{2}\right) + ic_2\sinh\left(\frac{\ln(x)}{2}\right)\right)
$$

Now the particular solution to this ODE is found

$$
x^2y'' - 4xy' + 6y = x^4\sin(x)
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1y_1 + u_2y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \left(x^5\right)^{\frac{2}{5}} \\
y_2 &= \left(x^5\right)^{\frac{3}{5}}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} (x^5)^{\frac{2}{5}} & (x^5)^{\frac{3}{5}} \\ \frac{d}{dx}\left((x^5)^{\frac{2}{5}}\right) & \frac{d}{dx}\left((x^5)^{\frac{3}{5}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} (x^5)^{\frac{2}{5}} & (x^5)^{\frac{3}{5}} \\ \frac{2x^4}{(x^5)^{\frac{3}{5}}} & \frac{3x^4}{(x^5)^{\frac{2}{5}}} \end{vmatrix}$$

Therefore

$$W = \left((x^5)^{\frac{2}{5}}\right)\left(\frac{3x^4}{(x^5)^{\frac{2}{5}}}\right) - \left((x^5)^{\frac{3}{5}}\right)\left(\frac{2x^4}{(x^5)^{\frac{3}{5}}}\right)$$

Which simplifies to

$$W = x^4$$

Which simplifies to

$$W = x^4$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{(x^5)^{\frac{3}{5}} x^4 \sin(x)}{x^6} \, dx$$

Which simplifies to

$$u_1 = -\int \frac{(x^5)^{\frac{3}{5}} \sin(x)}{x^2} dx$$

Hence

$$u_1 = \frac{(x^5)^{\frac{3}{5}} \cos(x)}{x^2} - \frac{(x^5)^{\frac{3}{5}} \sin(x)}{x^3}$$

And Eq. (3) becomes

$$u_2 = \int \frac{(x^5)^{\frac{2}{5}} x^4 \sin(x)}{x^6} \, dx$$

Which simplifies to

$$u_2 = \int \frac{(x^5)^{\frac{2}{5}} \sin(x)}{x^2} dx$$

Hence

$$u_2 = -\frac{(x^5)^{\frac{2}{5}} \cos(x)}{x^2}$$

Which simplifies to

$$u_1 = \frac{(-\sin(x) + \cos(x)\, x)\, (x^5)^{\frac{3}{5}}}{x^3}$$

$$u_2 = -\frac{(x^5)^{\frac{2}{5}} \cos(x)}{x^2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = (-\sin(x) + \cos(x)\, x)\, x^2 - \cos(x)\, x^3$$

Which simplifies to

$$y_p(x) = -x^2 \sin(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(x^{\frac{5}{2}}\left(c_1 \cosh\left(\frac{\ln(x)}{2}\right) + ic_2 \sinh\left(\frac{\ln(x)}{2}\right)\right)\right) + \left(-x^2 \sin(x)\right) \\ &= -x^2 \sin(x) + x^{\frac{5}{2}}\left(c_1 \cosh\left(\frac{\ln(x)}{2}\right) + ic_2 \sinh\left(\frac{\ln(x)}{2}\right)\right) \end{aligned}$$

Which simplifies to

$$y = i \sinh\left(\frac{\ln(x)}{2}\right) x^{\frac{5}{2}} c_2 + \cosh\left(\frac{\ln(x)}{2}\right) x^{\frac{5}{2}} c_1 - x^2 \sin(x)$$

Summary

The solution(s) found are the following

$$y = i \sinh\left(\frac{\ln(x)}{2}\right) x^{\frac{5}{2}} c_2 + \cosh\left(\frac{\ln(x)}{2}\right) x^{\frac{5}{2}} c_1 - x^2 \sin(x) \tag{1}$$

Verification of solutions

$$y = i \sinh\left(\frac{\ln(x)}{2}\right) x^{\frac{5}{2}} c_2 + \cosh\left(\frac{\ln(x)}{2}\right) x^{\frac{5}{2}} c_1 - x^2 \sin(x)$$

Verified OK.

10.5.5 Solving as second order change of variable on y method 1 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' - 4xy' + 6y = 0$$

In normal form the given ode is written as

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{4}{x} \\ q(x) &= \frac{6}{x^2} \end{aligned}$$

Calculating the Liouville ode invariant Q given by

$$\begin{aligned} Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\ &= \frac{6}{x^2} - \frac{\left(-\frac{4}{x}\right)'}{2} - \frac{\left(-\frac{4}{x}\right)^2}{4} \\ &= \frac{6}{x^2} - \frac{\left(\frac{4}{x^2}\right)}{2} - \frac{\left(\frac{16}{x^2}\right)}{4} \\ &= \frac{6}{x^2} - \left(\frac{2}{x^2}\right) - \frac{4}{x^2} \\ &= 0 \end{aligned}$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$y = v(x)\, z(x) \tag{3}$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$\begin{aligned} z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)} \\ &= e^{-\int \frac{-\frac{4}{x}}{2}} \\ &= x^2 \end{aligned} \tag{5}$$

Hence (3) becomes

$$y = v(x)\, x^2 \tag{4}$$

Applying this change of variable to the original ode results in

$$v''(x) = \sin(x)$$

Which is now solved for $v(x)$ Integrating once gives

$$v'(x) = -\cos(x) + c_1$$

Integrating again gives

$$v(x) = -\sin(x) + c_1 x + c_2$$

Now that $v(x)$ is known, then

$$\begin{aligned} y &= v(x)\, z(x) \\ &= (c_1 x - \sin(x) + c_2)\,(z(x)) \end{aligned} \tag{7}$$

But from (5)

$$z(x) = x^2$$

Hence (7) becomes

$$y = (c_1 x - \sin(x) + c_2)\, x^2$$

Therefore the homogeneous solution y_h is

$$y_h = (c_1 x - \sin(x) + c_2)\, x^2$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \left(x^5\right)^{\frac{2}{5}}$$
$$y_2 = \left(x^5\right)^{\frac{3}{5}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \left(x^5\right)^{\frac{2}{5}} & \left(x^5\right)^{\frac{3}{5}} \\ \frac{d}{dx}\left(\left(x^5\right)^{\frac{2}{5}}\right) & \frac{d}{dx}\left(\left(x^5\right)^{\frac{3}{5}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \left(x^5\right)^{\frac{2}{5}} & \left(x^5\right)^{\frac{3}{5}} \\ \frac{2x^4}{\left(x^5\right)^{\frac{3}{5}}} & \frac{3x^4}{\left(x^5\right)^{\frac{2}{5}}} \end{vmatrix}$$

Therefore

$$W = \left(\left(x^5\right)^{\frac{2}{5}}\right)\left(\frac{3x^4}{\left(x^5\right)^{\frac{2}{5}}}\right) - \left(\left(x^5\right)^{\frac{3}{5}}\right)\left(\frac{2x^4}{\left(x^5\right)^{\frac{3}{5}}}\right)$$

Which simplifies to

$$W = x^4$$

Which simplifies to

$$W = x^4$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{(x^5)^{\frac{3}{5}} x^4 \sin(x)}{x^6} dx$$

Which simplifies to

$$u_1 = -\int \frac{(x^5)^{\frac{3}{5}} \sin(x)}{x^2} dx$$

Hence

$$u_1 = \frac{(x^5)^{\frac{3}{5}} \cos(x)}{x^2} - \frac{(x^5)^{\frac{3}{5}} \sin(x)}{x^3}$$

And Eq. (3) becomes

$$u_2 = \int \frac{(x^5)^{\frac{2}{5}} x^4 \sin(x)}{x^6} dx$$

Which simplifies to

$$u_2 = \int \frac{(x^5)^{\frac{2}{5}} \sin(x)}{x^2} dx$$

Hence

$$u_2 = -\frac{(x^5)^{\frac{2}{5}} \cos(x)}{x^2}$$

Which simplifies to

$$u_1 = \frac{(-\sin(x) + \cos(x)\, x)\, (x^5)^{\frac{3}{5}}}{x^3}$$

$$u_2 = -\frac{(x^5)^{\frac{2}{5}} \cos(x)}{x^2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = (-\sin(x) + \cos(x)\, x)\, x^2 - \cos(x)\, x^3$$

Which simplifies to

$$y_p(x) = -x^2 \sin(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left((c_1 x - \sin(x) + c_2)\, x^2\right) + \left(-x^2 \sin(x)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = (c_1 x - \sin(x) + c_2)\, x^2 - x^2 \sin(x) \tag{1}$$

Verification of solutions

$$y = (c_1 x - \sin(x) + c_2)\, x^2 - x^2 \sin(x)$$

Verified OK.

10.5.6 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -4x, C = 6, f(x) = x^4 \sin(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2 y'' - 4xy' + 6y = 0$$

In normal form the ode

$$x^2 y'' - 4xy' + 6y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$
\begin{aligned}
p(x) &= -\frac{4}{x} \\
q(x) &= \frac{6}{x^2}
\end{aligned}
$$

Applying change of variables on the depndent variable $y = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$
v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}
$$

Let the coefficient of $v(x)$ above be zero. Hence

$$
\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}
$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$
\frac{n(n-1)}{x^2} - \frac{4n}{x^2} + \frac{6}{x^2} = 0 \tag{5}
$$

Solving (5) for n gives

$$
n = 3 \tag{6}
$$

Substituting this value in (3) gives

$$
\begin{aligned}
v''(x) + \frac{2v'(x)}{x} &= 0 \\
v''(x) + \frac{2v'(x)}{x} &= 0 \qquad (7)
\end{aligned}
$$

Using the substitution

$$
u(x) = v'(x)
$$

Then (7) becomes

$$
u'(x) + \frac{2u(x)}{x} = 0 \tag{8}
$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{2u}{x} \end{aligned}$$

Where $f(x) = -\frac{2}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= -\frac{2}{x}\,dx \\ \int \frac{1}{u}\,du &= \int -\frac{2}{x}\,dx \\ \ln(u) &= -2\ln(x) + c_1 \\ u &= \mathrm{e}^{-2\ln(x)+c_1} \\ &= \frac{c_1}{x^2} \end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\,dx + c_2 \\ &= -\frac{c_1}{x} + c_2 \end{aligned}$$

Hence

$$\begin{aligned} y &= v(x)\,x^n \\ &= \left(-\frac{c_1}{x} + c_2\right)x^3 \\ &= (c_2x - c_1)\,x^2 \end{aligned}$$

Now the particular solution to this ODE is found

$$x^2y'' - 4xy' + 6y = x^4\sin(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
y_1 = x^2
$$
$$
y_2 = x^3
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$
$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$
W = \begin{vmatrix} x^2 & x^3 \\ \frac{d}{dx}(x^2) & \frac{d}{dx}(x^3) \end{vmatrix}
$$

Which gives

$$
W = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix}
$$

Therefore

$$
W = (x^2)(3x^2) - (x^3)(2x)
$$

Which simplifies to

$$
W = x^4
$$

Which simplifies to

$$
W = x^4
$$

Therefore Eq. (2) becomes

$$
u_1 = -\int \frac{x^7 \sin(x)}{x^6} \, dx
$$

Which simplifies to

$$u_1 = -\int x \sin(x)\, dx$$

Hence

$$u_1 = -\sin(x) + \cos(x)\, x$$

And Eq. (3) becomes

$$u_2 = \int \frac{\sin(x)\, x^6}{x^6}\, dx$$

Which simplifies to

$$u_2 = \int \sin(x)\, dx$$

Hence

$$u_2 = -\cos(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = (-\sin(x) + \cos(x)\, x)\, x^2 - \cos(x)\, x^3$$

Which simplifies to

$$y_p(x) = -x^2 \sin(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\left(-\frac{c_1}{x} + c_2\right) x^3\right) + \left(-x^2 \sin(x)\right) \\ &= -x^2 \sin(x) + \left(-\frac{c_1}{x} + c_2\right) x^3 \end{aligned}$$

Which simplifies to

$$y = x^2(-\sin(x) + c_2 x - c_1)$$

Summary

The solution(s) found are the following

$$y = x^2(-\sin(x) + c_2 x - c_1) \tag{1}$$

Verification of solutions

$$y = x^2(-\sin(x) + c_2x - c_1)$$

Verified OK.

10.5.7 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' - 4xy' + 6y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -4x \\ C &= 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 304: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4x}{x^2}\,dx} \\
&= z_1 e^{2\ln(x)} \\
&= z_1\left(x^2\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^2
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{x^2}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{4\ln(x)}}{\left(y_1\right)^2}\,dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^2\right) + c_2\left(x^2(x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$
x^2y'' - 4xy' + 6y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_2 x^3 + c_1 x^2$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= x^2 \\ y_2 &= x^3 \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^2 & x^3 \\ \frac{d}{dx}(x^2) & \frac{d}{dx}(x^3) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix}$$

Therefore

$$W = (x^2)(3x^2) - (x^3)(2x)$$

Which simplifies to

$$W = x^4$$

Which simplifies to

$$W = x^4$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^7 \sin(x)}{x^6} dx$$

Which simplifies to

$$u_1 = -\int x \sin(x) \, dx$$

Hence

$$u_1 = -\sin(x) + \cos(x) x$$

And Eq. (3) becomes

$$u_2 = \int \frac{\sin(x) x^6}{x^6} dx$$

Which simplifies to

$$u_2 = \int \sin(x) \, dx$$

Hence

$$u_2 = -\cos(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = (-\sin(x) + \cos(x) x) x^2 - \cos(x) x^3$$

Which simplifies to

$$y_p(x) = -x^2 \sin(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_2 x^3 + c_1 x^2\right) + \left(-x^2 \sin(x)\right) \end{aligned}$$

Which simplifies to

$$y = x^2(c_2x + c_1) - x^2 \sin(x)$$

Summary

The solution(s) found are the following

$$y = x^2(c_2x + c_1) - x^2 \sin(x) \tag{1}$$

Verification of solutions

$$y = x^2(c_2x + c_1) - x^2 \sin(x)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   <- LODE of Euler type successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 17

```
dsolve(x^2*diff(y(x),x$2)-4*x*diff(y(x),x)+6*y(x)=x^4*sin(x),y(x), singsol=all)
```

$$y(x) = x^2(c_2x - \sin(x) + c_1)$$

✓ Solution by Mathematica

Time used: 0.031 (sec). Leaf size: 20

```
DSolve[x^2*y''[x]-4*x*y'[x]+6*y[x]==x^4*Sin[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^2(-\sin(x) + c_2x + c_1)$$

10.6 problem Problem 19

Internal problem ID [2807]
Internal file name [`OUTPUT/2299_Sunday_June_05_2022_02_58_19_AM_9664201/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.8, A Differential Equation with Nonconstant Coefficients. page 567
Problem number: Problem 19.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_1", "second_order_change_of_variable_on_y_method_2", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{x^2y'' + 6xy' + 6y = 4\,\mathrm{e}^{2x}}$$

10.6.1 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = 6x, C = 6, f(x) = 4\,e^{2x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' + 6xy' + 6y = 0$$

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} + 6xrx^{r-1} + 6x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r + 6r\,x^r + 6x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) + 6r + 6 = 0$$

Or

$$r^2 + 5r + 6 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -3$$
$$r_2 = -2$$

Since the roots are real and distinct, then the general solution is

$$y = c_1y_1 + c_2y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = \frac{c_1}{x^3} + \frac{c_2}{x^2}$$

Next, we find the particular solution to the ODE

$$x^2y'' + 6xy' + 6y = 4\,e^{2x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x^3}$$

$$y_2 = \frac{1}{x^2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x^3} & \frac{1}{x^2} \\ \frac{d}{dx}\left(\frac{1}{x^3}\right) & \frac{d}{dx}\left(\frac{1}{x^2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x^3} & \frac{1}{x^2} \\ -\frac{3}{x^4} & -\frac{2}{x^3} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x^3}\right)\left(-\frac{2}{x^3}\right) - \left(\frac{1}{x^2}\right)\left(-\frac{3}{x^4}\right)$$

Which simplifies to

$$W = \frac{1}{x^6}$$

Which simplifies to

$$W = \frac{1}{x^6}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{4\,e^{2x}}{x^2}}{\frac{1}{x^4}}\,dx$$

Which simplifies to

$$u_1 = -\int 4x^2 e^{2x} dx$$

Hence

$$u_1 = -\left(2x^2 - 2x + 1\right) e^{2x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{4\,e^{2x}}{x^3}}{\frac{1}{x^4}}\,dx$$

Which simplifies to

$$u_2 = \int 4x\,e^{2x} dx$$

Hence

$$u_2 = (2x - 1)\,e^{2x}$$

Which simplifies to

$$u_1 = \left(-2x^2 + 2x - 1\right) e^{2x}$$
$$u_2 = (2x - 1)\,e^{2x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{(-2x^2 + 2x - 1)\,e^{2x}}{x^3} + \frac{(2x - 1)\,e^{2x}}{x^2}$$

Which simplifies to

$$y_p(x) = \frac{e^{2x}(x - 1)}{x^3}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \frac{(x-1)\,\mathrm{e}^{2x} + c_2 x + c_1}{x^3}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{(x-1)\,\mathrm{e}^{2x} + c_2 x + c_1}{x^3} \tag{1}
$$

Verification of solutions

$$
y = \frac{(x-1)\,\mathrm{e}^{2x} + c_2 x + c_1}{x^3}
$$

Verified OK.

10.6.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$
y'' + p(x)\,y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)
$$

Where $p(x) = \frac{6}{x}$. Therefore, there is an integrating factor given by

$$
\begin{aligned}
M(x) &= e^{\frac{1}{2}\int p\,dx} \\
&= e^{\int \frac{6}{x}\,dx} \\
&= x^3
\end{aligned}
$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$
\begin{aligned}
(M(x)y)'' &= 4x\,\mathrm{e}^{2x} \\
\left(x^3 y\right)'' &= 4x\,\mathrm{e}^{2x}
\end{aligned}
$$

Integrating once gives

$$
\left(x^3 y\right)' = (2x-1)\,\mathrm{e}^{2x} + c_1
$$

Integrating again gives

$$
\left(x^3 y\right) = (x-1)\,\mathrm{e}^{2x} + c_1 x + c_2
$$

Hence the solution is

$$y = \frac{(x-1)\,e^{2x} + c_1 x + c_2}{x^3}$$

Or

$$y = \frac{e^{2x}}{x^2} + \frac{c_1}{x^2} - \frac{e^{2x}}{x^3} + \frac{c_2}{x^3}$$

Summary

The solution(s) found are the following

$$y = \frac{e^{2x}}{x^2} + \frac{c_1}{x^2} - \frac{e^{2x}}{x^3} + \frac{c_2}{x^3} \tag{1}$$

Verification of solutions

$$y = \frac{e^{2x}}{x^2} + \frac{c_1}{x^2} - \frac{e^{2x}}{x^3} + \frac{c_2}{x^3}$$

Verified OK.

10.6.3 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' + 6xy' + 6y = 0$$

In normal form the ode

$$x^2y'' + 6xy' + 6y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$p(x) = \frac{6}{x}$$
$$q(x) = \frac{6}{x^2}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)} dx \\
&= \int \mathrm{e}^{-\left(\int \frac{6}{x}dx\right)} dx \\
&= \int e^{-6\ln(x)}\, dx \\
&= \int \frac{1}{x^6} dx \\
&= -\frac{1}{5x^5}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{6}{x^2}}{\frac{1}{x^{12}}} \\
&= 6x^{10}
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0$$

$$\frac{d^2}{d\tau^2}y(\tau) + 6x^{10}y(\tau) = 0$$

But in terms of τ

$$6x^{10} = \frac{6}{25\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{6y(\tau)}{25\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$25\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + 6y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$25\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 6\tau^r = 0$$

Simplifying gives

$$25r(r-1)\,\tau^r + 0\,\tau^r + 6\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$25r(r-1) + 0 + 6 = 0$$

Or

$$25r^2 - 25r + 6 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{2}{5}$$
$$r_2 = \frac{3}{5}$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1y_1 + c_2y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = c_1\tau^{\frac{2}{5}} + c_2\tau^{\frac{3}{5}}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{c_1 5^{\frac{3}{5}}\left(-\frac{1}{x^5}\right)^{\frac{2}{5}}}{5} + \frac{c_2 5^{\frac{2}{5}}\left(-\frac{1}{x^5}\right)^{\frac{3}{5}}}{5}$$

Therefore the homogeneous solution y_h is

$$
y_h = \frac{c_1 5^{\frac{3}{5}} \left(-\frac{1}{x^5}\right)^{\frac{2}{5}}}{5} + \frac{c_2 5^{\frac{2}{5}} \left(-\frac{1}{x^5}\right)^{\frac{3}{5}}}{5}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
y_1 = \left(-\frac{1}{x^5}\right)^{\frac{2}{5}}
$$

$$
y_2 = \left(-\frac{1}{x^5}\right)^{\frac{3}{5}}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$
W = \begin{vmatrix} \left(-\frac{1}{x^5}\right)^{\frac{2}{5}} & \left(-\frac{1}{x^5}\right)^{\frac{3}{5}} \\ \frac{d}{dx}\left(\left(-\frac{1}{x^5}\right)^{\frac{2}{5}}\right) & \frac{d}{dx}\left(\left(-\frac{1}{x^5}\right)^{\frac{3}{5}}\right) \end{vmatrix}
$$

Which gives

$$
W = \begin{vmatrix} \left(-\frac{1}{x^5}\right)^{\frac{2}{5}} & \left(-\frac{1}{x^5}\right)^{\frac{3}{5}} \\ \frac{2}{\left(-\frac{1}{x^5}\right)^{\frac{3}{5}} x^6} & \frac{3}{\left(-\frac{1}{x^5}\right)^{\frac{2}{5}} x^6} \end{vmatrix}
$$

Therefore

$$W = \left(\left(-\frac{1}{x^5}\right)^{\frac{2}{5}}\right)\left(\frac{3}{\left(-\frac{1}{x^5}\right)^{\frac{2}{5}} x^6}\right) - \left(\left(-\frac{1}{x^5}\right)^{\frac{3}{5}}\right)\left(\frac{2}{\left(-\frac{1}{x^5}\right)^{\frac{3}{5}} x^6}\right)$$

Which simplifies to

$$W = \frac{1}{x^6}$$

Which simplifies to

$$W = \frac{1}{x^6}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{4\left(-\frac{1}{x^5}\right)^{\frac{3}{5}} e^{2x}}{\frac{1}{x^4}}\, dx$$

Which simplifies to

$$u_1 = -\int 4\left(-\frac{1}{x^5}\right)^{\frac{3}{5}} e^{2x} x^4 dx$$

Hence

$$u_1 = -(2x-1)\, x^3 \left(-\frac{1}{x^5}\right)^{\frac{3}{5}} e^{2x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{4\left(-\frac{1}{x^5}\right)^{\frac{2}{5}} e^{2x}}{\frac{1}{x^4}}\, dx$$

Which simplifies to

$$u_2 = \int 4\left(-\frac{1}{x^5}\right)^{\frac{2}{5}} e^{2x} x^4 dx$$

Hence

$$u_2 = \left(2x^2 - 2x + 1\right) x^2 \left(-\frac{1}{x^5}\right)^{\frac{2}{5}} e^{2x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{(2x-1)\,\mathrm{e}^{2x}}{x^2} - \frac{(2x^2-2x+1)\,\mathrm{e}^{2x}}{x^3}$$

Which simplifies to

$$y_p(x) = \frac{\mathrm{e}^{2x}(x-1)}{x^3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1 5^{\frac{3}{5}}\left(-\frac{1}{x^5}\right)^{\frac{2}{5}}}{5} + \frac{c_2 5^{\frac{2}{5}}\left(-\frac{1}{x^5}\right)^{\frac{3}{5}}}{5}\right) + \left(\frac{\mathrm{e}^{2x}(x-1)}{x^3}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 5^{\frac{3}{5}}\left(-\frac{1}{x^5}\right)^{\frac{2}{5}}}{5} + \frac{c_2 5^{\frac{2}{5}}\left(-\frac{1}{x^5}\right)^{\frac{3}{5}}}{5} + \frac{\mathrm{e}^{2x}(x-1)}{x^3} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 5^{\frac{3}{5}}\left(-\frac{1}{x^5}\right)^{\frac{2}{5}}}{5} + \frac{c_2 5^{\frac{2}{5}}\left(-\frac{1}{x^5}\right)^{\frac{3}{5}}}{5} + \frac{\mathrm{e}^{2x}(x-1)}{x^3}$$

Verified OK.

10.6.4 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = 6x, C = 6, f(x) = 4\,\mathrm{e}^{2x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' + 6xy' + 6y = 0$$

In normal form the ode

$$x^2y'' + 6xy' + 6y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned}
p(x) &= \frac{6}{x} \\
q(x) &= \frac{6}{x^2}
\end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}
\tau' &= \frac{1}{c}\sqrt{q} \\
&= \frac{\sqrt{6}\,\sqrt{\frac{1}{x^2}}}{c} \\
\tau'' &= -\frac{\sqrt{6}}{c\sqrt{\frac{1}{x^2}}\,x^3}
\end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{-\frac{\sqrt{6}}{c\sqrt{\frac{1}{x^2}}\,x^3} + \frac{6}{x}\frac{\sqrt{6}\,\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{6}\,\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\
&= \frac{5c\sqrt{6}}{6}
\end{aligned}$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2} y(\tau) + \frac{5c\sqrt{6}\left(\frac{d}{d\tau} y(\tau)\right)}{6} + c^2 y(\tau) &= 0
\end{aligned} \tag{7}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
y(\tau) = \mathrm{e}^{-\frac{5\sqrt{6}\, c\tau}{12}} \left(c_1 \cosh\left(\frac{\sqrt{6}\, c\tau}{12}\right) + i c_2 \sinh\left(\frac{\sqrt{6}\, c\tau}{12}\right) \right)
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c} \sqrt{q}\, dx \\
&= \frac{\int \sqrt{6}\, \sqrt{\frac{1}{x^2}} dx}{c} \\
&= \frac{\sqrt{6}\, \sqrt{\frac{1}{x^2}}\, x \ln(x)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = \frac{c_1 \cosh\left(\frac{\ln(x)}{2}\right) + i c_2 \sinh\left(\frac{\ln(x)}{2}\right)}{x^{\frac{5}{2}}}
$$

Now the particular solution to this ODE is found

$$
x^2 y'' + 6xy' + 6y = 4\,\mathrm{e}^{2x}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \left(-\frac{1}{x^5}\right)^{\frac{2}{5}}$$

$$y_2 = \left(-\frac{1}{x^5}\right)^{\frac{3}{5}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \left(-\frac{1}{x^5}\right)^{\frac{2}{5}} & \left(-\frac{1}{x^5}\right)^{\frac{3}{5}} \\ \frac{d}{dx}\left(\left(-\frac{1}{x^5}\right)^{\frac{2}{5}}\right) & \frac{d}{dx}\left(\left(-\frac{1}{x^5}\right)^{\frac{3}{5}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \left(-\frac{1}{x^5}\right)^{\frac{2}{5}} & \left(-\frac{1}{x^5}\right)^{\frac{3}{5}} \\ \frac{2}{\left(-\frac{1}{x^5}\right)^{\frac{3}{5}} x^6} & \frac{3}{\left(-\frac{1}{x^5}\right)^{\frac{2}{5}} x^6} \end{vmatrix}$$

Therefore

$$W = \left(\left(-\frac{1}{x^5}\right)^{\frac{2}{5}}\right)\left(\frac{3}{\left(-\frac{1}{x^5}\right)^{\frac{2}{5}} x^6}\right) - \left(\left(-\frac{1}{x^5}\right)^{\frac{3}{5}}\right)\left(\frac{2}{\left(-\frac{1}{x^5}\right)^{\frac{3}{5}} x^6}\right)$$

Which simplifies to

$$W = \frac{1}{x^6}$$

Which simplifies to

$$W = \frac{1}{x^6}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{4\left(-\frac{1}{x^5}\right)^{\frac{3}{5}} e^{2x}}{\frac{1}{x^4}} dx$$

Which simplifies to

$$u_1 = -\int 4\left(-\frac{1}{x^5}\right)^{\frac{3}{5}} e^{2x} x^4 dx$$

Hence

$$u_1 = -(2x-1)\, x^3 \left(-\frac{1}{x^5}\right)^{\frac{3}{5}} e^{2x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{4\left(-\frac{1}{x^5}\right)^{\frac{2}{5}} e^{2x}}{\frac{1}{x^4}} dx$$

Which simplifies to

$$u_2 = \int 4\left(-\frac{1}{x^5}\right)^{\frac{2}{5}} e^{2x} x^4 dx$$

Hence

$$u_2 = \left(2x^2 - 2x + 1\right) x^2 \left(-\frac{1}{x^5}\right)^{\frac{2}{5}} e^{2x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{(2x-1)\, e^{2x}}{x^2} - \frac{\left(2x^2 - 2x + 1\right) e^{2x}}{x^3}$$

Which simplifies to

$$y_p(x) = \frac{e^{2x}(x-1)}{x^3}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\frac{c_1 \cosh\left(\frac{\ln(x)}{2}\right) + ic_2 \sinh\left(\frac{\ln(x)}{2}\right)}{x^{\frac{5}{2}}}\right) + \left(\frac{e^{2x}(x-1)}{x^3}\right) \\
&= \frac{e^{2x}(x-1)}{x^3} + \frac{c_1 \cosh\left(\frac{\ln(x)}{2}\right) + ic_2 \sinh\left(\frac{\ln(x)}{2}\right)}{x^{\frac{5}{2}}}
\end{aligned}
$$

Which simplifies to

$$
y = \frac{e^{2x}(x-1)}{x^3} + \frac{c_1 \cosh\left(\frac{\ln(x)}{2}\right) + ic_2 \sinh\left(\frac{\ln(x)}{2}\right)}{x^{\frac{5}{2}}}
$$

Summary

The solution(s) found are the following

$$
y = \frac{e^{2x}(x-1)}{x^3} + \frac{c_1 \cosh\left(\frac{\ln(x)}{2}\right) + ic_2 \sinh\left(\frac{\ln(x)}{2}\right)}{x^{\frac{5}{2}}} \tag{1}
$$

Verification of solutions

$$
y = \frac{e^{2x}(x-1)}{x^3} + \frac{c_1 \cosh\left(\frac{\ln(x)}{2}\right) + ic_2 \sinh\left(\frac{\ln(x)}{2}\right)}{x^{\frac{5}{2}}}
$$

Verified OK.

10.6.5 Solving as second order change of variable on y method 1 ode

This is second order non-homogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$
x^2y'' + 6xy' + 6y = 0
$$

In normal form the given ode is written as

$$
y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}
$$

Where

$$
\begin{aligned}
p(x) &= \frac{6}{x} \\
q(x) &= \frac{6}{x^2}
\end{aligned}
$$

Calculating the Liouville ode invariant Q given by

$$
\begin{aligned}
Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\
&= \frac{6}{x^2} - \frac{\left(\frac{6}{x}\right)'}{2} - \frac{\left(\frac{6}{x}\right)^2}{4} \\
&= \frac{6}{x^2} - \frac{\left(-\frac{6}{x^2}\right)}{2} - \frac{\left(\frac{36}{x^2}\right)}{4} \\
&= \frac{6}{x^2} - \left(-\frac{3}{x^2}\right) - \frac{9}{x^2} \\
&= 0
\end{aligned}
$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$
y = v(x)\, z(x) \tag{3}
$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$
\begin{aligned}
z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)} \\
&= e^{-\int \frac{\frac{6}{x}}{2}} \\
&= \frac{1}{x^3} \qquad (5)
\end{aligned}
$$

Hence (3) becomes

$$
y = \frac{v(x)}{x^3} \tag{4}
$$

Applying this change of variable to the original ode results in

$$
v''(x) = 4x\, \mathrm{e}^{2x}
$$

Which is now solved for $v(x)$ Integrating once gives

$$
v'(x) = (2x - 1)\, \mathrm{e}^{2x} + c_1
$$

Integrating again gives

$$v(x) = (x-1)\,\mathrm{e}^{2x} + c_1 x + c_2$$

Now that $v(x)$ is known, then

$$\begin{aligned} y &= v(x)\,z(x) \\ &= \left((x-1)\,\mathrm{e}^{2x} + c_1 x + c_2\right)(z(x)) \end{aligned} \tag{7}$$

But from (5)

$$z(x) = \frac{1}{x^3}$$

Hence (7) becomes

$$y = \frac{(x-1)\,\mathrm{e}^{2x} + c_1 x + c_2}{x^3}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{(x-1)\,\mathrm{e}^{2x} + c_1 x + c_2}{x^3}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \left(-\frac{1}{x^5}\right)^{\frac{2}{5}}$$

$$y_2 = \left(-\frac{1}{x^5}\right)^{\frac{3}{5}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \left(-\frac{1}{x^5}\right)^{\frac{2}{5}} & \left(-\frac{1}{x^5}\right)^{\frac{3}{5}} \\ \frac{d}{dx}\left(\left(-\frac{1}{x^5}\right)^{\frac{2}{5}}\right) & \frac{d}{dx}\left(\left(-\frac{1}{x^5}\right)^{\frac{3}{5}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \left(-\frac{1}{x^5}\right)^{\frac{2}{5}} & \left(-\frac{1}{x^5}\right)^{\frac{3}{5}} \\ \frac{2}{\left(-\frac{1}{x^5}\right)^{\frac{3}{5}} x^6} & \frac{3}{\left(-\frac{1}{x^5}\right)^{\frac{2}{5}} x^6} \end{vmatrix}$$

Therefore

$$W = \left(\left(-\frac{1}{x^5}\right)^{\frac{2}{5}}\right)\left(\frac{3}{\left(-\frac{1}{x^5}\right)^{\frac{2}{5}} x^6}\right) - \left(\left(-\frac{1}{x^5}\right)^{\frac{3}{5}}\right)\left(\frac{2}{\left(-\frac{1}{x^5}\right)^{\frac{3}{5}} x^6}\right)$$

Which simplifies to

$$W = \frac{1}{x^6}$$

Which simplifies to

$$W = \frac{1}{x^6}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{4\left(-\frac{1}{x^5}\right)^{\frac{3}{5}} e^{2x}}{\frac{1}{x^4}}\, dx$$

Which simplifies to

$$u_1 = -\int 4\left(-\frac{1}{x^5}\right)^{\frac{3}{5}} e^{2x} x^4 dx$$

Hence

$$u_1 = -(2x-1)\, x^3 \left(-\frac{1}{x^5}\right)^{\frac{3}{5}} e^{2x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{4\left(-\frac{1}{x^5}\right)^{\frac{2}{5}} \mathrm{e}^{2x}}{\frac{1}{x^4}}dx$$

Which simplifies to

$$u_2 = \int 4\left(-\frac{1}{x^5}\right)^{\frac{2}{5}} \mathrm{e}^{2x} x^4 dx$$

Hence

$$u_2 = \left(2x^2 - 2x + 1\right) x^2 \left(-\frac{1}{x^5}\right)^{\frac{2}{5}} \mathrm{e}^{2x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{(2x-1)\,\mathrm{e}^{2x}}{x^2} - \frac{\left(2x^2-2x+1\right)\mathrm{e}^{2x}}{x^3}$$

Which simplifies to

$$y_p(x) = \frac{\mathrm{e}^{2x}(x-1)}{x^3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{(x-1)\,\mathrm{e}^{2x} + c_1 x + c_2}{x^3}\right) + \left(\frac{\mathrm{e}^{2x}(x-1)}{x^3}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{(x-1)\,\mathrm{e}^{2x} + c_1 x + c_2}{x^3} + \frac{\mathrm{e}^{2x}(x-1)}{x^3} \tag{1}$$

Verification of solutions

$$y = \frac{(x-1)\,\mathrm{e}^{2x} + c_1 x + c_2}{x^3} + \frac{\mathrm{e}^{2x}(x-1)}{x^3}$$

Verified OK.

10.6.6 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = 6x, C = 6, f(x) = 4\,\mathrm{e}^{2x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' + 6xy' + 6y = 0$$

In normal form the ode

$$x^2y'' + 6xy' + 6y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{6}{x} \\ q(x) &= \frac{6}{x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{6n}{x^2} + \frac{6}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = -2 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned}
v''(x) + \frac{2v'(x)}{x} &= 0 \\
v''(x) + \frac{2v'(x)}{x} &= 0 \qquad (7)
\end{aligned}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{2u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned}
u' &= F(x, u) \\
&= f(x)g(u) \\
&= -\frac{2u}{x}
\end{aligned}$$

Where $f(x) = -\frac{2}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\, du &= -\frac{2}{x}\, dx \\
\int \frac{1}{u}\, du &= \int -\frac{2}{x}\, dx \\
\ln(u) &= -2\ln(x) + c_1 \\
u &= \mathrm{e}^{-2\ln(x)+c_1} \\
&= \frac{c_1}{x^2}
\end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\, dx + c_2 \\
&= -\frac{c_1}{x} + c_2
\end{aligned}$$

Hence

$$
\begin{aligned}
y &= v(x)\, x^n \\
&= \frac{-\frac{c_1}{x} + c_2}{x^2} \\
&= \frac{c_2 x - c_1}{x^3}
\end{aligned}
$$

Now the particular solution to this ODE is found

$$
x^2 y'' + 6xy' + 6y = 4\,\mathrm{e}^{2x}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \frac{1}{x^3} \\
y_2 &= \frac{1}{x^2}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$
W = \begin{vmatrix} \frac{1}{x^3} & \frac{1}{x^2} \\ \frac{d}{dx}\left(\frac{1}{x^3}\right) & \frac{d}{dx}\left(\frac{1}{x^2}\right) \end{vmatrix}
$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x^3} & \frac{1}{x^2} \\ -\frac{3}{x^4} & -\frac{2}{x^3} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x^3}\right)\left(-\frac{2}{x^3}\right) - \left(\frac{1}{x^2}\right)\left(-\frac{3}{x^4}\right)$$

Which simplifies to

$$W = \frac{1}{x^6}$$

Which simplifies to

$$W = \frac{1}{x^6}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{4\,e^{2x}}{x^2}}{\frac{1}{x^4}}\,dx$$

Which simplifies to

$$u_1 = -\int 4x^2 e^{2x} dx$$

Hence

$$u_1 = -\left(2x^2 - 2x + 1\right) e^{2x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{4\,e^{2x}}{x^3}}{\frac{1}{x^4}}\,dx$$

Which simplifies to

$$u_2 = \int 4x\,e^{2x} dx$$

Hence

$$u_2 = (2x - 1)\,e^{2x}$$

Which simplifies to

$$
\begin{aligned}
u_1 &= \left(-2x^2+2x-1\right) e^{2x} \\
u_2 &= (2x-1)\, e^{2x}
\end{aligned}
$$

Therefore the particular solution, from equation (1) is

$$
y_p(x) = \frac{(-2x^2+2x-1)\, e^{2x}}{x^3} + \frac{(2x-1)\, e^{2x}}{x^2}
$$

Which simplifies to

$$
y_p(x) = \frac{e^{2x}(x-1)}{x^3}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\frac{-\frac{c_1}{x}+c_2}{x^2}\right) + \left(\frac{e^{2x}(x-1)}{x^3}\right) \\
&= \frac{e^{2x}(x-1)}{x^3} + \frac{-\frac{c_1}{x}+c_2}{x^2}
\end{aligned}
$$

Which simplifies to

$$
y = \frac{(x-1)\, e^{2x} + c_2 x - c_1}{x^3}
$$

Summary

The solution(s) found are the following

$$
y = \frac{(x-1)\, e^{2x} + c_2 x - c_1}{x^3} \tag{1}
$$

Verification of solutions

$$
y = \frac{(x-1)\, e^{2x} + c_2 x - c_1}{x^3}
$$

Verified OK.

10.6.7 Solving using Kovacic algorithm

Writing the ode as

$$
\begin{aligned}
x^2y'' + 6xy' + 6y &= 0 && (1)\\
Ay'' + By' + Cy &= 0 && (2)
\end{aligned}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= x^2\\
B &= 6x && (3)\\
C &= 6
\end{aligned}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = ye^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$
z''(x) = rz(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} && (5)\\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{0}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 0\\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = 0 \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 305: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{6x}{x^2}\,dx}
\end{aligned}
$$

$$
\begin{aligned}
&= z_1 e^{-3\ln(x)} \\
&= z_1\left(\frac{1}{x^3}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{1}{x^3}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{6x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-6\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{1}{x^3}\right) + c_2\left(\frac{1}{x^3}(x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
x^2y'' + 6xy' + 6y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = \frac{c_1}{x^3} + \frac{c_2}{x^2}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x^3}$$
$$y_2 = \frac{1}{x^2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x^3} & \frac{1}{x^2} \\ \frac{d}{dx}\left(\frac{1}{x^3}\right) & \frac{d}{dx}\left(\frac{1}{x^2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x^3} & \frac{1}{x^2} \\ -\frac{3}{x^4} & -\frac{2}{x^3} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x^3}\right)\left(-\frac{2}{x^3}\right) - \left(\frac{1}{x^2}\right)\left(-\frac{3}{x^4}\right)$$

Which simplifies to

$$W = \frac{1}{x^6}$$

Which simplifies to

$$W = \frac{1}{x^6}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{4\,e^{2x}}{x^2}}{\frac{1}{x^4}}\,dx$$

Which simplifies to

$$u_1 = -\int 4x^2 e^{2x} dx$$

Hence

$$u_1 = -\left(2x^2 - 2x + 1\right) e^{2x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{4\,e^{2x}}{x^3}}{\frac{1}{x^4}}\,dx$$

Which simplifies to

$$u_2 = \int 4x\,e^{2x} dx$$

Hence

$$u_2 = (2x - 1)\,e^{2x}$$

Which simplifies to

$$u_1 = \left(-2x^2 + 2x - 1\right) e^{2x}$$
$$u_2 = (2x - 1)\,e^{2x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{(-2x^2 + 2x - 1)\,e^{2x}}{x^3} + \frac{(2x - 1)\,e^{2x}}{x^2}$$

Which simplifies to

$$y_p(x) = \frac{e^{2x}(x - 1)}{x^3}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\frac{c_1}{x^3} + \frac{c_2}{x^2}\right) + \left(\frac{e^{2x}(x-1)}{x^3}\right)
\end{aligned}
$$

Which simplifies to

$$
y = \frac{c_2 x + c_1}{x^3} + \frac{e^{2x}(x-1)}{x^3}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_2 x + c_1}{x^3} + \frac{e^{2x}(x-1)}{x^3} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_2 x + c_1}{x^3} + \frac{e^{2x}(x-1)}{x^3}
$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(x^2*diff(y(x),x$2)+6*x*diff(y(x),x)+6*y(x)=4*exp(2*x),y(x), singsol=all)
```

$$
y(x) = \frac{(x-1)\,e^{2x} + c_2 x - c_1}{x^3}
$$

✓ Solution by Mathematica

Time used: 0.056 (sec). Leaf size: 25

```
DSolve[x^2*y''[x]+6*x*y'[x]+6*y[x]==4*Exp[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{2x}(x-1) + c_2 x + c_1}{x^3}$$

10.7 problem Problem 20

Internal problem ID [2808]
Internal file name [`OUTPUT/2300_Sunday_June_05_2022_02_58_21_AM_59318423/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.8, A Differential Equation with Nonconstant Coefficients. page 567
Problem number: Problem 20.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{x^2y'' - 3xy' + 4y = \frac{x^2}{\ln(x)}}$$

10.7.1 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -3x, C = 4, f(x) = \frac{x^2}{\ln(x)}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. Solving for y_h from

$$x^2y''-3xy'+4y=0$$

This is Euler second order ODE. Let the solution be $y=x^r$, then $y'=rx^{r-1}$ and $y''=r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2}-3xrx^{r-1}+4x^r=0$$

Simplifying gives

$$r(r-1)\,x^r-3r\,x^r+4x^r=0$$

Since $x^r\neq 0$ then dividing throughout by x^r gives

$$r(r-1)-3r+4=0$$

Or

$$r^2-4r+4=0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned}r_1&=2\\r_2&=2\end{aligned}$$

Since the roots are equal, then the general solution is

$$y=c_1y_1+c_2y_2$$

Where $y_1=x^r$ and $y_2=x^r\ln(x)$. Hence

$$y=c_1x^2+c_2x^2\ln(x)$$

Next, we find the particular solution to the ODE

$$x^2y''-3xy'+4y=\frac{x^2}{\ln(x)}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x)=u_1y_1+u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x^2$$
$$y_2 = \ln(x)\, x^2$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^2 & \ln(x)\, x^2 \\ \frac{d}{dx}(x^2) & \frac{d}{dx}(\ln(x)\, x^2) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^2 & \ln(x)\, x^2 \\ 2x & x + 2\ln(x)\, x \end{vmatrix}$$

Therefore

$$W = (x^2)(x + 2\ln(x)\, x) - (\ln(x)\, x^2)(2x)$$

Which simplifies to

$$W = x^3$$

Which simplifies to

$$W = x^3$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^4}{x^5}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{1}{x} dx$$

Hence

$$u_1 = -\ln(x)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{x^4}{\ln(x)}}{x^5} dx$$

Which simplifies to

$$u_2 = \int \frac{1}{\ln(x)\, x} dx$$

Hence

$$u_2 = \ln(\ln(x))$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\ln(x)\, x^2 + \ln(\ln(x)) \ln(x)\, x^2$$

Which simplifies to

$$y_p(x) = \ln(x)\, x^2(-1 + \ln(\ln(x)))$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= x^2(\ln(x) \ln(\ln(x)) + (c_2 - 1) \ln(x) + c_1) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = x^2(\ln(x) \ln(\ln(x)) + (c_2 - 1) \ln(x) + c_1) \tag{1}$$

Verification of solutions

$$y = x^2(\ln(x) \ln(\ln(x)) + (c_2 - 1) \ln(x) + c_1)$$

Verified OK.

10.7.2 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' - 3xy' + 4y = 0$$

In normal form the ode

$$x^2y'' - 3xy' + 4y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{3}{x} \\ q(x) &= \frac{4}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$
\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\
&= \int \mathrm{e}^{-\left(\int -\frac{3}{x}dx\right)}dx \\
&= \int e^{3\ln(x)}\,dx \\
&= \int x^3 dx \\
&= \frac{x^4}{4} \qquad (6)
\end{aligned}
$$

Using (6) to evaluate q_1 from (5) gives

$$
\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{4}{x^2}}{x^6} \\
&= \frac{4}{x^8} \qquad (7)
\end{aligned}
$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$
\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + \frac{4y(\tau)}{x^8} &= 0
\end{aligned}
$$

But in terms of τ

$$
\frac{4}{x^8} = \frac{1}{4\tau^2}
$$

Hence the above ode becomes

$$
\frac{d^2}{d\tau^2}y(\tau) + \frac{y(\tau)}{4\tau^2} = 0
$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$
4\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + y(\tau) = 0
$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$
4\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + \tau^r = 0
$$

Simplifying gives

$$4r(r-1)\,\tau^r + 0\,\tau^r + \tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$4r(r-1) + 0 + 1 = 0$$

Or

$$4r^2 - 4r + 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{2}$$
$$r_2 = \frac{1}{2}$$

Since the roots are equal, then the general solution is

$$y(\tau) = c_1 y_1 + c_2 y_2$$

Where $y_1 = \tau^r$ and $y_2 = \tau^r \ln(\tau)$. Hence

$$y(\tau) = c_1\sqrt{\tau} + c_2\sqrt{\tau}\,\ln(\tau)$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{\left(c_2 \ln(x^4) - 2c_2 \ln(2) + c_1\right)\sqrt{x^4}}{2}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{\left(c_2 \ln(x^4) - 2c_2 \ln(2) + c_1\right)\sqrt{x^4}}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \sqrt{x^4} \\
y_2 &= \frac{\ln(x^4)\sqrt{x^4}}{2} - \ln(2)\sqrt{x^4}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$
W = \begin{vmatrix} \sqrt{x^4} & \frac{\ln(x^4)\sqrt{x^4}}{2} - \ln(2)\sqrt{x^4} \\ \frac{d}{dx}\left(\sqrt{x^4}\right) & \frac{d}{dx}\left(\frac{\ln(x^4)\sqrt{x^4}}{2} - \ln(2)\sqrt{x^4}\right) \end{vmatrix}
$$

Which gives

$$
W = \begin{vmatrix} \sqrt{x^4} & \frac{\ln(x^4)\sqrt{x^4}}{2} - \ln(2)\sqrt{x^4} \\ \frac{2x^3}{\sqrt{x^4}} & \frac{2\sqrt{x^4}}{x} + \frac{\ln(x^4)x^3}{\sqrt{x^4}} - \frac{2\ln(2)x^3}{\sqrt{x^4}} \end{vmatrix}
$$

Therefore

$$
W = \left(\sqrt{x^4}\right)\left(\frac{2\sqrt{x^4}}{x} + \frac{\ln(x^4)x^3}{\sqrt{x^4}} - \frac{2\ln(2)x^3}{\sqrt{x^4}}\right) - \left(\frac{\ln(x^4)\sqrt{x^4}}{2} - \ln(2)\sqrt{x^4}\right)\left(\frac{2x^3}{\sqrt{x^4}}\right)
$$

Which simplifies to

$$
W = 2x^3
$$

Which simplifies to

$$
W = 2x^3
$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\left(\frac{\ln\left(x^4\right)\sqrt{x^4}}{2}-\ln(2)\sqrt{x^4}\right)x^2}{\ln(x)}}{2x^5}dx$$

Which simplifies for $0 < x$ to

$$u_1 = -\int \frac{\ln(x) - \frac{\ln(2)}{2}}{x\ln(x)}dx$$

Hence

$$u_1 = -\ln(x) + \frac{\ln(2)\ln(\ln(x))}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{\sqrt{x^4}\,x^2}{\ln(x)}}{2x^5}dx$$

Which simplifies for $0 < x$ to

$$u_2 = \int \frac{1}{2\ln(x)\,x}dx$$

Hence

$$u_2 = \frac{\ln(\ln(x))}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-\ln(x) + \frac{\ln(2)\ln(\ln(x))}{2}\right)\sqrt{x^4} + \frac{\ln(\ln(x))\left(\frac{\ln(x^4)\sqrt{x^4}}{2} - \ln(2)\sqrt{x^4}\right)}{2}$$

Which simplifies to

$$y_p(x) = \ln(x)\,x^2(-1 + \ln(\ln(x)))$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\frac{\left(c_2 \ln\left(x^4\right) - 2c_2 \ln\left(2\right) + c_1\right)\sqrt{x^4}}{2}\right) + \left(\ln\left(x\right) x^2 \left(-1 + \ln\left(\ln\left(x\right)\right)\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{\left(c_2 \ln\left(x^4\right) - 2c_2 \ln\left(2\right) + c_1\right)\sqrt{x^4}}{2} + \ln\left(x\right) x^2 \left(-1 + \ln\left(\ln\left(x\right)\right)\right) \tag{1}
$$

Verification of solutions

$$
y = \frac{\left(c_2 \ln\left(x^4\right) - 2c_2 \ln\left(2\right) + c_1\right)\sqrt{x^4}}{2} + \ln\left(x\right) x^2 \left(-1 + \ln\left(\ln\left(x\right)\right)\right)
$$

Verified OK. `{0 < x}`

10.7.3 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$
Ay''(x) + By'(x) + Cy(x) = f(x)
$$

Where $A = x^2, B = -3x, C = 4, f(x) = \frac{x^2}{\ln(x)}$. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$
x^2y'' - 3xy' + 4y = 0
$$

In normal form the ode

$$
x^2y'' - 3xy' + 4y = 0 \tag{1}
$$

Becomes

$$
y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}
$$

Where

$$
\begin{aligned}
p(x) &= -\frac{3}{x} \\
q(x) &= \frac{4}{x^2}
\end{aligned}
$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}\tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{2\sqrt{\frac{1}{x^2}}}{c} \\ \tau'' &= -\frac{2}{c\sqrt{\frac{1}{x^2}}\,x^3}\end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned}p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\ &= \frac{-\frac{2}{c\sqrt{\frac{1}{x^2}}\,x^3} - \frac{3}{x}\frac{2\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{2\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\ &= -2c\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) - 2c\left(\frac{d}{d\tau}y(\tau)\right) + c^2 y(\tau) &= 0\end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = e^{c\tau}c_1$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int 2\sqrt{\frac{1}{x^2}}dx}{c} \\
&= \frac{2\sqrt{\frac{1}{x^2}}\,x\ln(x)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = c_1x^2
$$

Now the particular solution to this ODE is found

$$
x^2y'' - 3xy' + 4y = \frac{x^2}{\ln(x)}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1y_1 + u_2y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \sqrt{x^4} \\
y_2 &= \frac{\ln(x^4)\sqrt{x^4}}{2} - \ln(2)\sqrt{x^4}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \sqrt{x^4} & \frac{\ln(x^4)\sqrt{x^4}}{2} - \ln(2)\sqrt{x^4} \\ \frac{d}{dx}\left(\sqrt{x^4}\right) & \frac{d}{dx}\left(\frac{\ln(x^4)\sqrt{x^4}}{2} - \ln(2)\sqrt{x^4}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \sqrt{x^4} & \frac{\ln(x^4)\sqrt{x^4}}{2} - \ln(2)\sqrt{x^4} \\ \frac{2x^3}{\sqrt{x^4}} & \frac{2\sqrt{x^4}}{x} + \frac{\ln(x^4)x^3}{\sqrt{x^4}} - \frac{2\ln(2)x^3}{\sqrt{x^4}} \end{vmatrix}$$

Therefore

$$W = \left(\sqrt{x^4}\right)\left(\frac{2\sqrt{x^4}}{x} + \frac{\ln(x^4)\,x^3}{\sqrt{x^4}} - \frac{2\ln(2)\,x^3}{\sqrt{x^4}}\right) - \left(\frac{\ln(x^4)\sqrt{x^4}}{2} - \ln(2)\sqrt{x^4}\right)\left(\frac{2x^3}{\sqrt{x^4}}\right)$$

Which simplifies to

$$W = 2x^3$$

Which simplifies to

$$W = 2x^3$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\left(\frac{\ln(x^4)\sqrt{x^4}}{2} - \ln(2)\sqrt{x^4}\right)x^2}{\ln(x)}}{2x^5}\,dx$$

Which simplifies for $0 < x$ to

$$u_1 = -\int \frac{\ln(x) - \frac{\ln(2)}{2}}{x\ln(x)}\,dx$$

Hence

$$u_1 = -\ln(x) + \frac{\ln(2)\ln(\ln(x))}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{\sqrt{x^4}\, x^2}{\ln(x)}}{2x^5}\, dx$$

Which simplifies for $0 < x$ to

$$u_2 = \int \frac{1}{2 \ln(x)\, x} dx$$

Hence

$$u_2 = \frac{\ln(\ln(x))}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-\ln(x) + \frac{\ln(2) \ln(\ln(x))}{2}\right) \sqrt{x^4} + \frac{\ln(\ln(x)) \left(\frac{\ln(x^4)\sqrt{x^4}}{2} - \ln(2) \sqrt{x^4}\right)}{2}$$

Which simplifies to

$$y_p(x) = \ln(x)\, x^2(-1 + \ln(\ln(x)))$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 x^2) + (\ln(x)\, x^2(-1 + \ln(\ln(x)))) \\ &= \ln(x)\, x^2(-1 + \ln(\ln(x))) + c_1 x^2 \end{aligned}$$

Which simplifies to

$$y = x^2(\ln(x)(-1 + \ln(\ln(x))) + c_1)$$

Summary
The solution(s) found are the following

$$y = x^2(\ln(x)(-1 + \ln(\ln(x))) + c_1) \tag{1}$$

Verification of solutions

$$y = x^2(\ln(x)(-1 + \ln(\ln(x))) + c_1)$$

Verified OK. `{0 < x}`

10.7.4 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -3x, C = 4, f(x) = \frac{x^2}{\ln(x)}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - 3xy' + 4y = 0$$

In normal form the ode

$$x^2y'' - 3xy' + 4y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{3}{x} \\ q(x) &= \frac{4}{x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{3n}{x^2} + \frac{4}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 2 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{v'(x)}{x} &= 0 \\ v''(x) + \frac{v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{u}{x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= -\frac{1}{x}\, dx \\ \int \frac{1}{u}\, du &= \int -\frac{1}{x}\, dx \\ \ln(u) &= -\ln(x) + c_1 \\ u &= \mathrm{e}^{-\ln(x) + c_1} \\ &= \frac{c_1}{x} \end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\, dx + c_2 \\ &= c_1 \ln(x) + c_2 \end{aligned}$$

Hence

$$
\begin{aligned}
y &= v(x)\, x^n \\
&= (c_1 \ln(x) + c_2)\, x^2 \\
&= (c_1 \ln(x) + c_2)\, x^2
\end{aligned}
$$

Now the particular solution to this ODE is found

$$
x^2 y'' - 3xy' + 4y = \frac{x^2}{\ln(x)}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= x^2 \\
y_2 &= \ln(x)\, x^2
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$
W = \begin{vmatrix} x^2 & \ln(x)\, x^2 \\ \frac{d}{dx}(x^2) & \frac{d}{dx}(\ln(x)\, x^2) \end{vmatrix}
$$

Which gives

$$W = \begin{vmatrix} x^2 & \ln(x)\, x^2 \\ 2x & x + 2\ln(x)\, x \end{vmatrix}$$

Therefore

$$W = (x^2)\,(x + 2\ln(x)\, x) - (\ln(x)\, x^2)\,(2x)$$

Which simplifies to

$$W = x^3$$

Which simplifies to

$$W = x^3$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^4}{x^5}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{1}{x} dx$$

Hence

$$u_1 = -\ln(x)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{x^4}{\ln(x)}}{x^5}\, dx$$

Which simplifies to

$$u_2 = \int \frac{1}{\ln(x)\, x} dx$$

Hence

$$u_2 = \ln(\ln(x))$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\ln(x)\, x^2 + \ln(\ln(x))\ln(x)\, x^2$$

Which simplifies to

$$y_p(x) = \ln(x)\, x^2(-1 + \ln(\ln(x)))$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p \\
&= \left((c_1 \ln(x) + c_2)\, x^2\right) + \left(\ln(x)\, x^2(-1 + \ln(\ln(x)))\right) \\
&= \ln(x)\, x^2(-1 + \ln(\ln(x))) + (c_1 \ln(x) + c_2)\, x^2
\end{aligned}$$

Which simplifies to

$$y = x^2(\ln(x)\ln(\ln(x)) + (c_1 - 1)\ln(x) + c_2)$$

Summary

The solution(s) found are the following

$$y = x^2(\ln(x)\ln(\ln(x)) + (c_1 - 1)\ln(x) + c_2) \tag{1}$$

Verification of solutions

$$y = x^2(\ln(x)\ln(\ln(x)) + (c_1 - 1)\ln(x) + c_2)$$

Verified OK. `{0 < x}`

10.7.5 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' - 3xy' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}
A &= x^2 \\
B &= -3x \\
C &= 4
\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 306: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + (-)\,(0) \\
&= \frac{1}{2x} \\
&= \frac{1}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \frac{1}{2x} dx} \\
&= \sqrt{x}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-3x}{x^2} \, dx} \\
&= z_1 e^{\frac{3 \ln(x)}{2}} \\
&= z_1 \left(x^{\frac{3}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x^2$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-3x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{3\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(\ln(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(x^2) + c_2(x^2(\ln(x))) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' - 3xy' + 4y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1x^2 + c_2x^2 \ln(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= x^2 \\ y_2 &= \ln(x)\, x^2 \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^2 & \ln(x)\, x^2 \\ \frac{d}{dx}(x^2) & \frac{d}{dx}(\ln(x)\, x^2) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^2 & \ln(x)\, x^2 \\ 2x & x + 2\ln(x)\, x \end{vmatrix}$$

Therefore

$$W = (x^2)\,(x + 2\ln(x)\, x) - (\ln(x)\, x^2)\,(2x)$$

Which simplifies to

$$W = x^3$$

Which simplifies to

$$W = x^3$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^4}{x^5}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{1}{x} dx$$

Hence

$$u_1 = -\ln(x)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{x^4}{\ln(x)}}{x^5}\, dx$$

Which simplifies to

$$u_2 = \int \frac{1}{\ln(x)\, x} dx$$

Hence

$$u_2 = \ln(\ln(x))$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\ln(x)\, x^2 + \ln(\ln(x)) \ln(x)\, x^2$$

Which simplifies to

$$y_p(x) = \ln(x)\, x^2(-1 + \ln(\ln(x)))$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 x^2 + c_2 x^2 \ln(x)\right) + \left(\ln(x)\, x^2(-1 + \ln(\ln(x)))\right) \end{aligned}$$

Which simplifies to

$$y = x^2(c_1 + c_2 \ln(x)) + \ln(x)\, x^2(-1 + \ln(\ln(x)))$$

Summary

The solution(s) found are the following

$$y = x^2(c_1 + c_2 \ln(x)) + \ln(x)\, x^2(-1 + \ln(\ln(x))) \tag{1}$$

Verification of solutions

$$y = x^2(c_1 + c_2 \ln(x)) + \ln(x)\, x^2(-1 + \ln(\ln(x)))$$

Verified OK. `{0 < x}`

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   <- LODE of Euler type successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 22

```
dsolve(x^2*diff(y(x),x$2)-3*x*diff(y(x),x)+4*y(x)=x^2/ln(x),y(x), singsol=all)
```

$$y(x) = x^2(\ln(x)\ln(\ln(x)) + (c_1 - 1)\ln(x) + c_2)$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 24

```
DSolve[x^2*y''[x]-3*x*y'[x]+4*y[x]==x^2/Log[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^2(\log(x)(\log(\log(x)) - 1 + 2c_2) + c_1)$$

10.8 problem Problem 21

Internal problem ID [2809]
Internal file name [`OUTPUT/2301_Sunday_June_05_2022_02_58_23_AM_79803279/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.8, A Differential Equation with Nonconstant Coefficients. page 567
Problem number: Problem 21.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{x^2y'' - (2m-1)\,xy' + m^2y = x^m \ln(x)^k}$$

10.8.1 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = (-2m+1)\,x, C = m^2, f(x) = x^m \ln(x)^k$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' + (-2m+1)\,y'x + m^2y = 0$$

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2}(-2m+1)\,xrx^{r-1} + m^2x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r(-2m+1)\,r\,x^r + m^2x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1)\,(-2m+1)\,r + m^2 = 0$$

Or

$$m^2 - 2rm + r^2 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = m$$
$$r_2 = m$$

Since the roots are equal, then the general solution is

$$y = c_1y_1 + c_2y_2$$

Where $y_1 = x^r$ and $y_2 = x^r \ln(x)$. Hence

$$y = c_1x^m + c_2x^m \ln(x)$$

Next, we find the particular solution to the ODE

$$x^2y'' + (-2m+1)\,y'x + m^2y = x^m \ln(x)^k$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x^m$$
$$y_2 = x^m \ln(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^m & x^m \ln(x) \\ \frac{d}{dx}(x^m) & \frac{d}{dx}(x^m \ln(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^m & x^m \ln(x) \\ \frac{x^m m}{x} & \frac{x^m m \ln(x)}{x} + \frac{x^m}{x} \end{vmatrix}$$

Therefore

$$W = (x^m)\left(\frac{x^m m \ln(x)}{x} + \frac{x^m}{x}\right) - (x^m \ln(x))\left(\frac{x^m m}{x}\right)$$

Which simplifies to

$$W = \frac{x^{2m}}{x}$$

Which simplifies to

$$W = x^{2m-1}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^{2m} \ln(x) \ln(x)^k}{x^2 x^{2m-1}} \, dx$$

Which simplifies to

$$u_1 = -\int \frac{\ln(x)^{1+k}}{x} dx$$

Hence

$$u_1 = -\frac{\ln(x)^{2+k}}{2+k}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^{2m} \ln(x)^k}{x^2 x^{2m-1}} \, dx$$

Which simplifies to

$$u_2 = \int \frac{\ln(x)^k}{x} dx$$

Hence

$$u_2 = \frac{\ln(x)^{1+k}}{1+k}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{\ln(x)^{2+k} x^m}{2+k} + \frac{\ln(x)^{1+k} x^m \ln(x)}{1+k}$$

Which simplifies to

$$y_p(x) = \frac{\ln(x)^{2+k} x^m}{(2+k)(1+k)}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= x^m \left(\frac{\ln(x)^2 \ln(x)^k}{(2+k)(1+k)} + c_1 + c_2 \ln(x) \right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = x^m \left(\frac{\ln(x)^2 \ln(x)^k}{(2+k)(1+k)} + c_1 + c_2 \ln(x) \right) \tag{1}$$

Verification of solutions

$$y = x^m \left(\frac{\ln(x)^2 \ln(x)^k}{(2+k)(1+k)} + c_1 + c_2 \ln(x) \right)$$

Verified OK.

10.8.2 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' + (-2mx + x)\,y' + m^2y = 0$$

In normal form the ode

$$x^2y'' + (-2mx + x)\,y' + m^2y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{-2m+1}{x} \\ q(x) &= \frac{m^2}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$
\begin{aligned}
\tau &= \int e^{-\left(\int p(x)dx\right)}dx \\
&= \int e^{-\left(\int \frac{-2m+1}{x}dx\right)}dx \\
&= \int e^{(2m-1)\ln(x)}\,dx \\
&= \int x^{2m-1}dx \\
&= \frac{x^{2m}}{2m}
\end{aligned}
\tag{6}
$$

Using (6) to evaluate q_1 from (5) gives

$$
\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{m^2}{x^2}}{x^{4m-2}} \\
&= m^2 x^{-4m}
\end{aligned}
\tag{7}
$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$
\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + m^2 x^{-4m} y(\tau) &= 0
\end{aligned}
$$

But in terms of τ

$$
m^2 x^{-4m} = \frac{1}{4\tau^2}
$$

Hence the above ode becomes

$$
\frac{d^2}{d\tau^2}y(\tau) + \frac{y(\tau)}{4\tau^2} = 0
$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$
4\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + y(\tau) = 0
$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$
4\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + \tau^r = 0
$$

Simplifying gives

$$4r(r-1)\,\tau^r + 0\,\tau^r + \tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$4r(r-1) + 0 + 1 = 0$$

Or

$$4r^2 - 4r + 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{2}$$
$$r_2 = \frac{1}{2}$$

Since the roots are equal, then the general solution is

$$y(\tau) = c_1 y_1 + c_2 y_2$$

Where $y_1 = \tau^r$ and $y_2 = \tau^r \ln(\tau)$. Hence

$$y(\tau) = c_1\sqrt{\tau} + c_2\sqrt{\tau}\,\ln(\tau)$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{\sqrt{2}\,\sqrt{\frac{x^{2m}}{m}}\left(c_1 + c_2 \ln\left(\frac{x^{2m}}{m}\right) - c_2 \ln(2)\right)}{2}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{\sqrt{2}\,\sqrt{\frac{x^{2m}}{m}}\left(c_1 + c_2 \ln\left(\frac{x^{2m}}{m}\right) - c_2 \ln(2)\right)}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \sqrt{\frac{x^{2m}}{m}}$$

$$y_2 = \frac{\sqrt{2}\sqrt{\frac{x^{2m}}{m}}\ln\left(\frac{x^{2m}}{m}\right)}{2} - \frac{\sqrt{2}\sqrt{\frac{x^{2m}}{m}}\ln(2)}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \sqrt{\frac{x^{2m}}{m}} & \frac{\sqrt{2}\sqrt{\frac{x^{2m}}{m}}\ln\left(\frac{x^{2m}}{m}\right)}{2} - \frac{\sqrt{2}\sqrt{\frac{x^{2m}}{m}}\ln(2)}{2} \\ \frac{d}{dx}\left(\sqrt{\frac{x^{2m}}{m}}\right) & \frac{d}{dx}\left(\frac{\sqrt{2}\sqrt{\frac{x^{2m}}{m}}\ln\left(\frac{x^{2m}}{m}\right)}{2} - \frac{\sqrt{2}\sqrt{\frac{x^{2m}}{m}}\ln(2)}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \sqrt{\frac{x^{2m}}{m}} & \frac{\sqrt{2}\sqrt{\frac{x^{2m}}{m}}\ln\left(\frac{x^{2m}}{m}\right)}{2} - \frac{\sqrt{2}\sqrt{\frac{x^{2m}}{m}}\ln(2)}{2} \\ \frac{x^{2m}}{\sqrt{\frac{x^{2m}}{m}}\,x} & \frac{\sqrt{2}\ln\left(\frac{x^{2m}}{m}\right)x^{2m}}{2\sqrt{\frac{x^{2m}}{m}}\,x} + \frac{\sqrt{2}\sqrt{\frac{x^{2m}}{m}}\,m}{x} - \frac{\sqrt{2}\ln(2)x^{2m}}{2\sqrt{\frac{x^{2m}}{m}}\,x} \end{vmatrix}$$

Therefore

$$W = \left(\sqrt{\frac{x^{2m}}{m}}\right)\left(\frac{\sqrt{2}\ln\left(\frac{x^{2m}}{m}\right)x^{2m}}{2\sqrt{\frac{x^{2m}}{m}}\,x} + \frac{\sqrt{2}\sqrt{\frac{x^{2m}}{m}}\,m}{x} - \frac{\sqrt{2}\ln(2)\,x^{2m}}{2\sqrt{\frac{x^{2m}}{m}}\,x}\right) - \left(\frac{\sqrt{2}\sqrt{\frac{x^{2m}}{m}}\ln\left(\frac{x^{2m}}{m}\right)}{2} - \frac{\sqrt{2}\sqrt{\frac{x^{2m}}{m}}\ln(2)}{2}\right)\left(\frac{x^{2m}}{\sqrt{\frac{x^{2m}}{m}}\,x}\right)$$

Which simplifies to

$$W = \frac{\sqrt{2}\, x^{2m}}{x}$$

Which simplifies to

$$W = x^{2m-1}\sqrt{2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\left(\frac{\sqrt{2}\,\sqrt{\frac{x^{2m}}{m}}\,\ln\left(\frac{x^{2m}}{m}\right)}{2} - \frac{\sqrt{2}\,\sqrt{\frac{x^{2m}}{m}}\,\ln(2)}{2}\right) x^m \ln(x)^k}{x^2 x^{2m-1}\sqrt{2}}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{x^{-1-m}\ln(x)^k \sqrt{\frac{x^{2m}}{m}}\left(-\ln(2) + \ln\left(\frac{x^{2m}}{m}\right)\right)}{2} dx$$

Hence

$$u_1 = -\left(\int_0^x \frac{\alpha^{-1-m}\ln(\alpha)^k \sqrt{\frac{\alpha^{2m}}{m}}\left(-\ln(2) + \ln\left(\frac{\alpha^{2m}}{m}\right)\right)}{2} d\alpha\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\sqrt{\frac{x^{2m}}{m}}\, x^m \ln(x)^k}{x^2 x^{2m-1}\sqrt{2}}\, dx$$

Which simplifies to

$$u_2 = \int \frac{\sqrt{2}\, x^{-1-m}\ln(x)^k \sqrt{\frac{x^{2m}}{m}}}{2} dx$$

Hence

$$u_2 = \frac{\sqrt{2}\,\sqrt{\frac{x^{2m}}{m}}\, x^{-m}\ln(x)\ln(x)^k}{2+2k}$$

Which simplifies to

$$u_1 = -\frac{\left(\int_0^x \alpha^{-1-m} \ln(\alpha)^k \sqrt{\frac{\alpha^{2m}}{m}} \left(-\ln(2) + \ln\left(\frac{\alpha^{2m}}{m}\right)\right) d\alpha\right)}{2}$$

$$u_2 = \frac{\sqrt{2}\, \ln(x)^{1+k}\, x^{-m} \sqrt{\frac{x^{2m}}{m}}}{2+2k}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{\left(\int_0^x \alpha^{-1-m} \ln(\alpha)^k \sqrt{\frac{\alpha^{2m}}{m}} \left(-\ln(2) + \ln\left(\frac{\alpha^{2m}}{m}\right)\right) d\alpha\right) \sqrt{\frac{x^{2m}}{m}}}{2} + \frac{\sqrt{2}\, \ln(x)^{1+k}\, x^{-m} \sqrt{\frac{x^{2m}}{m}} \left(\frac{\sqrt{2}\sqrt{\frac{x^{2m}}{m}}\ln\left(\frac{x^{2m}}{m}\right)}{2} - \frac{\sqrt{2}\sqrt{\frac{x^{2m}}{m}}\ln(2)}{2}\right)}{2+2k}$$

Which simplifies to

$$y_p(x) = \frac{\sqrt{\frac{x^{2m}}{m}}\, m(1+k) \left(\int_0^x \alpha^{-1-m} \ln(\alpha)^k \sqrt{\frac{\alpha^{2m}}{m}} \left(\ln(2) - \ln\left(\frac{\alpha^{2m}}{m}\right)\right) d\alpha\right) + \ln(x)^{1+k}\, x^m \left(-\ln(2) + \ln\left(\frac{x^{2m}}{m}\right)\right)}{2m(1+k)}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{\sqrt{2}\sqrt{\frac{x^{2m}}{m}}\left(c_1 + c_2 \ln\left(\frac{x^{2m}}{m}\right) - c_2 \ln(2)\right)}{2}\right) \\ &\quad + \left(\frac{\sqrt{\frac{x^{2m}}{m}}\, m(1+k) \left(\int_0^x \alpha^{-1-m} \ln(\alpha)^k \sqrt{\frac{\alpha^{2m}}{m}} \left(\ln(2) - \ln\left(\frac{\alpha^{2m}}{m}\right)\right) d\alpha\right) + \ln(x)^{1+k}\, x^m \left(-\ln(2) + \ln\left(\right.\right.}{2m(1+k)}\right. \end{aligned}$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & \frac{\sqrt{2}\,\sqrt{\frac{x^{2m}}{m}}\,\left(c_1 + c_2 \ln\left(\frac{x^{2m}}{m}\right) - c_2 \ln(2)\right)}{2} \\
& + \frac{\sqrt{\frac{x^{2m}}{m}}\, m(1+k) \left(\int_0^x \alpha^{-1-m} \ln(\alpha)^k \sqrt{\frac{\alpha^{2m}}{m}} \left(\ln(2) - \ln\left(\frac{\alpha^{2m}}{m}\right)\right) d\alpha\right) + \ln(x)^{1+k} x^m \left(-\ln(2) + \ln\left(\frac{x^{2m}}{m}\right.\right.}{2m(1+k)}
\end{aligned}
\tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = {} & \frac{\sqrt{2}\,\sqrt{\frac{x^{2m}}{m}}\,\left(c_1 + c_2 \ln\left(\frac{x^{2m}}{m}\right) - c_2 \ln(2)\right)}{2} \\
& + \frac{\sqrt{\frac{x^{2m}}{m}}\, m(1+k) \left(\int_0^x \alpha^{-1-m} \ln(\alpha)^k \sqrt{\frac{\alpha^{2m}}{m}} \left(\ln(2) - \ln\left(\frac{\alpha^{2m}}{m}\right)\right) d\alpha\right) + \ln(x)^{1+k} x^m \left(-\ln(2) + \ln\left(\frac{x^{2m}}{m}\right.\right.}{2m(1+k)}
\end{aligned}
$$

Verified OK.

10.8.3 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$
Ay''(x) + By'(x) + Cy(x) = f(x)
$$

Where $A = x^2, B = -2mx + x, C = m^2, f(x) = x^m \ln(x)^k$. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$
x^2y'' + (-2mx + x)\, y' + m^2y = 0
$$

In normal form the ode

$$
x^2y'' + (-2mx + x)\, y' + m^2y = 0 \tag{1}
$$

Becomes

$$
y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}
$$

Where

$$
\begin{aligned}
p(x) &= \frac{-2m+1}{x} \\
q(x) &= \frac{m^2}{x^2}
\end{aligned}
$$

Applying change of variables on the depndent variable $y = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$
v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}
$$

Let the coefficient of $v(x)$ above be zero. Hence

$$
\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}
$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$
\frac{n(n-1)}{x^2} + \frac{n(-2m+1)}{x^2} + \frac{m^2}{x^2} = 0 \tag{5}
$$

Solving (5) for n gives

$$
n = m \tag{6}
$$

Substituting this value in (3) gives

$$
\begin{aligned}
v''(x) + \left(\frac{2m}{x} + \frac{-2m+1}{x}\right) v'(x) &= 0 \\
v''(x) + \frac{v'(x)}{x} &= 0 \qquad (7)
\end{aligned}
$$

Using the substitution

$$
u(x) = v'(x)
$$

Then (7) becomes

$$
u'(x) + \frac{u(x)}{x} = 0 \tag{8}
$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned}
u' &= F(x, u) \\
&= f(x)g(u) \\
&= -\frac{u}{x}
\end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\, du &= -\frac{1}{x}\, dx \\
\int \frac{1}{u}\, du &= \int -\frac{1}{x}\, dx \\
\ln (u) &= -\ln (x) + c_1 \\
u &= \mathrm{e}^{-\ln(x)+c_1} \\
&= \frac{c_1}{x}
\end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\, dx + c_2 \\
&= c_1 \ln (x) + c_2
\end{aligned}$$

Hence

$$\begin{aligned}
y &= v(x)\, x^n \\
&= (c_1 \ln (x) + c_2)\, x^m \\
&= (c_1 \ln (x) + c_2)\, x^m
\end{aligned}$$

Now the particular solution to this ODE is found

$$x^2 y'' + (-2mx + x)\, y' + m^2 y = x^m \ln (x)^k$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x^m$$
$$y_2 = x^m \ln(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^m & x^m \ln(x) \\ \frac{d}{dx}(x^m) & \frac{d}{dx}(x^m \ln(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^m & x^m \ln(x) \\ \frac{x^m m}{x} & \frac{x^m m \ln(x)}{x} + \frac{x^m}{x} \end{vmatrix}$$

Therefore

$$W = (x^m)\left(\frac{x^m m \ln(x)}{x} + \frac{x^m}{x}\right) - (x^m \ln(x))\left(\frac{x^m m}{x}\right)$$

Which simplifies to

$$W = \frac{x^{2m}}{x}$$

Which simplifies to

$$W = x^{2m-1}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^{2m} \ln(x) \ln(x)^k}{x^2 x^{2m-1}} \, dx$$

Which simplifies to

$$u_1 = -\int \frac{\ln(x)^{1+k}}{x} dx$$

Hence

$$u_1 = -\frac{\ln(x)^{2+k}}{2+k}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^{2m} \ln(x)^k}{x^2 x^{2m-1}} dx$$

Which simplifies to

$$u_2 = \int \frac{\ln(x)^k}{x} dx$$

Hence

$$u_2 = \frac{\ln(x)^{1+k}}{1+k}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{\ln(x)^{2+k} x^m}{2+k} + \frac{\ln(x)^{1+k} x^m \ln(x)}{1+k}$$

Which simplifies to

$$y_p(x) = \frac{\ln(x)^{2+k} x^m}{(2+k)(1+k)}$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p \\
&= ((c_1 \ln(x) + c_2) x^m) + \left(\frac{\ln(x)^{2+k} x^m}{(2+k)(1+k)}\right) \\
&= \frac{\ln(x)^{2+k} x^m}{(2+k)(1+k)} + (c_1 \ln(x) + c_2) x^m
\end{aligned}$$

Which simplifies to

$$y = x^m \left(\frac{\ln(x)^2 \ln(x)^k}{(2+k)(1+k)} + c_1 \ln(x) + c_2 \right)$$

Summary

The solution(s) found are the following

$$y = x^m \left(\frac{\ln(x)^2 \ln(x)^k}{(2+k)(1+k)} + c_1 \ln(x) + c_2 \right) \tag{1}$$

Verification of solutions

$$y = x^m \left(\frac{\ln(x)^2 \ln(x)^k}{(2+k)(1+k)} + c_1 \ln(x) + c_2 \right)$$

Verified OK.

10.8.4 Solving using Kovacic algorithm

Writing the ode as

$$x^2 y'' + (-2mx + x) y' + m^2 y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2mx + x \\ C &= m^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 307: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + (-)(0) \\
&= \frac{1}{2x} \\
&= \frac{1}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-2mx+x}{x^2} \, dx} \\ &= z_1 e^{\frac{(2m-1)\ln(x)}{2}} \\ &= z_1 \left(x^{m-\frac{1}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x^m$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2mx+x}{x^2} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{(2m-1)\ln(x)}}{(y_1)^2} \, dx \\ &= y_1(\ln(x)) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x^m) + c_2(x^m(\ln(x)))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
x^2y'' + (-2mx + x)\,y' + m^2y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1x^m + c_2x^m \ln(x)
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1y_1 + u_2y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= x^m \\
y_2 &= x^m \ln(x)
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^m & x^m \ln(x) \\ \frac{d}{dx}(x^m) & \frac{d}{dx}(x^m \ln(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^m & x^m \ln(x) \\ \frac{x^m m}{x} & \frac{x^m m \ln(x)}{x} + \frac{x^m}{x} \end{vmatrix}$$

Therefore

$$W = (x^m)\left(\frac{x^m m \ln(x)}{x} + \frac{x^m}{x}\right) - (x^m \ln(x))\left(\frac{x^m m}{x}\right)$$

Which simplifies to

$$W = \frac{x^{2m}}{x}$$

Which simplifies to

$$W = x^{2m-1}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^{2m} \ln(x) \ln(x)^k}{x^2 x^{2m-1}} \, dx$$

Which simplifies to

$$u_1 = -\int \frac{\ln(x)^{1+k}}{x} dx$$

Hence

$$u_1 = -\frac{\ln(x)^{2+k}}{2+k}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^{2m} \ln(x)^k}{x^2 x^{2m-1}} \, dx$$

Which simplifies to

$$u_2 = \int \frac{\ln(x)^k}{x} dx$$

Hence

$$u_2 = \frac{\ln(x)^{1+k}}{1+k}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{\ln(x)^{2+k} x^m}{2+k} + \frac{\ln(x)^{1+k} x^m \ln(x)}{1+k}$$

Which simplifies to

$$y_p(x) = \frac{\ln(x)^{2+k} x^m}{(2+k)(1+k)}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 x^m + c_2 x^m \ln(x)) + \left(\frac{\ln(x)^{2+k} x^m}{(2+k)(1+k)}\right) \end{aligned}$$

Which simplifies to

$$y = (c_1 + c_2 \ln(x)) x^m + \frac{\ln(x)^{2+k} x^m}{(2+k)(1+k)}$$

Summary

The solution(s) found are the following

$$y = (c_1 + c_2 \ln(x)) x^m + \frac{\ln(x)^{2+k} x^m}{(2+k)(1+k)} \tag{1}$$

Verification of solutions

$$y = (c_1 + c_2 \ln(x)) x^m + \frac{\ln(x)^{2+k} x^m}{(2+k)(1+k)}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   <- LODE of Euler type successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 33

```
dsolve(x^2*diff(y(x),x$2)-(2*m-1)*x*diff(y(x),x)+m^2*y(x)=x^m*(ln(x))^k,y(x), singsol=all)
```

$$y(x) = x^{m}\left(c_{2} + \ln(x) c_{1} + \frac{\ln(x)^{2} \ln(x)^{k}}{k^{2} + 3k + 2}\right)$$

✓ Solution by Mathematica

Time used: 0.064 (sec). Leaf size: 35

```
DSolve[x^2*y''[x]-(2*m-1)*x*y'[x]+m^2*y[x]==x^m*(Log[x])^k,y[x],x,IncludeSingularSolutions -
```

$$y(x) \to x^{m}\left(\frac{\log^{k+2}(x)}{k^{2} + 3k + 2} + c_{2} m \log(x) + c_{1}\right)$$

10.9 problem Problem 22

Internal problem ID [2810]
Internal file name [`OUTPUT/2302_Sunday_June_05_2022_02_58_26_AM_54237912/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.8, A Differential Equation with Nonconstant Coefficients. page 567
Problem number: Problem 22.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{x^2y'' - xy' + 5y = 0}$$

With initial conditions

$$\left[y(1) = \sqrt{2}, y'(1) = 3\sqrt{2}\right]$$

10.9.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= \frac{5}{x^2} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - \frac{y'}{x} + \frac{5y}{x^2} = 0$$

The domain of $p(x) = -\frac{1}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = \frac{5}{x^2}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

10.9.2 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} - xrx^{r-1} + 5x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r - r\,x^r + 5x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) - r + 5 = 0$$

Or

$$r^2 - 2r + 5 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$
\begin{aligned}
r_1 &= 1 - 2i \\
r_2 &= 1 + 2i
\end{aligned}
$$

The roots are complex conjugate of each others. Let the roots be

$$
\begin{aligned}
r_1 &= \alpha + i\beta \\
r_2 &= \alpha - i\beta
\end{aligned}
$$

Where in this case $\alpha = 1$ and $\beta = -2$. Hence the solution becomes

$$
\begin{aligned}
y &= c_1 x^{r_1} + c_2 x^{r_2} \\
&= c_1 x^{\alpha+i\beta} + c_2 x^{\alpha-i\beta} \\
&= x^{\alpha}\left(c_1 x^{i\beta} + c_2 x^{-i\beta}\right) \\
&= x^{\alpha}\left(c_1 e^{\ln\left(x^{i\beta}\right)} + c_2 e^{\ln\left(x^{-i\beta}\right)}\right) \\
&= x^{\alpha}\left(c_1 e^{i(\beta\ln x)} + c_2 e^{-i(\beta\ln x)}\right)
\end{aligned}
$$

Using the values for $\alpha = 1, \beta = -2$, the above becomes

$$
y = x^1\left(c_1 e^{-2i\ln(x)} + c_2 e^{2i\ln(x)}\right)
$$

Using Euler relation, the expression $c_1 e^{iA} + c_2 e^{-iA}$ is transformed to $c_1 \cos A + c_1 \sin A$ where the constants are free to change. Applying this to the above result gives

$$
y = x(c_1 \cos(2\ln(x)) + c_2 \sin(2\ln(x)))
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = x(c_1 \cos(2\ln(x)) + c_2 \sin(2\ln(x))) \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = \sqrt{2}$ and $x = 1$ in the above gives

$$
\sqrt{2} = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$y' = c_1 \cos(2\ln(x)) + c_2 \sin(2\ln(x)) + x\left(-\frac{2c_1 \sin(2\ln(x))}{x} + \frac{2c_2 \cos(2\ln(x))}{x}\right)$$

substituting $y' = 3\sqrt{2}$ and $x = 1$ in the above gives

$$3\sqrt{2} = c_1 + 2c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \sqrt{2}$$
$$c_2 = \sqrt{2}$$

Substituting these values back in above solution results in

$$y = \sqrt{2}\cos(2\ln(x))\,x + \sin(2\ln(x))\sqrt{2}\,x$$

Which simplifies to

$$y = \sqrt{2}\,(\cos(2\ln(x)) + \sin(2\ln(x)))\,x$$

Summary

The solution(s) found are the following

$$y = \sqrt{2}\,(\cos(2\ln(x)) + \sin(2\ln(x)))\,x \tag{1}$$

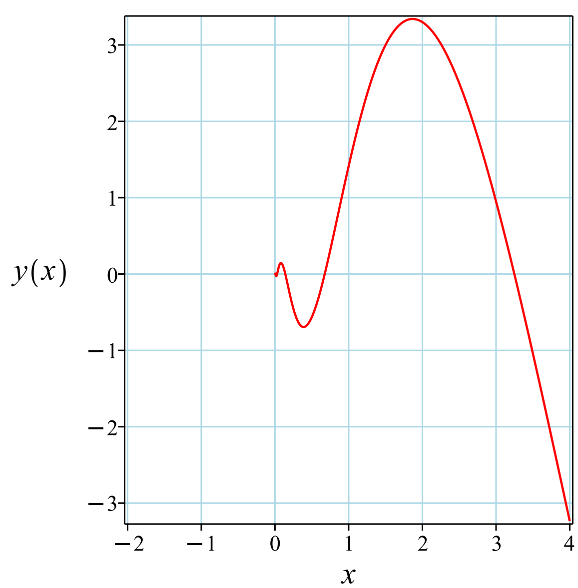


Figure 311: Solution plot

Verification of solutions

$$y = \sqrt{2}\,(\cos(2\ln(x)) + \sin(2\ln(x)))\,x$$

Verified OK.

10.9.3 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$x^2y'' - xy' + 5y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$p(x) = -\frac{1}{x}$$
$$q(x) = \frac{5}{x^2}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\
&= \int \mathrm{e}^{-\left(\int -\frac{1}{x}dx\right)}dx \\
&= \int e^{\ln(x)}\,dx \\
&= \int x dx \\
&= \frac{x^2}{2}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$
\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{5}{x^2}}{x^2} \\
&= \frac{5}{x^4}
\end{aligned} \tag{7}
$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$
\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + \frac{5y(\tau)}{x^4} &= 0
\end{aligned}
$$

But in terms of τ

$$
\frac{5}{x^4} = \frac{5}{4\tau^2}
$$

Hence the above ode becomes

$$
\frac{d^2}{d\tau^2}y(\tau) + \frac{5y(\tau)}{4\tau^2} = 0
$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$
4\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + 5y(\tau) = 0
$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$
4\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 5\tau^r = 0
$$

Simplifying gives

$$
4r(r-1)\,\tau^r + 0\,\tau^r + 5\tau^r = 0
$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$
4r(r-1) + 0 + 5 = 0
$$

Or

$$
4r^2 - 4r + 5 = 0 \tag{1}
$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$
\begin{aligned}
r_1 &= \frac{1}{2} - i \\
r_2 &= \frac{1}{2} + i
\end{aligned}
$$

The roots are complex conjugate of each others. Let the roots be

$$
\begin{aligned}
r_1 &= \alpha + i\beta \\
r_2 &= \alpha - i\beta
\end{aligned}
$$

Where in this case $\alpha = \frac{1}{2}$ and $\beta = -1$. Hence the solution becomes

$$
\begin{aligned}
y(\tau) &= c_1 \tau^{r_1} + c_2 \tau^{r_2} \\
&= c_1 \tau^{\alpha + i\beta} + c_2 \tau^{\alpha - i\beta} \\
&= \tau^{\alpha} \left(c_1 \tau^{i\beta} + c_2 \tau^{-i\beta}\right) \\
&= \tau^{\alpha} \left(c_1 e^{\ln\left(\tau^{i\beta}\right)} + c_2 e^{\ln\left(\tau^{-i\beta}\right)}\right) \\
&= \tau^{\alpha} \left(c_1 e^{i(\beta \ln \tau)} + c_2 e^{-i(\beta \ln \tau)}\right)
\end{aligned}
$$

Using the values for $\alpha = \frac{1}{2}, \beta = -1$, the above becomes

$$
y(\tau) = \tau^{\frac{1}{2}} \left(c_1 e^{-i \ln(\tau)} + c_2 e^{i \ln(\tau)}\right)
$$

Using Euler relation, the expression $c_1 e^{iA} + c_2 e^{-iA}$ is transformed to $c_1 \cos A + c_1 \sin A$ where the constants are free to change. Applying this to the above result gives

$$
y(\tau) = \sqrt{\tau}\left(c_1 \cos\left(\ln\left(\tau\right)\right) + c_2 \sin\left(\ln\left(\tau\right)\right)\right)
$$

The above solution is now transformed back to y using (6) which results in

$$
y = \frac{\sqrt{2}\, x \left(c_1 \cos\left(-\ln\left(2\right) + 2\ln\left(x\right)\right) + c_2 \sin\left(-\ln\left(2\right) + 2\ln\left(x\right)\right)\right)}{2}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = \frac{\sqrt{2}\, x \left(c_1 \cos\left(-\ln\left(2\right) + 2\ln\left(x\right)\right) + c_2 \sin\left(-\ln\left(2\right) + 2\ln\left(x\right)\right)\right)}{2} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = \sqrt{2}$ and $x = 1$ in the above gives

$$\sqrt{2} = \frac{\left(c_1 \cos\left(\ln\left(2\right)\right) - c_2 \sin\left(\ln\left(2\right)\right)\right)\sqrt{2}}{2} \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{\sqrt{2}\left(c_1 \cos\left(-\ln\left(2\right) + 2\ln\left(x\right)\right) + c_2 \sin\left(-\ln\left(2\right) + 2\ln\left(x\right)\right)\right)}{2} + \frac{\sqrt{2}\,x\left(-\frac{2c_1 \sin(-\ln(2)+2\ln(x))}{x} + \frac{2c_2 \cos(-\ln\cdots}{x}\cdots\right.}{2}$$

substituting $y' = 3\sqrt{2}$ and $x = 1$ in the above gives

$$3\sqrt{2} = \frac{\sqrt{2}\left(\left(c_1 + 2c_2\right)\cos\left(\ln\left(2\right)\right) + 2\sin\left(\ln\left(2\right)\right)\left(c_1 - \frac{c_2}{2}\right)\right)}{2} \tag{2A}$$

Equations $\{1A, 2A\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 2\cos\left(\ln\left(2\right)\right) + 2\sin\left(\ln\left(2\right)\right) \\ c_2 &= 2\cos\left(\ln\left(2\right)\right) - 2\sin\left(\ln\left(2\right)\right) \end{aligned}$$

Substituting these values back in above solution results in

$$y = \sqrt{2}\,\cos\left(2\ln\left(x\right)\right)x + \sin\left(2\ln\left(x\right)\right)\sqrt{2}\,x$$

Which simplifies to

$$y = \sqrt{2}\left(\cos\left(2\ln\left(x\right)\right) + \sin\left(2\ln\left(x\right)\right)\right)x$$

Summary

The solution(s) found are the following

$$y = \sqrt{2}\left(\cos\left(2\ln\left(x\right)\right) + \sin\left(2\ln\left(x\right)\right)\right)x \tag{1}$$

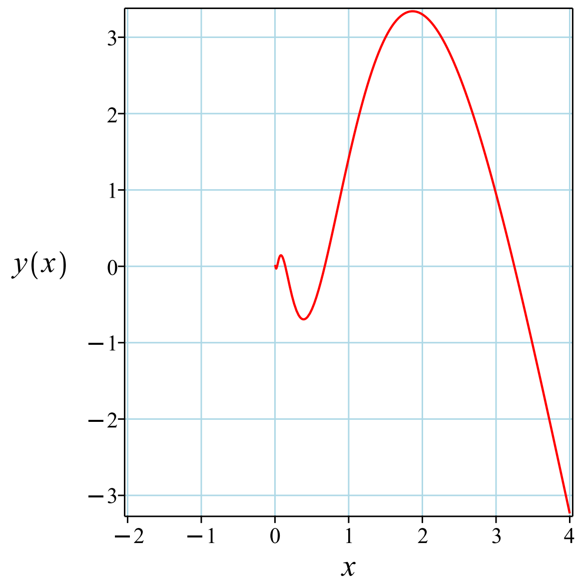


Figure 312: Solution plot

Verification of solutions

$$y = \sqrt{2}\,(\cos(2\ln(x)) + \sin(2\ln(x)))\,x$$

Verified OK.

10.9.4 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$x^2y'' - xy' + 5y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$p(x) = -\frac{1}{x}$$
$$q(x) = \frac{5}{x^2}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}\tau' &= \frac{1}{c}\sqrt{q}\\ &= \frac{\sqrt{5}\,\sqrt{\frac{1}{x^2}}}{c} \qquad (6)\\ \tau'' &= -\frac{\sqrt{5}}{c\sqrt{\frac{1}{x^2}}\,x^3}\end{aligned}$$

Substituting the above into (4) results in

$$\begin{aligned}p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2}\\ &= \frac{-\frac{\sqrt{5}}{c\sqrt{\frac{1}{x^2}}\,x^3} - \frac{1}{x}\frac{\sqrt{5}\,\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{5}\,\sqrt{\frac{1}{x^2}}}{c}\right)^2}\\ &= -\frac{2c\sqrt{5}}{5}\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0\\ \frac{d^2}{d\tau^2}y(\tau) - \frac{2c\sqrt{5}\left(\frac{d}{d\tau}y(\tau)\right)}{5} + c^2 y(\tau) &= 0 \qquad (7)\end{aligned}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = \mathrm{e}^{\frac{\sqrt{5}\,c\tau}{5}}\left(c_1 \cos\left(\frac{2\sqrt{5}\,c\tau}{5}\right) + c_2 \sin\left(\frac{2\sqrt{5}\,c\tau}{5}\right)\right)$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int \sqrt{5}\,\sqrt{\frac{1}{x^2}}dx}{c} \\
&= \frac{\sqrt{5}\,\sqrt{\frac{1}{x^2}}\,x\ln\left(x\right)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = x(c_1\cos\left(2\ln\left(x\right)\right) + c_2\sin\left(2\ln\left(x\right)\right))
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = x(c_1\cos\left(2\ln\left(x\right)\right) + c_2\sin\left(2\ln\left(x\right)\right)) \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = \sqrt{2}$ and $x = 1$ in the above gives

$$
\sqrt{2} = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = c_1\cos\left(2\ln\left(x\right)\right) + c_2\sin\left(2\ln\left(x\right)\right) + x\left(-\frac{2c_1\sin\left(2\ln\left(x\right)\right)}{x} + \frac{2c_2\cos\left(2\ln\left(x\right)\right)}{x}\right)
$$

substituting $y' = 3\sqrt{2}$ and $x = 1$ in the above gives

$$
3\sqrt{2} = c_1 + 2c_2 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= \sqrt{2} \\
c_2 &= \sqrt{2}
\end{aligned}
$$

Substituting these values back in above solution results in

$$
y = \sqrt{2}\,\cos\left(2\ln\left(x\right)\right)x + \sin\left(2\ln\left(x\right)\right)\sqrt{2}\,x
$$

Which simplifies to

$$y = \sqrt{2}\,(\cos(2\ln(x)) + \sin(2\ln(x)))\,x$$

Summary

The solution(s) found are the following

$$y = \sqrt{2}\,(\cos(2\ln(x)) + \sin(2\ln(x)))\,x \tag{1}$$

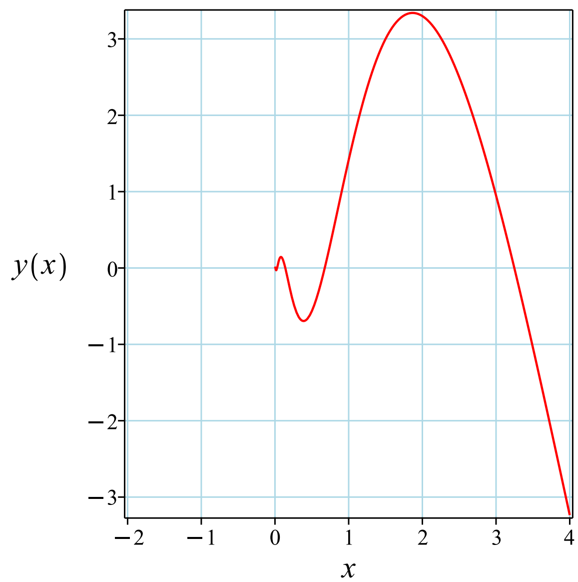


Figure 313: Solution plot

Verification of solutions

$$y = \sqrt{2}\,(\cos(2\ln(x)) + \sin(2\ln(x)))\,x$$

Verified OK.

10.9.5 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$x^2y'' - xy' + 5y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$
\begin{aligned}
p(x) &= -\frac{1}{x} \\
q(x) &= \frac{5}{x^2}
\end{aligned}
$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$
v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}
$$

Let the coefficient of $v(x)$ above be zero. Hence

$$
\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}
$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$
\frac{n(n-1)}{x^2} - \frac{n}{x^2} + \frac{5}{x^2} = 0 \tag{5}
$$

Solving (5) for n gives

$$
n = 1 + 2i \tag{6}
$$

Substituting this value in (3) gives

$$
\begin{aligned}
v''(x) + \left(\frac{2+4i}{x} - \frac{1}{x}\right) v'(x) &= 0 \\
v''(x) + \frac{(1+4i)\, v'(x)}{x} &= 0
\end{aligned} \tag{7}
$$

Using the substitution

$$
u(x) = v'(x)
$$

Then (7) becomes

$$
u'(x) + \frac{(1+4i)\, u(x)}{x} = 0 \tag{8}
$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x, u) \\
&= f(x)g(u) \\
&= \frac{(-1-4i)\,u}{x}
\end{aligned}
$$

Where $f(x) = \frac{-1-4i}{x}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= \frac{-1-4i}{x}\,dx \\
\int \frac{1}{u}\,du &= \int \frac{-1-4i}{x}\,dx \\
\ln(u) &= (-1-4i)\ln(x) + c_1 \\
u &= \mathrm{e}^{(-1-4i)\ln(x)+c_1} \\
&= c_1 \mathrm{e}^{(-1-4i)\ln(x)}
\end{aligned}
$$

Which simplifies to

$$
u(x) = \frac{c_1 x^{-4i}}{x}
$$

Now that $u(x)$ is known, then

$$
\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\,dx + c_2 \\
&= \frac{i c_1 x^{-4i}}{4} + c_2
\end{aligned}
$$

Hence

$$
\begin{aligned}
y &= v(x)\,x^n \\
&= \left(\frac{i c_1 x^{-4i}}{4} + c_2\right) x^{1+2i} \\
&= x^{1+2i} c_2 + \frac{i x^{1-2i} c_1}{4}
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = \left(\frac{i c_1 x^{-4i}}{4} + c_2\right) x^{1+2i} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = \sqrt{2}$ and $x = 1$ in the above gives

$$\sqrt{2} = \frac{ic_1}{4} + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{c_1 x^{-4i} x^{1+2i}}{x} + \frac{(1+2i)\left(\frac{ic_1 x^{-4i}}{4} + c_2\right) x^{1+2i}}{x}$$

substituting $y' = 3\sqrt{2}$ and $x = 1$ in the above gives

$$3\sqrt{2} = \left(\frac{1}{2} + \frac{i}{4}\right) c_1 + (1+2i)\, c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= (2-2i)\sqrt{2} \\ c_2 &= \left(\frac{1}{2} - \frac{i}{2}\right)\sqrt{2} \end{aligned}$$

Substituting these values back in above solution results in

$$y = \frac{i x^{1+2i}\sqrt{2}\, x^{-4i}}{2} - \frac{i x^{1+2i}\sqrt{2}}{2} + \frac{x^{1+2i}\sqrt{2}\, x^{-4i}}{2} + \frac{\sqrt{2}\, x^{1+2i}}{2}$$

Summary
The solution(s) found are the following

$$y = \frac{\left((1+i)\, x^{1-2i} + (1-i)\, x^{1+2i}\right)\sqrt{2}}{2} \tag{1}$$

Verification of solutions

$$y = \frac{\left((1+i)\, x^{1-2i} + (1-i)\, x^{1+2i}\right)\sqrt{2}}{2}$$

Verified OK.

10.9.6 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' - xy' + 5y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -x \\ C &= 5 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-17}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -17 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{17}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 308: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{17}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{17}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + 2i \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - 2i
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{17}{4x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{17}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + 2i \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - 2i
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{17}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2} + 2i$	$\frac{1}{2} - 2i$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2} + 2i$	$\frac{1}{2} - 2i$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_\infty^- = \frac{1}{2} - 2i$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= \frac{1}{2} - 2i - \left(\frac{1}{2} - 2i\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{\frac{1}{2} - 2i}{x} + (-)(0) \\
&= \frac{\frac{1}{2} - 2i}{x} \\
&= \frac{\frac{1}{2} - 2i}{x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{\frac{1}{2} - 2i}{x}\right)(0) + \left(\left(\frac{-\frac{1}{2} + 2i}{x^2} \right) + \left(\frac{\frac{1}{2} - 2i}{x} \right)^2 - \left(-\frac{17}{4x^2} \right) \right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \frac{\frac{1}{2} - 2i}{x} dx} \\
&= x^{\frac{1}{2} - 2i}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x}{x^2}\,dx} \\
&= z_1 e^{\frac{\ln(x)}{2}} \\
&= z_1\left(\sqrt{x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^{1-2i}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{x^2}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{\ln(x)}}{\left(y_1\right)^2}\,dx \\
&= y_1\left(-\frac{i x^{4i}}{4}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^{1-2i}\right) + c_2\left(x^{1-2i}\left(-\frac{i x^{4i}}{4}\right)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = x^{1-2i} c_1 - \frac{i c_2 x^{1+2i}}{4} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = \sqrt{2}$ and $x = 1$ in the above gives

$$\sqrt{2} = c_1 - \frac{ic_2}{4} \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{(1-2i)\, x^{1-2i} c_1}{x} + \frac{\left(\frac{1}{2} - \frac{i}{4}\right) c_2 x^{1+2i}}{x}$$

substituting $y' = 3\sqrt{2}$ and $x = 1$ in the above gives

$$3\sqrt{2} = (1-2i)\, c_1 + \left(\frac{1}{2} - \frac{i}{4}\right) c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \left(\frac{1}{2} + \frac{i}{2}\right) \sqrt{2}$$
$$c_2 = (2+2i)\, \sqrt{2}$$

Substituting these values back in above solution results in

$$y = \frac{i x^{1-2i} \sqrt{2}}{2} - \frac{i x^{1+2i} \sqrt{2}}{2} + \frac{x^{1-2i} \sqrt{2}}{2} + \frac{\sqrt{2}\, x^{1+2i}}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{\left((1+i)\, x^{1-2i} + (1-i)\, x^{1+2i}\right) \sqrt{2}}{2} \tag{1}$$

Verification of solutions

$$y = \frac{\left((1+i)\, x^{1-2i} + (1-i)\, x^{1+2i}\right) \sqrt{2}}{2}$$

Verified OK.

10.9.7 Maple step by step solution

Let's solve

$$\left[x^2y'' - xy' + 5y = 0, y(1) = \sqrt{2}, y'\Big|_{\{x=1\}} = 3\sqrt{2}\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{y'}{x} - \frac{5y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{y'}{x} + \frac{5y}{x^2} = 0$

- Multiply by denominators of the ODE

  $x^2y'' - xy' + 5y = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right)t'(x)$

  - Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2}y(t)\right)t'(x)^2 + t''(x)\left(\frac{d}{dt}y(t)\right)$

  - Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) - \frac{d}{dt}y(t) + 5y(t) = 0$

- Simplify

  $\frac{d^2}{dt^2}y(t) - 2\frac{d}{dt}y(t) + 5y(t) = 0$

- Characteristic polynomial of ODE

$r^2 - 2r + 5 = 0$

- Use quadratic formula to solve for r

  $r = \frac{2 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (1 - 2\,\mathrm{I}, 1 + 2\,\mathrm{I})$

- 1st solution of the ODE

  $y_1(t) = \mathrm{e}^t \cos(2t)$

- 2nd solution of the ODE

  $y_2(t) = \mathrm{e}^t \sin(2t)$

- General solution of the ODE

  $y(t) = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y(t) = c_1 \mathrm{e}^t \cos(2t) + c_2 \mathrm{e}^t \sin(2t)$

- Change variables back using $t = \ln(x)$

  $y = c_1 x \cos(2\ln(x)) + \sin(2\ln(x)) c_2 x$

- Simplify

  $y = x(c_1 \cos(2\ln(x)) + c_2 \sin(2\ln(x)))$

□ Check validity of solution $y = x(c_1 \cos(2\ln(x)) + c_2 \sin(2\ln(x)))$

  - Use initial condition $y(1) = \sqrt{2}$

    $\sqrt{2} = c_1$

  - Compute derivative of the solution

    $y' = c_1 \cos(2\ln(x)) + c_2 \sin(2\ln(x)) + x\left(-\frac{2c_1 \sin(2\ln(x))}{x} + \frac{2c_2 \cos(2\ln(x))}{x}\right)$

  - Use the initial condition $y'\Big|_{\{x=1\}} = 3\sqrt{2}$

    $3\sqrt{2} = c_1 + 2c_2$

  - Solve for c_1 and c_2

    $\{c_1 = \sqrt{2}, c_2 = \sqrt{2}\}$

  - Substitute constant values into general solution and simplify

    $y = \sqrt{2}\,(\cos(2\ln(x)) + \sin(2\ln(x)))\,x$

- Solution to the IVP

$$y = \sqrt{2}\left(\cos\left(2\ln\left(x\right)\right) + \sin\left(2\ln\left(x\right)\right)\right) x$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 20

```
dsolve([x^2*diff(y(x),x$2)-x*diff(y(x),x)+5*y(x)=0,y(1) = sqrt(2), D(y)(1) = 3*sqrt(2)],y(x)
```

$$y(x) = \sqrt{2}\, x(\sin\left(2\ln\left(x\right)\right) + \cos\left(2\ln\left(x\right)\right))$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 23

```
DSolve[{x^2*y''[x]-x*y'[x]+5*y[x]==0,{y[1]==Sqrt[2],y'[1]==3*Sqrt[2]}},y[x],x,IncludeSingula
```

$$y(x) \to \sqrt{2}x(\sin(2\log(x)) + \cos(2\log(x)))$$

10.10 problem Problem 23

Internal problem ID [2811]
Internal file name [`OUTPUT/2303_Sunday_June_05_2022_02_58_28_AM_51473416/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.8, A Differential Equation with Nonconstant Coefficients. page 567
Problem number: Problem 23.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler], [_2nd_order, _linear, `_with_symmetry_[0,F(
   x)]`]]
```

$$\boxed{t^2y'' + y't + 25y = 0}$$

With initial conditions

$$\left[y(1) = \frac{3\sqrt{3}}{2}, y'(1) = \frac{15}{2}\right]$$

10.10.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned}
p(t) &= \frac{1}{t} \\
q(t) &= \frac{25}{t^2} \\
F &= 0
\end{aligned}$$

Hence the ode is

$$y'' + \frac{y'}{t} + \frac{25y}{t^2} = 0$$

The domain of $p(t) = \frac{1}{t}$ is

$$\{t < 0 \lor 0 < t\}$$

And the point $t_0 = 1$ is inside this domain. The domain of $q(t) = \frac{25}{t^2}$ is

$$\{t < 0 \lor 0 < t\}$$

And the point $t_0 = 1$ is also inside this domain. Hence solution exists and is unique.

10.10.2 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = t^r$, then $y' = rt^{r-1}$ and $y'' = r(r-1)t^{r-2}$. Substituting these back into the given ODE gives

$$t^2(r(r-1))t^{r-2} + trt^{r-1} + 25t^r = 0$$

Simplifying gives

$$r(r-1)\,t^r + r\,t^r + 25t^r = 0$$

Since $t^r \neq 0$ then dividing throughout by t^r gives

$$r(r-1) + r + 25 = 0$$

Or

$$r^2 + 25 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$
\begin{aligned}
r_1 &= -5i \\
r_2 &= 5i
\end{aligned}
$$

The roots are complex conjugate of each others. Let the roots be

$$
\begin{aligned}
r_1 &= \alpha + i\beta \\
r_2 &= \alpha - i\beta
\end{aligned}
$$

Where in this case $\alpha = 0$ and $\beta = -5$. Hence the solution becomes

$$
\begin{aligned}
y &= c_1 t^{r_1} + c_2 t^{r_2} \\
&= c_1 t^{\alpha+i\beta} + c_2 t^{\alpha-i\beta} \\
&= t^{\alpha}\left(c_1 t^{i\beta} + c_2 t^{-i\beta}\right) \\
&= t^{\alpha}\left(c_1 e^{\ln\left(t^{i\beta}\right)} + c_2 e^{\ln\left(t^{-i\beta}\right)}\right) \\
&= t^{\alpha}\left(c_1 e^{i(\beta \ln t)} + c_2 e^{-i(\beta \ln t)}\right)
\end{aligned}
$$

Using the values for $\alpha = 0, \beta = -5$, the above becomes

$$
y = t^0\left(c_1 e^{-5i\ln(t)} + c_2 e^{5i\ln(t)}\right)
$$

Using Euler relation, the expression $c_1 e^{iA} + c_2 e^{-iA}$ is transformed to $c_1 \cos A + c_1 \sin A$ where the constants are free to change. Applying this to the above result gives

$$
y = c_1 \cos(5\ln(t)) + c_2 \sin(5\ln(t))
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \cos(5\ln(t)) + c_2 \sin(5\ln(t)) \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = \frac{3\sqrt{3}}{2}$ and $t = 1$ in the above gives

$$
\frac{3\sqrt{3}}{2} = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$y' = -\frac{5c_1 \sin(5\ln(t))}{t} + \frac{5c_2 \cos(5\ln(t))}{t}$$

substituting $y' = \frac{15}{2}$ and $t = 1$ in the above gives

$$\frac{15}{2} = 5c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{3\sqrt{3}}{2}$$
$$c_2 = \frac{3}{2}$$

Substituting these values back in above solution results in

$$y = \frac{3\cos(5\ln(t))\sqrt{3}}{2} + \frac{3\sin(5\ln(t))}{2}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{3\cos(5\ln(t))\sqrt{3}}{2} + \frac{3\sin(5\ln(t))}{2} \tag{1}$$

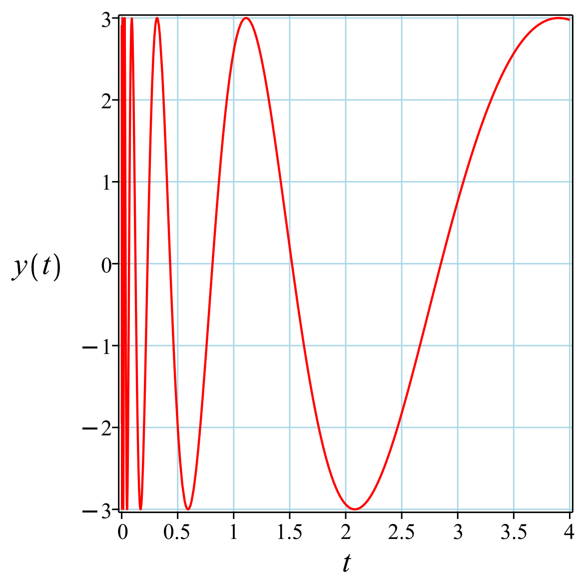


Figure 314: Solution plot

Verification of solutions

$$y = \frac{3\cos(5\ln(t))\sqrt{3}}{2} + \frac{3\sin(5\ln(t))}{2}$$

Verified OK.

10.10.3 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$t^2y'' + y't + 25y = 0 \tag{1}$$

Becomes

$$y'' + p(t)\, y' + q(t)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(t) &= \frac{1}{t} \\ q(t) &= \frac{25}{t^2} \end{aligned}$$

Applying change of variables $\tau = g(t)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(t) + p(t)\,\tau'(t)}{\tau'(t)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(t)}{\tau'(t)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(t) + p(t)\,\tau'(t) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int \mathrm{e}^{-\left(\int p(t)dt\right)} dt \\ &= \int \mathrm{e}^{-\left(\int \frac{1}{t}dt\right)} dt \\ &= \int e^{-\ln(t)}\, dt \\ &= \int \frac{1}{t} dt \\ &= \ln(t) \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(t)}{\tau'(t)^2} \\
&= \frac{\frac{25}{t^2}}{\frac{1}{t^2}} \\
&= 25
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}
\frac{d^2}{d\tau^2} y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2} y(\tau) + 25 y(\tau) &= 0
\end{aligned}$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = 25$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda\tau} + 25\,\mathrm{e}^{\lambda\tau} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$\lambda^2 + 25 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 25$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(25)} \\
&= \pm 5i
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= +5i \\
\lambda_2 &= -5i
\end{aligned}$$

Which simplifies to

$$\lambda_1 = 5i$$
$$\lambda_2 = -5i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 5$. Therefore the final solution, when using Euler relation, can be written as

$$y(\tau) = e^{\alpha\tau}(c_1 \cos(\beta\tau) + c_2 \sin(\beta\tau))$$

Which becomes

$$y(\tau) = e^0(c_1 \cos(5\tau) + c_2 \sin(5\tau))$$

Or

$$y(\tau) = c_1 \cos(5\tau) + c_2 \sin(5\tau)$$

The above solution is now transformed back to y using (6) which results in

$$y = c_1 \cos(5\ln(t)) + c_2 \sin(5\ln(t))$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \cos(5\ln(t)) + c_2 \sin(5\ln(t)) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = \frac{3\sqrt{3}}{2}$ and $t = 1$ in the above gives

$$\frac{3\sqrt{3}}{2} = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{5c_1 \sin(5\ln(t))}{t} + \frac{5c_2 \cos(5\ln(t))}{t}$$

substituting $y' = \frac{15}{2}$ and $t = 1$ in the above gives

$$\frac{15}{2} = 5c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{3\sqrt{3}}{2}$$
$$c_2 = \frac{3}{2}$$

Substituting these values back in above solution results in

$$y = \frac{3\cos(5\ln(t))\sqrt{3}}{2} + \frac{3\sin(5\ln(t))}{2}$$

Summary
The solution(s) found are the following

$$y = \frac{3\cos(5\ln(t))\sqrt{3}}{2} + \frac{3\sin(5\ln(t))}{2} \tag{1}$$

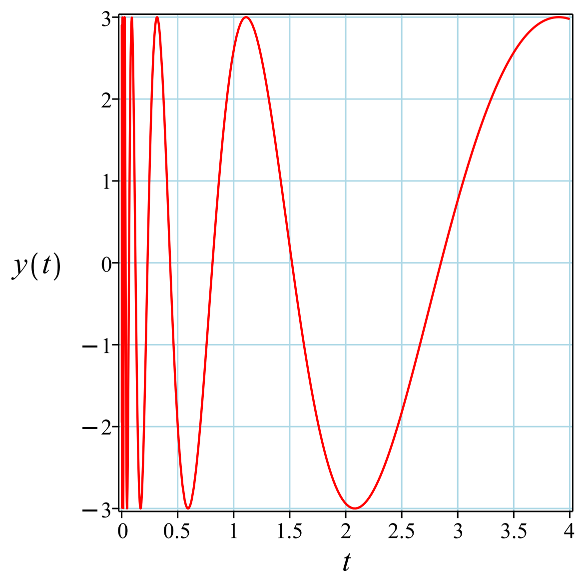


Figure 315: Solution plot

Verification of solutions

$$y = \frac{3\cos(5\ln(t))\sqrt{3}}{2} + \frac{3\sin(5\ln(t))}{2}$$

Verified OK.

10.10.4 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$t^2y'' + y't + 25y = 0 \tag{1}$$

Becomes

$$y'' + p(t)\, y' + q(t)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(t) &= \frac{1}{t} \\ q(t) &= \frac{25}{t^2} \end{aligned}$$

Applying change of variables $\tau = g(t)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(t) + p(t)\,\tau'(t)}{\tau'(t)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(t)}{\tau'(t)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{5\sqrt{\frac{1}{t^2}}}{c} \\ \tau'' &= -\frac{5}{c\sqrt{\frac{1}{t^2}}\, t^3} \end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned} p_1(\tau) &= \frac{\tau''(t) + p(t)\,\tau'(t)}{\tau'(t)^2} \\ &= \frac{-\frac{5}{c\sqrt{\frac{1}{t^2}}\, t^3} + \frac{1}{t}\frac{5\sqrt{\frac{1}{t^2}}}{c}}{\left(\frac{5\sqrt{\frac{1}{t^2}}}{c}\right)^2} \\ &= 0 \end{aligned}$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2} y(\tau) + c^2 y(\tau) &= 0
\end{aligned} \tag{7}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c} \sqrt{q}\, dt \\
&= \frac{\int 5\sqrt{\frac{1}{t^2}} dt}{c} \\
&= \frac{5\sqrt{\frac{1}{t^2}}\, t \ln(t)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = c_1 \cos(5\ln(t)) + c_2 \sin(5\ln(t))
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \cos(5\ln(t)) + c_2 \sin(5\ln(t)) \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = \frac{3\sqrt{3}}{2}$ and $t = 1$ in the above gives

$$
\frac{3\sqrt{3}}{2} = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -\frac{5c_1 \sin(5\ln(t))}{t} + \frac{5c_2 \cos(5\ln(t))}{t}
$$

substituting $y' = \frac{15}{2}$ and $t = 1$ in the above gives

$$
\frac{15}{2} = 5c_2 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= \frac{3\sqrt{3}}{2} \\
c_2 &= \frac{3}{2}
\end{aligned}
$$

Substituting these values back in above solution results in

$$
y = \frac{3\cos\left(5\ln\left(t\right)\right)\sqrt{3}}{2} + \frac{3\sin\left(5\ln\left(t\right)\right)}{2}
$$

Summary

The solution(s) found are the following

$$
y = \frac{3\cos\left(5\ln\left(t\right)\right)\sqrt{3}}{2} + \frac{3\sin\left(5\ln\left(t\right)\right)}{2} \tag{1}
$$

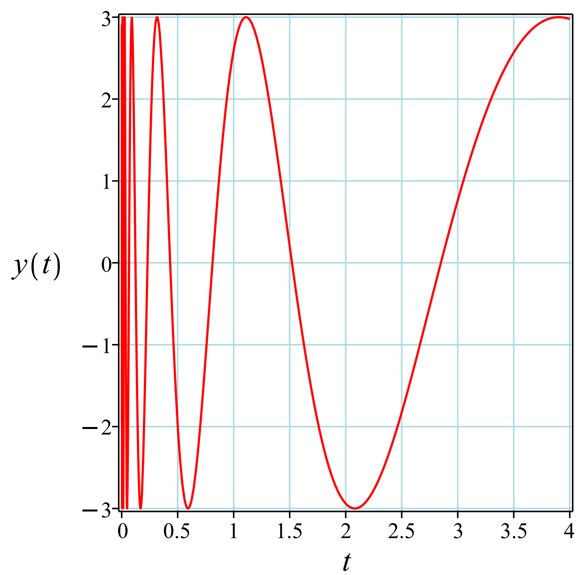


Figure 316: Solution plot

Verification of solutions

$$
y = \frac{3\cos\left(5\ln\left(t\right)\right)\sqrt{3}}{2} + \frac{3\sin\left(5\ln\left(t\right)\right)}{2}
$$

Verified OK.

10.10.5 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$t^2y'' + y't + 25y = 0 \tag{1}$$

Becomes

$$y'' + p(t)\, y' + q(t)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(t) &= \frac{1}{t} \\ q(t) &= \frac{25}{t^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(t)\, t^n$ to (2) gives the following ode where the dependent variables is $v(t)$ and not y.

$$v''(t) + \left(\frac{2n}{t} + p\right) v'(t) + \left(\frac{n(n-1)}{t^2} + \frac{np}{t} + q\right) v(t) = 0 \tag{3}$$

Let the coefficient of $v(t)$ above be zero. Hence

$$\frac{n(n-1)}{t^2} + \frac{np}{t} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(t)$ and $q(t)$ into (4) gives

$$\frac{n(n-1)}{t^2} + \frac{n}{t^2} + \frac{25}{t^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 5i \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(t) + \left(\frac{10i}{t} + \frac{1}{t}\right) v'(t) &= 0 \\ v''(t) + \frac{(1+10i)\, v'(t)}{t} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(t) = v'(t)$$

Then (7) becomes

$$u'(t)+\frac{(1+10i)\,u(t)}{t}=0 \tag{8}$$

The above is now solved for $u(t)$. In canonical form the ODE is

$$\begin{aligned}u' &= F(t,u)\\ &= f(t)g(u)\\ &= \frac{(-1-10i)\,u}{t}\end{aligned}$$

Where $f(t)=\frac{-1-10i}{t}$ and $g(u)=u$. Integrating both sides gives

$$\begin{aligned}\frac{1}{u}\,du &= \frac{-1-10i}{t}\,dt\\ \int\frac{1}{u}\,du &= \int\frac{-1-10i}{t}\,dt\\ \ln(u) &= (-1-10i)\ln(t)+c_1\\ u &= \mathrm{e}^{(-1-10i)\ln(t)+c_1}\\ &= c_1\mathrm{e}^{(-1-10i)\ln(t)}\end{aligned}$$

Which simplifies to

$$u(t)=\frac{c_1 t^{-10i}}{t}$$

Now that $u(t)$ is known, then

$$\begin{aligned}v'(t) &= u(t)\\ v(t) &= \int u(t)\,dt+c_2\\ &= \frac{i c_1 t^{-10i}}{10}+c_2\end{aligned}$$

Hence

$$\begin{aligned}y &= v(t)\,t^n\\ &= \left(\frac{i c_1 t^{-10i}}{10}+c_2\right)t^{5i}\\ &= t^{5i}c_2+\frac{i t^{-5i}c_1}{10}\end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \left(\frac{i c_1 t^{-10i}}{10} + c_2\right) t^{5i} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = \frac{3\sqrt{3}}{2}$ and $t = 1$ in the above gives

$$\frac{3\sqrt{3}}{2} = \frac{i c_1}{10} + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{c_1 t^{-10i} t^{5i}}{t} + \frac{5i\left(\frac{i c_1 t^{-10i}}{10} + c_2\right) t^{5i}}{t}$$

substituting $y' = \frac{15}{2}$ and $t = 1$ in the above gives

$$\frac{15}{2} = \frac{c_1}{2} + 5 i c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{15}{2} - \frac{15 i\sqrt{3}}{2}$$
$$c_2 = -\frac{3i}{4} + \frac{3\sqrt{3}}{4}$$

Substituting these values back in above solution results in

$$y = \frac{3 i t^{5i} t^{-10i}}{4} + \frac{3 t^{5i}\sqrt{3}\, t^{-10i}}{4} - \frac{3 i t^{5i}}{4} + \frac{3 t^{5i}\sqrt{3}}{4}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(3i + 3\sqrt{3}\right) t^{-5i}}{4} - \frac{3 t^{5i}\left(i - \sqrt{3}\right)}{4} \tag{1}$$

Verification of solutions

$$y = \frac{\left(3i + 3\sqrt{3}\right) t^{-5i}}{4} - \frac{3 t^{5i}\left(i - \sqrt{3}\right)}{4}$$

Verified OK.

10.10.6 Solving using Kovacic algorithm

Writing the ode as

$$
\begin{align*}
t^2y'' + y't + 25y &= 0 \tag{1} \\
Ay'' + By' + Cy &= 0 \tag{2}
\end{align*}
$$

Comparing (1) and (2) shows that

$$
\begin{align*}
A &= t^2 \\
B &= t \tag{3} \\
C &= 25
\end{align*}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(t) = ye^{\int \frac{B}{2A}\,dt}
$$

Then (2) becomes

$$
z''(t) = rz(t) \tag{4}
$$

Where r is given by

$$
\begin{align*}
r &= \frac{s}{t} \tag{5} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{align*}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-101}{4t^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{align*}
s &= -101 \\
t &= 4t^2
\end{align*}
$$

Therefore eq. (4) becomes

$$
z''(t) = \left(-\frac{101}{4t^2}\right) z(t) \tag{7}
$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$
y = z(t)\, e^{-\int \frac{B}{2A}\,dt}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 310: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{101}{4t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{101}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + 5i \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - 5i
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{t^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = -\frac{101}{4t^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{101}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + 5i \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - 5i
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = -\frac{101}{4t^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2} + 5i$	$\frac{1}{2} - 5i$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2} + 5i$	$\frac{1}{2} - 5i$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_{\infty}^{-} = \frac{1}{2} - 5i$ then

$$
\begin{aligned}
d &= \alpha_{\infty}^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= \frac{1}{2} - 5i - \left(\frac{1}{2} - 5i\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_{\infty}
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t - c_1} \right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{\frac{1}{2} - 5i}{t} + (-)\,(0) \\
&= \frac{\frac{1}{2} - 5i}{t} \\
&= \frac{\frac{1}{2} - 5i}{t}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(t) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{\frac{1}{2} - 5i}{t}\right)(0) + \left(\left(\frac{-\frac{1}{2} + 5i}{t^2} \right) + \left(\frac{\frac{1}{2} - 5i}{t} \right)^2 - \left(-\frac{101}{4t^2} \right) \right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= \mathrm{e}^{\int \frac{\frac{1}{2} - 5i}{t} dt} \\
&= t^{\frac{1}{2} - 5i}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{t}{t^2}\,dt} \\
&= z_1 e^{-\frac{\ln(t)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{t}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = t^{-5i}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{t}{t^2}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{-\ln(t)}}{(y_1)^2}\,dt \\
&= y_1 \left(-\frac{i t^{10i}}{10}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(t^{-5i}\right) + c_2 \left(t^{-5i}\left(-\frac{i t^{10i}}{10}\right)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = t^{-5i} c_1 - \frac{i c_2 t^{5i}}{10} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = \frac{3\sqrt{3}}{2}$ and $t = 1$ in the above gives

$$\frac{3\sqrt{3}}{2} = c_1 - \frac{ic_2}{10} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{5it^{-5i}c_1}{t} + \frac{c_2 t^{5i}}{2t}$$

substituting $y' = \frac{15}{2}$ and $t = 1$ in the above gives

$$\frac{15}{2} = -5ic_1 + \frac{c_2}{2} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{3i}{4} + \frac{3\sqrt{3}}{4}$$
$$c_2 = \frac{15}{2} + \frac{15i\sqrt{3}}{2}$$

Substituting these values back in above solution results in

$$y = \frac{3it^{-5i}}{4} + \frac{3\sqrt{3}\,t^{-5i}}{4} - \frac{3it^{5i}}{4} + \frac{3t^{5i}\sqrt{3}}{4}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(3i + 3\sqrt{3}\right) t^{-5i}}{4} - \frac{3t^{5i}\left(i - \sqrt{3}\right)}{4} \tag{1}$$

Verification of solutions

$$y = \frac{\left(3i + 3\sqrt{3}\right) t^{-5i}}{4} - \frac{3t^{5i}\left(i - \sqrt{3}\right)}{4}$$

Verified OK.

10.10.7 Maple step by step solution

Let's solve

$$\left[t^2y'' + y't + 25y = 0, y(1) = \frac{3\sqrt{3}}{2}, y'\Big|_{\{t=1\}} = \frac{15}{2}\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y'}{t} - \frac{25y}{t^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{t} + \frac{25y}{t^2} = 0$

- Multiply by denominators of the ODE

  $t^2y'' + y't + 25y = 0$

- Make a change of variables

  $s = \ln(t)$

- ☐ Substitute the change of variables back into the ODE
  - Calculate the 1st derivative of y with respect to t , using the chain rule

    $y' = \left(\frac{d}{ds}y(s)\right) s'(t)$

  - Compute derivative

    $y' = \frac{\frac{d}{ds}y(s)}{t}$

  - Calculate the 2nd derivative of y with respect to t , using the chain rule

    $y'' = \left(\frac{d^2}{ds^2}y(s)\right) s'(t)^2 + s''(t) \left(\frac{d}{ds}y(s)\right)$

  - Compute derivative

    $y'' = \frac{\frac{d^2}{ds^2}y(s)}{t^2} - \frac{\frac{d}{ds}y(s)}{t^2}$

  Substitute the change of variables back into the ODE

  $t^2\left(\frac{\frac{d^2}{ds^2}y(s)}{t^2} - \frac{\frac{d}{ds}y(s)}{t^2}\right) + \frac{d}{ds}y(s) + 25y(s) = 0$

- Simplify

  $\frac{d^2}{ds^2}y(s) + 25y(s) = 0$

- Characteristic polynomial of ODE

$r^2 + 25 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-100})}{2}$

- Roots of the characteristic polynomial

  $r = (-5\,\mathrm{I}, 5\,\mathrm{I})$

- 1st solution of the ODE

  $y_1(s) = \cos(5s)$

- 2nd solution of the ODE

  $y_2(s) = \sin(5s)$

- General solution of the ODE

  $y(s) = c_1 y_1(s) + c_2 y_2(s)$

- Substitute in solutions

  $y(s) = c_1 \cos(5s) + c_2 \sin(5s)$

- Change variables back using $s = \ln(t)$

  $y = c_1 \cos(5\ln(t)) + c_2 \sin(5\ln(t))$

☐ Check validity of solution $y = c_1 \cos(5\ln(t)) + c_2 \sin(5\ln(t))$

  - Use initial condition $y(1) = \frac{3\sqrt{3}}{2}$

    $\frac{3\sqrt{3}}{2} = c_1$

  - Compute derivative of the solution

    $y' = -\frac{5c_1 \sin(5\ln(t))}{t} + \frac{5c_2 \cos(5\ln(t))}{t}$

  - Use the initial condition $y'\Big|_{\{t=1\}} = \frac{15}{2}$

    $\frac{15}{2} = 5c_2$

  - Solve for c_1 and c_2

    $\left\{c_1 = \frac{3\sqrt{3}}{2}, c_2 = \frac{3}{2}\right\}$

  - Substitute constant values into general solution and simplify

    $y = \frac{3\cos(5\ln(t))\sqrt{3}}{2} + \frac{3\sin(5\ln(t))}{2}$

- Solution to the IVP

  $y = \frac{3\cos(5\ln(t))\sqrt{3}}{2} + \frac{3\sin(5\ln(t))}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 22

```
dsolve([t^2*diff(y(t),t$2)+t*diff(y(t),t)+25*y(t)=0,y(1) = 3/2*sqrt(3), D(y)(1) = 15/2],y(t)
```

$$y(t) = \frac{3\sin(5\ln(t))}{2} + \frac{3\sqrt{3}\cos(5\ln(t))}{2}$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 26

```
DSolve[{t^2*y''[t]+t*y'[t]+25*y[t]==0,{y[1]==3*Sqrt[3]/2,y'[1]==15/2}},y[t],t,IncludeSingula
```

$$y(t) \to \frac{3}{2}\left(\sin(5\log(t)) + \sqrt{3}\cos(5\log(t))\right)$$

11 Chapter 8, Linear differential equations of order n. Section 8.9, Reduction of Order. page 572

11.1 problem Problem 1

Internal problem ID [2812]
Internal file name [`OUTPUT/2304_Sunday_June_05_2022_02_58_31_AM_30789350/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.9, Reduction of Order. page 572
Problem number: Problem 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler], [_2nd_order, _linear, `_with_symmetry_[0,F(
   x)]`]]
```

$$\boxed{x^2y'' - 3xy' + 4y = 0}$$

Given that one solution of the ode is

$$y_1 = x^2$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1\left(\int \frac{\mathrm{e}^{-\left(\int p dx\right)}}{y_1^2} dx\right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y'' + p(x)\, y' + q(x)\, y = f(x)$$

Looking at the ode to solve shows that

$$p(x) = -\frac{3}{x}$$

Therefore

$$y_2(x) = x^2 \left(\int \frac{e^{-(\int -\frac{3}{x}dx)}}{x^4} dx\right)$$

$$y_2(x) = x^2 \int \frac{x^3}{x^4}, dx$$

$$y_2(x) = x^2 \left(\int \frac{1}{x} dx\right)$$

$$y_2(x) = \ln(x)\, x^2$$

Hence the solution is

$$\begin{aligned} y &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^2 + c_2 x^2 \ln(x) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x^2 + c_2 x^2 \ln(x) \tag{1}$$

Verification of solutions

$$y = c_1 x^2 + c_2 x^2 \ln(x)$$

Verified OK.

11.1.1 Maple step by step solution

Let's solve

$x^2 y'' - 3xy' + 4y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{3y'}{x} - \frac{4y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{3y'}{x} + \frac{4y}{x^2} = 0$
- Multiply by denominators of the ODE

  $x^2 y'' - 3xy' + 4y = 0$
- Make a change of variables

  $t = \ln(x)$

☐ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right) t'(x)$
  - Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$
  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2}y(t)\right) t'(x)^2 + t''(x) \left(\frac{d}{dt}y(t)\right)$
  - Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) - 3\frac{d}{dt}y(t) + 4y(t) = 0$
- Simplify

  $\frac{d^2}{dt^2}y(t) - 4\frac{d}{dt}y(t) + 4y(t) = 0$
- Characteristic polynomial of ODE

  $r^2 - 4r + 4 = 0$
- Factor the characteristic polynomial

  $(r-2)^2 = 0$
- Root of the characteristic polynomial

  $r = 2$
- 1st solution of the ODE

  $y_1(t) = e^{2t}$
- Repeated root, multiply $y_1(t)$ by t to ensure linear independence

$$y_2(t) = t\,e^{2t}$$

- General solution of the ODE

$$y(t) = c_1y_1(t) + c_2y_2(t)$$

- Substitute in solutions

$$y(t) = c_1e^{2t} + c_2t\,e^{2t}$$

- Change variables back using $t = \ln(x)$

$$y = c_1x^2 + c_2x^2\ln(x)$$

- Simplify

$$y = x^2(c_1 + c_2\ln(x))$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve([x^2*diff(y(x),x$2)-3*x*diff(y(x),x)+4*y(x)=0,x^2],singsol=all)
```

$$y(x) = x^2(c_2\ln(x) + c_1)$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 18

```
DSolve[x^2*y''[x]-3*x*y'[x]+4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^2(2c_2\log(x) + c_1)$$

11.2 problem Problem 2

Internal problem ID [2813]
Internal file name [`OUTPUT/2305_Sunday_June_05_2022_02_58_32_AM_13488917/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.9, Reduction of Order. page 572
Problem number: Problem 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{xy'' + (1-2x)\, y' + y(x-1) = 0}$$

Given that one solution of the ode is

$$y_1 = e^x$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1 \left(\int \frac{e^{-(\int p dx)}}{y_1^2} dx \right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y'' + p(x)\, y' + q(x)\, y = f(x)$$

Looking at the ode to solve shows that

$$p(x) = \frac{1-2x}{x}$$

Therefore

$$
\begin{aligned}
y_2(x) &= e^x \left(\int e^{-\left(\int \frac{1-2x}{x} dx\right)} e^{-2x} dx \right) \\
y_2(x) &= e^x \int \frac{e^{2x-\ln(x)}}{e^{2x}}, dx \\
y_2(x) &= e^x \left(\int \frac{1}{x} dx \right) \\
y_2(x) &= e^x \ln(x)
\end{aligned}
$$

Hence the solution is

$$
\begin{aligned}
y &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 e^x + c_2 e^x \ln(x)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 e^x + c_2 e^x \ln(x) \tag{1}
$$

Verification of solutions

$$
y = c_1 e^x + c_2 e^x \ln(x)
$$

Verified OK.

11.2.1 Maple step by step solution

Let's solve

$y''x + (1-2x)y' + y(x-1) = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x-1)y}{x} + \frac{(2x-1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(2x-1)y'}{x} + \frac{(x-1)y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

$\left[P_2(x) = -\frac{2x-1}{x}, P_3(x) = \frac{x-1}{x}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $$(x \cdot P_2(x))\Big|_{x=0} = 1$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$(x^2 \cdot P_3(x))\Big|_{x=0} = 0$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $$y''x + (1-2x)\, y' + y(x-1) = 0$$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k-m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

  $$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k- >k+1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r^2 x^{-1+r} + \left(a_1(1+r)^2 - a_0(1+2r)\right) x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+1+r)^2 - a_k(2k+2r+1) + a_{k-1}\right) x^{k}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r^2 = 0$
- Values of r that satisfy the indicial equation

  $r = 0$
- Each term must be 0

  $a_1(1+r)^2 - a_0(1+2r) = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1)^2 + (-2k-1)\, a_k + a_{k-1} = 0$
- Shift index using $k- >k+1$

  $a_{k+2}(k+2)^2 + (-2k-3)\, a_{k+1} + a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{2k a_{k+1} - a_k + 3a_{k+1}}{(k+2)^2}$
- Recursion relation for $r = 0$

  $a_{k+2} = \frac{2k a_{k+1} - a_k + 3a_{k+1}}{(k+2)^2}$
- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{2k a_{k+1} - a_k + 3a_{k+1}}{(k+2)^2}, a_1 - a_0 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve([x*diff(y(x),x$2)+(1-2*x)*diff(y(x),x)+(x-1)*y(x)=0,exp(x)],singsol=all)
```

$$y(x) = e^{x}(c_2 \ln(x) + c_1)$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 17

```
DSolve[x*y''[x]+(1-2*x)*y'[x]+(x-1)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{x}(c_2 \log(x) + c_1)$$

11.3 problem Problem 3

Internal problem ID [2814]
Internal file name [OUTPUT/2306_Sunday_June_05_2022_02_58_34_AM_54069979/index.tex]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.9, Reduction of Order. page 572
Problem number: Problem 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2 y'' - 2xy' + (x^2 + 2)\, y = 0}$$

Given that one solution of the ode is

$$y_1 = x \sin(x)$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1 \left(\int \frac{e^{-(\int p dx)}}{y_1^2} dx \right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y'' + p(x)\, y' + q(x)\, y = f(x)$$

Looking at the ode to solve shows that

$$p(x) = -\frac{2}{x}$$

Therefore

$$y_2(x) = x\sin(x)\left(\int \frac{e^{-\left(\int -\frac{2}{x}dx\right)}}{x^2\sin(x)^2}dx\right)$$

$$y_2(x) = x\sin(x)\int \frac{x^2}{\sin(x)^2 x^2}, dx$$

$$y_2(x) = x\sin(x)\left(\int \csc(x)^2\,dx\right)$$

$$y_2(x) = -x\sin(x)\cot(x)$$

Hence the solution is

$$\begin{aligned} y &= c_1y_1(x) + c_2y_2(x) \\ &= c_1x\sin(x) - c_2x\sin(x)\cot(x) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1x\sin(x) - c_2x\sin(x)\cot(x) \tag{1}$$

Verification of solutions

$$y = c_1x\sin(x) - c_2x\sin(x)\cot(x)$$

Verified OK.

11.3.1 Maple step by step solution

Let's solve

$x^2y'' - 2xy' + (x^2+2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x^2+2)y}{x^2} + \frac{2y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{2y'}{x} + \frac{(x^2+2)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

$\left[P_2(x) = -\frac{2}{x}, P_3(x) = \frac{x^2+2}{x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = -2$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = 2$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2y'' - 2xy' + (x^2 + 2)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- > k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k(k + r)\, x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k + r)\,(k + r - 1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1 + r)\,(-2 + r)\, x^r + a_1 r(-1 + r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k + r - 1)\,(k + r - 2) + a_{k-2})\, x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$(-1+r)(-2+r) = 0$

- Values of r that satisfy the indicial equation

$r \in \{1, 2\}$

- Each term must be 0

$a_1 r(-1+r) = 0$

- Solve for the dependent coefficient(s)

$a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

$a_k(k+r-1)(k+r-2) + a_{k-2} = 0$

- Shift index using $k- > k+2$

$a_{k+2}(k+1+r)(k+r) + a_k = 0$

- Recursion relation that defines series solution to ODE

$a_{k+2} = -\frac{a_k}{(k+1+r)(k+r)}$

- Recursion relation for $r = 1$

$a_{k+2} = -\frac{a_k}{(k+2)(k+1)}$

- Solution for $r = 1$

$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{a_k}{(k+2)(k+1)}, a_1 = 0\right]$

- Recursion relation for $r = 2$

$a_{k+2} = -\frac{a_k}{(k+3)(k+2)}$

- Solution for $r = 2$

$\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = -\frac{a_k}{(k+3)(k+2)}, a_1 = 0\right]$

- Combine solutions and rename parameters

$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{1+k}\right) + \left(\sum_{k=0}^{\infty} b_k x^{2+k}\right), a_{2+k} = -\frac{a_k}{(2+k)(1+k)}, a_1 = 0, b_{2+k} = -\frac{b_k}{(k+3)(2+k)}, b_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 15

```
dsolve([x^2*diff(y(x),x$2)-2*x*diff(y(x),x)+(x^2+2)*y(x)=0,x*sin(x)],singsol=all)
```

$$y(x) = x(c_1 \sin(x) + c_2 \cos(x))$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 33

```
DSolve[x^2*y''[x]-2*x*y'[x]+(x^2+2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{-ix} x - \frac{1}{2} i c_2 e^{ix} x$$

11.4 problem Problem 4

Internal problem ID [2815]
Internal file name [`OUTPUT/2307_Sunday_June_05_2022_02_58_35_AM_78314112/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.9, Reduction of Order. page 572
Problem number: Problem 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

`[_Gegenbauer]`

$$\boxed{(-x^2+1)\,y''-2xy'+2y=0}$$

Given that one solution of the ode is

$$y_1 = x$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1\left(\int \frac{\mathrm{e}^{-(\int p dx)}}{y_1^2}dx\right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y''+p(x)\,y'+q(x)\,y=f(x)$$

Looking at the ode to solve shows that

$$p(x) = -\frac{2x}{-x^2+1}$$

Therefore

$$y_2(x) = x\left(\int \frac{e^{-\left(\int -\frac{2x}{-x^2+1}dx\right)}}{x^2}dx\right)$$

$$y_2(x) = x\int \frac{e^{-\ln(x-1)-\ln(x+1)}}{x^2}, dx$$

$$y_2(x) = x\left(\int \frac{1}{x^2\,(x^2-1)}dx\right)$$

$$y_2(x) = x\left(-\frac{\ln(x+1)}{2}+\frac{1}{x}+\frac{\ln(x-1)}{2}\right)$$

Hence the solution is

$$\begin{aligned} y &= c_1y_1(x)+c_2y_2(x) \\ &= c_1x + c_2x\left(-\frac{\ln(x+1)}{2}+\frac{1}{x}+\frac{\ln(x-1)}{2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1x + c_2x\left(-\frac{\ln(x+1)}{2}+\frac{1}{x}+\frac{\ln(x-1)}{2}\right) \tag{1}$$

Verification of solutions

$$y = c_1x + c_2x\left(-\frac{\ln(x+1)}{2}+\frac{1}{x}+\frac{\ln(x-1)}{2}\right)$$

Verified OK.

11.4.1 Maple step by step solution

Let's solve

$(-x^2+1)\,y''-2xy'+2y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{2xy'}{x^2-1}+\frac{2y}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{2xy'}{x^2-1}-\frac{2y}{x^2-1}=0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{2x}{x^2-1}, P_3(x)=-\frac{2}{x^2-1}\right]$

  - $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

    $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1}=1$

  - $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1}=0$

  - $x=-1$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0=-1$

- Multiply by denominators

  $y''(x^2-1)+2xy'-2y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right)+(2u-2)\left(\frac{d}{du}y(u)\right)-2y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  - Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)(k+r-1)\,u^{k+r-2+m}$

- Shift index using $k- >k+2-m$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$$

  Rewrite ODE with series expansions

  $$-2a_0 r^2 u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2a_{k+1}(k+1+r)^2 + a_k(k+r+2)\,(k+r-1)\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r^2 = 0$

- Values of r that satisfy the indicial equation

  $r = 0$

- Each term in the series must be 0, giving the recursion relation

  $-2a_{k+1}(k+1)^2 + a_k(k+2)\,(k-1) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+2)(k-1)}{2(k+1)^2}$

- Recursion relation for $r = 0$; series terminates at $k = 1$

  $a_{k+1} = \frac{a_k(k+2)(k-1)}{2(k+1)^2}$

- Apply recursion relation for $k = 0$

  $a_1 = -a_0$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot (-u+1)$

- Revert the change of variables $u = x+1$

  $[y = -a_0 x]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 25

```
dsolve([(1-x^2)*diff(y(x),x$2)-2*x*diff(y(x),x)+2*y(x)=0,x],singsol=all)
```

$$y(x) = -\frac{c_2 \ln (x+1) x}{2} + \frac{c_2 \ln (x-1) x}{2} + c_1 x + c_2$$

✓ Solution by Mathematica
Time used: 0.021 (sec). Leaf size: 33

```
DSolve[(1-x^2)*y''[x]-2*x*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 x - \frac{1}{2} c_2 (x \log(1-x) - x \log(x+1) + 2)$$

11.5 problem Problem 5

Internal problem ID [2816]
Internal file name [`OUTPUT/2308_Sunday_June_05_2022_02_58_37_AM_97888430/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.9, Reduction of Order. page 572
Problem number: Problem 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "second_order_bessel_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler], [_2nd_order, _linear, `_with_symmetry_[0,F(
   x)]`]]
```

$$\boxed{y'' - \frac{y'}{x} + 4x^2 y = 0}$$

Given that one solution of the ode is

$$y_1 = \sin\left(x^2\right)$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1 \left(\int \frac{\mathrm{e}^{-\left(\int p dx\right)}}{y_1^2} dx \right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y'' + p(x)\, y' + q(x)\, y = f(x)$$

Looking at the ode to solve shows that

$$p(x) = -\frac{1}{x}$$

Therefore

$$y_2(x) = \sin\left(x^2\right)\left(\int \frac{\mathrm{e}^{-\left(\int -\frac{1}{x}dx\right)}}{\sin\left(x^2\right)^2}dx\right)$$

$$y_2(x) = \sin\left(x^2\right)\int \frac{x}{\sin\left(x^2\right)^2}, dx$$

$$y_2(x) = \sin\left(x^2\right)\left(\int \csc\left(x^2\right)^2 x dx\right)$$

$$y_2(x) = -\frac{\sin\left(x^2\right)\cot\left(x^2\right)}{2}$$

Hence the solution is

$$\begin{aligned} y &= c_1 y_1(x) + c_2 y_2(x) \\ &= \sin\left(x^2\right) c_1 - \frac{c_2 \sin\left(x^2\right)\cot\left(x^2\right)}{2} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \sin\left(x^2\right) c_1 - \frac{c_2 \sin\left(x^2\right)\cot\left(x^2\right)}{2} \tag{1}$$

Verification of solutions

$$y = \sin\left(x^2\right) c_1 - \frac{c_2 \sin\left(x^2\right)\cot\left(x^2\right)}{2}$$

Verified OK.

11.5.1 Maple step by step solution

Let's solve

$y'' - \frac{y'}{x} + 4x^2 y = 0$

- Highest derivative means the order of the ODE is 2

  y''

□ Check to see if $x_0 = 0$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = -\frac{1}{x}, P_3(x) = 4x^2\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = -1$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = 0$

- ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4yx^3 + y''x - y' = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $x^3 \cdot y$ to series expansion

  $x^3 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+3}$

- ◦ Shift index using $k- >k-3$

  $x^3 \cdot y = \sum_{k=3}^{\infty} a_{k-3} x^{k+r}$

- ◦ Convert y' to series expansion

$$y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$$

- Shift index using $k- >k+1$

  $$y' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

  $$x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k- >k+1$

  $$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(-2+r)\, x^{-1+r} + a_1(1+r)\,(-1+r)\, x^r + a_2(2+r)\, r\, x^{1+r} + a_3(3+r)\,(1+r)\, x^{2+r} + \left(\sum_{k=3}^{\infty} (a_k$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$

- The coefficients of each power of x must be 0

  $[a_1(1+r)\,(-1+r) = 0, a_2(2+r)\, r = 0, a_3(3+r)\,(1+r) = 0]$

- Solve for the dependent coefficient(s)

  $\{a_1 = 0, a_2 = 0, a_3 = 0\}$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)\,(k+r-1) + 4a_{k-3} = 0$

- Shift index using $k- >k+3$

  $a_{k+4}(k+4+r)\,(k+2+r) + 4a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+4} = -\frac{4a_k}{(k+4+r)(k+2+r)}$

- Recursion relation for $r = 0$

  $a_{k+4} = -\frac{4a_k}{(k+4)(k+2)}$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{4a_k}{(k+4)(k+2)}, a_1 = 0, a_2 = 0, a_3 = 0\right]$$

- Recursion relation for $r = 2$

$$a_{k+4} = -\frac{4a_k}{(k+6)(k+4)}$$

- Solution for $r = 2$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+4} = -\frac{4a_k}{(k+6)(k+4)}, a_1 = 0, a_2 = 0, a_3 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{2+k}\right), a_{k+4} = -\frac{4a_k}{(k+4)(2+k)}, a_1 = 0, a_2 = 0, a_3 = 0, b_{k+4} = -\frac{4b_k}{(k+6)(k+4)}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve([diff(y(x),x$2)-1/x*diff(y(x),x)+4*x^2*y(x)=0,sin(x^2)],singsol=all)
```

$$y(x) = c_1 \sin(x^2) + c_2 \cos(x^2)$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 20

```
DSolve[y''[x]-1/x*y'[x]+4*x^2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \cos(x^2) + c_2 \sin(x^2)$$

11.6 problem Problem 6

Internal problem ID [2817]
Internal file name [`OUTPUT/2309_Sunday_June_05_2022_02_58_39_AM_19257038/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.9, Reduction of Order. page 572
Problem number: Problem 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2y'' + 4xy' + (4x^2 - 1)\, y = 0}$$

Given that one solution of the ode is

$$y_1 = \frac{\sin(x)}{\sqrt{x}}$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1\left(\int \frac{e^{-(\int p dx)}}{y_1^2} dx\right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y'' + p(x)\, y' + q(x)\, y = f(x)$$

Looking at the ode to solve shows that

$$p(x) = \frac{1}{x}$$

Therefore

$$y_2(x) = \frac{\sin(x)\left(\int \frac{\mathrm{e}^{-\left(\int \frac{1}{x}dx\right)}x}{\sin(x)^2}dx\right)}{\sqrt{x}}$$

$$y_2(x) = \frac{\sin(x)}{\sqrt{x}}\int \frac{\frac{1}{x}}{\frac{\sin(x)^2}{x}}, dx$$

$$y_2(x) = \frac{\sin(x)\left(\int \csc(x)^2\, dx\right)}{\sqrt{x}}$$

$$y_2(x) = -\frac{\sin(x)\cot(x)}{\sqrt{x}}$$

Hence the solution is

$$\begin{aligned} y &= c_1 y_1(x) + c_2 y_2(x) \\ &= \frac{\sin(x)\, c_1}{\sqrt{x}} - \frac{c_2 \sin(x)\cot(x)}{\sqrt{x}} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{\sin(x)\, c_1}{\sqrt{x}} - \frac{c_2 \sin(x)\cot(x)}{\sqrt{x}} \tag{1}$$

Verification of solutions

$$y = \frac{\sin(x)\, c_1}{\sqrt{x}} - \frac{c_2 \sin(x)\cot(x)}{\sqrt{x}}$$

Verified OK.

11.6.1 Maple step by step solution

Let's solve

$4x^2y'' + 4xy' + (4x^2 - 1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(4x^2-1)y}{4x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{(4x^2-1)y}{4x^2} = 0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{4x^2-1}{4x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 1$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -\frac{1}{4}$
  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $4x^2y'' + 4xy' + (4x^2 - 1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r}$
  - Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r}$

Rewrite ODE with series expansions

$a_0(1+2r)(-1+2r)x^r + a_1(3+2r)(1+2r)x^{1+r} + \left(\sum_{k=2}^{\infty}\left(a_k(2k+2r+1)(2k+2r-1)+4a_{k-}\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)(-1+2r)=0$
- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$
- Each term must be 0

  $a_1(3+2r)(1+2r)=0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2+8kr+4r^2-1)+4a_{k-2}=0$
- Shift index using $k->k+2$

  $a_{k+2}\left(4(k+2)^2+8(k+2)r+4r^2-1\right)+4a_k=0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4a_k}{4k^2+8kr+4r^2+16k+16r+15}$
- Recursion relation for $r=-\frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+12k+8}$
- Solution for $r=-\frac{1}{2}$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1=0\right]$
- Recursion relation for $r=\frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+20k+24}$
- Solution for $r=\frac{1}{2}$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+20k+24}, a_1=0\right]$
- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{2+k} = -\frac{4a_k}{4k^2+12k+8}, a_1=0, b_{2+k} = -\frac{4b_k}{4k^2+20k+24}, b_1=0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 17

```
dsolve([4*x^2*diff(y(x),x$2)+4*x*diff(y(x),x)+(4*x^2-1)*y(x)=0,sin(x)/x^(1/2)],singsol=all)
```

$$y(x) = \frac{c_1 \sin(x) + c_2 \cos(x)}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.031 (sec). Leaf size: 39

```
DSolve[4*x^2*y''[x]+4*x*y'[x]+(4*x^2-1)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{e^{-ix}(2c_1 - ic_2 e^{2ix})}{2\sqrt{x}}$$

11.7 problem Problem 10

Internal problem ID [2818]
Internal file name [OUTPUT/2310_Sunday_June_05_2022_02_58_41_AM_27045713/index.tex]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.9, Reduction of Order. page 572
Problem number: Problem 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y = \csc(x)}$$

Given that one solution of the ode is

$$y_1 = \sin(x)$$

This is second order nonhomogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = \csc(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the inhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1 \left(\int \frac{e^{-\left(\int p dx\right)}}{y_1^2} dx \right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y'' + p(x)\, y' + q(x)\, y = f(x)$$

Looking at the ode to solve shows that

$$p(x) = 0$$

Therefore

$$y_2(x) = \sin(x) \left(\int \frac{e^{-\left(\int 0 dx\right)}}{\sin(x)^2} dx \right)$$

$$y_2(x) = \sin(x) \int \frac{1}{\sin(x)^2}, dx$$

$$y_2(x) = \sin(x) \left(\int \csc(x)^2 dx \right)$$

$$y_2(x) = -\cot(x) \sin(x)$$

Hence the solution is

$$\begin{aligned} y &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 \sin(x) - c_2 \cot(x) \sin(x) \end{aligned}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \sin(x) - c_2 \cot(x) \sin(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \sin(x) \\
y_2 &= -\cot(x)\sin(x)
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$
W = \begin{vmatrix} \sin(x) & -\cot(x)\sin(x) \\ \frac{d}{dx}(\sin(x)) & \frac{d}{dx}(-\cot(x)\sin(x)) \end{vmatrix}
$$

Which gives

$$
W = \begin{vmatrix} \sin(x) & -\cot(x)\sin(x) \\ \cos(x) & -(-1-\cot(x)^2)\sin(x) - \cos(x)\cot(x) \end{vmatrix}
$$

Therefore

$$
W = (\sin(x))\left(-(-1-\cot(x)^2)\sin(x) - \cos(x)\cot(x)\right) - (-\cot(x)\sin(x))(\cos(x))
$$

Which simplifies to

$$
W = \cot(x)^2\sin(x)^2 + \sin(x)^2
$$

Which simplifies to

$$
W = 1
$$

Therefore Eq. (2) becomes

$$
u_1 = -\int \frac{-\cot(x)\sin(x)\csc(x)}{1}\,dx
$$

Which simplifies to

$$u_1 = -\int -\cot (x)\, dx$$

Hence

$$u_1 = \ln (\sin (x))$$

And Eq. (3) becomes

$$u_2 = \int \frac{\sin (x) \csc (x)}{1}\, dx$$

Which simplifies to

$$u_2 = \int 1 dx$$

Hence

$$u_2 = x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \sin (x) \ln (\sin (x)) - x \sin (x) \cot (x)$$

Which simplifies to

$$y_p(x) = \sin (x) \ln (\sin (x)) - \cos (x)\, x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \sin (x) - c_2 \cot (x) \sin (x)) + (\sin (x) \ln (\sin (x)) - \cos (x)\, x) \end{aligned}$$

Which simplifies to

$$y = -c_2 \cos (x) + c_1 \sin (x) + \sin (x) \ln (\sin (x)) - \cos (x)\, x$$

Summary

The solution(s) found are the following

$$y = -c_2 \cos (x) + c_1 \sin (x) + \sin (x) \ln (\sin (x)) - \cos (x)\, x \tag{1}$$

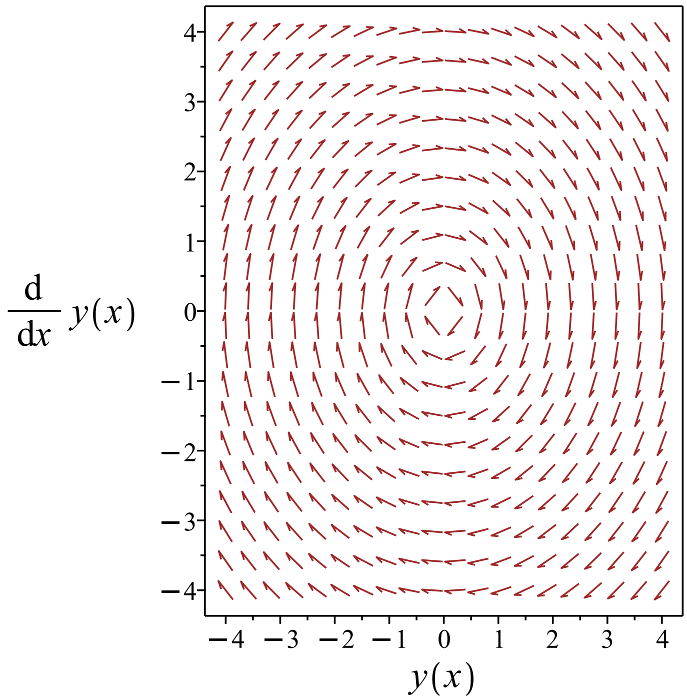


Figure 317: Slope field plot

Verification of solutions

$$y = -c_2 \cos(x) + c_1 \sin(x) + \sin(x) \ln(\sin(x)) - \cos(x) x$$

Verified OK.

11.7.1 Maple step by step solution

Let's solve

$y'' + y = \csc(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

$r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \csc(x)\right]$

- ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$

- ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = 1$

- ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\cos(x) \left(\int 1 dx\right) + \sin(x) \left(\int \cot(x) dx\right)$

- ◦ Compute integrals

  $y_p(x) = \sin(x) \ln(\sin(x)) - \cos(x) x$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + \sin(x) \ln(\sin(x)) - \cos(x) x$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 24

```
dsolve([diff(y(x),x$2)+y(x)=csc(x),sin(x)],singsol=all)
```

$$y(x) = -\ln(\csc(x))\sin(x) + (c_1 - x)\cos(x) + \sin(x)\, c_2$$

✓ Solution by Mathematica
Time used: 0.023 (sec). Leaf size: 24

```
DSolve[y''[x]+y[x]==Csc[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to (-x + c_1)\cos(x) + \sin(x)(\log(\sin(x)) + c_2)$$

11.8 problem Problem 11

Internal problem ID [2819]
Internal file name [`OUTPUT/2311_Sunday_June_05_2022_02_58_43_AM_14688829/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.9, Reduction of Order. page 572
Problem number: Problem 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{xy'' - (1+2x)\,y' + 2y = 8x^2 e^{2x}}$$

Given that one solution of the ode is

$$y_1 = e^{2x}$$

This is second order nonhomogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x, B = -1 - 2x, C = 2, f(x) = 8x^2 e^{2x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the inhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$xy'' + (-1 - 2x)\,y' + 2y = 0$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1 \left(\int \frac{e^{-\left(\int p dx\right)}}{y_1^2} dx \right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y'' + p(x)\,y' + q(x)\,y = f(x)$$

Looking at the ode to solve shows that

$$p(x) = \frac{-1-2x}{x}$$

Therefore

$$\begin{aligned}
y_2(x) &= \mathrm{e}^{2x}\left(\int \mathrm{e}^{-\left(\int \frac{-1-2x}{x}dx\right)}\mathrm{e}^{-4x}dx\right)\\
y_2(x) &= \mathrm{e}^{2x}\int \frac{\mathrm{e}^{2x+\ln(x)}}{\mathrm{e}^{4x}}, dx\\
y_2(x) &= \mathrm{e}^{2x}\left(\int x\,\mathrm{e}^{-2x}dx\right)\\
y_2(x) &= -\frac{\mathrm{e}^{2x}(1+2x)\,\mathrm{e}^{-2x}}{4}
\end{aligned}$$

Hence the solution is

$$\begin{aligned}
y &= c_1y_1(x) + c_2y_2(x)\\
&= \mathrm{e}^{2x}c_1 - \frac{c_2\mathrm{e}^{2x}(1+2x)\,\mathrm{e}^{-2x}}{4}
\end{aligned}$$

Therefore the homogeneous solution y_h is

$$y_h = \mathrm{e}^{2x}c_1 - \frac{c_2\mathrm{e}^{2x}(1+2x)\,\mathrm{e}^{-2x}}{4}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}
y_1 &= \mathrm{e}^{2x}\\
y_2 &= -\frac{\mathrm{e}^{2x}(1+2x)\,\mathrm{e}^{-2x}}{4}
\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{2x} & -\frac{e^{2x}(1+2x)e^{-2x}}{4} \\ \frac{d}{dx}(e^{2x}) & \frac{d}{dx}\left(-\frac{e^{2x}(1+2x)e^{-2x}}{4}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{2x} & -\frac{e^{2x}(1+2x)e^{-2x}}{4} \\ 2\,e^{2x} & -\frac{e^{2x}e^{-2x}}{2} \end{vmatrix}$$

Therefore

$$W = \left(e^{2x}\right)\left(-\frac{e^{2x}e^{-2x}}{2}\right) - \left(-\frac{e^{2x}(1+2x)\,e^{-2x}}{4}\right)\left(2\,e^{2x}\right)$$

Which simplifies to

$$W = e^{4x}e^{-2x}x$$

Which simplifies to

$$W = x\,e^{2x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-2\,e^{4x}(1+2x)\,e^{-2x}x^2}{x^2 e^{2x}}\,dx$$

Which simplifies to

$$u_1 = -\int (-2-4x)\,dx$$

Hence

$$u_1 = 2x^2 + 2x$$

And Eq. (3) becomes

$$u_2 = \int \frac{8x^2 e^{4x}}{x^2 e^{2x}}\,dx$$

Which simplifies to

$$u_2 = \int 8\,e^{2x} dx$$

Hence

$$u_2 = 4\,e^{2x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(2x^2 + 2x\right) e^{2x} - e^{4x}(1 + 2x)\, e^{-2x}$$

Which simplifies to

$$y_p(x) = e^{2x}\left(2x^2 - 1\right)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{2x} c_1 - \frac{c_2 e^{2x}(1 + 2x)\, e^{-2x}}{4}\right) + \left(e^{2x}\left(2x^2 - 1\right)\right) \end{aligned}$$

Which simplifies to

$$y = e^{2x} c_1 + \frac{(-1 - 2x)\, c_2}{4} + e^{2x}\left(2x^2 - 1\right)$$

Summary

The solution(s) found are the following

$$y = e^{2x} c_1 + \frac{(-1 - 2x)\, c_2}{4} + e^{2x}\left(2x^2 - 1\right) \tag{1}$$

Verification of solutions

$$y = e^{2x} c_1 + \frac{(-1 - 2x)\, c_2}{4} + e^{2x}\left(2x^2 - 1\right)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Reducible group (found another exponential solution)
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 25

```
dsolve([x*diff(y(x),x$2)-(2*x+1)*diff(y(x),x)+2*y(x)=8*x^2*exp(2*x),exp(2*x)],singsol=all)
```

$$y(x) = 2\,\mathrm{e}^{2x}x^2 + c_1\mathrm{e}^{2x} + 2c_2x + c_2$$

✓ Solution by Mathematica

Time used: 0.044 (sec). Leaf size: 32

```
DSolve[x*y''[x]-(2*x+1)*y'[x]+2*y[x]==8*x^2*Exp[2*x],y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to e^{2x}\left(2x^2 - 1 + c_1\right) - \frac{1}{4}c_2(2x+1)$$

11.9 problem Problem 12

Internal problem ID [2820]
Internal file name [`OUTPUT/2312_Sunday_June_05_2022_02_58_45_AM_2793161/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.9, Reduction of Order. page 572
Problem number: Problem 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - 3xy' + 4y = 8x^4}$$

Given that one solution of the ode is

$$y_1 = x^2$$

This is second order nonhomogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -3x, C = 4, f(x) = 8x^4$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the inhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' - 3xy' + 4y = 0$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1 \left(\int \frac{e^{-\left(\int p dx\right)}}{y_1^2} dx \right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y'' + p(x)\, y' + q(x)\, y = f(x)$$

Looking at the ode to solve shows that

$$p(x) = -\frac{3}{x}$$

Therefore

$$y_2(x) = x^2 \left(\int \frac{e^{-\left(\int -\frac{3}{x} dx\right)}}{x^4} dx \right)$$

$$y_2(x) = x^2 \int \frac{x^3}{x^4}, dx$$

$$y_2(x) = x^2 \left(\int \frac{1}{x} dx \right)$$

$$y_2(x) = \ln(x)\, x^2$$

Hence the solution is

$$\begin{aligned} y &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^2 + c_2 x^2 \ln(x) \end{aligned}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 x^2 + c_2 x^2 \ln(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x^2$$
$$y_2 = \ln(x)\, x^2$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^2 & \ln(x)\, x^2 \\ \frac{d}{dx}(x^2) & \frac{d}{dx}(\ln(x)\, x^2) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^2 & \ln(x)\, x^2 \\ 2x & x + 2\ln(x)\, x \end{vmatrix}$$

Therefore

$$W = (x^2)(x + 2\ln(x)\, x) - (\ln(x)\, x^2)(2x)$$

Which simplifies to

$$W = x^3$$

Which simplifies to

$$W = x^3$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{8\ln(x)\, x^6}{x^5}\, dx$$

Which simplifies to

$$u_1 = -\int 8\ln(x)\, x dx$$

Hence

$$u_1 = -4\ln(x)\, x^2 + 2x^2$$

And Eq. (3) becomes

$$u_2 = \int \frac{8x^6}{x^5}\, dx$$

Which simplifies to

$$u_2 = \int 8x dx$$

Hence

$$u_2 = 4x^2$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-4\ln(x)\, x^2 + 2x^2\right) x^2 + 4x^4 \ln(x)$$

Which simplifies to

$$y_p(x) = 2x^4$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 x^2 + c_2 x^2 \ln(x)\right) + \left(2x^4\right) \end{aligned}$$

Which simplifies to

$$y = x^2(c_1 + c_2 \ln(x)) + 2x^4$$

Summary

The solution(s) found are the following

$$y = x^2(c_1 + c_2 \ln(x)) + 2x^4 \tag{1}$$

Verification of solutions

$$y = x^2(c_1 + c_2 \ln(x)) + 2x^4$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   <- LODE of Euler type successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve([x^2*diff(y(x),x$2)-3*x*diff(y(x),x)+4*y(x)=8*x^4,x^2],singsol=all)
```

$$y(x) = x^2(\ln(x)\,c_1 + 2x^2 + c_2)$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 23

```
DSolve[x^2*y''[x]-3*x*y'[x]+4*y[x]==8*x^4,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^2(2x^2 + 2c_2\log(x) + c_1)$$

11.10 problem Problem 13

Internal problem ID [2821]
Internal file name [OUTPUT/2313_Sunday_June_05_2022_02_58_47_AM_39415246/index.tex]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.9, Reduction of Order. page 572
Problem number: Problem 13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 6y' + 9y = 15\,\mathrm{e}^{3x}\sqrt{x}}$$

Given that one solution of the ode is

$$y_1 = \mathrm{e}^{3x}$$

This is second order nonhomogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -6, C = 9, f(x) = 15\,\mathrm{e}^{3x}\sqrt{x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the inhomogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 6y' + 9y = 0$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1 \left(\int \frac{e^{-\left(\int p dx\right)}}{y_1^2} dx \right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y'' + p(x) y' + q(x) y = f(x)$$

Looking at the ode to solve shows that

$$p(x) = -6$$

Therefore

$$\begin{aligned}
y_2(x) &= e^{3x} \left(\int e^{-\left(\int (-6) dx\right)} e^{-6x} dx \right) \\
y_2(x) &= e^{3x} \int \frac{e^{6x}}{e^{6x}}, dx \\
y_2(x) &= e^{3x} \left(\int 1 dx \right) \\
y_2(x) &= x\, e^{3x}
\end{aligned}$$

Hence the solution is

$$\begin{aligned}
y &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 e^{3x} + c_2 x\, e^{3x}
\end{aligned}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{3x} + c_2 x\, e^{3x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \mathrm{e}^{3x}$$
$$y_2 = x\,\mathrm{e}^{3x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{3x} & x\,\mathrm{e}^{3x} \\ \frac{d}{dx}(\mathrm{e}^{3x}) & \frac{d}{dx}(x\,\mathrm{e}^{3x}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{3x} & x\,\mathrm{e}^{3x} \\ 3\,\mathrm{e}^{3x} & \mathrm{e}^{3x} + 3x\,\mathrm{e}^{3x} \end{vmatrix}$$

Therefore

$$W = \left(\mathrm{e}^{3x}\right)\left(\mathrm{e}^{3x} + 3x\,\mathrm{e}^{3x}\right) - \left(x\,\mathrm{e}^{3x}\right)\left(3\,\mathrm{e}^{3x}\right)$$

Which simplifies to

$$W = \mathrm{e}^{6x}$$

Which simplifies to

$$W = \mathrm{e}^{6x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{15x^{\frac{3}{2}}\mathrm{e}^{6x}}{\mathrm{e}^{6x}}\,dx$$

Which simplifies to

$$u_1 = -\int 15x^{\frac{3}{2}} dx$$

Hence

$$u_1 = -6x^{\frac{5}{2}}$$

And Eq. (3) becomes

$$u_2 = \int \frac{15\,\mathrm{e}^{6x}\sqrt{x}}{\mathrm{e}^{6x}} dx$$

Which simplifies to

$$u_2 = \int 15\sqrt{x} dx$$

Hence

$$u_2 = 10x^{\frac{3}{2}}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = 4x^{\frac{5}{2}}\mathrm{e}^{3x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{3x} + c_2 x\,\mathrm{e}^{3x}\right) + \left(4x^{\frac{5}{2}}\mathrm{e}^{3x}\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{3x}(c_2 x + c_1) + 4x^{\frac{5}{2}}\mathrm{e}^{3x}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{3x}(c_2 x + c_1) + 4x^{\frac{5}{2}}\mathrm{e}^{3x} \tag{1}$$

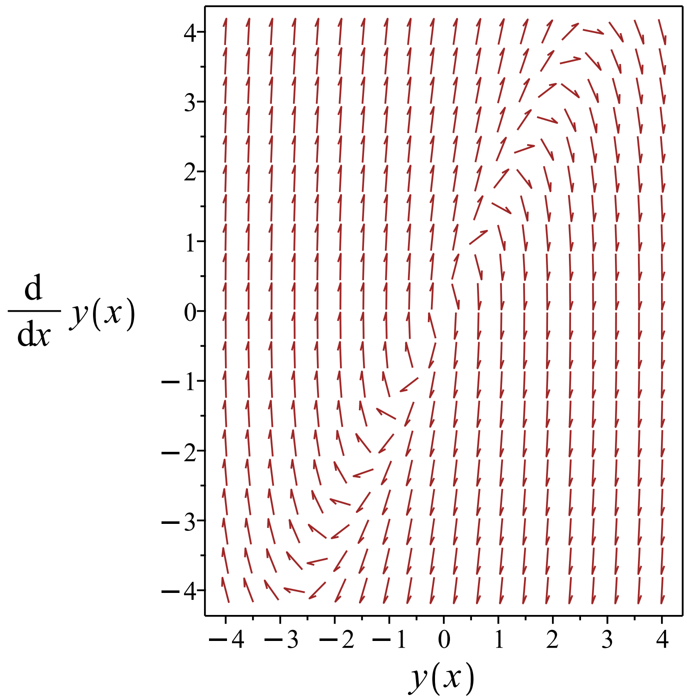


Figure 318: Slope field plot

Verification of solutions

$$y = e^{3x}(c_2x + c_1) + 4x^{\frac{5}{2}} e^{3x}$$

Verified OK.

11.10.1 Maple step by step solution

Let's solve

$y'' - 6y' + 9y = 15\, e^{3x} \sqrt{x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 6r + 9 = 0$

- Factor the characteristic polynomial

  $(r-3)^2 = 0$

- Root of the characteristic polynomial

$r = 3$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{3x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,\mathrm{e}^{3x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{3x} + c_2 x\,\mathrm{e}^{3x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 15\,\mathrm{e}^{3x}\sqrt{x}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{3x} & x\,\mathrm{e}^{3x} \\ 3\,\mathrm{e}^{3x} & \mathrm{e}^{3x} + 3x\,\mathrm{e}^{3x} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = \mathrm{e}^{6x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -15\,\mathrm{e}^{3x}\left(-\left(\int \sqrt{x}dx\right)x + \int x^{\frac{3}{2}}dx\right)$

  - Compute integrals

    $y_p(x) = 4x^{\frac{5}{2}}\mathrm{e}^{3x}$

- Substitute particular solution into general solution to ODE

  $y = 4x^{\frac{5}{2}}\mathrm{e}^{3x} + c_2 x\,\mathrm{e}^{3x} + c_1 \mathrm{e}^{3x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve([diff(y(x),x$2)-6*diff(y(x),x)+9*y(x)=15*exp(3*x)*sqrt(x),exp(3*x)],singsol=all)
```

$$y(x) = e^{3x}\left(c_2 + c_1x + 4x^{\frac{5}{2}}\right)$$

✓ Solution by Mathematica

Time used: 0.026 (sec). Leaf size: 25

```
DSolve[y''[x]-6*y'[x]+9*y[x]==15*Exp[3*x]*Sqrt[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{3x}\left(4x^{5/2} + c_2x + c_1\right)$$

11.11 problem Problem 14

Internal problem ID [2822]
Internal file name [`OUTPUT/2314_Sunday_June_05_2022_02_58_49_AM_58434158/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.9, Reduction of Order. page 572
Problem number: Problem 14.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 4y' + 4y = 4\,\mathrm{e}^{2x} \ln (x)}$$

Given that one solution of the ode is

$$y_1 = \mathrm{e}^{2x}$$

This is second order nonhomogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -4, C = 4, f(x) = 4\,\mathrm{e}^{2x} \ln (x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the inhomogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 4y' + 4y = 0$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1 \left(\int \frac{e^{-(\int p dx)}}{y_1^2} dx \right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y'' + p(x) y' + q(x) y = f(x)$$

Looking at the ode to solve shows that

$$p(x) = -4$$

Therefore

$$y_2(x) = e^{2x} \left(\int e^{-(\int (-4)dx)} e^{-4x} dx \right)$$

$$y_2(x) = e^{2x} \int \frac{e^{4x}}{e^{4x}}, dx$$

$$y_2(x) = e^{2x} \left(\int 1 dx \right)$$

$$y_2(x) = x e^{2x}$$

Hence the solution is

$$\begin{aligned} y &= c_1 y_1(x) + c_2 y_2(x) \\ &= e^{2x} c_1 + x e^{2x} c_2 \end{aligned}$$

Therefore the homogeneous solution y_h is

$$y_h = e^{2x} c_1 + x e^{2x} c_2$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \mathrm{e}^{2x} \\
y_2 &= x\,\mathrm{e}^{2x}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$
W = \begin{vmatrix} \mathrm{e}^{2x} & x\,\mathrm{e}^{2x} \\ \frac{d}{dx}\left(\mathrm{e}^{2x}\right) & \frac{d}{dx}\left(x\,\mathrm{e}^{2x}\right) \end{vmatrix}
$$

Which gives

$$
W = \begin{vmatrix} \mathrm{e}^{2x} & x\,\mathrm{e}^{2x} \\ 2\,\mathrm{e}^{2x} & \mathrm{e}^{2x} + 2x\,\mathrm{e}^{2x} \end{vmatrix}
$$

Therefore

$$
W = \left(\mathrm{e}^{2x}\right)\left(\mathrm{e}^{2x} + 2x\,\mathrm{e}^{2x}\right) - \left(x\,\mathrm{e}^{2x}\right)\left(2\,\mathrm{e}^{2x}\right)
$$

Which simplifies to

$$
W = \mathrm{e}^{4x}
$$

Which simplifies to

$$
W = \mathrm{e}^{4x}
$$

Therefore Eq. (2) becomes

$$
u_1 = -\int \frac{4x\,\mathrm{e}^{4x} \ln(x)}{\mathrm{e}^{4x}}\,dx
$$

Which simplifies to

$$u_1 = -\int 4 \ln(x)\, x dx$$

Hence

$$u_1 = -2 \ln(x)\, x^2 + x^2$$

And Eq. (3) becomes

$$u_2 = \int \frac{4\, e^{4x} \ln(x)}{e^{4x}} dx$$

Which simplifies to

$$u_2 = \int 4 \ln(x)\, dx$$

Hence

$$u_2 = 4 \ln(x)\, x - 4x$$

Which simplifies to

$$\begin{aligned} u_1 &= x^2(1 - 2\ln(x)) \\ u_2 &= 4\ln(x)\, x - 4x \end{aligned}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = x^2(1 - 2\ln(x))\, e^{2x} + (4\ln(x)\, x - 4x)\, x\, e^{2x}$$

Which simplifies to

$$y_p(x) = x^2 e^{2x}(2\ln(x) - 3)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{2x} c_1 + x\, e^{2x} c_2\right) + \left(x^2 e^{2x}(2\ln(x) - 3)\right) \end{aligned}$$

Which simplifies to

$$y = e^{2x}(c_2 x + c_1) + x^2 e^{2x}(2\ln(x) - 3)$$

Summary

The solution(s) found are the following

$$y = e^{2x}(c_2 x + c_1) + x^2 e^{2x}(2\ln(x) - 3) \tag{1}$$

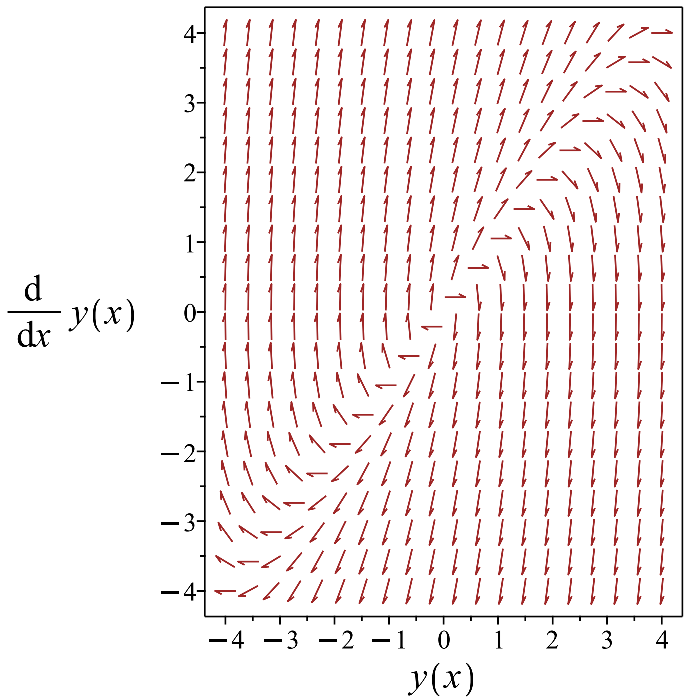


Figure 319: Slope field plot

Verification of solutions

$$y = e^{2x}(c_2 x + c_1) + x^2 e^{2x}(2\ln(x) - 3)$$

Verified OK.

11.11.1 Maple step by step solution

Let's solve

$y'' - 4y' + 4y = 4\,e^{2x}\ln(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 4r + 4 = 0$

- Factor the characteristic polynomial

  $(r-2)^2 = 0$
- Root of the characteristic polynomial

  $r = 2$
- 1st solution of the homogeneous ODE

  $y_1(x) = e^{2x}$
- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\, e^{2x}$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE

  $y = e^{2x} c_1 + x\, e^{2x} c_2 + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 4\, e^{2x} \ln(x)\right]$
  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{2x} & x\, e^{2x} \\ 2\, e^{2x} & e^{2x} + 2x\, e^{2x} \end{bmatrix}$
  - Compute Wronskian

    $W(y_1(x), y_2(x)) = e^{4x}$
  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -4\, e^{2x}\left(\int \ln(x)\, x dx - \left(\int \ln(x)\, dx\right) x\right)$
  - Compute integrals

    $y_p(x) = x^2 e^{2x}(2\ln(x) - 3)$
- Substitute particular solution into general solution to ODE

  $y = e^{2x} c_1 + x\, e^{2x} c_2 + x^2 e^{2x}(2\ln(x) - 3)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve([diff(y(x),x$2)-4*diff(y(x),x)+4*y(x)=4*exp(2*x)*ln(x),exp(2*x)],singsol=all)
```

$$y(x) = e^{2x}\left(2\ln(x)\,x^2 + c_1x - 3x^2 + c_2\right)$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 30

```
DSolve[y''[x]-4*y'[x]+4*y[x]==4*Exp[2*x]*Log[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{2x}\left(-3x^2 + 2x^2\log(x) + c_2x + c_1\right)$$

11.12 problem Problem 15

Internal problem ID [2823]
Internal file name [`OUTPUT/2315_Sunday_June_05_2022_02_58_51_AM_49321136/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.9, Reduction of Order. page 572
Problem number: Problem 15.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "second_order_euler_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{4x^2y'' + y = \sqrt{x}\,\ln(x)}$$

Given that one solution of the ode is

$$y_1 = \sqrt{x}$$

This is second order nonhomogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 4x^2, B = 0, C = 1, f(x) = \sqrt{x}\,\ln(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the inhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$4x^2y'' + y = 0$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1\left(\int \frac{e^{-(\int p dx)}}{y_1^2} dx\right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y'' + p(x)\,y' + q(x)\,y = f(x)$$

Looking at the ode to solve shows that

$$p(x) = 0$$

Therefore

$$\begin{aligned}
y_2(x) &= \sqrt{x}\left(\int \frac{\mathrm{e}^{-\left(\int 0dx\right)}}{x}dx\right)\\
y_2(x) &= \sqrt{x}\int \frac{1}{x}, dx\\
y_2(x) &= \sqrt{x}\left(\int \frac{1}{x}dx\right)\\
y_2(x) &= \sqrt{x}\,\ln\left(x\right)
\end{aligned}$$

Hence the solution is

$$\begin{aligned}
y &= c_1 y_1(x) + c_2 y_2(x)\\
&= c_1\sqrt{x} + c_2\sqrt{x}\,\ln\left(x\right)
\end{aligned}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1\sqrt{x} + c_2\sqrt{x}\,\ln\left(x\right)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}
y_1 &= \sqrt{x}\\
y_2 &= \sqrt{x}\,\ln\left(x\right)
\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \sqrt{x} & \sqrt{x}\,\ln(x) \\ \frac{d}{dx}(\sqrt{x}) & \frac{d}{dx}(\sqrt{x}\,\ln(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \sqrt{x} & \sqrt{x}\,\ln(x) \\ \frac{1}{2\sqrt{x}} & \frac{\ln(x)}{2\sqrt{x}} + \frac{1}{\sqrt{x}} \end{vmatrix}$$

Therefore

$$W = (\sqrt{x})\left(\frac{\ln(x)}{2\sqrt{x}} + \frac{1}{\sqrt{x}}\right) - (\sqrt{x}\,\ln(x))\left(\frac{1}{2\sqrt{x}}\right)$$

Which simplifies to

$$W = 1$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x \ln(x)^2}{4x^2}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{\ln(x)^2}{4x} dx$$

Hence

$$u_1 = -\frac{\ln(x)^3}{12}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\ln(x)\, x}{4x^2}\, dx$$

Which simplifies to

$$u_2 = \int \frac{\ln(x)}{4x} dx$$

Hence

$$u_2 = \frac{\ln(x)^2}{8}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\ln(x)^3 \sqrt{x}}{24}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\sqrt{x} + c_2\sqrt{x}\, \ln(x)\right) + \left(\frac{\ln(x)^3 \sqrt{x}}{24}\right) \end{aligned}$$

Which simplifies to

$$y = (c_1 + c_2 \ln(x)) \sqrt{x} + \frac{\ln(x)^3 \sqrt{x}}{24}$$

Summary

The solution(s) found are the following

$$y = (c_1 + c_2 \ln(x)) \sqrt{x} + \frac{\ln(x)^3 \sqrt{x}}{24} \tag{1}$$

Verification of solutions

$$y = (c_1 + c_2 \ln(x)) \sqrt{x} + \frac{\ln(x)^3 \sqrt{x}}{24}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   <- LODE of Euler type successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve([4*x^2*diff(y(x),x$2)+y(x)=sqrt(x)*ln(x),sqrt(x)],singsol=all)
```

$$y(x) = \left(c_2 + \ln(x)\, c_1 + \frac{\ln(x)^3}{24}\right)\sqrt{x}$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 29

```
DSolve[4*x^2*y''[x]+y[x]==Sqrt[x]*Log[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{24}\sqrt{x}\left(\log^3(x) + 12c_2\log(x) + 24c_1\right)$$

12 Chapter 8, Linear differential equations of order n. Section 8.10, Chapter review. page 575

12.1 problem Problem 7

Internal problem ID [2824]
Internal file name [`OUTPUT/2316_Sunday_June_05_2022_02_58_53_AM_69895820/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.10, Chapter review. page 575
Problem number: Problem 7.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' + 3y'' - 4y = 0}$$

The characteristic equation is

$$\lambda^3 + 3\lambda^2 - 4 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = -2$$
$$\lambda_3 = -2$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-2x} + c_2 x e^{-2x} + c_3 e^x$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-2x}$$
$$y_2 = x e^{-2x}$$
$$y_3 = e^x$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-2x} + c_2 x e^{-2x} + c_3 e^{x} \tag{1}$$

Verification of solutions

$$y = c_1 e^{-2x} + c_2 x e^{-2x} + c_3 e^{x}$$

Verified OK.

12.1.1 Maple step by step solution

Let's solve

$y''' + 3y'' - 4y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

□ Convert linear ODE into a system of first order ODEs

  - Define new variable $y_1(x)$

    $y_1(x) = y$

  - Define new variable $y_2(x)$

    $y_2(x) = y'$

  - Define new variable $y_3(x)$

    $y_3(x) = y''$

  - Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = -3y_3(x) + 4y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = -3y_3(x) + 4y_1(x)]$

- Define vector

  $\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$

- System to solve

$$\overrightarrow{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & 0 & -3 \end{bmatrix} \cdot \overrightarrow{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & 0 & -3 \end{bmatrix}$$

- Rewrite the system as

$$\overrightarrow{y}'(x) = A \cdot \overrightarrow{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right], \left[-2, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[-2, \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue -2

$$\overrightarrow{y}_1(x) = e^{-2x} \cdot \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\overrightarrow{p}$ is to be solved for, $\lambda = -2$ is the eigenvalue, a

$$\overrightarrow{y}_2(x) = e^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right)$$

- Note that the x multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obt
- Substitute $\overrightarrow{y}_2(x)$ into the homogeneous system

$$\lambda e^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right) + e^{\lambda x}\overrightarrow{v} = \left(e^{\lambda x} A\right) \cdot \left(x\overrightarrow{v} + \overrightarrow{p}\right)$$

- Use the fact that $\overrightarrow{v}$ is an eigenvector of A

  $\lambda\,\mathrm{e}^{\lambda x}\left(x\overrightarrow{v}+\overrightarrow{p}\right)+\mathrm{e}^{\lambda x}\overrightarrow{v}=\mathrm{e}^{\lambda x}\left(\lambda x\overrightarrow{v}+A\cdot\overrightarrow{p}\right)$

- Simplify equation

  $\lambda\overrightarrow{p}+\overrightarrow{v}=A\cdot\overrightarrow{p}$

- Make use of the identity matrix I

  $(\lambda\cdot I)\cdot\overrightarrow{p}+\overrightarrow{v}=A\cdot\overrightarrow{p}$

- Condition $\overrightarrow{p}$ must meet for $\overrightarrow{y}_2(x)$ to be a solution to the homogeneous system

  $(A-\lambda\cdot I)\cdot\overrightarrow{p}=\overrightarrow{v}$

- Choose $\overrightarrow{p}$ to use in the second solution to the homogeneous system from eigenvalue -2

  $$\left(\left[\begin{array}{ccc}0 & 1 & 0\\ 0 & 0 & 1\\ 4 & 0 & -3\end{array}\right]-(-2)\cdot\left[\begin{array}{ccc}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{array}\right]\right)\cdot\overrightarrow{p}=\left[\begin{array}{c}\frac{1}{4}\\ -\frac{1}{2}\\ 1\end{array}\right]$$

- Choice of $\overrightarrow{p}$

  $$\overrightarrow{p}=\left[\begin{array}{c}\frac{1}{8}\\ 0\\ 0\end{array}\right]$$

- Second solution from eigenvalue -2

  $$\overrightarrow{y}_2(x)=\mathrm{e}^{-2x}\cdot\left(x\cdot\left[\begin{array}{c}\frac{1}{4}\\ -\frac{1}{2}\\ 1\end{array}\right]+\left[\begin{array}{c}\frac{1}{8}\\ 0\\ 0\end{array}\right]\right)$$

- Consider eigenpair

  $$\left[1,\left[\begin{array}{c}1\\ 1\\ 1\end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

  $$\overrightarrow{y}_3=\mathrm{e}^{x}\cdot\left[\begin{array}{c}1\\ 1\\ 1\end{array}\right]$$

- General solution to the system of ODEs
  $\overrightarrow{y} = c_1 \overrightarrow{y}_1(x) + c_2 \overrightarrow{y}_2(x) + c_3 \overrightarrow{y}_3$
- Substitute solutions into the general solution
  $$\overrightarrow{y} = c_1 \mathrm{e}^{-2x} \cdot \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{-2x} \cdot \left(x \cdot \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{8} \\ 0 \\ 0 \end{bmatrix} \right) + c_3 \mathrm{e}^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$
- First component of the vector is the solution to the ODE
  $y = \frac{(8\,\mathrm{e}^{3x} c_3 + 2c_2 x + 2c_1 + c_2)\,\mathrm{e}^{-2x}}{8}$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 20

```
dsolve(diff(y(x),x$3)+3*diff(y(x),x$2)-4*y(x)=0,y(x), singsol=all)
```

$$y(x) = \left(c_1 \mathrm{e}^{3x} + c_3 x + c_2\right) \mathrm{e}^{-2x}$$

✓ Solution by Mathematica
Time used: 0.003 (sec). Leaf size: 26

```
DSolve[y'''[x]+3*y''[x]-4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-2x}\left(c_2 x + c_3 e^{3x} + c_1\right)$$

12.2 problem Problem 8

Internal problem ID [2825]
Internal file name [`OUTPUT/2317_Sunday_June_05_2022_02_58_54_AM_85631151/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.10, Chapter review. page 575
Problem number: Problem 8.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' + 11y'' + 36y' + 26y = 0}$$

The characteristic equation is

$$\lambda^3 + 11\lambda^2 + 36\lambda + 26 = 0$$

The roots of the above equation are

$$\lambda_1 = -1$$
$$\lambda_2 = -5 - i$$
$$\lambda_3 = -5 + i$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-x} + \mathrm{e}^{(-5-i)x} c_2 + \mathrm{e}^{(-5+i)x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = \mathrm{e}^{-x}$$
$$y_2 = \mathrm{e}^{(-5-i)x}$$
$$y_3 = \mathrm{e}^{(-5+i)x}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-x} + e^{(-5-i)x} c_2 + e^{(-5+i)x} c_3 \tag{1}$$

Verification of solutions

$$y = c_1 e^{-x} + e^{(-5-i)x} c_2 + e^{(-5+i)x} c_3$$

Verified OK.

12.2.1 Maple step by step solution

Let's solve

$y''' + 11y'' + 36y' + 26y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

□ Convert linear ODE into a system of first order ODEs

  ○ Define new variable $y_1(x)$

    $y_1(x) = y$

  ○ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ○ Define new variable $y_3(x)$

    $y_3(x) = y''$

  ○ Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = -11y_3(x) - 36y_2(x) - 26y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = -11y_3(x) - 36y_2(x) - 26y_1(x)]$

- Define vector

  $\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$

- System to solve

$$\vec{y}'(x) = \left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -26 & -36 & -11 \end{array}\right] \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -26 & -36 & -11 \end{array}\right]$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]\right], \left[-5-\mathrm{I}, \left[\begin{array}{c} \frac{6}{169}-\frac{5\,\mathrm{I}}{338} \\ -\frac{5}{26}+\frac{\mathrm{I}}{26} \\ 1 \end{array}\right]\right], \left[-5+\mathrm{I}, \left[\begin{array}{c} \frac{6}{169}+\frac{5\,\mathrm{I}}{338} \\ -\frac{5}{26}-\frac{\mathrm{I}}{26} \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[-1, \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{-x} \cdot \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-5-\mathrm{I}, \left[\begin{array}{c} \frac{6}{169}-\frac{5\,\mathrm{I}}{338} \\ -\frac{5}{26}+\frac{\mathrm{I}}{26} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$e^{(-5-I)x} \cdot \begin{bmatrix} \frac{6}{169} - \frac{5\,I}{338} \\ -\frac{5}{26} + \frac{I}{26} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$e^{-5x} \cdot (\cos(x) - I\sin(x)) \cdot \begin{bmatrix} \frac{6}{169} - \frac{5\,I}{338} \\ -\frac{5}{26} + \frac{I}{26} \\ 1 \end{bmatrix}$$

- Simplify expression

$$e^{-5x} \cdot \begin{bmatrix} \left(\frac{6}{169} - \frac{5\,I}{338}\right)(\cos(x) - I\sin(x)) \\ \left(-\frac{5}{26} + \frac{I}{26}\right)(\cos(x) - I\sin(x)) \\ \cos(x) - I\sin(x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_2(x) = e^{-5x} \cdot \begin{bmatrix} \frac{6\cos(x)}{169} - \frac{5\sin(x)}{338} \\ -\frac{5\cos(x)}{26} + \frac{\sin(x)}{26} \\ \cos(x) \end{bmatrix}, \vec{y}_3(x) = e^{-5x} \cdot \begin{bmatrix} -\frac{6\sin(x)}{169} - \frac{5\cos(x)}{338} \\ \frac{5\sin(x)}{26} + \frac{\cos(x)}{26} \\ -\sin(x) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x)$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_2 e^{-5x} \cdot \begin{bmatrix} \frac{6\cos(x)}{169} - \frac{5\sin(x)}{338} \\ -\frac{5\cos(x)}{26} + \frac{\sin(x)}{26} \\ \cos(x) \end{bmatrix} + e^{-5x} c_3 \cdot \begin{bmatrix} -\frac{6\sin(x)}{169} - \frac{5\cos(x)}{338} \\ \frac{5\sin(x)}{26} + \frac{\cos(x)}{26} \\ -\sin(x) \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{\left((12c_2 - 5c_3)\cos(x) - 5\left(c_2 + \frac{12c_3}{5}\right)\sin(x)\right)e^{-5x}}{338} + c_1 e^{-x}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(diff(y(x),x$3)+11*diff(y(x),x$2)+36*diff(y(x),x)+26*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{-x}c_1 + c_2e^{-5x}\sin(x) + c_3e^{-5x}\cos(x)$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 30

```
DSolve[y'''[x]+11*y''[x]+36*y'[x]+26*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-5x}\left(c_3e^{4x} + c_2\cos(x) + c_1\sin(x)\right)$$

12.3 problem Problem 18

Internal problem ID [2826]
Internal file name [`OUTPUT/2318_Sunday_June_05_2022_02_58_56_AM_58418777/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.10, Chapter review. page 575
Problem number: Problem 18.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 6y' + 9y = 4\,\mathrm{e}^{-3x}}$$

12.3.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 6, C = 9, f(x) = 4\,\mathrm{e}^{-3x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 6y' + 9y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 6, C = 9$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 6\lambda\, \mathrm{e}^{\lambda x} + 9\, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 6\lambda + 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 6, C = 9$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-6}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{(6)^2 - (4)\,(1)\,(9)}\\ &= -3\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 3$. Therefore the solution is

$$y = c_1 \mathrm{e}^{-3x} + c_2 x\, \mathrm{e}^{-3x} \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{-3x} + c_2 x\, \mathrm{e}^{-3x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$4\, \mathrm{e}^{-3x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{-3x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\, \mathrm{e}^{-3x}, \mathrm{e}^{-3x}\}$$

Since e^{-3x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,e^{-3x}\}]$$

Since $x\,e^{-3x}$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2e^{-3x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1x^2e^{-3x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1e^{-3x} = 4\,e^{-3x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2x^2e^{-3x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1e^{-3x} + c_2x\,e^{-3x}\right) + \left(2x^2e^{-3x}\right) \end{aligned}$$

Which simplifies to

$$y = e^{-3x}(c_2x + c_1) + 2x^2e^{-3x}$$

Summary
The solution(s) found are the following

$$y = e^{-3x}(c_2x + c_1) + 2x^2e^{-3x} \tag{1}$$

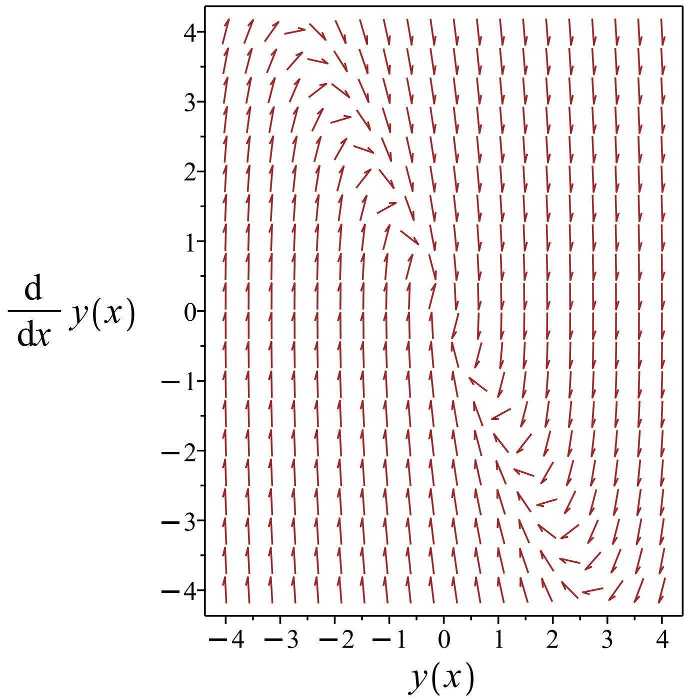


Figure 320: Slope field plot

Verification of solutions

$$y = e^{-3x}(c_2x + c_1) + 2x^2e^{-3x}$$

Verified OK.

12.3.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{(p(x)^2 + p'(x))\, y}{2} = f(x)$$

Where $p(x) = 6$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\, dx} \\ &= e^{\int 6\, dx} \\ &= e^{3x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$(M(x)y)'' = 4\,\mathrm{e}^{3x}\mathrm{e}^{-3x}$$
$$(\mathrm{e}^{3x}y)'' = 4\,\mathrm{e}^{3x}\mathrm{e}^{-3x}$$

Integrating once gives

$$(\mathrm{e}^{3x}y)' = 4x + c_1$$

Integrating again gives

$$(\mathrm{e}^{3x}y) = x(c_1 + 2x) + c_2$$

Hence the solution is

$$y = \frac{x(c_1 + 2x) + c_2}{\mathrm{e}^{3x}}$$

Or

$$y = c_1 x\,\mathrm{e}^{-3x} + 2x^2\mathrm{e}^{-3x} + c_2\mathrm{e}^{-3x}$$

Summary

The solution(s) found are the following

$$y = c_1 x\,\mathrm{e}^{-3x} + 2x^2\mathrm{e}^{-3x} + c_2\mathrm{e}^{-3x} \tag{1}$$

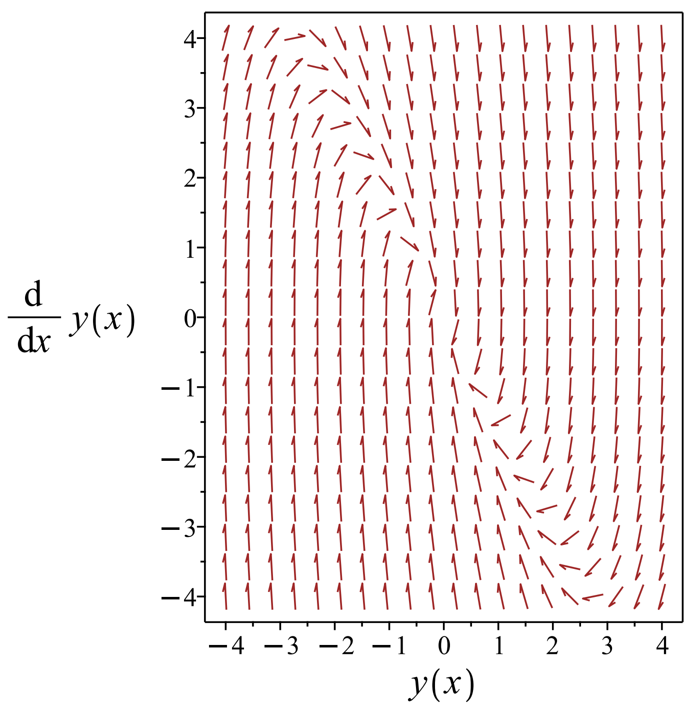


Figure 321: Slope field plot

Verification of solutions

$$y = c_1 x \,\mathrm{e}^{-3x} + 2x^2 \mathrm{e}^{-3x} + c_2 \mathrm{e}^{-3x}$$

Verified OK.

12.3.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 6y' + 9y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 6 \\ C &= 9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 323: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{6}{1}\,dx} \\
&= z_1 e^{-3x} \\
&= z_1 \left(\mathrm{e}^{-3x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-3x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{6}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-6x}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-3x}\right) + c_2\left(\mathrm{e}^{-3x}(x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 6y' + 9y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{-3x} + c_2 x \, e^{-3x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$4 \, e^{-3x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-3x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x \, e^{-3x}, e^{-3x}\}$$

Since e^{-3x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x \, e^{-3x}\}]$$

Since $x \, e^{-3x}$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2 e^{-3x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^2 e^{-3x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 e^{-3x} = 4 \, e^{-3x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2x^2 e^{-3x}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \mathrm{e}^{-3x} + c_2 x \,\mathrm{e}^{-3x}\right) + \left(2x^2 \mathrm{e}^{-3x}\right)
\end{aligned}
$$

Which simplifies to

$$
y = \mathrm{e}^{-3x}(c_2 x + c_1) + 2x^2 \mathrm{e}^{-3x}
$$

Summary

The solution(s) found are the following

$$
y = \mathrm{e}^{-3x}(c_2 x + c_1) + 2x^2 \mathrm{e}^{-3x} \tag{1}
$$

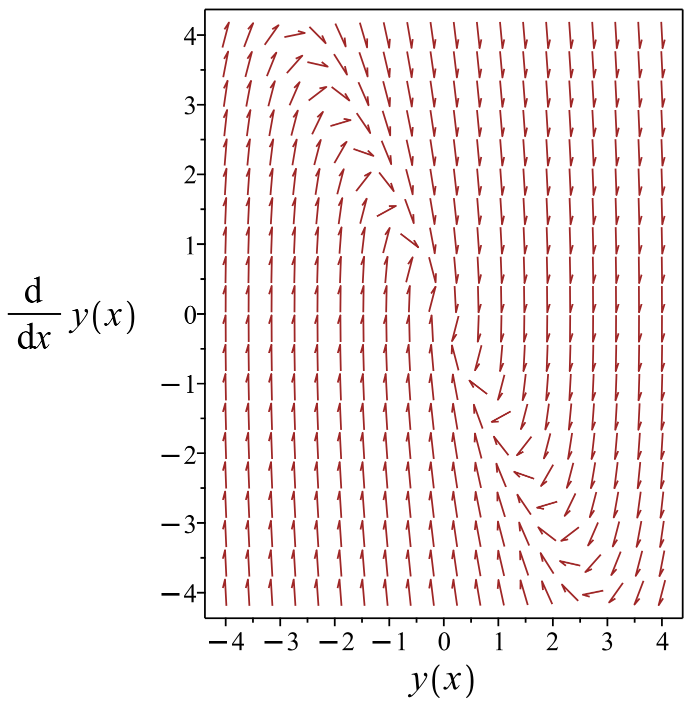


Figure 322: Slope field plot

Verification of solutions

$$
y = \mathrm{e}^{-3x}(c_2 x + c_1) + 2x^2 \mathrm{e}^{-3x}
$$

Verified OK.

12.3.4 Maple step by step solution

Let's solve

$y'' + 6y' + 9y = 4\,\mathrm{e}^{-3x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 6r + 9 = 0$

- Factor the characteristic polynomial

  $(r+3)^2 = 0$

- Root of the characteristic polynomial

  $r = -3$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-3x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,\mathrm{e}^{-3x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-3x} + c_2 x\,\mathrm{e}^{-3x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 4\,\mathrm{e}^{-3x}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-3x} & x\,\mathrm{e}^{-3x} \\ -3\,\mathrm{e}^{-3x} & \mathrm{e}^{-3x} - 3x\,\mathrm{e}^{-3x} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = \mathrm{e}^{-6x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -4\,\mathrm{e}^{-3x}\left(\int x dx - \left(\int 1 dx\right) x\right)$

- Compute integrals

  $y_p(x) = 2x^2 e^{-3x}$

- Substitute particular solution into general solution to ODE

  $y = c_2 x\, e^{-3x} + 2x^2 e^{-3x} + c_1 e^{-3x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 19

```
dsolve(diff(y(x),x$2)+6*diff(y(x),x)+9*y(x)=4*exp(-3*x),y(x), singsol=all)
```

$$y(x) = e^{-3x}(c_1 x + 2x^2 + c_2)$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 23

```
DSolve[y''[x]+6*y'[x]+9*y[x]==4*Exp[-3*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-3x}(2x^2 + c_2 x + c_1)$$

12.4 problem Problem 19

Internal problem ID [2827]
Internal file name [`OUTPUT/2319_Sunday_June_05_2022_02_58_58_AM_80909504/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.10, Chapter review. page 575
Problem number: Problem 19.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 6y' + 9y = 4\,\mathrm{e}^{-2x}}$$

12.4.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 6, C = 9, f(x) = 4\,\mathrm{e}^{-2x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 6y' + 9y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 6, C = 9$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 6\lambda\, e^{\lambda x} + 9\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 6\lambda + 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 6, C = 9$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-6}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{(6)^2 - (4)\,(1)\,(9)}\\ &= -3\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 3$. Therefore the solution is

$$y = c_1 e^{-3x} + c_2 x\, e^{-3x} \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{-3x} + c_2 x\, e^{-3x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$4\, e^{-2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\, e^{-3x}, e^{-3x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 \mathrm{e}^{-2x}
$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
A_1 \mathrm{e}^{-2x} = 4\,\mathrm{e}^{-2x}
$$

Solving for the unknowns by comparing coefficients results in

$$
[A_1 = 4]
$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$
y_p = 4\,\mathrm{e}^{-2x}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \mathrm{e}^{-3x} + c_2 x\,\mathrm{e}^{-3x}\right) + \left(4\,\mathrm{e}^{-2x}\right)
\end{aligned}
$$

Which simplifies to

$$
y = \mathrm{e}^{-3x}(c_2 x + c_1) + 4\,\mathrm{e}^{-2x}
$$

Summary

The solution(s) found are the following

$$
y = \mathrm{e}^{-3x}(c_2 x + c_1) + 4\,\mathrm{e}^{-2x} \tag{1}
$$

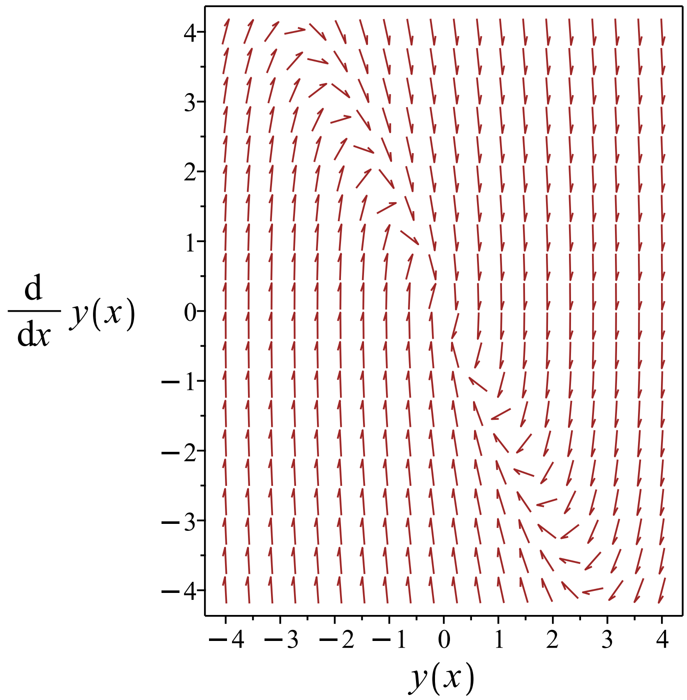


Figure 323: Slope field plot

Verification of solutions

$$y = e^{-3x}(c_2 x + c_1) + 4\,e^{-2x}$$

Verified OK.

12.4.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\,y' + \frac{\left(p(x)^2 + p'(x)\right)y}{2} = f(x)$$

Where $p(x) = 6$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int 6\,dx} \\ &= e^{3x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$(M(x)y)'' = 4\,\mathrm{e}^{3x}\mathrm{e}^{-2x}$$
$$\left(\mathrm{e}^{3x}y\right)'' = 4\,\mathrm{e}^{3x}\mathrm{e}^{-2x}$$

Integrating once gives

$$\left(\mathrm{e}^{3x}y\right)' = 4\,\mathrm{e}^{x} + c_1$$

Integrating again gives

$$\left(\mathrm{e}^{3x}y\right) = c_1 x + 4\,\mathrm{e}^{x} + c_2$$

Hence the solution is

$$y = \frac{c_1 x + 4\,\mathrm{e}^{x} + c_2}{\mathrm{e}^{3x}}$$

Or

$$y = c_1 x\,\mathrm{e}^{-3x} + c_2\mathrm{e}^{-3x} + 4\,\mathrm{e}^{-2x}$$

Summary

The solution(s) found are the following

$$y = c_1 x\,\mathrm{e}^{-3x} + c_2\mathrm{e}^{-3x} + 4\,\mathrm{e}^{-2x} \tag{1}$$

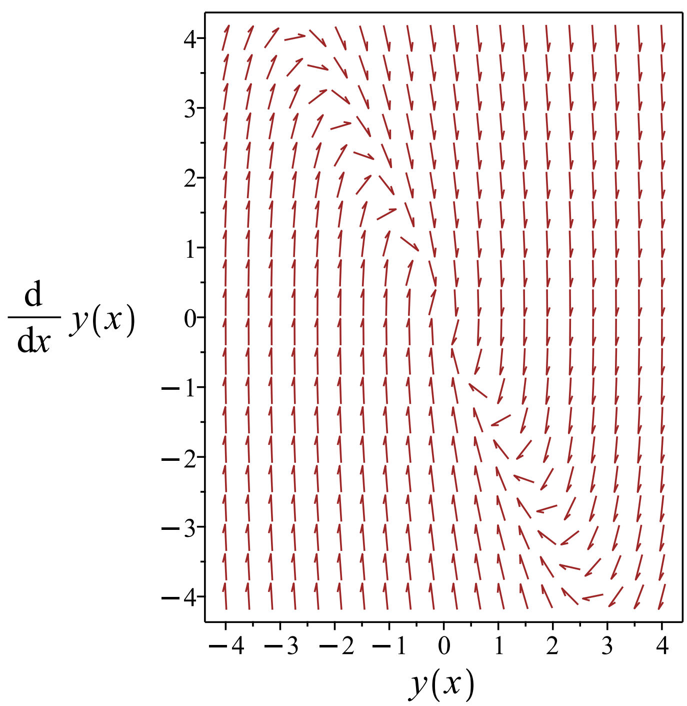


Figure 324: Slope field plot

Verification of solutions

$$y = c_1 x \,\mathrm{e}^{-3x} + c_2 \mathrm{e}^{-3x} + 4\,\mathrm{e}^{-2x}$$

Verified OK.

12.4.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 6y' + 9y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 6 \\ C &= 9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 325: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{6}{1}\,dx} \\
&= z_1 e^{-3x} \\
&= z_1\left(\mathrm{e}^{-3x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-3x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{6}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-6x}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-3x}\right) + c_2\left(\mathrm{e}^{-3x}(x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 6y' + 9y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-3x} + c_2 x \,\mathrm{e}^{-3x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$4 \,\mathrm{e}^{-2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{-2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x \,\mathrm{e}^{-3x}, \mathrm{e}^{-3x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \mathrm{e}^{-2x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 \mathrm{e}^{-2x} = 4 \,\mathrm{e}^{-2x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 4]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 4 \,\mathrm{e}^{-2x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-3x} + c_2 x \,\mathrm{e}^{-3x}\right) + \left(4 \,\mathrm{e}^{-2x}\right) \end{aligned}$$

Which simplifies to

$$y = e^{-3x}(c_2 x + c_1) + 4 e^{-2x}$$

Summary
The solution(s) found are the following

$$y = e^{-3x}(c_2 x + c_1) + 4 e^{-2x} \tag{1}$$

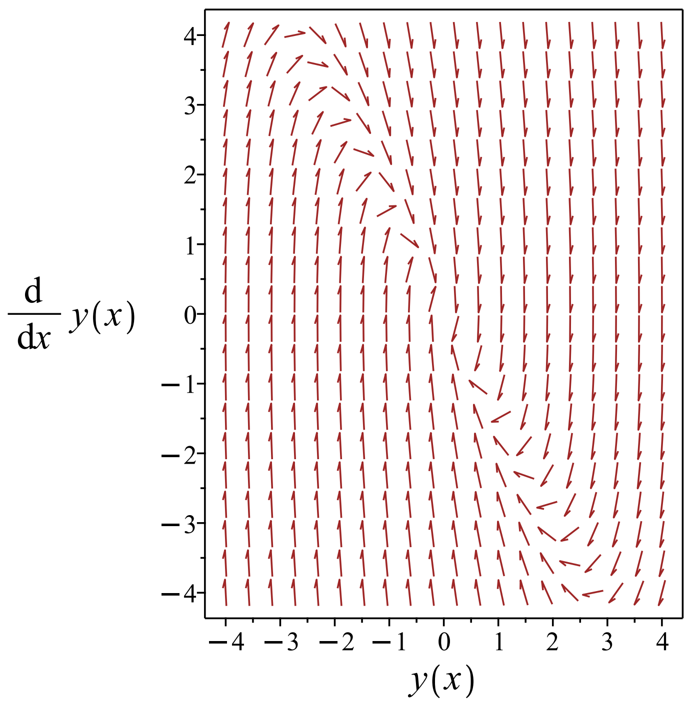


Figure 325: Slope field plot

Verification of solutions

$$y = e^{-3x}(c_2 x + c_1) + 4 e^{-2x}$$

Verified OK.

12.4.4 Maple step by step solution

Let's solve

$y'' + 6y' + 9y = 4\,e^{-2x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 6r + 9 = 0$

- Factor the characteristic polynomial

  $(r+3)^2 = 0$

- Root of the characteristic polynomial

  $r = -3$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-3x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,e^{-3x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-3x} + c_2 x\,e^{-3x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x),y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x),y_2(x))} dx\right), f(x) = 4\,e^{-2x}\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-3x} & x\,e^{-3x} \\ -3\,e^{-3x} & e^{-3x} - 3x\,e^{-3x} \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = e^{-6x}$

- ○ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -4\,e^{-3x}\left(\int x\,e^{x} dx - \left(\int e^{x} dx\right) x\right)$

- Compute integrals

  $y_p(x) = 4\,e^{-2x}$

- Substitute particular solution into general solution to ODE

  $y = c_2 x\,e^{-3x} + c_1 e^{-3x} + 4\,e^{-2x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 21

```
dsolve(diff(y(x),x$2)+6*diff(y(x),x)+9*y(x)=4*exp(-2*x),y(x), singsol=all)
```

$$y(x) = (c_1 x + c_2)\,e^{-3x} + 4\,e^{-2x}$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 23

```
DSolve[y''[x]+6*y'[x]+9*y[x]==4*Exp[-2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-3x}(4e^x + c_2 x + c_1)$$

12.5 problem Problem 20

Internal problem ID [2828]
Internal file name [OUTPUT/2320_Sunday_June_05_2022_02_59_01_AM_7048863/index.tex]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.10, Chapter review. page 575
Problem number: Problem 20.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_y]]
```

$$\boxed{y''' - 6y'' + 25y' = x^2}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - 6y'' + 25y' = 0$$

The characteristic equation is

$$\lambda^3 - 6\lambda^2 + 25\lambda = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 3 + 4i$$
$$\lambda_3 = 3 - 4i$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 + e^{(3+4i)x}c_2 + e^{(3-4i)x}c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = 1$$
$$y_2 = e^{(3+4i)x}$$
$$y_3 = e^{(3-4i)x}$$

Now the particular solution to the given ODE is found

$$y''' - 6y'' + 25y' = x^2$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^{(3-4i)x}, e^{(3+4i)x}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x, x^2, x^3\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_3x^3 + A_2x^2 + A_1x$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$75x^2A_3 + 50xA_2 - 36xA_3 + 25A_1 - 12A_2 + 6A_3 = x^2$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{22}{15625}, A_2 = \frac{6}{625}, A_3 = \frac{1}{75}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{1}{75}x^3 + \frac{6}{625}x^2 + \frac{22}{15625}x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 + \mathrm{e}^{(3+4i)x}c_2 + \mathrm{e}^{(3-4i)x}c_3\right) + \left(\frac{1}{75}x^3 + \frac{6}{625}x^2 + \frac{22}{15625}x\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 + \mathrm{e}^{(3+4i)x}c_2 + \mathrm{e}^{(3-4i)x}c_3 + \frac{x^3}{75} + \frac{6x^2}{625} + \frac{22x}{15625} \tag{1}$$

Verification of solutions

$$y = c_1 + \mathrm{e}^{(3+4i)x}c_2 + \mathrm{e}^{(3-4i)x}c_3 + \frac{x^3}{75} + \frac{6x^2}{625} + \frac{22x}{15625}$$

Verified OK.

12.5.1 Maple step by step solution

Let's solve

$y''' - 6y'' + 25y' = x^2$

- Highest derivative means the order of the ODE is 3

  y'''

□ Convert linear ODE into a system of first order ODEs

  - Define new variable $y_1(x)$

    $y_1(x) = y$

  - Define new variable $y_2(x)$

    $y_2(x) = y'$

- Define new variable $y_3(x)$

  $y_3(x) = y''$

- Isolate for $y_3'(x)$ using original ODE

  $y_3'(x) = x^2 + 6y_3(x) - 25y_2(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = x^2 + 6y_3(x) - 25y_2(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -25 & 6 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ x^2 \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ x^2 \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -25 & 6 \end{bmatrix}$$

- Rewrite the system as

  $\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right], \left[3-4\,\mathrm{I}, \begin{bmatrix} -\frac{7}{625}+\frac{24\,\mathrm{I}}{625} \\ \frac{3}{25}+\frac{4\,\mathrm{I}}{25} \\ 1 \end{bmatrix}\right], \left[3+4\,\mathrm{I}, \begin{bmatrix} -\frac{7}{625}-\frac{24\,\mathrm{I}}{625} \\ \frac{3}{25}-\frac{4\,\mathrm{I}}{25} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[3-4\,\mathrm{I}, \begin{bmatrix} -\frac{7}{625}+\frac{24\,\mathrm{I}}{625} \\ \frac{3}{25}+\frac{4\,\mathrm{I}}{25} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{(3-4\,\mathrm{I})x} \cdot \begin{bmatrix} -\frac{7}{625}+\frac{24\,\mathrm{I}}{625} \\ \frac{3}{25}+\frac{4\,\mathrm{I}}{25} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\mathrm{e}^{3x} \cdot (\cos(4x) - \mathrm{I}\sin(4x)) \cdot \begin{bmatrix} -\frac{7}{625}+\frac{24\,\mathrm{I}}{625} \\ \frac{3}{25}+\frac{4\,\mathrm{I}}{25} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\mathrm{e}^{3x} \cdot \begin{bmatrix} \left(-\frac{7}{625}+\frac{24\,\mathrm{I}}{625}\right)(\cos(4x) - \mathrm{I}\sin(4x)) \\ \left(\frac{3}{25}+\frac{4\,\mathrm{I}}{25}\right)(\cos(4x) - \mathrm{I}\sin(4x)) \\ \cos(4x) - \mathrm{I}\sin(4x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_2(x) = e^{3x} \cdot \begin{bmatrix} -\frac{7\cos(4x)}{625} + \frac{24\sin(4x)}{625} \\ \frac{3\cos(4x)}{25} + \frac{4\sin(4x)}{25} \\ \cos(4x) \end{bmatrix}, \vec{y}_3(x) = e^{3x} \cdot \begin{bmatrix} \frac{7\sin(4x)}{625} + \frac{24\cos(4x)}{625} \\ -\frac{3\sin(4x)}{25} + \frac{4\cos(4x)}{25} \\ -\sin(4x) \end{bmatrix}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + \vec{y}_p(x)$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} 1 & e^{3x}\left(-\frac{7\cos(4x)}{625} + \frac{24\sin(4x)}{625}\right) & e^{3x}\left(\frac{7\sin(4x)}{625} + \frac{24\cos(4x)}{625}\right) \\ 0 & e^{3x}\left(\frac{3\cos(4x)}{25} + \frac{4\sin(4x)}{25}\right) & e^{3x}\left(-\frac{3\sin(4x)}{25} + \frac{4\cos(4x)}{25}\right) \\ 0 & e^{3x}\cos(4x) & -e^{3x}\sin(4x) \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

$\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} 1 & e^{3x}\left(-\frac{7\cos(4x)}{625} + \frac{24\sin(4x)}{625}\right) & e^{3x}\left(\frac{7\sin(4x)}{625} + \frac{24\cos(4x)}{625}\right) \\ 0 & e^{3x}\left(\frac{3\cos(4x)}{25} + \frac{4\sin(4x)}{25}\right) & e^{3x}\left(-\frac{3\sin(4x)}{25} + \frac{4\cos(4x)}{25}\right) \\ 0 & e^{3x}\cos(4x) & -e^{3x}\sin(4x) \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 1 & -\frac{7}{625} & \frac{24}{625} \\ 0 & \frac{3}{25} & \frac{4}{25} \\ 0 & 1 & 0 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} 1 & -\frac{6}{25} + \frac{e^{3x}(7\sin(4x)+24\cos(4x))}{100} & \frac{1}{25} + \frac{(3\sin(4x)-4\cos(4x))e^{3x}}{100} \\ 0 & \frac{e^{3x}(-3\sin(4x)+4\cos(4x))}{4} & \frac{e^{3x}\sin(4x)}{4} \\ 0 & -\frac{25\,e^{3x}\sin(4x)}{4} & \frac{e^{3x}(4\cos(4x)+3\sin(4x))}{4} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

$\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$

- Take the derivative of the particular solution

$\overrightarrow{y}'_p(x) = \Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$

- Cancel like terms

  $\Phi(x) \cdot \overrightarrow{v}'(x) = \overrightarrow{f}(x)$

- Multiply by the inverse of the fundamental matrix

  $\overrightarrow{v}'(x) = \frac{1}{\Phi(x)} \cdot \overrightarrow{f}(x)$

- Integrate to solve for $\overrightarrow{v}(x)$

  $\overrightarrow{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds$

- Plug $\overrightarrow{v}(x)$ into the equation for the particular solution

  $\overrightarrow{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\overrightarrow{y}_p(x) = \left[\begin{array}{c} -\frac{168}{390625} + \frac{(336\cos(4x)-527\sin(4x))\mathrm{e}^{3x}}{781250} + \frac{x^3}{75} + \frac{6x^2}{625} + \frac{22x}{15625} \\ \frac{(-44\cos(4x)-117\sin(4x))\mathrm{e}^{3x}}{31250} + \frac{x^2}{25} + \frac{12x}{625} + \frac{22}{15625} \\ \frac{(-7\sin(4x)-24\cos(4x))\mathrm{e}^{3x}}{1250} + \frac{2x}{25} + \frac{12}{625} \end{array}\right]$$

- Plug particular solution back into general solution

$$\overrightarrow{y}(x) = c_1 \overrightarrow{y}_1 + c_2 \overrightarrow{y}_2(x) + c_3 \overrightarrow{y}_3(x) + \left[\begin{array}{c} -\frac{168}{390625} + \frac{(336\cos(4x)-527\sin(4x))\mathrm{e}^{3x}}{781250} + \frac{x^3}{75} + \frac{6x^2}{625} + \frac{22x}{15625} \\ \frac{(-44\cos(4x)-117\sin(4x))\mathrm{e}^{3x}}{31250} + \frac{x^2}{25} + \frac{12x}{625} + \frac{22}{15625} \\ \frac{(-7\sin(4x)-24\cos(4x))\mathrm{e}^{3x}}{1250} + \frac{2x}{25} + \frac{12}{625} \end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y = -\frac{168}{390625} + \frac{\left((168-4375c_2+15000c_3)\cos(4x)+15000\sin(4x)\left(c_2+\frac{7c_3}{24}-\frac{527}{30000}\right)\right)\mathrm{e}^{3x}}{390625} + \frac{x^3}{75} + \frac{6x^2}{625} + \frac{22x}{15625} + c_1$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
-> Calling odsolve with the ODE`, diff(diff(_b(_a), _a), _a) = _a^2+6*(diff(_b(_a), _a))-25*
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
   trying a double symmetry of the form [xi=0, eta=F(x)]
   -> Try solving first the homogeneous part of the ODE
      checking if the LODE has constant coefficients
      <- constant coefficients successful
   <- solving first the homogeneous part of the ODE successful
<- differential order: 3; linear nonhomogeneous with symmetry [0,1] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 49

```
dsolve(diff(y(x),x$3)-6*diff(y(x),x$2)+25*diff(y(x),x)=x^2,y(x), singsol=all)
```

$$y(x) = \frac{\left((3c_1 - 4c_2)\cos(4x) + 4\sin(4x)\left(c_1 + \frac{3c_2}{4}\right)\right)e^{3x}}{25} + \frac{x^3}{75} + \frac{6x^2}{625} + \frac{22x}{15625} + c_3$$

✓ Solution by Mathematica

Time used: 0.272 (sec). Leaf size: 71

```
DSolve[y'''[x]-6*y''[x]+25*y'[x]==x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^3}{75} + \frac{6x^2}{625} + \frac{22x}{15625} - \frac{1}{25}(4c_1 - 3c_2)e^{3x}\cos(4x) + \frac{1}{25}(3c_1 + 4c_2)e^{3x}\sin(4x) + c_3$$

12.6 problem Problem 21

Internal problem ID [2829]
Internal file name [`OUTPUT/2321_Sunday_June_05_2022_02_59_04_AM_48998655/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.10, Chapter review. page 575
Problem number: Problem 21.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_y]]
```

$$\boxed{y''' - 6y'' + 25y' = \sin(4x)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - 6y'' + 25y' = 0$$

The characteristic equation is

$$\lambda^3 - 6\lambda^2 + 25\lambda = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 3 + 4i$$
$$\lambda_3 = 3 - 4i$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 + e^{(3+4i)x}c_2 + e^{(3-4i)x}c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = 1$$
$$y_2 = e^{(3+4i)x}$$
$$y_3 = e^{(3-4i)x}$$

Now the particular solution to the given ODE is found

$$y''' - 6y'' + 25y' = \sin(4x)$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(4x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(4x), \sin(4x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^{(3-4i)x}, e^{(3+4i)x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(4x) + A_2 \sin(4x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-36A_1 \sin(4x) + 36A_2 \cos(4x) + 96A_1 \cos(4x) + 96A_2 \sin(4x) = \sin(4x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{292}, A_2 = \frac{2}{219}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{\cos(4x)}{292} + \frac{2\sin(4x)}{219}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 + e^{(3+4i)x} c_2 + e^{(3-4i)x} c_3\right) + \left(-\frac{\cos(4x)}{292} + \frac{2\sin(4x)}{219}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 + e^{(3+4i)x} c_2 + e^{(3-4i)x} c_3 - \frac{\cos(4x)}{292} + \frac{2\sin(4x)}{219} \tag{1}$$

Verification of solutions

$$y = c_1 + e^{(3+4i)x} c_2 + e^{(3-4i)x} c_3 - \frac{\cos(4x)}{292} + \frac{2\sin(4x)}{219}$$

Verified OK.

12.6.1 Maple step by step solution

Let's solve

$y''' - 6y'' + 25y' = \sin(4x)$

- Highest derivative means the order of the ODE is 3

  y'''

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$

  ◦ Isolate for $y_3'(x)$ using original ODE

$y_3'(x) = \sin(4x) + 6y_3(x) - 25y_2(x)$

Convert linear ODE into a system of first order ODEs

$[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = \sin(4x) + 6y_3(x) - 25y_2(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -25 & 6 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ \sin(4x) \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ \sin(4x) \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -25 & 6 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right], \left[3-4\,\mathrm{I}, \begin{bmatrix} -\frac{7}{625}+\frac{24\,\mathrm{I}}{625} \\ \frac{3}{25}+\frac{4\,\mathrm{I}}{25} \\ 1 \end{bmatrix}\right], \left[3+4\,\mathrm{I}, \begin{bmatrix} -\frac{7}{625}-\frac{24\,\mathrm{I}}{625} \\ \frac{3}{25}-\frac{4\,\mathrm{I}}{25} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[3 - 4\,\mathrm{I}, \begin{bmatrix} -\frac{7}{625} + \frac{24\,\mathrm{I}}{625} \\ \frac{3}{25} + \frac{4\,\mathrm{I}}{25} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{(3-4\,\mathrm{I})x} \cdot \begin{bmatrix} -\frac{7}{625} + \frac{24\,\mathrm{I}}{625} \\ \frac{3}{25} + \frac{4\,\mathrm{I}}{25} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\mathrm{e}^{3x} \cdot (\cos(4x) - \mathrm{I}\sin(4x)) \cdot \begin{bmatrix} -\frac{7}{625} + \frac{24\,\mathrm{I}}{625} \\ \frac{3}{25} + \frac{4\,\mathrm{I}}{25} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\mathrm{e}^{3x} \cdot \begin{bmatrix} \left(-\frac{7}{625} + \frac{24\,\mathrm{I}}{625}\right)(\cos(4x) - \mathrm{I}\sin(4x)) \\ \left(\frac{3}{25} + \frac{4\,\mathrm{I}}{25}\right)(\cos(4x) - \mathrm{I}\sin(4x)) \\ \cos(4x) - \mathrm{I}\sin(4x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_2(x) = \mathrm{e}^{3x} \cdot \begin{bmatrix} -\frac{7\cos(4x)}{625} + \frac{24\sin(4x)}{625} \\ \frac{3\cos(4x)}{25} + \frac{4\sin(4x)}{25} \\ \cos(4x) \end{bmatrix}, \vec{y}_3(x) = \mathrm{e}^{3x} \cdot \begin{bmatrix} \frac{7\sin(4x)}{625} + \frac{24\cos(4x)}{625} \\ -\frac{3\sin(4x)}{25} + \frac{4\cos(4x)}{25} \\ -\sin(4x) \end{bmatrix}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\overrightarrow{y}_p($

$\overrightarrow{y}(x) = c_1\overrightarrow{y}_1 + c_2\overrightarrow{y}_2(x) + c_3\overrightarrow{y}_3(x) + \overrightarrow{y}_p(x)$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} 1 & e^{3x}\left(-\frac{7\cos(4x)}{625} + \frac{24\sin(4x)}{625}\right) & e^{3x}\left(\frac{7\sin(4x)}{625} + \frac{24\cos(4x)}{625}\right) \\ 0 & e^{3x}\left(\frac{3\cos(4x)}{25} + \frac{4\sin(4x)}{25}\right) & e^{3x}\left(-\frac{3\sin(4x)}{25} + \frac{4\cos(4x)}{25}\right) \\ 0 & e^{3x}\cos(4x) & -e^{3x}\sin(4x) \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

$\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} 1 & e^{3x}\left(-\frac{7\cos(4x)}{625} + \frac{24\sin(4x)}{625}\right) & e^{3x}\left(\frac{7\sin(4x)}{625} + \frac{24\cos(4x)}{625}\right) \\ 0 & e^{3x}\left(\frac{3\cos(4x)}{25} + \frac{4\sin(4x)}{25}\right) & e^{3x}\left(-\frac{3\sin(4x)}{25} + \frac{4\cos(4x)}{25}\right) \\ 0 & e^{3x}\cos(4x) & -e^{3x}\sin(4x) \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 1 & -\frac{7}{625} & \frac{24}{625} \\ 0 & \frac{3}{25} & \frac{4}{25} \\ 0 & 1 & 0 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} 1 & -\frac{6}{25} + \frac{e^{3x}(7\sin(4x)+24\cos(4x))}{100} & \frac{1}{25} + \frac{(3\sin(4x)-4\cos(4x))e^{3x}}{100} \\ 0 & \frac{e^{3x}(-3\sin(4x)+4\cos(4x))}{4} & \frac{e^{3x}\sin(4x)}{4} \\ 0 & -\frac{25\,e^{3x}\sin(4x)}{4} & \frac{e^{3x}(4\cos(4x)+3\sin(4x))}{4} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(x)$ and solve for $\overrightarrow{v}(x)$

$\overrightarrow{y}_p(x) = \Phi(x) \cdot \overrightarrow{v}(x)$

- Take the derivative of the particular solution

$\overrightarrow{y}_p'(x) = \Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

$\Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

$$A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$$

- Cancel like terms

  $$\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$$

- Multiply by the inverse of the fundamental matrix

  $$\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$$

- Integrate to solve for $\vec{v}(x)$

  $$\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$$

- Plug $\vec{v}(x)$ into the equation for the particular solution

  $$\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$$

- Plug in the fundamental matrix and the forcing function and compute

  $$\vec{y}_p(x) = \left[\begin{array}{c} \frac{1}{100} + \frac{\left(-25-48\,\mathrm{e}^{3x}\right)\cos(4x)}{7300} - \frac{23\,\mathrm{e}^{3x}\sin(4x)}{5475} + \frac{2\sin(4x)}{219} \\ \frac{\left(-8\,\mathrm{e}^{3x}+8\right)\cos(4x)}{219} + \frac{\sin(4x)\left(\mathrm{e}^{3x}+1\right)}{73} \\ \frac{4\left(-\mathrm{e}^{3x}+1\right)\cos(4x)}{73} + \frac{41\,\mathrm{e}^{3x}\sin(4x)}{219} - \frac{32\sin(4x)}{219} \end{array}\right]$$

- Plug particular solution back into general solution

  $$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + \left[\begin{array}{c} \frac{1}{100} + \frac{\left(-25-48\,\mathrm{e}^{3x}\right)\cos(4x)}{7300} - \frac{23\,\mathrm{e}^{3x}\sin(4x)}{5475} + \frac{2\sin(4x)}{219} \\ \frac{\left(-8\,\mathrm{e}^{3x}+8\right)\cos(4x)}{219} + \frac{\sin(4x)\left(\mathrm{e}^{3x}+1\right)}{73} \\ \frac{4\left(-\mathrm{e}^{3x}+1\right)\cos(4x)}{73} + \frac{41\,\mathrm{e}^{3x}\sin(4x)}{219} - \frac{32\sin(4x)}{219} \end{array}\right]$$

- First component of the vector is the solution to the ODE

  $$y = \frac{1}{100} + \frac{\left(\left(-300-511c_2+1752c_3\right)\cos(4x)+1752\left(c_2+\frac{7c_3}{24}-\frac{575}{5256}\right)\sin(4x)\right)\mathrm{e}^{3x}}{45625} + c_1 - \frac{\cos(4x)}{292} + \frac{2\sin(4x)}{219}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
-> Calling odsolve with the ODE`, diff(diff(_b(_a), _a), _a) = 6*(diff(_b(_a), _a))-25*_b(_a
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
   trying a double symmetry of the form [xi=0, eta=F(x)]
   -> Try solving first the homogeneous part of the ODE
      checking if the LODE has constant coefficients
      <- constant coefficients successful
   <- solving first the homogeneous part of the ODE successful
<- differential order: 3; linear nonhomogeneous with symmetry [0,1] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 48

```
dsolve(diff(y(x),x$3)-6*diff(y(x),x$2)+25*diff(y(x),x)=sin(4*x),y(x), singsol=all)
```

$$y(x) = \frac{\left((3c_1 - 4c_2)\cos(4x) + 4\sin(4x)\left(c_1 + \frac{3c_2}{4}\right)\right)e^{3x}}{25} + c_3 - \frac{\cos(4x)}{292} + \frac{2\sin(4x)}{219}$$

✓ Solution by Mathematica

Time used: 0.686 (sec). Leaf size: 60

```
DSolve[y'''[x]-6*y''[x]+25*y'[x]==Sin[4*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{(25 + 292(4c_1 - 3c_2)e^{3x})\cos(4x)}{7300} + \frac{(50 + 219(3c_1 + 4c_2)e^{3x})\sin(4x)}{5475} + c_3$$

12.7 problem Problem 22

Internal problem ID [2830]
Internal file name [`OUTPUT/2322_Sunday_June_05_2022_02_59_06_AM_37579000/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.10, Chapter review. page 575
Problem number: Problem 22.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _with_linear_symmetries]]
```

$$\boxed{y''' + 9y'' + 24y' + 16y = 8\,\mathrm{e}^{-x} + 1}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' + 9y'' + 24y' + 16y = 0$$

The characteristic equation is

$$\lambda^3 + 9\lambda^2 + 24\lambda + 16 = 0$$

The roots of the above equation are

$$\lambda_1 = -1$$
$$\lambda_2 = -4$$
$$\lambda_3 = -4$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + c_2 e^{-4x} + x\, e^{-4x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned} y_1 &= e^{-x} \\ y_2 &= e^{-4x} \\ y_3 &= x\, e^{-4x} \end{aligned}$$

Now the particular solution to the given ODE is found

$$y''' + 9y'' + 24y' + 16y = 8\, e^{-x} + 1$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$8\, e^{-x} + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}, \{e^{-x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\, e^{-4x}, e^{-4x}, e^{-x}\}$$

Since e^{-x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{1\}, \{x\, e^{-x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 + A_2 x\, e^{-x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$9A_2 e^{-x} + 16A_1 = 8\, e^{-x} + 1$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{16}, A_2 = \frac{8}{9}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{1}{16} + \frac{8x\,\mathrm{e}^{-x}}{9}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{-4x} + x\,\mathrm{e}^{-4x} c_3\right) + \left(\frac{1}{16} + \frac{8x\,\mathrm{e}^{-x}}{9}\right) \end{aligned}$$

Which simplifies to

$$y = (c_3 x + c_2)\,\mathrm{e}^{-4x} + c_1 \mathrm{e}^{-x} + \frac{1}{16} + \frac{8x\,\mathrm{e}^{-x}}{9}$$

Summary

The solution(s) found are the following

$$y = (c_3 x + c_2)\,\mathrm{e}^{-4x} + c_1 \mathrm{e}^{-x} + \frac{1}{16} + \frac{8x\,\mathrm{e}^{-x}}{9} \tag{1}$$

Verification of solutions

$$y = (c_3 x + c_2)\,\mathrm{e}^{-4x} + c_1 \mathrm{e}^{-x} + \frac{1}{16} + \frac{8x\,\mathrm{e}^{-x}}{9}$$

Verified OK.

12.7.1 Maple step by step solution

Let's solve

$y''' + 9y'' + 24y' + 16y = 8\,\mathrm{e}^{-x} + 1$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Convert linear ODE into a system of first order ODEs

- Define new variable $y_1(x)$

  $y_1(x) = y$
- Define new variable $y_2(x)$

  $y_2(x) = y'$
- Define new variable $y_3(x)$

  $y_3(x) = y''$
- Isolate for $y_3'(x)$ using original ODE

  $y_3'(x) = 8\,\mathrm{e}^{-x} + 1 - 9y_3(x) - 24y_2(x) - 16y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x)\,, y_3(x) = y_2'(x)\,, y_3'(x) = 8\,\mathrm{e}^{-x} + 1 - 9y_3(x) - 24y_2(x) - 16y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -16 & -24 & -9 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 8\,\mathrm{e}^{-x} + 1 \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ 8\,\mathrm{e}^{-x} + 1 \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -16 & -24 & -9 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-4, \left[\begin{array}{c} \frac{1}{16} \\ -\frac{1}{4} \\ 1 \end{array}\right]\right], \left[-4, \left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right]\right], \left[-1, \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[-4, \left[\begin{array}{c} \frac{1}{16} \\ -\frac{1}{4} \\ 1 \end{array}\right]\right]$$

- First solution from eigenvalue -4

$$\vec{y}_1(x) = \mathrm{e}^{-4x} \cdot \left[\begin{array}{c} \frac{1}{16} \\ -\frac{1}{4} \\ 1 \end{array}\right]$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = -4$ is the eigenvalue, a

$$\vec{y}_2(x) = \mathrm{e}^{\lambda x}\left(x\vec{v} + \vec{p}\right)$$

- Note that the x multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obt
- Substitute $\vec{y}_2(x)$ into the homogeneous system

$$\lambda\,\mathrm{e}^{\lambda x}\left(x\vec{v} + \vec{p}\right) + \mathrm{e}^{\lambda x}\vec{v} = \left(\mathrm{e}^{\lambda x}A\right) \cdot \left(x\vec{v} + \vec{p}\right)$$

- Use the fact that $\vec{v}$ is an eigenvector of A

$$\lambda\,\mathrm{e}^{\lambda x}\left(x\vec{v} + \vec{p}\right) + \mathrm{e}^{\lambda x}\vec{v} = \mathrm{e}^{\lambda x}\left(\lambda x\vec{v} + A \cdot \vec{p}\right)$$

- Simplify equation

$$\lambda\vec{p} + \vec{v} = A \cdot \vec{p}$$

- Make use of the identity matrix I

$$(\lambda \cdot I) \cdot \vec{p} + \vec{v} = A \cdot \vec{p}$$

- Condition $\vec{p}$ must meet for $\vec{y}_2(x)$ to be a solution to the homogeneous system

$$(A - \lambda \cdot I) \cdot \vec{p} = \vec{v}$$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue -4

$$\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -16 & -24 & -9 \end{bmatrix} - (-4) \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \cdot \vec{p} = \begin{bmatrix} \frac{1}{16} \\ -\frac{1}{4} \\ 1 \end{bmatrix}$$

- Choice of $\vec{p}$

$$\vec{p} = \begin{bmatrix} \frac{1}{64} \\ 0 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue -4

$$\vec{y}_2(x) = \mathrm{e}^{-4x} \cdot \left(x \cdot \begin{bmatrix} \frac{1}{16} \\ -\frac{1}{4} \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{64} \\ 0 \\ 0 \end{bmatrix}\right)$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = \mathrm{e}^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$\vec{y}(x) = c_1\vec{y}_1(x) + c_2\vec{y}_2(x) + c_3\vec{y}_3 + \vec{y}_p(x)$

□ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} \frac{\mathrm{e}^{-4x}}{16} & \mathrm{e}^{-4x}\left(\frac{x}{16} + \frac{1}{64}\right) & \mathrm{e}^{-x} \\ -\frac{\mathrm{e}^{-4x}}{4} & -\frac{x\,\mathrm{e}^{-4x}}{4} & -\mathrm{e}^{-x} \\ \mathrm{e}^{-4x} & x\,\mathrm{e}^{-4x} & \mathrm{e}^{-x} \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

$\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \left[\begin{array}{ccc} \frac{e^{-4x}}{16} & e^{-4x}\left(\frac{x}{16}+\frac{1}{64}\right) & e^{-x} \\ -\frac{e^{-4x}}{4} & -\frac{x\,e^{-4x}}{4} & -e^{-x} \\ e^{-4x} & x\,e^{-4x} & e^{-x} \end{array}\right] \cdot \frac{1}{\left[\begin{array}{ccc} \frac{1}{16} & \frac{1}{64} & 1 \\ -\frac{1}{4} & 0 & -1 \\ 1 & 0 & 1 \end{array}\right]}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \left[\begin{array}{ccc} e^{-4x}(4x+1) & \frac{4\,e^{-4x}}{3}+5x\,e^{-4x}-\frac{4\,e^{-x}}{3} & \frac{e^{-4x}}{3}+x\,e^{-4x}-\frac{e^{-x}}{3} \\ -16x\,e^{-4x} & -\frac{e^{-4x}}{3}-20x\,e^{-4x}+\frac{4\,e^{-x}}{3} & -\frac{e^{-4x}}{3}-4x\,e^{-4x}+\frac{e^{-x}}{3} \\ 64x\,e^{-4x} & \frac{4\,e^{-4x}}{3}+80x\,e^{-4x}-\frac{4\,e^{-x}}{3} & \frac{4\,e^{-4x}}{3}+16x\,e^{-4x}-\frac{e^{-x}}{3} \end{array}\right]$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

  $\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$

- Take the derivative of the particular solution

  $\vec{y}_p'(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- Cancel like terms

  $\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$

- Integrate to solve for $\vec{v}(x)$

  $\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$

- Plug $\vec{v}(x)$ into the equation for the particular solution

$$\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \begin{bmatrix} -\frac{3}{16} + \frac{(-420x-277)e^{-4x}}{144} + \frac{(19-24x)e^{-x}}{9} \\ \frac{(105x+43)e^{-4x}}{9} + \frac{e^{-x}(24x-43)}{9} \\ 1 + \frac{4(-105x-43)e^{-4x}}{9} + \frac{(-24x+163)e^{-x}}{9} \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1 \vec{y}_1(x) + c_2 \vec{y}_2(x) + c_3 \vec{y}_3 + \begin{bmatrix} -\frac{3}{16} + \frac{(-420x-277)e^{-4x}}{144} + \frac{(19-24x)e^{-x}}{9} \\ \frac{(105x+43)e^{-4x}}{9} + \frac{e^{-x}(24x-43)}{9} \\ 1 + \frac{4(-105x-43)e^{-4x}}{9} + \frac{(-24x+163)e^{-x}}{9} \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{((36c_2-1680)x+36c_1+9c_2-1108)e^{-4x}}{576} - \frac{3}{16} + \frac{(-1536x+576c_3+1216)e^{-x}}{576}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 30

```
dsolve(diff(y(x),x$3)+9*diff(y(x),x$2)+24*diff(y(x),x)+16*y(x)=8*exp(-x)+1,y(x), singsol=all
```

$$y(x) = \frac{1}{16} + \frac{(-16 + 24x + 27c_2)\, e^{-x}}{27} + (c_3 x + c_1)\, e^{-4x}$$

✓ Solution by Mathematica

Time used: 0.076 (sec). Leaf size: 39

```
DSolve[y'''[x]+9*y''[x]+24*y'[x]+16*y[x]==8*Exp[-x]+1,y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to e^{-4x}(c_2x + c_1) + e^{-x}\left(\frac{8x}{9} - \frac{16}{27} + c_3\right) + \frac{1}{16}$$

12.8 problem Problem 27

Internal problem ID [2831]
Internal file name [`OUTPUT/2323_Sunday_June_05_2022_02_59_08_AM_34903683/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.10, Chapter review. page 575
Problem number: Problem 27.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 4y = 5\,\mathrm{e}^{x}}$$

12.8.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = -4, f(x) = 5\,\mathrm{e}^{x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 4\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(-4)} \\ &= \pm 2\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +2 \\ \lambda_2 &= -2\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2 \\ \lambda_2 &= -2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(2)x} + c_2 e^{(-2)x}\end{aligned}$$

Or

$$y = \mathrm{e}^{2x} c_1 + c_2 \mathrm{e}^{-2x}$$

Therefore the homogeneous solution y_h is

$$y_h = \mathrm{e}^{2x} c_1 + c_2 \mathrm{e}^{-2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$5\,\mathrm{e}^{x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\mathrm{e}^{-2x}, \mathrm{e}^{2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1\mathrm{e}^{x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-3A_1\mathrm{e}^{x} = 5\,\mathrm{e}^{x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{5}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{5\,\mathrm{e}^{x}}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\mathrm{e}^{2x}c_1 + c_2\mathrm{e}^{-2x}\right) + \left(-\frac{5\,\mathrm{e}^{x}}{3}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \mathrm{e}^{2x}c_1 + c_2\mathrm{e}^{-2x} - \frac{5\,\mathrm{e}^{x}}{3} \tag{1}$$

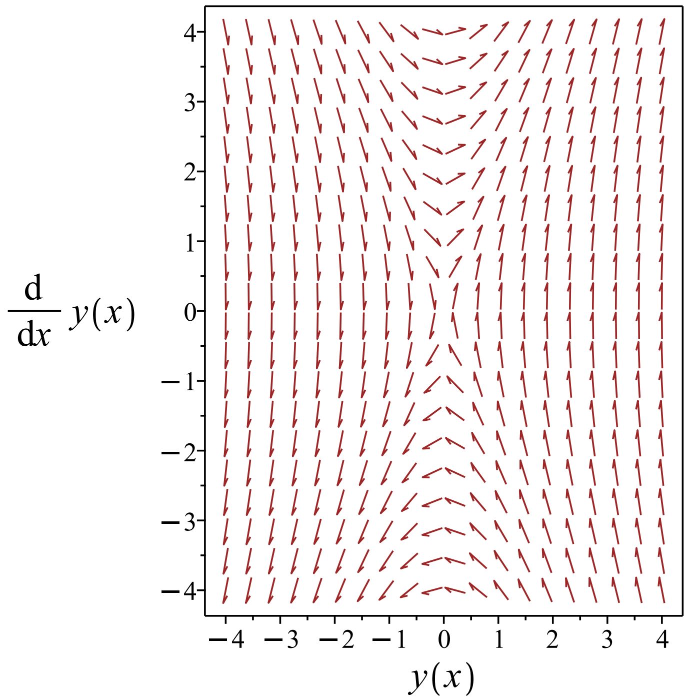


Figure 326: Slope field plot

Verification of solutions

$$y = e^{2x}c_1 + c_2e^{-2x} - \frac{5\,e^x}{3}$$

Verified OK.

12.8.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 4y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 330: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 0\\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = e^{-2x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}y_1 &= z_1\\ &= e^{-2x}\end{aligned}$$

Which simplifies to

$$y_1 = e^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}y_2 &= y_1 \int \frac{1}{y_1^2}\,dx\\ &= e^{-2x} \int \frac{1}{e^{-4x}}\,dx\\ &= e^{-2x}\left(\frac{e^{4x}}{4}\right)\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-2x}\right) + c_2 \left(\mathrm{e}^{-2x} \left(\frac{\mathrm{e}^{4x}}{4}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 4y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-2x} + \frac{c_2 \mathrm{e}^{2x}}{4}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
5\,\mathrm{e}^{x}
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\mathrm{e}^{x}\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{\frac{\mathrm{e}^{2x}}{4}, \mathrm{e}^{-2x}\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 \mathrm{e}^{x}
$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-3A_1 e^x = 5 e^x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{5}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{5 e^x}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-2x} + \frac{c_2 e^{2x}}{4}\right) + \left(-\frac{5 e^x}{3}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-2x} + \frac{c_2 e^{2x}}{4} - \frac{5 e^x}{3} \tag{1}$$

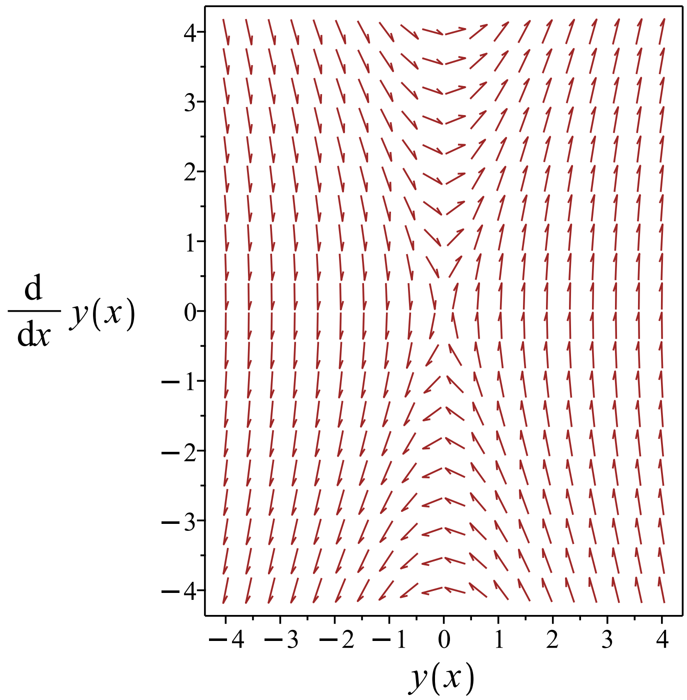


Figure 327: Slope field plot

Verification of solutions

$$y = c_1 e^{-2x} + \frac{c_2 e^{2x}}{4} - \frac{5\,e^{x}}{3}$$

Verified OK.

12.8.3 Maple step by step solution

Let's solve

$y'' - 4y = 5\,e^{x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 4 = 0$

- Factor the characteristic polynomial

  $(r-2)(r+2) = 0$

- Roots of the characteristic polynomial

$r = (-2, 2)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-2x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{2x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 5\,\mathrm{e}^{x}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-2x} & \mathrm{e}^{2x} \\ -2\,\mathrm{e}^{-2x} & 2\,\mathrm{e}^{2x} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 4$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\frac{5\,\mathrm{e}^{-2x}\left(\int \mathrm{e}^{3x} dx\right)}{4} + \frac{5\,\mathrm{e}^{2x}\left(\int \mathrm{e}^{-x} dx\right)}{4}$

  - Compute integrals

    $y_p(x) = -\frac{5\,\mathrm{e}^{x}}{3}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{2x} - \frac{5\,\mathrm{e}^{x}}{3}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 27

```
dsolve(diff(y(x),x$2)-4*y(x)=5*exp(x),y(x), singsol=all)
```

$$y(x) = -\frac{(-3\,\mathrm{e}^{4x}c_1 + 5\,\mathrm{e}^{3x} - 3c_2)\,\mathrm{e}^{-2x}}{3}$$

✓ Solution by Mathematica
Time used: 0.017 (sec). Leaf size: 29

```
DSolve[y''[x]-4*y[x]==5*Exp[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{5e^x}{3} + c_1 e^{2x} + c_2 e^{-2x}$$

12.9 problem Problem 28

Internal problem ID [2832]
Internal file name [`OUTPUT/2324_Sunday_June_05_2022_02_59_10_AM_4803815/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.10, Chapter review. page 575
Problem number: Problem 28.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 2y' + y = 2x\,\mathrm{e}^{-x}}$$

12.9.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 2, C = 1, f(x) = 2x\,\mathrm{e}^{-x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2y' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 2, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 2\lambda e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{(2)^2 - (4)\,(1)\,(1)}\\ &= -1\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 1$. Therefore the solution is

$$y = c_1 e^{-x} + c_2 x\, e^{-x} \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{-x} + c_2 x\, e^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2x\, e^{-x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\, e^{-x}, e^{-x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\, e^{-x}, e^{-x}\}$$

Since e^{-x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,\mathrm{e}^{-x}, \mathrm{e}^{-x}x^2\}]$$

Since $x\,\mathrm{e}^{-x}$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^3\mathrm{e}^{-x}, \mathrm{e}^{-x}x^2\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1x^3\mathrm{e}^{-x} + A_2\mathrm{e}^{-x}x^2$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$6A_1x\,\mathrm{e}^{-x} + 2A_2\mathrm{e}^{-x} = 2x\,\mathrm{e}^{-x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{3}, A_2 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x^3\mathrm{e}^{-x}}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{-x} + c_2x\,\mathrm{e}^{-x}\right) + \left(\frac{x^3\mathrm{e}^{-x}}{3}\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{-x}(c_2x + c_1) + \frac{x^3\mathrm{e}^{-x}}{3}$$

Summary
The solution(s) found are the following

$$y = e^{-x}(c_2 x + c_1) + \frac{x^3 e^{-x}}{3} \tag{1}$$

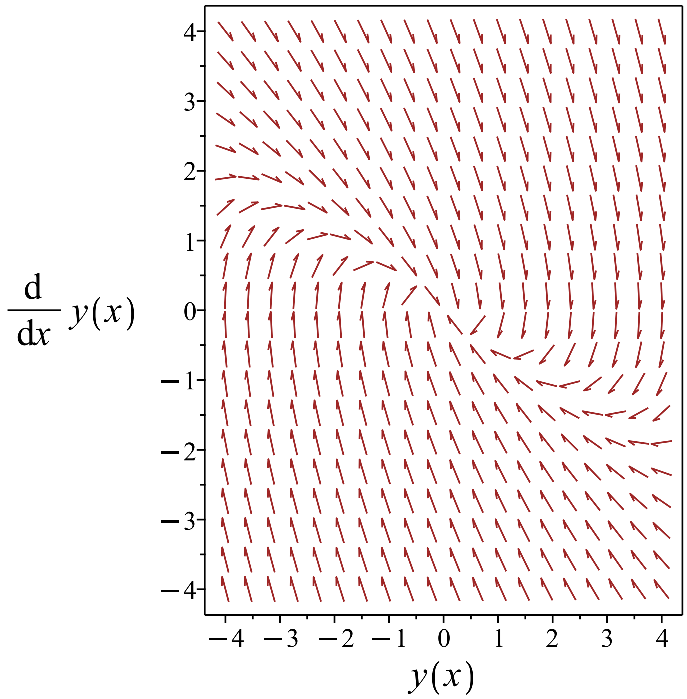


Figure 328: Slope field plot

Verification of solutions

$$y = e^{-x}(c_2 x + c_1) + \frac{x^3 e^{-x}}{3}$$

Verified OK.

12.9.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = 2$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int 2\,dx} \\ &= \mathrm{e}^{x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)y)'' &= 2\,\mathrm{e}^{x} x\,\mathrm{e}^{-x} \\ (\mathrm{e}^{x} y)'' &= 2\,\mathrm{e}^{x} x\,\mathrm{e}^{-x} \end{aligned}$$

Integrating once gives

$$(\mathrm{e}^{x} y)' = x^2 + c_1$$

Integrating again gives

$$(\mathrm{e}^{x} y) = \frac{1}{3}x^3 + c_1 x + c_2$$

Hence the solution is

$$y = \frac{\frac{1}{3}x^3 + c_1 x + c_2}{\mathrm{e}^{x}}$$

Or

$$y = \frac{x^3 \mathrm{e}^{-x}}{3} + c_1 x\,\mathrm{e}^{-x} + c_2 \mathrm{e}^{-x}$$

Summary

The solution(s) found are the following

$$y = \frac{x^3 \mathrm{e}^{-x}}{3} + c_1 x\,\mathrm{e}^{-x} + c_2 \mathrm{e}^{-x} \tag{1}$$

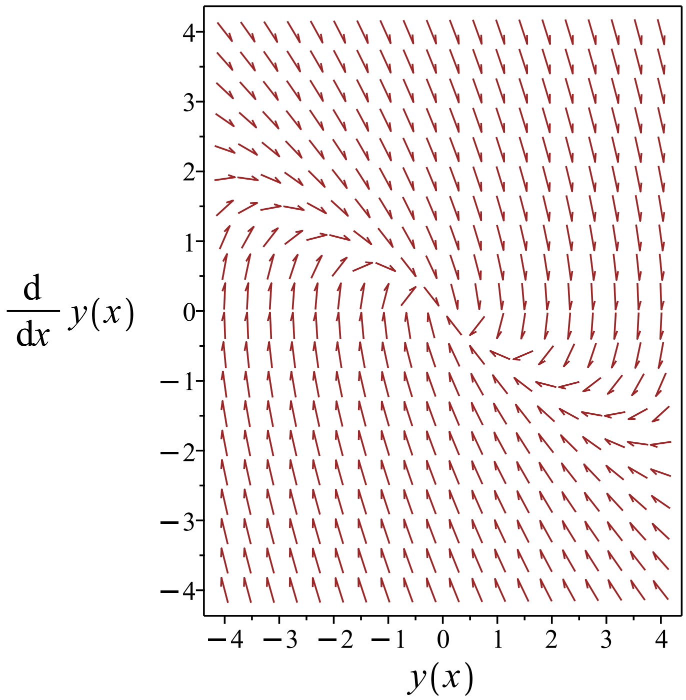


Figure 329: Slope field plot

Verification of solutions

$$y = \frac{x^3 e^{-x}}{3} + c_1 x e^{-x} + c_2 e^{-x}$$

Verified OK.

12.9.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 332: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dx} \\
&= z_1 e^{-x} \\
&= z_1\left(\mathrm{e}^{-x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2x}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-x}\right) + c_2\left(\mathrm{e}^{-x}(x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$
y'' + 2y' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-x} + c_2 x\,\mathrm{e}^{-x}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
2x\,\mathrm{e}^{-x}
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{x\,\mathrm{e}^{-x}, \mathrm{e}^{-x}\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{x\,\mathrm{e}^{-x}, \mathrm{e}^{-x}\}
$$

Since e^{-x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$
[\{x\,\mathrm{e}^{-x}, \mathrm{e}^{-x}x^2\}]
$$

Since $x\,\mathrm{e}^{-x}$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$
[\{x^3\mathrm{e}^{-x}, \mathrm{e}^{-x}x^2\}]
$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^3 \mathrm{e}^{-x} + A_2 \mathrm{e}^{-x} x^2$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$6A_1 x\,\mathrm{e}^{-x} + 2A_2 \mathrm{e}^{-x} = 2x\,\mathrm{e}^{-x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{3}, A_2 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x^3 \mathrm{e}^{-x}}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-x} + c_2 x\,\mathrm{e}^{-x}\right) + \left(\frac{x^3 \mathrm{e}^{-x}}{3}\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{-x}(c_2 x + c_1) + \frac{x^3 \mathrm{e}^{-x}}{3}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-x}(c_2 x + c_1) + \frac{x^3 \mathrm{e}^{-x}}{3} \tag{1}$$

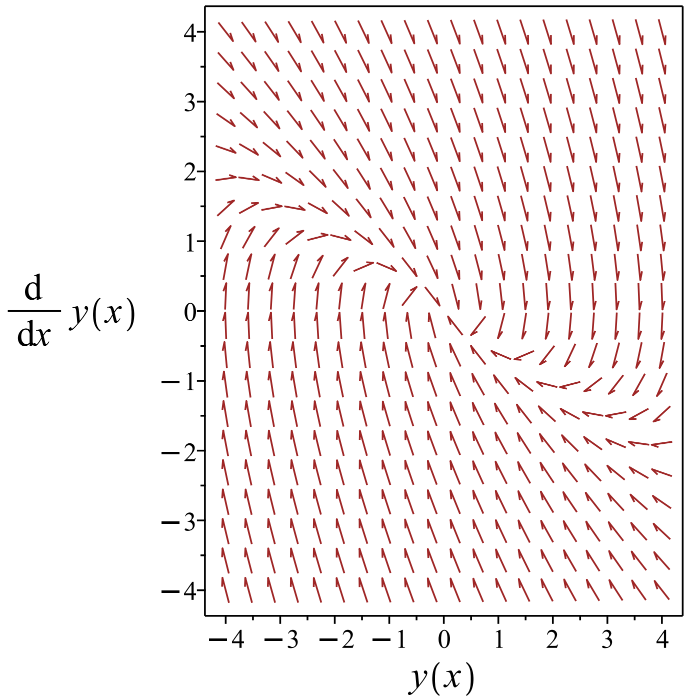


Figure 330: Slope field plot

Verification of solutions

$$y = e^{-x}(c_2 x + c_1) + \frac{x^3 e^{-x}}{3}$$

Verified OK.

12.9.4 Maple step by step solution

Let's solve

$y'' + 2y' + y = 2x\,e^{-x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 2r + 1 = 0$

- Factor the characteristic polynomial

  $(r+1)^2 = 0$

- Root of the characteristic polynomial

$r = -1$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,e^{-x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-x} + c_2 x\,e^{-x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 2x\,e^{-x}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \left[\begin{array}{cc} e^{-x} & x\,e^{-x} \\ -e^{-x} & e^{-x} - x\,e^{-x} \end{array}\right]$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = e^{-2x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -2\,e^{-x}\left(\int x^2 dx - \left(\int x dx\right) x\right)$

  - Compute integrals

    $y_p(x) = \frac{x^3 e^{-x}}{3}$

- Substitute particular solution into general solution to ODE

  $y = c_2 x\,e^{-x} + c_1 e^{-x} + \frac{x^3 e^{-x}}{3}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(diff(y(x),x$2)+2*diff(y(x),x)+y(x)=2*x*exp(-x),y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{-x}\left(c_2 + c_1 x + \frac{1}{3}x^3\right)$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 27

```
DSolve[y''[x]+2*y'[x]+y[x]==2*x*Exp[-x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{3}e^{-x}\left(x^3 + 3c_2 x + 3c_1\right)$$

12.10 problem Problem 29

Internal problem ID [2833]
Internal file name [`OUTPUT/2325_Sunday_June_05_2022_02_59_13_AM_75321258/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.10, Chapter review. page 575
Problem number: Problem 29.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - y = 4\,\mathrm{e}^{x}}$$

12.10.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = -1, f(x) = 4\,\mathrm{e}^{x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-1)}\\ &= \pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +1\\ \lambda_2 &= -1\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1\\ \lambda_2 &= -1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(1)x} + c_2 e^{(-1)x}\end{aligned}$$

Or

$$y = c_1 e^x + c_2 e^{-x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^x + c_2 e^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$4\,\mathrm{e}^{x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\mathrm{e}^{x}, \mathrm{e}^{-x}\}$$

Since e^{x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,\mathrm{e}^{x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\,\mathrm{e}^{x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1\mathrm{e}^{x} = 4\,\mathrm{e}^{x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2x\,\mathrm{e}^{x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{x} + c_2\mathrm{e}^{-x}\right) + \left(2x\,\mathrm{e}^{x}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-x} + 2x\,\mathrm{e}^{x} \tag{1}$$

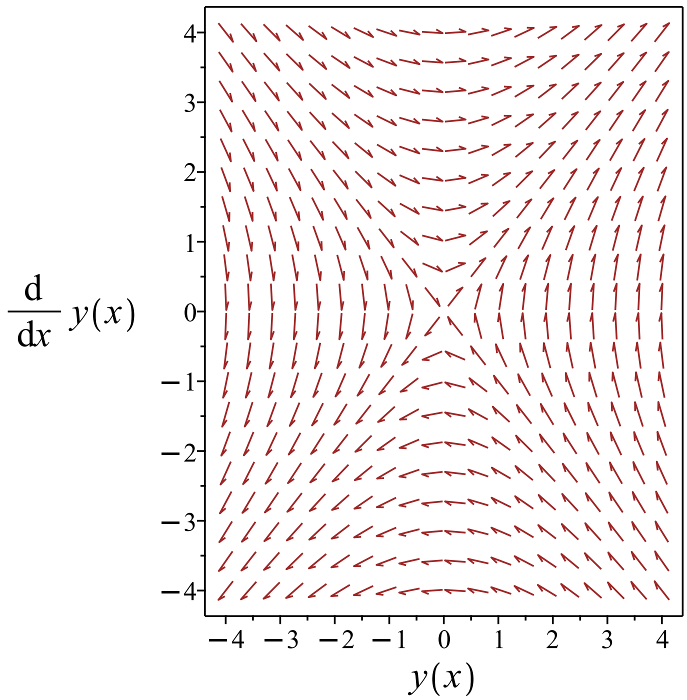


Figure 331: Slope field plot

Verification of solutions

$$y = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-x} + 2x\,\mathrm{e}^{x}$$

Verified OK.

12.10.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 334: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-x}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \mathrm{e}^{-x}
\end{aligned}
$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{\mathrm{e}^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= \mathrm{e}^{-x} \int \frac{1}{\mathrm{e}^{-2x}}\, dx \\
&= \mathrm{e}^{-x} \left(\frac{\mathrm{e}^{2x}}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-x}\right) + c_2 \left(\mathrm{e}^{-x} \left(\frac{\mathrm{e}^{2x}}{2}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{x}}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \mathrm{e}^{-x}$$
$$y_2 = \frac{\mathrm{e}^{x}}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{-x} & \frac{\mathrm{e}^{x}}{2} \\ \frac{d}{dx}(\mathrm{e}^{-x}) & \frac{d}{dx}\left(\frac{\mathrm{e}^{x}}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{-x} & \frac{\mathrm{e}^{x}}{2} \\ -\mathrm{e}^{-x} & \frac{\mathrm{e}^{x}}{2} \end{vmatrix}$$

Therefore

$$W = (\mathrm{e}^{-x})\left(\frac{\mathrm{e}^{x}}{2}\right) - \left(\frac{\mathrm{e}^{x}}{2}\right)(-\mathrm{e}^{-x})$$

Which simplifies to

$$W = \mathrm{e}^{-x}\mathrm{e}^{x}$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{2\,\mathrm{e}^{2x}}{1}\,dx$$

Which simplifies to

$$u_1 = -\int 2\,\mathrm{e}^{2x}dx$$

Hence

$$u_1 = -\mathrm{e}^{2x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{4\,\mathrm{e}^{-x}\mathrm{e}^{x}}{1}\,dx$$

Which simplifies to

$$u_2 = \int 4dx$$

Hence

$$u_2 = 4x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\mathrm{e}^{2x}\mathrm{e}^{-x} + 2x\,\mathrm{e}^{x}$$

Which simplifies to

$$y_p(x) = \mathrm{e}^{x}(2x - 1)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{-x} + \frac{c_2\mathrm{e}^{x}}{2}\right) + (\mathrm{e}^{x}(2x - 1)) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{x}}{2} + \mathrm{e}^{x}(2x - 1) \tag{1}$$

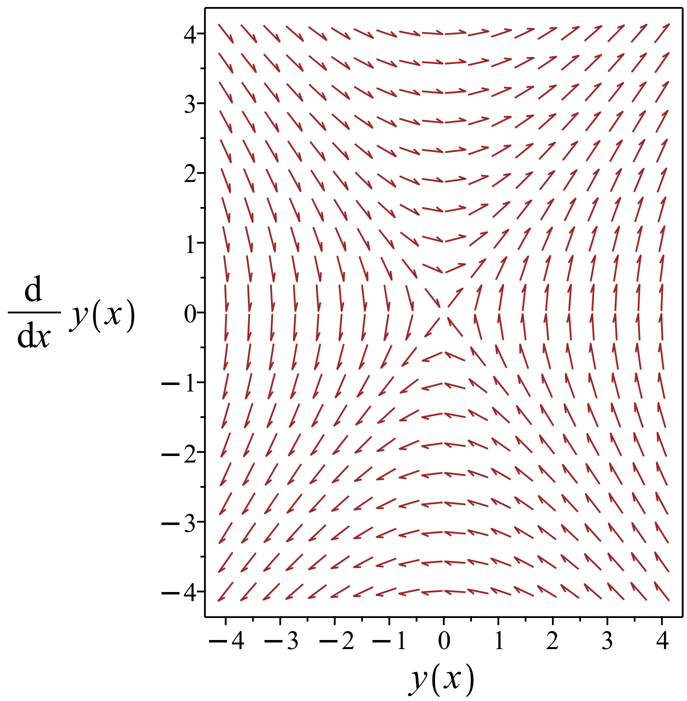


Figure 332: Slope field plot

Verification of solutions

$$y = c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{x}}{2} + \mathrm{e}^{x}(2x - 1)$$

Verified OK.

12.10.3 Maple step by step solution

Let's solve

$y'' - y = 4\,\mathrm{e}^{x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2 - 1 = 0$

- Factor the characteristic polynomial

  $(r-1)(r+1) = 0$

- Roots of the characteristic polynomial

  $r = (-1, 1)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x),y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x),y_2(x))} dx\right), f(x) = 4\,\mathrm{e}^{x}\right]$

  ◦ Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-x} & \mathrm{e}^{x} \\ -\mathrm{e}^{-x} & \mathrm{e}^{x} \end{bmatrix}$

  ◦ Compute Wronskian

    $W(y_1(x), y_2(x)) = 2$

  ◦ Substitute functions into equation for $y_p(x)$

    $y_p(x) = -2\,\mathrm{e}^{-x}\left(\int \mathrm{e}^{2x} dx\right) + 2\,\mathrm{e}^{x}\left(\int 1 dx\right)$

  ◦ Compute integrals

    $y_p(x) = \mathrm{e}^{x}(2x - 1)$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + \mathrm{e}^{x}(2x - 1)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 20

```
dsolve(diff(y(x),x$2)-y(x)=4*exp(x),y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{-x} c_2 + 2\,\mathrm{e}^{x}\left(x + \frac{c_1}{2}\right)$$

✓ Solution by Mathematica

Time used: 0.017 (sec). Leaf size: 25

```
DSolve[y''[x]-y[x]==4*Exp[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^x(2x - 1 + c_1) + c_2 e^{-x}$$

12.11 problem Problem 30

Internal problem ID [2834]
Internal file name [`OUTPUT/2326_Sunday_June_05_2022_02_59_15_AM_6621393/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.10, Chapter review. page 575
Problem number: Problem 30.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_airy", "second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + yx = \sin(x)}$$

12.11.1 Solving as second order airy ode

This is Airy ODE. It has the general form

$$ay'' + by' + cyx = F(x)$$

Where in this case

$$\begin{aligned} a &= 1 \\ b &= 0 \\ c &= 1 \\ F &= \sin(x) \end{aligned}$$

Therefore the solution to the homogeneous Airy ODE becomes

$$y = \mathrm{e}^{-\frac{bx}{2a}} \left(c_1 \operatorname{AiryAi}\left(-\frac{\left(\frac{c}{a}\right)^{\frac{1}{3}} \left(cxa - \frac{b^2}{4}\right)}{ca} \right) + c_2 \operatorname{AiryBi}\left(-\frac{\left(\frac{c}{a}\right)^{\frac{1}{3}} \left(cxa - \frac{b^2}{4}\right)}{ca} \right) \right)$$

Substituting the values for a, b, c gives

$$y = c_1 \operatorname{AiryAi}(-x) + c_2 \operatorname{AiryBi}(-x)$$

Since this is inhomogeneous Airy ODE, then we need to find the particular solution and add that to the homogeneous above. The particular solution is found using variation of parameters. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \operatorname{AiryAi}(-x)$$
$$y_2 = \operatorname{AiryBi}(-x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \operatorname{AiryAi}(-x) & \operatorname{AiryBi}(-x) \\ \frac{d}{dx}(\operatorname{AiryAi}(-x)) & \frac{d}{dx}(\operatorname{AiryBi}(-x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \operatorname{AiryAi}(-x) & \operatorname{AiryBi}(-x) \\ -\operatorname{AiryAi}(1, -x) & -\operatorname{AiryBi}(1, -x) \end{vmatrix}$$

Therefore

$$W = (\operatorname{AiryAi}(-x))(-\operatorname{AiryBi}(1, -x)) - (\operatorname{AiryBi}(-x))(-\operatorname{AiryAi}(1, -x))$$

Which simplifies to

$$W = -\operatorname{AiryAi}(-x)\operatorname{AiryBi}(1,-x) + \operatorname{AiryBi}(-x)\operatorname{AiryAi}(1,-x)$$

Which simplifies to

$$W = -\frac{1}{\pi}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\operatorname{AiryBi}(-x)\sin(x)}{-\frac{1}{\pi}}dx$$

Which simplifies to

$$u_1 = -\int -\operatorname{AiryBi}(-x)\sin(x)\pi dx$$

Hence

$$u_1 = -\left(\int_0^x -\operatorname{AiryBi}(-\alpha)\sin(\alpha)\pi d\alpha\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\operatorname{AiryAi}(-x)\sin(x)}{-\frac{1}{\pi}}dx$$

Which simplifies to

$$u_2 = \int -\operatorname{AiryAi}(-x)\sin(x)\pi dx$$

Hence

$$u_2 = \int_0^x -\operatorname{AiryAi}(-\alpha)\sin(\alpha)\pi d\alpha$$

Which simplifies to

$$u_1 = \pi\left(\int_0^x \operatorname{AiryBi}(-\alpha)\sin(\alpha)\,d\alpha\right)$$
$$u_2 = -\pi\left(\int_0^x \operatorname{AiryAi}(-\alpha)\sin(\alpha)\,d\alpha\right)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \pi\left(\int_0^x \operatorname{AiryBi}(-\alpha)\sin(\alpha)\,d\alpha\right)\operatorname{AiryAi}(-x) - \pi\left(\int_0^x \operatorname{AiryAi}(-\alpha)\sin(\alpha)\,d\alpha\right)\operatorname{AiryBi}(-x)$$

Which simplifies to

$$y_p(x) = \pi\left(\left(\int_0^x \operatorname{AiryBi}(-\alpha)\sin(\alpha)\,d\alpha\right)\operatorname{AiryAi}(-x) - \left(\int_0^x \operatorname{AiryAi}(-\alpha)\sin(\alpha)\,d\alpha\right)\operatorname{AiryBi}(-x)\right)$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p \\
&= (c_1 \operatorname{AiryAi}(-x) + c_2 \operatorname{AiryBi}(-x)) + \left(\pi\left(\left(\int_0^x \operatorname{AiryBi}(-\alpha)\sin(\alpha)\,d\alpha\right)\operatorname{AiryAi}(-x) - \left(\int_0^x \operatorname{AiryAi}(-\alpha)\sin(\alpha)\,d\alpha\right)\operatorname{AiryBi}(-x)\right)\right) \\
&= \pi\left(\left(\int_0^x \operatorname{AiryBi}(-\alpha)\sin(\alpha)\,d\alpha\right)\operatorname{AiryAi}(-x) - \left(\int_0^x \operatorname{AiryAi}(-\alpha)\sin(\alpha)\,d\alpha\right)\operatorname{AiryBi}(-x)\right) + c_1 \operatorname{AiryAi}(-x) + c_2 \operatorname{AiryBi}(-x)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \pi\left(\left(\int_0^x \operatorname{AiryBi}(-\alpha)\sin(\alpha)\,d\alpha\right)\operatorname{AiryAi}(-x) - \left(\int_0^x \operatorname{AiryAi}(-\alpha)\sin(\alpha)\,d\alpha\right)\operatorname{AiryBi}(-x)\right) + c_1 \operatorname{AiryAi}(-x) + c_2 \operatorname{AiryBi}(-x) \tag{1}$$

Verification of solutions

$$y = \pi\left(\left(\int_0^x \operatorname{AiryBi}(-\alpha)\sin(\alpha)\,d\alpha\right)\operatorname{AiryAi}(-x) - \left(\int_0^x \operatorname{AiryAi}(-\alpha)\sin(\alpha)\,d\alpha\right)\operatorname{AiryBi}(-x)\right) + c_1 \operatorname{AiryAi}(-x) + c_2 \operatorname{AiryBi}(-x)$$

Verified OK.

12.11.2 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' + yx^3 = x^2 \sin(x) \tag{1}$$

Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE and y_p is a particular solution to the non-homogeneous ODE. Bessel ode has the form

$$x^2y'' + xy' + (-n^2 + x^2)\, y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1-2\alpha)\, xy' + (\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2)\, y = 0 \tag{3}$$

With the standard solution

$$y = x^\alpha(c_1 \operatorname{BesselJ}(n, \beta\, x^\gamma) + c_2 \operatorname{BesselY}(n, \beta\, x^\gamma)) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned}
\alpha &= \frac{1}{2} \\
\beta &= \frac{2}{3} \\
n &= \frac{1}{3} \\
\gamma &= \frac{3}{2}
\end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = c_1\sqrt{x}\, \operatorname{BesselJ}\left(\frac{1}{3}, \frac{2x^{\frac{3}{2}}}{3}\right) + c_2\sqrt{x}\, \operatorname{BesselY}\left(\frac{1}{3}, \frac{2x^{\frac{3}{2}}}{3}\right)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1\sqrt{x}\, \operatorname{BesselJ}\left(\frac{1}{3}, \frac{2x^{\frac{3}{2}}}{3}\right) + c_2\sqrt{x}\, \operatorname{BesselY}\left(\frac{1}{3}, \frac{2x^{\frac{3}{2}}}{3}\right)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \operatorname{AiryAi}(-x)$$
$$y_2 = \operatorname{AiryBi}(-x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \operatorname{AiryAi}(-x) & \operatorname{AiryBi}(-x) \\ \frac{d}{dx}(\operatorname{AiryAi}(-x)) & \frac{d}{dx}(\operatorname{AiryBi}(-x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \operatorname{AiryAi}(-x) & \operatorname{AiryBi}(-x) \\ -\operatorname{AiryAi}(1, -x) & -\operatorname{AiryBi}(1, -x) \end{vmatrix}$$

Therefore

$$W = (\operatorname{AiryAi}(-x))(-\operatorname{AiryBi}(1, -x)) - (\operatorname{AiryBi}(-x))(-\operatorname{AiryAi}(1, -x))$$

Which simplifies to

$$W = -\operatorname{AiryAi}(-x)\operatorname{AiryBi}(1, -x) + \operatorname{AiryBi}(-x)\operatorname{AiryAi}(1, -x)$$

Which simplifies to

$$W = -\frac{1}{\pi}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\operatorname{AiryBi}(-x)\, x^2 \sin(x)}{-\frac{x^2}{\pi}} dx$$

Which simplifies to

$$u_1 = -\int -\operatorname{AiryBi}(-x) \sin(x)\, \pi dx$$

Hence

$$u_1 = -\left(\int_0^x -\operatorname{AiryBi}(-\alpha) \sin(\alpha)\, \pi d\alpha\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\operatorname{AiryAi}(-x)\, x^2 \sin(x)}{-\frac{x^2}{\pi}} dx$$

Which simplifies to

$$u_2 = \int -\operatorname{AiryAi}(-x) \sin(x)\, \pi dx$$

Hence

$$u_2 = \int_0^x -\operatorname{AiryAi}(-\alpha) \sin(\alpha)\, \pi d\alpha$$

Which simplifies to

$$u_1 = \pi\left(\int_0^x \operatorname{AiryBi}(-\alpha) \sin(\alpha)\, d\alpha\right)$$
$$u_2 = -\pi\left(\int_0^x \operatorname{AiryAi}(-\alpha) \sin(\alpha)\, d\alpha\right)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \pi\left(\int_0^x \operatorname{AiryBi}(-\alpha) \sin(\alpha)\, d\alpha\right) \operatorname{AiryAi}(-x) - \pi\left(\int_0^x \operatorname{AiryAi}(-\alpha) \sin(\alpha)\, d\alpha\right) \operatorname{AiryBi}(-x)$$

Which simplifies to

$$y_p(x) = \pi\left(\left(\int_0^x \text{AiryBi}\left(-\alpha\right)\sin\left(\alpha\right)d\alpha\right)\text{AiryAi}\left(-x\right) - \left(\int_0^x \text{AiryAi}\left(-\alpha\right)\sin\left(\alpha\right)d\alpha\right)\text{AiryBi}\left(-x\right)\right)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\sqrt{x}\,\text{BesselJ}\left(\frac{1}{3}, \frac{2x^{\frac{3}{2}}}{3}\right) + c_2\sqrt{x}\,\text{BesselY}\left(\frac{1}{3}, \frac{2x^{\frac{3}{2}}}{3}\right)\right) \\ &\quad + \left(\pi\left(\left(\int_0^x \text{AiryBi}\left(-\alpha\right)\sin\left(\alpha\right)d\alpha\right)\text{AiryAi}\left(-x\right) - \left(\int_0^x \text{AiryAi}\left(-\alpha\right)\sin\left(\alpha\right)d\alpha\right)\text{AiryBi}\left(-x\right)\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y &= c_1\sqrt{x}\,\text{BesselJ}\left(\frac{1}{3}, \frac{2x^{\frac{3}{2}}}{3}\right) + c_2\sqrt{x}\,\text{BesselY}\left(\frac{1}{3}, \frac{2x^{\frac{3}{2}}}{3}\right) \\ &\quad + \pi\left(\left(\int_0^x \text{AiryBi}\left(-\alpha\right)\sin\left(\alpha\right)d\alpha\right)\text{AiryAi}\left(-x\right) - \left(\int_0^x \text{AiryAi}\left(-\alpha\right)\sin\left(\alpha\right)d\alpha\right)\text{AiryBi}\left(-x\right)\right) \end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned} y &= c_1\sqrt{x}\,\text{BesselJ}\left(\frac{1}{3}, \frac{2x^{\frac{3}{2}}}{3}\right) + c_2\sqrt{x}\,\text{BesselY}\left(\frac{1}{3}, \frac{2x^{\frac{3}{2}}}{3}\right) \\ &\quad + \pi\left(\left(\int_0^x \text{AiryBi}\left(-\alpha\right)\sin\left(\alpha\right)d\alpha\right)\text{AiryAi}\left(-x\right) - \left(\int_0^x \text{AiryAi}\left(-\alpha\right)\sin\left(\alpha\right)d\alpha\right)\text{AiryBi}\left(-x\right)\right) \end{aligned}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 48

```
dsolve(diff(y(x),x$2)+x*y(x)=sin(x),y(x), singsol=all)
```

$$
\begin{aligned}
y(x) = {} & \pi\left(\int \operatorname{AiryBi}(-x)\sin(x)\,dx\right)\operatorname{AiryAi}(-x) \\
& - \pi\left(\int \operatorname{AiryAi}(-x)\sin(x)\,dx\right)\operatorname{AiryBi}(-x) + \operatorname{AiryBi}(-x)\,c_1 + \operatorname{AiryAi}(-x)\,c_2
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 105.448 (sec). Leaf size: 99

```
DSolve[y''[x]+x*y[x]==Sin[x],y[x],x,IncludeSingularSolutions -> True]
```

$$
\begin{aligned}
y(x) \to \operatorname{AiryAi}\left(\sqrt[3]{-1}x\right) \int_1^x (-1)^{2/3}\pi \operatorname{AiryBi}\left(\sqrt[3]{-1}K[1]\right) \sin(K[1])dK[1] \\
+ \operatorname{AiryBi}\left(\sqrt[3]{-1}x\right) \int_1^x -(-1)^{2/3}\pi \operatorname{AiryAi}\left(\sqrt[3]{-1}K[2]\right) \sin(K[2])dK[2] \\
+ c_1 \operatorname{AiryAi}\left(\sqrt[3]{-1}x\right) + c_2 \operatorname{AiryBi}\left(\sqrt[3]{-1}x\right)
\end{aligned}
$$

12.12 problem Problem 31

Internal problem ID [2835]
Internal file name [`OUTPUT/2327_Sunday_June_05_2022_02_59_17_AM_17275767/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.10, Chapter review. page 575
Problem number: Problem 31.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 4y = \ln(x)}$$

12.12.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 4, f(x) = \ln(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 4\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(4)} \\ &= \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +2i \\ \lambda_2 &= -2i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2i \\ \lambda_2 &= -2i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(2x) + c_2 \sin(2x))$$

Or

$$y = c_1 \cos(2x) + c_2 \sin(2x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(2x) + c_2 \sin(2x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= \cos(2x) \\ y_2 &= \sin(2x) \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(2x) & \sin(2x) \\ \frac{d}{dx}(\cos(2x)) & \frac{d}{dx}(\sin(2x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(2x) & \sin(2x) \\ -2\sin(2x) & 2\cos(2x) \end{vmatrix}$$

Therefore

$$W = (\cos(2x))(2\cos(2x)) - (\sin(2x))(-2\sin(2x))$$

Which simplifies to

$$W = 2\cos(2x)^2 + 2\sin(2x)^2$$

Which simplifies to

$$W = 2$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(2x)\ln(x)}{2}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{\sin(2x)\ln(x)}{2}dx$$

Hence

$$u_1 = \frac{i\pi(\operatorname{csgn}(x)-1)\operatorname{csgn}(ix)}{8} + \frac{\cos(2x)\ln(x)}{4} + \frac{\gamma}{4} + \frac{\ln(2)}{4} - \frac{\operatorname{Ci}(2x)}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(2x)\ln(x)}{2}\,dx$$

Which simplifies to

$$u_2 = \int \frac{\cos(2x)\ln(x)}{2}dx$$

Hence

$$u_2 = \frac{\sin(2x)\ln(x)}{4} + \frac{\pi}{8} - \frac{\operatorname{Si}(2x)}{4}$$

Which simplifies to

$$u_1 = \frac{\cos(2x)\ln(x)}{4} + \frac{\gamma}{4} + \frac{\ln(2)}{4} - \frac{\operatorname{Ci}(2x)}{4}$$
$$u_2 = \frac{\sin(2x)\ln(x)}{4} + \frac{\pi}{8} - \frac{\operatorname{Si}(2x)}{4}$$

Therefore the particular solution, from equation (1) is

$$
\begin{aligned}
y_p(x) = {} & \left(\frac{\cos (2x) \ln (x)}{4}+\frac{\gamma}{4}+\frac{\ln (2)}{4}-\frac{\operatorname{Ci}(2x)}{4}\right) \cos (2x) \\
& + \left(\frac{\sin (2x) \ln (x)}{4}+\frac{\pi}{8}-\frac{\operatorname{Si}(2x)}{4}\right) \sin (2x)
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
y_p(x) = {} & \frac{\ln (x) \cos (2x)^2}{4}+\frac{(2\gamma+2\ln (2)-2 \operatorname{Ci}(2x)) \cos (2x)}{8} \\
& + \frac{\sin (2x) (2\sin (2x) \ln (x)-2 \operatorname{Si}(2x)+\pi)}{8}
\end{aligned}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 \cos (2x)+c_2 \sin (2x)) + \left(\frac{\ln (x) \cos (2x)^2}{4}+\frac{(2\gamma+2\ln (2)-2 \operatorname{Ci}(2x)) \cos (2x)}{8}\right. \\
&\qquad \left. + \frac{\sin (2x) (2\sin (2x) \ln (x)-2 \operatorname{Si}(2x)+\pi)}{8}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & c_1 \cos (2x)+c_2 \sin (2x)+\frac{\ln (x) \cos (2x)^2}{4}+\frac{(2\gamma+2\ln (2)-2 \operatorname{Ci}(2x)) \cos (2x)}{8} \\
& + \frac{\sin (2x) (2\sin (2x) \ln (x)-2 \operatorname{Si}(2x)+\pi)}{8}
\end{aligned} \tag{1}
$$

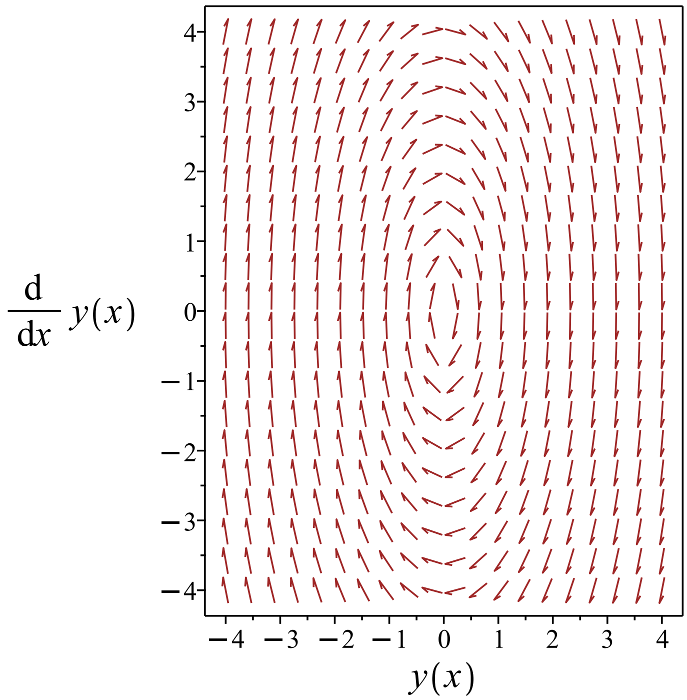


Figure 333: Slope field plot

Verification of solutions

$$y = c_1 \cos(2x) + c_2 \sin(2x) + \frac{\ln(x)\cos(2x)^2}{4} + \frac{(2\gamma + 2\ln(2) - 2\,\mathrm{Ci}(2x))\cos(2x)}{8} + \frac{\sin(2x)(2\sin(2x)\ln(x) - 2\,\mathrm{Si}(2x) + \pi)}{8}$$

Verified OK. `{0 < x}`

12.12.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 336: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(2x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(2x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(2x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \cos(2x) \int \frac{1}{\cos(2x)^2}\, dx \\ &= \cos(2x) \left(\frac{\tan(2x)}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(2x)) + c_2\left(\cos(2x)\left(\frac{\tan(2x)}{2}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(2x)$$
$$y_2 = \frac{\sin(2x)}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(2x) & \frac{\sin(2x)}{2} \\ \frac{d}{dx}(\cos(2x)) & \frac{d}{dx}\left(\frac{\sin(2x)}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(2x) & \frac{\sin(2x)}{2} \\ -2\sin(2x) & \cos(2x) \end{vmatrix}$$

Therefore

$$W = (\cos(2x))(\cos(2x)) - \left(\frac{\sin(2x)}{2}\right)(-2\sin(2x))$$

Which simplifies to

$$W = \cos(2x)^2 + \sin(2x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\sin(2x)\ln(x)}{2}}{1} dx$$

Which simplifies to

$$u_1 = -\int \frac{\sin(2x)\ln(x)}{2} dx$$

Hence

$$u_1 = \frac{i\pi(\operatorname{csgn}(x) - 1)\operatorname{csgn}(ix)}{8} + \frac{\cos(2x)\ln(x)}{4} + \frac{\gamma}{4} + \frac{\ln(2)}{4} - \frac{\operatorname{Ci}(2x)}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(2x)\ln(x)}{1} dx$$

Which simplifies to

$$u_2 = \int \cos(2x)\ln(x)\, dx$$

Hence

$$u_2 = \frac{\sin(2x)\ln(x)}{2} + \frac{\pi}{4} - \frac{\operatorname{Si}(2x)}{2}$$

Which simplifies to

$$u_1 = \frac{\cos(2x)\ln(x)}{4} + \frac{\gamma}{4} + \frac{\ln(2)}{4} - \frac{\operatorname{Ci}(2x)}{4}$$
$$u_2 = \frac{\sin(2x)\ln(x)}{2} + \frac{\pi}{4} - \frac{\operatorname{Si}(2x)}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(\frac{\cos(2x)\ln(x)}{4} + \frac{\gamma}{4} + \frac{\ln(2)}{4} - \frac{\operatorname{Ci}(2x)}{4}\right)\cos(2x) + \frac{\left(\frac{\sin(2x)\ln(x)}{2} + \frac{\pi}{4} - \frac{\operatorname{Si}(2x)}{2}\right)\sin(2x)}{2}$$

Which simplifies to

$$y_p(x) = \frac{\ln(x)\cos(2x)^2}{4} + \frac{(2\gamma + 2\ln(2) - 2\,\operatorname{Ci}(2x))\cos(2x)}{8} + \frac{\sin(2x)(2\sin(2x)\ln(x) - 2\,\operatorname{Si}(2x) + \pi)}{8}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\cos(2x) + \frac{c_2\sin(2x)}{2}\right) + \left(\frac{\ln(x)\cos(2x)^2}{4} + \frac{(2\gamma + 2\ln(2) - 2\,\operatorname{Ci}(2x))\cos(2x)}{8} + \frac{\sin(2x)(2\sin(2x)\ln(x) - 2\,\operatorname{Si}(2x) + \pi)}{8}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\cos(2x) + \frac{c_2\sin(2x)}{2} + \frac{\ln(x)\cos(2x)^2}{4} + \frac{(2\gamma + 2\ln(2) - 2\,\operatorname{Ci}(2x))\cos(2x)}{8} + \frac{\sin(2x)(2\sin(2x)\ln(x) - 2\,\operatorname{Si}(2x) + \pi)}{8} \tag{1}$$

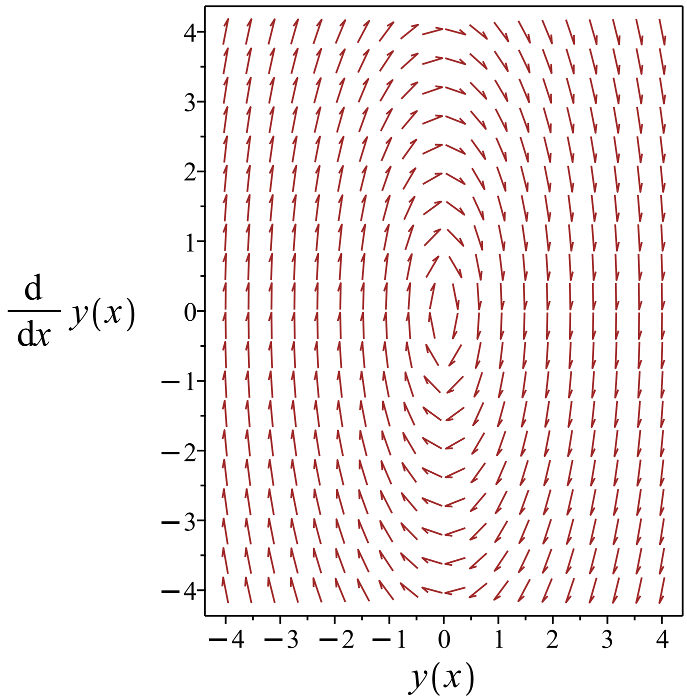


Figure 334: Slope field plot

Verification of solutions

$$y = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2} + \frac{\ln(x)\cos(2x)^2}{4} + \frac{(2\gamma + 2\ln(2) - 2\,\mathrm{Ci}(2x))\cos(2x)}{8} + \frac{\sin(2x)(2\sin(2x)\ln(x) - 2\,\mathrm{Si}(2x) + \pi)}{8}$$

Verified OK. `{0 < x}`

12.12.3 Maple step by step solution

Let's solve

$y'' + 4y = \ln(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4 = 0$

- Use quadratic formula to solve for r

$r = \frac{0 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (-2\,\mathrm{I}, 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(2x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(2x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(2x) + c_2 \sin(2x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \ln(x)\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(2x) & \sin(2x) \\ -2\sin(2x) & 2\cos(2x) \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = 2$

- ○ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\frac{\cos(2x)\left(\int \sin(2x) \ln(x) dx\right)}{2} + \frac{\sin(2x)\left(\int \cos(2x) \ln(x) dx\right)}{2}$

- ○ Compute integrals

  $y_p(x) = \frac{\mathrm{I} \cos(2x)(\mathrm{csgn}(x)-1)\pi\, \mathrm{csgn}(\mathrm{I}x)}{8} - \frac{\cos(2x)\mathrm{Ci}(2x)}{4} + \frac{(\pi\, \mathrm{csgn}(x) - 2\,\mathrm{Si}(2x))\sin(2x)}{8} + \frac{\ln(x)}{4}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(2x) + c_2 \sin(2x) + \frac{\mathrm{I} \cos(2x)(\mathrm{csgn}(x)-1)\pi\, \mathrm{csgn}(\mathrm{I}x)}{8} - \frac{\cos(2x)\mathrm{Ci}(2x)}{4} + \frac{(\pi\, \mathrm{csgn}(x) - 2\,\mathrm{Si}(2x))\sin(2x)}{8} + \frac{\ln(}{4}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 62

```
dsolve(diff(y(x),x$2)+4*y(x)=ln(x),y(x), singsol=all)
```

$$
\begin{aligned}
y(x) = & \frac{i \cos(2x) \pi(-1 + \operatorname{csgn}(x)) \operatorname{csgn}(ix)}{8} + \frac{(8c_1 - 2 \operatorname{Ci}(2x)) \cos(2x)}{8} \\
& + \frac{(\pi \operatorname{csgn}(x) + 8c_2 - 2 \operatorname{Si}(2x)) \sin(2x)}{8} + \frac{\ln(x)}{4}
\end{aligned}
$$

✓ Solution by Mathematica
Time used: 0.056 (sec). Leaf size: 48

```
DSolve[y''[x]+4*y[x]==Log[x],y[x],x,IncludeSingularSolutions -> True]
```

$$
y(x) \to \frac{1}{4}(-\operatorname{CosIntegral}(2x)\cos(2x) - \operatorname{Si}(2x)\sin(2x) + \log(x) + 4c_1\cos(2x) + 4c_2\sin(2x))
$$

12.13 problem Problem 32

Internal problem ID [2836]
Internal file name [`OUTPUT/2328_Sunday_June_05_2022_02_59_21_AM_46981541/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.10, Chapter review. page 575
Problem number: Problem 32.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 2y' - 3y = 5\,\mathrm{e}^{x}}$$

12.13.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 2, C = -3, f(x) = 5\,\mathrm{e}^{x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2y' - 3y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 2, C = -3$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 2\lambda\, \mathrm{e}^{\lambda x} - 3\, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2\lambda - 3 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = -3$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{2^2 - (4)\,(1)\,(-3)}\\ &= -1 \pm 2\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -1 + 2\\ \lambda_2 &= -1 - 2\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1\\ \lambda_2 &= -3\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(1)x} + c_2 e^{(-3)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-3x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-3x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$5\,\mathrm{e}^{x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\mathrm{e}^{x}, \mathrm{e}^{-3x}\}$$

Since e^{x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,\mathrm{e}^{x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\,\mathrm{e}^{x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4A_1\mathrm{e}^{x} = 5\,\mathrm{e}^{x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{5}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{5x\,\mathrm{e}^{x}}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{x} + c_2\mathrm{e}^{-3x}\right) + \left(\frac{5x\,\mathrm{e}^{x}}{4}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^x + c_2 e^{-3x} + \frac{5x e^x}{4} \tag{1}$$

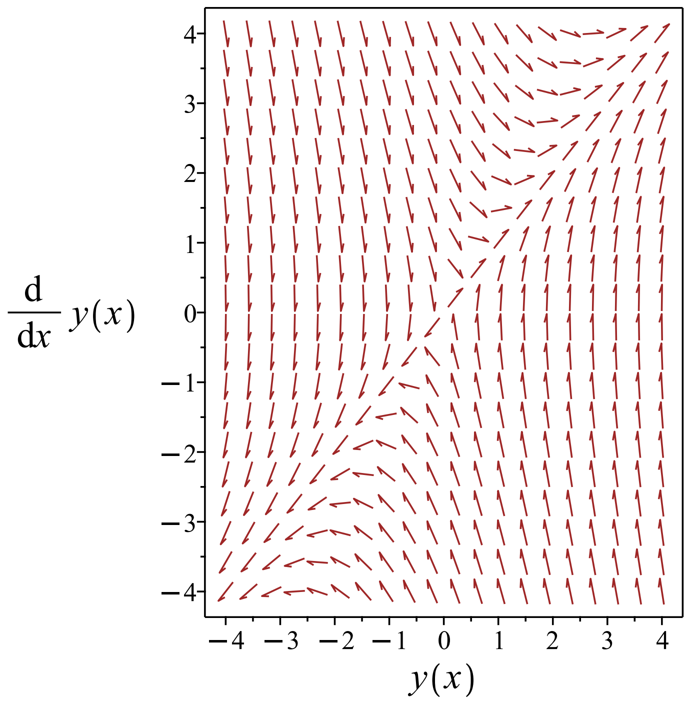


Figure 335: Slope field plot

Verification of solutions

$$y = c_1 e^x + c_2 e^{-3x} + \frac{5x e^x}{4}$$

Verified OK.

12.13.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y' - 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= 2 \\
C &= -3
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A} \, dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{4}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 4 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = 4z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x) \, e^{-\int \frac{B}{2A} \, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 338: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-2x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dx} \\
&= z_1 e^{-x} \\
&= z_1 \left(\mathrm{e}^{-x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = e^{-3x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-2x}}{(y_1)^2} \, dx \\ &= y_1 \left(\frac{e^{4x}}{4} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(e^{-3x} \right) + c_2 \left(e^{-3x} \left(\frac{e^{4x}}{4} \right) \right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2y' - 3y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{-3x} + \frac{c_2 e^x}{4}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \mathrm{e}^{-3x}$$
$$y_2 = \frac{\mathrm{e}^{x}}{4}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{-3x} & \frac{\mathrm{e}^{x}}{4} \\ \frac{d}{dx}\left(\mathrm{e}^{-3x}\right) & \frac{d}{dx}\left(\frac{\mathrm{e}^{x}}{4}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{-3x} & \frac{\mathrm{e}^{x}}{4} \\ -3\,\mathrm{e}^{-3x} & \frac{\mathrm{e}^{x}}{4} \end{vmatrix}$$

Therefore

$$W = \left(\mathrm{e}^{-3x}\right)\left(\frac{\mathrm{e}^{x}}{4}\right) - \left(\frac{\mathrm{e}^{x}}{4}\right)\left(-3\,\mathrm{e}^{-3x}\right)$$

Which simplifies to

$$W = \mathrm{e}^{-3x}\mathrm{e}^{x}$$

Which simplifies to

$$W = \mathrm{e}^{-2x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{5\,\mathrm{e}^{2x}}{4}}{\mathrm{e}^{-2x}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{5\,\mathrm{e}^{4x}}{4} dx$$

Hence

$$u_1 = -\frac{5\,\mathrm{e}^{4x}}{16}$$

And Eq. (3) becomes

$$u_2 = \int \frac{5\,\mathrm{e}^{-3x}\mathrm{e}^{x}}{\mathrm{e}^{-2x}}\,dx$$

Which simplifies to

$$u_2 = \int 5 dx$$

Hence

$$u_2 = 5x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{5\,\mathrm{e}^{-3x}\mathrm{e}^{4x}}{16} + \frac{5x\,\mathrm{e}^{x}}{4}$$

Which simplifies to

$$y_p(x) = \frac{5\,\mathrm{e}^{x}(-1+4x)}{16}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{-3x} + \frac{c_2\mathrm{e}^{x}}{4}\right) + \left(\frac{5\,\mathrm{e}^{x}(-1+4x)}{16}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-3x} + \frac{c_2 e^{x}}{4} + \frac{5 e^{x}(-1+4x)}{16} \tag{1}$$

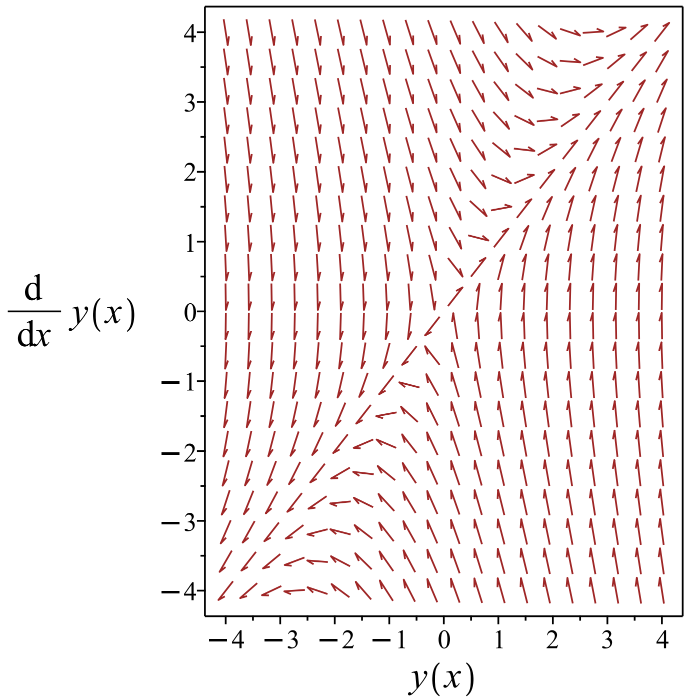


Figure 336: Slope field plot

Verification of solutions

$$y = c_1 e^{-3x} + \frac{c_2 e^{x}}{4} + \frac{5 e^{x}(-1+4x)}{16}$$

Verified OK.

12.13.3 Maple step by step solution

Let's solve

$y'' + 2y' - 3y = 5 e^{x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2 + 2r - 3 = 0$

- Factor the characteristic polynomial

  $(r+3)(r-1) = 0$

- Roots of the characteristic polynomial

  $r = (-3, 1)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-3x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-3x} + c_2 \mathrm{e}^{x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 5\,\mathrm{e}^{x}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-3x} & \mathrm{e}^{x} \\ -3\,\mathrm{e}^{-3x} & \mathrm{e}^{x} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 4\,\mathrm{e}^{-2x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = \frac{5\left(\mathrm{e}^{4x}\left(\int 1dx\right) - \left(\int \mathrm{e}^{4x}dx\right)\right)\mathrm{e}^{-3x}}{4}$

  - Compute integrals

    $y_p(x) = \frac{5\,\mathrm{e}^{x}(-1+4x)}{16}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-3x} + c_2 \mathrm{e}^{x} + \frac{5\,\mathrm{e}^{x}(-1+4x)}{16}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 28

```
dsolve(diff(y(x),x$2)+2*diff(y(x),x)-3*y(x)=5*exp(x),y(x), singsol=all)
```

$$y(x) = \frac{(5x + 4c_1)\,e^{-3x}e^{4x}}{4} + e^{-3x}c_2$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 29

```
DSolve[y''[x]+2*y'[x]-3*y[x]==5*Exp[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{-3x} + e^x\left(\frac{5x}{4} - \frac{5}{16} + c_2\right)$$

12.14 problem Problem 33

Internal problem ID [2837]
Internal file name [`OUTPUT/2329_Sunday_June_05_2022_02_59_23_AM_29122666/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.10, Chapter review. page 575
Problem number: Problem 33.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y = \tan(x)}$$

12.14.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = \tan(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\
&= \pm i
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= +i \\
\lambda_2 &= -i
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= i \\
\lambda_2 &= -i
\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{0}(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x)$$
$$y_2 = \sin(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ \frac{d}{dx}(\cos(x)) & \frac{d}{dx}(\sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

Therefore

$$W = (\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))$$

Which simplifies to

$$W = \cos(x)^2 + \sin(x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\tan(x)\sin(x)}{1}\,dx$$

Which simplifies to

$$u_1 = -\int \tan(x)\sin(x)\,dx$$

Hence

$$u_1 = \sin(x) - \ln(\sec(x) + \tan(x))$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(x)\tan(x)}{1}\,dx$$

Which simplifies to

$$u_2 = \int \sin(x)\,dx$$

Hence

$$u_2 = -\cos(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = (\sin(x) - \ln(\sec(x) + \tan(x)))\cos(x) - \sin(x)\cos(x)$$

Which simplifies to

$$y_p(x) = -\ln(\sec(x) + \tan(x))\cos(x)$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 \cos(x) + c_2 \sin(x)) + (-\ln(\sec(x) + \tan(x)) \cos(x))
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \cos(x) + c_2 \sin(x) - \ln(\sec(x) + \tan(x)) \cos(x) \tag{1}
$$

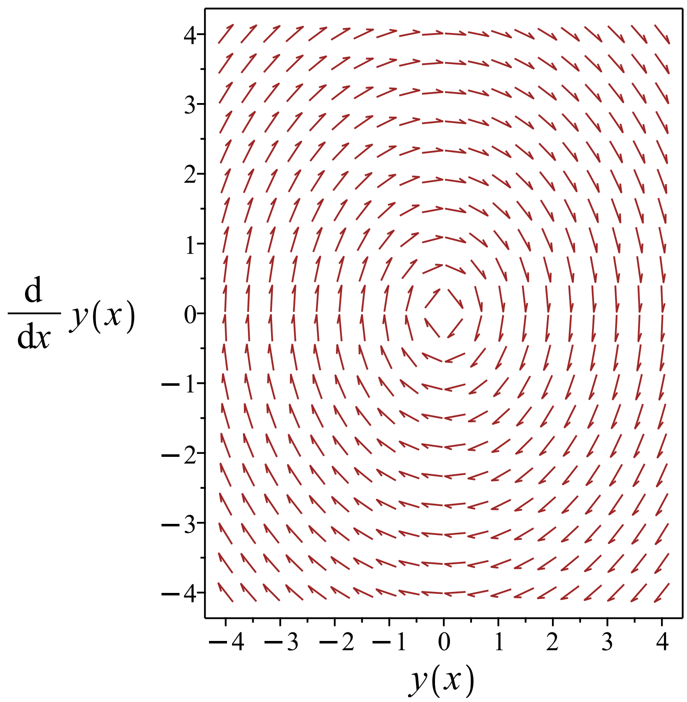


Figure 337: Slope field plot

Verification of solutions

$$
y = c_1 \cos(x) + c_2 \sin(x) - \ln(\sec(x) + \tan(x)) \cos(x)
$$

Verified OK.

12.14.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 340: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \cos(x) \int \frac{1}{\cos(x)^2}\, dx \\ &= \cos(x)\,(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(x)) + c_2(\cos(x)\,(\tan(x))) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x)$$
$$y_2 = \sin(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ \frac{d}{dx}(\cos(x)) & \frac{d}{dx}(\sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

Therefore

$$W = (\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))$$

Which simplifies to

$$W = \cos(x)^2 + \sin(x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\tan (x) \sin (x)}{1} dx$$

Which simplifies to

$$u_1 = -\int \tan (x) \sin (x) dx$$

Hence

$$u_1 = \sin (x) - \ln (\sec (x) + \tan (x))$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos (x) \tan (x)}{1} dx$$

Which simplifies to

$$u_2 = \int \sin (x) dx$$

Hence

$$u_2 = -\cos (x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = (\sin (x) - \ln (\sec (x) + \tan (x))) \cos (x) - \sin (x) \cos (x)$$

Which simplifies to

$$y_p(x) = -\ln (\sec (x) + \tan (x)) \cos (x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos (x) + c_2 \sin (x)) + (-\ln (\sec (x) + \tan (x)) \cos (x)) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \cos(x) + c_2 \sin(x) - \ln(\sec(x) + \tan(x)) \cos(x) \tag{1}$$

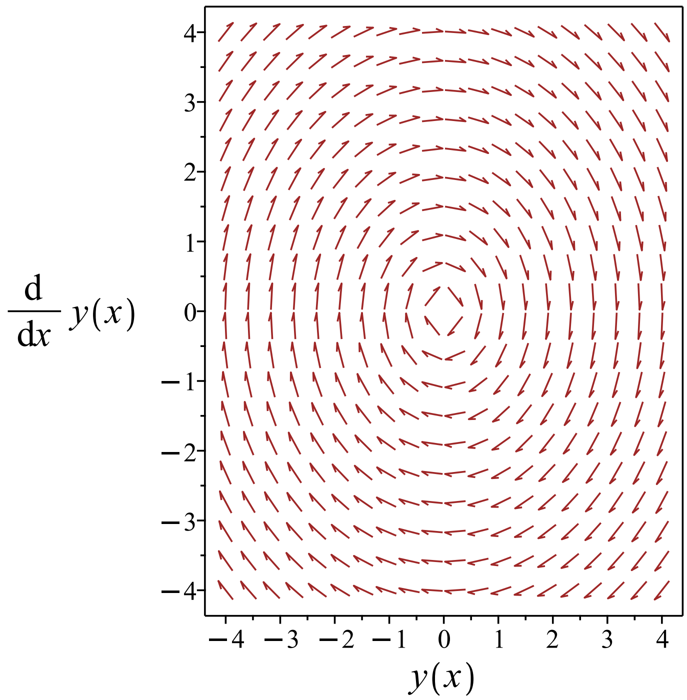


Figure 338: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x) - \ln(\sec(x) + \tan(x)) \cos(x)$$

Verified OK.

12.14.3 Maple step by step solution

Let's solve

$y'' + y = \tan(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$
- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$
- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$
- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \tan(x)\right]$
  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$
  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 1$
  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\cos(x)\left(\int \tan(x)\sin(x)\, dx\right) + \sin(x)\left(\int \sin(x)\, dx\right)$
  - Compute integrals

    $y_p(x) = -\ln(\sec(x) + \tan(x))\cos(x)$
- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(x) + c_2 \sin(x) - \ln(\sec(x) + \tan(x))\cos(x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(y(x),x$2)+y(x)=tan(x),y(x), singsol=all)
```

$$y(x) = \sin(x) c_2 + \cos(x) c_1 - \cos(x) \ln(\sec(x) + \tan(x))$$

✓ Solution by Mathematica

Time used: 0.031 (sec). Leaf size: 23

```
DSolve[y''[x]+y[x]==Tan[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \cos(x)(-\operatorname{arctanh}(\sin(x))) + c_1 \cos(x) + c_2 \sin(x)$$

12.15 problem Problem 34

Internal problem ID [2838]
Internal file name [`OUTPUT/2330_Sunday_June_05_2022_02_59_25_AM_93733018/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 8, Linear differential equations of order n. Section 8.10, Chapter review. page 575
Problem number: Problem 34.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y = 4\cos(2x) + 3\,\mathrm{e}^{x}}$$

12.15.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = 4\cos(2x) + 3\,\mathrm{e}^{x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i \\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i \\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$4\cos(2x) + 3\,\mathrm{e}^x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^x\}, \{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \mathrm{e}^x + A_2 \cos(2x) + A_3 \sin(2x)$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 \mathrm{e}^x - 3A_2 \cos(2x) - 3A_3 \sin(2x) = 4\cos(2x) + 3\,\mathrm{e}^x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{3}{2}, A_2 = -\frac{4}{3}, A_3 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{3\,\mathrm{e}^x}{2} - \frac{4\cos(2x)}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) + \left(\frac{3\,\mathrm{e}^x}{2} - \frac{4\cos(2x)}{3}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \cos(x) + c_2 \sin(x) + \frac{3 e^x}{2} - \frac{4 \cos(2x)}{3} \tag{1}$$

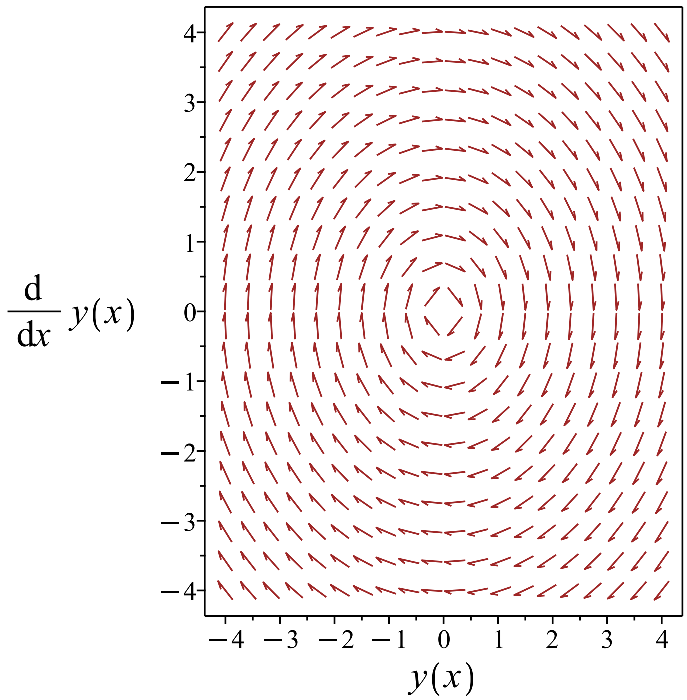


Figure 339: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x) + \frac{3 e^x}{2} - \frac{4 \cos(2x)}{3}$$

Verified OK.

12.15.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= 0 \\
C &= 1
\end{aligned}
\tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A} \, dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-1}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -1 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = -z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x) \, e^{-\int \frac{B}{2A} \, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 342: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \cos(x)
\end{aligned}
$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \cos(x) \int \frac{1}{\cos(x)^2}\,dx \\ &= \cos(x)\,(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(x)) + c_2(\cos(x)\,(\tan(x))) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$4\cos(2x) + 3\,\mathrm{e}^{x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^x\}, \{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^x + A_2 \cos(2x) + A_3 \sin(2x)$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 e^x - 3A_2 \cos(2x) - 3A_3 \sin(2x) = 4\cos(2x) + 3\,e^x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{3}{2}, A_2 = -\frac{4}{3}, A_3 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{3\,e^x}{2} - \frac{4\cos(2x)}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) + \left(\frac{3\,e^x}{2} - \frac{4\cos(2x)}{3}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(x) + c_2 \sin(x) + \frac{3\,e^x}{2} - \frac{4\cos(2x)}{3} \tag{1}$$

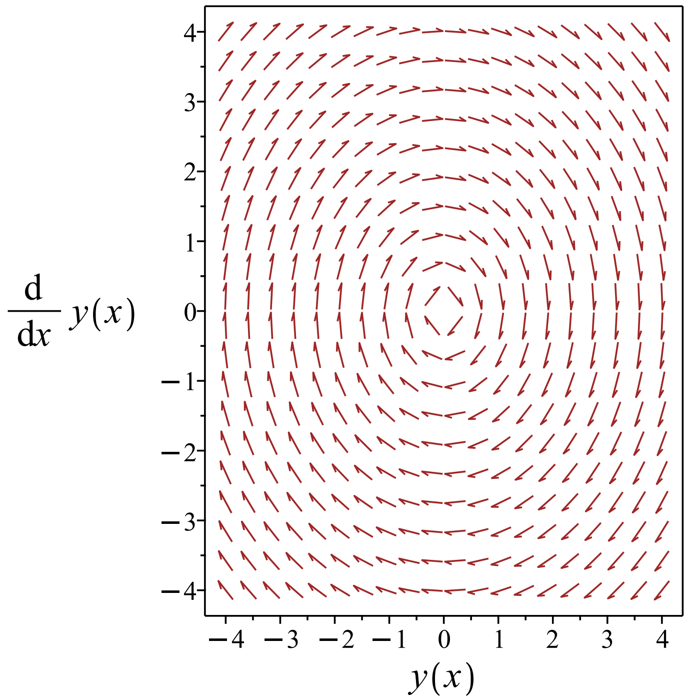


Figure 340: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x) + \frac{3\,\mathrm{e}^x}{2} - \frac{4\cos(2x)}{3}$$

Verified OK.

12.15.3 Maple step by step solution

Let's solve

$y'' + y = 4\cos(2x) + 3\,\mathrm{e}^x$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

$r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 4\cos(2x) + 3\,\mathrm{e}^{x}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 1$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\cos(x) \left(\int \sin(x) (4\cos(2x) + 3\,\mathrm{e}^{x})\, dx\right) + \sin(x) \left(\int \cos(x) (4\cos(2x) + 3\,\mathrm{e}^{x})\, dx\right)$

  - Compute integrals

    $y_p(x) = \frac{3\,\mathrm{e}^{x}}{2} - \frac{4\cos(2x)}{3}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + \frac{3\,\mathrm{e}^{x}}{2} - \frac{4\cos(2x)}{3}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(y(x),x$2)+y(x)=4*cos(2*x)+3*exp(x),y(x), singsol=all)
```

$$y(x) = \sin\left(x\right) c_2 + \cos\left(x\right) c_1 - \frac{4\cos\left(2x\right)}{3} + \frac{3\,\mathrm{e}^{x}}{2}$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 30

```
DSolve[y''[x]+y[x]==4*Cos[x]*3*Exp[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{12}{5}e^{x}(2\sin(x) + \cos(x)) + c_1\cos(x) + c_2\sin(x)$$

13 Chapter 10, The Laplace Transform and Some Elementary Applications. Exercises for 10.4. page 689

13.1 problem Problem 1

Internal problem ID [2839]
Internal file name [`OUTPUT/2331_Sunday_June_05_2022_02_59_29_AM_68412706/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 10, The Laplace Transform and Some Elementary Applications. Exercises for 10.4. page 689
Problem number: Problem 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' - 2y = 6\,\mathrm{e}^{5t}}$$

With initial conditions

$$[y(0) = 3]$$

13.1.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = -2$$
$$q(t) = 6\,\mathrm{e}^{5t}$$

Hence the ode is

$$y' - 2y = 6\,e^{5t}$$

The domain of $p(t) = -2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 6\,e^{5t}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

13.1.2 Solving as laplace ode

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(y') = sY(s) - y(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$sY(s) - y(0) - 2Y(s) = \frac{6}{s-5} \tag{1}$$

Replacing initial condition gives

$$sY(s) - 3 - 2Y(s) = \frac{6}{s-5}$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{-9+3s}{(s-5)(s-2)}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{1}{s-2} + \frac{2}{s-5}$$

The inverse Laplace of each term above is now found, which gives

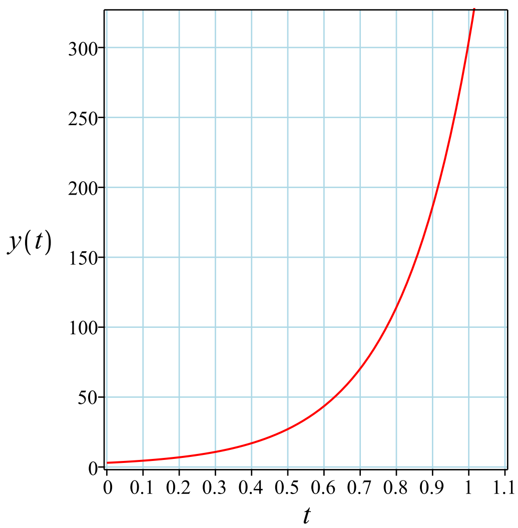
$$\mathcal{L}^{-1}\left(\frac{1}{s-2}\right) = e^{2t}$$
$$\mathcal{L}^{-1}\left(\frac{2}{s-5}\right) = 2\,e^{5t}$$

Adding the above results and simplifying gives

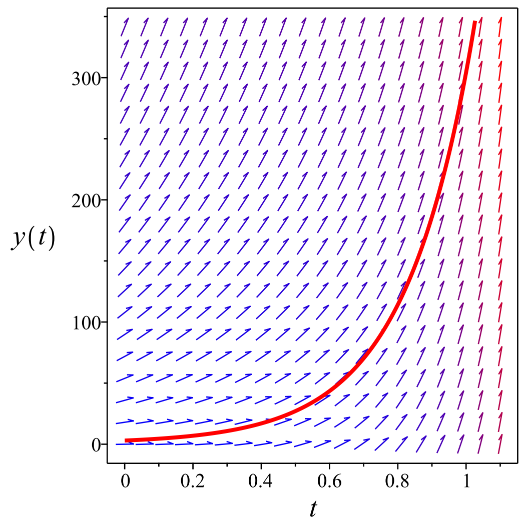
$$y = 2\,e^{5t} + e^{2t}$$

Summary
The solution(s) found are the following

$$y = 2\,e^{5t} + e^{2t} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 2\,e^{5t} + e^{2t}$$

Verified OK.

13.1.3 Maple step by step solution

Let's solve

$[y' - 2y = 6\,e^{5t}, y(0) = 3]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = 2y + 6\,e^{5t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - 2y = 6\,\mathrm{e}^{5t}$
- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\,(y' - 2y) = 6\mu(t)\,\mathrm{e}^{5t}$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,y)$

  $\mu(t)\,(y' - 2y) = \mu'(t)\,y + \mu(t)\,y'$
- Isolate $\mu'(t)$

  $\mu'(t) = -2\mu(t)$
- Solve to find the integrating factor

  $\mu(t) = \mathrm{e}^{-2t}$
- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\,y)\right) dt = \int 6\mu(t)\,\mathrm{e}^{5t} dt + c_1$
- Evaluate the integral on the lhs

  $\mu(t)\,y = \int 6\mu(t)\,\mathrm{e}^{5t} dt + c_1$
- Solve for y

  $y = \frac{\int 6\mu(t)\mathrm{e}^{5t} dt + c_1}{\mu(t)}$
- Substitute $\mu(t) = \mathrm{e}^{-2t}$

  $y = \frac{\int 6\,\mathrm{e}^{-2t}\mathrm{e}^{5t} dt + c_1}{\mathrm{e}^{-2t}}$
- Evaluate the integrals on the rhs

  $y = \frac{2\,\mathrm{e}^{3t} + c_1}{\mathrm{e}^{-2t}}$
- Simplify

  $y = \mathrm{e}^{2t}(2\,\mathrm{e}^{3t} + c_1)$
- Use initial condition $y(0) = 3$

  $3 = 2 + c_1$
- Solve for c_1

  $c_1 = 1$
- Substitute $c_1 = 1$ into general solution and simplify

  $y = \mathrm{e}^{2t}(2\,\mathrm{e}^{3t} + 1)$
- Solution to the IVP

$$y = e^{2t}(2\,e^{3t} + 1)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 2.891 (sec). Leaf size: 15

```
dsolve([diff(y(t),t)-2*y(t)=6*exp(5*t),y(0) = 3],y(t), singsol=all)
```

$$y(t) = 2\,e^{5t} + e^{2t}$$

✓ Solution by Mathematica

Time used: 0.046 (sec). Leaf size: 18

```
DSolve[{y'[t]-2*y[t]==6*Exp[5*t],{y[0]==3}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to e^{2t} + 2e^{5t}$$

13.2 problem Problem 2

Internal problem ID [2840]
Internal file name [`OUTPUT/2332_Sunday_June_05_2022_02_59_31_AM_54659613/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 10, The Laplace Transform and Some Elementary Applications. Exercises for 10.4. page 689
Problem number: Problem 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' + y = 8\,\mathrm{e}^{3t}}$$

With initial conditions

$$[y(0) = 2]$$

13.2.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$\begin{aligned} p(t) &= 1 \\ q(t) &= 8\,\mathrm{e}^{3t} \end{aligned}$$

Hence the ode is

$$y' + y = 8\,e^{3t}$$

The domain of $p(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 8\,e^{3t}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

13.2.2 Solving as laplace ode

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(y') = sY(s) - y(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$sY(s) - y(0) + Y(s) = \frac{8}{-3+s} \tag{1}$$

Replacing initial condition gives

$$sY(s) - 2 + Y(s) = \frac{8}{-3+s}$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{2}{-3+s}$$

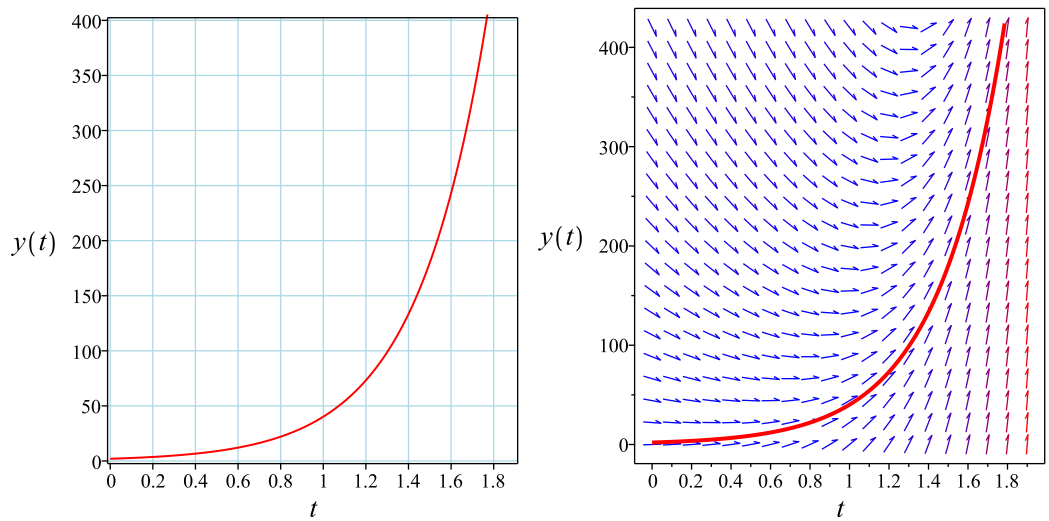
Taking the inverse Laplace transform gives

$$\begin{aligned} y &= \mathcal{L}^{-1}(Y(s)) \\ &= \mathcal{L}^{-1}\left(\frac{2}{-3+s}\right) \\ &= 2\,e^{3t} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = 2\,e^{3t} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = 2\,e^{3t}$$

Verified OK.

13.2.3 Maple step by step solution

Let's solve

$[y' + y = 8\,e^{3t}, y(0) = 2]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -y + 8\,e^{3t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + y = 8\,e^{3t}$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

$\mu(t)\,(y'+y) = 8\mu(t)\,\mathrm{e}^{3t}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,y)$

  $\mu(t)\,(y'+y) = \mu'(t)\,y + \mu(t)\,y'$

- Isolate $\mu'(t)$

  $\mu'(t) = \mu(t)$

- Solve to find the integrating factor

  $\mu(t) = \mathrm{e}^{t}$

- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\,y)\right) dt = \int 8\mu(t)\,\mathrm{e}^{3t}dt + c_1$

- Evaluate the integral on the lhs

  $\mu(t)\,y = \int 8\mu(t)\,\mathrm{e}^{3t}dt + c_1$

- Solve for y

  $y = \frac{\int 8\mu(t)\mathrm{e}^{3t}dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = \mathrm{e}^{t}$

  $y = \frac{\int 8\,\mathrm{e}^{t}\mathrm{e}^{3t}dt + c_1}{\mathrm{e}^{t}}$

- Evaluate the integrals on the rhs

  $y = \frac{2\,\mathrm{e}^{4t} + c_1}{\mathrm{e}^{t}}$

- Simplify

  $y = 2\,\mathrm{e}^{3t} + c_1\mathrm{e}^{-t}$

- Use initial condition $y(0) = 2$

  $2 = 2 + c_1$

- Solve for c_1

  $c_1 = 0$

- Substitute $c_1 = 0$ into general solution and simplify

  $y = 2\,\mathrm{e}^{3t}$

- Solution to the IVP

  $y = 2\,\mathrm{e}^{3t}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 2.703 (sec). Leaf size: 10

```
dsolve([diff(y(t),t)+y(t)=8*exp(3*t),y(0) = 2],y(t), singsol=all)
```

$$y(t) = 2\,\mathrm{e}^{3t}$$

✓ Solution by Mathematica

Time used: 0.051 (sec). Leaf size: 12

```
DSolve[{y'[t]+y[t]==8*Exp[3*t],{y[0]==2}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to 2e^{3t}$$

13.3 problem Problem 3

Internal problem ID [2841]
Internal file name [`OUTPUT/2333_Sunday_June_05_2022_02_59_33_AM_65921000/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 10, The Laplace Transform and Some Elementary Applications. Exercises for 10.4. page 689
Problem number: Problem 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' + 3y = 2\,\mathrm{e}^{-t}}$$

With initial conditions

$$[y(0) = 3]$$

13.3.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = 3$$
$$q(t) = 2\,\mathrm{e}^{-t}$$

Hence the ode is

$$y' + 3y = 2\,\mathrm{e}^{-t}$$

The domain of $p(t) = 3$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 2\,\mathrm{e}^{-t}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

13.3.2 Solving as laplace ode

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(y') = sY(s) - y(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$sY(s) - y(0) + 3Y(s) = \frac{2}{1+s} \tag{1}$$

Replacing initial condition gives

$$sY(s) - 3 + 3Y(s) = \frac{2}{1+s}$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{5+3s}{(1+s)\,(s+3)}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{2}{s+3} + \frac{1}{1+s}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(\frac{2}{s+3}\right) = 2\,\mathrm{e}^{-3t}$$
$$\mathcal{L}^{-1}\left(\frac{1}{1+s}\right) = \mathrm{e}^{-t}$$

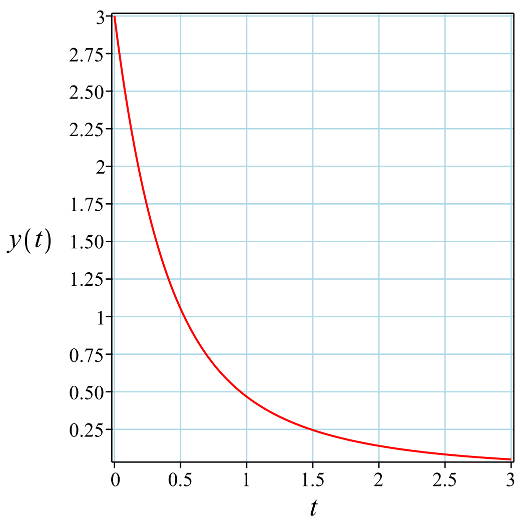
Adding the above results and simplifying gives

$$y = 2\,e^{-3t} + e^{-t}$$

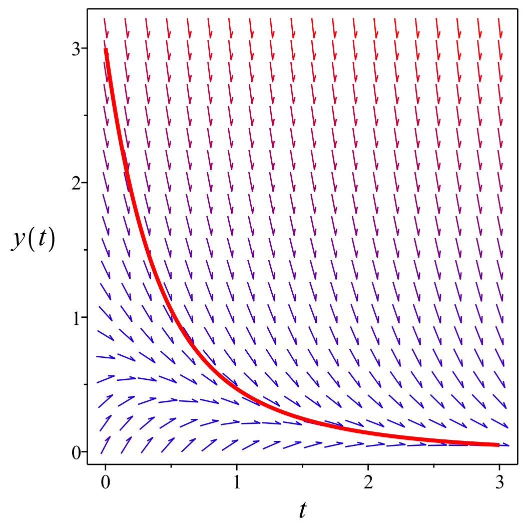
Summary

The solution(s) found are the following

$$y = 2\,e^{-3t} + e^{-t} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 2\,e^{-3t} + e^{-t}$$

Verified OK.

13.3.3 Maple step by step solution

Let's solve

$[y' + 3y = 2\,e^{-t}, y(0) = 3]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -3y + 2\,e^{-t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + 3y = 2\,\mathrm{e}^{-t}$
- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\,(y' + 3y) = 2\mu(t)\,\mathrm{e}^{-t}$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,y)$

  $\mu(t)\,(y' + 3y) = \mu'(t)\,y + \mu(t)\,y'$
- Isolate $\mu'(t)$

  $\mu'(t) = 3\mu(t)$
- Solve to find the integrating factor

  $\mu(t) = \mathrm{e}^{3t}$
- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\,y)\right) dt = \int 2\mu(t)\,\mathrm{e}^{-t} dt + c_1$
- Evaluate the integral on the lhs

  $\mu(t)\,y = \int 2\mu(t)\,\mathrm{e}^{-t} dt + c_1$
- Solve for y

  $y = \frac{\int 2\mu(t)\mathrm{e}^{-t} dt + c_1}{\mu(t)}$
- Substitute $\mu(t) = \mathrm{e}^{3t}$

  $y = \frac{\int 2\,\mathrm{e}^{3t}\mathrm{e}^{-t} dt + c_1}{\mathrm{e}^{3t}}$
- Evaluate the integrals on the rhs

  $y = \frac{\mathrm{e}^{2t} + c_1}{\mathrm{e}^{3t}}$
- Simplify

  $y = \mathrm{e}^{-3t}(\mathrm{e}^{2t} + c_1)$
- Use initial condition $y(0) = 3$

  $3 = c_1 + 1$
- Solve for c_1

  $c_1 = 2$
- Substitute $c_1 = 2$ into general solution and simplify

  $y = \mathrm{e}^{-3t}(\mathrm{e}^{2t} + 2)$
- Solution to the IVP

$$y = e^{-3t}(e^{2t} + 2)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 2.937 (sec). Leaf size: 15

```
dsolve([diff(y(t),t)+3*y(t)=2*exp(-t),y(0) = 3],y(t), singsol=all)
```

$$y(t) = e^{-t} + 2\,e^{-3t}$$

✓ Solution by Mathematica

Time used: 0.051 (sec). Leaf size: 18

```
DSolve[{y'[t]+3*y[t]==2*Exp[-t],{y[0]==3}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to e^{-3t}\left(e^{2t} + 2\right)$$

13.4 problem Problem 4

Internal problem ID [2842]
Internal file name [`OUTPUT/2334_Sunday_June_05_2022_02_59_35_AM_75677719/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 10, The Laplace Transform and Some Elementary Applications. Exercises for 10.4. page 689
Problem number: Problem 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' + 2y = 4t}$$

With initial conditions

$$[y(0) = 1]$$

13.4.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$\begin{aligned} p(t) &= 2 \\ q(t) &= 4t \end{aligned}$$

Hence the ode is

$$y' + 2y = 4t$$

The domain of $p(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 4t$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

13.4.2 Solving as laplace ode

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(y') = sY(s) - y(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$sY(s) - y(0) + 2Y(s) = \frac{4}{s^2} \tag{1}$$

Replacing initial condition gives

$$sY(s) - 1 + 2Y(s) = \frac{4}{s^2}$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{s^2 + 4}{s^2 (s + 2)}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{2}{s^2} - \frac{1}{s} + \frac{2}{s + 2}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(\frac{2}{s^2}\right) = 2t$$
$$\mathcal{L}^{-1}\left(-\frac{1}{s}\right) = -1$$
$$\mathcal{L}^{-1}\left(\frac{2}{s+2}\right) = 2\,\mathrm{e}^{-2t}$$

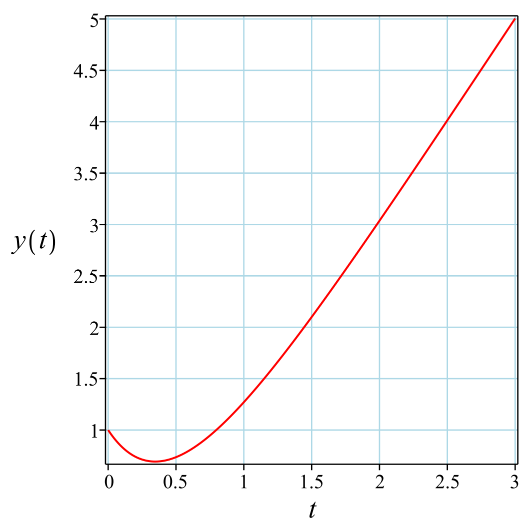
Adding the above results and simplifying gives

$$y = 2t - 1 + 2\,\mathrm{e}^{-2t}$$

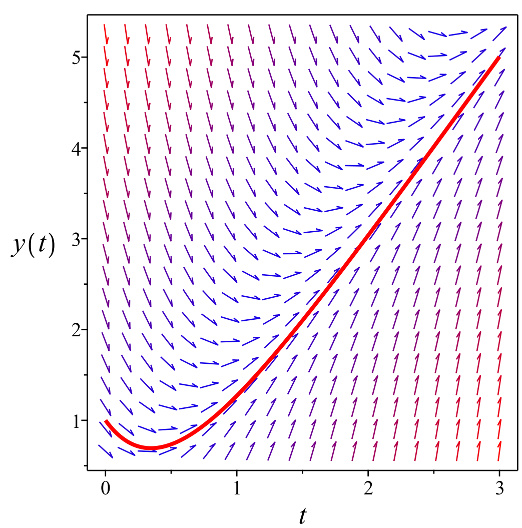
Summary

The solution(s) found are the following

$$y = 2t - 1 + 2\,\mathrm{e}^{-2t} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 2t - 1 + 2\,\mathrm{e}^{-2t}$$

Verified OK.

13.4.3 Maple step by step solution

Let's solve

$[y' + 2y = 4t, y(0) = 1]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -2y + 4t$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + 2y = 4t$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\,(y' + 2y) = 4\mu(t)\,t$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,y)$

  $\mu(t)\,(y' + 2y) = \mu'(t)\,y + \mu(t)\,y'$

- Isolate $\mu'(t)$

  $\mu'(t) = 2\mu(t)$

- Solve to find the integrating factor

  $\mu(t) = \mathrm{e}^{2t}$

- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\,y)\right) dt = \int 4\mu(t)\,t dt + c_1$

- Evaluate the integral on the lhs

  $\mu(t)\,y = \int 4\mu(t)\,t dt + c_1$

- Solve for y

  $y = \frac{\int 4\mu(t)t dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = \mathrm{e}^{2t}$

  $y = \frac{\int 4\,\mathrm{e}^{2t} t dt + c_1}{\mathrm{e}^{2t}}$

- Evaluate the integrals on the rhs

  $y = \frac{(2t-1)\mathrm{e}^{2t} + c_1}{\mathrm{e}^{2t}}$

- Simplify

  $y = 2t - 1 + c_1\mathrm{e}^{-2t}$

- Use initial condition $y(0) = 1$

  $1 = c_1 - 1$
- Solve for c_1

  $c_1 = 2$
- Substitute $c_1 = 2$ into general solution and simplify

  $y = 2t - 1 + 2\,\mathrm{e}^{-2t}$
- Solution to the IVP

  $y = 2t - 1 + 2\,\mathrm{e}^{-2t}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 1.703 (sec). Leaf size: 15

```
dsolve([diff(y(t),t)+2*y(t)=4*t,y(0) = 1],y(t), singsol=all)
```

$$y(t) = 2t + 2\,\mathrm{e}^{-2t} - 1$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 17

```
DSolve[{y'[t]+2*y[t]==4*t,{y[0]==1}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to 2t + 2e^{-2t} - 1$$

13.5 problem Problem 5

Internal problem ID [2843]
Internal file name [`OUTPUT/2335_Sunday_June_05_2022_02_59_38_AM_59406789/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 10, The Laplace Transform and Some Elementary Applications. Exercises for 10.4. page 689
Problem number: Problem 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' - y = 6\cos(t)}$$

With initial conditions

$$[y(0) = 2]$$

13.5.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$\begin{aligned} p(t) &= -1 \\ q(t) &= 6\cos(t) \end{aligned}$$

Hence the ode is

$$y' - y = 6\cos(t)$$

The domain of $p(t) = -1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 6\cos(t)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

13.5.2 Solving as laplace ode

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(y') = sY(s) - y(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$sY(s) - y(0) - Y(s) = \frac{6s}{s^2+1} \tag{1}$$

Replacing initial condition gives

$$sY(s) - 2 - Y(s) = \frac{6s}{s^2+1}$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{2s^2+6s+2}{(s^2+1)(s-1)}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{5}{s-1} + \frac{-\frac{3}{2}-\frac{3i}{2}}{s-i} + \frac{-\frac{3}{2}+\frac{3i}{2}}{s+i}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(\frac{5}{s-1}\right) = 5\,\mathrm{e}^{t}$$
$$\mathcal{L}^{-1}\left(\frac{-\frac{3}{2}-\frac{3i}{2}}{s-i}\right) = \left(-\frac{3}{2}-\frac{3i}{2}\right)\mathrm{e}^{it}$$
$$\mathcal{L}^{-1}\left(\frac{-\frac{3}{2}+\frac{3i}{2}}{s+i}\right) = \left(-\frac{3}{2}+\frac{3i}{2}\right)\mathrm{e}^{-it}$$

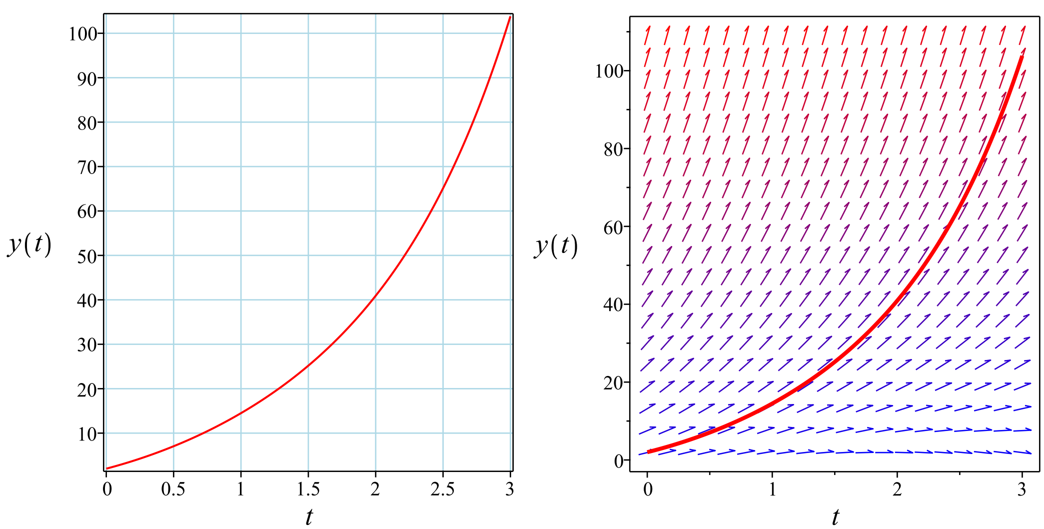
Adding the above results and simplifying gives

$$y = 5\,\mathrm{e}^{t} - 3\cos(t) + 3\sin(t)$$

Summary

The solution(s) found are the following

$$y = 5\,\mathrm{e}^{t} - 3\cos(t) + 3\sin(t) \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = 5\,\mathrm{e}^{t} - 3\cos(t) + 3\sin(t)$$

Verified OK.

13.5.3 Maple step by step solution

Let's solve

$[y' - y = 6\cos(t), y(0) = 2]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = y + 6\cos(t)$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - y = 6\cos(t)$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)(y' - y) = 6\mu(t)\cos(t)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\, y)$

  $\mu(t)(y' - y) = \mu'(t)\, y + \mu(t)\, y'$

- Isolate $\mu'(t)$

  $\mu'(t) = -\mu(t)$

- Solve to find the integrating factor

  $\mu(t) = \mathrm{e}^{-t}$

- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\, y)\right) dt = \int 6\mu(t)\cos(t)\, dt + c_1$

- Evaluate the integral on the lhs

  $\mu(t)\, y = \int 6\mu(t)\cos(t)\, dt + c_1$

- Solve for y

  $y = \frac{\int 6\mu(t)\cos(t)dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = \mathrm{e}^{-t}$

  $y = \frac{\int 6\,\mathrm{e}^{-t}\cos(t)dt + c_1}{\mathrm{e}^{-t}}$

- Evaluate the integrals on the rhs

  $y = \frac{-3\,\mathrm{e}^{-t}\cos(t) + 3\,\mathrm{e}^{-t}\sin(t) + c_1}{\mathrm{e}^{-t}}$

- Simplify

  $y = c_1\mathrm{e}^{t} - 3\cos(t) + 3\sin(t)$

- Use initial condition $y(0) = 2$

  $2 = c_1 - 3$
- Solve for c_1

  $c_1 = 5$
- Substitute $c_1 = 5$ into general solution and simplify

  $y = 5\,e^t - 3\cos(t) + 3\sin(t)$
- Solution to the IVP

  $y = 5\,e^t - 3\cos(t) + 3\sin(t)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 1.89 (sec). Leaf size: 17

```
dsolve([diff(y(t),t)-y(t)=6*cos(t),y(0) = 2],y(t), singsol=all)
```

$$y(t) = 5\,e^t - 3\cos(t) + 3\sin(t)$$

✓ Solution by Mathematica

Time used: 0.051 (sec). Leaf size: 19

```
DSolve[{y'[t]-y[t]==6*Cos[t],{y[0]==2}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to 5e^t + 3\sin(t) - 3\cos(t)$$

13.6 problem Problem 6

Internal problem ID [2844]
Internal file name [`OUTPUT/2336_Sunday_June_05_2022_02_59_41_AM_78593161/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 10, The Laplace Transform and Some Elementary Applications. Exercises for 10.4. page 689
Problem number: Problem 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' - y = 5\sin(2t)}$$

With initial conditions

$$[y(0) = -1]$$

13.6.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = -1$$
$$q(t) = 5\sin(2t)$$

Hence the ode is

$$y' - y = 5 \sin (2t)$$

The domain of $p(t) = -1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 5 \sin (2t)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

13.6.2 Solving as laplace ode

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(y') = sY(s) - y(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$sY(s) - y(0) - Y(s) = \frac{10}{s^2 + 4} \tag{1}$$

Replacing initial condition gives

$$sY(s) + 1 - Y(s) = \frac{10}{s^2 + 4}$$

Solving for $Y(s)$ gives

$$Y(s) = -\frac{s^2 - 6}{(s^2 + 4)(s - 1)}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{-1 + \frac{i}{2}}{s - 2i} + \frac{-1 - \frac{i}{2}}{s + 2i} + \frac{1}{s - 1}$$

The inverse Laplace of each term above is now found, which gives

$$
\begin{aligned}
\mathcal{L}^{-1}\left(\frac{-1+\frac{i}{2}}{s-2i}\right) &= \left(-1+\frac{i}{2}\right) \mathrm{e}^{2it} \\
\mathcal{L}^{-1}\left(\frac{-1-\frac{i}{2}}{s+2i}\right) &= \left(-1-\frac{i}{2}\right) \mathrm{e}^{-2it} \\
\mathcal{L}^{-1}\left(\frac{1}{s-1}\right) &= \mathrm{e}^{t}
\end{aligned}
$$

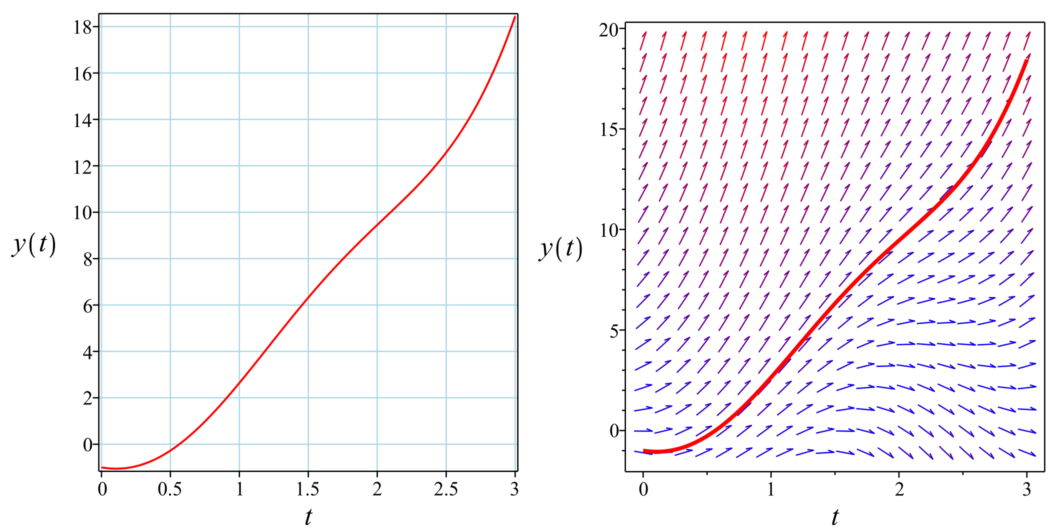
Adding the above results and simplifying gives

$$
y = \mathrm{e}^{t} - 2\cos(2t) - \sin(2t)
$$

Summary

The solution(s) found are the following

$$
y = \mathrm{e}^{t} - 2\cos(2t) - \sin(2t) \tag{1}
$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$
y = \mathrm{e}^{t} - 2\cos(2t) - \sin(2t)
$$

Verified OK.

13.6.3 Maple step by step solution

Let's solve

$[y' - y = 5\sin(2t), y(0) = -1]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = y + 5\sin(2t)$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - y = 5\sin(2t)$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)(y' - y) = 5\mu(t)\sin(2t)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,y)$

  $\mu(t)(y' - y) = \mu'(t)\,y + \mu(t)\,y'$

- Isolate $\mu'(t)$

  $\mu'(t) = -\mu(t)$

- Solve to find the integrating factor

  $\mu(t) = \mathrm{e}^{-t}$

- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\,y)\right) dt = \int 5\mu(t)\sin(2t)\,dt + c_1$

- Evaluate the integral on the lhs

  $\mu(t)\,y = \int 5\mu(t)\sin(2t)\,dt + c_1$

- Solve for y

  $y = \frac{\int 5\mu(t)\sin(2t)dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = \mathrm{e}^{-t}$

  $y = \frac{\int 5\,\mathrm{e}^{-t}\sin(2t)dt + c_1}{\mathrm{e}^{-t}}$

- Evaluate the integrals on the rhs

  $y = \frac{-2\,\mathrm{e}^{-t}\cos(2t) - \mathrm{e}^{-t}\sin(2t) + c_1}{\mathrm{e}^{-t}}$

- Simplify

  $y = c_1\mathrm{e}^{t} - \sin(2t) - 2\cos(2t)$

- Use initial condition $y(0) = -1$

  $-1 = c_1 - 2$
- Solve for c_1

  $c_1 = 1$
- Substitute $c_1 = 1$ into general solution and simplify

  $y = \mathrm{e}^t - 2\cos(2t) - \sin(2t)$
- Solution to the IVP

  $y = \mathrm{e}^t - 2\cos(2t) - \sin(2t)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 2.984 (sec). Leaf size: 19

```
dsolve([diff(y(t),t)-y(t)=5*sin(2*t),y(0) = -1],y(t), singsol=all)
```

$$y(t) = -2\cos(2t) - \sin(2t) + \mathrm{e}^t$$

✓ Solution by Mathematica

Time used: 0.09 (sec). Leaf size: 21

```
DSolve[{y'[t]-y[t]==5*Sin[2*t],{y[0]==-1}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to e^t - \sin(2t) - 2\cos(2t)$$

13.7 problem Problem 7

Internal problem ID [2845]
Internal file name [`OUTPUT/2337_Sunday_June_05_2022_02_59_43_AM_1185400/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 10, The Laplace Transform and Some Elementary Applications. Exercises for 10.4. page 689
Problem number: Problem 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first__order__ode__lie__symmetry__lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' + y = 5\,\mathrm{e}^{t} \sin(t)}$$

With initial conditions

$$[y(0) = 1]$$

13.7.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$\begin{aligned} p(t) &= 1 \\ q(t) &= 5\,\mathrm{e}^{t} \sin(t) \end{aligned}$$

Hence the ode is

$$y' + y = 5\,\mathrm{e}^{t} \sin(t)$$

The domain of $p(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 5\,\mathrm{e}^{t} \sin(t)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

13.7.2 Solving as laplace ode

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(y') = sY(s) - y(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$sY(s) - y(0) + Y(s) = \frac{5}{(s-1)^2 + 1} \tag{1}$$

Replacing initial condition gives

$$sY(s) - 1 + Y(s) = \frac{5}{(s-1)^2 + 1}$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{s^2 - 2s + 7}{(s^2 - 2s + 2)(s + 1)}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{-\frac{1}{2} - i}{s - 1 - i} + \frac{-\frac{1}{2} + i}{s - 1 + i} + \frac{2}{s + 1}$$

The inverse Laplace of each term above is now found, which gives

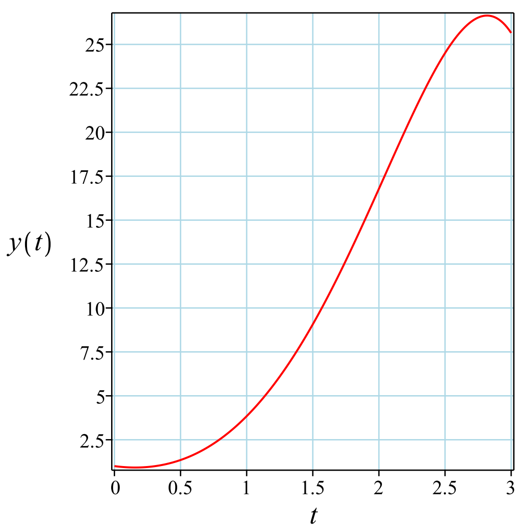
$$
\begin{aligned}
\mathcal{L}^{-1}\left(\frac{-\frac{1}{2}-i}{s-1-i}\right) &= \left(-\frac{1}{2}-i\right) \mathrm{e}^{(1+i)t} \\
\mathcal{L}^{-1}\left(\frac{-\frac{1}{2}+i}{s-1+i}\right) &= \left(-\frac{1}{2}+i\right) \mathrm{e}^{(1-i)t} \\
\mathcal{L}^{-1}\left(\frac{2}{s+1}\right) &= 2\,\mathrm{e}^{-t}
\end{aligned}
$$

Adding the above results and simplifying gives

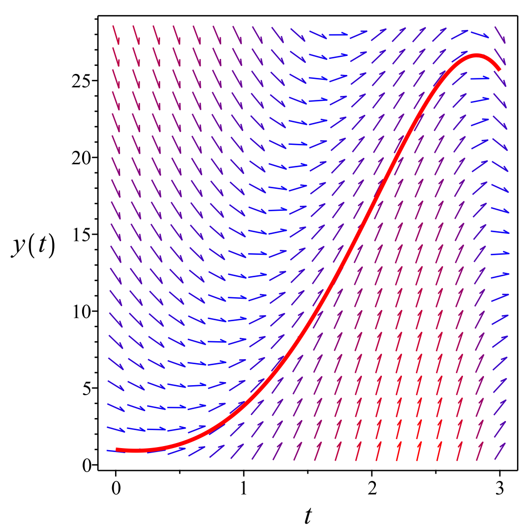
$$
y = (-\cos(t) + 2\sin(t))\,\mathrm{e}^{t} + 2\,\mathrm{e}^{-t}
$$

Summary
The solution(s) found are the following

$$
y = (-\cos(t) + 2\sin(t))\,\mathrm{e}^{t} + 2\,\mathrm{e}^{-t} \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$
y = (-\cos(t) + 2\sin(t))\,\mathrm{e}^{t} + 2\,\mathrm{e}^{-t}
$$

Verified OK.

13.7.3 Maple step by step solution

Let's solve

$[y' + y = 5\,\mathrm{e}^{t} \sin(t), y(0) = 1]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -y + 5\,\mathrm{e}^{t} \sin(t)$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + y = 5\,\mathrm{e}^{t} \sin(t)$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\,(y' + y) = 5\mu(t)\,\mathrm{e}^{t} \sin(t)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,y)$

  $\mu(t)\,(y' + y) = \mu'(t)\,y + \mu(t)\,y'$

- Isolate $\mu'(t)$

  $\mu'(t) = \mu(t)$

- Solve to find the integrating factor

  $\mu(t) = \mathrm{e}^{t}$

- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\,y)\right) dt = \int 5\mu(t)\,\mathrm{e}^{t} \sin(t)\,dt + c_1$

- Evaluate the integral on the lhs

  $\mu(t)\,y = \int 5\mu(t)\,\mathrm{e}^{t} \sin(t)\,dt + c_1$

- Solve for y

  $y = \frac{\int 5\mu(t)\mathrm{e}^{t} \sin(t)dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = \mathrm{e}^{t}$

  $y = \frac{\int 5(\mathrm{e}^{t})^{2} \sin(t)dt + c_1}{\mathrm{e}^{t}}$

- Evaluate the integrals on the rhs

  $y = \frac{-\mathrm{e}^{2t} \cos(t) + 2\sin(t)\mathrm{e}^{2t} + c_1}{\mathrm{e}^{t}}$

- Simplify

  $y = c_1 \mathrm{e}^{-t} - (\cos(t) - 2\sin(t))\,\mathrm{e}^{t}$

- Use initial condition $y(0) = 1$

  $1 = c_1 - 1$

- Solve for c_1

  $c_1 = 2$

- Substitute $c_1 = 2$ into general solution and simplify

  $y = (-\cos(t) + 2\sin(t))\,\mathrm{e}^{t} + 2\,\mathrm{e}^{-t}$

- Solution to the IVP

  $y = (-\cos(t) + 2\sin(t))\,\mathrm{e}^{t} + 2\,\mathrm{e}^{-t}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 3.125 (sec). Leaf size: 23

```
dsolve([diff(y(t),t)+y(t)=5*exp(t)*sin(t),y(0) = 1],y(t), singsol=all)
```

$$y(t) = \mathrm{e}^{t}(2\sin(t) - \cos(t)) + 2\,\mathrm{e}^{-t}$$

✓ Solution by Mathematica

Time used: 0.073 (sec). Leaf size: 27

```
DSolve[{y'[t]+y[t]==5*Exp[t]*Sin[t],{y[0]==1}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to 2e^{-t} + 2e^{t}\sin(t) - e^{t}\cos(t)$$

13.8 problem Problem 8

Internal problem ID [2846]
Internal file name [`OUTPUT/2338_Sunday_June_05_2022_02_59_45_AM_22223891/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 10, The Laplace Transform and Some Elementary Applications. Exercises for 10.4. page 689
Problem number: Problem 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _missing_x]]`

$$\boxed{y'' + y' - 2y = 0}$$

With initial conditions

$$[y(0) = 1, y'(0) = 4]$$

13.8.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 1 \\ q(t) &= -2 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + y' - 2y = 0$$

The domain of $p(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = -2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0) \\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + sY(s) - y(0) - 2Y(s) = 0 \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 1 \\ y'(0) &= 4\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 5 - s + sY(s) - 2Y(s) = 0$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{s+5}{s^2+s-2}$$

Applying partial fractions decomposition results in

$$Y(s) = -\frac{1}{s+2} + \frac{2}{s-1}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(-\frac{1}{s+2}\right) = -\mathrm{e}^{-2t}$$
$$\mathcal{L}^{-1}\left(\frac{2}{s-1}\right) = 2\,\mathrm{e}^{t}$$

Adding the above results and simplifying gives

$$y = 2\,\mathrm{e}^{t} - \mathrm{e}^{-2t}$$

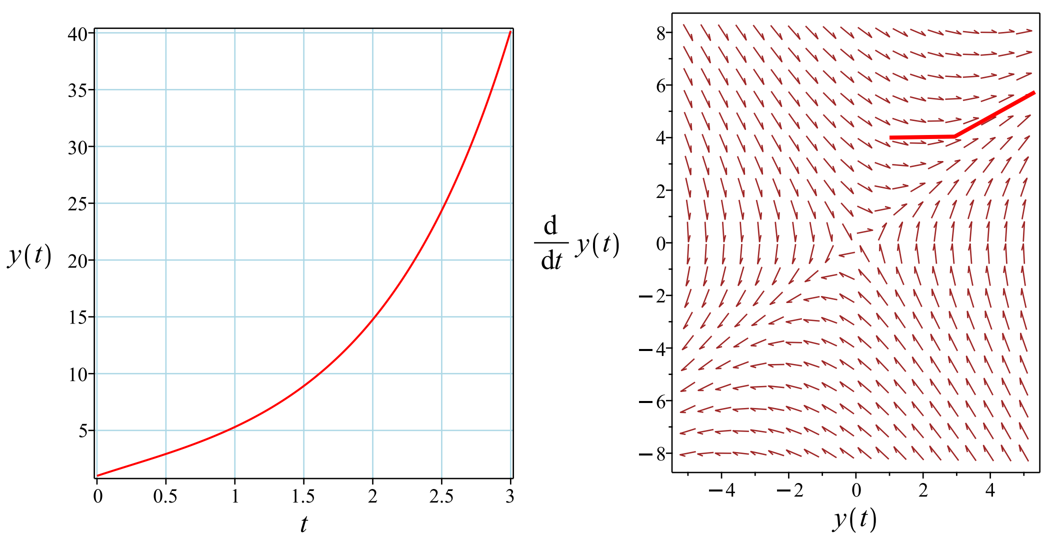
Simplifying the solution gives

$$y = \left(2\,\mathrm{e}^{3t} - 1\right)\mathrm{e}^{-2t}$$

Summary

The solution(s) found are the following

$$y = \left(2\,\mathrm{e}^{3t} - 1\right)\mathrm{e}^{-2t} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \left(2\,\mathrm{e}^{3t} - 1\right)\mathrm{e}^{-2t}$$

Verified OK.

13.8.2 Maple step by step solution

Let's solve

$$\left[y''+y'-2y=0, y(0)=1, y'\Big|_{\{t=0\}}=4\right]$$

- Highest derivative means the order of the ODE is 2

  y''
- Characteristic polynomial of ODE

  $r^2+r-2=0$
- Factor the characteristic polynomial

  $(r+2)(r-1)=0$
- Roots of the characteristic polynomial

  $r=(-2,1)$
- 1st solution of the ODE

  $y_1(t)=\mathrm{e}^{-2t}$
- 2nd solution of the ODE

  $y_2(t)=\mathrm{e}^{t}$
- General solution of the ODE

  $y=c_1y_1(t)+c_2y_2(t)$
- Substitute in solutions

  $y=c_1\mathrm{e}^{-2t}+c_2\mathrm{e}^{t}$

□ Check validity of solution $y=c_1\mathrm{e}^{-2t}+c_2\mathrm{e}^{t}$

  - Use initial condition $y(0)=1$

    $1=c_1+c_2$
  - Compute derivative of the solution

    $y'=-2c_1\mathrm{e}^{-2t}+c_2\mathrm{e}^{t}$
  - Use the initial condition $y'\Big|_{\{t=0\}}=4$

    $4=-2c_1+c_2$
  - Solve for c_1 and c_2

    $\{c_1=-1, c_2=2\}$

- Substitute constant values into general solution and simplify

  $$y = (2\,e^{3t} - 1)\,e^{-2t}$$

- Solution to the IVP

  $$y = (2\,e^{3t} - 1)\,e^{-2t}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 1.75 (sec). Leaf size: 15

```
dsolve([diff(y(t),t$2)+diff(y(t),t)-2*y(t)=0,y(0) = 1, D(y)(0) = 4],y(t), singsol=all)
```

$$y(t) = \left(2\,e^{3t} - 1\right) e^{-2t}$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 18

```
DSolve[{y''[t]+y'[t]-2*y[t]==0,{y[0]==1,y'[0]==4}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to 2e^{t} - e^{-2t}$$

13.9 problem Problem 9

Internal problem ID [2847]
Internal file name [`OUTPUT/2339_Sunday_June_05_2022_02_59_47_AM_86262074/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 10, The Laplace Transform and Some Elementary Applications. Exercises for 10.4. page 689
Problem number: Problem 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 4y = 0}$$

With initial conditions

$$[y(0) = 5, y'(0) = 1]$$

13.9.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned}
p(t) &= 0 \\
q(t) &= 4 \\
F &= 0
\end{aligned}$$

Hence the ode is

$$y'' + 4y = 0$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 4Y(s) = 0 \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 5\\ y'(0) &= 1\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 1 - 5s + 4Y(s) = 0$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{5s + 1}{s^2 + 4}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{\frac{5}{2} - \frac{i}{4}}{s - 2i} + \frac{\frac{5}{2} + \frac{i}{4}}{s + 2i}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(\frac{\frac{5}{2}-\frac{i}{4}}{s-2i}\right) = \left(\frac{5}{2}-\frac{i}{4}\right)\mathrm{e}^{2it}$$
$$\mathcal{L}^{-1}\left(\frac{\frac{5}{2}+\frac{i}{4}}{s+2i}\right) = \left(\frac{5}{2}+\frac{i}{4}\right)\mathrm{e}^{-2it}$$

Adding the above results and simplifying gives

$$y = 5\cos(2t) + \frac{\sin(2t)}{2}$$

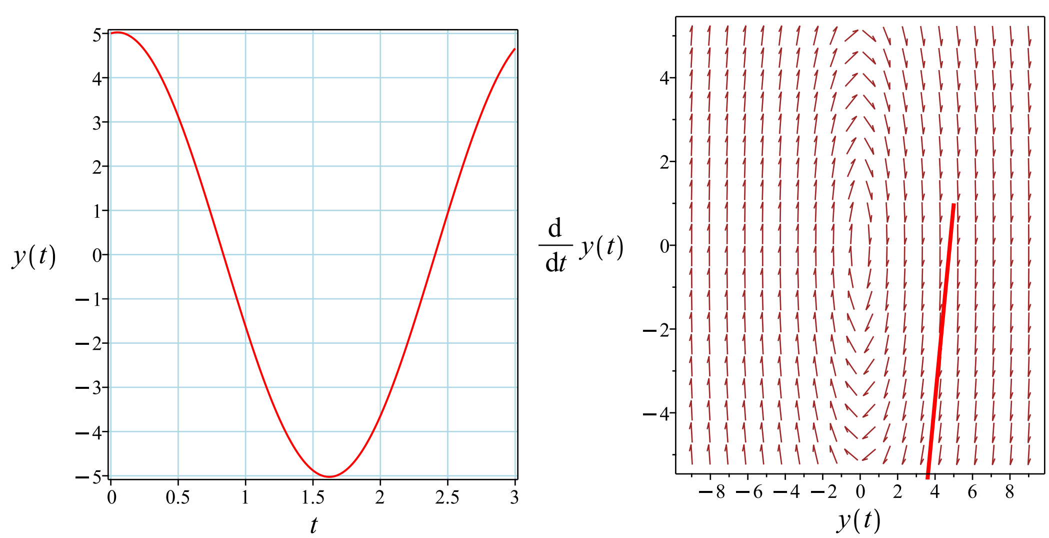
Simplifying the solution gives

$$y = 5\cos(2t) + \frac{\sin(2t)}{2}$$

Summary

The solution(s) found are the following

$$y = 5\cos(2t) + \frac{\sin(2t)}{2} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = 5\cos(2t) + \frac{\sin(2t)}{2}$$

Verified OK.

13.9.2 Maple step by step solution

Let's solve

$\left[y'' + 4y = 0, y(0) = 5, y'\Big|_{\{t=0\}} = 1\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + 4 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (-2\,\mathrm{I}, 2\,\mathrm{I})$

- 1st solution of the ODE

  $y_1(t) = \cos(2t)$

- 2nd solution of the ODE

  $y_2(t) = \sin(2t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y = c_1 \cos(2t) + c_2 \sin(2t)$

☐ Check validity of solution $y = c_1 \cos(2t) + c_2 \sin(2t)$

  - Use initial condition $y(0) = 5$

    $5 = c_1$

  - Compute derivative of the solution

    $y' = -2c_1 \sin(2t) + 2c_2 \cos(2t)$

  - Use the initial condition $y'\Big|_{\{t=0\}} = 1$

    $1 = 2c_2$

  - Solve for c_1 and c_2

    $\left\{c_1 = 5, c_2 = \frac{1}{2}\right\}$

- Substitute constant values into general solution and simplify

  $y = 5\cos(2t) + \frac{\sin(2t)}{2}$

- Solution to the IVP

  $y = 5\cos(2t) + \frac{\sin(2t)}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 1.938 (sec). Leaf size: 17

```
dsolve([diff(y(t),t$2)+4*y(t)=0,y(0) = 5, D(y)(0) = 1],y(t), singsol=all)
```

$$y(t) = 5\cos(2t) + \frac{\sin(2t)}{2}$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 17

```
DSolve[{y''[t]+4*y[t]==0,{y[0]==5,y'[0]==1}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to 5\cos(2t) + \sin(t)\cos(t)$$

13.10 problem Problem 10

Internal problem ID [2848]
Internal file name [`OUTPUT/2340_Sunday_June_05_2022_02_59_49_AM_35004514/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 10, The Laplace Transform and Some Elementary Applications. Exercises for 10.4. page 689
Problem number: Problem 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 3y' + 2y = 4}$$

With initial conditions

$$[y(0) = 0, y'(0) = 1]$$

13.10.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= -3 \\ q(t) &= 2 \\ F &= 4 \end{aligned}$$

Hence the ode is

$$y'' - 3y' + 2y = 4$$

The domain of $p(t) = -3$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) - 3sY(s) + 3y(0) + 2Y(s) = \frac{4}{s} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 0\\ y'(0) &= 1\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 1 - 3sY(s) + 2Y(s) = \frac{4}{s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{4+s}{s\,(s^2 - 3s + 2)}$$

Applying partial fractions decomposition results in

$$Y(s) = -\frac{5}{s-1} + \frac{3}{s-2} + \frac{2}{s}$$

The inverse Laplace of each term above is now found, which gives

$$\begin{aligned}
\mathcal{L}^{-1}\left(-\frac{5}{s-1}\right) &= -5\,\mathrm{e}^{t} \\
\mathcal{L}^{-1}\left(\frac{3}{s-2}\right) &= 3\,\mathrm{e}^{2t} \\
\mathcal{L}^{-1}\left(\frac{2}{s}\right) &= 2
\end{aligned}$$

Adding the above results and simplifying gives

$$y = 2 - 5\,\mathrm{e}^{t} + 3\,\mathrm{e}^{2t}$$

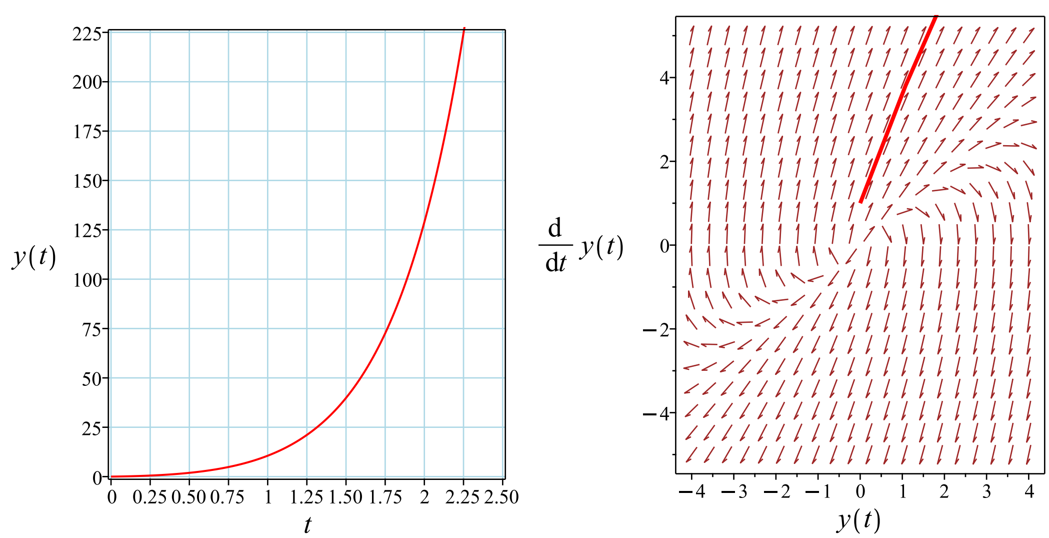
Simplifying the solution gives

$$y = 2 - 5\,\mathrm{e}^{t} + 3\,\mathrm{e}^{2t}$$

Summary

The solution(s) found are the following

$$y = 2 - 5\,\mathrm{e}^{t} + 3\,\mathrm{e}^{2t} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = 2 - 5\,\mathrm{e}^{t} + 3\,\mathrm{e}^{2t}$$

Verified OK.

13.10.2 Maple step by step solution

Let's solve

$\left[y'' - 3y' + 2y = 4, y(0) = 0, y'\Big|_{\{t=0\}} = 1\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 3r + 2 = 0$

- Factor the characteristic polynomial

  $(r-1)(r-2) = 0$

- Roots of the characteristic polynomial

  $r = (1, 2)$

- 1st solution of the homogeneous ODE

  $y_1(t) = \mathrm{e}^{t}$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \mathrm{e}^{2t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{t} + c_2 \mathrm{e}^{2t} + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = 4\right]$

  - Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \begin{bmatrix} e^t & e^{2t} \\ e^t & 2\,e^{2t} \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(t), y_2(t)) = e^{3t}$

- Substitute functions into equation for $y_p(t)$

  $y_p(t) = -4\,e^t\left(\int e^{-t}dt\right) + 4\,e^{2t}\left(\int e^{-2t}dt\right)$

- Compute integrals

  $y_p(t) = 2$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^t + c_2 e^{2t} + 2$

☐ Check validity of solution $y = c_1 e^t + c_2 e^{2t} + 2$

- Use initial condition $y(0) = 0$

  $0 = c_1 + c_2 + 2$

- Compute derivative of the solution

  $y' = c_1 e^t + 2c_2 e^{2t}$

- Use the initial condition $y'\Big|_{\{t=0\}} = 1$

  $1 = c_1 + 2c_2$

- Solve for c_1 and c_2

  $\{c_1 = -5, c_2 = 3\}$

- Substitute constant values into general solution and simplify

  $y = 2 - 5\,e^t + 3\,e^{2t}$

- Solution to the IVP

  $y = 2 - 5\,e^t + 3\,e^{2t}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 1.703 (sec). Leaf size: 16

```
dsolve([diff(y(t),t$2)-3*diff(y(t),t)+2*y(t)=4,y(0) = 0, D(y)(0) = 1],y(t), singsol=all)
```

$$y(t) = 3\,\mathrm{e}^{2t} - 5\,\mathrm{e}^{t} + 2$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 19

```
DSolve[{y''[t]-3*y'[t]+2*y[t]==4,{y[0]==0,y'[0]==1}},y[t],t,IncludeSingularSolutions -> True
```

$$y(t) \to -5e^{t} + 3e^{2t} + 2$$

13.11 problem Problem 11

Internal problem ID [2849]
Internal file name [`OUTPUT/2341_Sunday_June_05_2022_02_59_51_AM_39782151/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 10, The Laplace Transform and Some Elementary Applications. Exercises for 10.4. page 689
Problem number: Problem 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - y' - 12y = 36}$$

With initial conditions

$$[y(0) = 0, y'(0) = 12]$$

13.11.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= -1 \\ q(t) &= -12 \\ F &= 36 \end{aligned}$$

Hence the ode is

$$y'' - y' - 12y = 36$$

The domain of $p(t) = -1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = -12$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 36$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) - sY(s) + y(0) - 12Y(s) = \frac{36}{s} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 0\\ y'(0) &= 12\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 12 - sY(s) - 12Y(s) = \frac{36}{s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{12}{(s-4)\,s}$$

Applying partial fractions decomposition results in

$$Y(s) = -\frac{3}{s} + \frac{3}{s-4}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(-\frac{3}{s}\right) = -3$$
$$\mathcal{L}^{-1}\left(\frac{3}{s-4}\right) = 3\,\mathrm{e}^{4t}$$

Adding the above results and simplifying gives

$$y = 3\,\mathrm{e}^{4t} - 3$$

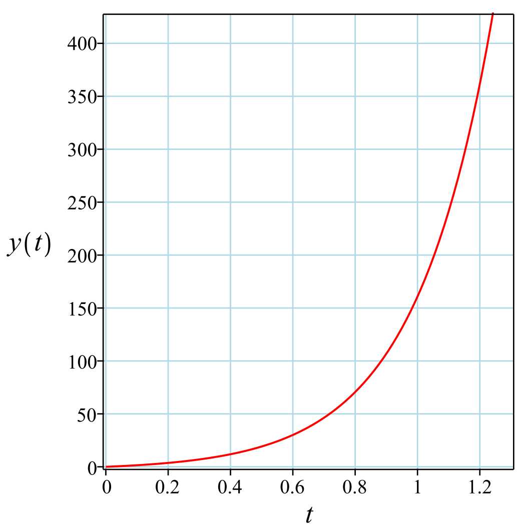
Simplifying the solution gives

$$y = 3\,\mathrm{e}^{4t} - 3$$

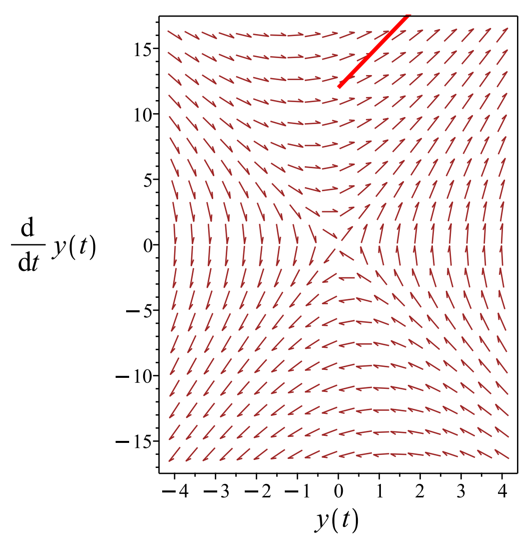
Summary

The solution(s) found are the following

$$y = 3\,\mathrm{e}^{4t} - 3 \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 3\,\mathrm{e}^{4t} - 3$$

Verified OK.

13.11.2 Maple step by step solution

Let's solve

$\left[y''-y'-12y=36, y(0)=0, y'\Big|_{\{t=0\}}=12\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2-r-12=0$

- Factor the characteristic polynomial

  $(r+3)(r-4)=0$

- Roots of the characteristic polynomial

  $r=(-3,4)$

- 1st solution of the homogeneous ODE

  $y_1(t)=e^{-3t}$

- 2nd solution of the homogeneous ODE

  $y_2(t)=e^{4t}$

- General solution of the ODE

  $y=c_1y_1(t)+c_2y_2(t)+y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y=c_1e^{-3t}+c_2e^{4t}+y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t)=-y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right)+y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t)=36\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(t),y_2(t))=\begin{bmatrix} e^{-3t} & e^{4t} \\ -3\,e^{-3t} & 4\,e^{4t} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(t),y_2(t))=7\,e^{t}$

  - Substitute functions into equation for $y_p(t)$

$y_p(t) = \frac{36\left(\mathrm{e}^{7t}\left(\int \mathrm{e}^{-4t}dt\right)-\left(\int \mathrm{e}^{3t}dt\right)\right)\mathrm{e}^{-3t}}{7}$

- Compute integrals

  $y_p(t) = -3$

- Substitute particular solution into general solution to ODE

  $y = c_1\mathrm{e}^{-3t} + c_2\mathrm{e}^{4t} - 3$

☐ Check validity of solution $y = c_1\mathrm{e}^{-3t} + c_2\mathrm{e}^{4t} - 3$

- Use initial condition $y(0) = 0$

  $0 = c_1 + c_2 - 3$

- Compute derivative of the solution

  $y' = -3c_1\mathrm{e}^{-3t} + 4c_2\mathrm{e}^{4t}$

- Use the initial condition $y'\big|_{\{t=0\}} = 12$

  $12 = -3c_1 + 4c_2$

- Solve for c_1 and c_2

  $\{c_1 = 0, c_2 = 3\}$

- Substitute constant values into general solution and simplify

  $y = 3\,\mathrm{e}^{4t} - 3$

- Solution to the IVP

  $y = 3\,\mathrm{e}^{4t} - 3$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 1.813 (sec). Leaf size: 12

```
dsolve([diff(y(t),t$2)-diff(y(t),t)-12*y(t)=36,y(0) = 0, D(y)(0) = 12],y(t), singsol=all)
```

$$y(t) = 3\,\mathrm{e}^{4t} - 3$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 14

```
DSolve[{y''[t]-y'[t]-12*y[t]==36,{y[0]==0,y'[0]==12}},y[t],t,IncludeSingularSolutions -> Tru
```

$$y(t) \to 3\left(e^{4t} - 1\right)$$

13.12 problem Problem 12

Internal problem ID [2850]
Internal file name [`OUTPUT/2342_Sunday_June_05_2022_02_59_53_AM_95848996/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 10, The Laplace Transform and Some Elementary Applications. Exercises for 10.4. page 689
Problem number: Problem 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + y' - 2y = 10\,\mathrm{e}^{-t}}$$

With initial conditions

$$[y(0) = 0, y'(0) = 1]$$

13.12.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 1 \\ q(t) &= -2 \\ F &= 10\,\mathrm{e}^{-t} \end{aligned}$$

Hence the ode is

$$y'' + y' - 2y = 10\,\mathrm{e}^{-t}$$

The domain of $p(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = -2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 10\,\mathrm{e}^{-t}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + sY(s) - y(0) - 2Y(s) = \frac{10}{1+s} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 0\\ y'(0) &= 1\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 1 + sY(s) - 2Y(s) = \frac{10}{1+s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{11+s}{(1+s)\,(s^2+s-2)}$$

Applying partial fractions decomposition results in

$$
Y(s) = \frac{2}{s-1} + \frac{3}{s+2} - \frac{5}{1+s}
$$

The inverse Laplace of each term above is now found, which gives

$$
\begin{aligned}
\mathcal{L}^{-1}\left(\frac{2}{s-1}\right) &= 2\,\mathrm{e}^{t} \\
\mathcal{L}^{-1}\left(\frac{3}{s+2}\right) &= 3\,\mathrm{e}^{-2t} \\
\mathcal{L}^{-1}\left(-\frac{5}{1+s}\right) &= -5\,\mathrm{e}^{-t}
\end{aligned}
$$

Adding the above results and simplifying gives

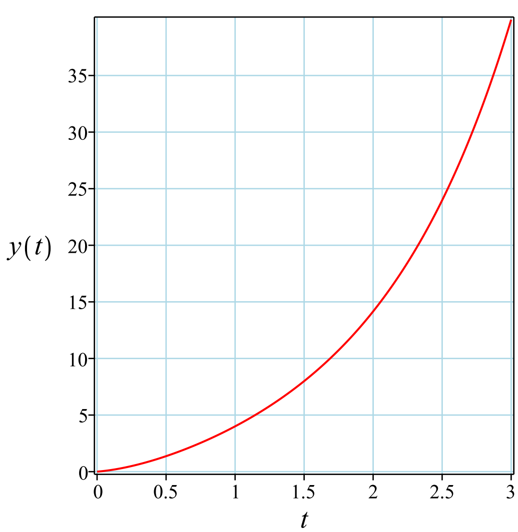
$$
y = -3\cosh(t) + 7\sinh(t) + 3\,\mathrm{e}^{-2t}
$$

Simplifying the solution gives

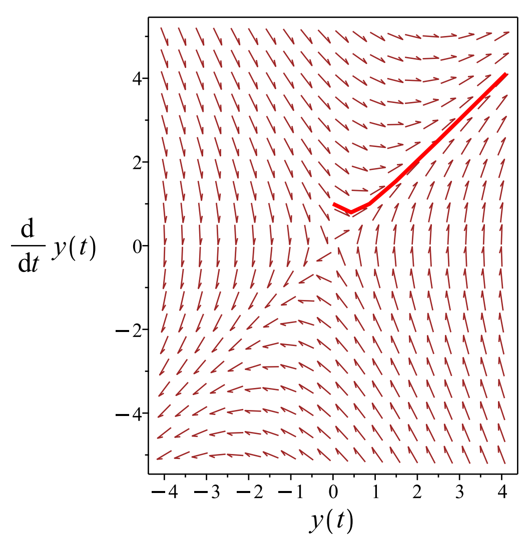
$$
y = -3\cosh(t) + 7\sinh(t) + 3\,\mathrm{e}^{-2t}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = -3\cosh(t) + 7\sinh(t) + 3\,\mathrm{e}^{-2t} \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -3\cosh(t) + 7\sinh(t) + 3\,\mathrm{e}^{-2t}$$

Verified OK.

13.12.2 Maple step by step solution

Let's solve

$\left[y'' + y' - 2y = 10\,\mathrm{e}^{-t}, y(0) = 0, y'\Big|_{\{t=0\}} = 1\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + r - 2 = 0$

- Factor the characteristic polynomial

  $(r+2)(r-1) = 0$

- Roots of the characteristic polynomial

  $r = (-2, 1)$

- 1st solution of the homogeneous ODE

  $y_1(t) = \mathrm{e}^{-2t}$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \mathrm{e}^{t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1\mathrm{e}^{-2t} + c_2\mathrm{e}^{t} + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

- ◦ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = 10\,\mathrm{e}^{-t}\right]$

- ◦ Wronskian of solutions of the homogeneous equation

$$W(y_1(t)\,,y_2(t)) = \begin{bmatrix} e^{-2t} & e^{t} \\ -2\,e^{-2t} & e^{t} \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(t)\,,y_2(t)) = 3\,e^{-t}$

- Substitute functions into equation for $y_p(t)$

  $y_p(t) = \frac{10\left(e^{3t}\left(\int e^{-2t}dt\right)-\left(\int e^{t}dt\right)\right)e^{-2t}}{3}$

- Compute integrals

  $y_p(t) = -5\,e^{-t}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-2t} + c_2 e^{t} - 5\,e^{-t}$

- ☐ Check validity of solution $y = c_1 e^{-2t} + c_2 e^{t} - 5e^{-t}$
  - Use initial condition $y(0) = 0$

    $0 = c_1 + c_2 - 5$

  - Compute derivative of the solution

    $y' = -2c_1 e^{-2t} + c_2 e^{t} + 5\,e^{-t}$

  - Use the initial condition $y'\Big|_{\{t=0\}} = 1$

    $1 = -2c_1 + c_2 + 5$

  - Solve for c_1 and c_2

    $\{c_1 = 3, c_2 = 2\}$

  - Substitute constant values into general solution and simplify

    $y = \left(2\,e^{3t} - 5\,e^{t} + 3\right)e^{-2t}$

- Solution to the IVP

  $y = \left(2\,e^{3t} - 5\,e^{t} + 3\right)e^{-2t}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 2.094 (sec). Leaf size: 19

```
dsolve([diff(y(t),t$2)+diff(y(t),t)-2*y(t)=10*exp(-t),y(0) = 0, D(y)(0) = 1],y(t), singsol=a
```

$$y(t) = -3\cosh(t) + 7\sinh(t) + 3\,\mathrm{e}^{-2t}$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 25

```
DSolve[{y''[t]+y'[t]-2*y[t]==10*Exp[-t],{y[0]==0,y'[0]==1}},y[t],t,IncludeSingularSolutions
```

$$y(t) \to e^{-2t}\left(-5e^t + 2e^{3t} + 3\right)$$

13.13 problem Problem 13

Internal problem ID [2851]
Internal file name [`OUTPUT/2343_Sunday_June_05_2022_02_59_56_AM_12307777/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 10, The Laplace Transform and Some Elementary Applications. Exercises for 10.4. page 689
Problem number: Problem 13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 3y' + 2y = 4\,\mathrm{e}^{3t}}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

13.13.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= -3 \\ q(t) &= 2 \\ F &= 4\,\mathrm{e}^{3t} \end{aligned}$$

Hence the ode is

$$y'' - 3y' + 2y = 4\,e^{3t}$$

The domain of $p(t) = -3$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 4\,e^{3t}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) - 3sY(s) + 3y(0) + 2Y(s) = \frac{4}{-3+s} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 0\\ y'(0) &= 0\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 3sY(s) + 2Y(s) = \frac{4}{-3+s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{4}{(-3+s)\,(s^2 - 3s + 2)}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{2}{s-1} + \frac{2}{-3+s} - \frac{4}{s-2}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(\frac{2}{s-1}\right) = 2\,\mathrm{e}^{t}$$
$$\mathcal{L}^{-1}\left(\frac{2}{-3+s}\right) = 2\,\mathrm{e}^{3t}$$
$$\mathcal{L}^{-1}\left(-\frac{4}{s-2}\right) = -4\,\mathrm{e}^{2t}$$

Adding the above results and simplifying gives

$$y = 2\,\mathrm{e}^{t} + 2\,\mathrm{e}^{3t} - 4\,\mathrm{e}^{2t}$$

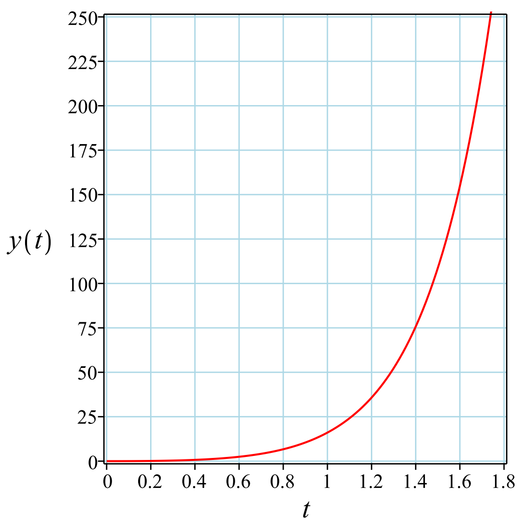
Simplifying the solution gives

$$y = 2\,\mathrm{e}^{t} + 2\,\mathrm{e}^{3t} - 4\,\mathrm{e}^{2t}$$

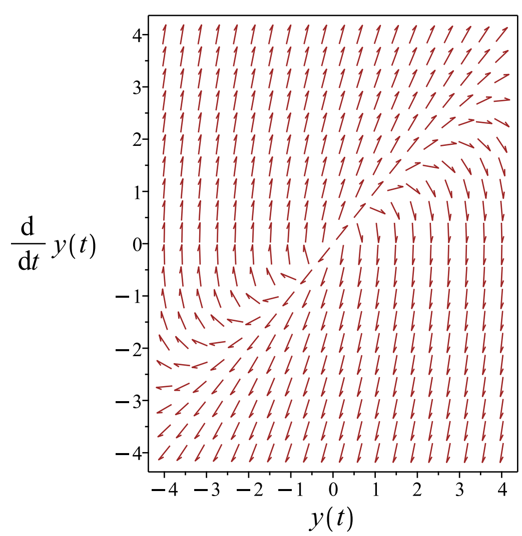
Summary

The solution(s) found are the following

$$y = 2\,\mathrm{e}^{t} + 2\,\mathrm{e}^{3t} - 4\,\mathrm{e}^{2t} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 2\,\mathrm{e}^{t} + 2\,\mathrm{e}^{3t} - 4\,\mathrm{e}^{2t}$$

Verified OK.

13.13.2 Maple step by step solution

Let's solve

$\left[y'' - 3y' + 2y = 4\,\mathrm{e}^{3t}, y(0) = 0, y'\Big|_{\{t=0\}} = 0\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 3r + 2 = 0$

- Factor the characteristic polynomial

  $(r-1)(r-2) = 0$

- Roots of the characteristic polynomial

  $r = (1, 2)$

- 1st solution of the homogeneous ODE

  $y_1(t) = \mathrm{e}^{t}$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \mathrm{e}^{2t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{t} + c_2 \mathrm{e}^{2t} + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = 4\,\mathrm{e}^{3t}\right]$

  - Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \begin{bmatrix} e^t & e^{2t} \\ e^t & 2\,e^{2t} \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(t), y_2(t)) = e^{3t}$

- Substitute functions into equation for $y_p(t)$

  $y_p(t) = -4\,e^t\left(\int e^{2t}dt\right) + 4\,e^{2t}\left(\int e^t dt\right)$

- Compute integrals

  $y_p(t) = 2\,e^{3t}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^t + c_2 e^{2t} + 2\,e^{3t}$

☐ Check validity of solution $y = c_1 e^t + c_2 e^{2t} + 2e^{3t}$

- Use initial condition $y(0) = 0$

  $0 = c_1 + c_2 + 2$

- Compute derivative of the solution

  $y' = c_1 e^t + 2c_2 e^{2t} + 6\,e^{3t}$

- Use the initial condition $y'\Big|_{\{t=0\}} = 0$

  $0 = c_1 + 2c_2 + 6$

- Solve for c_1 and c_2

  $\{c_1 = 2, c_2 = -4\}$

- Substitute constant values into general solution and simplify

  $y = 2\,e^t + 2\,e^{3t} - 4\,e^{2t}$

- Solution to the IVP

  $y = 2\,e^t + 2\,e^{3t} - 4\,e^{2t}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 2.203 (sec). Leaf size: 21

```
dsolve([diff(y(t),t$2)-3*diff(y(t),t)+2*y(t)=4*exp(3*t),y(0) = 0, D(y)(0) = 0],y(t), singsol
```

$$y(t) = 2\,e^{3t} - 4\,e^{2t} + 2\,e^{t}$$

✓ Solution by Mathematica

Time used: 0.016 (sec). Leaf size: 17

```
DSolve[{y''[t]-3*y'[t]+2*y[t]==4*Exp[3*t],{y[0]==0,y'[0]==0}},y[t],t,IncludeSingularSolution
```

$$y(t) \to 2e^{t}\left(e^{t} - 1\right)^{2}$$

13.14 problem Problem 14

Internal problem ID [2852]
Internal file name [`OUTPUT/2344_Sunday_June_05_2022_02_59_58_AM_93310471/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 10, The Laplace Transform and Some Elementary Applications. Exercises for 10.4. page 689
Problem number: Problem 14.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{y'' - 2y' = 30\,\mathrm{e}^{-3t}}$$

With initial conditions

$$[y(0) = 1, y'(0) = 0]$$

13.14.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= -2 \\ q(t) &= 0 \\ F &= 30\,\mathrm{e}^{-3t} \end{aligned}$$

Hence the ode is

$$y'' - 2y' = 30\,\mathrm{e}^{-3t}$$

The domain of $p(t) = -2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $F = 30\,\mathrm{e}^{-3t}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}
\mathcal{L}(y') &= sY(s) - y(0) \\
\mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)
\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) - 2sY(s) + 2y(0) = \frac{30}{3+s} \tag{1}$$

But the initial conditions are

$$\begin{aligned}
y(0) &= 1 \\
y'(0) &= 0
\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + 2 - s - 2sY(s) = \frac{30}{3+s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{s^2 + s + 24}{(3+s)\,s\,(s-2)}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{3}{s-2} + \frac{2}{3+s} - \frac{4}{s}$$

The inverse Laplace of each term above is now found, which gives

$$
\begin{aligned}
\mathcal{L}^{-1}\left(\frac{3}{s-2}\right) &= 3\,\mathrm{e}^{2t} \\
\mathcal{L}^{-1}\left(\frac{2}{3+s}\right) &= 2\,\mathrm{e}^{-3t} \\
\mathcal{L}^{-1}\left(-\frac{4}{s}\right) &= -4
\end{aligned}
$$

Adding the above results and simplifying gives

$$
y = -4 + 2\,\mathrm{e}^{-3t} + 3\,\mathrm{e}^{2t}
$$

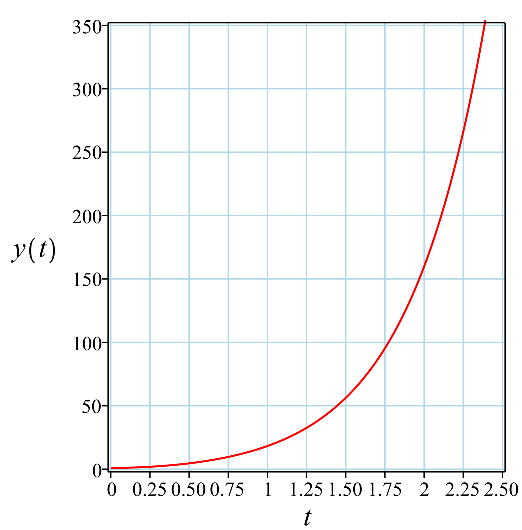
Simplifying the solution gives

$$
y = \left(3\,\mathrm{e}^{5t} - 4\,\mathrm{e}^{3t} + 2\right)\mathrm{e}^{-3t}
$$

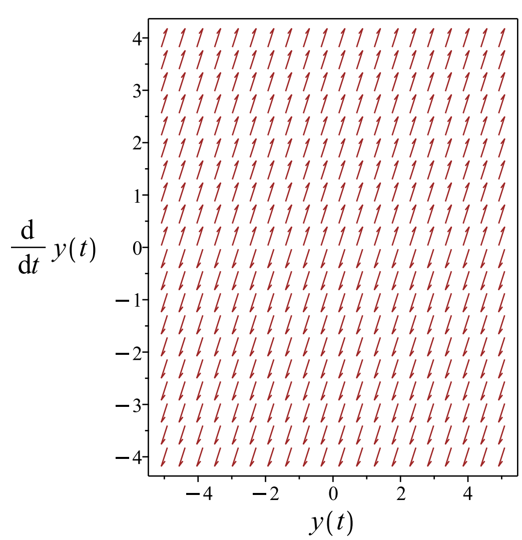
Summary

The solution(s) found are the following

$$
y = \left(3\,\mathrm{e}^{5t} - 4\,\mathrm{e}^{3t} + 2\right)\mathrm{e}^{-3t} \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$
y = \left(3\,\mathrm{e}^{5t} - 4\,\mathrm{e}^{3t} + 2\right)\mathrm{e}^{-3t}
$$

Verified OK.

13.14.2 Maple step by step solution

Let's solve

$$\left[y''-2y'=30\,\mathrm{e}^{-3t}, y(0)=1, y'\Big|_{\{t=0\}}=0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2-2r=0$

- Factor the characteristic polynomial

  $r(r-2)=0$

- Roots of the characteristic polynomial

  $r=(0,2)$

- 1st solution of the homogeneous ODE

  $y_1(t)=1$

- 2nd solution of the homogeneous ODE

  $y_2(t)=\mathrm{e}^{2t}$

- General solution of the ODE

  $y=c_1y_1(t)+c_2y_2(t)+y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y=c_1+c_2\mathrm{e}^{2t}+y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $$\left[y_p(t)=-y_1(t)\left(\int\frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right)+y_2(t)\left(\int\frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t)=30\,\mathrm{e}^{-3t}\right]$$

  - Wronskian of solutions of the homogeneous equation

    $$W(y_1(t)\,,y_2(t))=\left[\begin{array}{cc}1 & \mathrm{e}^{2t}\\ 0 & 2\,\mathrm{e}^{2t}\end{array}\right]$$

  - Compute Wronskian

    $W(y_1(t)\,,y_2(t))=2\,\mathrm{e}^{2t}$

  - Substitute functions into equation for $y_p(t)$

$y_p(t) = -15\left(\int e^{-3t}dt\right) + 15\,e^{2t}\left(\int e^{-5t}dt\right)$

- Compute integrals

  $y_p(t) = 2\,e^{-3t}$

- Substitute particular solution into general solution to ODE

  $y = c_1 + c_2 e^{2t} + 2\,e^{-3t}$

□ Check validity of solution $y = c_1 + c_2 e^{2t} + 2e^{-3t}$

- Use initial condition $y(0) = 1$

  $1 = c_1 + c_2 + 2$

- Compute derivative of the solution

  $y' = 2c_2 e^{2t} - 6\,e^{-3t}$

- Use the initial condition $y'\Big|_{\{t=0\}} = 0$

  $0 = 2c_2 - 6$

- Solve for c_1 and c_2

  $\{c_1 = -4, c_2 = 3\}$

- Substitute constant values into general solution and simplify

  $y = (3\,e^{5t} - 4\,e^{3t} + 2)\,e^{-3t}$

- Solution to the IVP

  $y = (3\,e^{5t} - 4\,e^{3t} + 2)\,e^{-3t}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = 2*_b(_a)+30*exp(-3*_a), _b(_a)`   *** S
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 2.781 (sec). Leaf size: 18

```
dsolve([diff(y(t),t$2)-2*diff(y(t),t)=30*exp(-3*t),y(0) = 1, D(y)(0) = 0],y(t), singsol=all)
```

$$y(t) = \left(3\,\mathrm{e}^{5t} - 4\,\mathrm{e}^{3t} + 2\right)\mathrm{e}^{-3t}$$

✓ Solution by Mathematica

Time used: 0.071 (sec). Leaf size: 21

```
DSolve[{y''[t]-2*y'[t]==30*Exp[-3*t],{y[0]==1,y'[0]==0}},y[t],t,IncludeSingularSolutions ->
```

$$y(t) \to 2e^{-3t} + 3e^{2t} - 4$$

13.15 problem Problem 15

Internal problem ID [2853]
Internal file name [`OUTPUT/2345_Sunday_June_05_2022_02_59_59_AM_98155230/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 10, The Laplace Transform and Some Elementary Applications. Exercises for 10.4. page 689
Problem number: Problem 15.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' - y = 12\,\mathrm{e}^{2t}}$$

With initial conditions

$$[y(0) = 1, y'(0) = 1]$$

13.15.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned}
p(t) &= 0 \\
q(t) &= -1 \\
F &= 12\,\mathrm{e}^{2t}
\end{aligned}$$

Hence the ode is

$$y'' - y = 12\,\mathrm{e}^{2t}$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = -1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 12\,\mathrm{e}^{2t}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) - Y(s) = \frac{12}{s-2} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 1\\ y'(0) &= 1\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 1 - s - Y(s) = \frac{12}{s-2}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{s^2 - s + 10}{(s-2)\,(s^2-1)}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{2}{s+1} - \frac{5}{s-1} + \frac{4}{s-2}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(\frac{2}{s+1}\right) = 2\,\mathrm{e}^{-t}$$
$$\mathcal{L}^{-1}\left(-\frac{5}{s-1}\right) = -5\,\mathrm{e}^{t}$$
$$\mathcal{L}^{-1}\left(\frac{4}{s-2}\right) = 4\,\mathrm{e}^{2t}$$

Adding the above results and simplifying gives

$$y = -3\cosh(t) - 7\sinh(t) + 4\,\mathrm{e}^{2t}$$

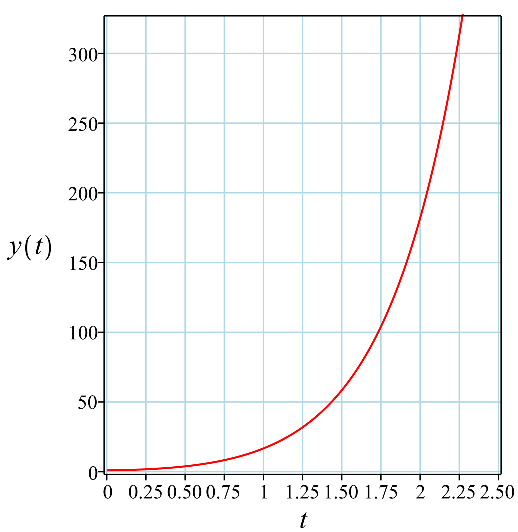
Simplifying the solution gives

$$y = -3\cosh(t) - 7\sinh(t) + 4\,\mathrm{e}^{2t}$$

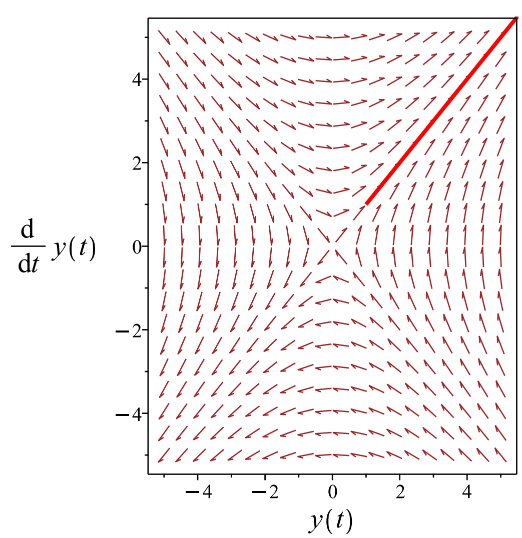
Summary

The solution(s) found are the following

$$y = -3\cosh(t) - 7\sinh(t) + 4\,\mathrm{e}^{2t} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -3\cosh(t) - 7\sinh(t) + 4\,\mathrm{e}^{2t}$$

Verified OK.

13.15.2 Maple step by step solution

Let's solve

$\left[y'' - y = 12\,\mathrm{e}^{2t}, y(0) = 1, y'\Big|_{\{t=0\}} = 1\right]$

- Highest derivative means the order of the ODE is 2

  y''
- Characteristic polynomial of homogeneous ODE

  $r^2 - 1 = 0$
- Factor the characteristic polynomial

  $(r-1)(r+1) = 0$
- Roots of the characteristic polynomial

  $r = (-1, 1)$
- 1st solution of the homogeneous ODE

  $y_1(t) = \mathrm{e}^{-t}$
- 2nd solution of the homogeneous ODE

  $y_2(t) = \mathrm{e}^{t}$
- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$
- Substitute in solutions of the homogeneous ODE

  $y = c_1\mathrm{e}^{-t} + c_2\mathrm{e}^{t} + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = 12\,\mathrm{e}^{2t}\right]$
- ○ Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \begin{bmatrix} e^{-t} & e^t \\ -e^{-t} & e^t \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(t), y_2(t)) = 2$

- Substitute functions into equation for $y_p(t)$

  $y_p(t) = -6\, e^{-t}\left(\int e^{3t} dt\right) + 6\, e^t\left(\int e^t dt\right)$

- Compute integrals

  $y_p(t) = 4\, e^{2t}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-t} + c_2 e^t + 4\, e^{2t}$

□ Check validity of solution $y = c_1 e^{-t} + c_2 e^t + 4e^{2t}$

- Use initial condition $y(0) = 1$

  $1 = c_1 + c_2 + 4$

- Compute derivative of the solution

  $y' = -c_1 e^{-t} + c_2 e^t + 8\, e^{2t}$

- Use the initial condition $y'\Big|_{\{t=0\}} = 1$

  $1 = -c_1 + c_2 + 8$

- Solve for c_1 and c_2

  $\{c_1 = 2, c_2 = -5\}$

- Substitute constant values into general solution and simplify

  $y = 2\, e^{-t} - 5\, e^t + 4\, e^{2t}$

- Solution to the IVP

  $y = 2\, e^{-t} - 5\, e^t + 4\, e^{2t}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 2.953 (sec). Leaf size: 19

```
dsolve([diff(y(t),t$2)-y(t)=12*exp(2*t),y(0) = 1, D(y)(0) = 1],y(t), singsol=all)
```

$$y(t) = -3\cosh(t) - 7\sinh(t) + 4\,\mathrm{e}^{2t}$$

✓ Solution by Mathematica

Time used: 0.017 (sec). Leaf size: 25

```
DSolve[{y''[t]-y[t]==12*Exp[2*t],{y[0]==1,y'[0]==1}},y[t],t,IncludeSingularSolutions -> True
```

$$y(t) \to 2e^{-t} - 5e^{t} + 4e^{2t}$$

13.16 problem Problem 16

Internal problem ID [2854]
Internal file name [`OUTPUT/2346_Sunday_June_05_2022_03_00_02_AM_77690956/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 10, The Laplace Transform and Some Elementary Applications. Exercises for 10.4. page 689
Problem number: Problem 16.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 4y = 10\,\mathrm{e}^{-t}}$$

With initial conditions

$$[y(0) = 4, y'(0) = 0]$$

13.16.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= 4 \\ F &= 10\,\mathrm{e}^{-t} \end{aligned}$$

Hence the ode is

$$y'' + 4y = 10\,\mathrm{e}^{-t}$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 10\,\mathrm{e}^{-t}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 4Y(s) = \frac{10}{1+s} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 4\\ y'(0) &= 0\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 4s + 4Y(s) = \frac{10}{1+s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{4s^2 + 4s + 10}{(1+s)\,(s^2+4)}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{1-\frac{i}{2}}{s-2i} + \frac{1+\frac{i}{2}}{s+2i} + \frac{2}{1+s}$$

The inverse Laplace of each term above is now found, which gives

$$\begin{aligned}
\mathcal{L}^{-1}\left(\frac{1-\frac{i}{2}}{s-2i}\right) &= \left(1-\frac{i}{2}\right) e^{2it} \\
\mathcal{L}^{-1}\left(\frac{1+\frac{i}{2}}{s+2i}\right) &= \left(1+\frac{i}{2}\right) e^{-2it} \\
\mathcal{L}^{-1}\left(\frac{2}{1+s}\right) &= 2\,e^{-t}
\end{aligned}$$

Adding the above results and simplifying gives

$$y = 2\,e^{-t} + 2\cos(2t) + \sin(2t)$$

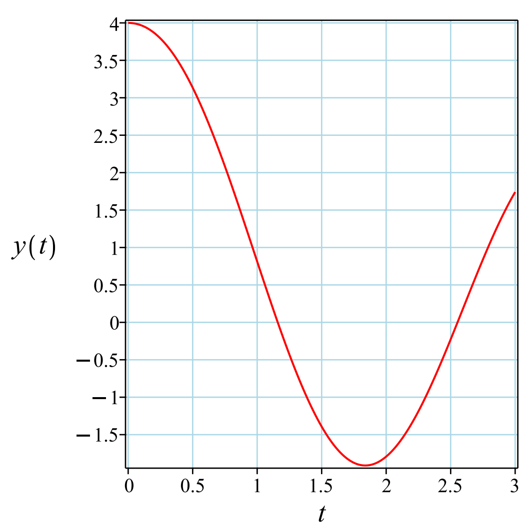
Simplifying the solution gives

$$y = 2\,e^{-t} + 2\cos(2t) + \sin(2t)$$

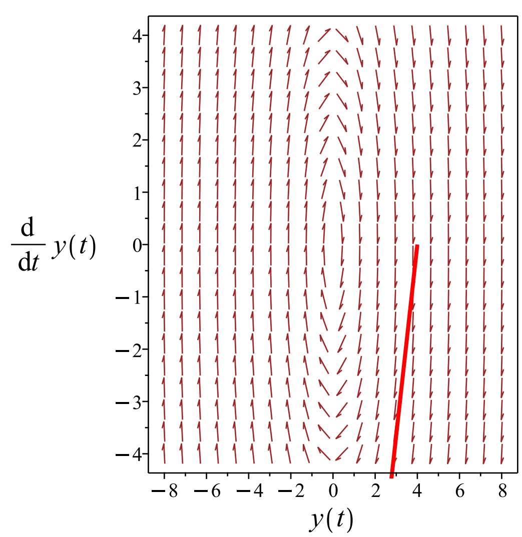
Summary

The solution(s) found are the following

$$y = 2\,e^{-t} + 2\cos(2t) + \sin(2t) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 2\,\mathrm{e}^{-t} + 2\cos(2t) + \sin(2t)$$

Verified OK.

13.16.2 Maple step by step solution

Let's solve

$\left[y'' + 4y = 10\,\mathrm{e}^{-t}, y(0) = 4, y'\Big|_{\{t=0\}} = 0\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (-2\,\mathrm{I}, 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t) = \cos(2t)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \sin(2t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(2t) + c_2 \sin(2t) + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = 10\,\mathrm{e}^{-t}\right]$

  ◦ Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \begin{bmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(t), y_2(t)) = 2$

- Substitute functions into equation for $y_p(t)$

  $y_p(t) = -5\cos(2t)\left(\int \mathrm{e}^{-t}\sin(2t)\,dt\right) + 5\sin(2t)\left(\int \mathrm{e}^{-t}\cos(2t)\,dt\right)$

- Compute integrals

  $y_p(t) = 2\,\mathrm{e}^{-t}$

• Substitute particular solution into general solution to ODE

  $y = c_1\cos(2t) + c_2\sin(2t) + 2\,\mathrm{e}^{-t}$

□ Check validity of solution $y = c_1\cos(2t) + c_2\sin(2t) + 2\mathrm{e}^{-t}$

- Use initial condition $y(0) = 4$

  $4 = 2 + c_1$

- Compute derivative of the solution

  $y' = -2c_1\sin(2t) + 2c_2\cos(2t) - 2\,\mathrm{e}^{-t}$

- Use the initial condition $y'\Big|_{\{t=0\}} = 0$

  $0 = -2 + 2c_2$

- Solve for c_1 and c_2

  $\{c_1 = 2, c_2 = 1\}$

- Substitute constant values into general solution and simplify

  $y = 2\,\mathrm{e}^{-t} + 2\cos(2t) + \sin(2t)$

• Solution to the IVP

  $y = 2\,\mathrm{e}^{-t} + 2\cos(2t) + \sin(2t)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 2.25 (sec). Leaf size: 21

```
dsolve([diff(y(t),t$2)+4*y(t)=10*exp(-t),y(0) = 4, D(y)(0) = 0],y(t), singsol=all)
```

$$y(t) = 2\,\mathrm{e}^{-t} + 2\cos(2t) + \sin(2t)$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 23

```
DSolve[{y''[t]+4*y[t]==10*Exp[-t],{y[0]==4,y'[0]==0}},y[t],t,IncludeSingularSolutions -> Tru
```

$$y(t) \to 2e^{-t} + \sin(2t) + 2\cos(2t)$$

13.17 problem Problem 17

Internal problem ID [2855]
Internal file name [`OUTPUT/2347_Sunday_June_05_2022_03_00_04_AM_15118246/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 10, The Laplace Transform and Some Elementary Applications. Exercises for 10.4. page 689
Problem number: Problem 17.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' - y' - 6y = 12 - 6\,\mathrm{e}^{t}}$$

With initial conditions

$$[y(0) = 5, y'(0) = -3]$$

13.17.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= -1 \\ q(t) &= -6 \\ F &= 12 - 6\,\mathrm{e}^{t} \end{aligned}$$

Hence the ode is

$$y'' - y' - 6y = 12 - 6\,e^{t}$$

The domain of $p(t) = -1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = -6$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 12 - 6\,e^{t}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) - sY(s) + y(0) - 6Y(s) = \frac{6s - 12}{s\,(s-1)} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 5\\ y'(0) &= -3\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + 8 - 5s - sY(s) - 6Y(s) = \frac{6s - 12}{s\,(s-1)}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{5s^3 - 13s^2 + 14s - 12}{s\,(s-1)\,(s^2 - s - 6)}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{1}{s-1} + \frac{8}{5(s-3)} + \frac{22}{5(s+2)} - \frac{2}{s}$$

The inverse Laplace of each term above is now found, which gives

$$\begin{aligned}
\mathcal{L}^{-1}\left(\frac{1}{s-1}\right) &= \mathrm{e}^{t} \\
\mathcal{L}^{-1}\left(\frac{8}{5(s-3)}\right) &= \frac{8\,\mathrm{e}^{3t}}{5} \\
\mathcal{L}^{-1}\left(\frac{22}{5(s+2)}\right) &= \frac{22\,\mathrm{e}^{-2t}}{5} \\
\mathcal{L}^{-1}\left(-\frac{2}{s}\right) &= -2
\end{aligned}$$

Adding the above results and simplifying gives

$$y = -2 + \mathrm{e}^{t} + \frac{22\,\mathrm{e}^{-2t}}{5} + \frac{8\,\mathrm{e}^{3t}}{5}$$

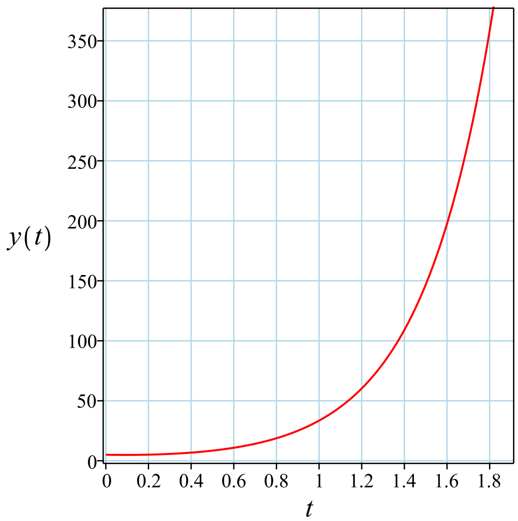
Simplifying the solution gives

$$y = \frac{\left(8\,\mathrm{e}^{5t} + 5\,\mathrm{e}^{3t} - 10\,\mathrm{e}^{2t} + 22\right)\mathrm{e}^{-2t}}{5}$$

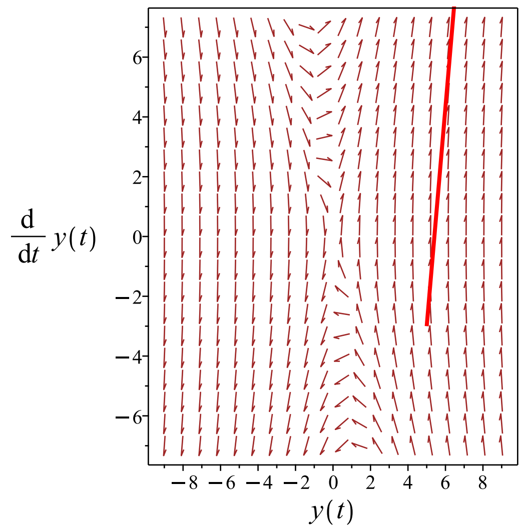
Summary

The solution(s) found are the following

$$y = \frac{\left(8\,\mathrm{e}^{5t} + 5\,\mathrm{e}^{3t} - 10\,\mathrm{e}^{2t} + 22\right)\mathrm{e}^{-2t}}{5} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{(8\,e^{5t} + 5\,e^{3t} - 10\,e^{2t} + 22)\,e^{-2t}}{5}$$

Verified OK.

13.17.2 Maple step by step solution

Let's solve

$$\left[y'' - y' - 6y = 12 - 6\,e^{t}, y(0) = 5, y'\Big|_{\{t=0\}} = -3\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - r - 6 = 0$

- Factor the characteristic polynomial

  $(r + 2)(r - 3) = 0$

- Roots of the characteristic polynomial

  $r = (-2, 3)$

- 1st solution of the homogeneous ODE

$y_1(t) = \mathrm{e}^{-2t}$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \mathrm{e}^{3t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{3t} + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t),y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t),y_2(t))} dt\right), f(t) = 12 - 6\,\mathrm{e}^{t}\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(t), y_2(t)) = \begin{bmatrix} \mathrm{e}^{-2t} & \mathrm{e}^{3t} \\ -2\,\mathrm{e}^{-2t} & 3\,\mathrm{e}^{3t} \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(t), y_2(t)) = 5\,\mathrm{e}^{t}$

- ○ Substitute functions into equation for $y_p(t)$

  $y_p(t) = \frac{6\left(-\mathrm{e}^{5t}\left(\int (-2+\mathrm{e}^{t})\mathrm{e}^{-3t} dt\right) + \int (-2+\mathrm{e}^{t})\mathrm{e}^{2t} dt\right)\mathrm{e}^{-2t}}{5}$

- ○ Compute integrals

  $y_p(t) = -2 + \mathrm{e}^{t}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{3t} - 2 + \mathrm{e}^{t}$

□ Check validity of solution $y = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{3t} - 2 + \mathrm{e}^{t}$

- ○ Use initial condition $y(0) = 5$

  $5 = c_1 + c_2 - 1$

- ○ Compute derivative of the solution

  $y' = -2c_1 \mathrm{e}^{-2t} + 3c_2 \mathrm{e}^{3t} + \mathrm{e}^{t}$

- ○ Use the initial condition $y'\Big|_{\{t=0\}} = -3$

  $-3 = -2c_1 + 3c_2 + 1$

- Solve for c_1 and c_2

  $\{c_1 = \frac{22}{5}, c_2 = \frac{8}{5}\}$

- Substitute constant values into general solution and simplify

  $y = \frac{(8\,e^{5t}+5\,e^{3t}-10\,e^{2t}+22)\,e^{-2t}}{5}$

- Solution to the IVP

  $y = \frac{(8\,e^{5t}+5\,e^{3t}-10\,e^{2t}+22)\,e^{-2t}}{5}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 1.859 (sec). Leaf size: 20

```
dsolve([diff(y(t),t$2)-diff(y(t),t)-6*y(t)=6*(2-exp(t)),y(0) = 5, D(y)(0) = -3],y(t), singso
```

$$y(t) = \frac{(8\,e^{5t} + 5\,e^{3t} - 10\,e^{2t} + 22)\,e^{-2t}}{5}$$

✓ Solution by Mathematica

Time used: 0.067 (sec). Leaf size: 28

```
DSolve[{y''[t]-y'[t]-6*y[t]==6*(2-Exp[t]),{y[0]==5,y'[0]==-3}},y[t],t,IncludeSingularSolutio
```

$$y(t) \to \frac{22e^{-2t}}{5} + e^t + \frac{8e^{3t}}{5} - 2$$

13.18 problem Problem 18

Internal problem ID [2856]
Internal file name [`OUTPUT/2348_Sunday_June_05_2022_03_00_05_AM_82525205/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 10, The Laplace Transform and Some Elementary Applications. Exercises for 10.4. page 689
Problem number: Problem 18.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' - y = 6\cos(t)}$$

With initial conditions

$$[y(0) = 0, y'(0) = 4]$$

13.18.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= -1 \\ F &= 6\cos(t) \end{aligned}$$

Hence the ode is

$$y'' - y = 6\cos(t)$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = -1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 6\cos(t)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) - Y(s) = \frac{6s}{s^2+1} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 0\\ y'(0) &= 4\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 4 - Y(s) = \frac{6s}{s^2+1}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{4s^2+6s+4}{(s^2+1)(s^2-1)}$$

Applying partial fractions decomposition results in

$$Y(s) = -\frac{3}{2(s-i)} - \frac{3}{2(s+i)} - \frac{1}{2(s+1)} + \frac{7}{2(s-1)}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(-\frac{3}{2(s-i)}\right) = -\frac{3\,\mathrm{e}^{it}}{2}$$
$$\mathcal{L}^{-1}\left(-\frac{3}{2(s+i)}\right) = -\frac{3\,\mathrm{e}^{-it}}{2}$$
$$\mathcal{L}^{-1}\left(-\frac{1}{2(s+1)}\right) = -\frac{\mathrm{e}^{-t}}{2}$$
$$\mathcal{L}^{-1}\left(\frac{7}{2(s-1)}\right) = \frac{7\,\mathrm{e}^{t}}{2}$$

Adding the above results and simplifying gives

$$y = 4\sinh(t) - 3\cos(t) + 3\cosh(t)$$

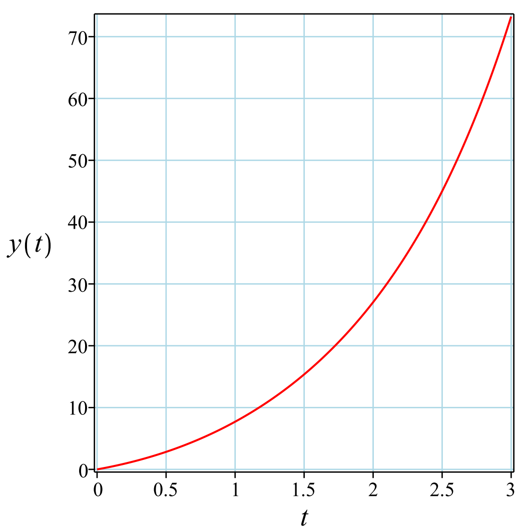
Simplifying the solution gives

$$y = 4\sinh(t) - 3\cos(t) + 3\cosh(t)$$

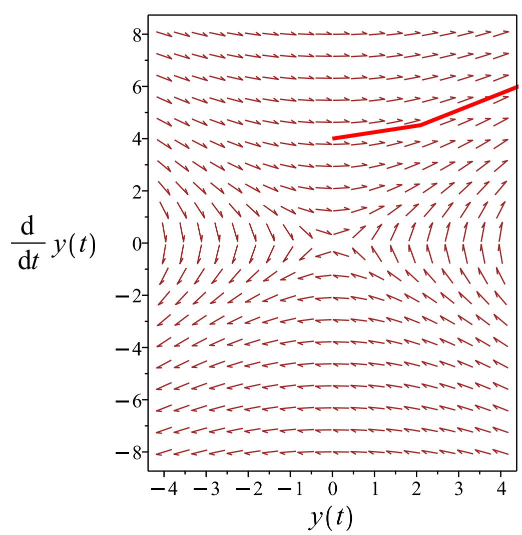
Summary

The solution(s) found are the following

$$y = 4\sinh(t) - 3\cos(t) + 3\cosh(t) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 4 \sinh(t) - 3\cos(t) + 3\cosh(t)$$

Verified OK.

13.18.2 Maple step by step solution

Let's solve

$\left[y'' - y = 6\cos(t), y(0) = 0, y'\Big|_{\{t=0\}} = 4\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 1 = 0$

- Factor the characteristic polynomial

  $(r-1)(r+1) = 0$

- Roots of the characteristic polynomial

  $r = (-1, 1)$

- 1st solution of the homogeneous ODE

  $y_1(t) = \mathrm{e}^{-t}$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \mathrm{e}^{t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{t} + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

  ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = 6\cos(t)\right]$

  ○ Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \begin{bmatrix} e^{-t} & e^t \\ -e^{-t} & e^t \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(t), y_2(t)) = 2$

- Substitute functions into equation for $y_p(t)$

  $y_p(t) = -3\,e^{-t}\left(\int e^t \cos(t)\,dt\right) + 3\,e^t\left(\int e^{-t} \cos(t)\,dt\right)$

- Compute integrals

  $y_p(t) = -3\cos(t)$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-t} + c_2 e^t - 3\cos(t)$

☐ Check validity of solution $y = c_1 e^{-t} + c_2 e^t - 3\cos(t)$

- Use initial condition $y(0) = 0$

  $0 = c_1 + c_2 - 3$

- Compute derivative of the solution

  $y' = -c_1 e^{-t} + c_2 e^t + 3\sin(t)$

- Use the initial condition $y'\Big|_{\{t=0\}} = 4$

  $4 = -c_1 + c_2$

- Solve for c_1 and c_2

  $\left\{c_1 = -\frac{1}{2}, c_2 = \frac{7}{2}\right\}$

- Substitute constant values into general solution and simplify

  $y = -\frac{e^{-t}}{2} + \frac{7\,e^t}{2} - 3\cos(t)$

- Solution to the IVP

  $y = -\frac{e^{-t}}{2} + \frac{7\,e^t}{2} - 3\cos(t)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 1.859 (sec). Leaf size: 17

```
dsolve([diff(y(t),t$2)-y(t)=6*cos(t),y(0) = 0, D(y)(0) = 4],y(t), singsol=all)
```

$$y(t) = 4\sinh(t) - 3\cos(t) + 3\cosh(t)$$

✓ Solution by Mathematica

Time used: 0.018 (sec). Leaf size: 26

```
DSolve[{y''[t]-y[t]==6*Cos[t],{y[0]==0,y'[0]==4}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{2}\left(-e^{-t} + 7e^{t} - 6\cos(t)\right)$$

13.19 problem Problem 19

Internal problem ID [2857]
Internal file name [`OUTPUT/2349_Sunday_June_05_2022_03_00_07_AM_36585125/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 10, The Laplace Transform and Some Elementary Applications. Exercises for 10.4. page 689
Problem number: Problem 19.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' - 9y = 13\sin(2t)}$$

With initial conditions

$$[y(0) = 3, y'(0) = 1]$$

13.19.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= -9 \\ F &= 13\sin(2t) \end{aligned}$$

Hence the ode is

$$y'' - 9y = 13\sin(2t)$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = -9$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 13\sin(2t)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) - 9Y(s) = \frac{26}{s^2+4} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 3\\ y'(0) &= 1\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 1 - 3s - 9Y(s) = \frac{26}{s^2+4}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{3s^3 + s^2 + 12s + 30}{(s^2+4)(s^2-9)}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{2}{s-3} + \frac{i}{2s-4i} - \frac{i}{2\,(s+2i)} + \frac{1}{s+3}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(\frac{2}{s-3}\right) = 2\,\mathrm{e}^{3t}$$

$$\mathcal{L}^{-1}\left(\frac{i}{2s-4i}\right) = \frac{i\mathrm{e}^{2it}}{2}$$

$$\mathcal{L}^{-1}\left(-\frac{i}{2\,(s+2i)}\right) = -\frac{i\mathrm{e}^{-2it}}{2}$$

$$\mathcal{L}^{-1}\left(\frac{1}{s+3}\right) = \mathrm{e}^{-3t}$$

Adding the above results and simplifying gives

$$y = \mathrm{e}^{-3t} - \sin\left(2t\right) + 2\,\mathrm{e}^{3t}$$

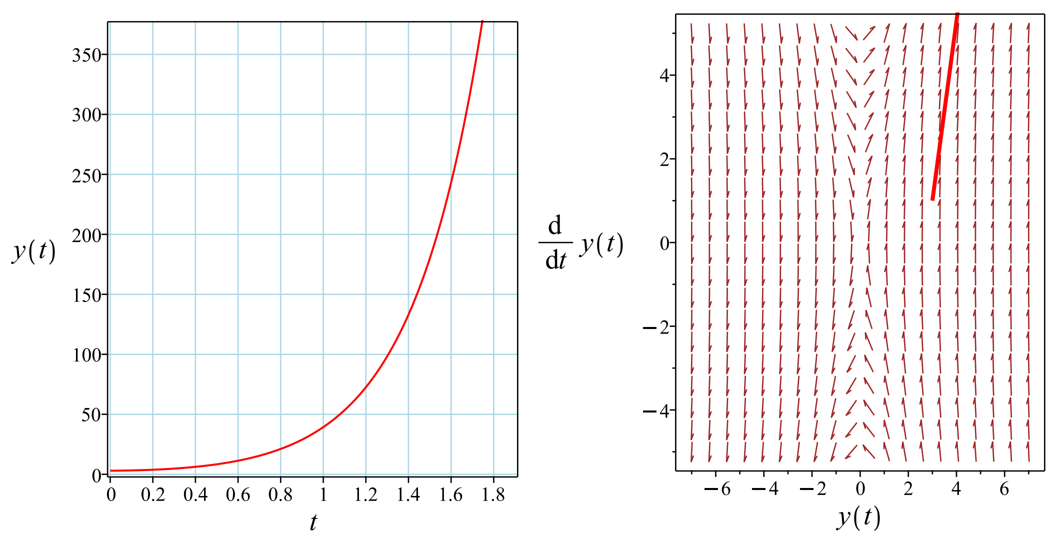
Simplifying the solution gives

$$y = \mathrm{e}^{-3t} - \sin\left(2t\right) + 2\,\mathrm{e}^{3t}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-3t} - \sin\left(2t\right) + 2\,\mathrm{e}^{3t} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = e^{-3t} - \sin(2t) + 2\,e^{3t}$$

Verified OK.

13.19.2 Maple step by step solution

Let's solve

$\left[y'' - 9y = 13\sin(2t), y(0) = 3, y'\Big|_{\{t=0\}} = 1\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 9 = 0$

- Factor the characteristic polynomial

  $(r-3)(r+3) = 0$

- Roots of the characteristic polynomial

  $r = (-3, 3)$

- 1st solution of the homogeneous ODE

  $y_1(t) = e^{-3t}$

- 2nd solution of the homogeneous ODE

  $y_2(t) = e^{3t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-3t} + c_2 e^{3t} + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = 13\sin(2t)\right]$

  ◦ Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \begin{bmatrix} e^{-3t} & e^{3t} \\ -3\,e^{-3t} & 3\,e^{3t} \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(t), y_2(t)) = 6$

- Substitute functions into equation for $y_p(t)$

  $y_p(t) = -\frac{13\,e^{-3t}\left(\int e^{3t}\sin(2t)dt\right)}{6} + \frac{13\,e^{3t}\left(\int e^{-3t}\sin(2t)dt\right)}{6}$

- Compute integrals

  $y_p(t) = -\sin(2t)$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-3t} + c_2 e^{3t} - \sin(2t)$

☐ Check validity of solution $y = c_1 e^{-3t} + c_2 e^{3t} - \sin(2t)$

- Use initial condition $y(0) = 3$

  $3 = c_1 + c_2$

- Compute derivative of the solution

  $y' = -3c_1 e^{-3t} + 3c_2 e^{3t} - 2\cos(2t)$

- Use the initial condition $y'\Big|_{\{t=0\}} = 1$

  $1 = -3c_1 + 3c_2 - 2$

- Solve for c_1 and c_2

  $\{c_1 = 1, c_2 = 2\}$

- Substitute constant values into general solution and simplify

  $y = e^{-3t} - \sin(2t) + 2\,e^{3t}$

- Solution to the IVP

  $y = e^{-3t} - \sin(2t) + 2\,e^{3t}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple
Time used: 2.75 (sec). Leaf size: 21

```
dsolve([diff(y(t),t$2)-9*y(t)=13*sin(2*t),y(0) = 3, D(y)(0) = 1],y(t), singsol=all)
```

$$y(t) = -\sin(2t) + 2\,\mathrm{e}^{3t} + \mathrm{e}^{-3t}$$

✓ Solution by Mathematica
Time used: 0.019 (sec). Leaf size: 24

```
DSolve[{y''[t]-9*y[t]==13*Sin[2*t],{y[0]==3,y'[0]==1}},y[t],t,IncludeSingularSolutions -> Tr
```

$$y(t) \to e^{-3t} + 2e^{3t} - \sin(2t)$$

13.20 problem Problem 20

Internal problem ID [2858]
Internal file name [`OUTPUT/2350_Sunday_June_05_2022_03_00_09_AM_52079326/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 10, The Laplace Transform and Some Elementary Applications. Exercises for 10.4. page 689
Problem number: Problem 20.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - y = 8 \sin(t) - 6 \cos(t)}$$

With initial conditions

$$[y(0) = 2, y'(0) = -1]$$

13.20.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= -1 \\ F &= 8 \sin(t) - 6 \cos(t) \end{aligned}$$

Hence the ode is

$$y'' - y = 8 \sin(t) - 6 \cos(t)$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = -1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 8 \sin(t) - 6 \cos(t)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) - Y(s) = -\frac{2(-4 + 3s)}{s^2 + 1} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 2\\ y'(0) &= -1\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + 1 - 2s - Y(s) = -\frac{2(-4 + 3s)}{s^2 + 1}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{2s^3 - s^2 - 4s + 7}{(s^2 + 1)(s^2 - 1)}$$

Applying partial fractions decomposition results in

$$Y(s) = -\frac{2}{s+1} + \frac{1}{s-1} + \frac{\frac{3}{2}+2i}{s-i} + \frac{\frac{3}{2}-2i}{s+i}$$

The inverse Laplace of each term above is now found, which gives

$$\begin{aligned}
\mathcal{L}^{-1}\left(-\frac{2}{s+1}\right) &= -2\,\mathrm{e}^{-t} \\
\mathcal{L}^{-1}\left(\frac{1}{s-1}\right) &= \mathrm{e}^{t} \\
\mathcal{L}^{-1}\left(\frac{\frac{3}{2}+2i}{s-i}\right) &= \left(\frac{3}{2}+2i\right)\mathrm{e}^{it} \\
\mathcal{L}^{-1}\left(\frac{\frac{3}{2}-2i}{s+i}\right) &= \left(\frac{3}{2}-2i\right)\mathrm{e}^{-it}
\end{aligned}$$

Adding the above results and simplifying gives

$$y = -4\sin(t) + 3\sinh(t) - \cosh(t) + 3\cos(t)$$

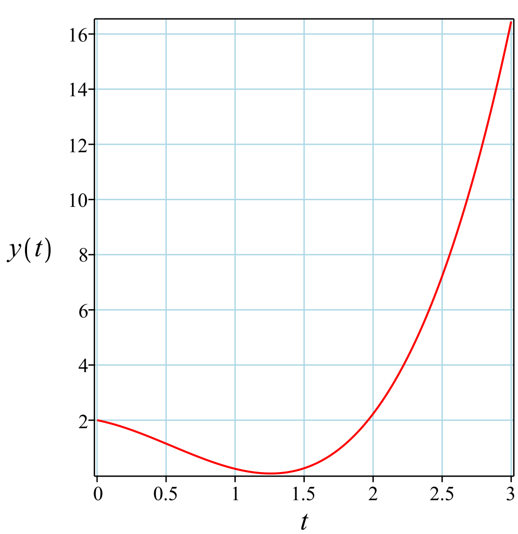
Simplifying the solution gives

$$y = -4\sin(t) + 3\sinh(t) - \cosh(t) + 3\cos(t)$$

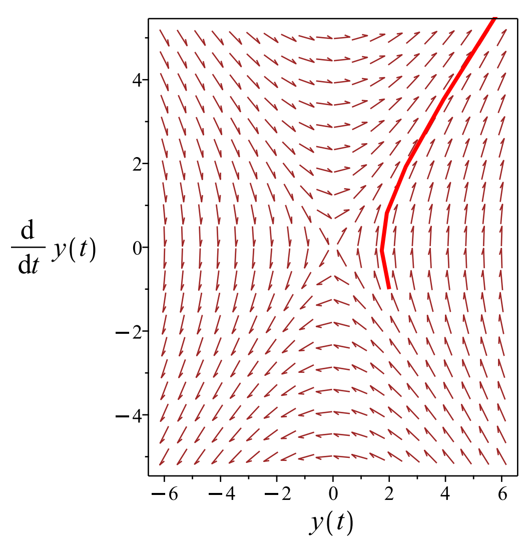
Summary

The solution(s) found are the following

$$y = -4\sin(t) + 3\sinh(t) - \cosh(t) + 3\cos(t) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -4\sin(t) + 3\sinh(t) - \cosh(t) + 3\cos(t)$$

Verified OK.

13.20.2 Maple step by step solution

Let's solve

$\left[y'' - y = 8\sin(t) - 6\cos(t), y(0) = 2, y'\Big|_{\{t=0\}} = -1\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 1 = 0$

- Factor the characteristic polynomial

  $(r-1)(r+1) = 0$

- Roots of the characteristic polynomial

  $r = (-1, 1)$

- 1st solution of the homogeneous ODE

  $y_1(t) = \mathrm{e}^{-t}$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \mathrm{e}^{t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{t} + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = 8\sin(t) - 6\cos(t)\right]$

  ◦ Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \begin{bmatrix} e^{-t} & e^{t} \\ -e^{-t} & e^{t} \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(t), y_2(t)) = 2$

- Substitute functions into equation for $y_p(t)$

  $y_p(t) = e^{-t}\left(\int (-4\sin(t) + 3\cos(t))\, e^{t} dt\right) - e^{t}\left(\int (-4\sin(t) + 3\cos(t))\, e^{-t} dt\right)$

- Compute integrals

  $y_p(t) = -4\sin(t) + 3\cos(t)$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-t} + c_2 e^{t} - 4\sin(t) + 3\cos(t)$

□ Check validity of solution $y = c_1 e^{-t} + c_2 e^{t} - 4\sin(t) + 3\cos(t)$

- Use initial condition $y(0) = 2$

  $2 = c_1 + c_2 + 3$

- Compute derivative of the solution

  $y' = -c_1 e^{-t} + c_2 e^{t} - 4\cos(t) - 3\sin(t)$

- Use the initial condition $y'\Big|_{\{t=0\}} = -1$

  $-1 = -c_1 + c_2 - 4$

- Solve for c_1 and c_2

  $\{c_1 = -2, c_2 = 1\}$

- Substitute constant values into general solution and simplify

  $y = -2\,e^{-t} + e^{t} - 4\sin(t) + 3\cos(t)$

- Solution to the IVP

  $y = -2\,e^{-t} + e^{t} - 4\sin(t) + 3\cos(t)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 1.953 (sec). Leaf size: 21

```
dsolve([diff(y(t),t$2)-y(t)=8*sin(t)-6*cos(t),y(0) = 2, D(y)(0) = -1],y(t), singsol=all)
```

$$y(t) = -4\sin(t) + 3\cos(t) + 3\sinh(t) - \cosh(t)$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 24

```
DSolve[{y''[t]-y[t]==8*Sin[t]-6*Cos[t],{y[0]==2,y'[0]==-1}},y[t],t,IncludeSingularSolutions
```

$$y(t) \to -2e^{-t} + e^{t} - 4\sin(t) + 3\cos(t)$$

13.21 problem Problem 21

Internal problem ID [2859]
Internal file name [`OUTPUT/2351_Sunday_June_05_2022_03_00_11_AM_52807654/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 10, The Laplace Transform and Some Elementary Applications. Exercises for 10.4. page 689
Problem number: Problem 21.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' - y' - 2y = 10\cos(t)}$$

With initial conditions

$$[y(0) = 0, y'(0) = -1]$$

13.21.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= -1 \\ q(t) &= -2 \\ F &= 10\cos(t) \end{aligned}$$

Hence the ode is

$$y'' - y' - 2y = 10\cos(t)$$

The domain of $p(t) = -1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = -2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 10\cos(t)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) - sY(s) + y(0) - 2Y(s) = \frac{10s}{s^2+1} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 0\\ y'(0) &= -1\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + 1 - sY(s) - 2Y(s) = \frac{10s}{s^2+1}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = -\frac{s^2 - 10s + 1}{(s^2+1)(s^2 - s - 2)}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{1}{s-2} + \frac{2}{s+1} + \frac{-\frac{3}{2}+\frac{i}{2}}{s-i} + \frac{-\frac{3}{2}-\frac{i}{2}}{s+i}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(\frac{1}{s-2}\right) = e^{2t}$$
$$\mathcal{L}^{-1}\left(\frac{2}{s+1}\right) = 2\,e^{-t}$$
$$\mathcal{L}^{-1}\left(\frac{-\frac{3}{2}+\frac{i}{2}}{s-i}\right) = \left(-\frac{3}{2}+\frac{i}{2}\right)e^{it}$$
$$\mathcal{L}^{-1}\left(\frac{-\frac{3}{2}-\frac{i}{2}}{s+i}\right) = \left(-\frac{3}{2}-\frac{i}{2}\right)e^{-it}$$

Adding the above results and simplifying gives

$$y = -3\cos(t) - \sin(t) + 2\,e^{-t} + e^{2t}$$

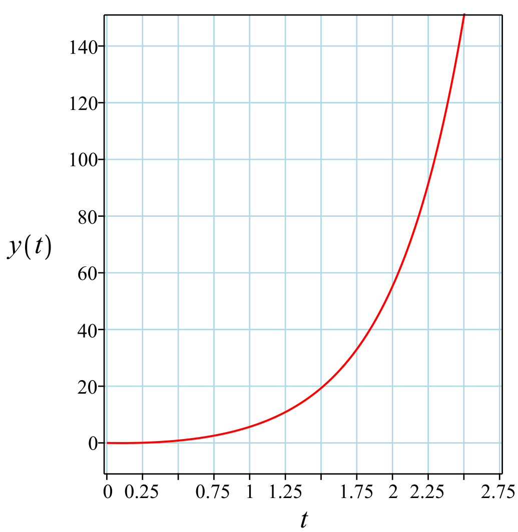
Simplifying the solution gives

$$y = -3\cos(t) - \sin(t) + 2\,e^{-t} + e^{2t}$$

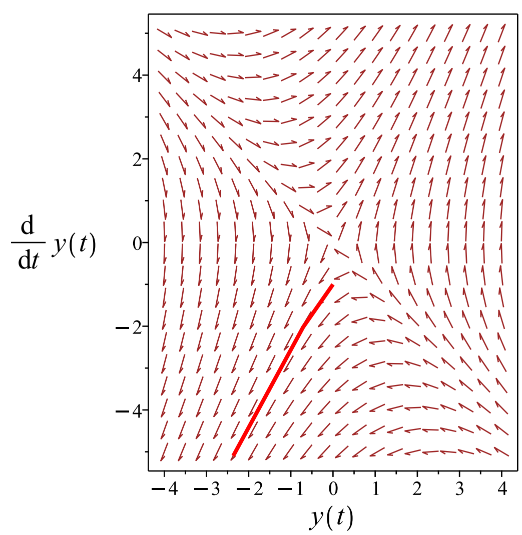
Summary

The solution(s) found are the following

$$y = -3\cos(t) - \sin(t) + 2\,e^{-t} + e^{2t} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -3\cos(t) - \sin(t) + 2\,\mathrm{e}^{-t} + \mathrm{e}^{2t}$$

Verified OK.

13.21.2 Maple step by step solution

Let's solve

$$\left[y'' - y' - 2y = 10\cos(t), y(0) = 0, y'\Big|_{\{t=0\}} = -1\right]$$

- Highest derivative means the order of the ODE is 2
  y''
- Characteristic polynomial of homogeneous ODE
  $r^2 - r - 2 = 0$
- Factor the characteristic polynomial
  $(r+1)(r-2) = 0$
- Roots of the characteristic polynomial
  $r = (-1, 2)$
- 1st solution of the homogeneous ODE
  $y_1(t) = \mathrm{e}^{-t}$
- 2nd solution of the homogeneous ODE
  $y_2(t) = \mathrm{e}^{2t}$
- General solution of the ODE
  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$
- Substitute in solutions of the homogeneous ODE
  $y = c_1\mathrm{e}^{-t} + c_2\mathrm{e}^{2t} + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function
  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = 10\cos(t)\right]$
- ○ Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \begin{bmatrix} e^{-t} & e^{2t} \\ -e^{-t} & 2e^{2t} \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(t), y_2(t)) = 3e^t$

- Substitute functions into equation for $y_p(t)$

  $y_p(t) = -\frac{10e^{-t}(\int e^t \cos(t)dt)}{3} + \frac{10e^{2t}(\int \cos(t)e^{-2t}dt)}{3}$

- Compute integrals

  $y_p(t) = -3\cos(t) - \sin(t)$

- Substitute particular solution into general solution to ODE

  $y = c_1e^{-t} + c_2e^{2t} - 3\cos(t) - \sin(t)$

□ Check validity of solution $y = c_1e^{-t} + c_2e^{2t} - 3\cos(t) - \sin(t)$

- Use initial condition $y(0) = 0$

  $0 = c_1 + c_2 - 3$

- Compute derivative of the solution

  $y' = -c_1e^{-t} + 2c_2e^{2t} + 3\sin(t) - \cos(t)$

- Use the initial condition $y'\Big|_{\{t=0\}} = -1$

  $-1 = -c_1 + 2c_2 - 1$

- Solve for c_1 and c_2

  $\{c_1 = 2, c_2 = 1\}$

- Substitute constant values into general solution and simplify

  $y = -3\cos(t) - \sin(t) + 2e^{-t} + e^{2t}$

- Solution to the IVP

  $y = -3\cos(t) - \sin(t) + 2e^{-t} + e^{2t}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 1.891 (sec). Leaf size: 23

```
dsolve([diff(y(t),t$2)-diff(y(t),t)-2*y(t)=10*cos(t),y(0) = 0, D(y)(0) = -1],y(t), singsol=a
```

$$y(t) = 2\,e^{-t} + e^{2t} - 3\cos(t) - \sin(t)$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 26

```
DSolve[{y''[t]-y'[t]-2*y[t]==10*Cos[t],{y[0]==0,y'[0]==-1}},y[t],t,IncludeSingularSolutions
```

$$y(t) \to 2e^{-t} + e^{2t} - \sin(t) - 3\cos(t)$$

13.22 problem Problem 22

Internal problem ID [2860]
Internal file name [`OUTPUT/2352_Sunday_June_05_2022_03_00_13_AM_40746384/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 10, The Laplace Transform and Some Elementary Applications. Exercises for 10.4. page 689
Problem number: Problem 22.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 5y' + 4y = 20 \sin(2t)}$$

With initial conditions

$$[y(0) = -1, y'(0) = 2]$$

13.22.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 5 \\ q(t) &= 4 \\ F &= 20 \sin(2t) \end{aligned}$$

Hence the ode is

$$y'' + 5y' + 4y = 20\sin(2t)$$

The domain of $p(t) = 5$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 20\sin(2t)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}
\mathcal{L}(y') &= sY(s) - y(0) \\
\mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)
\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 5sY(s) - 5y(0) + 4Y(s) = \frac{40}{s^2+4} \tag{1}$$

But the initial conditions are

$$\begin{aligned}
y(0) &= -1 \\
y'(0) &= 2
\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + 3 + s + 5sY(s) + 4Y(s) = \frac{40}{s^2+4}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = -\frac{s^3 + 3s^2 + 4s - 28}{(s^2+4)(s^2+5s+4)}$$

Applying partial fractions decomposition results in

$$Y(s) = -\frac{1}{s-2i} - \frac{1}{s+2i} - \frac{1}{s+4} + \frac{2}{s+1}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(-\frac{1}{s-2i}\right) = -e^{2it}$$
$$\mathcal{L}^{-1}\left(-\frac{1}{s+2i}\right) = -e^{-2it}$$
$$\mathcal{L}^{-1}\left(-\frac{1}{s+4}\right) = -e^{-4t}$$
$$\mathcal{L}^{-1}\left(\frac{2}{s+1}\right) = 2\,e^{-t}$$

Adding the above results and simplifying gives

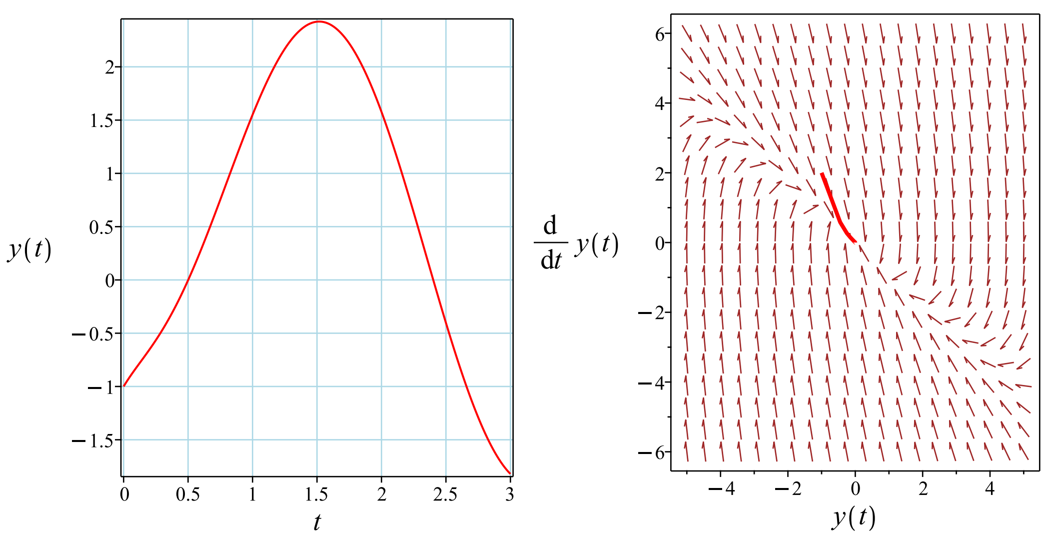
$$y = -e^{-4t} - 2\cos(2t) + 2\,e^{-t}$$

Simplifying the solution gives

$$y = -e^{-4t} - 2\cos(2t) + 2\,e^{-t}$$

Summary
The solution(s) found are the following

$$y = -e^{-4t} - 2\cos(2t) + 2\,e^{-t} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = -\mathrm{e}^{-4t} - 2\cos(2t) + 2\,\mathrm{e}^{-t}$$

Verified OK.

13.22.2 Maple step by step solution

Let's solve

$$\left[y'' + 5y' + 4y = 20\sin(2t)\,, y(0) = -1, y'\Big|_{\{t=0\}} = 2\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 5r + 4 = 0$

- Factor the characteristic polynomial

  $(r+4)(r+1) = 0$

- Roots of the characteristic polynomial

  $r = (-4, -1)$

- 1st solution of the homogeneous ODE

  $y_1(t) = \mathrm{e}^{-4t}$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \mathrm{e}^{-t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-4t} + c_2 \mathrm{e}^{-t} + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = 20\sin(2t)\right]$

  - Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \begin{bmatrix} e^{-4t} & e^{-t} \\ -4\,e^{-4t} & -e^{-t} \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(t), y_2(t)) = 3\,e^{-5t}$

- Substitute functions into equation for $y_p(t)$

  $y_p(t) = -\frac{20\,e^{-4t}\left(\int \sin(2t)e^{4t}dt\right)}{3} + \frac{20\,e^{-t}\left(\int \sin(2t)e^{t}dt\right)}{3}$

- Compute integrals

  $y_p(t) = -2\cos(2t)$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-4t} + c_2 e^{-t} - 2\cos(2t)$

☐ Check validity of solution $y = c_1 e^{-4t} + c_2 e^{-t} - 2\cos(2t)$

- Use initial condition $y(0) = -1$

  $-1 = c_1 + c_2 - 2$

- Compute derivative of the solution

  $y' = -4c_1 e^{-4t} - c_2 e^{-t} + 4\sin(2t)$

- Use the initial condition $y'\Big|_{\{t=0\}} = 2$

  $2 = -4c_1 - c_2$

- Solve for c_1 and c_2

  $\{c_1 = -1, c_2 = 2\}$

- Substitute constant values into general solution and simplify

  $y = -e^{-4t} - 2\cos(2t) + 2\,e^{-t}$

- Solution to the IVP

  $y = -e^{-4t} - 2\cos(2t) + 2\,e^{-t}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple
Time used: 2.875 (sec). Leaf size: 23

```
dsolve([diff(y(t),t$2)+5*diff(y(t),t)+4*y(t)=20*sin(2*t),y(0) = -1, D(y)(0) = 2],y(t), sings
```

$$y(t) = 2\,\mathrm{e}^{-t} - \mathrm{e}^{-4t} - 2\cos(2t)$$

✓ Solution by Mathematica
Time used: 0.021 (sec). Leaf size: 27

```
DSolve[{y''[t]+5*y'[t]+4*y[t]==20*Sin[2*t],{y[0]==-1,y'[0]==2}},y[t],t,IncludeSingularSoluti
```

$$y(t) \to e^{-4t}\left(2e^{3t} - 1\right) - 2\cos(2t)$$

13.23 problem Problem 23

Internal problem ID [2861]
Internal file name [`OUTPUT/2353_Sunday_June_05_2022_03_00_15_AM_14461231/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 10, The Laplace Transform and Some Elementary Applications. Exercises for 10.4. page 689
Problem number: Problem 23.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 5y' + 4y = 20 \sin (2t)}$$

With initial conditions

$$[y(0) = 1, y'(0) = -2]$$

13.23.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 5 \\ q(t) &= 4 \\ F &= 20 \sin (2t) \end{aligned}$$

Hence the ode is

$$y'' + 5y' + 4y = 20\sin(2t)$$

The domain of $p(t) = 5$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 20\sin(2t)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}
\mathcal{L}(y') &= sY(s) - y(0) \\
\mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)
\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 5sY(s) - 5y(0) + 4Y(s) = \frac{40}{s^2+4} \tag{1}$$

But the initial conditions are

$$\begin{aligned}
y(0) &= 1 \\
y'(0) &= -2
\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 3 - s + 5sY(s) + 4Y(s) = \frac{40}{s^2+4}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{s^3 + 3s^2 + 4s + 52}{(s^2+4)(s^2+5s+4)}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{10}{3(s+1)} - \frac{1}{3(s+4)} - \frac{1}{s-2i} - \frac{1}{s+2i}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(\frac{10}{3(s+1)}\right) = \frac{10\,\mathrm{e}^{-t}}{3}$$
$$\mathcal{L}^{-1}\left(-\frac{1}{3(s+4)}\right) = -\frac{\mathrm{e}^{-4t}}{3}$$
$$\mathcal{L}^{-1}\left(-\frac{1}{s-2i}\right) = -\mathrm{e}^{2it}$$
$$\mathcal{L}^{-1}\left(-\frac{1}{s+2i}\right) = -\mathrm{e}^{-2it}$$

Adding the above results and simplifying gives

$$y = -\frac{\mathrm{e}^{-4t}}{3} - 2\cos(2t) + \frac{10\,\mathrm{e}^{-t}}{3}$$

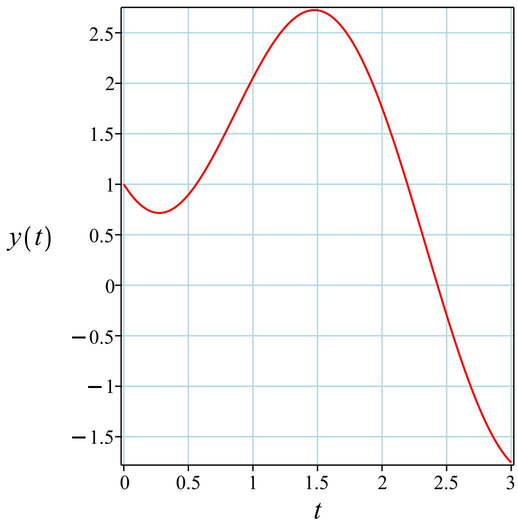
Simplifying the solution gives

$$y = -\frac{\mathrm{e}^{-4t}}{3} - 2\cos(2t) + \frac{10\,\mathrm{e}^{-t}}{3}$$

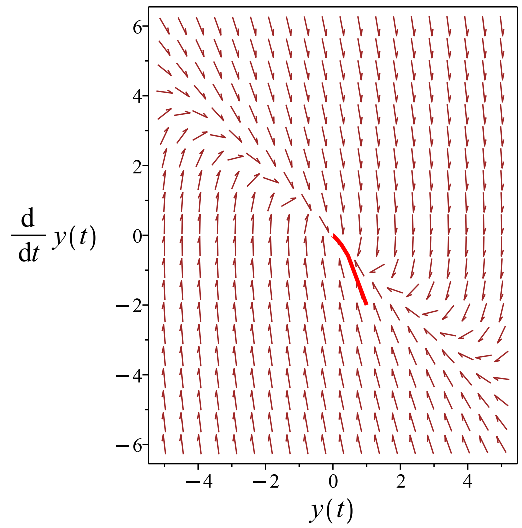
Summary

The solution(s) found are the following

$$y = -\frac{\mathrm{e}^{-4t}}{3} - 2\cos(2t) + \frac{10\,\mathrm{e}^{-t}}{3} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{e^{-4t}}{3} - 2\cos(2t) + \frac{10\,e^{-t}}{3}$$

Verified OK.

13.23.2 Maple step by step solution

Let's solve

$$\left[y'' + 5y' + 4y = 20\sin(2t), y(0) = 1, y'\Big|_{\{t=0\}} = -2\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 5r + 4 = 0$

- Factor the characteristic polynomial

  $(r+4)(r+1) = 0$

- Roots of the characteristic polynomial

  $r = (-4, -1)$

- 1st solution of the homogeneous ODE

$y_1(t) = e^{-4t}$

- 2nd solution of the homogeneous ODE

  $y_2(t) = e^{-t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-4t} + c_2 e^{-t} + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = 20\sin(2t)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(t), y_2(t)) = \begin{bmatrix} e^{-4t} & e^{-t} \\ -4\,e^{-4t} & -e^{-t} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(t), y_2(t)) = 3\,e^{-5t}$

  - Substitute functions into equation for $y_p(t)$

    $y_p(t) = -\frac{20\,e^{-4t}\left(\int \sin(2t)e^{4t}dt\right)}{3} + \frac{20\,e^{-t}\left(\int \sin(2t)e^{t}dt\right)}{3}$

  - Compute integrals

    $y_p(t) = -2\cos(2t)$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-4t} + c_2 e^{-t} - 2\cos(2t)$

☐ Check validity of solution $y = c_1 e^{-4t} + c_2 e^{-t} - 2\cos(2t)$

  - Use initial condition $y(0) = 1$

    $1 = c_1 + c_2 - 2$

  - Compute derivative of the solution

    $y' = -4c_1 e^{-4t} - c_2 e^{-t} + 4\sin(2t)$

  - Use the initial condition $y'\Big|_{\{t=0\}} = -2$

    $-2 = -4c_1 - c_2$

- Solve for c_1 and c_2

  $\left\{c_1 = -\frac{1}{3}, c_2 = \frac{10}{3}\right\}$
- Substitute constant values into general solution and simplify

  $y = -\frac{e^{-4t}}{3} - 2\cos(2t) + \frac{10\,e^{-t}}{3}$

- Solution to the IVP

  $y = -\frac{e^{-4t}}{3} - 2\cos(2t) + \frac{10\,e^{-t}}{3}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 1.562 (sec). Leaf size: 23

```
dsolve([diff(y(t),t$2)+5*diff(y(t),t)+4*y(t)=20*sin(2*t),y(0) = 1, D(y)(0) = -2],y(t), sings
```

$$y(t) = \frac{10\,e^{-t}}{3} - \frac{e^{-4t}}{3} - 2\cos(2t)$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 30

```
DSolve[{y''[t]+5*y'[t]+4*y[t]==20*Sin[2*t],{y[0]==1,y'[0]==-2}},y[t],t,IncludeSingularSoluti
```

$$y(t) \to \frac{1}{3}e^{-4t}\left(10e^{3t} - 1\right) - 2\cos(2t)$$

13.24 problem Problem 24

Internal problem ID [2862]
Internal file name [`OUTPUT/2354_Sunday_June_05_2022_03_00_17_AM_33117395/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 10, The Laplace Transform and Some Elementary Applications. Exercises for 10.4. page 689
Problem number: Problem 24.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' - 3y' + 2y = 3\cos(t) + \sin(t)}$$

With initial conditions

$$[y(0) = 1, y'(0) = 1]$$

13.24.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= -3 \\ q(t) &= 2 \\ F &= 3\cos(t) + \sin(t) \end{aligned}$$

Hence the ode is

$$y'' - 3y' + 2y = 3\cos(t) + \sin(t)$$

The domain of $p(t) = -3$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 3\cos(t) + \sin(t)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) - 3sY(s) + 3y(0) + 2Y(s) = \frac{3s+1}{s^2+1} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 1\\ y'(0) &= 1\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + 2 - s - 3sY(s) + 2Y(s) = \frac{3s+1}{s^2+1}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{s^3 - 2s^2 + 4s - 1}{(s^2+1)(s^2-3s+2)}$$

Applying partial fractions decomposition results in

$$
Y(s) = -\frac{1}{s-1} + \frac{\frac{3}{10}+\frac{2i}{5}}{s-i} + \frac{\frac{3}{10}-\frac{2i}{5}}{s+i} + \frac{7}{5(s-2)}
$$

The inverse Laplace of each term above is now found, which gives

$$
\begin{aligned}
\mathcal{L}^{-1}\left(-\frac{1}{s-1}\right) &= -\mathrm{e}^{t} \\
\mathcal{L}^{-1}\left(\frac{\frac{3}{10}+\frac{2i}{5}}{s-i}\right) &= \left(\frac{3}{10}+\frac{2i}{5}\right)\mathrm{e}^{it} \\
\mathcal{L}^{-1}\left(\frac{\frac{3}{10}-\frac{2i}{5}}{s+i}\right) &= \left(\frac{3}{10}-\frac{2i}{5}\right)\mathrm{e}^{-it} \\
\mathcal{L}^{-1}\left(\frac{7}{5(s-2)}\right) &= \frac{7\,\mathrm{e}^{2t}}{5}
\end{aligned}
$$

Adding the above results and simplifying gives

$$
y = -\mathrm{e}^{t} + \frac{3\cos(t)}{5} - \frac{4\sin(t)}{5} + \frac{7\,\mathrm{e}^{2t}}{5}
$$

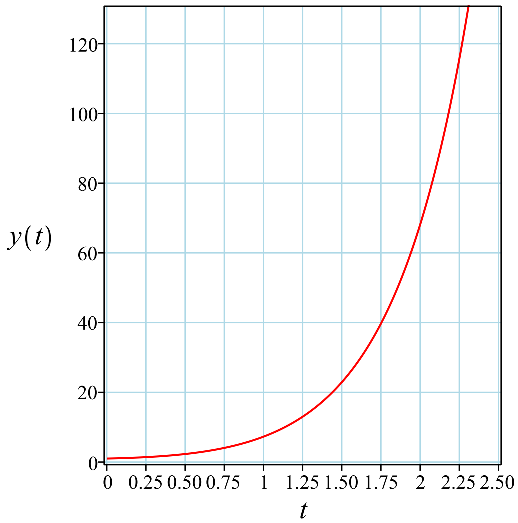
Simplifying the solution gives

$$
y = -\mathrm{e}^{t} + \frac{3\cos(t)}{5} - \frac{4\sin(t)}{5} + \frac{7\,\mathrm{e}^{2t}}{5}
$$

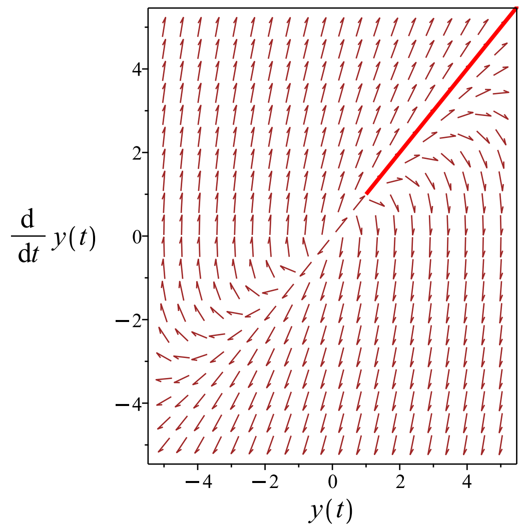
Summary

The solution(s) found are the following

$$
y = -\mathrm{e}^{t} + \frac{3\cos(t)}{5} - \frac{4\sin(t)}{5} + \frac{7\,\mathrm{e}^{2t}}{5} \tag{1}
$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = -e^{t} + \frac{3\cos(t)}{5} - \frac{4\sin(t)}{5} + \frac{7\,e^{2t}}{5}$$

Verified OK.

13.24.2 Maple step by step solution

Let's solve

$$\left[y'' - 3y' + 2y = 3\cos(t) + \sin(t), y(0) = 1, y'\Big|_{\{t=0\}} = 1\right]$$

- Highest derivative means the order of the ODE is 2
  y''
- Characteristic polynomial of homogeneous ODE
  $r^2 - 3r + 2 = 0$
- Factor the characteristic polynomial
  $(r - 1)(r - 2) = 0$
- Roots of the characteristic polynomial
  $r = (1, 2)$
- 1st solution of the homogeneous ODE

$y_1(t) = e^t$

- 2nd solution of the homogeneous ODE

  $y_2(t) = e^{2t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^t + c_2 e^{2t} + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t) \left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t) \left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = 3\cos(t) + \sin(t)\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(t), y_2(t)) = \begin{bmatrix} e^t & e^{2t} \\ e^t & 2e^{2t} \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(t), y_2(t)) = e^{3t}$

- ○ Substitute functions into equation for $y_p(t)$

  $y_p(t) = -e^t \left(\int (3\cos(t) + \sin(t)) e^{-t} dt\right) + e^{2t} \left(\int (3\cos(t) + \sin(t)) e^{-2t} dt\right)$

- ○ Compute integrals

  $y_p(t) = \frac{3\cos(t)}{5} - \frac{4\sin(t)}{5}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^t + c_2 e^{2t} + \frac{3\cos(t)}{5} - \frac{4\sin(t)}{5}$

☐ Check validity of solution $y = c_1 e^t + c_2 e^{2t} + \frac{3\cos(t)}{5} - \frac{4\sin(t)}{5}$

- ○ Use initial condition $y(0) = 1$

  $1 = c_1 + c_2 + \frac{3}{5}$

- ○ Compute derivative of the solution

  $y' = c_1 e^t + 2c_2 e^{2t} - \frac{3\sin(t)}{5} - \frac{4\cos(t)}{5}$

- ○ Use the initial condition $y'\Big|_{\{t=0\}} = 1$

  $1 = c_1 + 2c_2 - \frac{4}{5}$

- Solve for c_1 and c_2

  $\{c_1 = -1, c_2 = \frac{7}{5}\}$
- Substitute constant values into general solution and simplify

  $y = -e^t + \frac{3\cos(t)}{5} - \frac{4\sin(t)}{5} + \frac{7\,e^{2t}}{5}$

- Solution to the IVP

  $y = -e^t + \frac{3\cos(t)}{5} - \frac{4\sin(t)}{5} + \frac{7\,e^{2t}}{5}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 2.0 (sec). Leaf size: 23

```
dsolve([diff(y(t),t$2)-3*diff(y(t),t)+2*y(t)=3*cos(t)+sin(t),y(0) = 1, D(y)(0) = 1],y(t), si
```

$$y(t) = \frac{7\,e^{2t}}{5} - e^t - \frac{4\sin(t)}{5} + \frac{3\cos(t)}{5}$$

✓ Solution by Mathematica

Time used: 0.112 (sec). Leaf size: 29

```
DSolve[{y''[t]-3*y'[t]+2*y[t]==3*Cos[t]+Sin[t],{y[0]==1,y'[0]==1}},y[t],t,IncludeSingularSol
```

$$y(t) \to \frac{1}{5}\left(e^t\left(7e^t - 5\right) - 4\sin(t) + 3\cos(t)\right)$$

13.25 problem Problem 25

Internal problem ID [2863]
Internal file name [`OUTPUT/2355_Sunday_June_05_2022_03_00_19_AM_47489344/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 10, The Laplace Transform and Some Elementary Applications. Exercises for 10.4. page 689
Problem number: Problem 25.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 4y = 9\sin(t)}$$

With initial conditions

$$[y(0) = 1, y'(0) = -1]$$

13.25.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= 4 \\ F &= 9\sin(t) \end{aligned}$$

Hence the ode is

$$y'' + 4y = 9\sin(t)$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 9\sin(t)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 4Y(s) = \frac{9}{s^2 + 1} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 1\\ y'(0) &= -1\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + 1 - s + 4Y(s) = \frac{9}{s^2 + 1}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{s^3 - s^2 + s + 8}{(s^2 + 1)(s^2 + 4)}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{\frac{1}{2}+i}{s-2i} + \frac{\frac{1}{2}-i}{s+2i} - \frac{3i}{2\,(s-i)} + \frac{3i}{2\,(s+i)}$$

The inverse Laplace of each term above is now found, which gives

$$\begin{aligned}
\mathcal{L}^{-1}\left(\frac{\frac{1}{2}+i}{s-2i}\right) &= \left(\frac{1}{2}+i\right)\mathrm{e}^{2it} \\
\mathcal{L}^{-1}\left(\frac{\frac{1}{2}-i}{s+2i}\right) &= \left(\frac{1}{2}-i\right)\mathrm{e}^{-2it} \\
\mathcal{L}^{-1}\left(-\frac{3i}{2\,(s-i)}\right) &= -\frac{3i\mathrm{e}^{it}}{2} \\
\mathcal{L}^{-1}\left(\frac{3i}{2\,(s+i)}\right) &= \frac{3i\mathrm{e}^{-it}}{2}
\end{aligned}$$

Adding the above results and simplifying gives

$$y = \cos(2t) - 2\sin(2t) + 3\sin(t)$$

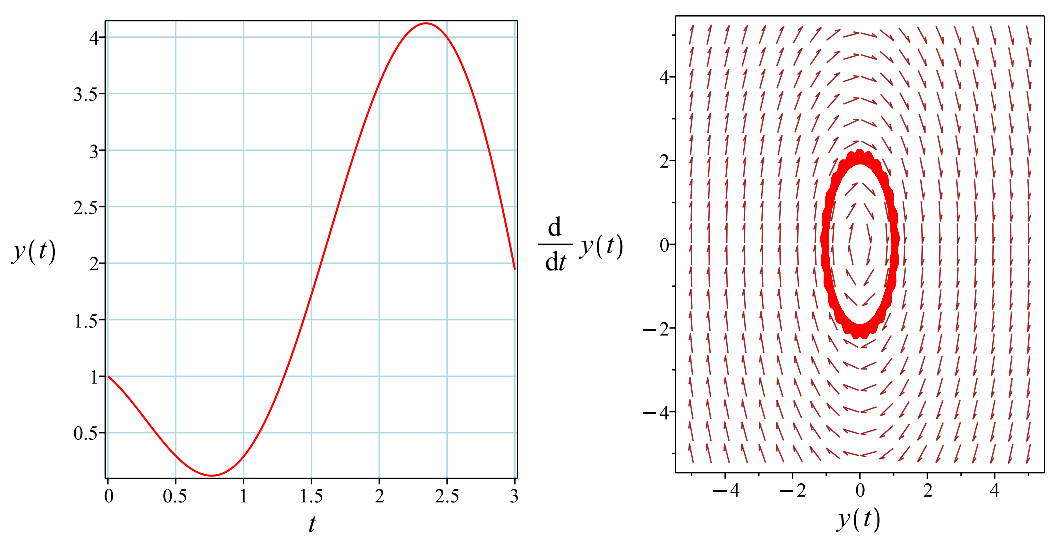
Simplifying the solution gives

$$y = \cos(2t) - 2\sin(2t) + 3\sin(t)$$

Summary

The solution(s) found are the following

$$y = \cos(2t) - 2\sin(2t) + 3\sin(t) \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \cos(2t) - 2\sin(2t) + 3\sin(t)$$

Verified OK.

13.25.2 Maple step by step solution

Let's solve

$\left[y'' + 4y = 9\sin(t), y(0) = 1, y'\Big|_{\{t=0\}} = -1\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (-2\,\mathrm{I}, 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t) = \cos(2t)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \sin(2t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(2t) + c_2 \sin(2t) + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

  ∘ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = 9\sin(t)\right]$

  ∘ Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \begin{bmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(t), y_2(t)) = 2$

- Substitute functions into equation for $y_p(t)$

  $y_p(t) = -9\cos(2t)\left(\int \cos(t)\sin(t)^2\,dt\right) + \frac{9\sin(2t)\left(\int (\sin(3t) - \sin(t))dt\right)}{4}$

- Compute integrals

  $y_p(t) = 3\sin(t)$

- Substitute particular solution into general solution to ODE

  $y = c_1\cos(2t) + c_2\sin(2t) + 3\sin(t)$

☐ Check validity of solution $y = c_1\cos(2t) + c_2\sin(2t) + 3\sin(t)$

- Use initial condition $y(0) = 1$

  $1 = c_1$

- Compute derivative of the solution

  $y' = -2c_1\sin(2t) + 2c_2\cos(2t) + 3\cos(t)$

- Use the initial condition $y'\Big|_{\{t=0\}} = -1$

  $-1 = 3 + 2c_2$

- Solve for c_1 and c_2

  $\{c_1 = 1, c_2 = -2\}$

- Substitute constant values into general solution and simplify

  $y = \cos(2t) - 2\sin(2t) + 3\sin(t)$

- Solution to the IVP

  $y = \cos(2t) - 2\sin(2t) + 3\sin(t)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 2.047 (sec). Leaf size: 19

```
dsolve([diff(y(t),t$2)+4*y(t)=9*sin(t),y(0) = 1, D(y)(0) = -1],y(t), singsol=all)
```

$$y(t) = \cos(2t) - 2\sin(2t) + 3\sin(t)$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 20

```
DSolve[{y''[t]+4*y[t]==9*Sin[t],{y[0]==1,y'[0]==-1}},y[t],t,IncludeSingularSolutions -> True
```

$$y(t) \to 3\sin(t) - 2\sin(2t) + \cos(2t)$$

13.26 problem Problem 26

Internal problem ID [2864]
Internal file name [`OUTPUT/2356_Sunday_June_05_2022_03_00_21_AM_92629855/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 10, The Laplace Transform and Some Elementary Applications. Exercises for 10.4. page 689
Problem number: Problem 26.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y = 6\cos(2t)}$$

With initial conditions

$$[y(0) = 0, y'(0) = 2]$$

13.26.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= 1 \\ F &= 6\cos(2t) \end{aligned}$$

Hence the ode is

$$y'' + y = 6\cos(2t)$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 6\cos(2t)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + Y(s) = \frac{6s}{s^2+4} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 0\\ y'(0) &= 2\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 2 + Y(s) = \frac{6s}{s^2+4}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{2s^2 + 6s + 8}{(s^2+4)(s^2+1)}$$

Applying partial fractions decomposition results in

$$Y(s) = -\frac{1}{s-2i} - \frac{1}{s+2i} + \frac{1-i}{s-i} + \frac{1+i}{s+i}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(-\frac{1}{s-2i}\right) = -e^{2it}$$
$$\mathcal{L}^{-1}\left(-\frac{1}{s+2i}\right) = -e^{-2it}$$
$$\mathcal{L}^{-1}\left(\frac{1-i}{s-i}\right) = (1-i)\,e^{it}$$
$$\mathcal{L}^{-1}\left(\frac{1+i}{s+i}\right) = (1+i)\,e^{-it}$$

Adding the above results and simplifying gives

$$y = -2\cos(2t) + 2\cos(t) + 2\sin(t)$$

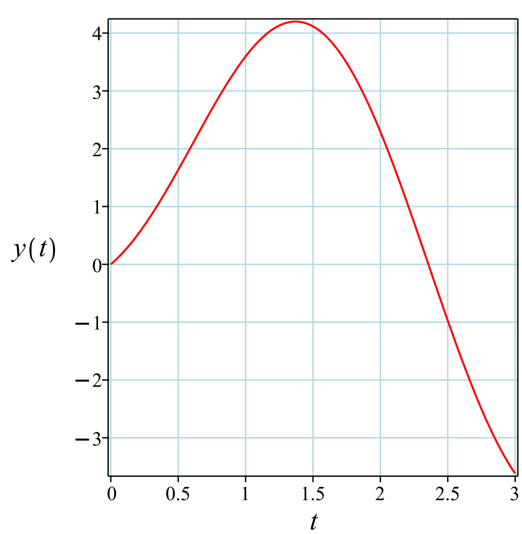
Simplifying the solution gives

$$y = -2\cos(2t) + 2\cos(t) + 2\sin(t)$$

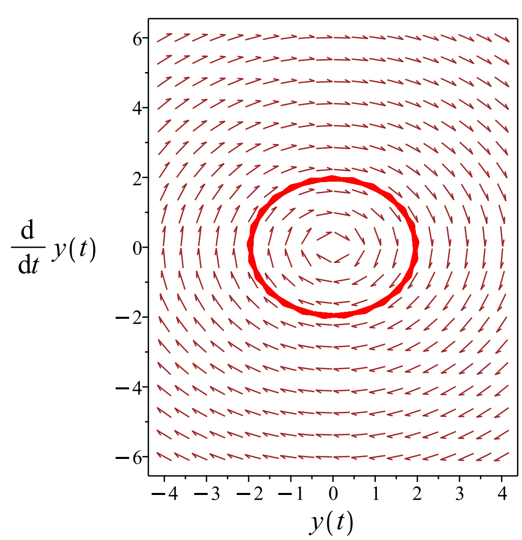
Summary

The solution(s) found are the following

$$y = -2\cos(2t) + 2\cos(t) + 2\sin(t) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -2\cos(2t) + 2\cos(t) + 2\sin(t)$$

Verified OK.

13.26.2 Maple step by step solution

Let's solve

$\left[y'' + y = 6\cos(2t), y(0) = 0, y'\Big|_{\{t=0\}} = 2\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t) = \cos(t)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \sin(t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1\cos(t) + c_2\sin(t) + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = 6\cos(2t)\right]$

  - Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(t), y_2(t)) = 1$

- Substitute functions into equation for $y_p(t)$

  $y_p(t) = -3\cos(t)\left(\int (\sin(3t) - \sin(t))\, dt\right) + 3\sin(t)\left(\int (\cos(t) + \cos(3t))\, dt\right)$

- Compute integrals

  $y_p(t) = -2\cos(2t)$

- Substitute particular solution into general solution to ODE

  $y = c_1\cos(t) + c_2\sin(t) - 2\cos(2t)$

☐ Check validity of solution $y = c_1\cos(t) + c_2\sin(t) - 2\cos(2t)$

- Use initial condition $y(0) = 0$

  $0 = c_1 - 2$

- Compute derivative of the solution

  $y' = -c_1\sin(t) + c_2\cos(t) + 4\sin(2t)$

- Use the initial condition $y'\Big|_{\{t=0\}} = 2$

  $2 = c_2$

- Solve for c_1 and c_2

  $\{c_1 = 2, c_2 = 2\}$

- Substitute constant values into general solution and simplify

  $y = -2\cos(2t) + 2\cos(t) + 2\sin(t)$

- Solution to the IVP

  $y = -2\cos(2t) + 2\cos(t) + 2\sin(t)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 1.906 (sec). Leaf size: 19

```
dsolve([diff(y(t),t$2)+y(t)=6*cos(2*t),y(0) = 0, D(y)(0) = 2],y(t), singsol=all)
```

$$y(t) = -2\cos(2t) + 2\cos(t) + 2\sin(t)$$

✓ Solution by Mathematica

Time used: 0.017 (sec). Leaf size: 18

```
DSolve[{y''[t]+y[t]==6*Cos[2*t],{y[0]==0,y'[0]==2}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to 2(\sin(t) + \cos(t) - \cos(2t))$$

13.27 problem Problem 27

Internal problem ID [2865]
Internal file name [`OUTPUT/2357_Sunday_June_05_2022_03_00_23_AM_86244952/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 10, The Laplace Transform and Some Elementary Applications. Exercises for 10.4. page 689
Problem number: Problem 27.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 9y = 7\sin(4t) + 14\cos(4t)}$$

With initial conditions

$$[y(0) = 1, y'(0) = 2]$$

13.27.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= 9 \\ F &= 7\sin(4t) + 14\cos(4t) \end{aligned}$$

Hence the ode is

$$y'' + 9y = 7\sin(4t) + 14\cos(4t)$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 9$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 7\sin(4t)+14\cos(4t)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 9Y(s) = \frac{14s + 28}{s^2 + 16} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 1\\ y'(0) &= 2\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 2 - s + 9Y(s) = \frac{14s + 28}{s^2 + 16}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{s^3 + 2s^2 + 30s + 60}{(s^2 + 16)(s^2 + 9)}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{-1+\frac{i}{2}}{s-4i} + \frac{-1-\frac{i}{2}}{s+4i} + \frac{\frac{3}{2}-i}{s-3i} + \frac{\frac{3}{2}+i}{s+3i}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(\frac{-1+\frac{i}{2}}{s-4i}\right) = \left(-1+\frac{i}{2}\right) e^{4it}$$

$$\mathcal{L}^{-1}\left(\frac{-1-\frac{i}{2}}{s+4i}\right) = \left(-1-\frac{i}{2}\right) e^{-4it}$$

$$\mathcal{L}^{-1}\left(\frac{\frac{3}{2}-i}{s-3i}\right) = \left(\frac{3}{2}-i\right) e^{3it}$$

$$\mathcal{L}^{-1}\left(\frac{\frac{3}{2}+i}{s+3i}\right) = \left(\frac{3}{2}+i\right) e^{-3it}$$

Adding the above results and simplifying gives

$$y = 3\cos(3t) + 2\sin(3t) - 2\cos(4t) - \sin(4t)$$

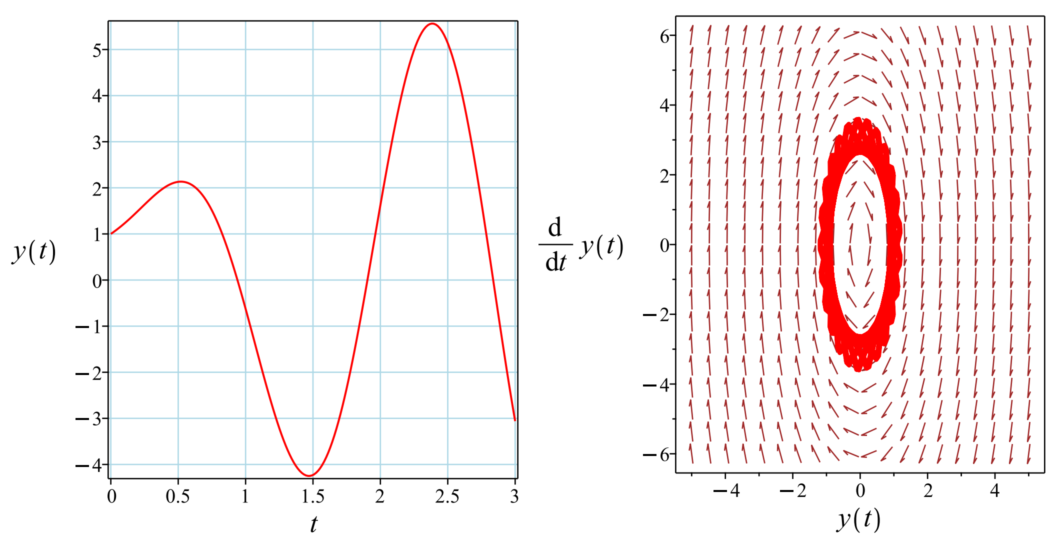
Simplifying the solution gives

$$y = 3\cos(3t) + 2\sin(3t) - 2\cos(4t) - \sin(4t)$$

<u>Summary</u>

The solution(s) found are the following

$$y = 3\cos(3t) + 2\sin(3t) - 2\cos(4t) - \sin(4t) \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = 3\cos(3t) + 2\sin(3t) - 2\cos(4t) - \sin(4t)$$

Verified OK.

13.27.2 Maple step by step solution

Let's solve

$$\left[y'' + 9y = 7\sin(4t) + 14\cos(4t), y(0) = 1, y'\Big|_{\{t=0\}} = 2\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 9 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-36})}{2}$

- Roots of the characteristic polynomial

  $r = (-3\,\mathrm{I}, 3\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t) = \cos(3t)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \sin(3t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(3t) + c_2 \sin(3t) + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = 7\sin(4t) + 14\cos(4t)\right]$

  ◦ Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \begin{bmatrix} \cos(3t) & \sin(3t) \\ -3\sin(3t) & 3\cos(3t) \end{bmatrix}$$

- ◦ Compute Wronskian

  $W(y_1(t), y_2(t)) = 3$

- ◦ Substitute functions into equation for $y_p(t)$

  $y_p(t) = -\frac{7\cos(3t)\left(\int \sin(3t)(\sin(4t)+2\cos(4t))dt\right)}{3} + \frac{7\sin(3t)\left(\int \cos(3t)(\sin(4t)+2\cos(4t))dt\right)}{3}$

- ◦ Compute integrals

  $y_p(t) = -\sin(4t) - 2\cos(4t)$

- • Substitute particular solution into general solution to ODE

  $y = c_1 \cos(3t) + c_2 \sin(3t) - \sin(4t) - 2\cos(4t)$

- □ Check validity of solution $y = c_1 \cos(3t) + c_2 \sin(3t) - \sin(4t) - 2\cos(4t)$

- ◦ Use initial condition $y(0) = 1$

  $1 = c_1 - 2$

- ◦ Compute derivative of the solution

  $y' = -3c_1 \sin(3t) + 3c_2 \cos(3t) - 4\cos(4t) + 8\sin(4t)$

- ◦ Use the initial condition $y'\Big|_{\{t=0\}} = 2$

  $2 = -4 + 3c_2$

- ◦ Solve for c_1 and c_2

  $\{c_1 = 3, c_2 = 2\}$

- ◦ Substitute constant values into general solution and simplify

  $y = 3\cos(3t) + 2\sin(3t) - 2\cos(4t) - \sin(4t)$

- • Solution to the IVP

  $y = 3\cos(3t) + 2\sin(3t) - 2\cos(4t) - \sin(4t)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 3.047 (sec). Leaf size: 29

```
dsolve([diff(y(t),t$2)+9*y(t)=7*sin(4*t)+14*cos(4*t),y(0) = 1, D(y)(0) = 2],y(t), singsol=al
```

$$y(t) = -2\cos(4t) - \sin(4t) + 3\cos(3t) + 2\sin(3t)$$

✓ Solution by Mathematica

Time used: 0.028 (sec). Leaf size: 49

```
DSolve[{y''[t]+8*y[t]==7*Sin[4*t]+14*Cos[4*t],{y[0]==1,y'[0]==2}},y[t],t,IncludeSingularSolu
```

$$y(t) \to \frac{1}{8}\left(-7\sin(4t) + 11\sqrt{2}\sin\left(2\sqrt{2}t\right) - 14\cos(4t) + 22\cos\left(2\sqrt{2}t\right)\right)$$

13.28 problem Problem 28

Internal problem ID [2866]
Internal file name [`OUTPUT/2358_Sunday_June_05_2022_03_00_26_AM_76402982/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 10, The Laplace Transform and Some Elementary Applications. Exercises for 10.4. page 689
Problem number: Problem 28.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - y = 0}$$

With initial conditions

$$[y(0) = A, y'(0) = B]$$

13.28.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$p(t) = 0$$
$$q(t) = -1$$
$$F = 0$$

Hence the ode is

$$y'' - y = 0$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = -1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0) \\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) - Y(s) = 0 \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= A \\ y'(0) &= B\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - B - sA - Y(s) = 0$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{sA + B}{s^2 - 1}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{\frac{A}{2} - \frac{B}{2}}{s + 1} + \frac{\frac{A}{2} + \frac{B}{2}}{s - 1}$$

The inverse Laplace of each term above is now found, which gives

$$
\begin{aligned}
\mathcal{L}^{-1}\left(\frac{\frac{A}{2}-\frac{B}{2}}{s+1}\right) &= \frac{(A-B)\,\mathrm{e}^{-t}}{2} \\
\mathcal{L}^{-1}\left(\frac{\frac{A}{2}+\frac{B}{2}}{s-1}\right) &= \frac{(A+B)\,\mathrm{e}^{t}}{2}
\end{aligned}
$$

Adding the above results and simplifying gives

$$
y = A\cosh(t) + B\sinh(t)
$$

Simplifying the solution gives

$$
y = A\cosh(t) + B\sinh(t)
$$

Summary

The solution(s) found are the following

$$
y = A\cosh(t) + B\sinh(t) \tag{1}
$$

Verification of solutions

$$
y = A\cosh(t) + B\sinh(t)
$$

Verified OK.

13.28.2 Maple step by step solution

Let's solve

$\left[y'' - y = 0, y(0) = A, y'\Big|_{\{t=0\}} = B\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 1 = 0$

- Factor the characteristic polynomial

  $(r-1)(r+1) = 0$

- Roots of the characteristic polynomial

  $r = (-1, 1)$

- 1st solution of the ODE

  $y_1(t) = e^{-t}$
- 2nd solution of the ODE

  $y_2(t) = e^{t}$
- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t)$
- Substitute in solutions

  $y = c_1 e^{-t} + c_2 e^{t}$

□ Check validity of solution $y = c_1 e^{-t} + c_2 e^{t}$

  - Use initial condition $y(0) = A$

    $A = c_1 + c_2$
  - Compute derivative of the solution

    $y' = -c_1 e^{-t} + c_2 e^{t}$
  - Use the initial condition $y'\Big|_{\{t=0\}} = B$

    $B = -c_1 + c_2$
  - Solve for c_1 and c_2

    $\left\{c_1 = \frac{A}{2} - \frac{B}{2}, c_2 = \frac{A}{2} + \frac{B}{2}\right\}$
  - Substitute constant values into general solution and simplify

    $y = \frac{(A-B)e^{-t}}{2} + \frac{(A+B)e^{t}}{2}$

- Solution to the IVP

  $y = \frac{(A-B)e^{-t}}{2} + \frac{(A+B)e^{t}}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 1.703 (sec). Leaf size: 13

```
dsolve([diff(y(t),t$2)-y(t)=0,y(0) = A, D(y)(0) = B],y(t), singsol=all)
```

$$y(t) = A\cosh(t) + B\sinh(t)$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 33

```
DSolve[{y''[t]-y[t]==0,{y[0]==a,y'[0]==b}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{2}e^{-t}\left(a\left(e^{2t}+1\right)+b\left(e^{2t}-1\right)\right)$$

14 Chapter 10, The Laplace Transform and Some Elementary Applications. Exercises for 10.7. page 704

14.1 problem Problem 27

Internal problem ID [2867]
Internal file name [`OUTPUT/2359_Sunday_June_05_2022_03_00_28_AM_89142793/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 10, The Laplace Transform and Some Elementary Applications. Exercises for 10.7. page 704
Problem number: Problem 27.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first__order__ode__lie__symmetry__lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' + 2y = 2\,\mathrm{Heaviside}\,(t-1)}$$

With initial conditions

$$[y(0) = 1]$$

14.1.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$\begin{aligned} p(t) &= 2 \\ q(t) &= 2\,\mathrm{Heaviside}\,(t-1) \end{aligned}$$

Hence the ode is

$$y' + 2y = 2\,\text{Heaviside}\,(t-1)$$

The domain of $p(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 2\,\text{Heaviside}\,(t-1)$ is

$$\{t < 1 \vee 1 < t\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

14.1.2 Solving as laplace ode

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(y') = sY(s) - y(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$sY(s) - y(0) + 2Y(s) = \frac{2\,\mathrm{e}^{-s}}{s} \tag{1}$$

Replacing initial condition gives

$$sY(s) - 1 + 2Y(s) = \frac{2\,\mathrm{e}^{-s}}{s}$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{2\,\mathrm{e}^{-s} + s}{s\,(s+2)}$$

Taking the inverse Laplace transform gives

$$\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(\frac{2\,\mathrm{e}^{-s} + s}{s\,(s+2)}\right) \\
&= \text{Heaviside}\,(t-1)\left(1 - \mathrm{e}^{-2t+2}\right) + \mathrm{e}^{-2t}
\end{aligned}$$

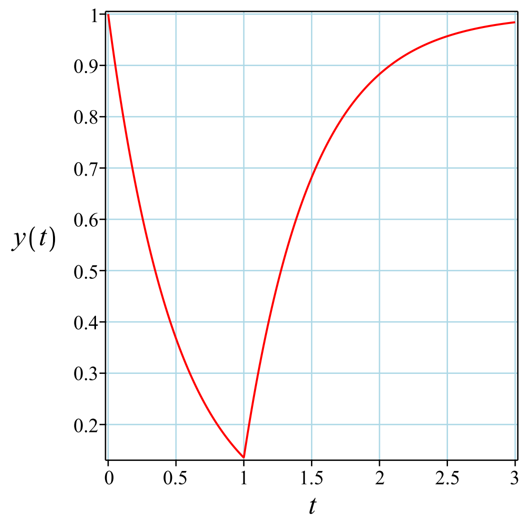
Hence the final solution is

$$y = \text{Heaviside}\,(t-1)\left(1-\mathrm{e}^{-2t+2}\right)+\mathrm{e}^{-2t}$$

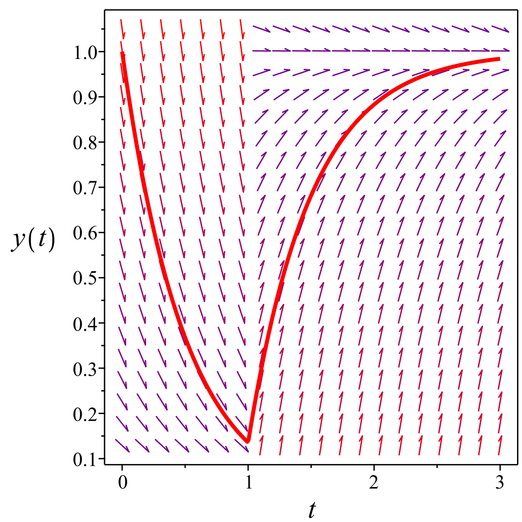
Summary

The solution(s) found are the following

$$y = \text{Heaviside}\,(t-1)\left(1-\mathrm{e}^{-2t+2}\right)+\mathrm{e}^{-2t} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \text{Heaviside}\,(t-1)\left(1-\mathrm{e}^{-2t+2}\right)+\mathrm{e}^{-2t}$$

Verified OK.

14.1.3 Maple step by step solution

Let's solve

$[y' + 2y = 2\mathit{Heaviside}(t-1)\,, y(0) = 1]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -2y + 2\mathit{Heaviside}(t-1)$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + 2y = 2\mathit{Heaviside}(t-1)$
- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\,(y' + 2y) = 2\mu(t)\,\mathit{Heaviside}(t-1)$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,y)$

  $\mu(t)\,(y' + 2y) = \mu'(t)\,y + \mu(t)\,y'$
- Isolate $\mu'(t)$

  $\mu'(t) = 2\mu(t)$
- Solve to find the integrating factor

  $\mu(t) = \mathrm{e}^{2t}$
- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\,y)\right) dt = \int 2\mu(t)\,\mathit{Heaviside}(t-1)\,dt + c_1$
- Evaluate the integral on the lhs

  $\mu(t)\,y = \int 2\mu(t)\,\mathit{Heaviside}(t-1)\,dt + c_1$
- Solve for y

  $y = \frac{\int 2\mu(t)\mathit{Heaviside}(t-1)dt + c_1}{\mu(t)}$
- Substitute $\mu(t) = \mathrm{e}^{2t}$

  $y = \frac{\int 2\,\mathrm{e}^{2t}\mathit{Heaviside}(t-1)dt + c_1}{\mathrm{e}^{2t}}$
- Evaluate the integrals on the rhs

  $y = \frac{\mathrm{e}^{2t}\mathit{Heaviside}(t-1) - \mathit{Heaviside}(t-1)\mathrm{e}^{2} + c_1}{\mathrm{e}^{2t}}$
- Simplify

  $y = \mathit{Heaviside}(t-1) - \mathit{Heaviside}(t-1)\,\mathrm{e}^{-2t+2} + c_1\mathrm{e}^{-2t}$
- Use initial condition $y(0) = 1$

  $1 = c_1$
- Solve for c_1

  $c_1 = 1$
- Substitute $c_1 = 1$ into general solution and simplify

  $y = \mathit{Heaviside}(t-1) - \mathit{Heaviside}(t-1)\,\mathrm{e}^{-2t+2} + \mathrm{e}^{-2t}$
- Solution to the IVP

$$y = \mathit{Heaviside}(t-1) - \mathit{Heaviside}(t-1)\,\mathrm{e}^{-2t+2} + \mathrm{e}^{-2t}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 2.328 (sec). Leaf size: 24

```
dsolve([diff(y(t),t)+2*y(t)=2*Heaviside(t-1),y(0) = 1],y(t), singsol=all)
```

$$y(t) = -\operatorname{Heaviside}(t-1)\,\mathrm{e}^{-2t+2} + \operatorname{Heaviside}(t-1) + \mathrm{e}^{-2t}$$

✓ Solution by Mathematica

Time used: 0.044 (sec). Leaf size: 26

```
DSolve[{y'[t]-y[t]==2*UnitStep[t-1],{y[0]==1}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \begin{cases} e^t & t \leq 1 \\ -2 + 2e^{t-1} + e^t & \text{True} \end{cases}$$

14.2 problem Problem 28

Internal problem ID [2868]
Internal file name [`OUTPUT/2360_Sunday_June_05_2022_03_00_33_AM_15085754/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 10, The Laplace Transform and Some Elementary Applications. Exercises for 10.7. page 704
Problem number: Problem 28.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' - 2y = \text{Heaviside}\,(t-2)\,\mathrm{e}^{t-2}}$$

With initial conditions

$$[y(0) = 2]$$

14.2.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$\begin{aligned} p(t) &= -2 \\ q(t) &= \text{Heaviside}\,(t-2)\,\mathrm{e}^{t-2} \end{aligned}$$

Hence the ode is

$$y' - 2y = \text{Heaviside}\,(t-2)\,\mathrm{e}^{t-2}$$

The domain of $p(t) = -2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = \text{Heaviside}\,(t-2)\,\mathrm{e}^{t-2}$ is

$$\{t < 2 \lor 2 < t\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

14.2.2 Solving as laplace ode

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(y') = sY(s) - y(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$sY(s) - y(0) - 2Y(s) = \frac{\mathrm{e}^{-2s}}{s-1} \tag{1}$$

Replacing initial condition gives

$$sY(s) - 2 - 2Y(s) = \frac{\mathrm{e}^{-2s}}{s-1}$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{\mathrm{e}^{-2s} + 2s - 2}{(s-1)(s-2)}$$

Taking the inverse Laplace transform gives

$$\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(\frac{\mathrm{e}^{-2s} + 2s - 2}{(s-1)(s-2)}\right) \\
&= \left(\mathrm{e}^{t-2} - \mathrm{e}^{2t-4}\right) \text{Heaviside}\,(-t+2) + 2\,\mathrm{e}^{2t} - \mathrm{e}^{t-2} + \mathrm{e}^{2t-4}
\end{aligned}$$

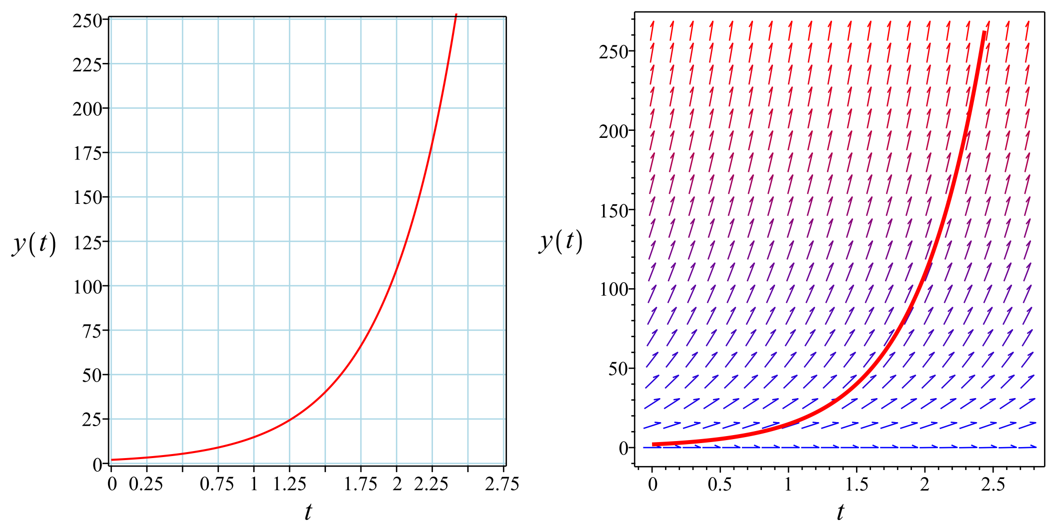
Hence the final solution is

$$y = \left(e^{t-2} - e^{2t-4}\right) \text{Heaviside}\left(-t+2\right) + 2\,e^{2t} - e^{t-2} + e^{2t-4}$$

Summary

The solution(s) found are the following

$$y = \left(e^{t-2} - e^{2t-4}\right) \text{Heaviside}\left(-t+2\right) + 2\,e^{2t} - e^{t-2} + e^{2t-4} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \left(e^{t-2} - e^{2t-4}\right) \text{Heaviside}\left(-t+2\right) + 2\,e^{2t} - e^{t-2} + e^{2t-4}$$

Verified OK.

14.2.3 Maple step by step solution

Let's solve

$\left[y' - 2y = \mathit{Heaviside}(t-2)\,e^{t-2}, y(0) = 2\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = 2y + \mathit{Heaviside}(t-2)\,e^{t-2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - 2y = \mathit{Heaviside}(t-2)\,\mathrm{e}^{t-2}$
- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\,(y' - 2y) = \mu(t)\,\mathit{Heaviside}(t-2)\,\mathrm{e}^{t-2}$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,y)$

  $\mu(t)\,(y' - 2y) = \mu'(t)\,y + \mu(t)\,y'$
- Isolate $\mu'(t)$

  $\mu'(t) = -2\mu(t)$
- Solve to find the integrating factor

  $\mu(t) = \mathrm{e}^{-2t}$
- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\,y)\right) dt = \int \mu(t)\,\mathit{Heaviside}(t-2)\,\mathrm{e}^{t-2} dt + c_1$
- Evaluate the integral on the lhs

  $\mu(t)\,y = \int \mu(t)\,\mathit{Heaviside}(t-2)\,\mathrm{e}^{t-2} dt + c_1$
- Solve for y

  $y = \frac{\int \mu(t)\mathit{Heaviside}(t-2)\mathrm{e}^{t-2}dt+c_1}{\mu(t)}$
- Substitute $\mu(t) = \mathrm{e}^{-2t}$

  $y = \frac{\int \mathrm{e}^{-2t}\mathit{Heaviside}(t-2)\mathrm{e}^{t-2}dt+c_1}{\mathrm{e}^{-2t}}$
- Evaluate the integrals on the rhs

  $y = \frac{-\mathrm{e}^{-t-2}\mathit{Heaviside}(t-2)+\mathit{Heaviside}(t-2)\mathrm{e}^{-4}+c_1}{\mathrm{e}^{-2t}}$
- Simplify

  $y = \mathrm{e}^{2t}\left(-\mathrm{e}^{-t-2}\mathit{Heaviside}(t-2) + \mathit{Heaviside}(t-2)\,\mathrm{e}^{-4} + c_1\right)$
- Use initial condition $y(0) = 2$

  $2 = c_1$
- Solve for c_1

  $c_1 = 2$
- Substitute $c_1 = 2$ into general solution and simplify

  $y = \mathrm{e}^{2t}\left(-\mathrm{e}^{-t-2}\mathit{Heaviside}(t-2) + \mathit{Heaviside}(t-2)\,\mathrm{e}^{-4} + 2\right)$
- Solution to the IVP

$$y = \mathrm{e}^{2t}(-\mathrm{e}^{-t-2} Heaviside(t-2) + Heaviside(t-2)\,\mathrm{e}^{-4} + 2)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 2.297 (sec). Leaf size: 43

```
dsolve([diff(y(t),t)-2*y(t)=Heaviside(t-2)*exp(t-2),y(0) = 2],y(t), singsol=all)
```

$$y(t) = -\operatorname{Heaviside}(t-2)\,\mathrm{e}^{t-2} + \operatorname{Heaviside}(t-2)\,\mathrm{e}^{-4+2t} + 2\,\mathrm{e}^{2t}$$

✓ Solution by Mathematica

Time used: 0.088 (sec). Leaf size: 40

```
DSolve[{y'[t]-2*y[t]==UnitStep[t-2]*Exp[t-2],{y[0]==2}},y[t],t,IncludeSingularSolutions -> T
```

$$y(t) \to \begin{cases} 2e^{2t} & t \leq 2 \\ e^{t-4}(-e^2 + e^t + 2e^{t+4}) & \text{True} \end{cases}$$

14.3 problem Problem 29

Internal problem ID [2869]
Internal file name [`OUTPUT/2361_Sunday_June_05_2022_03_00_39_AM_56517254/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 10, The Laplace Transform and Some Elementary Applications. Exercises for 10.7. page 704
Problem number: Problem 29.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' - y = 4\,\text{Heaviside}\left(t - \frac{\pi}{4}\right)\sin\left(t + \frac{\pi}{4}\right)}$$

With initial conditions

$$[y(0) = 1]$$

14.3.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$\begin{aligned} p(t) &= -1 \\ q(t) &= 4\,\text{Heaviside}\left(t - \frac{\pi}{4}\right)\sin\left(t + \frac{\pi}{4}\right) \end{aligned}$$

Hence the ode is

$$y' - y = 4\,\mathrm{Heaviside}\left(t - \frac{\pi}{4}\right)\sin\left(t + \frac{\pi}{4}\right)$$

The domain of $p(t) = -1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 4\,\mathrm{Heaviside}\left(t - \frac{\pi}{4}\right)\sin\left(t + \frac{\pi}{4}\right)$ is

$$\left\{t < \frac{\pi}{4} \vee \frac{\pi}{4} < t\right\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

14.3.2 Solving as laplace ode

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(y') = sY(s) - y(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$sY(s) - y(0) - Y(s) = \frac{4\,\mathrm{e}^{-\frac{s\pi}{4}} s}{s^2 + 1} \tag{1}$$

Replacing initial condition gives

$$sY(s) - 1 - Y(s) = \frac{4\,\mathrm{e}^{-\frac{s\pi}{4}} s}{s^2 + 1}$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{4\,\mathrm{e}^{-\frac{s\pi}{4}} s + s^2 + 1}{(s^2 + 1)(s - 1)}$$

Taking the inverse Laplace transform gives

$$\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(\frac{4\,\mathrm{e}^{-\frac{s\pi}{4}} s + s^2 + 1}{(s^2 + 1)(s - 1)}\right) \\
&= -2\,\mathrm{Heaviside}\left(t - \frac{\pi}{4}\right)\cos(t)\sqrt{2} + \mathrm{e}^t + 2\left(1 - \mathrm{Heaviside}\left(-t + \frac{\pi}{4}\right)\right)\mathrm{e}^{t - \frac{\pi}{4}}
\end{aligned}$$

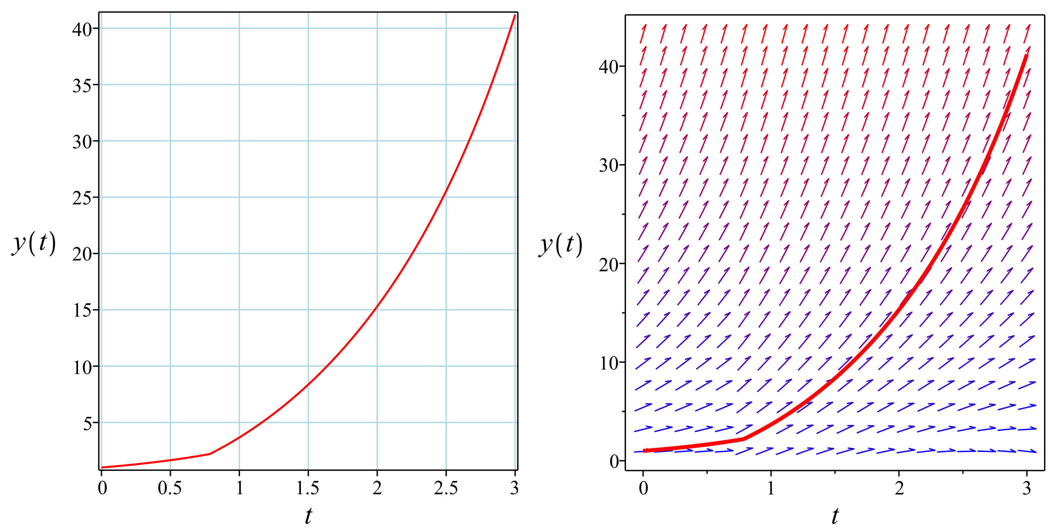
Hence the final solution is

$$y = -2\,\mathrm{Heaviside}\left(t-\frac{\pi}{4}\right)\cos\left(t\right)\sqrt{2}+\mathrm{e}^{t}+2\left(1-\mathrm{Heaviside}\left(-t+\frac{\pi}{4}\right)\right)\mathrm{e}^{t-\frac{\pi}{4}}$$

Summary

The solution(s) found are the following

$$y = -2\,\mathrm{Heaviside}\left(t-\frac{\pi}{4}\right)\cos\left(t\right)\sqrt{2}+\mathrm{e}^{t}+2\left(1-\mathrm{Heaviside}\left(-t+\frac{\pi}{4}\right)\right)\mathrm{e}^{t-\frac{\pi}{4}} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -2\,\mathrm{Heaviside}\left(t-\frac{\pi}{4}\right)\cos\left(t\right)\sqrt{2}+\mathrm{e}^{t}+2\left(1-\mathrm{Heaviside}\left(-t+\frac{\pi}{4}\right)\right)\mathrm{e}^{t-\frac{\pi}{4}}$$

Verified OK.

14.3.3 Maple step by step solution

Let's solve

$\left[y'-y=4\mathit{Heaviside}\left(t-\frac{\pi}{4}\right)\sin\left(t+\frac{\pi}{4}\right), y(0)=1\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

$$y' = y + 4\mathit{Heaviside}\left(t - \frac{\pi}{4}\right) \sin\left(t + \frac{\pi}{4}\right)$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

$$y' - y = 4\mathit{Heaviside}\left(t - \frac{\pi}{4}\right) \sin\left(t + \frac{\pi}{4}\right)$$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

$$\mu(t)\left(y' - y\right) = 4\mu(t)\,\mathit{Heaviside}\left(t - \frac{\pi}{4}\right) \sin\left(t + \frac{\pi}{4}\right)$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,y)$

$$\mu(t)\left(y' - y\right) = \mu'(t)\,y + \mu(t)\,y'$$

- Isolate $\mu'(t)$

$$\mu'(t) = -\mu(t)$$

- Solve to find the integrating factor

$$\mu(t) = \mathrm{e}^{-t}$$

- Integrate both sides with respect to t

$$\int \left(\frac{d}{dt}(\mu(t)\,y)\right) dt = \int 4\mu(t)\,\mathit{Heaviside}\left(t - \frac{\pi}{4}\right) \sin\left(t + \frac{\pi}{4}\right) dt + c_1$$

- Evaluate the integral on the lhs

$$\mu(t)\,y = \int 4\mu(t)\,\mathit{Heaviside}\left(t - \frac{\pi}{4}\right) \sin\left(t + \frac{\pi}{4}\right) dt + c_1$$

- Solve for y

$$y = \frac{\int 4\mu(t)\mathit{Heaviside}\left(t - \frac{\pi}{4}\right) \sin\left(t + \frac{\pi}{4}\right) dt + c_1}{\mu(t)}$$

- Substitute $\mu(t) = \mathrm{e}^{-t}$

$$y = \frac{\int 4\,\mathrm{e}^{-t}\mathit{Heaviside}\left(t - \frac{\pi}{4}\right) \sin\left(t + \frac{\pi}{4}\right) dt + c_1}{\mathrm{e}^{-t}}$$

- Evaluate the integrals on the rhs

$$y = \frac{4\left(-\frac{\mathrm{e}^{-t}\cos\left(t + \frac{\pi}{4}\right)}{2} - \frac{\mathrm{e}^{-t}\sin\left(t + \frac{\pi}{4}\right)}{2}\right)\mathit{Heaviside}\left(t - \frac{\pi}{4}\right) + 2\mathit{Heaviside}\left(t - \frac{\pi}{4}\right)\mathrm{e}^{-\frac{\pi}{4}} + c_1}{\mathrm{e}^{-t}}$$

- Simplify

$$y = \left(-2\cos\left(t + \frac{\pi}{4}\right) + 2\,\mathrm{e}^{t - \frac{\pi}{4}} - 2\sin\left(t + \frac{\pi}{4}\right)\right)\mathit{Heaviside}\left(t - \frac{\pi}{4}\right) + c_1\mathrm{e}^{t}$$

- Use initial condition $y(0) = 1$

$$1 = c_1$$

- Solve for c_1

$$c_1 = 1$$

- Substitute $c_1 = 1$ into general solution and simplify

$$y = \left(-2\cos\left(t+\frac{\pi}{4}\right) + 2\,\mathrm{e}^{t-\frac{\pi}{4}} - 2\sin\left(t+\frac{\pi}{4}\right)\right) \mathit{Heaviside}\left(t-\frac{\pi}{4}\right) + \mathrm{e}^{t}$$

- Solution to the IVP

$$y = \left(-2\cos\left(t+\frac{\pi}{4}\right) + 2\,\mathrm{e}^{t-\frac{\pi}{4}} - 2\sin\left(t+\frac{\pi}{4}\right)\right) \mathit{Heaviside}\left(t-\frac{\pi}{4}\right) + \mathrm{e}^{t}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 2.329 (sec). Leaf size: 40

```
dsolve([diff(y(t),t)-y(t)=4*Heaviside(t-Pi/4)*cos(t-Pi/4),y(0) = 1],y(t), singsol=all)
```

$$y(t) = \left(-2\cos(t)\sqrt{2} + 2\,\mathrm{e}^{t-\frac{\pi}{4}}\right) \mathrm{Heaviside}\left(t-\frac{\pi}{4}\right) + \mathrm{e}^{t}$$

✓ Solution by Mathematica

Time used: 0.112 (sec). Leaf size: 40

```
DSolve[{y'[t]-y[t]==4*UnitStep[t-Pi/4]*Cos[t-Pi/4],{y[0]==1}},y[t],t,IncludeSingularSolution
```

$$y(t) \to \begin{cases} e^{t} & 4t \leq \pi \\ -2\sqrt{2}\cos(t) + e^{t} + 2e^{t-\frac{\pi}{4}} & \text{True} \end{cases}$$

14.4 problem Problem 30

Internal problem ID [2870]
Internal file name [`OUTPUT/2362_Sunday_June_05_2022_03_00_46_AM_64087347/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 10, The Laplace Transform and Some Elementary Applications. Exercises for 10.7. page 704
Problem number: Problem 30.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' + 2y = \text{Heaviside}\left(t-\pi\right)\sin\left(2t\right)}$$

With initial conditions

$$[y(0) = 3]$$

14.4.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$\begin{aligned}p(t) &= 2\\ q(t) &= \text{Heaviside}\left(t-\pi\right)\sin\left(2t\right)\end{aligned}$$

Hence the ode is

$$y' + 2y = \text{Heaviside}\,(t - \pi) \sin(2t)$$

The domain of $p(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = \text{Heaviside}\,(t - \pi) \sin(2t)$ is

$$\{t < \pi \vee \pi < t\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

14.4.2 Solving as laplace ode

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(y') = sY(s) - y(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$sY(s) - y(0) + 2Y(s) = \frac{2\,\mathrm{e}^{-s\pi}}{s^2 + 4} \tag{1}$$

Replacing initial condition gives

$$sY(s) - 3 + 2Y(s) = \frac{2\,\mathrm{e}^{-s\pi}}{s^2 + 4}$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{3s^2 + 2\,\mathrm{e}^{-s\pi} + 12}{(s^2 + 4)(s + 2)}$$

Taking the inverse Laplace transform gives

$$\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(\frac{3s^2 + 2\,\mathrm{e}^{-s\pi} + 12}{(s^2 + 4)(s + 2)}\right) \\
&= 3\,\mathrm{e}^{-2t} + \frac{\text{Heaviside}\,(t - \pi)\left(\mathrm{e}^{-t+\pi} \cosh(t - \pi) - \cos(t)^2 + \sin(t)\cos(t)\right)}{2}
\end{aligned}$$

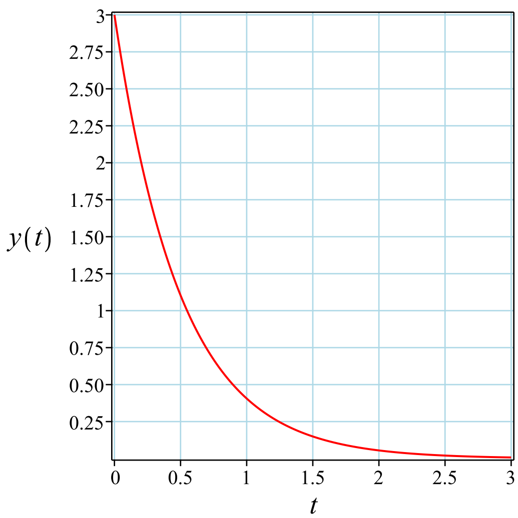
Hence the final solution is

$$y = 3\,\mathrm{e}^{-2t} + \frac{\mathrm{Heaviside}\,(t-\pi)\left(\mathrm{e}^{-t+\pi}\cosh\,(t-\pi) - \cos\,(t)^{2} + \sin\,(t)\cos\,(t)\right)}{2}$$

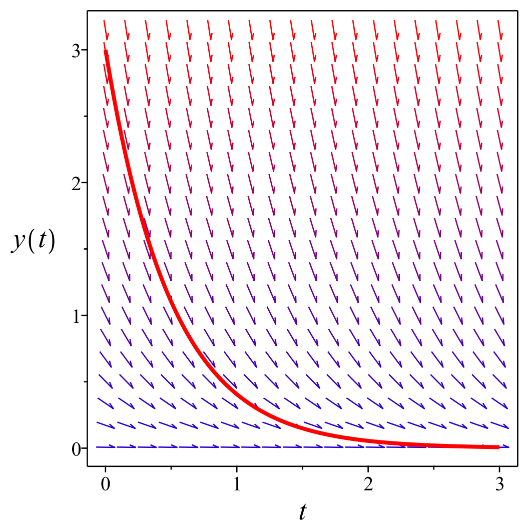
Summary

The solution(s) found are the following

$$y = 3\,\mathrm{e}^{-2t} + \frac{\mathrm{Heaviside}\,(t-\pi)\left(\mathrm{e}^{-t+\pi}\cosh\,(t-\pi) - \cos\,(t)^{2} + \sin\,(t)\cos\,(t)\right)}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 3\,\mathrm{e}^{-2t} + \frac{\mathrm{Heaviside}\,(t-\pi)\left(\mathrm{e}^{-t+\pi}\cosh\,(t-\pi) - \cos\,(t)^{2} + \sin\,(t)\cos\,(t)\right)}{2}$$

Verified OK.

14.4.3 Maple step by step solution

Let's solve

$[y' + 2y = \mathit{Heaviside}(t-\pi)\sin\,(2t)\,, y(0) = 3]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -2y + \mathit{Heaviside}(t-\pi)\sin(2t)$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + 2y = \mathit{Heaviside}(t-\pi)\sin(2t)$
- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)(y' + 2y) = \mu(t)\,\mathit{Heaviside}(t-\pi)\sin(2t)$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,y)$

  $\mu(t)(y' + 2y) = \mu'(t)\,y + \mu(t)\,y'$
- Isolate $\mu'(t)$

  $\mu'(t) = 2\mu(t)$
- Solve to find the integrating factor

  $\mu(t) = e^{2t}$
- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\,y)\right) dt = \int \mu(t)\,\mathit{Heaviside}(t-\pi)\sin(2t)\,dt + c_1$
- Evaluate the integral on the lhs

  $\mu(t)\,y = \int \mu(t)\,\mathit{Heaviside}(t-\pi)\sin(2t)\,dt + c_1$
- Solve for y

  $y = \frac{\int \mu(t)\mathit{Heaviside}(t-\pi)\sin(2t)dt + c_1}{\mu(t)}$
- Substitute $\mu(t) = e^{2t}$

  $y = \frac{\int e^{2t}\mathit{Heaviside}(t-\pi)\sin(2t)dt + c_1}{e^{2t}}$
- Evaluate the integrals on the rhs

  $y = \frac{\left(-\frac{e^{2t}\cos(2t)}{4} + \frac{e^{2t}\sin(2t)}{4}\right)\mathit{Heaviside}(t-\pi) + \frac{\mathit{Heaviside}(t-\pi)e^{2\pi}}{4} + c_1}{e^{2t}}$
- Simplify

  $y = \frac{e^{-2t+2\pi}\mathit{Heaviside}(t-\pi)}{4} + \frac{\mathit{Heaviside}(t-\pi)(-\cos(2t)+\sin(2t))}{4} + c_1 e^{-2t}$
- Use initial condition $y(0) = 3$

  $3 = c_1$
- Solve for c_1

  $c_1 = 3$
- Substitute $c_1 = 3$ into general solution and simplify

$$y = \frac{e^{-2t+2\pi} Heaviside(t-\pi)}{4} + \frac{Heaviside(t-\pi)(-\cos(2t)+\sin(2t))}{4} + 3\,e^{-2t}$$

- Solution to the IVP

$$y = \frac{e^{-2t+2\pi} Heaviside(t-\pi)}{4} + \frac{Heaviside(t-\pi)(-\cos(2t)+\sin(2t))}{4} + 3\,e^{-2t}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 2.312 (sec). Leaf size: 43

```
dsolve([diff(y(t),t)+2*y(t)=Heaviside(t-Pi)*sin(2*t),y(0) = 3],y(t), singsol=all)
```

$$y(t) = \frac{\text{Heaviside}\,(t-\pi)\,e^{-2t+2\pi}}{4} + \frac{\text{Heaviside}\,(t-\pi)\,(-\cos\,(2t)+\sin\,(2t))}{4} + 3\,e^{-2t}$$

✓ Solution by Mathematica

Time used: 0.117 (sec). Leaf size: 55

```
DSolve[{y'[t]+2*y[t]==UnitStep[t-Pi]*Sin[2*t],{y[0]==3}},y[t],t,IncludeSingularSolutions ->
```

$$y(t) \to \begin{cases} 3e^{-2t} & t \leq \pi \\ \frac{1}{4}e^{-2t}(-e^{2t}\cos(2t) + e^{2t}\sin(2t) + e^{2\pi} + 12) & \text{True} \end{cases}$$

14.5 problem Problem 31

Internal problem ID [2871]
Internal file name [`OUTPUT/2363_Sunday_June_05_2022_03_00_53_AM_61815175/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 10, The Laplace Transform and Some Elementary Applications. Exercises for 10.7. page 704
Problem number: Problem 31.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' + 3y = \begin{cases} 1 & 0 \leq t < 1 \\ 0 & 1 \leq t \end{cases}}$$

With initial conditions

$$[y(0) = 1]$$

14.5.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$
\begin{aligned}
p(t) &= 3 \\
q(t) &= \begin{cases} 0 & t < 0 \\ 1 & t < 1 \\ 0 & 1 \leq t \end{cases}
\end{aligned}
$$

Hence the ode is

$$
y' + 3y = \begin{cases} 0 & t < 0 \\ 1 & t < 1 \\ 0 & 1 \leq t \end{cases}
$$

The domain of $p(t) = 3$ is

$$
\{-\infty < t < \infty\}
$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = \begin{cases} 0 & t < 0 \\ 1 & t < 1 \\ 0 & 1 \leq t \end{cases}$ is

$$
\{0 \leq t \leq 1, 1 \leq t \leq \infty, -\infty \leq t \leq 0\}
$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

14.5.2 Solving as laplace ode

Solving using the Laplace transform method. Let

$$
\mathcal{L}(y) = Y(s)
$$

Taking the Laplace transform of the ode and using the relations that

$$
\mathcal{L}(y') = sY(s) - y(0)
$$

The given ode now becomes an algebraic equation in the Laplace domain

$$
sY(s) - y(0) + 3Y(s) = \frac{1 - \mathrm{e}^{-s}}{s} \tag{1}
$$

Replacing initial condition gives

$$sY(s) - 1 + 3Y(s) = \frac{1 - e^{-s}}{s}$$

Solving for $Y(s)$ gives

$$Y(s) = -\frac{-1 + e^{-s} - s}{s(s+3)}$$

Taking the inverse Laplace transform gives

$$\begin{aligned} y &= \mathcal{L}^{-1}(Y(s)) \\ &= \mathcal{L}^{-1}\left(-\frac{-1 + e^{-s} - s}{s(s+3)}\right) \\ &= \frac{\text{Heaviside}(1-t)}{3} + \frac{\text{Heaviside}(t-1)\, e^{-3t+3}}{3} + \frac{2\, e^{-3t}}{3} \end{aligned}$$

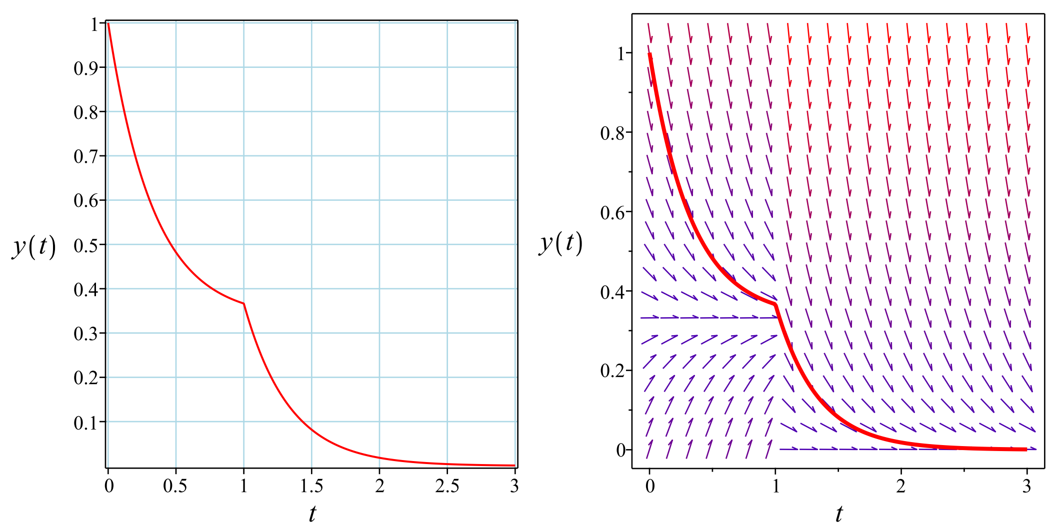
Hence the final solution is

$$y = \frac{\text{Heaviside}(1-t)}{3} + \frac{\text{Heaviside}(t-1)\, e^{-3t+3}}{3} + \frac{2\, e^{-3t}}{3}$$

Summary

The solution(s) found are the following

$$y = \frac{\text{Heaviside}(1-t)}{3} + \frac{\text{Heaviside}(t-1)\, e^{-3t+3}}{3} + \frac{2\, e^{-3t}}{3} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{\text{Heaviside}\left(1-t\right)}{3}+\frac{\text{Heaviside}\left(t-1\right) \mathrm{e}^{-3t+3}}{3}+\frac{2\,\mathrm{e}^{-3t}}{3}$$

Verified OK.

14.5.3 Maple step by step solution

Let's solve

$$\left[y'+3y=\left\{\begin{array}{cc} 1 & 0\le t<1 \\ 0 & 1\le t \end{array}\right., y(0)=1\right]$$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $$y'=-3y+\left\{\begin{array}{cc} 1 & 0\le t<1 \\ 0 & 1\le t \end{array}\right.$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $$y'+3y=\left\{\begin{array}{cc} 1 & 0\le t<1 \\ 0 & 1\le t \end{array}\right.$$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $$\mu(t)\left(y'+3y\right)=\mu(t)\left(\left\{\begin{array}{cc} 1 & 0\le t<1 \\ 0 & 1\le t \end{array}\right.\right)$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,y)$

  $\mu(t)\left(y'+3y\right)=\mu'(t)\,y+\mu(t)\,y'$

- Isolate $\mu'(t)$

  $\mu'(t)=3\mu(t)$

- Solve to find the integrating factor

  $\mu(t)=\mathrm{e}^{3t}$

- Integrate both sides with respect to t

$$\int \left(\frac{d}{dt}(\mu(t)\, y)\right) dt = \int \mu(t) \left(\left\{\begin{array}{cc} 1 & 0 \le t < 1 \\ 0 & 1 \le t \end{array}\right.\right) dt + c_1$$

- Evaluate the integral on the lhs

$$\mu(t)\, y = \int \mu(t) \left(\left\{\begin{array}{cc} 1 & 0 \le t < 1 \\ 0 & 1 \le t \end{array}\right.\right) dt + c_1$$

- Solve for y

$$y = \frac{\int \mu(t) \left(\left\{\begin{array}{cc} 1 & 0 \le t < 1 \\ 0 & 1 \le t \end{array}\right.\right) dt + c_1}{\mu(t)}$$

- Substitute $\mu(t) = \mathrm{e}^{3t}$

$$y = \frac{\int \mathrm{e}^{3t} \left(\left\{\begin{array}{cc} 1 & 0 \le t < 1 \\ 0 & 1 \le t \end{array}\right.\right) dt + c_1}{\mathrm{e}^{3t}}$$

- Evaluate the integrals on the rhs

$$y = \frac{\left\{\begin{array}{cc} 0 & t \le 0 \\ \frac{\mathrm{e}^{3t}}{3} - \frac{1}{3} & t \le 1 \\ \frac{\mathrm{e}^{3}}{3} - \frac{1}{3} & 1 < t \end{array}\right. + c_1}{\mathrm{e}^{3t}}$$

- Simplify

$$y = -\frac{\mathrm{e}^{-3t} \left(\left\{\begin{array}{cc} 0 & t \le 0 \\ 1 - \mathrm{e}^{3t} & t \le 1 \\ 1 - \mathrm{e}^{3} & 1 < t \end{array}\right. - 3c_1\right)}{3}$$

- Use initial condition $y(0) = 1$

$$1 = c_1$$

- Solve for c_1

$$c_1 = 1$$

- Substitute $c_1 = 1$ into general solution and simplify

$$y = -\frac{\mathrm{e}^{-3t}\left(\left\{\begin{array}{cc} 0 & t \le 0 \\ 1-\mathrm{e}^{3t} & t \le 1 \\ 1-\mathrm{e}^{3} & 1 < t \end{array}\right. -3\right)}{3}$$

- Solution to the IVP

$$y = -\frac{\mathrm{e}^{-3t}\left(\left\{\begin{array}{cc} 0 & t \le 0 \\ 1-\mathrm{e}^{3t} & t \le 1 \\ 1-\mathrm{e}^{3} & 1 < t \end{array}\right. -3\right)}{3}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 3.688 (sec). Leaf size: 43

```
dsolve([diff(y(t),t)+3*y(t)=piecewise(0<=t and t<1,1,t>=1,0),y(0) = 1],y(t), singsol=all)
```

$$y(t) = \frac{\left(\left\{\begin{array}{cc} 1+2\,\mathrm{e}^{-3t} & t < 1 \\ 2\,\mathrm{e}^{-3}+2 & t = 1 \\ 2\,\mathrm{e}^{-3t}+\mathrm{e}^{-3t+3} & 1 < t \end{array}\right.\right)}{3}$$

✓ Solution by Mathematica

Time used: 0.069 (sec). Leaf size: 47

```
DSolve[{y'[t]+3*y[t]==Piecewise[{{1,0<=t<1},{0,t >= 1}}],{y[0]==1}},y[t],t,IncludeSingularSo
```

$$y(t) \to \begin{cases} e^{-3t} & t \leq 0 \\ \frac{1}{3}e^{-3t}(2+e^3) & t > 1 \\ \frac{1}{3}+\frac{2e^{-3t}}{3} & \text{True} \end{cases}$$

14.6 problem Problem 32

Internal problem ID [2872]
Internal file name [`OUTPUT/2364_Sunday_June_05_2022_03_00_59_AM_2867957/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 10, The Laplace Transform and Some Elementary Applications. Exercises for 10.7. page 704
Problem number: Problem 32.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' - 3y = \begin{cases} \sin(t) & 0 \leq t < \frac{\pi}{2} \\ 1 & \frac{\pi}{2} \leq t \end{cases}}$$

With initial conditions

$$[y(0) = 2]$$

14.6.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = -3$$

$$q(t) = \begin{cases} 0 & t < 0 \\ \sin(t) & t < \frac{\pi}{2} \\ 1 & \frac{\pi}{2} \leq t \end{cases}$$

Hence the ode is

$$y' - 3y = \begin{cases} 0 & t < 0 \\ \sin(t) & t < \frac{\pi}{2} \\ 1 & \frac{\pi}{2} \leq t \end{cases}$$

The domain of $p(t) = -3$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = \begin{cases} 0 & t < 0 \\ \sin(t) & t < \frac{\pi}{2} \\ 1 & \frac{\pi}{2} \leq t \end{cases}$ is

$$\left\{0 \leq t \leq \frac{\pi}{2}, \frac{\pi}{2} \leq t \leq \infty, -\infty \leq t \leq 0\right\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

14.6.2 Solving as laplace ode

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(y') = sY(s) - y(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$sY(s) - y(0) - 3Y(s) = \frac{e^{-\frac{s\pi}{2}} + s}{(s^2 + 1)\, s} \tag{1}$$

Replacing initial condition gives

$$sY(s) - 2 - 3Y(s) = \frac{\mathrm{e}^{-\frac{s\pi}{2}} + s}{(s^2+1)\, s}$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{2s^3 + \mathrm{e}^{-\frac{s\pi}{2}} + 3s}{(s^2+1)\, s\, (s-3)}$$

Taking the inverse Laplace transform gives

$$\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(\frac{2s^3 + \mathrm{e}^{-\frac{s\pi}{2}} + 3s}{(s^2+1)\, s\, (s-3)}\right) \\
&= -\frac{\mathrm{Heaviside}\left(t - \frac{\pi}{2}\right)}{3} + \frac{21\,\mathrm{e}^{3t}}{10} + \frac{\mathrm{e}^{-\frac{3\pi}{2}+3t}}{30} - \frac{\mathrm{Heaviside}\left(\frac{\pi}{2} - t\right)\left(3\cos(t) + \mathrm{e}^{-\frac{3\pi}{2}+3t} + 9\sin(t)\right)}{30}
\end{aligned}$$

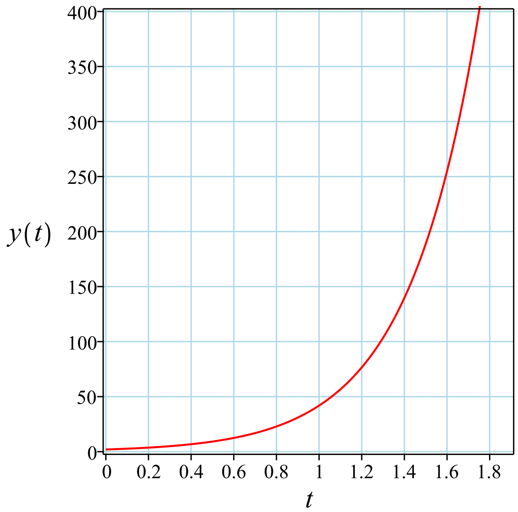
Hence the final solution is

$$\begin{aligned}
y = &-\frac{\mathrm{Heaviside}\left(t - \frac{\pi}{2}\right)}{3} + \frac{21\,\mathrm{e}^{3t}}{10} + \frac{\mathrm{e}^{-\frac{3\pi}{2}+3t}}{30} \\
&- \frac{\mathrm{Heaviside}\left(\frac{\pi}{2} - t\right)\left(3\cos(t) + \mathrm{e}^{-\frac{3\pi}{2}+3t} + 9\sin(t)\right)}{30}
\end{aligned}$$

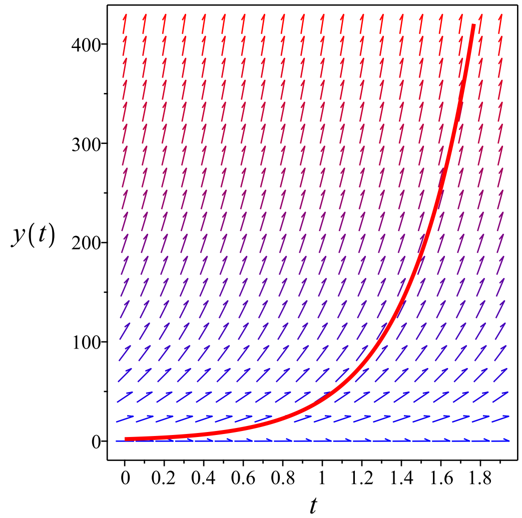
Summary

The solution(s) found are the following

$$\begin{aligned}
y = &-\frac{\mathrm{Heaviside}\left(t - \frac{\pi}{2}\right)}{3} + \frac{21\,\mathrm{e}^{3t}}{10} + \frac{\mathrm{e}^{-\frac{3\pi}{2}+3t}}{30} \\
&- \frac{\mathrm{Heaviside}\left(\frac{\pi}{2} - t\right)\left(3\cos(t) + \mathrm{e}^{-\frac{3\pi}{2}+3t} + 9\sin(t)\right)}{30}
\end{aligned} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$
y = -\frac{\text{Heaviside}\left(t-\frac{\pi}{2}\right)}{3} + \frac{21\,\mathrm{e}^{3t}}{10} + \frac{\mathrm{e}^{-\frac{3\pi}{2}+3t}}{30} - \frac{\text{Heaviside}\left(\frac{\pi}{2}-t\right)\left(3\cos(t) + \mathrm{e}^{-\frac{3\pi}{2}+3t} + 9\sin(t)\right)}{30}
$$

Verified OK.

14.6.3 Maple step by step solution

Let's solve

$$
\left[y' - 3y = \begin{cases} \sin(t) & 0 \le t < \frac{\pi}{2} \\ 1 & \frac{\pi}{2} \le t \end{cases}, y(0) = 2\right]
$$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $$
  y' = 3y + \begin{cases} \sin(t) & 0 \le t < \frac{\pi}{2} \\ 1 & \frac{\pi}{2} \le t \end{cases}
  $$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

$$y' - 3y = \left\{\begin{array}{cc} \sin \left(t \right) & 0\le t <\frac{\pi}{2} \\ 1 & \frac{\pi}{2}\le t \end{array}\right.$$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

$$\mu(t)\left(y' - 3y\right) = \mu(t)\left(\left\{\begin{array}{cc} \sin \left(t \right) & 0\le t <\frac{\pi}{2} \\ 1 & \frac{\pi}{2}\le t \end{array}\right.\right)$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\, y)$

$$\mu(t)\left(y' - 3y\right) = \mu'(t)\, y + \mu(t)\, y'$$

- Isolate $\mu'(t)$

$$\mu'(t) = -3\mu(t)$$

- Solve to find the integrating factor

$$\mu(t) = \mathrm{e}^{-3t}$$

- Integrate both sides with respect to t

$$\int \left(\frac{d}{dt}(\mu(t)\, y)\right) dt = \int \mu(t)\left(\left\{\begin{array}{cc} \sin \left(t \right) & 0\le t <\frac{\pi}{2} \\ 1 & \frac{\pi}{2}\le t \end{array}\right.\right) dt + c_1$$

- Evaluate the integral on the lhs

$$\mu(t)\, y = \int \mu(t)\left(\left\{\begin{array}{cc} \sin \left(t \right) & 0\le t <\frac{\pi}{2} \\ 1 & \frac{\pi}{2}\le t \end{array}\right.\right) dt + c_1$$

- Solve for y

$$y = \frac{\int \mu(t)\left(\left\{\begin{array}{cc} \sin \left(t \right) & 0\le t <\frac{\pi}{2} \\ 1 & \frac{\pi}{2}\le t \end{array}\right.\right) dt + c_1}{\mu(t)}$$

- Substitute $\mu(t) = \mathrm{e}^{-3t}$

$$y = \frac{\int \mathrm{e}^{-3t}\left(\left\{\begin{array}{cc} \sin \left(t \right) & 0\le t <\frac{\pi}{2} \\ 1 & \frac{\pi}{2}\le t \end{array}\right.\right) dt + c_1}{\mathrm{e}^{-3t}}$$

- Evaluate the integrals on the rhs

$$y = \frac{\left\{\begin{array}{cc} 0 & t \leq 0 \\ -\frac{e^{-3t}\cos(t)}{10} - \frac{3\,e^{-3t}\sin(t)}{10} + \frac{1}{10} & t \leq \frac{\pi}{2} \\ -\frac{e^{-3t}}{3} + \frac{1}{10} + \frac{e^{-\frac{3\pi}{2}}}{30} & \frac{\pi}{2} < t \end{array}\right. + c_1}{e^{-3t}}$$

- Simplify

$$y = -\frac{e^{3t}\left(\left(\left\{\begin{array}{cc} 0 & t \leq 0 \\ -1 + (\cos(t) + 3\sin(t))\,e^{-3t} & t \leq \frac{\pi}{2} \\ \frac{10\,e^{-3t}}{3} - 1 - \frac{e^{-\frac{3\pi}{2}}}{3} & \frac{\pi}{2} < t \end{array}\right.\right) - 10c_1\right)}{10}$$

- Use initial condition $y(0) = 2$

$$2 = c_1$$

- Solve for c_1

$$c_1 = 2$$

- Substitute $c_1 = 2$ into general solution and simplify

$$y = \left\{\begin{array}{cc} 2\,e^{3t} & t \leq 0 \\ \frac{21\,e^{3t}}{10} - \frac{\cos(t)}{10} - \frac{3\sin(t)}{10} & t \leq \frac{\pi}{2} \\ \frac{21\,e^{3t}}{10} + \frac{e^{-\frac{3\pi}{2}+3t}}{30} - \frac{1}{3} & \frac{\pi}{2} < t \end{array}\right.$$

- Solution to the IVP

$$y = \left\{\begin{array}{cc} 2\,e^{3t} & t \leq 0 \\ \frac{21\,e^{3t}}{10} - \frac{\cos(t)}{10} - \frac{3\sin(t)}{10} & t \leq \frac{\pi}{2} \\ \frac{21\,e^{3t}}{10} + \frac{e^{-\frac{3\pi}{2}+3t}}{30} - \frac{1}{3} & \frac{\pi}{2} < t \end{array}\right.$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 3.703 (sec). Leaf size: 61

```
dsolve([diff(y(t),t)-3*y(t)=piecewise(0<=t and t<Pi/2,sin(t),t>=Pi/2,1),y(0) = 2],y(t), sing
```

$$y(t) = \frac{\left(\begin{cases} 21\,\mathrm{e}^{3t} - \cos\left(t\right) - 3\sin\left(t\right) & t < \frac{\pi}{2} \\ -\frac{19}{3} + 21\,\mathrm{e}^{\frac{3\pi}{2}} & t = \frac{\pi}{2} \\ 21\,\mathrm{e}^{3t} + \frac{\mathrm{e}^{3t - \frac{3\pi}{2}}}{3} - \frac{10}{3} & \frac{\pi}{2} < t \end{cases}\right)}{10}$$

✓ Solution by Mathematica

Time used: 0.1 (sec). Leaf size: 68

```
DSolve[{y'[t]-3*y[t]==Piecewise[{{Sin[t],0<=t<Pi/2},{1,t >= Pi/2}}],{y[0]==2}},y[t],t,Includ
```

$$y(t) \to \begin{cases} 2e^{3t} & t \leq 0 \\ \frac{1}{30}\left(-10 + 63e^{3t} + e^{3t - \frac{3\pi}{2}}\right) & 2t > \pi \\ \frac{1}{10}(-\cos(t) + 21e^{3t} - 3\sin(t)) & \text{True} \end{cases}$$

14.7 problem Problem 33

Internal problem ID [2873]
Internal file name [`OUTPUT/2365_Sunday_June_05_2022_03_01_07_AM_93783359/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 10, The Laplace Transform and Some Elementary Applications. Exercises for 10.7. page 704
Problem number: Problem 33.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first__order__ode__lie__symmetry__lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' - 3y = -10\,\mathrm{e}^{-t+a} \sin\left(-2t + 2a\right) \mathrm{Heaviside}\left(t - a\right)}$$

With initial conditions

$$[y(0) = 5]$$

14.7.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$\begin{aligned} p(t) &= -3 \\ q(t) &= -10\,\mathrm{e}^{-t+a} \sin\left(-2t + 2a\right) \mathrm{Heaviside}\left(t - a\right) \end{aligned}$$

Hence the ode is

$$y' - 3y = -10\,\mathrm{e}^{-t+a} \sin\left(-2t + 2a\right) \mathrm{Heaviside}\left(t - a\right)$$

The domain of $p(t) = -3$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = -10\,\mathrm{e}^{-t+a} \sin\left(-2t + 2a\right) \mathrm{Heaviside}\left(t - a\right)$ is

$$\{t < a \vee a < t\}$$

But the point $t_0 = 0$ is not inside this domain. Hence existence and uniqueness theorem does not apply. There could be infinite number of solutions, or one solution or no solution at all.

14.7.2 Solving as laplace ode

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(y') = sY(s) - y(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$sY(s) - y(0) - 3Y(s) = 10\,\mathrm{laplace}\left(-\mathrm{e}^{-t+a} \sin\left(-2t + 2a\right) \mathrm{Heaviside}\left(t - a\right), t, s\right) \tag{1}$$

Replacing initial condition gives

$$sY(s) - 5 - 3Y(s) = 10\,\mathrm{laplace}\left(-\mathrm{e}^{-t+a} \sin\left(-2t + 2a\right) \mathrm{Heaviside}\left(t - a\right), t, s\right)$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{10\,\mathrm{laplace}\left(-\mathrm{e}^{-t+a} \sin\left(-2t + 2a\right) \mathrm{Heaviside}\left(t - a\right), t, s\right) + 5}{s - 3}$$

Taking the inverse Laplace transform gives

$$\begin{aligned} y &= \mathcal{L}^{-1}(Y(s)) \\ &= \mathcal{L}^{-1}\left(\frac{10\,\mathrm{laplace}\left(-\mathrm{e}^{-t+a} \sin\left(-2t + 2a\right) \mathrm{Heaviside}\left(t - a\right), t, s\right) + 5}{s - 3}\right) \\ &= 5\,\mathrm{e}^{3t} - \mathrm{e}^{-t+a} \mathrm{Heaviside}\left(t - a\right)\left(\cos\left(-2t + 2a\right) - 2\sin\left(-2t + 2a\right)\right) + \mathrm{Heaviside}\left(-a\right)\left(-\mathrm{e}^{-3a+3t} + \mathrm{e}^{a+3t}\right. \end{aligned}$$

Hence the final solution is

$$
\begin{aligned}
y = 5\,\mathrm{e}^{3t} &- \mathrm{e}^{-t+a}\,\mathrm{Heaviside}\,(t-a)\,(\cos\,(-2t+2a) - 2\sin\,(-2t+2a)) \\
&+ \mathrm{Heaviside}\,(-a)\left(-\mathrm{e}^{-3a+3t} + \mathrm{e}^{a+3t}(\cos\,(2a) - 2\sin\,(2a))\right) \\
&+ (1 - \mathrm{Heaviside}\,(-t+a))\,\mathrm{e}^{-3a+3t}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = 5\,\mathrm{e}^{3t} &- \mathrm{e}^{-t+a}\,\mathrm{Heaviside}\,(t-a)\,(\cos\,(-2t+2a) - 2\sin\,(-2t+2a)) \\
&+ \mathrm{Heaviside}\,(-a)\left(-\mathrm{e}^{-3a+3t} + \mathrm{e}^{a+3t}(\cos\,(2a) - 2\sin\,(2a))\right) \\
&+ (1 - \mathrm{Heaviside}\,(-t+a))\,\mathrm{e}^{-3a+3t}
\end{aligned}
\tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = 5\,\mathrm{e}^{3t} &- \mathrm{e}^{-t+a}\,\mathrm{Heaviside}\,(t-a)\,(\cos\,(-2t+2a) - 2\sin\,(-2t+2a)) \\
&+ \mathrm{Heaviside}\,(-a)\left(-\mathrm{e}^{-3a+3t} + \mathrm{e}^{a+3t}(\cos\,(2a) - 2\sin\,(2a))\right) \\
&+ (1 - \mathrm{Heaviside}\,(-t+a))\,\mathrm{e}^{-3a+3t}
\end{aligned}
$$

Verified OK.

14.7.3 Maple step by step solution

Let's solve

$[y' - 3y = -10\,\mathrm{e}^{-t+a}\sin\,(-2t+2a)\,\mathit{Heaviside}(t-a)\,, y(0) = 5]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = 3y - 10\,\mathrm{e}^{-t+a}\sin\,(-2t+2a)\,\mathit{Heaviside}(t-a)$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - 3y = -10\,\mathrm{e}^{-t+a}\sin\,(-2t+2a)\,\mathit{Heaviside}(t-a)$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\,(y' - 3y) = -10\mu(t)\,\mathrm{e}^{-t+a}\sin\,(-2t+2a)\,\mathit{Heaviside}(t-a)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,y)$

  $\mu(t)\,(y' - 3y) = \mu'(t)\,y + \mu(t)\,y'$

- Isolate $\mu'(t)$

  $\mu'(t) = -3\mu(t)$

- Solve to find the integrating factor

  $\mu(t) = \mathrm{e}^{-3t}$
- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\, y)\right) dt = \int -10\mu(t)\, \mathrm{e}^{-t+a} \sin(-2t+2a)\, \mathit{Heaviside}(t-a)\, dt + c_1$
- Evaluate the integral on the lhs

  $\mu(t)\, y = \int -10\mu(t)\, \mathrm{e}^{-t+a} \sin(-2t+2a)\, \mathit{Heaviside}(t-a)\, dt + c_1$
- Solve for y

  $y = \frac{\int -10\mu(t)\mathrm{e}^{-t+a} \sin(-2t+2a)\mathit{Heaviside}(t-a)dt+c_1}{\mu(t)}$
- Substitute $\mu(t) = \mathrm{e}^{-3t}$

  $y = \frac{\int -10\,\mathrm{e}^{-3t}\mathrm{e}^{-t+a} \sin(-2t+2a)\mathit{Heaviside}(t-a)dt+c_1}{\mathrm{e}^{-3t}}$
- Evaluate the integrals on the rhs

  $y = \frac{10\left(\frac{\mathrm{e}^{a} \cos(a)^2 \mathrm{e}^{-4t}(-4\sin(2t)-2\cos(2t))}{10} - \frac{\mathrm{e}^{a}\mathrm{e}^{-4t}(-4\sin(2t)-2\cos(2t))}{20} - 4\,\mathrm{e}^{a}\cos(a)\sin(a)\left(\frac{(-4\cos(t)+2\sin(t))\mathrm{e}^{-4t}\cos(t)}{20} - \frac{1}{40(\mathrm{e}^{t})^4}\right) - \mathrm{e}^{}\cdots\right.}{}$
- Simplify

  $y = \left(c_1\mathrm{e}^{3a+4t} - 4\left(\left(\frac{1}{4} + \left(\cos(t)^2 + 2\sin(t)\cos(t) - \frac{1}{2}\right)\cos(a)^2 + \sin(a)\left(\sin(t)\cos(t) - 2\cos(t)^2\right.\right.\right.\right.$
- Use initial condition $y(0) = 5$

  $5 = \left(c_1\mathrm{e}^{3a} - 4\left(\left(-\frac{1}{4} + \frac{\cos(a)^2}{2} - \sin(a)\cos(a)\right)\mathrm{e}^{4a} - \frac{1}{4}\right)\mathit{Heaviside}(-a)\right)\mathrm{e}^{-3a}$
- Solve for c_1

  $c_1 = \frac{2\cos(a)^2\mathrm{e}^{-3a}\mathit{Heaviside}(-a)\mathrm{e}^{4a} - 4\cos(a)\sin(a)\mathrm{e}^{-3a}\mathit{Heaviside}(-a)\mathrm{e}^{4a} - \mathrm{e}^{-3a}\mathit{Heaviside}(-a)\mathrm{e}^{4a} - \mathrm{e}^{-3a}\mathit{Heaviside}(-a)+5}{\mathrm{e}^{3a}\mathrm{e}^{-3a}}$
- Substitute $c_1 = \frac{2\cos(a)^2\mathrm{e}^{-3a}\mathit{Heaviside}(-a)\mathrm{e}^{4a} - 4\cos(a)\sin(a)\mathrm{e}^{-3a}\mathit{Heaviside}(-a)\mathrm{e}^{4a} - \mathrm{e}^{-3a}\mathit{Heaviside}(-a)\mathrm{e}^{4a} - \mathrm{e}^{-3a}\mathit{Heaviside}(}{\mathrm{e}^{3a}\mathrm{e}^{-3a}}$

  $y = \mathrm{e}^{-6a-t}\left(-2(\mathit{Heaviside}(a)-1)\left(\cos(a)^2 - 2\sin(a)\cos(a) - \frac{1}{2}\right)\mathrm{e}^{7a+4t} + (\mathit{Heaviside}(t-a) + \mathit{Hea}\right.$
- Solution to the IVP

  $y = \mathrm{e}^{-6a-t}\left(-2(\mathit{Heaviside}(a)-1)\left(\cos(a)^2 - 2\sin(a)\cos(a) - \frac{1}{2}\right)\mathrm{e}^{7a+4t} + (\mathit{Heaviside}(t-a) + \mathit{Hea}\right.$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 3.907 (sec). Leaf size: 97

```
dsolve([diff(y(t),t)-3*y(t)=10*exp(-(t-a))*sin(2*(t-a))*Heaviside(t-a),y(0) = 5],y(t), sings
```

$$
\begin{aligned}
y(t) = {} & (\mathrm{Heaviside}\,(t-a)+\mathrm{Heaviside}\,(a)-1)\,\mathrm{e}^{-3a+3t}-\Bigg((\cos\,(2t)+2\sin\,(2t))\cos\,(2a) \\
& -2\sin\,(2a)\left(\cos\,(2t)-\frac{\sin\,(2t)}{2}\right)\Bigg)\,\mathrm{e}^{-t+a}\,\mathrm{Heaviside}\,(t-a) \\
& -(\mathrm{Heaviside}\,(a)-1)\,(\cos\,(2a)-2\sin\,(2a))\,\mathrm{e}^{3t+a}+5\,\mathrm{e}^{3t}
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.461 (sec). Leaf size: 103

```
DSolve[{y'[t]-3*y[t]==10*Exp[-(t-a)]*Sin[2*(t-a)]*UnitStep[t-a],{y[0]==5}},y[t],t,IncludeSin
```

$$
\begin{aligned}
y(t) \to e^{-3a-t}\big(& e^{4t}\theta(-a)\left(-2e^{4a}\sin(2a)+e^{4a}\cos(2a)-1\right) \\
& +\theta(t-a)\left(2e^{4a}\sin(2(a-t))-e^{4a}\cos(2(a-t))+e^{4t}\right)+5e^{3a+4t}\big)
\end{aligned}
$$

14.8 problem Problem 34

Internal problem ID [2874]
Internal file name [`OUTPUT/2366_Sunday_June_05_2022_03_01_21_AM_82718483/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 10, The Laplace Transform and Some Elementary Applications. Exercises for 10.7. page 704
Problem number: Problem 34.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - y = \text{Heaviside}\left(t-1\right)}$$

With initial conditions

$$[y(0) = 1, y'(0) = 0]$$

14.8.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= -1 \\ F &= \text{Heaviside}\left(t-1\right) \end{aligned}$$

Hence the ode is

$$y'' - y = \text{Heaviside}(t-1)$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = -1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \text{Heaviside}(t-1)$ is

$$\{t < 1 \vee 1 < t\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) - Y(s) = \frac{e^{-s}}{s} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 1\\ y'(0) &= 0\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - s - Y(s) = \frac{e^{-s}}{s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{s^2 + e^{-s}}{s\,(s^2-1)}$$

Taking the inverse Laplace transform gives

$$
\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(\frac{s^2+\mathrm{e}^{-s}}{s\,(s^2-1)}\right) \\
&= \cosh(t) + 2\,\mathrm{Heaviside}(t-1)\sinh\left(\frac{t}{2}-\frac{1}{2}\right)^2
\end{aligned}
$$

Hence the final solution is

$$
y = \cosh(t) + 2\,\mathrm{Heaviside}(t-1)\sinh\left(\frac{t}{2}-\frac{1}{2}\right)^2
$$

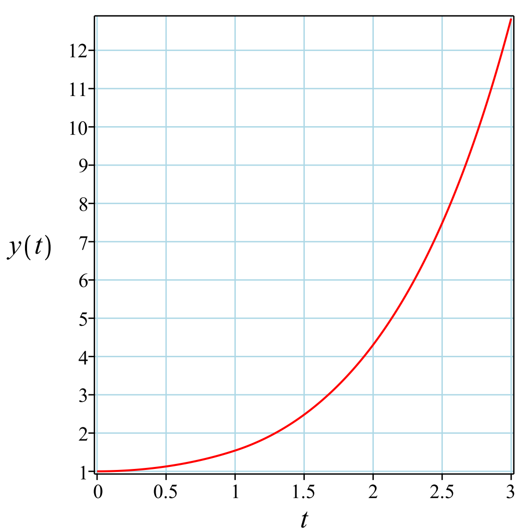
Simplifying the solution gives

$$
y = \cosh(t) + \mathrm{Heaviside}(t-1)\,(-1+\cosh(t-1))
$$

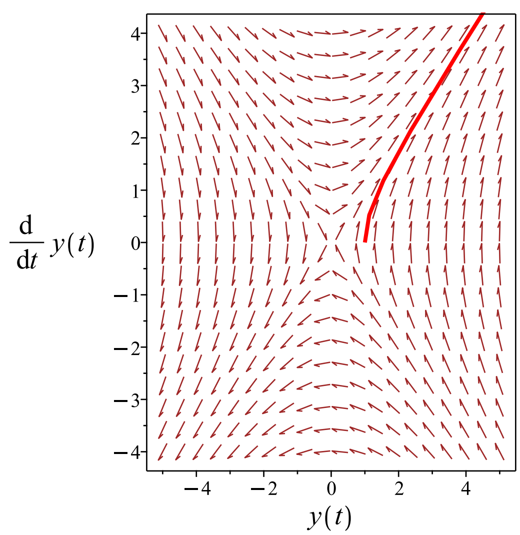
Summary

The solution(s) found are the following

$$
y = \cosh(t) + \mathrm{Heaviside}(t-1)\,(-1+\cosh(t-1)) \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$
y = \cosh(t) + \mathrm{Heaviside}(t-1)\,(-1+\cosh(t-1))
$$

Verified OK.

14.8.2 Maple step by step solution

Let's solve

$$\left[y'' - y = \mathit{Heaviside}(t-1), y(0) = 1, y'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 1 = 0$

- Factor the characteristic polynomial

  $(r-1)(r+1) = 0$

- Roots of the characteristic polynomial

  $r = (-1, 1)$

- 1st solution of the homogeneous ODE

  $y_1(t) = \mathrm{e}^{-t}$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \mathrm{e}^{t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{t} + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $$\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = \mathit{Heaviside}(t-1)\right]$$

  ◦ Wronskian of solutions of the homogeneous equation

  $$W(y_1(t), y_2(t)) = \begin{bmatrix} \mathrm{e}^{-t} & \mathrm{e}^{t} \\ -\mathrm{e}^{-t} & \mathrm{e}^{t} \end{bmatrix}$$

  ◦ Compute Wronskian

  $W(y_1(t), y_2(t)) = 2$

  ◦ Substitute functions into equation for $y_p(t)$

$$y_p(t) = -\frac{\mathrm{e}^{-t}\left(\int \mathrm{e}^{t} \mathit{Heaviside}(t-1)dt\right)}{2} + \frac{\mathrm{e}^{t}\left(\int \mathrm{e}^{-t} \mathit{Heaviside}(t-1)dt\right)}{2}$$

- Compute integrals

  $$y_p(t) = \frac{\mathit{Heaviside}(t-1)\left(-2+\mathrm{e}^{1-t}+\mathrm{e}^{t-1}\right)}{2}$$

- Substitute particular solution into general solution to ODE

  $$y = c_1\mathrm{e}^{-t} + c_2\mathrm{e}^{t} + \frac{\mathit{Heaviside}(t-1)\left(-2+\mathrm{e}^{1-t}+\mathrm{e}^{t-1}\right)}{2}$$

□ Check validity of solution $y = c_1\mathrm{e}^{-t} + c_2\mathrm{e}^{t} + \frac{\mathit{Heaviside}(t-1)\left(-2+\mathrm{e}^{1-t}+\mathrm{e}^{t-1}\right)}{2}$

- Use initial condition $y(0) = 1$

  $$1 = c_1 + c_2$$

- Compute derivative of the solution

  $$y' = -c_1\mathrm{e}^{-t} + c_2\mathrm{e}^{t} + \frac{\mathit{Dirac}(t-1)\left(-2+\mathrm{e}^{1-t}+\mathrm{e}^{t-1}\right)}{2} + \frac{\mathit{Heaviside}(t-1)\left(-\mathrm{e}^{1-t}+\mathrm{e}^{t-1}\right)}{2}$$

- Use the initial condition $y'\Big|_{\{t=0\}} = 0$

  $$0 = -c_1 + c_2$$

- Solve for c_1 and c_2

  $$\left\{c_1 = \frac{1}{2}, c_2 = \frac{1}{2}\right\}$$

- Substitute constant values into general solution and simplify

  $$y = \frac{\mathit{Heaviside}(t-1)\mathrm{e}^{1-t}}{2} + \frac{\left(\mathrm{e}^{t-1}-2\right)\mathit{Heaviside}(t-1)}{2} + \frac{\mathrm{e}^{t}}{2} + \frac{\mathrm{e}^{-t}}{2}$$

- Solution to the IVP

  $$y = \frac{\mathit{Heaviside}(t-1)\mathrm{e}^{1-t}}{2} + \frac{\left(\mathrm{e}^{t-1}-2\right)\mathit{Heaviside}(t-1)}{2} + \frac{\mathrm{e}^{t}}{2} + \frac{\mathrm{e}^{-t}}{2}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 2.203 (sec). Leaf size: 21

```
dsolve([diff(y(t),t$2)-y(t)=Heaviside(t-1),y(0) = 1, D(y)(0) = 0],y(t), singsol=all)
```

$$y(t) = \cosh(t) + \text{Heaviside}(t-1)(-1+\cosh(t-1))$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 57

```
DSolve[{y''[t]-y[t]==UnitStep[t-1],{y[0]==1,y'[0]==0}},y[t],t,IncludeSingularSolutions -> Tr
```

$$y(t) \to \frac{1}{2}e^{-t-1}\left((e-e^t)^2(-\theta(1-t)) + e^{2t} - 2e^{t+1} + e^{2t+1} + e^2 + e\right)$$

14.9 problem Problem 35

Internal problem ID [2875]
Internal file name [`OUTPUT/2367_Sunday_June_05_2022_03_01_27_AM_4958839/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 10, The Laplace Transform and Some Elementary Applications. Exercises for 10.7. page 704
Problem number: Problem 35.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - y' - 2y = 1 - 3\,\mathrm{Heaviside}\,(t-2)}$$

With initial conditions

$$[y(0) = 1, y'(0) = -2]$$

14.9.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= -1 \\ q(t) &= -2 \\ F &= 1 - 3\,\mathrm{Heaviside}\,(t-2) \end{aligned}$$

Hence the ode is

$$y'' - y' - 2y = 1 - 3\,\text{Heaviside}\,(t - 2)$$

The domain of $p(t) = -1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = -2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 1 - 3\,\text{Heaviside}\,(t - 2)$ is

$$\{t < 2 \lor 2 < t\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) - sY(s) + y(0) - 2Y(s) = \frac{1 - 3\,\mathrm{e}^{-2s}}{s} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 1\\ y'(0) &= -2\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + 3 - s - sY(s) - 2Y(s) = \frac{1 - 3\,\mathrm{e}^{-2s}}{s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = -\frac{-s^2 + 3\,\mathrm{e}^{-2s} + 3s - 1}{s\,(s^2 - s - 2)}$$

Taking the inverse Laplace transform gives

$$
\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(-\frac{-s^2+3\,\mathrm{e}^{-2s}+3s-1}{s\,(s^2-s-2)}\right) \\
&= -\frac{1}{2}+\frac{5\,\mathrm{e}^{-t}}{3}-\frac{\mathrm{e}^{2t}}{6}+\frac{\mathrm{e}^{2t-4}(-1+\mathrm{Heaviside}\,(-t+2))}{2}+\frac{\mathrm{Heaviside}\,(t-2)\,(3-2\,\mathrm{e}^{-t+2})}{2}
\end{aligned}
$$

Hence the final solution is

$$
y = -\frac{1}{2}+\frac{5\,\mathrm{e}^{-t}}{3}-\frac{\mathrm{e}^{2t}}{6}+\frac{\mathrm{e}^{2t-4}(-1+\mathrm{Heaviside}\,(-t+2))}{2}+\frac{\mathrm{Heaviside}\,(t-2)\,(3-2\,\mathrm{e}^{-t+2})}{2}
$$

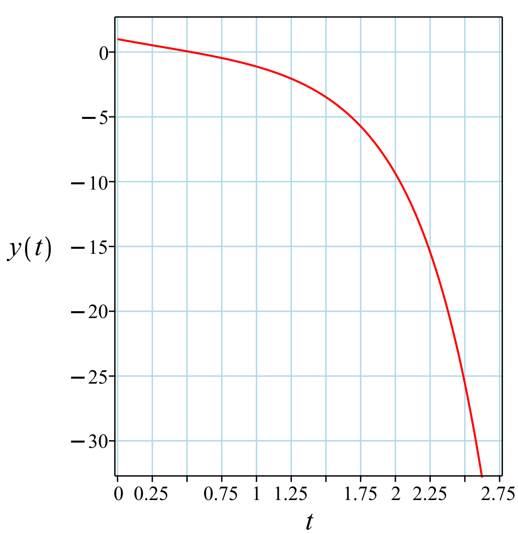
Simplifying the solution gives

$$
\begin{aligned}
y = -\frac{1}{2}+\frac{5\,\mathrm{e}^{-t}}{3}-\frac{\mathrm{e}^{2t}}{6}-\frac{\mathrm{e}^{2t-4}\,\mathrm{Heaviside}\,(t-2)}{2} \\
-\mathrm{Heaviside}\,(t-2)\,\mathrm{e}^{-t+2}+\frac{3\,\mathrm{Heaviside}\,(t-2)}{2}
\end{aligned}
$$

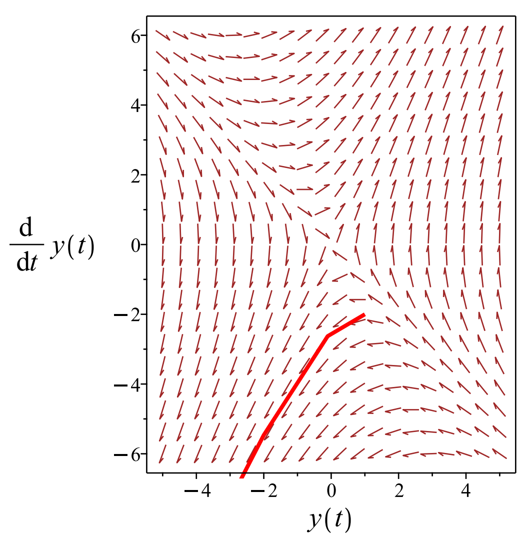
Summary

The solution(s) found are the following

$$
\begin{aligned}
y = -\frac{1}{2}+\frac{5\,\mathrm{e}^{-t}}{3}-\frac{\mathrm{e}^{2t}}{6}-\frac{\mathrm{e}^{2t-4}\,\mathrm{Heaviside}\,(t-2)}{2} \\
-\mathrm{Heaviside}\,(t-2)\,\mathrm{e}^{-t+2}+\frac{3\,\mathrm{Heaviside}\,(t-2)}{2}
\end{aligned} \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{1}{2}+\frac{5\,e^{-t}}{3}-\frac{e^{2t}}{6}-\frac{e^{2t-4}\,\mathrm{Heaviside}\,(t-2)}{2} - \mathrm{Heaviside}\,(t-2)\,e^{-t+2}+\frac{3\,\mathrm{Heaviside}\,(t-2)}{2}$$

Verified OK.

14.9.2 Maple step by step solution

Let's solve

$$\left[y''-y'-2y=1-3\mathit{Heaviside}(t-2)\,, y(0)=1, y'\Big|_{\{t=0\}}=-2\right]$$

- Highest derivative means the order of the ODE is 2

  y''
- Characteristic polynomial of homogeneous ODE

  $r^2-r-2=0$
- Factor the characteristic polynomial

  $(r+1)\,(r-2)=0$
- Roots of the characteristic polynomial

  $r=(-1,2)$
- 1st solution of the homogeneous ODE

  $y_1(t)=e^{-t}$
- 2nd solution of the homogeneous ODE

  $y_2(t)=e^{2t}$
- General solution of the ODE

  $y=c_1y_1(t)+c_2y_2(t)+y_p(t)$
- Substitute in solutions of the homogeneous ODE

  $y=c_1e^{-t}+c_2e^{2t}+y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

- Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t)=-y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right)+y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t)=1-3\mathit{Heaviside}(t-2)\right]$
- Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \begin{bmatrix} e^{-t} & e^{2t} \\ -e^{-t} & 2\,e^{2t} \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(t), y_2(t)) = 3\,e^{t}$

- Substitute functions into equation for $y_p(t)$

  $y_p(t) = \frac{e^{-t}\left(\int e^{t}(-1+3\mathit{Heaviside}(t-2))dt\right)}{3} - \frac{e^{2t}\left(\int e^{-2t}(-1+3\mathit{Heaviside}(t-2))dt\right)}{3}$

- Compute integrals

  $y_p(t) = \frac{3\mathit{Heaviside}(t-2)}{2} - \mathit{Heaviside}(t-2)\,e^{-t+2} - \frac{1}{2} - \frac{e^{2t-4}\mathit{Heaviside}(t-2)}{2}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-t} + c_2 e^{2t} + \frac{3\mathit{Heaviside}(t-2)}{2} - \mathit{Heaviside}(t-2)\,e^{-t+2} - \frac{1}{2} - \frac{e^{2t-4}\mathit{Heaviside}(t-2)}{2}$

□ Check validity of solution $y = c_1 e^{-t} + c_2 e^{2t} + \frac{3\mathit{Heaviside}(t-2)}{2} - \mathit{Heaviside}(t-2)\,e^{-t+2} - \frac{1}{2} - \frac{e^{2t-4}\mathit{H}}{}$

- Use initial condition $y(0) = 1$

  $1 = c_1 + c_2 - \frac{1}{2}$

- Compute derivative of the solution

  $y' = -c_1 e^{-t} + 2c_2 e^{2t} + \frac{3\mathit{Dirac}(t-2)}{2} - \mathit{Dirac}(t-2)\,e^{-t+2} + \mathit{Heaviside}(t-2)\,e^{-t+2} - e^{2t-4}\mathit{Heaviside}$

- Use the initial condition $y'\Big|_{\{t=0\}} = -2$

  $-2 = -c_1 + 2c_2$

- Solve for c_1 and c_2

  $\left\{c_1 = \frac{5}{3}, c_2 = -\frac{1}{6}\right\}$

- Substitute constant values into general solution and simplify

  $y = -\frac{1}{2} + \frac{5\,e^{-t}}{3} - \frac{e^{2t}}{6} - \frac{e^{2t-4}\mathit{Heaviside}(t-2)}{2} - \mathit{Heaviside}(t-2)\,e^{-t+2} + \frac{3\mathit{Heaviside}(t-2)}{2}$

- Solution to the IVP

  $y = -\frac{1}{2} + \frac{5\,e^{-t}}{3} - \frac{e^{2t}}{6} - \frac{e^{2t-4}\mathit{Heaviside}(t-2)}{2} - \mathit{Heaviside}(t-2)\,e^{-t+2} + \frac{3\mathit{Heaviside}(t-2)}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 2.218 (sec). Leaf size: 50

```
dsolve([diff(y(t),t$2)-diff(y(t),t)-2*y(t)=1-3*Heaviside(t-2),y(0) = 1, D(y)(0) = -2],y(t),
```

$$y(t) = -\frac{1}{2} + \frac{5\,\mathrm{e}^{-t}}{3} - \frac{\mathrm{e}^{2t}}{6} - \frac{\mathrm{Heaviside}\,(t-2)\,\mathrm{e}^{-4+2t}}{2} - \mathrm{Heaviside}\,(t-2)\,\mathrm{e}^{2-t} + \frac{3\,\mathrm{Heaviside}\,(t-2)}{2}$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 70

```
DSolve[{y''[t]-y'[t]-2*y[t]==1-3*UnitStep[t-2],{y[0]==1,y'[0]==-2}},y[t],t,IncludeSingularSo
```

$$y(t) \to \begin{cases} -\frac{1}{6}e^{-t}(-10+3e^t+e^{3t}) & t \leq 2 \\ \frac{1}{6}(6-6e^{2-t}+10e^{-t}-e^{2t}-3e^{2t-4}) & \text{True} \end{cases}$$

14.10 problem Problem 36

Internal problem ID [2876]
Internal file name [`OUTPUT/2368_Sunday_June_05_2022_03_01_34_AM_567848/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 10, The Laplace Transform and Some Elementary Applications. Exercises for 10.7. page 704
Problem number: Problem 36.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 4y = \text{Heaviside}\,(t-1) - \text{Heaviside}\,(t-2)}$$

With initial conditions

$$[y(0) = 0, y'(0) = 4]$$

14.10.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= -4 \\ F &= \text{Heaviside}\,(t-1) - \text{Heaviside}\,(t-2) \end{aligned}$$

Hence the ode is

$$y'' - 4y = \text{Heaviside}\,(t-1) - \text{Heaviside}\,(t-2)$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = -4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \text{Heaviside}\,(t-1) - \text{Heaviside}\,(t-2)$ is

$$\{1 \leq t \leq 2, 2 \leq t \leq \infty, -\infty \leq t \leq 1\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0) \\ \mathcal{L}(y'') &= s^2 Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2 Y(s) - y'(0) - sy(0) - 4Y(s) = \frac{\mathrm{e}^{-s} - \mathrm{e}^{-2s}}{s} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 0 \\ y'(0) &= 4\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2 Y(s) - 4 - 4Y(s) = \frac{\mathrm{e}^{-s} - \mathrm{e}^{-2s}}{s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{\mathrm{e}^{-s} - \mathrm{e}^{-2s} + 4s}{s\,(s^2 - 4)}$$

Taking the inverse Laplace transform gives

$$
\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(\frac{\mathrm{e}^{-s}-\mathrm{e}^{-2s}+4s}{s\left(s^{2}-4\right)}\right) \\
&= \frac{\operatorname{Heaviside}(t-1)\sinh(t-1)^{2}}{2}-\frac{\operatorname{Heaviside}(t-2)\sinh(t-2)^{2}}{2}+2\sinh(2t)
\end{aligned}
$$

Hence the final solution is

$$
y=\frac{\operatorname{Heaviside}(t-1)\sinh(t-1)^{2}}{2}-\frac{\operatorname{Heaviside}(t-2)\sinh(t-2)^{2}}{2}+2\sinh(2t)
$$

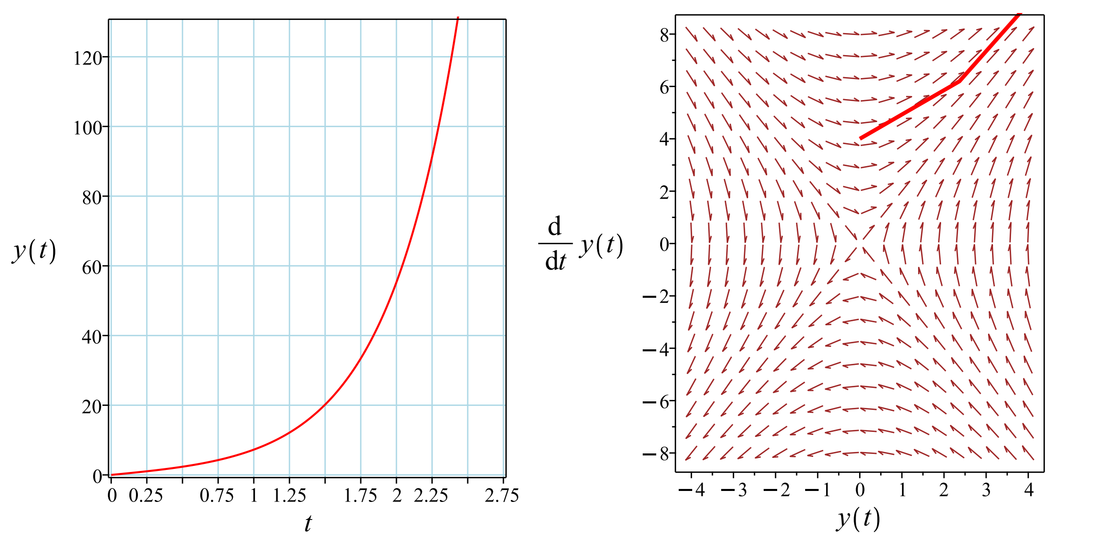
Simplifying the solution gives

$$
y=\frac{\operatorname{Heaviside}(t-1)\sinh(t-1)^{2}}{2}-\frac{\operatorname{Heaviside}(t-2)\sinh(t-2)^{2}}{2}+2\sinh(2t)
$$

Summary

The solution(s) found are the following

$$
y=\frac{\operatorname{Heaviside}(t-1)\sinh(t-1)^{2}}{2}-\frac{\operatorname{Heaviside}(t-2)\sinh(t-2)^{2}}{2}+2\sinh(2t) \tag{1}
$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{\text{Heaviside}\,(t-1)\sinh\,(t-1)^2}{2} - \frac{\text{Heaviside}\,(t-2)\sinh\,(t-2)^2}{2} + 2\sinh\,(2t)$$

Verified OK.

14.10.2 Maple step by step solution

Let's solve

$$\left[y'' - 4y = Heaviside(t-1) - Heaviside(t-2)\,, y(0) = 0, y'\Big|_{\{t=0\}} = 4\right]$$

- Highest derivative means the order of the ODE is 2

  y''
- Characteristic polynomial of homogeneous ODE

  $r^2 - 4 = 0$
- Factor the characteristic polynomial

  $(r-2)\,(r+2) = 0$
- Roots of the characteristic polynomial

  $r = (-2, 2)$
- 1st solution of the homogeneous ODE

  $y_1(t) = \mathrm{e}^{-2t}$
- 2nd solution of the homogeneous ODE

  $y_2(t) = \mathrm{e}^{2t}$
- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$
- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{2t} + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

- Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = Heaviside(t-1) - Heavisi\right.$
- Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \begin{bmatrix} e^{-2t} & e^{2t} \\ -2\,e^{-2t} & 2\,e^{2t} \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(t), y_2(t)) = 4$

- Substitute functions into equation for $y_p(t)$

  $y_p(t) = -\frac{e^{-2t}\left(\int e^{2t}(\mathit{Heaviside}(t-1)-\mathit{Heaviside}(t-2))dt\right)}{4} + \frac{e^{2t}\left(\int e^{-2t}(\mathit{Heaviside}(t-1)-\mathit{Heaviside}(t-2))dt\right)}{4}$

- Compute integrals

  $y_p(t) = -\frac{\mathit{Heaviside}(t-1)}{4} + \frac{\mathit{Heaviside}(t-1)e^{-2t+2}}{8} + \frac{\mathit{Heaviside}(t-2)}{4} - \frac{\mathit{Heaviside}(t-2)e^{-2t+4}}{8} + \frac{\mathit{Heaviside}(t-1)e^{2t-2}}{8} -$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-2t} + c_2 e^{2t} - \frac{\mathit{Heaviside}(t-1)}{4} + \frac{\mathit{Heaviside}(t-1)e^{-2t+2}}{8} + \frac{\mathit{Heaviside}(t-2)}{4} - \frac{\mathit{Heaviside}(t-2)e^{-2t+4}}{8} + \frac{\mathit{Heaviside}}{}$

□ Check validity of solution $y = c_1 e^{-2t} + c_2 e^{2t} - \frac{\mathit{Heaviside}(t-1)}{4} + \frac{\mathit{Heaviside}(t-1)e^{-2t+2}}{8} + \frac{\mathit{Heaviside}(t-2)}{4} -$

- Use initial condition $y(0) = 0$

  $0 = c_1 + c_2$

- Compute derivative of the solution

  $y' = -2c_1 e^{-2t} + 2c_2 e^{2t} - \frac{\mathit{Dirac}(t-1)}{4} + \frac{\mathit{Dirac}(t-1)e^{-2t+2}}{8} - \frac{\mathit{Heaviside}(t-1)e^{-2t+2}}{4} + \frac{\mathit{Dirac}(t-2)}{4} - \frac{\mathit{Dirac}(t-2)e^{-2}}{8}$

- Use the initial condition $y'\Big|_{\{t=0\}} = 4$

  $4 = -2c_1 + 2c_2$

- Solve for c_1 and c_2

  $\{c_1 = -1, c_2 = 1\}$

- Substitute constant values into general solution and simplify

  $y = -e^{-2t} + e^{2t} - \frac{\mathit{Heaviside}(t-1)}{4} + \frac{\mathit{Heaviside}(t-1)e^{-2t+2}}{8} + \frac{\mathit{Heaviside}(t-2)}{4} - \frac{\mathit{Heaviside}(t-2)e^{-2t+4}}{8} + \frac{\mathit{Heaviside}(t-}{8}$

- Solution to the IVP

  $y = -e^{-2t} + e^{2t} - \frac{\mathit{Heaviside}(t-1)}{4} + \frac{\mathit{Heaviside}(t-1)e^{-2t+2}}{8} + \frac{\mathit{Heaviside}(t-2)}{4} - \frac{\mathit{Heaviside}(t-2)e^{-2t+4}}{8} + \frac{\mathit{Heaviside}(t-}{8}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 2.579 (sec). Leaf size: 35

```
dsolve([diff(y(t),t$2)-4*y(t)=Heaviside(t-1)-Heaviside(t-2),y(0) = 0, D(y)(0) = 4],y(t), sin
```

$$y(t) = \frac{\text{Heaviside}\,(t-1)\sinh\,(t-1)^2}{2} - \frac{\text{Heaviside}\,(t-2)\sinh\,(t-2)^2}{2} + 2\sinh\,(2t)$$

✓ Solution by Mathematica

Time used: 0.04 (sec). Leaf size: 113

```
DSolve[{y''[t]-4*y[t]==UnitStep[t-1]-UnitStep[t-2],{y[0]==0,y'[0]==4}},y[t],t,IncludeSingula
```

$$y(t) \to \begin{cases} e^{-2t}(-1+e^{4t}) & t \leq 1 \\ \frac{1}{8}(-2+e^{2-2t}-8e^{-2t}+8e^{2t}+e^{2t-2}) & 1 < t \leq 2 \\ \frac{1}{8}e^{-2(t+2)}(-8e^4+e^6-e^8-e^{4t}+e^{4t+2}+8e^{4t+4}) & \text{True} \end{cases}$$

14.11 problem Problem 37

Internal problem ID [2877]
Internal file name [`OUTPUT/2369_Sunday_June_05_2022_03_01_43_AM_94568775/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 10, The Laplace Transform and Some Elementary Applications. Exercises for 10.7. page 704
Problem number: Problem 37.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' + y = t - \text{Heaviside}\,(t-1)\,(t-1)}$$

With initial conditions

$$[y(0) = 2, y'(0) = 1]$$

14.11.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= 1 \\ F &= -\text{Heaviside}\,(t-1)\,t + \text{Heaviside}\,(t-1) + t \end{aligned}$$

Hence the ode is

$$y'' + y = -\text{Heaviside}\,(t-1)\,t + \text{Heaviside}\,(t-1) + t$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = -\text{Heaviside}\,(t-1)\,t + \text{Heaviside}\,(t-1) + t$ is

$$\{t < 1 \lor 1 < t\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + Y(s) = \frac{1-\mathrm{e}^{-s}}{s^2} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 2\\ y'(0) &= 1\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 1 - 2s + Y(s) = \frac{1-\mathrm{e}^{-s}}{s^2}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = -\frac{-2s^3 - s^2 + \mathrm{e}^{-s} - 1}{s^2\,(s^2+1)}$$

Taking the inverse Laplace transform gives

$$
\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(-\frac{-2s^3 - s^2 + \mathrm{e}^{-s} - 1}{s^2\left(s^2+1\right)}\right) \\
&= 2\cos(t) - \mathrm{Heaviside}(t-1)\,(t-1-\sin(t-1)) + t
\end{aligned}
$$

Hence the final solution is

$$
y = 2\cos(t) - \mathrm{Heaviside}(t-1)\,(t-1-\sin(t-1)) + t
$$

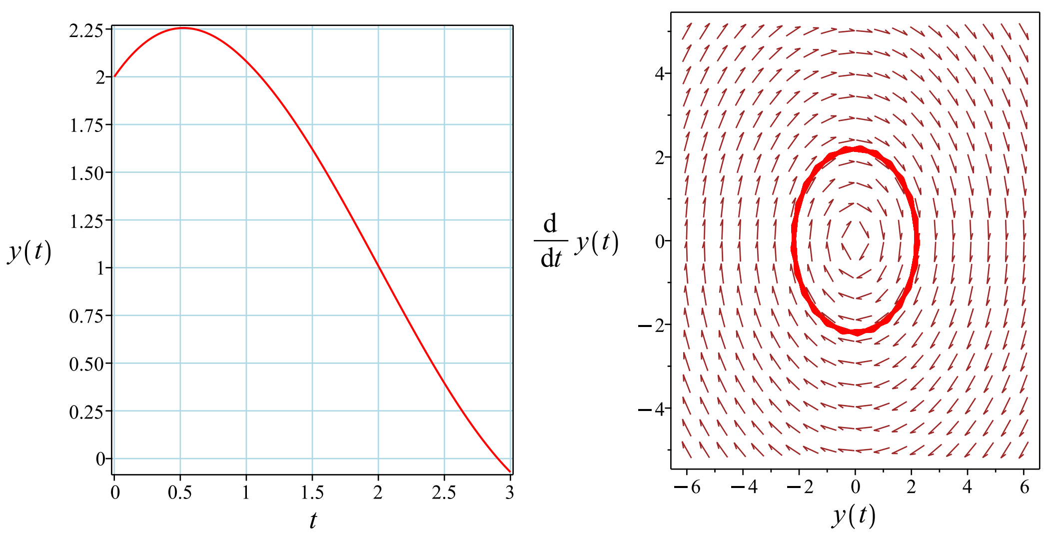
Simplifying the solution gives

$$
y = (-t + 1 + \sin(t-1))\,\mathrm{Heaviside}(t-1) + t + 2\cos(t)
$$

Summary

The solution(s) found are the following

$$
y = (-t + 1 + \sin(t-1))\,\mathrm{Heaviside}(t-1) + t + 2\cos(t) \tag{1}
$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$
y = (-t + 1 + \sin(t-1))\,\mathrm{Heaviside}(t-1) + t + 2\cos(t)
$$

Verified OK.

14.11.2 Maple step by step solution

Let's solve

$\left[y'' + y = -Heaviside(t-1)\, t + Heaviside(t-1) + t, y(0) = 2, y'\Big|_{\{t=0\}} = 1\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t) = \cos(t)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \sin(t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(t) + c_2 \sin(t) + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = -Heaviside(t-1)\, t + Hea\right.$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(t), y_2(t)) = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(t), y_2(t)) = 1$

  ◦ Substitute functions into equation for $y_p(t)$

$y_p(t) = \cos(t)\left(\int (-t + Heaviside(t-1)(t-1))\sin(t)\,dt\right) - \sin(t)\left(\int (-t + Heaviside(t-1)(t$

- Compute integrals

  $y_p(t) = (-\cos(t)\sin(1) + \sin(t)\cos(1) - t + 1)\,Heaviside(t-1) + t$

- Substitute particular solution into general solution to ODE

  $y = c_1\cos(t) + c_2\sin(t) + (-\cos(t)\sin(1) + \sin(t)\cos(1) - t + 1)\,Heaviside(t-1) + t$

☐ Check validity of solution $y = c_1\cos(t) + c_2\sin(t) + (-\cos(t)\sin(1) + \sin(t)\cos(1) - t + 1)\,H$

- Use initial condition $y(0) = 2$

  $2 = c_1$

- Compute derivative of the solution

  $y' = -c_1\sin(t) + c_2\cos(t) + (\sin(t)\sin(1) + \cos(t)\cos(1) - 1)\,Heaviside(t-1) + (-\cos(t)\sin$

- Use the initial condition $y'\Big|_{\{t=0\}} = 1$

  $1 = c_2 + 1$

- Solve for c_1 and c_2

  $\{c_1 = 2, c_2 = 0\}$

- Substitute constant values into general solution and simplify

  $y = 2\cos(t) + (-\cos(t)\sin(1) + \sin(t)\cos(1) - t + 1)\,Heaviside(t-1) + t$

- Solution to the IVP

  $y = 2\cos(t) + (-\cos(t)\sin(1) + \sin(t)\cos(1) - t + 1)\,Heaviside(t-1) + t$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 2.156 (sec). Leaf size: 25

```
dsolve([diff(y(t),t$2)+y(t)=t-Heaviside(t-1)*(t-1),y(0) = 2, D(y)(0) = 1],y(t), singsol=all)
```

$$y(t) = (-t + \sin(t-1) + 1) \operatorname{Heaviside}(t-1) + t + 2\cos(t)$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 31

```
DSolve[{y''[t]+y[t]==t-UnitStep[t-1]*(t-1),{y[0]==2,y'[0]==1}},y[t],t,IncludeSingularSolutio
```

$$y(t) \to \begin{cases} t + 2\cos(t) & t \leq 1 \\ 2\cos(t) - \sin(1-t) + 1 & \text{True} \end{cases}$$

14.12 problem Problem 38

Internal problem ID [2878]
Internal file name [`OUTPUT/2370_Sunday_June_05_2022_03_01_48_AM_51361726/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 10, The Laplace Transform and Some Elementary Applications. Exercises for 10.7. page 704
Problem number: Problem 38.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 3y' + 2y = -10\,\text{Heaviside}\left(t - \frac{\pi}{4}\right)\cos\left(t + \frac{\pi}{4}\right)}$$

With initial conditions

$$[y(0) = 1, y'(0) = 0]$$

14.12.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned}
p(t) &= 3 \\
q(t) &= 2 \\
F &= -10\,\text{Heaviside}\left(t - \frac{\pi}{4}\right)\cos\left(t + \frac{\pi}{4}\right)
\end{aligned}$$

Hence the ode is

$$y'' + 3y' + 2y = -10\,\text{Heaviside}\left(t - \frac{\pi}{4}\right)\cos\left(t + \frac{\pi}{4}\right)$$

The domain of $p(t) = 3$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = -10\,\text{Heaviside}\left(t - \frac{\pi}{4}\right)\cos\left(t + \frac{\pi}{4}\right)$ is

$$\left\{t < \frac{\pi}{4} \vee \frac{\pi}{4} < t\right\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 3sY(s) - 3y(0) + 2Y(s) = \frac{10\,\mathrm{e}^{-\frac{s\pi}{4}}}{s^2+1} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 1\\ y'(0) &= 0\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 3 - s + 3sY(s) + 2Y(s) = \frac{10\,\mathrm{e}^{-\frac{s\pi}{4}}}{s^2+1}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{s^3 + 3s^2 + 10\,\mathrm{e}^{-\frac{s\pi}{4}} + s + 3}{(s^2+1)(s^2+3s+2)}$$

Taking the inverse Laplace transform gives

$$\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(\frac{s^3 + 3s^2 + 10\,\mathrm{e}^{-\frac{s\pi}{4}} + s + 3}{(s^2+1)(s^2+3s+2)}\right) \\
&= -\mathrm{e}^{-2t} + 2\,\mathrm{e}^{-t} + \frac{\sqrt{2}\left(5\sqrt{2}\,\mathrm{e}^{-t+\frac{\pi}{4}} - 2\sqrt{2}\,\mathrm{e}^{-2t+\frac{\pi}{2}} - 2\sin(t) - 4\cos(t)\right)\mathrm{Heaviside}\left(t - \frac{\pi}{4}\right)}{2}
\end{aligned}$$

Hence the final solution is

$$y = -\mathrm{e}^{-2t} + 2\,\mathrm{e}^{-t} + \frac{\sqrt{2}\left(5\sqrt{2}\,\mathrm{e}^{-t+\frac{\pi}{4}} - 2\sqrt{2}\,\mathrm{e}^{-2t+\frac{\pi}{2}} - 2\sin(t) - 4\cos(t)\right)\mathrm{Heaviside}\left(t - \frac{\pi}{4}\right)}{2}$$

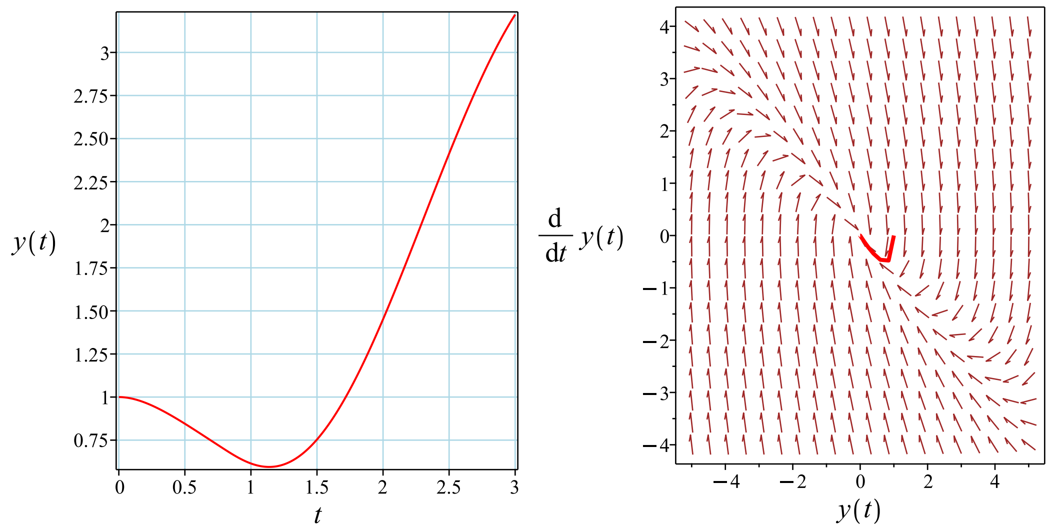
Simplifying the solution gives

$$\begin{aligned}
y = &-2\,\mathrm{Heaviside}\left(t - \frac{\pi}{4}\right)\mathrm{e}^{-2t+\frac{\pi}{2}} + 5\,\mathrm{Heaviside}\left(t - \frac{\pi}{4}\right)\mathrm{e}^{-t+\frac{\pi}{4}} \\
&- 2\left(\cos(t) + \frac{\sin(t)}{2}\right)\sqrt{2}\,\mathrm{Heaviside}\left(t - \frac{\pi}{4}\right) - \mathrm{e}^{-2t} + 2\,\mathrm{e}^{-t}
\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}
y = &-2\,\mathrm{Heaviside}\left(t - \frac{\pi}{4}\right)\mathrm{e}^{-2t+\frac{\pi}{2}} + 5\,\mathrm{Heaviside}\left(t - \frac{\pi}{4}\right)\mathrm{e}^{-t+\frac{\pi}{4}} \\
&- 2\left(\cos(t) + \frac{\sin(t)}{2}\right)\sqrt{2}\,\mathrm{Heaviside}\left(t - \frac{\pi}{4}\right) - \mathrm{e}^{-2t} + 2\,\mathrm{e}^{-t}
\end{aligned} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = -2\,\mathrm{Heaviside}\left(t-\frac{\pi}{4}\right) \mathrm{e}^{-2t+\frac{\pi}{2}} + 5\,\mathrm{Heaviside}\left(t-\frac{\pi}{4}\right) \mathrm{e}^{-t+\frac{\pi}{4}} - 2\left(\cos\left(t\right)+\frac{\sin\left(t\right)}{2}\right)\sqrt{2}\,\mathrm{Heaviside}\left(t-\frac{\pi}{4}\right) - \mathrm{e}^{-2t} + 2\,\mathrm{e}^{-t}$$

Verified OK.

14.12.2 Maple step by step solution

Let's solve

$$\left[y''+3y'+2y=-10\mathit{Heaviside}\left(t-\frac{\pi}{4}\right)\cos\left(t+\frac{\pi}{4}\right), y(0)=1, y'\Big|_{\{t=0\}}=0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2+3r+2=0$

- Factor the characteristic polynomial

  $(r+2)(r+1)=0$

- Roots of the characteristic polynomial

  $r=(-2,-1)$

- 1st solution of the homogeneous ODE

  $y_1(t) = \mathrm{e}^{-2t}$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \mathrm{e}^{-t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{-t} + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t) \left(\int \frac{y_2(t) f(t)}{W(y_1(t),y_2(t))} dt\right) + y_2(t) \left(\int \frac{y_1(t) f(t)}{W(y_1(t),y_2(t))} dt\right), f(t) = -10 \mathit{Heaviside}\left(t - \frac{\pi}{4}\right) \cos \left(t\right.\right.$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(t), y_2(t)) = \begin{bmatrix} \mathrm{e}^{-2t} & \mathrm{e}^{-t} \\ -2\,\mathrm{e}^{-2t} & -\mathrm{e}^{-t} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(t), y_2(t)) = \mathrm{e}^{-3t}$

  - Substitute functions into equation for $y_p(t)$

    $y_p(t) = 10\,\mathrm{e}^{-2t} \left(\int \mathit{Heaviside}\left(t - \frac{\pi}{4}\right) \cos\left(t + \frac{\pi}{4}\right) \mathrm{e}^{2t} dt\right) - 10\,\mathrm{e}^{-t} \left(\int \mathit{Heaviside}\left(t - \frac{\pi}{4}\right) \cos\left(t + \frac{\pi}{4}\right) \mathrm{e}^{t} d\right.$

  - Compute integrals

    $y_p(t) = \mathit{Heaviside}\left(t - \frac{\pi}{4}\right) \left(-\cos\left(t + \frac{\pi}{4}\right) - 3\sin\left(t + \frac{\pi}{4}\right) - 2\,\mathrm{e}^{-2t+\frac{\pi}{2}} + 5\,\mathrm{e}^{-t+\frac{\pi}{4}}\right)$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{-t} + \mathit{Heaviside}\left(t - \frac{\pi}{4}\right) \left(-\cos\left(t + \frac{\pi}{4}\right) - 3\sin\left(t + \frac{\pi}{4}\right) - 2\,\mathrm{e}^{-2t+\frac{\pi}{2}} + 5\,\mathrm{e}^{-t+\frac{\pi}{4}}\right)$

☐ Check validity of solution $y = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{-t} + \mathit{Heaviside}\left(t - \frac{\pi}{4}\right) \left(-\cos\left(t + \frac{\pi}{4}\right) - 3\sin\left(t + \frac{\pi}{4}\right) -\right.$

  - Use initial condition $y(0) = 1$

    $1 = c_1 + c_2$

  - Compute derivative of the solution

    $y' = -2c_1 \mathrm{e}^{-2t} - c_2 \mathrm{e}^{-t} + \mathit{Dirac}\left(t - \frac{\pi}{4}\right) \left(-\cos\left(t + \frac{\pi}{4}\right) - 3\sin\left(t + \frac{\pi}{4}\right) - 2\,\mathrm{e}^{-2t+\frac{\pi}{2}} + 5\,\mathrm{e}^{-t+\frac{\pi}{4}}\right) + \mathit{He}$

  - Use the initial condition $y'\Big|_{\{t=0\}} = 0$

$$0 = -2c_1 - c_2$$

- Solve for c_1 and c_2

  $$\{c_1 = -1, c_2 = 2\}$$

- Substitute constant values into general solution and simplify

  $$y = \left(-\cos\left(t + \frac{\pi}{4}\right) - 3\sin\left(t + \frac{\pi}{4}\right)\right) Heaviside\left(t - \frac{\pi}{4}\right) - 2Heaviside\left(t - \frac{\pi}{4}\right) e^{-2t+\frac{\pi}{2}} + 5Heaviside($$

- Solution to the IVP

  $$y = \left(-\cos\left(t + \frac{\pi}{4}\right) - 3\sin\left(t + \frac{\pi}{4}\right)\right) Heaviside\left(t - \frac{\pi}{4}\right) - 2Heaviside\left(t - \frac{\pi}{4}\right) e^{-2t+\frac{\pi}{2}} + 5Heaviside($$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 2.375 (sec). Leaf size: 63

```
dsolve([diff(y(t),t$2)+3*diff(y(t),t)+2*y(t)=10*Heaviside(t-Pi/4)*sin(t-Pi/4),y(0) = 1, D(y)
```

$$y(t) = -2\,\mathrm{Heaviside}\left(t - \frac{\pi}{4}\right) e^{\frac{\pi}{2}-2t} + 5\,\mathrm{Heaviside}\left(t - \frac{\pi}{4}\right) e^{-t+\frac{\pi}{4}} - 2\sqrt{2}\left(\cos(t) + \frac{\sin(t)}{2}\right) \mathrm{Heaviside}\left(t - \frac{\pi}{4}\right) - e^{-2t} + 2\,e^{-t}$$

✓ Solution by Mathematica

Time used: 0.143 (sec). Leaf size: 87

```
DSolve[{y''[t]+3*y'[t]+2*y[t]==10*UnitStep[t-Pi/4]*Sin[t-Pi/4],{y[0]==1,y'[0]==0}},y[t],t,In
```

$$y(t) \to \begin{cases} e^{-2t}(-1 + 2e^t) & 4t \leq \pi \\ -e^{-2t}\left(2\sqrt{2}e^{2t}\cos(t) - 2e^t - 5e^{t+\frac{\pi}{4}} + \sqrt{2}e^{2t}\sin(t) + 2e^{\pi/2} + 1\right) & \text{True} \end{cases}$$

14.13 problem Problem 39

Internal problem ID [2879]
Internal file name [`OUTPUT/2371_Sunday_June_05_2022_03_01_56_AM_85053664/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 10, The Laplace Transform and Some Elementary Applications. Exercises for 10.7. page 704
Problem number: Problem 39.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' + y' - 6y = 30 \,\mathrm{Heaviside}\left(t-1\right) \mathrm{e}^{1-t}}$$

With initial conditions

$$[y(0) = 3, y'(0) = -4]$$

14.13.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned}
p(t) &= 1 \\
q(t) &= -6 \\
F &= 30 \,\mathrm{Heaviside}\left(t-1\right) \mathrm{e}^{1-t}
\end{aligned}$$

Hence the ode is

$$y'' + y' - 6y = 30\,\text{Heaviside}\,(t-1)\,\mathrm{e}^{1-t}$$

The domain of $p(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = -6$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 30\,\text{Heaviside}\,(t-1)\,\mathrm{e}^{1-t}$ is

$$\{t < 1 \lor 1 < t\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + sY(s) - y(0) - 6Y(s) = \frac{30\,\mathrm{e}^{-s}}{1+s} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 3\\ y'(0) &= -4\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + 1 - 3s + sY(s) - 6Y(s) = \frac{30\,\mathrm{e}^{-s}}{1+s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{3s^2 + 30\,\mathrm{e}^{-s} + 2s - 1}{(1+s)\,(s^2+s-6)}$$

Taking the inverse Laplace transform gives

$$
\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(\frac{3s^2+30\,\mathrm{e}^{-s}+2s-1}{(1+s)\,(s^2+s-6)}\right) \\
&= 2\,\mathrm{e}^{-3t}+\mathrm{e}^{2t}+2(1-\text{Heaviside}\,(1-t))\,\mathrm{e}^{2t-2}+\left(-5\,\mathrm{e}^{1-t}+3\,\mathrm{e}^{-3t+3}\right)\text{Heaviside}\,(t-1)
\end{aligned}
$$

Hence the final solution is

$$
y = 2\,\mathrm{e}^{-3t}+\mathrm{e}^{2t}+2(1-\text{Heaviside}\,(1-t))\,\mathrm{e}^{2t-2}+\left(-5\,\mathrm{e}^{1-t}+3\,\mathrm{e}^{-3t+3}\right)\text{Heaviside}\,(t-1)
$$

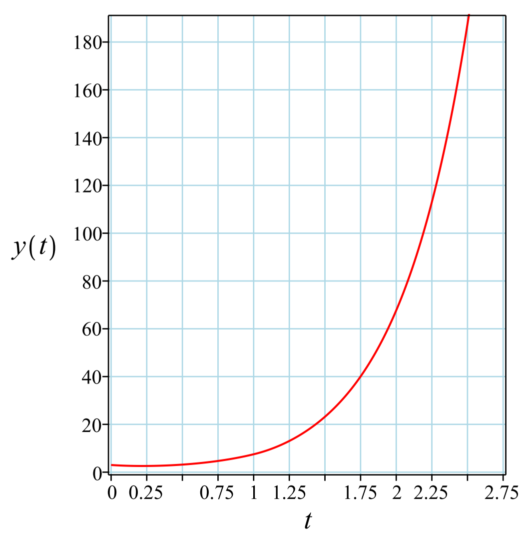
Simplifying the solution gives

$$
y = \left(-5\,\mathrm{e}^{2t+1}\,\text{Heaviside}\,(t-1)+3\,\mathrm{e}^{3}\,\text{Heaviside}\,(t-1)+2\,\mathrm{e}^{5t-2}\,\text{Heaviside}\,(t-1)+\mathrm{e}^{5t}+2\right)\mathrm{e}^{-3t}
$$

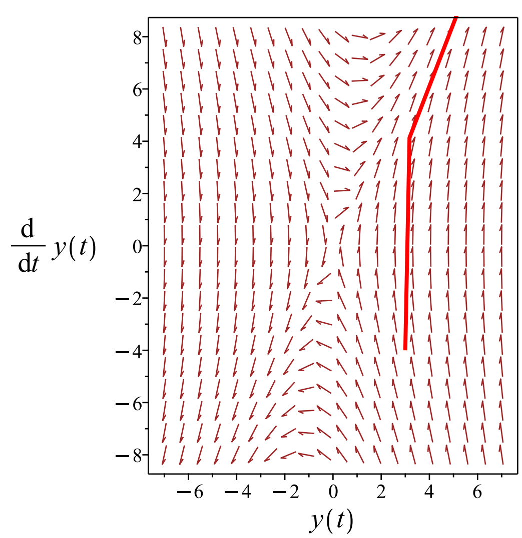
Summary

The solution(s) found are the following

$$
y = \left(-5\,\mathrm{e}^{2t+1}\,\text{Heaviside}\,(t-1)+3\,\mathrm{e}^{3}\,\text{Heaviside}\,(t-1)+2\,\mathrm{e}^{5t-2}\,\text{Heaviside}\,(t-1)+\mathrm{e}^{5t}+2\right)\mathrm{e}^{-3t} \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \left(-5\,\mathrm{e}^{2t+1}\,\mathrm{Heaviside}\,(t-1) + 3\,\mathrm{e}^{3}\,\mathrm{Heaviside}\,(t-1) + 2\,\mathrm{e}^{5t-2}\,\mathrm{Heaviside}\,(t-1) + \mathrm{e}^{5t} + 2\right)\mathrm{e}^{-3t}$$

Verified OK.

14.13.2 Maple step by step solution

Let's solve

$$\left[y'' + y' - 6y = 30 \mathit{Heaviside}(t-1)\,\mathrm{e}^{1-t}, y(0) = 3, y'\Big|_{\{t=0\}} = -4\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + r - 6 = 0$

- Factor the characteristic polynomial

  $(r+3)\,(r-2) = 0$

- Roots of the characteristic polynomial

  $r = (-3, 2)$

- 1st solution of the homogeneous ODE

  $y_1(t) = \mathrm{e}^{-3t}$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \mathrm{e}^{2t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-3t} + c_2 \mathrm{e}^{2t} + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = 30\mathit{Heaviside}(t-1)\,\mathrm{e}^{1-t}\right]$

  - Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \begin{bmatrix} e^{-3t} & e^{2t} \\ -3\,e^{-3t} & 2\,e^{2t} \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(t), y_2(t)) = 5\,e^{-t}$

- Substitute functions into equation for $y_p(t)$

  $y_p(t) = -6\left(-e^{5t}\left(\int \mathit{Heaviside}(t-1)\,e^{-3t+1}dt\right) + \int e^{2t+1}\mathit{Heaviside}(t-1)\,dt\right)e^{-3t}$

- Compute integrals

  $y_p(t) = \mathit{Heaviside}(t-1)\,e^{-3t}(-5\,e^{2t+1} + 3\,e^3 + 2\,e^{5t-2})$

- Substitute particular solution into general solution to ODE

  $y = c_1e^{-3t} + c_2e^{2t} + \mathit{Heaviside}(t-1)\,e^{-3t}(-5\,e^{2t+1} + 3\,e^3 + 2\,e^{5t-2})$

□ Check validity of solution $y = c_1e^{-3t} + c_2e^{2t} + \mathit{Heaviside}(t-1)\,e^{-3t}(-5e^{2t+1} + 3e^3 + 2e^{5t-2})$

- Use initial condition $y(0) = 3$

  $3 = c_1 + c_2$

- Compute derivative of the solution

  $y' = -3c_1e^{-3t} + 2c_2e^{2t} + \mathit{Dirac}(t-1)\,e^{-3t}(-5\,e^{2t+1} + 3\,e^3 + 2\,e^{5t-2}) - 3\mathit{Heaviside}(t-1)\,e^{-3t}(-5$

- Use the initial condition $y'\Big|_{\{t=0\}} = -4$

  $-4 = -3c_1 + 2c_2$

- Solve for c_1 and c_2

  $\{c_1 = 2, c_2 = 1\}$

- Substitute constant values into general solution and simplify

  $y = (-5\,e^{2t+1}\mathit{Heaviside}(t-1) + 3\,e^3\mathit{Heaviside}(t-1) + 2\,e^{5t-2}\mathit{Heaviside}(t-1) + e^{5t} + 2)\,e^{-3t}$

- Solution to the IVP

  $y = (-5\,e^{2t+1}\mathit{Heaviside}(t-1) + 3\,e^3\mathit{Heaviside}(t-1) + 2\,e^{5t-2}\mathit{Heaviside}(t-1) + e^{5t} + 2)\,e^{-3t}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 2.265 (sec). Leaf size: 55

```
dsolve([diff(y(t),t$2)+diff(y(t),t)-6*y(t)=30*Heaviside(t-1)*exp(-(t-1)),y(0) = 3, D(y)(0) =
```

$$y(t) = \left(-5\,\mathrm{Heaviside}\,(t-1)\,\mathrm{e}^{1+2t} + 3\,\mathrm{Heaviside}\,(t-1)\,\mathrm{e}^{3} + 2\,\mathrm{e}^{-2+5t}\,\mathrm{Heaviside}\,(t-1) + \mathrm{e}^{5t} + 2\right)\mathrm{e}^{-3t}$$

✓ Solution by Mathematica

Time used: 0.087 (sec). Leaf size: 66

```
DSolve[{y''[t]+y'[t]-6*y[t]==30*UnitStep[t-1]*Exp[-(t-1)],{y[0]==3,y'[0]==-4}},y[t],t,Includ
```

$$y(t) \to \begin{cases} e^{-3t}(2+e^{5t}) & t \leq 1 \\ e^{-3t-2}(2e^2+3e^5+2e^{5t}-5e^{2t+3}+e^{5t+2}) & \text{True} \end{cases}$$

14.14 problem Problem 40

Internal problem ID [2880]
Internal file name [`OUTPUT/2372_Sunday_June_05_2022_03_02_03_AM_30140179/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 10, The Laplace Transform and Some Elementary Applications. Exercises for 10.7. page 704
Problem number: Problem 40.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 4y' + 5y = 5\,\mathrm{Heaviside}\,(t - 3)}$$

With initial conditions

$$[y(0) = 2, y'(0) = 1]$$

14.14.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 4 \\ q(t) &= 5 \\ F &= 5\,\mathrm{Heaviside}\,(t - 3) \end{aligned}$$

Hence the ode is

$$y'' + 4y' + 5y = 5\,\text{Heaviside}\,(t-3)$$

The domain of $p(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 5$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 5\,\text{Heaviside}\,(t-3)$ is

$$\{t < 3 \lor 3 < t\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 4sY(s) - 4y(0) + 5Y(s) = \frac{5\,\mathrm{e}^{-3s}}{s} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 2\\ y'(0) &= 1\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 9 - 2s + 4sY(s) + 5Y(s) = \frac{5\,\mathrm{e}^{-3s}}{s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{2s^2 + 5\,\mathrm{e}^{-3s} + 9s}{s\,(s^2 + 4s + 5)}$$

Taking the inverse Laplace transform gives

$$
\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(\frac{2s^2 + 5\,\mathrm{e}^{-3s} + 9s}{s\,(s^2 + 4s + 5)}\right) \\
&= \mathrm{e}^{-2t}(2\cos(t) + 5\sin(t)) + \left(\frac{1}{10} + \frac{i}{5}\right)\left(2 - 4i - 5\,\mathrm{e}^{(-2-i)(t-3)} + (3+4i)\,\mathrm{e}^{(-2+i)(t-3)}\right) \mathrm{Heaviside}(t-3)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= \mathrm{e}^{-2t}(2\cos(t) + 5\sin(t)) \\
&\quad + \left(\frac{1}{10} + \frac{i}{5}\right)\left(2 - 4i - 5\,\mathrm{e}^{(-2-i)(t-3)} + (3+4i)\,\mathrm{e}^{(-2+i)(t-3)}\right) \mathrm{Heaviside}(t-3)
\end{aligned}
$$

Simplifying the solution gives

$$
\begin{aligned}
y &= \left(-\frac{1}{2} - i\right) \mathrm{Heaviside}(t-3)\,\mathrm{e}^{(-2-i)(t-3)} + \left(-\frac{1}{2} + i\right) \mathrm{Heaviside}(t-3)\,\mathrm{e}^{(-2+i)(t-3)} \\
&\quad + \mathrm{Heaviside}(t-3) + \mathrm{e}^{-2t}(2\cos(t) + 5\sin(t))
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= \left(-\frac{1}{2} - i\right) \mathrm{Heaviside}(t-3)\,\mathrm{e}^{(-2-i)(t-3)} + \left(-\frac{1}{2} + i\right) \mathrm{Heaviside}(t-3)\,\mathrm{e}^{(-2+i)(t-3)} \\
&\quad + \mathrm{Heaviside}(t-3) + \mathrm{e}^{-2t}(2\cos(t) + 5\sin(t))
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= \left(-\frac{1}{2} - i\right) \mathrm{Heaviside}(t-3)\,\mathrm{e}^{(-2-i)(t-3)} + \left(-\frac{1}{2} + i\right) \mathrm{Heaviside}(t-3)\,\mathrm{e}^{(-2+i)(t-3)} \\
&\quad + \mathrm{Heaviside}(t-3) + \mathrm{e}^{-2t}(2\cos(t) + 5\sin(t))
\end{aligned}
$$

Verified OK.

14.14.2 Maple step by step solution

Let's solve

$$\left[y''+4y'+5y=5\mathit{Heaviside}(t-3)\,,y(0)=2,y'\Big|_{\{t=0\}}=1\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2+4r+5=0$

- Use quadratic formula to solve for r

  $r=\frac{(-4)\pm(\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r=(-2-\mathrm{I},-2+\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t)=\cos(t)\,\mathrm{e}^{-2t}$

- 2nd solution of the homogeneous ODE

  $y_2(t)=\sin(t)\,\mathrm{e}^{-2t}$

- General solution of the ODE

  $y=c_1y_1(t)+c_2y_2(t)+y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y=c_1\cos(t)\,\mathrm{e}^{-2t}+c_2\sin(t)\,\mathrm{e}^{-2t}+y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t)=-y_1(t)\left(\int\frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right)+y_2(t)\left(\int\frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right),f(t)=5\mathit{Heaviside}(t-3)\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(t)\,,y_2(t))=\begin{bmatrix}\cos(t)\,\mathrm{e}^{-2t} & \sin(t)\,\mathrm{e}^{-2t}\\ -\sin(t)\,\mathrm{e}^{-2t}-2\cos(t)\,\mathrm{e}^{-2t} & \cos(t)\,\mathrm{e}^{-2t}-2\sin(t)\,\mathrm{e}^{-2t}\end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(t)\,,y_2(t))=\mathrm{e}^{-4t}$

- ○ Substitute functions into equation for $y_p(t)$

$y_p(t) = -5\,\mathrm{e}^{-2t}\left(\cos(t)\left(\int \mathit{Heaviside}(t-3)\sin(t)\,\mathrm{e}^{2t}dt\right) - \sin(t)\left(\int \mathit{Heaviside}(t-3)\cos(t)\,\mathrm{e}^{2t}dt\right)\right.$

- Compute integrals

  $y_p(t) = -\mathit{Heaviside}(t-3)\left(-1+\left((\cos(t)+2\sin(t))\cos(3)-2\sin(3)\left(\cos(t)-\frac{\sin(t)}{2}\right)\right)\mathrm{e}^{-2t+6}\right.$

- Substitute particular solution into general solution to ODE

  $y = c_1\cos(t)\,\mathrm{e}^{-2t} + c_2\sin(t)\,\mathrm{e}^{-2t} - \mathit{Heaviside}(t-3)\left(-1+\left((\cos(t)+2\sin(t))\cos(3)-2\sin(3\right.\right.$

☐ Check validity of solution $y = c_1\cos(t)\,\mathrm{e}^{-2t} + c_2\sin(t)\,\mathrm{e}^{-2t} - \mathit{Heaviside}(t-3)\left(-1+\left((\cos(t)\right.\right.$

- Use initial condition $y(0) = 2$

  $2 = c_1$

- Compute derivative of the solution

  $y' = -c_1\sin(t)\,\mathrm{e}^{-2t} - 2c_1\cos(t)\,\mathrm{e}^{-2t} + c_2\cos(t)\,\mathrm{e}^{-2t} - 2c_2\sin(t)\,\mathrm{e}^{-2t} - \mathit{Dirac}(t-3)\left(-1+\left((\mathrm{co}\right.\right.$

- Use the initial condition $y'\Big|_{\{t=0\}} = 1$

  $1 = -2c_1 + c_2$

- Solve for c_1 and c_2

  $\{c_1 = 2, c_2 = 5\}$

- Substitute constant values into general solution and simplify

  $y = -\left((\cos(3)-2\sin(3))\cos(t)+2\left(\cos(3)+\frac{\sin(3)}{2}\right)\sin(t)\right)\mathit{Heaviside}(t-3)\,\mathrm{e}^{-2t+6} + \mathit{Heavisi}$

- Solution to the IVP

  $y = -\left((\cos(3)-2\sin(3))\cos(t)+2\left(\cos(3)+\frac{\sin(3)}{2}\right)\sin(t)\right)\mathit{Heaviside}(t-3)\,\mathrm{e}^{-2t+6} + \mathit{Heavisi}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 4.297 (sec). Leaf size: 53

```
dsolve([diff(y(t),t$2)+4*diff(y(t),t)+5*y(t)=5*Heaviside(t-3),y(0) = 2, D(y)(0) = 1],y(t), s
```

$$
\begin{aligned}
y(t) = &\left(-\frac{1}{2}-i\right) \operatorname{Heaviside}(-3+t) \mathrm{e}^{(-2-i)(-3+t)} \\
&+\left(-\frac{1}{2}+i\right) \operatorname{Heaviside}(-3+t) \mathrm{e}^{(-2+i)(-3+t)} \\
&+\operatorname{Heaviside}(-3+t)+\mathrm{e}^{-2t}(2\cos(t)+5\sin(t))
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.037 (sec). Leaf size: 68

```
DSolve[{y''[t]+4*y'[t]+5*y[t]==5*UnitStep[t-3],{y[0]==2,y'[0]==1}},y[t],t,IncludeSingularSol
```

$$
y(t) \rightarrow \begin{cases} e^{-2t}(2\cos(t)+5\sin(t)) & t \leq 3 \\ e^{-2t}(-e^{6}\cos(3-t)+e^{2t}+2\cos(t)+2e^{6}\sin(3-t)+5\sin(t)) & \text{True} \end{cases}
$$

14.15 problem Problem 41

Internal problem ID [2881]
Internal file name [`OUTPUT/2373_Sunday_June_05_2022_03_02_12_AM_772570/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 10, The Laplace Transform and Some Elementary Applications. Exercises for 10.7. page 704
Problem number: Problem 41.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 2y' + 5y = 2\sin(t) + \text{Heaviside}\left(t - \frac{\pi}{2}\right)(1 + \cos(t))}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

14.15.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned}
p(t) &= -2 \\
q(t) &= 5 \\
F &= \text{Heaviside}\left(t - \frac{\pi}{2}\right)\cos(t) + 2\sin(t) + \text{Heaviside}\left(t - \frac{\pi}{2}\right)
\end{aligned}$$

Hence the ode is

$$y'' - 2y' + 5y = \text{Heaviside}\left(t - \frac{\pi}{2}\right)\cos(t) + 2\sin(t) + \text{Heaviside}\left(t - \frac{\pi}{2}\right)$$

The domain of $p(t) = -2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 5$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \text{Heaviside}\left(t - \frac{\pi}{2}\right)\cos(t) + 2\sin(t) + \text{Heaviside}\left(t - \frac{\pi}{2}\right)$ is

$$\left\{t < \frac{\pi}{2} \vee \frac{\pi}{2} < t\right\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) - 2sY(s) + 2y(0) + 5Y(s) = \frac{e^{-\frac{s\pi}{2}}}{s} + \frac{2 - e^{-\frac{s\pi}{2}}}{s^2 + 1} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 0\\ y'(0) &= 0\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 2sY(s) + 5Y(s) = \frac{e^{-\frac{s\pi}{2}}}{s} + \frac{2 - e^{-\frac{s\pi}{2}}}{s^2 + 1}$$

Solving the above equation for $Y(s)$ results in

$$Y(s)=\frac{\mathrm{e}^{-\frac{s\pi}{2}} s^{2}-s\,\mathrm{e}^{-\frac{s\pi}{2}}+\mathrm{e}^{-\frac{s\pi}{2}}+2s}{s\left(s^{2}+1\right)\left(s^{2}-2s+5\right)}$$

Taking the inverse Laplace transform gives

$$\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s))\\
&= \mathcal{L}^{-1}\left(\frac{\mathrm{e}^{-\frac{s\pi}{2}} s^{2}-s\,\mathrm{e}^{-\frac{s\pi}{2}}+\mathrm{e}^{-\frac{s\pi}{2}}+2s}{s\left(s^{2}+1\right)\left(s^{2}-2s+5\right)}\right)\\
&= \frac{\cos(t)}{5}+\frac{2\sin(t)}{5}-\frac{\mathrm{e}^{t}(2\cos(2t)+\sin(2t))}{10}+\frac{(-2\cos(2t)+3\sin(2t))\left(\mathrm{Heaviside}\left(\frac{\pi}{2}-t\right)-1\right)\mathrm{e}^{t-\frac{\pi}{2}}}{20}
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y = {} & \frac{\cos(t)}{5}+\frac{2\sin(t)}{5}-\frac{\mathrm{e}^{t}(2\cos(2t)+\sin(2t))}{10}\\
&+\frac{(-2\cos(2t)+3\sin(2t))\left(\mathrm{Heaviside}\left(\frac{\pi}{2}-t\right)-1\right)\mathrm{e}^{t-\frac{\pi}{2}}}{20}\\
&+\frac{\mathrm{Heaviside}\left(t-\frac{\pi}{2}\right)(-\sin(t)+2\cos(t)+2)}{10}
\end{aligned}$$

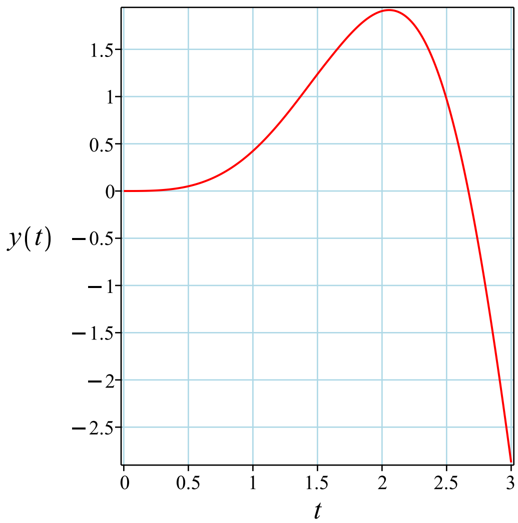
Simplifying the solution gives

$$\begin{aligned}
y = {} & \frac{\left(\mathrm{e}^{t-\frac{\pi}{2}}\left(2\cos(t)^{2}-3\sin(t)\cos(t)-1\right)+2\cos(t)-\sin(t)+2\right)\mathrm{Heaviside}\left(t-\frac{\pi}{2}\right)}{10}\\
&-\frac{2\cos(t)^{2}\mathrm{e}^{t}}{5}-\frac{\sin(t)\cos(t)\mathrm{e}^{t}}{5}+\frac{\cos(t)}{5}+\frac{\mathrm{e}^{t}}{5}+\frac{2\sin(t)}{5}
\end{aligned}$$

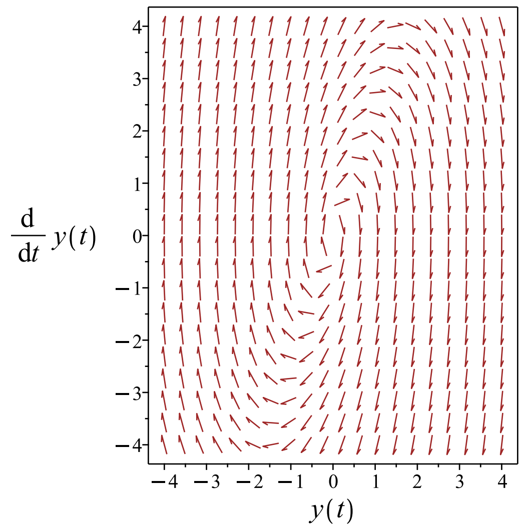
Summary

The solution(s) found are the following

$$\begin{aligned}
y = {} & \frac{\left(\mathrm{e}^{t-\frac{\pi}{2}}\left(2\cos(t)^{2}-3\sin(t)\cos(t)-1\right)+2\cos(t)-\sin(t)+2\right)\mathrm{Heaviside}\left(t-\frac{\pi}{2}\right)}{10}\\
&-\frac{2\cos(t)^{2}\mathrm{e}^{t}}{5}-\frac{\sin(t)\cos(t)\mathrm{e}^{t}}{5}+\frac{\cos(t)}{5}+\frac{\mathrm{e}^{t}}{5}+\frac{2\sin(t)}{5}
\end{aligned} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{\left(e^{t-\frac{\pi}{2}}\left(2\cos(t)^2 - 3\sin(t)\cos(t) - 1\right) + 2\cos(t) - \sin(t) + 2\right)\text{Heaviside}\left(t - \frac{\pi}{2}\right)}{10} - \frac{2\cos(t)^2 e^t}{5} - \frac{\sin(t)\cos(t)e^t}{5} + \frac{\cos(t)}{5} + \frac{e^t}{5} + \frac{2\sin(t)}{5}$$

Verified OK.

14.15.2 Maple step by step solution

Let's solve

$$\left[y'' - 2y' + 5y = \mathit{Heaviside}\left(t - \frac{\pi}{2}\right)\cos(t) + 2\sin(t) + \mathit{Heaviside}\left(t - \frac{\pi}{2}\right), y(0) = 0, y'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 2r + 5 = 0$

- Use quadratic formula to solve for r

  $r = \frac{2\pm(\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

$r = (1-2\mathrm{I}, 1+2\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t) = \cos(2t)\,\mathrm{e}^t$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \sin(2t)\,\mathrm{e}^t$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(2t)\,\mathrm{e}^t + c_2 \sin(2t)\,\mathrm{e}^t + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = \mathit{Heaviside}\left(t-\frac{\pi}{2}\right)\cos(t) + 2\right.$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(t), y_2(t)) = \begin{bmatrix} \cos(2t)\,\mathrm{e}^t & \sin(2t)\,\mathrm{e}^t \\ -2\sin(2t)\,\mathrm{e}^t + \cos(2t)\,\mathrm{e}^t & 2\cos(2t)\,\mathrm{e}^t + \sin(2t)\,\mathrm{e}^t \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(t), y_2(t)) = 2\,\mathrm{e}^{2t}$

  - Substitute functions into equation for $y_p(t)$

    $y_p(t) = \frac{\mathrm{e}^t\left(\sin(2t)\left(\int \cos(2t)\left(\mathit{Heaviside}\left(t-\frac{\pi}{2}\right)\cos(t)+2\sin(t)+\mathit{Heaviside}\left(t-\frac{\pi}{2}\right)\right)\mathrm{e}^{-t}dt\right)-2\cos(2t)\left(\int \mathrm{e}^{-t}\sin(t)\cos(t)\left(\mathit{Heaviside}\left(t-\right.\right.\right.\right.}{2}$

  - Compute integrals

    $y_p(t) = \frac{\left(\mathrm{e}^{t-\frac{\pi}{2}}\left(2\cos(t)^2-3\sin(t)\cos(t)-1\right)+2\cos(t)-\sin(t)+2\right)\mathit{Heaviside}\left(t-\frac{\pi}{2}\right)}{10} + \frac{\cos(t)}{5} + \frac{2\sin(t)}{5}$

- Substitute particular solution into general solution to ODE

  $y = c_1\cos(2t)\,\mathrm{e}^t + c_2\sin(2t)\,\mathrm{e}^t + \frac{\left(\mathrm{e}^{t-\frac{\pi}{2}}\left(2\cos(t)^2-3\sin(t)\cos(t)-1\right)+2\cos(t)-\sin(t)+2\right)\mathit{Heaviside}\left(t-\frac{\pi}{2}\right)}{10} + \frac{\cos(t)}{5}$

☐ Check validity of solution $y = c_1\cos(2t)\,\mathrm{e}^t + c_2\sin(2t)\,\mathrm{e}^t + \frac{\left(\mathrm{e}^{t-\frac{\pi}{2}}\left(2\cos(t)^2-3\sin(t)\cos(t)-1\right)+2\cos(t)-\sin\right.}{10}$

  - Use initial condition $y(0) = 0$

    $0 = c_1 + \frac{1}{5}$

  - Compute derivative of the solution

$$y' = -2c_1 \sin(2t)\, e^t + c_1 \cos(2t)\, e^t + 2c_2 \cos(2t)\, e^t + c_2 \sin(2t)\, e^t + \frac{\left(e^{t-\frac{\pi}{2}}\left(2\cos(t)^2 - 3\sin(t)\cos(t) - 1\right) + e\right.}{}$$

- Use the initial condition $y'\Big|_{\{t=0\}} = 0$

$$0 = \frac{2}{5} + c_1 + 2c_2$$

- Solve for c_1 and c_2

$$\left\{c_1 = -\frac{1}{5}, c_2 = -\frac{1}{10}\right\}$$

- Substitute constant values into general solution and simplify

$$y = \frac{\left(e^{t-\frac{\pi}{2}}\left(2\cos(t)^2 - 3\sin(t)\cos(t) - 1\right) + 2\cos(t) - \sin(t) + 2\right) \mathit{Heaviside}\left(t - \frac{\pi}{2}\right)}{10} - \frac{2\cos(t)^2 e^t}{5} - \frac{\sin(t)\cos(t) e^t}{5} + \frac{\cos(t)}{5} + \frac{e^t}{5}$$

- Solution to the IVP

$$y = \frac{\left(e^{t-\frac{\pi}{2}}\left(2\cos(t)^2 - 3\sin(t)\cos(t) - 1\right) + 2\cos(t) - \sin(t) + 2\right) \mathit{Heaviside}\left(t - \frac{\pi}{2}\right)}{10} - \frac{2\cos(t)^2 e^t}{5} - \frac{\sin(t)\cos(t) e^t}{5} + \frac{\cos(t)}{5} + \frac{e^t}{5}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 2.515 (sec). Leaf size: 77

```
dsolve([diff(y(t),t$2)-2*diff(y(t),t)+5*y(t)=2*sin(t)+Heaviside(t-Pi/2)*(1-sin(t-Pi/2)),y(0)
```

$$y(t) = \frac{\left(\left(2\cos(t)^2 - 3\cos(t)\sin(t) - 1\right) e^{t-\frac{\pi}{2}} + 2\cos(t) - \sin(t) + 2\right) \mathrm{Heaviside}\left(t - \frac{\pi}{2}\right)}{10} - \frac{2 e^t \cos(t)^2}{5} - \frac{\sin(t)\cos(t) e^t}{5} + \frac{\cos(t)}{5} + \frac{e^t}{5} + \frac{2\sin(t)}{5}$$

✓ Solution by Mathematica

Time used: 0.502 (sec). Leaf size: 98

```
DSolve[{y''[t]-2*y'[t]+5*y[t]==2*Sin[t]+UnitStep[t-Pi/2]*(1-Sin[t-Pi/2]),{y[0]==0,y'[0]==0}}
```

$y(t)$

$$\rightarrow \begin{cases} \frac{1}{5}(-e^t\sin(t)\cos(t)+\cos(t)-e^t\cos(2t)+2\sin(t)) & 2t\leq\pi \\ \frac{1}{20}\left(8\cos(t)+2e^t\left(-2+e^{-\pi/2}\right)\cos(2t)+6\sin(t)-2e^t\sin(2t)-3e^{t-\frac{\pi}{2}}\sin(2t)+4\right) & \text{True} \end{cases}$$

14.16 problem Problem 46 part a

Internal problem ID [2882]
Internal file name [`OUTPUT/2374_Sunday_June_05_2022_03_02_20_AM_82997815/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 10, The Laplace Transform and Some Elementary Applications. Exercises for 10.7. page 704
Problem number: Problem 46 part a.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' - y = \begin{cases} 2 & 0 \le t < 1 \\ -1 & 1 \le t \end{cases}}$$

With initial conditions

$$[y(0) = 1]$$

14.16.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$
\begin{aligned}
p(t) &= -1 \\
q(t) &= \begin{cases} 0 & t<0 \\ 2 & t<1 \\ -1 & 1\le t \end{cases}
\end{aligned}
$$

Hence the ode is

$$
y' - y = \begin{cases} 0 & t<0 \\ 2 & t<1 \\ -1 & 1\le t \end{cases}
$$

The domain of $p(t) = -1$ is

$$
\{-\infty < t < \infty\}
$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = \begin{cases} 0 & t<0 \\ 2 & t<1 \\ -1 & 1\le t \end{cases}$ is

$$
\{0 \le t \le 1, 1 \le t \le \infty, -\infty \le t \le 0\}
$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

14.16.2 Solving as laplace ode

Solving using the Laplace transform method. Let

$$
\mathcal{L}(y) = Y(s)
$$

Taking the Laplace transform of the ode and using the relations that

$$
\mathcal{L}(y') = sY(s) - y(0)
$$

The given ode now becomes an algebraic equation in the Laplace domain

$$
sY(s) - y(0) - Y(s) = \frac{2 - 3\,\mathrm{e}^{-s}}{s} \tag{1}
$$

Replacing initial condition gives

$$sY(s) - 1 - Y(s) = \frac{2 - 3\,e^{-s}}{s}$$

Solving for $Y(s)$ gives

$$Y(s) = -\frac{-2 + 3\,e^{-s} - s}{s\,(s-1)}$$

Taking the inverse Laplace transform gives

$$\begin{aligned} y &= \mathcal{L}^{-1}(Y(s)) \\ &= \mathcal{L}^{-1}\left(-\frac{-2 + 3\,e^{-s} - s}{s\,(s-1)}\right) \\ &= -2 + 3\,\text{Heaviside}\,(t-1) + 3\,e^{t} + 3\,e^{t-1}(-1 + \text{Heaviside}\,(1-t)) \end{aligned}$$

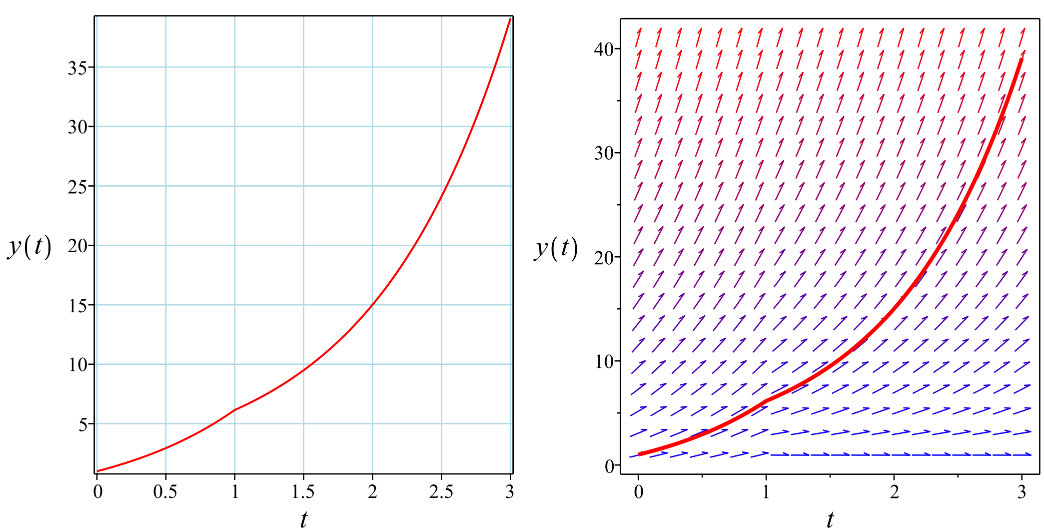
Hence the final solution is

$$y = -2 + 3\,\text{Heaviside}\,(t-1) + 3\,e^{t} + 3\,e^{t-1}(-1 + \text{Heaviside}\,(1-t))$$

Summary

The solution(s) found are the following

$$y = -2 + 3\,\text{Heaviside}\,(t-1) + 3\,e^{t} + 3\,e^{t-1}(-1 + \text{Heaviside}\,(1-t)) \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = -2 + 3\,\text{Heaviside}\,(t-1) + 3\,e^{t} + 3\,e^{t-1}(-1 + \text{Heaviside}\,(1-t))$$

Verified OK.

14.16.3 Maple step by step solution

Let's solve

$$\left[y'-y=\left\{\begin{array}{cc}2 & 0\le t<1\\ -1 & 1\le t\end{array}\right., y(0)=1\right]$$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $$y'=y+\left\{\begin{array}{cc}2 & 0\le t<1\\ -1 & 1\le t\end{array}\right.$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $$y'-y=\left\{\begin{array}{cc}2 & 0\le t<1\\ -1 & 1\le t\end{array}\right.$$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $$\mu(t)\,(y'-y)=\mu(t)\left(\left\{\begin{array}{cc}2 & 0\le t<1\\ -1 & 1\le t\end{array}\right.\right)$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,y)$

  $\mu(t)\,(y'-y)=\mu'(t)\,y+\mu(t)\,y'$

- Isolate $\mu'(t)$

  $\mu'(t)=-\mu(t)$

- Solve to find the integrating factor

  $\mu(t)=\mathrm{e}^{-t}$

- Integrate both sides with respect to t

  $$\int\left(\frac{d}{dt}(\mu(t)\,y)\right)dt=\int\mu(t)\left(\left\{\begin{array}{cc}2 & 0\le t<1\\ -1 & 1\le t\end{array}\right.\right)dt+c_1$$

- Evaluate the integral on the lhs

  $$\mu(t)\,y=\int\mu(t)\left(\left\{\begin{array}{cc}2 & 0\le t<1\\ -1 & 1\le t\end{array}\right.\right)dt+c_1$$

- Solve for y

$$y = \frac{\int \mu(t) \left(\left\{\begin{array}{cc} 2 & 0 \le t < 1 \\ -1 & 1 \le t \end{array}\right.\right) dt + c_1}{\mu(t)}$$

- Substitute $\mu(t) = \mathrm{e}^{-t}$

$$y = \frac{\int \mathrm{e}^{-t} \left(\left\{\begin{array}{cc} 2 & 0 \le t < 1 \\ -1 & 1 \le t \end{array}\right.\right) dt + c_1}{\mathrm{e}^{-t}}$$

- Evaluate the integrals on the rhs

$$y = \frac{\left\{\begin{array}{cc} 0 & t \le 0 \\ -2\,\mathrm{e}^{-t} + 2 & t \le 1 \\ \mathrm{e}^{-t} - 3\,\mathrm{e}^{-1} + 2 & 1 < t \end{array}\right. + c_1}{\mathrm{e}^{-t}}$$

- Simplify

$$y = \mathrm{e}^{t} \left(\left\{\begin{array}{cc} 0 & t \le 0 \\ -2\,\mathrm{e}^{-t} + 2 & t \le 1 \\ \mathrm{e}^{-t} - 3\,\mathrm{e}^{-1} + 2 & 1 < t \end{array}\right. + c_1\right)$$

- Use initial condition $y(0) = 1$

$1 = c_1$

- Solve for c_1

$c_1 = 1$

- Substitute $c_1 = 1$ into general solution and simplify

$$y = \left\{\begin{array}{cc} \mathrm{e}^{t} & t \le 0 \\ -2 + 3\,\mathrm{e}^{t} & t \le 1 \\ 1 + 3\,\mathrm{e}^{t} - 3\,\mathrm{e}^{t-1} & 1 < t \end{array}\right.$$

- Solution to the IVP

$$y = \left\{\begin{array}{cc} \mathrm{e}^{t} & t \le 0 \\ -2 + 3\,\mathrm{e}^{t} & t \le 1 \\ 1 + 3\,\mathrm{e}^{t} - 3\,\mathrm{e}^{t-1} & 1 < t \end{array}\right.$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 2.938 (sec). Leaf size: 38

```
dsolve([diff(y(t),t)-y(t)=piecewise(0<=t and t<1,2,t>=1,-1),y(0) = 1],y(t), singsol=all)
```

$$y(t) = \begin{cases} -2+3\,\mathrm{e}^{t} & t<1 \\ 1+3\,\mathrm{e} & t=1 \\ 1+3\,\mathrm{e}^{t}-3\,\mathrm{e}^{t-1} & 1<t \end{cases}$$

✓ Solution by Mathematica

Time used: 0.072 (sec). Leaf size: 42

```
DSolve[{y'[t]-y[t]==Piecewise[{{2,0<=t<1},{-1,t>=1}}],{y[0]==1}},y[t],t,IncludeSingularSolut
```

$$y(t) \to \{ \begin{array}{cc} e^{t} & t\leq 0 \\ -2+3e^{t} & 0<t\leq 1 \\ 1-3e^{t-1}+3e^{t} & \text{True} \end{array}$$

14.17 problem Problem 46 part b

Internal problem ID [2883]
Internal file name [`OUTPUT/2375_Sunday_June_05_2022_03_02_26_AM_22789862/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 10, The Laplace Transform and Some Elementary Applications. Exercises for 10.7. page 704
Problem number: Problem 46 part b.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' - y = \begin{cases} 2 & 0 \le t < 1 \\ -1 & 1 \le t \end{cases}}$$

With initial conditions

$$[y(0) = 1]$$

14.17.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$
\begin{aligned}
p(t) &= -1 \\
q(t) &= \left\{ \begin{array}{cc} 0 & t < 0 \\ 2 & t < 1 \\ -1 & 1 \leq t \end{array} \right.
\end{aligned}
$$

Hence the ode is

$$
y' - y = \left\{ \begin{array}{cc} 0 & t < 0 \\ 2 & t < 1 \\ -1 & 1 \leq t \end{array} \right.
$$

The domain of $p(t) = -1$ is

$$
\{-\infty < t < \infty\}
$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = \left\{ \begin{array}{cc} 0 & t < 0 \\ 2 & t < 1 \\ -1 & 1 \leq t \end{array} \right.$ is

$$
\{0 \leq t \leq 1, 1 \leq t \leq \infty, -\infty \leq t \leq 0\}
$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

14.17.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int (-1)dt} \\
&= e^{-t}
\end{aligned}
$$

The ode becomes

$$\frac{d}{dt}(\mu y) = (\mu)\left(\left(\begin{cases} 0 & t<0 \\ 2 & t<1 \\ -1 & 1\le t \end{cases}\right)\right)$$

$$\frac{d}{dt}\left(e^{-t} y\right) = \left(e^{-t}\right)\left(\left(\begin{cases} 0 & t<0 \\ 2 & t<1 \\ -1 & 1\le t \end{cases}\right)\right)$$

$$d\left(e^{-t} y\right) = \left(\left(\left(\begin{cases} 0 & t<0 \\ 2 & t<1 \\ -1 & 1\le t \end{cases}\right)\right) e^{-t}\right) dt$$

Integrating gives

$$e^{-t} y = \int \left(\left(\begin{cases} 0 & t<0 \\ 2 & t<1 \\ -1 & 1\le t \end{cases}\right)\right) e^{-t}\, dt$$

$$e^{-t} y = \begin{cases} 0 & t\le 0 \\ -2\,e^{-t}+2 & t\le 1 \\ e^{-t}-3\,e^{-1}+2 & 1<t \end{cases} + c_1$$

Dividing both sides by the integrating factor $\mu = e^{-t}$ results in

$$y = e^{t}\left(\left(\begin{cases} 0 & t\le 0 \\ -2\,e^{-t}+2 & t\le 1 \\ e^{-t}-3\,e^{-1}+2 & 1<t \end{cases}\right)\right) + c_1 e^{t}$$

which simplifies to

$$y = e^{t}\left(\left(\left(\begin{cases} 0 & t\le 0 \\ -2\,e^{-t}+2 & t\le 1 \\ e^{-t}-3\,e^{-1}+2 & 1<t \end{cases}\right)\right) + c_1\right)$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1$$

$$c_1 = 1$$

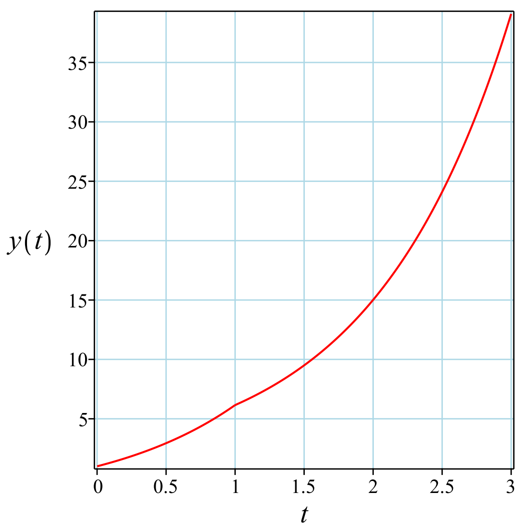
Substituting c_1 found above in the general solution gives

$$y = \begin{cases} e^t & t \leq 0 \\ -2+3\,e^t & t \leq 1 \\ 1+3\,e^t-3\,e^{t-1} & 1<t \\ e^t & \text{otherwise} \end{cases}$$

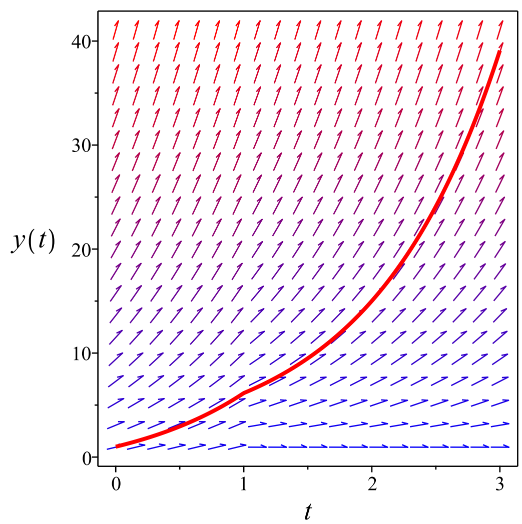
Summary

The solution(s) found are the following

$$y = \begin{cases} e^t & t \leq 0 \\ -2+3\,e^t & t \leq 1 \\ 1+3\,e^t-3\,e^{t-1} & 1<t \\ e^t & \text{otherwise} \end{cases} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \begin{cases} e^{t} & t \le 0 \\ -2+3\,e^{t} & t \le 1 \\ 1+3\,e^{t}-3\,e^{t-1} & 1 < t \\ e^{t} & \text{otherwise} \end{cases}$$

Verified OK.

14.17.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = y + \left(\begin{cases} 2 & 0 \le t < 1 \\ -1 & 1 \le t \end{cases} \right)$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 388: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= 0 \\
\eta(t, y) &= \mathrm{e}^t
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^t} dy \end{aligned}$$

Which results in

$$S = e^{-t} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,y) S_y}{R_t + \omega(t,y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t,y)$ is the right hand side of the original ode given by

$$\omega(t,y) = y + \left(\begin{cases} 2 & 0 \leq t < 1 \\ -1 & 1 \leq t \end{cases} \right)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= -e^{-t} y \\ S_y &= e^{-t} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \left(\begin{cases} 0 & t < 0 \\ 2 & t < 1 \\ -1 & 1 \leq t \end{cases} \right) e^{-t} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \left(\begin{cases} 0 & R < 0 \\ 2 & R < 1 \\ -1 & 1 \leq R \end{cases} \right) e^{-R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \begin{cases} c_1 & R < 0 \\ c_1 - 2\,\mathrm{e}^{-R} + 2 & R < 1 \\ c_1 + \mathrm{e}^{-R} - 3\,\mathrm{e}^{-1} + 2 & 1 \leq R \end{cases} \tag{4}
$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$
\mathrm{e}^{-t} y = \begin{cases} c_1 & t < 0 \\ c_1 - 2\,\mathrm{e}^{-t} + 2 & t < 1 \\ c_1 + \mathrm{e}^{-t} - 3\,\mathrm{e}^{-1} + 2 & 1 \leq t \end{cases}
$$

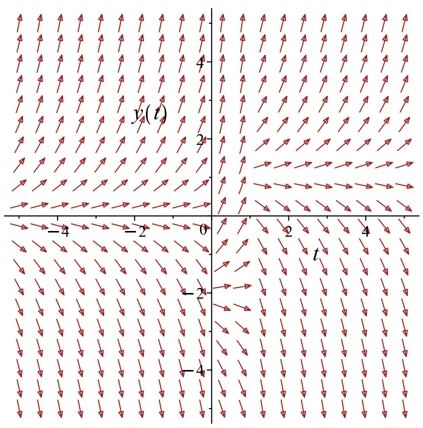
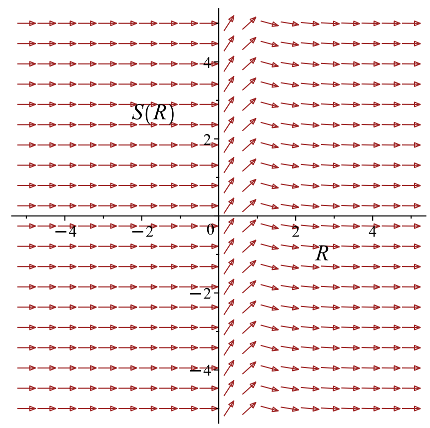
Which simplifies to

$$
\mathrm{e}^{-t} y = \begin{cases} c_1 & t < 0 \\ c_1 - 2\,\mathrm{e}^{-t} + 2 & t < 1 \\ c_1 + \mathrm{e}^{-t} - 3\,\mathrm{e}^{-1} + 2 & 1 \leq t \end{cases}
$$

Which gives

$$
y = \begin{cases} [c_1 \mathrm{e}^{t}] & t < 0 \\ [-2 + (2 + c_1)\,\mathrm{e}^{t}] & t < 1 \\ [-3\,\mathrm{e}^{t-1} + 1 + (2 + c_1)\,\mathrm{e}^{t}] & 1 \leq t \end{cases}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = y + \left(\left\{\begin{array}{cc} 2 & 0 \le t < 1 \\ -1 & 1 \le t \end{array}\right.\right)$ 	$R = t$ $S = \mathrm{e}^{-t} y$	$\frac{dS}{dR} = \left(\left\{\begin{array}{cc} 0 & R < 0 \\ 2 & R < 1 \\ -1 & 1 \le R \end{array}\right.\right) \mathrm{e}^{-R}$ 

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = [c_1]$$

Unable to solve for constant of integration. Verification of solutions N/A

14.17.4 Maple step by step solution

Let's solve

$$\left[y' - y = \left\{\begin{array}{cc} 2 & 0 \le t < 1 \\ -1 & 1 \le t \end{array}\right., y(0) = 1\right]$$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

$$y' = y + \left\{\begin{array}{cc} 2 & 0 \le t < 1 \\ -1 & 1 \le t \end{array}\right.$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

$$y' - y = \left\{\begin{array}{cc} 2 & 0 \le t < 1 \\ -1 & 1 \le t \end{array}\right.$$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

$$\mu(t)\left(y' - y\right) = \mu(t)\left(\left\{\begin{array}{cc} 2 & 0 \le t < 1 \\ -1 & 1 \le t \end{array}\right.\right)$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\, y)$

$$\mu(t)\left(y' - y\right) = \mu'(t)\, y + \mu(t)\, y'$$

- Isolate $\mu'(t)$

$$\mu'(t) = -\mu(t)$$

- Solve to find the integrating factor

$$\mu(t) = \mathrm{e}^{-t}$$

- Integrate both sides with respect to t

$$\int \left(\frac{d}{dt}(\mu(t)\, y)\right) dt = \int \mu(t)\left(\left\{\begin{array}{cc} 2 & 0 \le t < 1 \\ -1 & 1 \le t \end{array}\right.\right) dt + c_1$$

- Evaluate the integral on the lhs

$$\mu(t)\, y = \int \mu(t)\left(\left\{\begin{array}{cc} 2 & 0 \le t < 1 \\ -1 & 1 \le t \end{array}\right.\right) dt + c_1$$

- Solve for y

$$y = \frac{\int \mu(t)\left(\left\{\begin{array}{cc} 2 & 0 \le t < 1 \\ -1 & 1 \le t \end{array}\right.\right) dt + c_1}{\mu(t)}$$

- Substitute $\mu(t) = \mathrm{e}^{-t}$

$$y = \frac{\int \mathrm{e}^{-t}\left(\left\{\begin{array}{cc} 2 & 0 \le t < 1 \\ -1 & 1 \le t \end{array}\right.\right) dt + c_1}{\mathrm{e}^{-t}}$$

- Evaluate the integrals on the rhs

$$y = \frac{\left\{\begin{array}{cc} 0 & t \leq 0 \\ -2\,\mathrm{e}^{-t}+2 & t \leq 1 \\ \mathrm{e}^{-t}-3\,\mathrm{e}^{-1}+2 & 1<t \end{array}\right. + c_1}{\mathrm{e}^{-t}}$$

- Simplify

$$y = \mathrm{e}^{t}\left(\left\{\begin{array}{cc} 0 & t \leq 0 \\ -2\,\mathrm{e}^{-t}+2 & t \leq 1 \\ \mathrm{e}^{-t}-3\,\mathrm{e}^{-1}+2 & 1<t \end{array}\right. + c_1\right)$$

- Use initial condition $y(0) = 1$

$$1 = c_1$$

- Solve for c_1

$$c_1 = 1$$

- Substitute $c_1 = 1$ into general solution and simplify

$$y = \left\{\begin{array}{cc} \mathrm{e}^{t} & t \leq 0 \\ -2+3\,\mathrm{e}^{t} & t \leq 1 \\ 1+3\,\mathrm{e}^{t}-3\,\mathrm{e}^{t-1} & 1<t \end{array}\right.$$

- Solution to the IVP

$$y = \left\{\begin{array}{cc} \mathrm{e}^{t} & t \leq 0 \\ -2+3\,\mathrm{e}^{t} & t \leq 1 \\ 1+3\,\mathrm{e}^{t}-3\,\mathrm{e}^{t-1} & 1<t \end{array}\right.$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.344 (sec). Leaf size: 34

```
dsolve([diff(y(t),t)-y(t)=piecewise(0<=t and t<1,2,t>=1,-1),y(0) = 1],y(t), singsol=all)
```

$$y(t) = \begin{cases} e^{t} & t < 0 \\ -2+3\,e^{t} & t < 1 \\ 1+3\,e^{t}-3\,e^{t-1} & 1 \leq t \end{cases}$$

✓ Solution by Mathematica

Time used: 0.071 (sec). Leaf size: 42

```
DSolve[{y'[t]-y[t]==Piecewise[{{2,0<=t<1},{-1,t>=1}}],{y[0]==1}},y[t],t,IncludeSingularSolut
```

$$y(t) \to \left\{ \begin{array}{cc} e^{t} & t \leq 0 \\ -2+3e^{t} & 0 < t \leq 1 \\ 1-3e^{t-1}+3e^{t} & \text{True} \end{array} \right.$$

15 Chapter 10, The Laplace Transform and Some Elementary Applications. Exercises for 10.8. page 710

15.1 problem Problem 1

Internal problem ID [2884]
Internal file name [`OUTPUT/2376_Sunday_June_05_2022_03_02_37_AM_41964257/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 10, The Laplace Transform and Some Elementary Applications. Exercises for 10.8. page 710
Problem number: Problem 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' + y = \delta(t - 5)}$$

With initial conditions

$$[y(0) = 3]$$

15.1.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$\begin{aligned} p(t) &= 1 \\ q(t) &= \delta(t - 5) \end{aligned}$$

Hence the ode is

$$y' + y = \delta(t - 5)$$

The domain of $p(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = \delta(t - 5)$ is

$$\{t < 5 \lor 5 < t\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

15.1.2 Solving as laplace ode

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(y') = sY(s) - y(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$sY(s) - y(0) + Y(s) = e^{-5s} \tag{1}$$

Replacing initial condition gives

$$sY(s) - 3 + Y(s) = e^{-5s}$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{e^{-5s} + 3}{s + 1}$$

Taking the inverse Laplace transform gives

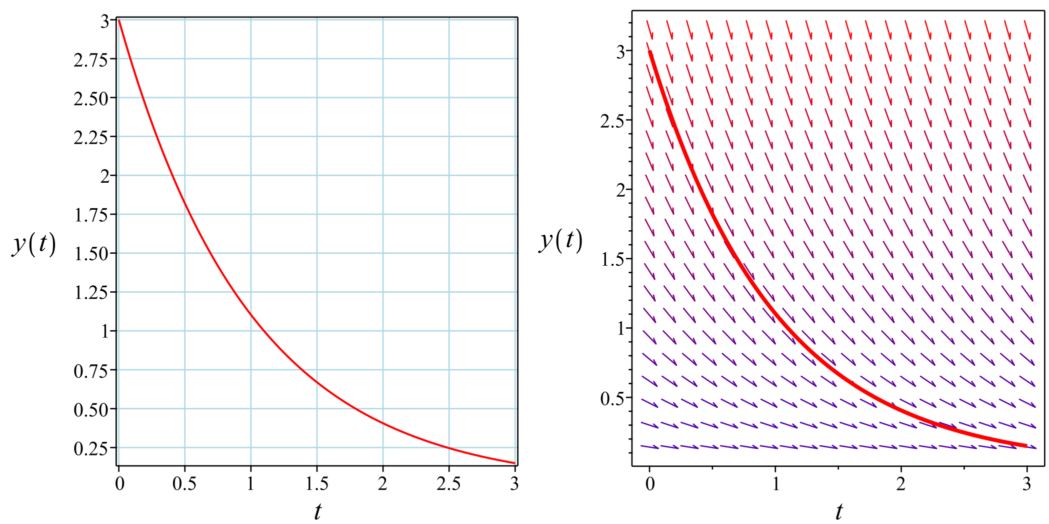
$$\begin{aligned} y &= \mathcal{L}^{-1}(Y(s)) \\ &= \mathcal{L}^{-1}\left(\frac{e^{-5s} + 3}{s + 1}\right) \\ &= \text{Heaviside}\,(t - 5)\, e^{-t+5} + 3\, e^{-t} \end{aligned}$$

Hence the final solution is

$$y = \text{Heaviside}\,(t-5)\,\mathrm{e}^{-t+5} + 3\,\mathrm{e}^{-t}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \text{Heaviside}\,(t-5)\,\mathrm{e}^{-t+5} + 3\,\mathrm{e}^{-t} \tag{1}$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$y = \text{Heaviside}\,(t-5)\,\mathrm{e}^{-t+5} + 3\,\mathrm{e}^{-t}$$

Verified OK.

15.1.3 Maple step by step solution

Let's solve

$[y' + y = Dirac(t-5)\,, y(0) = 3]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -y + Dirac(t-5)$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE
  $y' + y = \mathit{Dirac}(t-5)$
- The ODE is linear; multiply by an integrating factor $\mu(t)$
  $\mu(t)\,(y' + y) = \mu(t)\,\mathit{Dirac}(t-5)$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,y)$
  $\mu(t)\,(y' + y) = \mu'(t)\,y + \mu(t)\,y'$
- Isolate $\mu'(t)$
  $\mu'(t) = \mu(t)$
- Solve to find the integrating factor
  $\mu(t) = \mathrm{e}^{t}$
- Integrate both sides with respect to t
  $\int \left(\frac{d}{dt}(\mu(t)\,y)\right) dt = \int \mu(t)\,\mathit{Dirac}(t-5)\,dt + c_1$
- Evaluate the integral on the lhs
  $\mu(t)\,y = \int \mu(t)\,\mathit{Dirac}(t-5)\,dt + c_1$
- Solve for y
  $y = \frac{\int \mu(t)\mathit{Dirac}(t-5)dt + c_1}{\mu(t)}$
- Substitute $\mu(t) = \mathrm{e}^{t}$
  $y = \frac{\int \mathrm{e}^{t}\,\mathit{Dirac}(t-5)dt + c_1}{\mathrm{e}^{t}}$
- Evaluate the integrals on the rhs
  $y = \frac{\mathit{Heaviside}(t-5)\mathrm{e}^{5} + c_1}{\mathrm{e}^{t}}$
- Simplify
  $y = \mathrm{e}^{-t}(\mathit{Heaviside}(t-5)\,\mathrm{e}^{5} + c_1)$
- Use initial condition $y(0) = 3$
  $3 = c_1$
- Solve for c_1
  $c_1 = 3$
- Substitute $c_1 = 3$ into general solution and simplify
  $y = \mathrm{e}^{-t}(\mathit{Heaviside}(t-5)\,\mathrm{e}^{5} + 3)$
- Solution to the IVP

$$y = e^{-t}(\mathit{Heaviside}(t-5)\,e^{5} + 3)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 2.219 (sec). Leaf size: 22

```
dsolve([diff(y(t),t)+y(t)=Dirac(t-5),y(0) = 3],y(t), singsol=all)
```

$$y(t) = \text{Heaviside}\,(t-5)\,e^{-t+5} + 3\,e^{-t}$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 21

```
DSolve[{y'[t]+y[t]==DiracDelta[t-5],{y[0]==3}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to e^{-t}\left(e^{5}\theta(t-5)+3\right)$$

15.2 problem Problem 2

Internal problem ID [2885]
Internal file name [`OUTPUT/2377_Sunday_June_05_2022_03_02_43_AM_57791878/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 10, The Laplace Transform and Some Elementary Applications. Exercises for 10.8. page 710
Problem number: Problem 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' - 2y = \delta(t-2)}$$

With initial conditions

$$[y(0) = 1]$$

15.2.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = -2$$
$$q(t) = \delta(t-2)$$

Hence the ode is

$$y' - 2y = \delta(t - 2)$$

The domain of $p(t) = -2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = \delta(t - 2)$ is

$$\{t < 2 \lor 2 < t\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

15.2.2 Solving as laplace ode

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(y') = sY(s) - y(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$sY(s) - y(0) - 2Y(s) = \mathrm{e}^{-2s} \tag{1}$$

Replacing initial condition gives

$$sY(s) - 1 - 2Y(s) = \mathrm{e}^{-2s}$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{\mathrm{e}^{-2s} + 1}{s - 2}$$

Taking the inverse Laplace transform gives

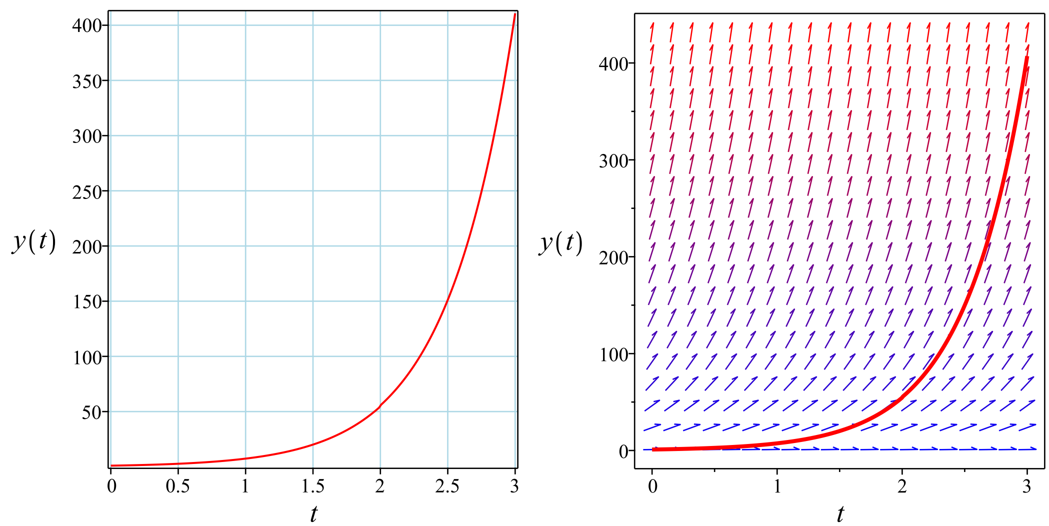
$$\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(\frac{\mathrm{e}^{-2s} + 1}{s - 2}\right) \\
&= \mathrm{e}^{2t} + \mathrm{e}^{2t-4}(-\operatorname{Heaviside}(-t + 2) + 1)
\end{aligned}$$

Hence the final solution is

$$y = e^{2t} + e^{2t-4}(-\operatorname{Heaviside}(-t+2)+1)$$

Summary
The solution(s) found are the following

$$y = e^{2t} + e^{2t-4}(-\operatorname{Heaviside}(-t+2)+1) \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = e^{2t} + e^{2t-4}(-\operatorname{Heaviside}(-t+2)+1)$$

Verified OK.

15.2.3 Maple step by step solution

Let's solve

$[y' - 2y = Dirac(t-2)\,, y(0) = 1]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = 2y + Dirac(t-2)$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - 2y = \mathit{Dirac}(t - 2)$
- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\,(y' - 2y) = \mu(t)\,\mathit{Dirac}(t - 2)$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,y)$

  $\mu(t)\,(y' - 2y) = \mu'(t)\,y + \mu(t)\,y'$
- Isolate $\mu'(t)$

  $\mu'(t) = -2\mu(t)$
- Solve to find the integrating factor

  $\mu(t) = \mathrm{e}^{-2t}$
- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\,y)\right) dt = \int \mu(t)\,\mathit{Dirac}(t - 2)\,dt + c_1$
- Evaluate the integral on the lhs

  $\mu(t)\,y = \int \mu(t)\,\mathit{Dirac}(t - 2)\,dt + c_1$
- Solve for y

  $y = \frac{\int \mu(t)\mathit{Dirac}(t-2)dt + c_1}{\mu(t)}$
- Substitute $\mu(t) = \mathrm{e}^{-2t}$

  $y = \frac{\int \mathrm{e}^{-2t}\mathit{Dirac}(t-2)dt + c_1}{\mathrm{e}^{-2t}}$
- Evaluate the integrals on the rhs

  $y = \frac{\mathit{Heaviside}(t-2)\mathrm{e}^{-4} + c_1}{\mathrm{e}^{-2t}}$
- Simplify

  $y = \mathrm{e}^{2t}(\mathit{Heaviside}(t - 2)\,\mathrm{e}^{-4} + c_1)$
- Use initial condition $y(0) = 1$

  $1 = c_1$
- Solve for c_1

  $c_1 = 1$
- Substitute $c_1 = 1$ into general solution and simplify

  $y = \mathrm{e}^{2t}(\mathit{Heaviside}(t - 2)\,\mathrm{e}^{-4} + 1)$
- Solution to the IVP

$$y = \mathrm{e}^{2t}(Heaviside(t-2)\,\mathrm{e}^{-4} + 1)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 2.234 (sec). Leaf size: 26

```
dsolve([diff(y(t),t)-2*y(t)=Dirac(t-2),y(0) = 1],y(t), singsol=all)
```

$$y(t) = \mathrm{Heaviside}\,(t-2)\,\mathrm{e}^{-4+2t} + \mathrm{e}^{2t}$$

✓ Solution by Mathematica

Time used: 0.038 (sec). Leaf size: 23

```
DSolve[{y'[t]-2*y[t]==DiracDelta[t-2],{y[0]==3}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to e^{2t-4}(\theta(t-2) + 3e^4)$$

15.3 problem Problem 3

Internal problem ID [2886]
Internal file name [`OUTPUT/2378_Sunday_June_05_2022_03_02_49_AM_12555487/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 10, The Laplace Transform and Some Elementary Applications. Exercises for 10.8. page 710
Problem number: Problem 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first__order__ode__lie__symmetry__lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' + 4y = 3\delta(t-1)}$$

With initial conditions

$$[y(0) = 2]$$

15.3.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$\begin{aligned} p(t) &= 4 \\ q(t) &= 3\delta(t-1) \end{aligned}$$

Hence the ode is

$$y' + 4y = 3\delta(t - 1)$$

The domain of $p(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 3\delta(t - 1)$ is

$$\{t < 1 \lor 1 < t\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

15.3.2 Solving as laplace ode

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(y') = sY(s) - y(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$sY(s) - y(0) + 4Y(s) = 3\,\mathrm{e}^{-s} \tag{1}$$

Replacing initial condition gives

$$sY(s) - 2 + 4Y(s) = 3\,\mathrm{e}^{-s}$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{3\,\mathrm{e}^{-s} + 2}{s + 4}$$

Taking the inverse Laplace transform gives

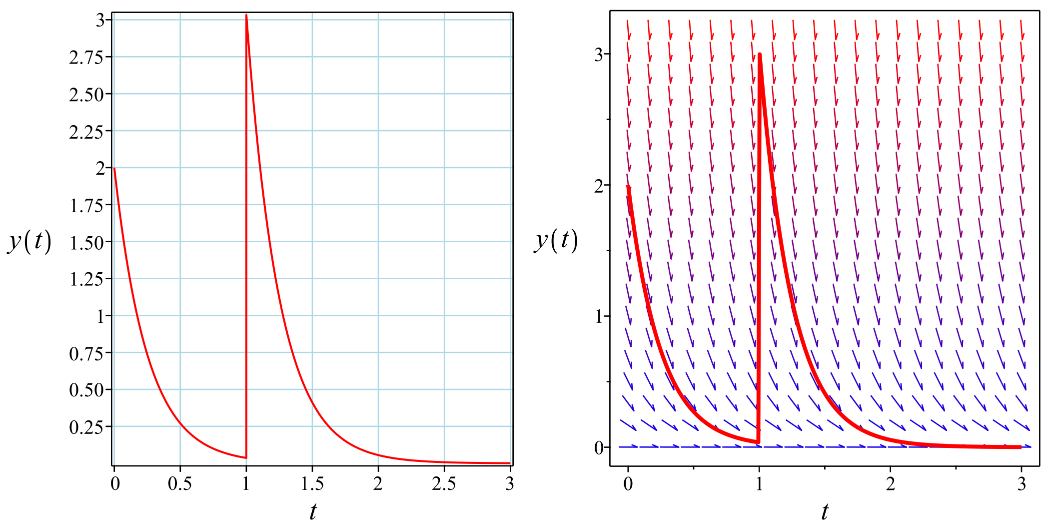
$$\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(\frac{3\,\mathrm{e}^{-s} + 2}{s + 4}\right) \\
&= 3\,\mathrm{Heaviside}\,(t - 1)\,\mathrm{e}^{-4t+4} + 2\,\mathrm{e}^{-4t}
\end{aligned}$$

Hence the final solution is

$$y = 3\,\text{Heaviside}\left(t-1\right) \mathrm{e}^{-4t+4}+2\,\mathrm{e}^{-4t}$$

Summary
The solution(s) found are the following

$$y = 3\,\text{Heaviside}\left(t-1\right) \mathrm{e}^{-4t+4}+2\,\mathrm{e}^{-4t} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = 3\,\text{Heaviside}\left(t-1\right) \mathrm{e}^{-4t+4}+2\,\mathrm{e}^{-4t}$$

Verified OK.

15.3.3 Maple step by step solution

Let's solve

$[y' + 4y = 3\mathit{Dirac}(t-1)\,, y(0) = 2]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -4y + 3\mathit{Dirac}(t-1)$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + 4y = 3\mathit{Dirac}(t-1)$
- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\,(y' + 4y) = 3\mu(t)\,\mathit{Dirac}(t-1)$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,y)$

  $\mu(t)\,(y' + 4y) = \mu'(t)\,y + \mu(t)\,y'$
- Isolate $\mu'(t)$

  $\mu'(t) = 4\mu(t)$
- Solve to find the integrating factor

  $\mu(t) = \mathrm{e}^{4t}$
- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\,y)\right) dt = \int 3\mu(t)\,\mathit{Dirac}(t-1)\,dt + c_1$
- Evaluate the integral on the lhs

  $\mu(t)\,y = \int 3\mu(t)\,\mathit{Dirac}(t-1)\,dt + c_1$
- Solve for y

  $y = \frac{\int 3\mu(t)\mathit{Dirac}(t-1)dt + c_1}{\mu(t)}$
- Substitute $\mu(t) = \mathrm{e}^{4t}$

  $y = \frac{\int 3\,\mathrm{e}^{4t}\,\mathit{Dirac}(t-1)dt + c_1}{\mathrm{e}^{4t}}$
- Evaluate the integrals on the rhs

  $y = \frac{3\mathit{Heaviside}(t-1)\mathrm{e}^{4} + c_1}{\mathrm{e}^{4t}}$
- Simplify

  $y = \mathrm{e}^{-4t}(3\mathit{Heaviside}(t-1)\,\mathrm{e}^{4} + c_1)$
- Use initial condition $y(0) = 2$

  $2 = c_1$
- Solve for c_1

  $c_1 = 2$
- Substitute $c_1 = 2$ into general solution and simplify

  $y = 3\mathit{Heaviside}(t-1)\,\mathrm{e}^{-4t}\mathrm{e}^{4} + 2\,\mathrm{e}^{-4t}$
- Solution to the IVP

$$y = 3 Heaviside(t-1)\,\mathrm{e}^{-4t}\mathrm{e}^{4} + 2\,\mathrm{e}^{-4t}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 2.422 (sec). Leaf size: 23

```
dsolve([diff(y(t),t)+4*y(t)=3*Dirac(t-1),y(0) = 2],y(t), singsol=all)
```

$$y(t) = 3\,\mathrm{Heaviside}\,(t-1)\,\mathrm{e}^{-4t+4} + 2\,\mathrm{e}^{-4t}$$

✓ Solution by Mathematica

Time used: 0.032 (sec). Leaf size: 22

```
DSolve[{y'[t]+4*y[t]==3*DiracDelta[t-1],{y[0]==2}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to e^{-4t}\left(3e^{4}\theta(t-1)+2\right)$$

15.4 problem Problem 4

Internal problem ID [2887]
Internal file name [`OUTPUT/2379_Sunday_June_05_2022_03_02_55_AM_59375699/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 10, The Laplace Transform and Some Elementary Applications. Exercises for 10.8. page 710
Problem number: Problem 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' - 5y = 2\,\mathrm{e}^{-t} + \delta(t-3)}$$

With initial conditions

$$[y(0) = 0]$$

15.4.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$\begin{aligned} p(t) &= -5 \\ q(t) &= 2\,\mathrm{e}^{-t} + \delta(t-3) \end{aligned}$$

Hence the ode is

$$y' - 5y = 2\,e^{-t} + \delta(t-3)$$

The domain of $p(t) = -5$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 2\,e^{-t} + \delta(t-3)$ is

$$\{t < 3 \lor 3 < t\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

15.4.2 Solving as laplace ode

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(y') = sY(s) - y(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$sY(s) - y(0) - 5Y(s) = \frac{2}{1+s} + e^{-3s} \tag{1}$$

Replacing initial condition gives

$$sY(s) - 5Y(s) = \frac{2}{1+s} + e^{-3s}$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{e^{-3s}s + e^{-3s} + 2}{(1+s)(s-5)}$$

Taking the inverse Laplace transform gives

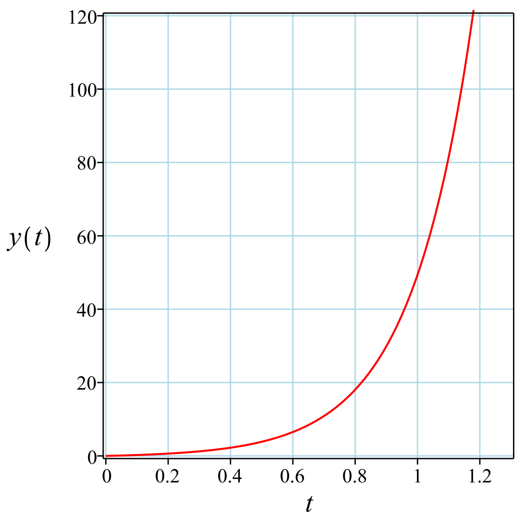
$$\begin{aligned} y &= \mathcal{L}^{-1}(Y(s)) \\ &= \mathcal{L}^{-1}\left(\frac{e^{-3s}s + e^{-3s} + 2}{(1+s)(s-5)}\right) \\ &= \frac{2\,e^{2t}\sinh(3t)}{3} + e^{-15+5t}(-\operatorname{Heaviside}(-t+3) + 1) \end{aligned}$$

Hence the final solution is

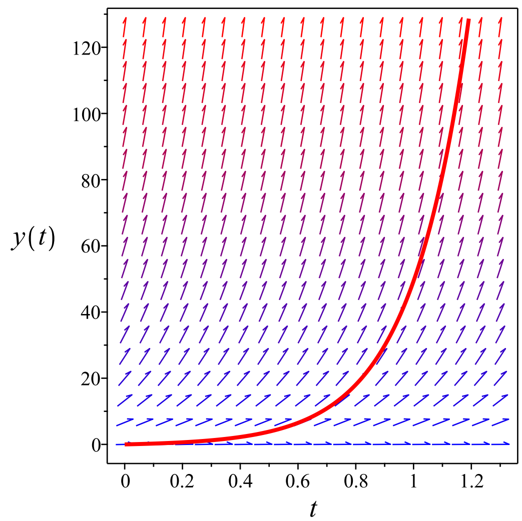
$$y = \frac{2\,\mathrm{e}^{2t}\sinh(3t)}{3} + \mathrm{e}^{-15+5t}(-\operatorname{Heaviside}(-t+3)+1)$$

Summary
The solution(s) found are the following

$$y = \frac{2\,\mathrm{e}^{2t}\sinh(3t)}{3} + \mathrm{e}^{-15+5t}(-\operatorname{Heaviside}(-t+3)+1) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{2\,\mathrm{e}^{2t}\sinh(3t)}{3} + \mathrm{e}^{-15+5t}(-\operatorname{Heaviside}(-t+3)+1)$$

Verified OK.

15.4.3 Maple step by step solution

Let's solve

$[y' - 5y = 2\,\mathrm{e}^{-t} + \mathit{Dirac}(t-3)\,, y(0) = 0]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = 5y + 2\,\mathrm{e}^{-t} + \mathit{Dirac}(t-3)$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - 5y = 2\,\mathrm{e}^{-t} + \mathit{Dirac}(t-3)$
- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\,(y' - 5y) = \mu(t)\,(2\,\mathrm{e}^{-t} + \mathit{Dirac}(t-3))$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,y)$

  $\mu(t)\,(y' - 5y) = \mu'(t)\,y + \mu(t)\,y'$
- Isolate $\mu'(t)$

  $\mu'(t) = -5\mu(t)$
- Solve to find the integrating factor

  $\mu(t) = \mathrm{e}^{-5t}$
- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\,y)\right) dt = \int \mu(t)\,(2\,\mathrm{e}^{-t} + \mathit{Dirac}(t-3))\,dt + c_1$
- Evaluate the integral on the lhs

  $\mu(t)\,y = \int \mu(t)\,(2\,\mathrm{e}^{-t} + \mathit{Dirac}(t-3))\,dt + c_1$
- Solve for y

  $y = \frac{\int \mu(t)(2\,\mathrm{e}^{-t}+\mathit{Dirac}(t-3))dt+c_1}{\mu(t)}$
- Substitute $\mu(t) = \mathrm{e}^{-5t}$

  $y = \frac{\int \mathrm{e}^{-5t}(2\,\mathrm{e}^{-t}+\mathit{Dirac}(t-3))dt+c_1}{\mathrm{e}^{-5t}}$
- Evaluate the integrals on the rhs

  $y = \frac{\mathit{Heaviside}(t-3)\mathrm{e}^{-15}-\frac{\mathrm{e}^{-6t}}{3}+c_1}{\mathrm{e}^{-5t}}$
- Simplify

  $y = c_1\mathrm{e}^{5t} + \mathrm{e}^{-15+5t}\mathit{Heaviside}(t-3) - \frac{\mathrm{e}^{-t}}{3}$
- Use initial condition $y(0) = 0$

  $0 = c_1 - \frac{1}{3}$
- Solve for c_1

  $c_1 = \frac{1}{3}$
- Substitute $c_1 = \frac{1}{3}$ into general solution and simplify

$y = \frac{e^{5t}}{3} + e^{-15+5t} \mathit{Heaviside}(t-3) - \frac{e^{-t}}{3}$

- Solution to the IVP

  $y = \frac{e^{5t}}{3} + e^{-15+5t} \mathit{Heaviside}(t-3) - \frac{e^{-t}}{3}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 2.578 (sec). Leaf size: 32

```
dsolve([diff(y(t),t)-5*y(t)=2*exp(-t)+Dirac(t-3),y(0) = 0],y(t), singsol=all)
```

$$y(t) = \frac{2\,e^{2t} \sinh(3t)}{3} + \mathrm{Heaviside}(-3+t)\,e^{5t-15}$$

✓ Solution by Mathematica

Time used: 0.093 (sec). Leaf size: 34

```
DSolve[{y'[t]-5*y[t]==2*Exp[-t]+DiracDelta[t-3],{y[0]==0}},y[t],t,IncludeSingularSolutions -
```

$$y(t) \to \frac{1}{3} e^{-t} \left(3e^{6t-15} \theta(t-3) + e^{6t} - 1\right)$$

15.5 problem Problem 5

Internal problem ID [2888]
Internal file name [`OUTPUT/2380_Sunday_June_05_2022_03_03_13_AM_99697359/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 10, The Laplace Transform and Some Elementary Applications. Exercises for 10.8. page 710
Problem number: Problem 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 3y' + 2y = \delta(t-1)}$$

With initial conditions

$$[y(0) = 1, y'(0) = 0]$$

15.5.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= -3 \\ q(t) &= 2 \\ F &= \delta(t-1) \end{aligned}$$

Hence the ode is

$$y'' - 3y' + 2y = \delta(t-1)$$

The domain of $p(t) = -3$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \delta(t-1)$ is

$$\{t < 1 \lor 1 < t\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) - 3sY(s) + 3y(0) + 2Y(s) = \mathrm{e}^{-s} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 1\\ y'(0) &= 0\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + 3 - s - 3sY(s) + 2Y(s) = \mathrm{e}^{-s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{\mathrm{e}^{-s} + s - 3}{s^2 - 3s + 2}$$

Taking the inverse Laplace transform gives

$$
\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(\frac{e^{-s}+s-3}{s^2-3s+2}\right) \\
&= \left(-e^{2t-2}+e^{t-1}\right) \text{Heaviside}\,(1-t) + 2\,e^{t} - e^{2t} + e^{2t-2} - e^{t-1}
\end{aligned}
$$

Hence the final solution is

$$
y = \left(-e^{2t-2}+e^{t-1}\right) \text{Heaviside}\,(1-t) + 2\,e^{t} - e^{2t} + e^{2t-2} - e^{t-1}
$$

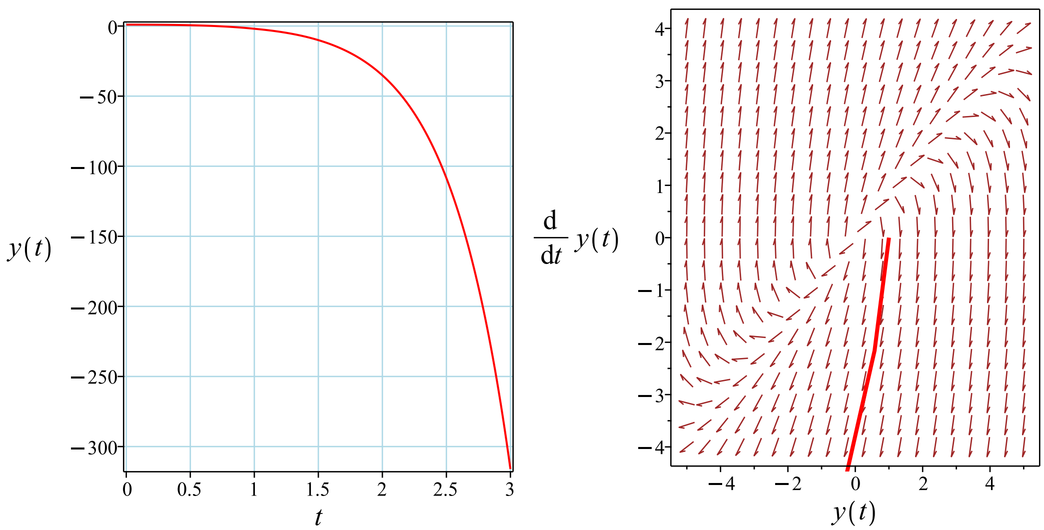
Simplifying the solution gives

$$
y = \text{Heaviside}\,(t-1)\,e^{2t-2} - \text{Heaviside}\,(t-1)\,e^{t-1} + 2\,e^{t} - e^{2t}
$$

Summary

The solution(s) found are the following

$$
y = \text{Heaviside}\,(t-1)\,e^{2t-2} - \text{Heaviside}\,(t-1)\,e^{t-1} + 2\,e^{t} - e^{2t} \tag{1}
$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$
y = \text{Heaviside}\,(t-1)\,e^{2t-2} - \text{Heaviside}\,(t-1)\,e^{t-1} + 2\,e^{t} - e^{2t}
$$

Verified OK.

15.5.2 Maple step by step solution

Let's solve

$$\left[y'' - 3y' + 2y = \mathit{Dirac}(t-1)\,, y(0) = 1, y'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 3r + 2 = 0$

- Factor the characteristic polynomial

  $(r-1)(r-2) = 0$

- Roots of the characteristic polynomial

  $r = (1, 2)$

- 1st solution of the homogeneous ODE

  $y_1(t) = \mathrm{e}^t$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \mathrm{e}^{2t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^t + c_2 \mathrm{e}^{2t} + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t),y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t),y_2(t))} dt\right), f(t) = \mathit{Dirac}(t-1)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(t)\,, y_2(t)) = \begin{bmatrix} \mathrm{e}^t & \mathrm{e}^{2t} \\ \mathrm{e}^t & 2\,\mathrm{e}^{2t} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(t)\,, y_2(t)) = \mathrm{e}^{3t}$

  - Substitute functions into equation for $y_p(t)$

$y_p(t) = \left(\int Dirac(t-1)\,dt\right)(e^{2t-2} - e^{t-1})$

- Compute integrals

  $y_p(t) = Heaviside(t-1)\,(e^{2t-2} - e^{t-1})$

- Substitute particular solution into general solution to ODE

  $y = c_1e^t + c_2e^{2t} + Heaviside(t-1)\,(e^{2t-2} - e^{t-1})$

□ Check validity of solution $y = c_1e^t + c_2e^{2t} + Heaviside(t-1)\,(e^{2t-2} - e^{t-1})$

- Use initial condition $y(0) = 1$

  $1 = c_1 + c_2$

- Compute derivative of the solution

  $y' = c_1e^t + 2c_2e^{2t} + Dirac(t-1)\,(e^{2t-2} - e^{t-1}) + Heaviside(t-1)\,(2\,e^{2t-2} - e^{t-1})$

- Use the initial condition $y'\Big|_{\{t=0\}} = 0$

  $0 = c_1 + 2c_2$

- Solve for c_1 and c_2

  $\{c_1 = 2, c_2 = -1\}$

- Substitute constant values into general solution and simplify

  $y = Heaviside(t-1)\,e^{2t-2} - Heaviside(t-1)\,e^{t-1} + 2\,e^t - e^{2t}$

- Solution to the IVP

  $y = Heaviside(t-1)\,e^{2t-2} - Heaviside(t-1)\,e^{t-1} + 2\,e^t - e^{2t}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 2.375 (sec). Leaf size: 47

```
dsolve([diff(y(t),t$2)-3*diff(y(t),t)+2*y(t)=Dirac(t-1),y(0) = 1, D(y)(0) = 0],y(t), singsol
```

$$y(t) = -\operatorname{Heaviside}(t-1)\,\mathrm{e}^{t-1} + \operatorname{Heaviside}(t-1)\,\mathrm{e}^{2t-2} - \mathrm{e}^{2t} + 2\,\mathrm{e}^{t}$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 31

```
DSolve[{y''[t]-3*y'[t]+2*y[t]==DiracDelta[t-1],{y[0]==1,y'[0]==0}},y[t],t,IncludeSingularSol
```

$$y(t) \to e^t\left(\frac{(e^t - e)\,\theta(t-1)}{e^2} - e^t + 2\right)$$

15.6 problem Problem 6

Internal problem ID [2889]
Internal file name [`OUTPUT/2381_Sunday_June_05_2022_03_03_19_AM_18570981/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 10, The Laplace Transform and Some Elementary Applications. Exercises for 10.8. page 710
Problem number: Problem 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 4y = \delta(t - 3)}$$

With initial conditions

$$[y(0) = 0, y'(0) = 1]$$

15.6.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= -4 \\ F &= \delta(t - 3) \end{aligned}$$

Hence the ode is

$$y'' - 4y = \delta(t-3)$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = -4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \delta(t-3)$ is

$$\{t < 3 \vee 3 < t\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0) \\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) - 4Y(s) = e^{-3s} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 0 \\ y'(0) &= 1\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 1 - 4Y(s) = e^{-3s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{e^{-3s} + 1}{s^2 - 4}$$

Taking the inverse Laplace transform gives

$$
\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(\frac{\mathrm{e}^{-3s}+1}{s^2-4}\right) \\
&= \frac{\text{Heaviside}\,(t-3)\sinh\,(2t-6)}{2} + \frac{\sinh\,(2t)}{2}
\end{aligned}
$$

Hence the final solution is

$$
y = \frac{\text{Heaviside}\,(t-3)\sinh\,(2t-6)}{2} + \frac{\sinh\,(2t)}{2}
$$

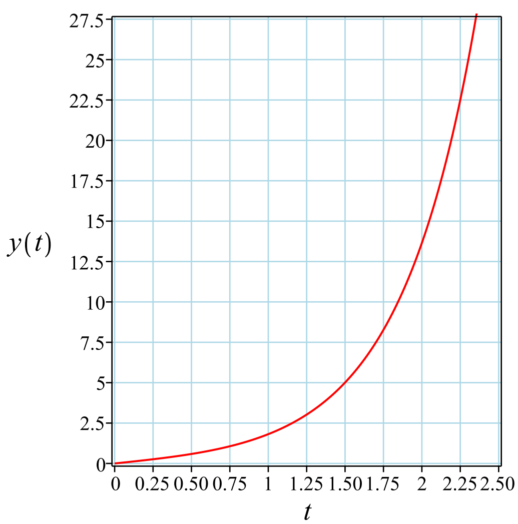
Simplifying the solution gives

$$
y = \frac{\text{Heaviside}\,(t-3)\sinh\,(2t-6)}{2} + \frac{\sinh\,(2t)}{2}
$$

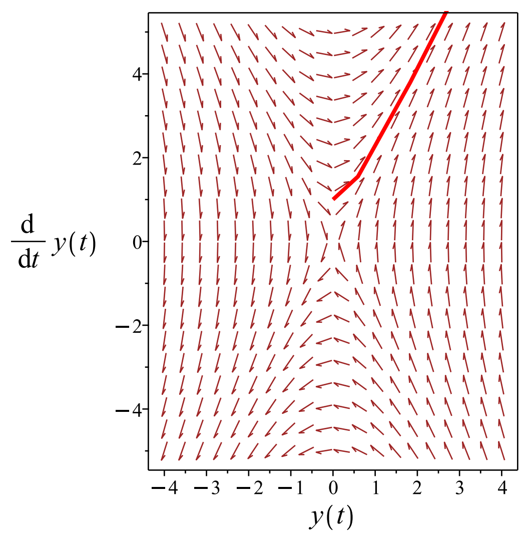
Summary

The solution(s) found are the following

$$
y = \frac{\text{Heaviside}\,(t-3)\sinh\,(2t-6)}{2} + \frac{\sinh\,(2t)}{2} \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$
y = \frac{\text{Heaviside}\,(t-3)\sinh\,(2t-6)}{2} + \frac{\sinh\,(2t)}{2}
$$

Verified OK.

15.6.2 Maple step by step solution

Let's solve

$$\left[y'' - 4y = Dirac(t-3), y(0) = 0, y'\Big|_{\{t=0\}} = 1\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 4 = 0$

- Factor the characteristic polynomial

  $(r-2)(r+2) = 0$

- Roots of the characteristic polynomial

  $r = (-2, 2)$

- 1st solution of the homogeneous ODE

  $y_1(t) = \mathrm{e}^{-2t}$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \mathrm{e}^{2t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{2t} + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = Dirac(t-3)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(t), y_2(t)) = \begin{bmatrix} \mathrm{e}^{-2t} & \mathrm{e}^{2t} \\ -2\,\mathrm{e}^{-2t} & 2\,\mathrm{e}^{2t} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(t), y_2(t)) = 4$

  - Substitute functions into equation for $y_p(t)$

$$y_p(t) = -\frac{\left(\int \mathit{Dirac}(t-3)dt\right)\left(\mathrm{e}^{-2t+6}-\mathrm{e}^{2t-6}\right)}{4}$$

- Compute integrals

$$y_p(t) = -\frac{\mathit{Heaviside}(t-3)\left(\mathrm{e}^{-2t+6}-\mathrm{e}^{2t-6}\right)}{4}$$

- Substitute particular solution into general solution to ODE

$$y = c_1\mathrm{e}^{-2t} + c_2\mathrm{e}^{2t} - \frac{\mathit{Heaviside}(t-3)\left(\mathrm{e}^{-2t+6}-\mathrm{e}^{2t-6}\right)}{4}$$

□ Check validity of solution $y = c_1\mathrm{e}^{-2t} + c_2\mathrm{e}^{2t} - \frac{\mathit{Heaviside}(t-3)\left(\mathrm{e}^{-2t+6}-\mathrm{e}^{2t-6}\right)}{4}$

- Use initial condition $y(0) = 0$

$$0 = c_1 + c_2$$

- Compute derivative of the solution

$$y' = -2c_1\mathrm{e}^{-2t} + 2c_2\mathrm{e}^{2t} - \frac{\mathit{Dirac}(t-3)\left(\mathrm{e}^{-2t+6}-\mathrm{e}^{2t-6}\right)}{4} - \frac{\mathit{Heaviside}(t-3)\left(-2\,\mathrm{e}^{-2t+6}-2\,\mathrm{e}^{2t-6}\right)}{4}$$

- Use the initial condition $y'\Big|_{\{t=0\}} = 1$

$$1 = -2c_1 + 2c_2$$

- Solve for c_1 and c_2

$$\left\{c_1 = -\frac{1}{4}, c_2 = \frac{1}{4}\right\}$$

- Substitute constant values into general solution and simplify

$$y = -\frac{\mathrm{e}^{-2t}}{4} + \frac{\mathrm{e}^{2t}}{4} - \frac{\mathit{Heaviside}(t-3)\mathrm{e}^{-2t+6}}{4} + \frac{\mathit{Heaviside}(t-3)\mathrm{e}^{2t-6}}{4}$$

- Solution to the IVP

$$y = -\frac{\mathrm{e}^{-2t}}{4} + \frac{\mathrm{e}^{2t}}{4} - \frac{\mathit{Heaviside}(t-3)\mathrm{e}^{-2t+6}}{4} + \frac{\mathit{Heaviside}(t-3)\mathrm{e}^{2t-6}}{4}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 2.594 (sec). Leaf size: 23

```
dsolve([diff(y(t),t$2)-4*y(t)=Dirac(t-3),y(0) = 0, D(y)(0) = 1],y(t), singsol=all)
```

$$y(t) = \frac{\text{Heaviside}\,(-3+t)\sinh\,(2t-6)}{2} + \frac{\sinh\,(2t)}{2}$$

✓ Solution by Mathematica

Time used: 0.032 (sec). Leaf size: 44

```
DSolve[{y''[t]-4*y[t]==DiracDelta[t-3],{y[0]==0,y'[0]==1}},y[t],t,IncludeSingularSolutions -
```

$$y(t) \to \frac{1}{4}e^{-2(t+3)}\left(\left(e^{4t}-e^{12}\right)\theta(t-3)+e^{6}\left(e^{4t}-1\right)\right)$$

15.7 problem Problem 7

Internal problem ID [2890]
Internal file name [`OUTPUT/2382_Sunday_June_05_2022_03_03_24_AM_94482554/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 10, The Laplace Transform and Some Elementary Applications. Exercises for 10.8. page 710
Problem number: Problem 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 2y' + 5y = \delta\left(t - \frac{\pi}{2}\right)}$$

With initial conditions

$$[y(0) = 0, y'(0) = 2]$$

15.7.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 2 \\ q(t) &= 5 \\ F &= \delta\left(t - \frac{\pi}{2}\right) \end{aligned}$$

Hence the ode is

$$y'' + 2y' + 5y = \delta\left(t - \frac{\pi}{2}\right)$$

The domain of $p(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 5$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \delta\left(t - \frac{\pi}{2}\right)$ is

$$\left\{t < \frac{\pi}{2} \vee \frac{\pi}{2} < t\right\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 2sY(s) - 2y(0) + 5Y(s) = \mathrm{e}^{-\frac{s\pi}{2}} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 0\\ y'(0) &= 2\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 2 + 2sY(s) + 5Y(s) = \mathrm{e}^{-\frac{s\pi}{2}}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{\mathrm{e}^{-\frac{s\pi}{2}} + 2}{s^2 + 2s + 5}$$

Taking the inverse Laplace transform gives

$$
\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(\frac{e^{-\frac{s\pi}{2}}+2}{s^2+2s+5}\right) \\
&= \frac{\sin(2t)\left(-\text{Heaviside}\left(t-\frac{\pi}{2}\right)e^{\frac{\pi}{2}-t}+2\,e^{-t}\right)}{2}
\end{aligned}
$$

Converting the above solution to piecewise it becomes

$$
y = \begin{cases} e^{-t}\sin(2t) & t \le \frac{\pi}{2} \\ \frac{\sin(2t)\left(2\,e^{-t}-e^{\frac{\pi}{2}-t}\right)}{2} & \frac{\pi}{2} < t \end{cases}
$$

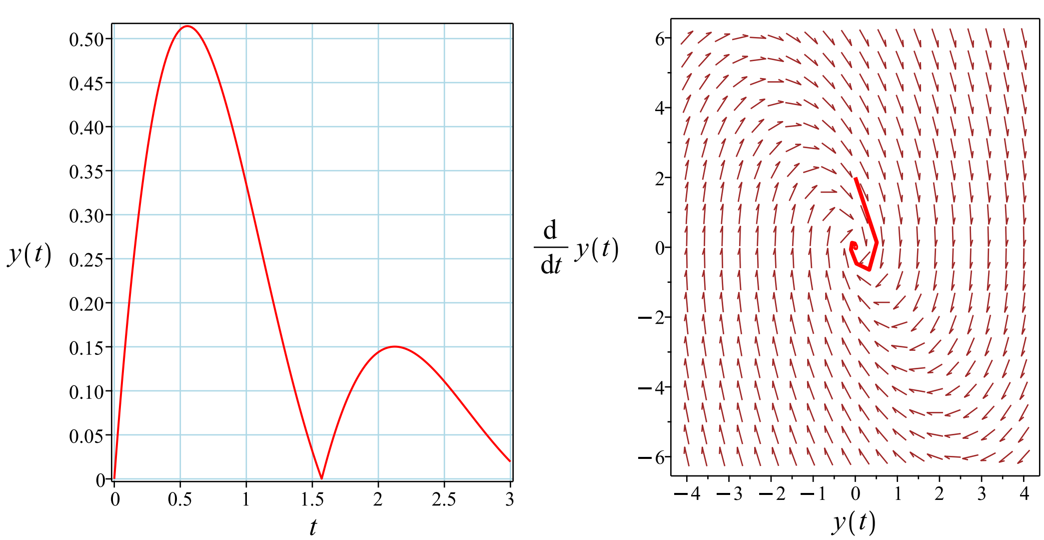
Simplifying the solution gives

$$
y = \sin(2t)\left(e^{-t} - \left(\begin{cases} 0 & t \le \frac{\pi}{2} \\ \frac{e^{\frac{\pi}{2}-t}}{2} & \frac{\pi}{2} < t \end{cases}\right)\right)
$$

Summary

The solution(s) found are the following

$$
y = \sin(2t)\left(e^{-t} - \left(\begin{cases} 0 & t \le \frac{\pi}{2} \\ \frac{e^{\frac{\pi}{2}-t}}{2} & \frac{\pi}{2} < t \end{cases}\right)\right) \tag{1}
$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \sin(2t)\left(\mathrm{e}^{-t} - \left(\left\{\begin{array}{cc} 0 & t \le \frac{\pi}{2} \\ \frac{\mathrm{e}^{\frac{\pi}{2}-t}}{2} & \frac{\pi}{2} < t \end{array}\right.\right)\right)$$

Verified OK.

15.7.2 Maple step by step solution

Let's solve

$\left[y'' + 2y' + 5y = \mathit{Dirac}\left(t - \frac{\pi}{2}\right), y(0) = 0, y'\Big|_{\{t=0\}} = 2\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 2r + 5 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-2)\pm(\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (-1 - 2\,\mathrm{I}, -1 + 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t) = \mathrm{e}^{-t}\cos(2t)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \mathrm{e}^{-t}\sin(2t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-t}\cos(2t) + c_2 \mathrm{e}^{-t}\sin(2t) + y_p(t)$

- □ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = \mathit{Dirac}\left(t - \frac{\pi}{2}\right)\right]$

  - Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \begin{bmatrix} e^{-t}\cos(2t) & e^{-t}\sin(2t) \\ -e^{-t}\cos(2t) - 2e^{-t}\sin(2t) & -e^{-t}\sin(2t) + 2e^{-t}\cos(2t) \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(t), y_2(t)) = 2e^{-2t}$

- Substitute functions into equation for $y_p(t)$

  $y_p(t) = -\frac{\left(\int Dirac\left(t-\frac{\pi}{2}\right)dt\right)\sin(2t)e^{\frac{\pi}{2}-t}}{2}$

- Compute integrals

  $y_p(t) = -\frac{Heaviside\left(t-\frac{\pi}{2}\right)\sin(2t)e^{\frac{\pi}{2}-t}}{2}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t) - \frac{Heaviside\left(t-\frac{\pi}{2}\right)\sin(2t)e^{\frac{\pi}{2}-t}}{2}$

☐ Check validity of solution $y = c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t) - \frac{Heaviside\left(t-\frac{\pi}{2}\right)\sin(2t)e^{\frac{\pi}{2}-t}}{2}$

- Use initial condition $y(0) = 0$

  $0 = c_1$

- Compute derivative of the solution

  $y' = -c_1 e^{-t}\cos(2t) - 2c_1 e^{-t}\sin(2t) - c_2 e^{-t}\sin(2t) + 2c_2 e^{-t}\cos(2t) - \frac{Dirac\left(t-\frac{\pi}{2}\right)\sin(2t)e^{\frac{\pi}{2}-t}}{2} - He$

- Use the initial condition $y'\Big|_{\{t=0\}} = 2$

  $2 = -c_1 + 2c_2$

- Solve for c_1 and c_2

  $\{c_1 = 0, c_2 = 1\}$

- Substitute constant values into general solution and simplify

  $y = -\frac{\sin(2t)\left(Heaviside\left(t-\frac{\pi}{2}\right)e^{\frac{\pi}{2}-t} - 2e^{-t}\right)}{2}$

- Solution to the IVP

  $y = -\frac{\sin(2t)\left(Heaviside\left(t-\frac{\pi}{2}\right)e^{\frac{\pi}{2}-t} - 2e^{-t}\right)}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 2.204 (sec). Leaf size: 33

```
dsolve([diff(y(t),t$2)+2*diff(y(t),t)+5*y(t)=Dirac(t-Pi/2),y(0) = 0, D(y)(0) = 2],y(t), sing
```

$$y(t) = \sin(2t)\left(-\frac{\text{Heaviside}\left(t-\frac{\pi}{2}\right)\mathrm{e}^{-t+\frac{\pi}{2}}}{2}+\mathrm{e}^{-t}\right)$$

✓ Solution by Mathematica

Time used: 0.123 (sec). Leaf size: 34

```
DSolve[{y''[t]+2*y'[t]+5*y[t]==DiracDelta[t-Pi/2],{y[0]==0,y'[0]==2}},y[t],t,IncludeSingular
```

$$y(t)\to -e^{-t}\left(e^{\pi/2}\theta(2t-\pi)-2\right)\sin(t)\cos(t)$$

15.8 problem Problem 8

Internal problem ID [2891]
Internal file name [`OUTPUT/2383_Sunday_June_05_2022_03_03_31_AM_69290317/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 10, The Laplace Transform and Some Elementary Applications. Exercises for 10.8. page 710
Problem number: Problem 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' - 4y' + 13y = \delta\left(t - \frac{\pi}{4}\right)}$$

With initial conditions

$$[y(0) = 3, y'(0) = 0]$$

15.8.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= -4 \\ q(t) &= 13 \\ F &= \delta\left(t - \frac{\pi}{4}\right) \end{aligned}$$

Hence the ode is

$$y'' - 4y' + 13y = \delta\left(t - \frac{\pi}{4}\right)$$

The domain of $p(t) = -4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 13$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \delta\left(t - \frac{\pi}{4}\right)$ is

$$\left\{t < \frac{\pi}{4} \vee \frac{\pi}{4} < t\right\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) - 4sY(s) + 4y(0) + 13Y(s) = \mathrm{e}^{-\frac{s\pi}{4}} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 3\\ y'(0) &= 0\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + 12 - 3s - 4sY(s) + 13Y(s) = \mathrm{e}^{-\frac{s\pi}{4}}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{\mathrm{e}^{-\frac{s\pi}{4}} + 3s - 12}{s^2 - 4s + 13}$$

Taking the inverse Laplace transform gives

$$
\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(\frac{e^{-\frac{s\pi}{4}} + 3s - 12}{s^2 - 4s + 13}\right) \\
&= \frac{e^{2t-\frac{\pi}{2}} \sin\left(3t + \frac{\pi}{4}\right)\left(-1 + \text{Heaviside}\left(-t + \frac{\pi}{4}\right)\right)}{3} + e^{2t}(3\cos(3t) - 2\sin(3t))
\end{aligned}
$$

Hence the final solution is

$$
y = \frac{e^{2t-\frac{\pi}{2}} \sin\left(3t + \frac{\pi}{4}\right)\left(-1 + \text{Heaviside}\left(-t + \frac{\pi}{4}\right)\right)}{3} + e^{2t}(3\cos(3t) - 2\sin(3t))
$$

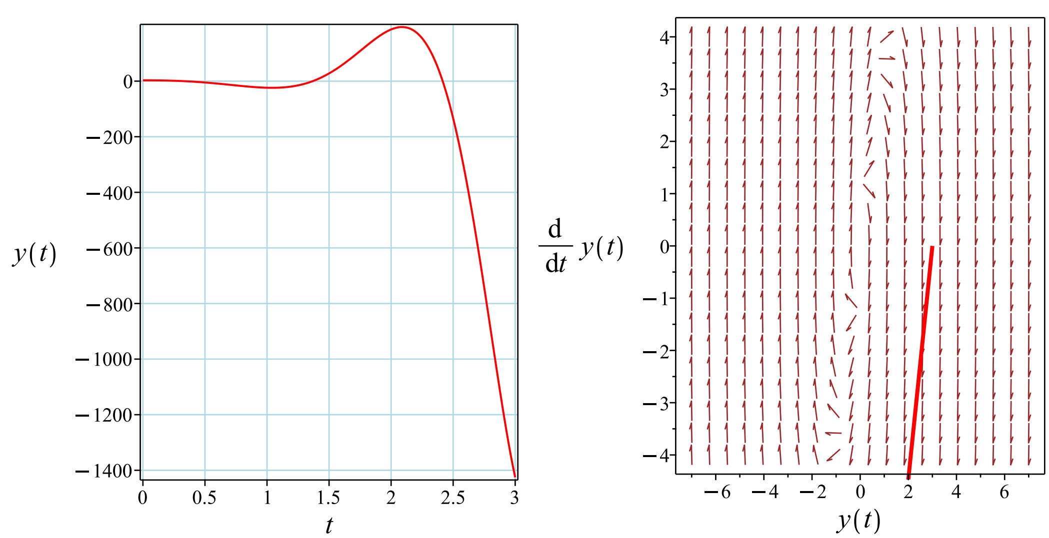
Simplifying the solution gives

$$
y = -\frac{\sqrt{2}\,(\sin(3t) + \cos(3t))\,\text{Heaviside}\left(t - \frac{\pi}{4}\right) e^{2t-\frac{\pi}{2}}}{6} + 3\,e^{2t}\left(\cos(3t) - \frac{2\sin(3t)}{3}\right)
$$

Summary

The solution(s) found are the following

$$
y = -\frac{\sqrt{2}\,(\sin(3t) + \cos(3t))\,\text{Heaviside}\left(t - \frac{\pi}{4}\right) e^{2t-\frac{\pi}{2}}}{6} + 3\,e^{2t}\left(\cos(3t) - \frac{2\sin(3t)}{3}\right) \tag{1}
$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = -\frac{\sqrt{2}\,(\sin(3t)+\cos(3t))\,\mathrm{Heaviside}\left(t-\frac{\pi}{4}\right)\mathrm{e}^{2t-\frac{\pi}{2}}}{6} + 3\,\mathrm{e}^{2t}\left(\cos(3t)-\frac{2\sin(3t)}{3}\right)$$

Verified OK.

15.8.2 Maple step by step solution

Let's solve

$\left[y''-4y'+13y = \mathit{Dirac}\left(t-\frac{\pi}{4}\right), y(0)=3, y'\Big|_{\{t=0\}}=0\right]$

- Highest derivative means the order of the ODE is 2
  y''
- Characteristic polynomial of homogeneous ODE
  $r^2-4r+13=0$
- Use quadratic formula to solve for r
  $r=\frac{4\pm(\sqrt{-36})}{2}$
- Roots of the characteristic polynomial
  $r=(2-3\,\mathrm{I}, 2+3\,\mathrm{I})$
- 1st solution of the homogeneous ODE
  $y_1(t)=\mathrm{e}^{2t}\cos(3t)$
- 2nd solution of the homogeneous ODE
  $y_2(t)=\mathrm{e}^{2t}\sin(3t)$
- General solution of the ODE
  $y=c_1y_1(t)+c_2y_2(t)+y_p(t)$
- Substitute in solutions of the homogeneous ODE
  $y=c_1\mathrm{e}^{2t}\cos(3t)+c_2\mathrm{e}^{2t}\sin(3t)+y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE
  - Use variation of parameters to find y_p here $f(t)$ is the forcing function
    $\left[y_p(t)=-y_1(t)\left(\int\frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right)+y_2(t)\left(\int\frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t)=\mathit{Dirac}\left(t-\frac{\pi}{4}\right)\right]$
  - Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \begin{bmatrix} e^{2t}\cos(3t) & e^{2t}\sin(3t) \\ 2e^{2t}\cos(3t) - 3e^{2t}\sin(3t) & 2e^{2t}\sin(3t) + 3e^{2t}\cos(3t) \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(t), y_2(t)) = 3e^{4t}$

- Substitute functions into equation for $y_p(t)$

  $y_p(t) = -\frac{\sqrt{2}\, e^{2t-\frac{\pi}{2}}(\sin(3t)+\cos(3t))\left(\int Dirac\left(t-\frac{\pi}{4}\right)dt\right)}{6}$

- Compute integrals

  $y_p(t) = -\frac{\sqrt{2}(\sin(3t)+\cos(3t))\, Heaviside\left(t-\frac{\pi}{4}\right) e^{2t-\frac{\pi}{2}}}{6}$

• Substitute particular solution into general solution to ODE

  $y = c_1 e^{2t}\cos(3t) + c_2 e^{2t}\sin(3t) - \frac{\sqrt{2}(\sin(3t)+\cos(3t))\, Heaviside\left(t-\frac{\pi}{4}\right) e^{2t-\frac{\pi}{2}}}{6}$

☐ Check validity of solution $y = c_1 e^{2t}\cos(3t) + c_2 e^{2t}\sin(3t) - \frac{\sqrt{2}(\sin(3t)+\cos(3t))\, Heaviside\left(t-\frac{\pi}{4}\right) e^{2t-\frac{\pi}{2}}}{6}$

- Use initial condition $y(0) = 3$

  $3 = c_1$

- Compute derivative of the solution

  $y' = 2c_1 e^{2t}\cos(3t) - 3c_1 e^{2t}\sin(3t) + 2c_2 e^{2t}\sin(3t) + 3c_2 e^{2t}\cos(3t) - \frac{\sqrt{2}(3\cos(3t)-3\sin(3t))\, Heaviside(t}{6}$

- Use the initial condition $y'\Big|_{\{t=0\}} = 0$

  $0 = 2c_1 + 3c_2$

- Solve for c_1 and c_2

  $\{c_1 = 3, c_2 = -2\}$

- Substitute constant values into general solution and simplify

  $y = -\frac{\sqrt{2}(\sin(3t)+\cos(3t))\, Heaviside\left(t-\frac{\pi}{4}\right) e^{2t-\frac{\pi}{2}}}{6} + 3e^{2t}\left(\cos(3t) - \frac{2\sin(3t)}{3}\right)$

• Solution to the IVP

  $y = -\frac{\sqrt{2}(\sin(3t)+\cos(3t))\, Heaviside\left(t-\frac{\pi}{4}\right) e^{2t-\frac{\pi}{2}}}{6} + 3e^{2t}\left(\cos(3t) - \frac{2\sin(3t)}{3}\right)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 3.344 (sec). Leaf size: 51

```
dsolve([diff(y(t),t$2)-4*diff(y(t),t)+13*y(t)=Dirac(t-Pi/4),y(0) = 3, D(y)(0) = 0],y(t), sin
```

$$y(t) = -\frac{\sqrt{2}\,\mathrm{e}^{-\frac{\pi}{2}+2t}\,\mathrm{Heaviside}\left(t-\frac{\pi}{4}\right)\left(\sin\left(3t\right)+\cos\left(3t\right)\right)}{6} + 3\,\mathrm{e}^{2t}\left(\cos\left(3t\right)-\frac{2\sin\left(3t\right)}{3}\right)$$

✓ Solution by Mathematica

Time used: 0.211 (sec). Leaf size: 61

```
DSolve[{y''[t]-4*y'[t]+13*y[t]==DiracDelta[t-Pi/4],{y[0]==3,y'[0]==0}},y[t],t,IncludeSingula
```

$$y(t) \to \frac{1}{6}e^{2t}\left(6(3\cos(3t)-2\sin(3t))-\sqrt{2}e^{-\pi/2}\theta(12t-3\pi)(\sin(3t)+\cos(3t))\right)$$

15.9 problem Problem 9

Internal problem ID [2892]
Internal file name [`OUTPUT/2384_Sunday_June_05_2022_03_03_38_AM_53352669/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 10, The Laplace Transform and Some Elementary Applications. Exercises for 10.8. page 710
Problem number: Problem 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' + 4y' + 3y = \delta(t - 2)}$$

With initial conditions

$$[y(0) = 1, y'(0) = -1]$$

15.9.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 4 \\ q(t) &= 3 \\ F &= \delta(t - 2) \end{aligned}$$

Hence the ode is

$$y'' + 4y' + 3y = \delta(t - 2)$$

The domain of $p(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 3$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \delta(t - 2)$ is

$$\{t < 2 \vee 2 < t\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 4sY(s) - 4y(0) + 3Y(s) = e^{-2s} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 1\\ y'(0) &= -1\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 3 - s + 4sY(s) + 3Y(s) = e^{-2s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{e^{-2s} + s + 3}{s^2 + 4s + 3}$$

Taking the inverse Laplace transform gives

$$\begin{aligned}y &= \mathcal{L}^{-1}(Y(s)) \\ &= \mathcal{L}^{-1}\left(\frac{e^{-2s} + s + 3}{s^2 + 4s + 3}\right) \\ &= \text{Heaviside}\,(t-2)\,e^{-2t+4}\sinh\,(t-2) + e^{-t}\end{aligned}$$

Hence the final solution is

$$y = \text{Heaviside}\,(t-2)\,e^{-2t+4}\sinh\,(t-2) + e^{-t}$$

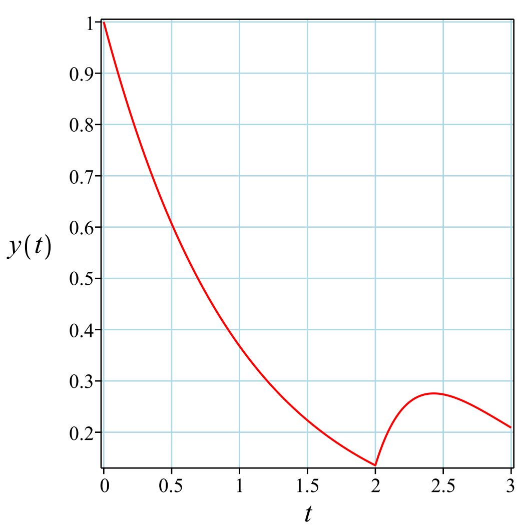
Simplifying the solution gives

$$y = \text{Heaviside}\,(t-2)\,e^{-2t+4}\sinh\,(t-2) + e^{-t}$$

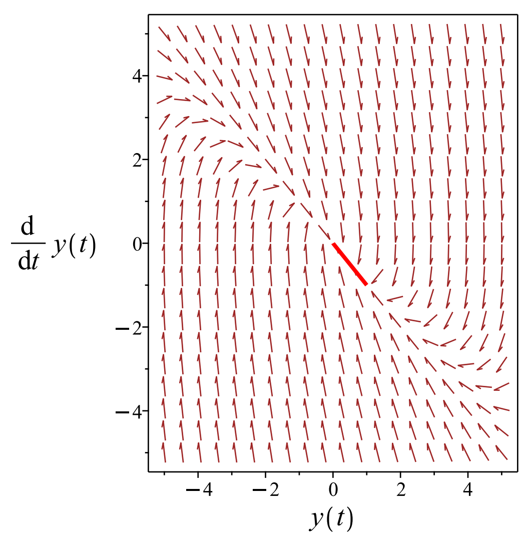
Summary

The solution(s) found are the following

$$y = \text{Heaviside}\,(t-2)\,e^{-2t+4}\sinh\,(t-2) + e^{-t} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \text{Heaviside}\,(t-2)\,e^{-2t+4}\sinh\,(t-2) + e^{-t}$$

Verified OK.

15.9.2 Maple step by step solution

Let's solve

$$\left[y''+4y'+3y=\mathit{Dirac}(t-2)\,,y(0)=1,y'\Big|_{\{t=0\}}=-1\right]$$

- Highest derivative means the order of the ODE is 2

  y''
- Characteristic polynomial of homogeneous ODE

  $r^2+4r+3=0$
- Factor the characteristic polynomial

  $(r+3)\,(r+1)=0$
- Roots of the characteristic polynomial

  $r=(-3,-1)$
- 1st solution of the homogeneous ODE

  $y_1(t)=\mathrm{e}^{-3t}$
- 2nd solution of the homogeneous ODE

  $y_2(t)=\mathrm{e}^{-t}$
- General solution of the ODE

  $y=c_1y_1(t)+c_2y_2(t)+y_p(t)$
- Substitute in solutions of the homogeneous ODE

  $y=c_1\mathrm{e}^{-3t}+c_2\mathrm{e}^{-t}+y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $$\left[y_p(t)=-y_1(t)\left(\int\frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right)+y_2(t)\left(\int\frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right),f(t)=\mathit{Dirac}(t-2)\right]$$
- ○ Wronskian of solutions of the homogeneous equation

  $$W(y_1(t)\,,y_2(t))=\left[\begin{array}{cc}\mathrm{e}^{-3t} & \mathrm{e}^{-t}\\ -3\,\mathrm{e}^{-3t} & -\mathrm{e}^{-t}\end{array}\right]$$
- ○ Compute Wronskian

  $W(y_1(t)\,,y_2(t))=2\,\mathrm{e}^{-4t}$
- ○ Substitute functions into equation for $y_p(t)$

$y_p(t) = -\frac{\left(\int \mathit{Dirac}(t-2)dt\right)\left(\mathrm{e}^{-3t+6}-\mathrm{e}^{-t+2}\right)}{2}$

- Compute integrals

  $y_p(t) = -\frac{\mathit{Heaviside}(t-2)\left(\mathrm{e}^{-3t+6}-\mathrm{e}^{-t+2}\right)}{2}$

- Substitute particular solution into general solution to ODE

  $y = c_1\mathrm{e}^{-3t} + c_2\mathrm{e}^{-t} - \frac{\mathit{Heaviside}(t-2)\left(\mathrm{e}^{-3t+6}-\mathrm{e}^{-t+2}\right)}{2}$

- □ Check validity of solution $y = c_1\mathrm{e}^{-3t} + c_2\mathrm{e}^{-t} - \frac{\mathit{Heaviside}(t-2)\left(\mathrm{e}^{-3t+6}-\mathrm{e}^{-t+2}\right)}{2}$
  - Use initial condition $y(0) = 1$

    $1 = c_1 + c_2$

  - Compute derivative of the solution

    $y' = -3c_1\mathrm{e}^{-3t} - c_2\mathrm{e}^{-t} - \frac{\mathit{Dirac}(t-2)\left(\mathrm{e}^{-3t+6}-\mathrm{e}^{-t+2}\right)}{2} - \frac{\mathit{Heaviside}(t-2)\left(-3\,\mathrm{e}^{-3t+6}+\mathrm{e}^{-t+2}\right)}{2}$

  - Use the initial condition $y'\Big|_{\{t=0\}} = -1$

    $-1 = -3c_1 - c_2$

  - Solve for c_1 and c_2

    $\{c_1 = 0, c_2 = 1\}$

  - Substitute constant values into general solution and simplify

    $y = \frac{\mathit{Heaviside}(t-2)\mathrm{e}^{-t+2}}{2} - \frac{\mathit{Heaviside}(t-2)\mathrm{e}^{-3t+6}}{2} + \mathrm{e}^{-t}$

- Solution to the IVP

  $y = \frac{\mathit{Heaviside}(t-2)\mathrm{e}^{-t+2}}{2} - \frac{\mathit{Heaviside}(t-2)\mathrm{e}^{-3t+6}}{2} + \mathrm{e}^{-t}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 2.406 (sec). Leaf size: 24

```
dsolve([diff(y(t),t$2)+4*diff(y(t),t)+3*y(t)=Dirac(t-2),y(0) = 1, D(y)(0) = -1],y(t), singso
```

$$y(t) = \text{Heaviside}\,(t-2)\, \mathrm{e}^{-2t+4} \sinh\,(t-2) + \mathrm{e}^{-t}$$

✓ Solution by Mathematica

Time used: 0.044 (sec). Leaf size: 37

```
DSolve[{y''[t]+4*y'[t]+3*y[t]==DiracDelta[t-2],{y[0]==1,y'[0]==-1}},y[t],t,IncludeSingularSo
```

$$y(t) \to \frac{1}{2} e^{2-3t} \left(e^{2t} - e^4\right) \theta(t-2) + e^{-t}$$

15.10 problem Problem 10

Internal problem ID [2893]
Internal file name [`OUTPUT/2385_Sunday_June_05_2022_03_03_45_AM_83974142/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 10, The Laplace Transform and Some Elementary Applications. Exercises for 10.8. page 710
Problem number: Problem 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 6y' + 13y = \delta\left(t - \frac{\pi}{4}\right)}$$

With initial conditions

$$[y(0) = 5, y'(0) = 5]$$

15.10.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 6 \\ q(t) &= 13 \\ F &= \delta\left(t - \frac{\pi}{4}\right) \end{aligned}$$

Hence the ode is

$$y'' + 6y' + 13y = \delta\left(t - \frac{\pi}{4}\right)$$

The domain of $p(t) = 6$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 13$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \delta\left(t - \frac{\pi}{4}\right)$ is

$$\left\{t < \frac{\pi}{4} \vee \frac{\pi}{4} < t\right\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 6sY(s) - 6y(0) + 13Y(s) = \mathrm{e}^{-\frac{s\pi}{4}} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 5\\ y'(0) &= 5\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 35 - 5s + 6sY(s) + 13Y(s) = \mathrm{e}^{-\frac{s\pi}{4}}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{\mathrm{e}^{-\frac{s\pi}{4}} + 5s + 35}{s^2 + 6s + 13}$$

Taking the inverse Laplace transform gives

$$
\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(\frac{e^{-\frac{s\pi}{4}} + 5s + 35}{s^2 + 6s + 13}\right) \\
&= -\frac{\text{Heaviside}\left(t - \frac{\pi}{4}\right) e^{-3t + \frac{3\pi}{4}} \cos(2t)}{2} + 5\,e^{-3t}(\cos(2t) + 2\sin(2t))
\end{aligned}
$$

Hence the final solution is

$$
y = -\frac{\text{Heaviside}\left(t - \frac{\pi}{4}\right) e^{-3t + \frac{3\pi}{4}} \cos(2t)}{2} + 5\,e^{-3t}(\cos(2t) + 2\sin(2t))
$$

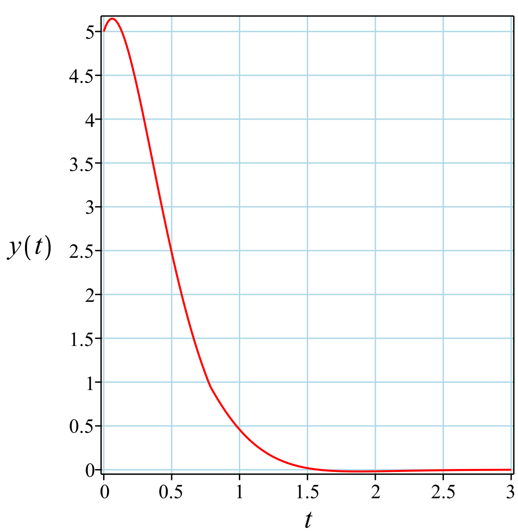
Simplifying the solution gives

$$
y = -\frac{\text{Heaviside}\left(t - \frac{\pi}{4}\right) e^{-3t + \frac{3\pi}{4}} \cos(2t)}{2} + 5\,e^{-3t}(\cos(2t) + 2\sin(2t))
$$

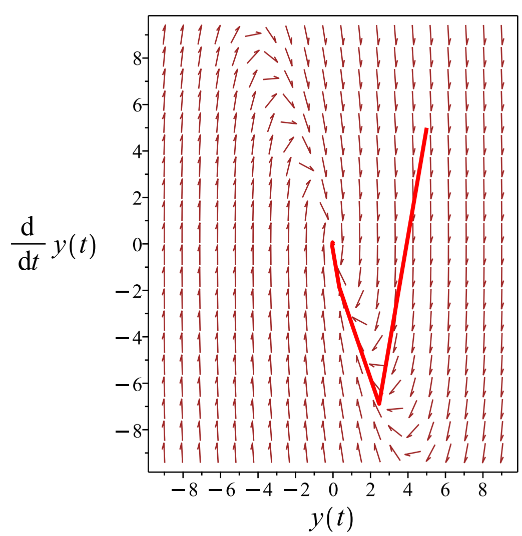
Summary

The solution(s) found are the following

$$
y = -\frac{\text{Heaviside}\left(t - \frac{\pi}{4}\right) e^{-3t + \frac{3\pi}{4}} \cos(2t)}{2} + 5\,e^{-3t}(\cos(2t) + 2\sin(2t)) \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{\text{Heaviside}\left(t - \frac{\pi}{4}\right) \mathrm{e}^{-3t+\frac{3\pi}{4}} \cos(2t)}{2} + 5\,\mathrm{e}^{-3t}(\cos(2t) + 2\sin(2t))$$

Verified OK.

15.10.2 Maple step by step solution

Let's solve

$\left[y'' + 6y' + 13y = \mathit{Dirac}\left(t - \frac{\pi}{4}\right), y(0) = 5, y'\Big|_{\{t=0\}} = 5\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 6r + 13 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-6)\pm(\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (-3 - 2\,\mathrm{I}, -3 + 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t) = \mathrm{e}^{-3t}\cos(2t)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \mathrm{e}^{-3t}\sin(2t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-3t}\cos(2t) + c_2 \mathrm{e}^{-3t}\sin(2t) + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = \mathit{Dirac}\left(t - \frac{\pi}{4}\right)\right]$

  ◦ Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \begin{bmatrix} e^{-3t}\cos(2t) & e^{-3t}\sin(2t) \\ -3e^{-3t}\cos(2t) - 2e^{-3t}\sin(2t) & -3e^{-3t}\sin(2t) + 2e^{-3t}\cos(2t) \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(t), y_2(t)) = 2e^{-6t}$

- Substitute functions into equation for $y_p(t)$

  $y_p(t) = -\frac{\left(\int Dirac\left(t-\frac{\pi}{4}\right)dt\right)\cos(2t)e^{-3t+\frac{3\pi}{4}}}{2}$

- Compute integrals

  $y_p(t) = -\frac{Heaviside\left(t-\frac{\pi}{4}\right)e^{-3t+\frac{3\pi}{4}}\cos(2t)}{2}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-3t}\cos(2t) + c_2 e^{-3t}\sin(2t) - \frac{Heaviside\left(t-\frac{\pi}{4}\right)e^{-3t+\frac{3\pi}{4}}\cos(2t)}{2}$

□ Check validity of solution $y = c_1 e^{-3t}\cos(2t) + c_2 e^{-3t}\sin(2t) - \frac{Heaviside\left(t-\frac{\pi}{4}\right)e^{-3t+\frac{3\pi}{4}}\cos(2t)}{2}$

- Use initial condition $y(0) = 5$

  $5 = c_1$

- Compute derivative of the solution

  $y' = -3c_1 e^{-3t}\cos(2t) - 2c_1 e^{-3t}\sin(2t) - 3c_2 e^{-3t}\sin(2t) + 2c_2 e^{-3t}\cos(2t) - \frac{Dirac\left(t-\frac{\pi}{4}\right)e^{-3t+\frac{3\pi}{4}}\cos(}{2}$

- Use the initial condition $y'\Big|_{\{t=0\}} = 5$

  $5 = -3c_1 + 2c_2$

- Solve for c_1 and c_2

  $\{c_1 = 5, c_2 = 10\}$

- Substitute constant values into general solution and simplify

  $y = -\frac{Heaviside\left(t-\frac{\pi}{4}\right)e^{-3t+\frac{3\pi}{4}}\cos(2t)}{2} + 5e^{-3t}(\cos(2t) + 2\sin(2t))$

- Solution to the IVP

  $y = -\frac{Heaviside\left(t-\frac{\pi}{4}\right)e^{-3t+\frac{3\pi}{4}}\cos(2t)}{2} + 5e^{-3t}(\cos(2t) + 2\sin(2t))$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple
Time used: 2.531 (sec). Leaf size: 42

```
dsolve([diff(y(t),t$2)+6*diff(y(t),t)+13*y(t)=Dirac(t-Pi/4),y(0) = 5, D(y)(0) = 5],y(t), sin
```

$$y(t) = -\frac{\mathrm{Heaviside}\left(t-\frac{\pi}{4}\right)\cos(2t)\,\mathrm{e}^{\frac{3\pi}{4}-3t}}{2}+5\,\mathrm{e}^{-3t}(\cos(2t)+2\sin(2t))$$

✓ Solution by Mathematica
Time used: 0.287 (sec). Leaf size: 121

```
DSolve[{y''[t]+46*y'[t]+13*y[t]==DiracDelta[t-Pi/4],{y[0]==1,y'[0]==-1}},y[t],t,IncludeSingu
```

$$y(t) \to \frac{1}{516}e^{-2\sqrt{129}t-23t-\frac{\sqrt{129}\pi}{2}}\left(2e^{\frac{\sqrt{129}\pi}{2}}\left(\left(129+11\sqrt{129}\right)e^{4\sqrt{129}t}+129-11\sqrt{129}\right)\right.$$
$$\left.-\sqrt{129}e^{23\pi/4}\left(e^{\sqrt{129}\pi}-e^{4\sqrt{129}t}\right)\theta(4t-\pi)\right)$$

15.11 problem Problem 11

Internal problem ID [2894]
Internal file name [`OUTPUT/2386_Sunday_June_05_2022_03_03_52_AM_56924303/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 10, The Laplace Transform and Some Elementary Applications. Exercises for 10.8. page 710
Problem number: Problem 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' + 9y = 15\sin(2t) + \delta\left(t - \frac{\pi}{6}\right)}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

15.11.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= 9 \\ F &= 15\sin(2t) + \delta\left(t - \frac{\pi}{6}\right) \end{aligned}$$

Hence the ode is

$$y'' + 9y = 15\sin(2t) + \delta\left(t - \frac{\pi}{6}\right)$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 9$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 15\sin(2t) + \delta\left(t - \frac{\pi}{6}\right)$ is

$$\left\{t < \frac{\pi}{6} \lor \frac{\pi}{6} < t\right\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 9Y(s) = \frac{30}{s^2+4} + \mathrm{e}^{-\frac{s\pi}{6}} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 0\\ y'(0) &= 0\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + 9Y(s) = \frac{30}{s^2+4} + \mathrm{e}^{-\frac{s\pi}{6}}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{\mathrm{e}^{-\frac{s\pi}{6}}s^2 + 4\,\mathrm{e}^{-\frac{s\pi}{6}} + 30}{(s^2+4)(s^2+9)}$$

Taking the inverse Laplace transform gives

$$
\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(\frac{e^{-\frac{s\pi}{6}} s^2 + 4\,e^{-\frac{s\pi}{6}} + 30}{(s^2+4)(s^2+9)}\right) \\
&= -\frac{\text{Heaviside}\left(t-\frac{\pi}{6}\right)\cos(3t)}{3} - 2\sin(3t) + 3\sin(2t)
\end{aligned}
$$

Hence the final solution is

$$
y = -\frac{\text{Heaviside}\left(t-\frac{\pi}{6}\right)\cos(3t)}{3} - 2\sin(3t) + 3\sin(2t)
$$

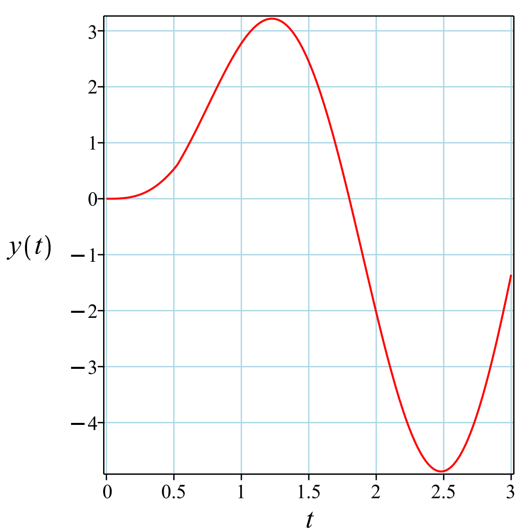
Simplifying the solution gives

$$
y = -\frac{\text{Heaviside}\left(t-\frac{\pi}{6}\right)\cos(3t)}{3} - 2\sin(3t) + 3\sin(2t)
$$

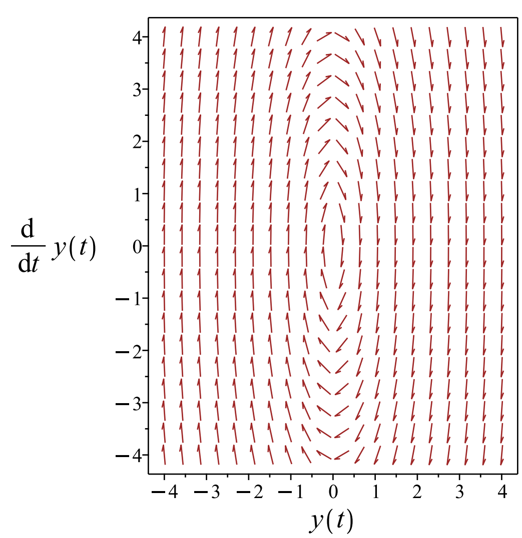
<u>Summary</u>

The solution(s) found are the following

$$
y = -\frac{\text{Heaviside}\left(t-\frac{\pi}{6}\right)\cos(3t)}{3} - 2\sin(3t) + 3\sin(2t) \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y=-\frac{\text{Heaviside}\left(t-\frac{\pi}{6}\right)\cos(3t)}{3}-2\sin(3t)+3\sin(2t)$$

Verified OK.

15.11.2 Maple step by step solution

Let's solve

$\left[y''+9y=15\sin(2t)+\mathit{Dirac}\left(t-\frac{\pi}{6}\right), y(0)=0, y'\Big|_{\{t=0\}}=0\right]$

- Highest derivative means the order of the ODE is 2

  y''
- Characteristic polynomial of homogeneous ODE

  $r^2+9=0$
- Use quadratic formula to solve for r

  $r=\frac{0\pm(\sqrt{-36})}{2}$
- Roots of the characteristic polynomial

  $r=(-3\,\mathrm{I}, 3\,\mathrm{I})$
- 1st solution of the homogeneous ODE

  $y_1(t)=\cos(3t)$
- 2nd solution of the homogeneous ODE

  $y_2(t)=\sin(3t)$
- General solution of the ODE

  $y=c_1y_1(t)+c_2y_2(t)+y_p(t)$
- Substitute in solutions of the homogeneous ODE

  $y=c_1\cos(3t)+c_2\sin(3t)+y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t)=-y_1(t)\left(\int\frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right)+y_2(t)\left(\int\frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t)=15\sin(2t)+\mathit{Dirac}\left(t-\frac{\pi}{6}\right)\right]$
- ○ Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \begin{bmatrix} \cos(3t) & \sin(3t) \\ -3\sin(3t) & 3\cos(3t) \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(t), y_2(t)) = 3$

- Substitute functions into equation for $y_p(t)$

  $$y_p(t) = -\frac{\cos(3t)\left(\int\left(-120\cos(t)^5+150\cos(t)^3+Dirac\left(t-\frac{\pi}{6}\right)-30\cos(t)\right)dt\right)}{3} + \frac{5\sin(3t)\left(\int(\sin(5t)-\sin(t))dt\right)}{2}$$

- Compute integrals

  $$y_p(t) = -\frac{\cos(t)\left(\left(4\cos(t)^2-3\right)Heaviside\left(t-\frac{\pi}{6}\right)-18\sin(t)\right)}{3}$$

- Substitute particular solution into general solution to ODE

  $$y = c_1\cos(3t) + c_2\sin(3t) - \frac{\cos(t)\left(\left(4\cos(t)^2-3\right)Heaviside\left(t-\frac{\pi}{6}\right)-18\sin(t)\right)}{3}$$

☐ Check validity of solution $y = c_1\cos(3t) + c_2\sin(3t) - \frac{\cos(t)\left(\left(4\cos(t)^2-3\right)Heaviside\left(t-\frac{\pi}{6}\right)-18\sin(t)\right)}{3}$

- Use initial condition $y(0) = 0$

  $0 = c_1$

- Compute derivative of the solution

  $$y' = -3c_1\sin(3t) + 3c_2\cos(3t) + \frac{\sin(t)\left(\left(4\cos(t)^2-3\right)Heaviside\left(t-\frac{\pi}{6}\right)-18\sin(t)\right)}{3} - \frac{\cos(t)\left(-8\sin(t)\cos(t)Heavisid\right.}{}$$

- Use the initial condition $y'\Big|_{\{t=0\}} = 0$

  $0 = 6 + 3c_2$

- Solve for c_1 and c_2

  $\{c_1 = 0, c_2 = -2\}$

- Substitute constant values into general solution and simplify

  $$y = \frac{\left(-4\cos(t)^3+3\cos(t)\right)Heaviside\left(t-\frac{\pi}{6}\right)}{3} - 8(\cos(t)-1)\sin(t)\left(\cos(t)+\frac{1}{4}\right)$$

- Solution to the IVP

  $$y = \frac{\left(-4\cos(t)^3+3\cos(t)\right)Heaviside\left(t-\frac{\pi}{6}\right)}{3} - 8(\cos(t)-1)\sin(t)\left(\cos(t)+\frac{1}{4}\right)$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 3.438 (sec). Leaf size: 29

```
dsolve([diff(y(t),t$2)+9*y(t)=15*sin(2*t)+Dirac(t-Pi/6),y(0) = 0, D(y)(0) = 0],y(t), singsol
```

$$y(t) = -\frac{\cos(3t)\,\mathrm{Heaviside}\left(t-\frac{\pi}{6}\right)}{3} - 2\sin(3t) + 3\sin(2t)$$

✓ Solution by Mathematica

Time used: 0.075 (sec). Leaf size: 34

```
DSolve[{y''[t]+9*y[t]==15*Sin[2*t]+DiracDelta[t-Pi/6],{y[0]==0,y'[0]==0}},y[t],t,IncludeSing
```

$$y(t) \to -\frac{1}{3}\theta(6t-\pi)\cos(3t) + 3\sin(2t) - 2\sin(3t)$$

15.12 problem Problem 12

Internal problem ID [2895]
Internal file name [`OUTPUT/2387_Sunday_June_05_2022_03_03_59_AM_82386455/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 10, The Laplace Transform and Some Elementary Applications. Exercises for 10.8. page 710
Problem number: Problem 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 16y = 4\cos(3t) + \delta\left(t - \frac{\pi}{3}\right)}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

15.12.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= 16 \\ F &= 4\cos(3t) + \delta\left(t - \frac{\pi}{3}\right) \end{aligned}$$

Hence the ode is

$$y'' + 16y = 4\cos(3t) + \delta\left(t - \frac{\pi}{3}\right)$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 16$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 4\cos(3t) + \delta\left(t - \frac{\pi}{3}\right)$ is

$$\left\{t < \frac{\pi}{3} \vee \frac{\pi}{3} < t\right\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}
\mathcal{L}(y') &= sY(s) - y(0) \\
\mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)
\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 16Y(s) = \frac{4s}{s^2+9} + \mathrm{e}^{-\frac{s\pi}{3}} \tag{1}$$

But the initial conditions are

$$\begin{aligned}
y(0) &= 0 \\
y'(0) &= 0
\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + 16Y(s) = \frac{4s}{s^2+9} + \mathrm{e}^{-\frac{s\pi}{3}}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{\mathrm{e}^{-\frac{s\pi}{3}} s^2 + 9\,\mathrm{e}^{-\frac{s\pi}{3}} + 4s}{(s^2+9)(s^2+16)}$$

Taking the inverse Laplace transform gives

$$
\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(\frac{\mathrm{e}^{-\frac{s\pi}{3}} s^2 + 9\,\mathrm{e}^{-\frac{s\pi}{3}} + 4s}{(s^2+9)(s^2+16)}\right) \\
&= \frac{\mathrm{Heaviside}\left(t-\frac{\pi}{3}\right)\cos\left(4t+\frac{\pi}{6}\right)}{4} + \frac{4\cos(3t)}{7} - \frac{4\cos(4t)}{7}
\end{aligned}
$$

Hence the final solution is

$$
y = \frac{\mathrm{Heaviside}\left(t-\frac{\pi}{3}\right)\cos\left(4t+\frac{\pi}{6}\right)}{4} + \frac{4\cos(3t)}{7} - \frac{4\cos(4t)}{7}
$$

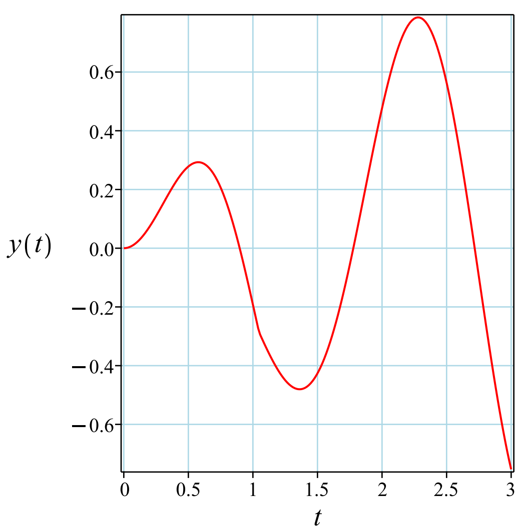
Simplifying the solution gives

$$
y = \frac{\mathrm{Heaviside}\left(t-\frac{\pi}{3}\right)\left(\cos(4t)\sqrt{3}-\sin(4t)\right)}{8} + \frac{4\cos(3t)}{7} - \frac{4\cos(4t)}{7}
$$

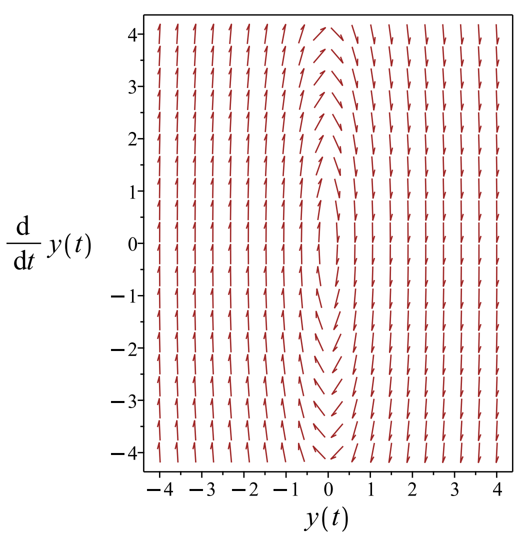
Summary

The solution(s) found are the following

$$
y = \frac{\mathrm{Heaviside}\left(t-\frac{\pi}{3}\right)\left(\cos(4t)\sqrt{3}-\sin(4t)\right)}{8} + \frac{4\cos(3t)}{7} - \frac{4\cos(4t)}{7} \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{\mathrm{Heaviside}\left(t-\frac{\pi}{3}\right)\left(\cos\left(4t\right)\sqrt{3}-\sin\left(4t\right)\right)}{8}+\frac{4\cos\left(3t\right)}{7}-\frac{4\cos\left(4t\right)}{7}$$

Verified OK.

15.12.2 Maple step by step solution

Let's solve

$\left[y''+16y = 4\cos\left(3t\right)+\mathit{Dirac}\left(t-\frac{\pi}{3}\right), y\left(0\right)=0, y'\Big|_{\{t=0\}}=0\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2+16=0$

- Use quadratic formula to solve for r

  $r = \frac{0\pm\left(\sqrt{-64}\right)}{2}$

- Roots of the characteristic polynomial

  $r = \left(-4\,\mathrm{I}, 4\,\mathrm{I}\right)$

- 1st solution of the homogeneous ODE

  $y_1(t) = \cos\left(4t\right)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \sin\left(4t\right)$

- General solution of the ODE

  $y = c_1y_1(t)+c_2y_2(t)+y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1\cos\left(4t\right)+c_2\sin\left(4t\right)+y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right)+y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = 4\cos\left(3t\right)+\mathit{Dirac}\left(t-\frac{\pi}{3}\right)\right]$

  - Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \begin{bmatrix} \cos(4t) & \sin(4t) \\ -4\sin(4t) & 4\cos(4t) \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(t), y_2(t)) = 4$

- Substitute functions into equation for $y_p(t)$

  $$y_p(t) = -\frac{\cos(4t)\left(\int\left(-\sqrt{3}\,Dirac(t-\frac{\pi}{3})+8\sin(4t)\cos(3t)\right)dt\right)}{8} + \frac{\sin(4t)\left(\int\left(8\cos(4t)\cos(3t)-Dirac(t-\frac{\pi}{3})\right)dt\right)}{8}$$

- Compute integrals

  $$y_p(t) = \frac{\left(\left(8\cos(t)^4-8\cos(t)^2+1\right)\sqrt{3}-8\sin(t)\cos(t)^3+4\sin(t)\cos(t)\right)Heaviside(t-\frac{\pi}{3})}{8} + \frac{16\cos(t)^3}{7} - \frac{12\cos(t)}{7}$$

- Substitute particular solution into general solution to ODE

  $$y = c_1\cos(4t) + c_2\sin(4t) + \frac{\left(\left(8\cos(t)^4-8\cos(t)^2+1\right)\sqrt{3}-8\sin(t)\cos(t)^3+4\sin(t)\cos(t)\right)Heaviside(t-\frac{\pi}{3})}{8} + \frac{16\cos(t)^3}{7}$$

☐ Check validity of solution $y = c_1\cos(4t) + c_2\sin(4t) + \frac{\left(\left(8\cos(t)^4-8\cos(t)^2+1\right)\sqrt{3}-8\sin(t)\cos(t)^3+4\sin(t)\right.}{8}$

- Use initial condition $y(0) = 0$

  $0 = c_1 + \frac{4}{7}$

- Compute derivative of the solution

  $$y' = -4c_1\sin(4t) + 4c_2\cos(4t) + \frac{\left(\left(-32\sin(t)\cos(t)^3+16\sin(t)\cos(t)\right)\sqrt{3}-8\cos(t)^4+24\sin(t)^2\cos(t)^2+4\cos(t)^2-4\right.}{8}$$

- Use the initial condition $y'\Big|_{\{t=0\}} = 0$

  $0 = 4c_2$

- Solve for c_1 and c_2

  $\left\{c_1 = -\frac{4}{7}, c_2 = 0\right\}$

- Substitute constant values into general solution and simplify

  $$y = -\frac{4}{7} + \frac{\left(\left(8\cos(t)^4-8\cos(t)^2+1\right)\sqrt{3}-8\sin(t)\cos(t)^3+4\sin(t)\cos(t)\right)Heaviside(t-\frac{\pi}{3})}{8} - \frac{32\cos(t)^4}{7} + \frac{16\cos(t)^3}{7} + \frac{32\cos}{7}$$

- Solution to the IVP

  $$y = -\frac{4}{7} + \frac{\left(\left(8\cos(t)^4-8\cos(t)^2+1\right)\sqrt{3}-8\sin(t)\cos(t)^3+4\sin(t)\cos(t)\right)Heaviside(t-\frac{\pi}{3})}{8} - \frac{32\cos(t)^4}{7} + \frac{16\cos(t)^3}{7} + \frac{32\cos}{7}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 3.609 (sec). Leaf size: 33

```
dsolve([diff(y(t),t$2)+16*y(t)=4*cos(3*t)+Dirac(t-Pi/3),y(0) = 0, D(y)(0) = 0],y(t), singsol
```

$$y(t) = \frac{\left(\cos \left(4t\right) \sqrt{3}-\sin \left(4t\right)\right) \operatorname{Heaviside}\left(t-\frac{\pi}{3}\right)}{8}-\frac{4 \cos \left(4t\right)}{7}+\frac{4 \cos \left(3t\right)}{7}$$

✓ Solution by Mathematica

Time used: 0.159 (sec). Leaf size: 50

```
DSolve[{y''[t]+16*y[t]==4*Cos[3*t]+DiracDelta[t-Pi/3],{y[0]==0,y'[0]==0}},y[t],t,IncludeSing
```

$$y(t) \to \frac{1}{8}\theta(3t-\pi)\left(\sqrt{3}\cos(4t)-\sin(4t)\right)+\frac{4}{7}(\cos(3t)-\cos(4t))$$

15.13 problem Problem 13

Internal problem ID [2896]
Internal file name [`OUTPUT/2388_Sunday_June_05_2022_03_04_08_AM_19570783/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 10, The Laplace Transform and Some Elementary Applications. Exercises for 10.8. page 710
Problem number: Problem 13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 2y' + 5y = 4\sin(t) + \delta\left(t - \frac{\pi}{6}\right)}$$

With initial conditions

$$[y(0) = 0, y'(0) = 1]$$

15.13.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 2 \\ q(t) &= 5 \\ F &= 4\sin(t) + \delta\left(t - \frac{\pi}{6}\right) \end{aligned}$$

Hence the ode is

$$y'' + 2y' + 5y = 4\sin(t) + \delta\left(t - \frac{\pi}{6}\right)$$

The domain of $p(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 5$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 4\sin(t) + \delta\left(t - \frac{\pi}{6}\right)$ is

$$\left\{t < \frac{\pi}{6} \vee \frac{\pi}{6} < t\right\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}
\mathcal{L}(y') &= sY(s) - y(0) \\
\mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)
\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 2sY(s) - 2y(0) + 5Y(s) = \frac{4}{s^2+1} + e^{-\frac{s\pi}{6}} \tag{1}$$

But the initial conditions are

$$\begin{aligned}
y(0) &= 0 \\
y'(0) &= 1
\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 1 + 2sY(s) + 5Y(s) = \frac{4}{s^2+1} + e^{-\frac{s\pi}{6}}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{e^{-\frac{s\pi}{6}}s^2 + s^2 + e^{-\frac{s\pi}{6}} + 5}{(s^2+1)(s^2+2s+5)}$$

Taking the inverse Laplace transform gives

$$
\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(\frac{e^{-\frac{s\pi}{6}}s^2 + s^2 + e^{-\frac{s\pi}{6}} + 5}{(s^2+1)(s^2+2s+5)}\right) \\
&= -\frac{\text{Heaviside}\left(t-\frac{\pi}{6}\right) e^{-t+\frac{\pi}{6}} \cos\left(2t+\frac{\pi}{6}\right)}{2} - \frac{2\cos(t)}{5} + \frac{4\sin(t)}{5} + \frac{e^{-t}(4\cos(2t)+3\sin(2t))}{10}
\end{aligned}
$$

Hence the final solution is

$$
y = -\frac{\text{Heaviside}\left(t-\frac{\pi}{6}\right) e^{-t+\frac{\pi}{6}} \cos\left(2t+\frac{\pi}{6}\right)}{2} - \frac{2\cos(t)}{5} + \frac{4\sin(t)}{5} + \frac{e^{-t}(4\cos(2t)+3\sin(2t))}{10}
$$

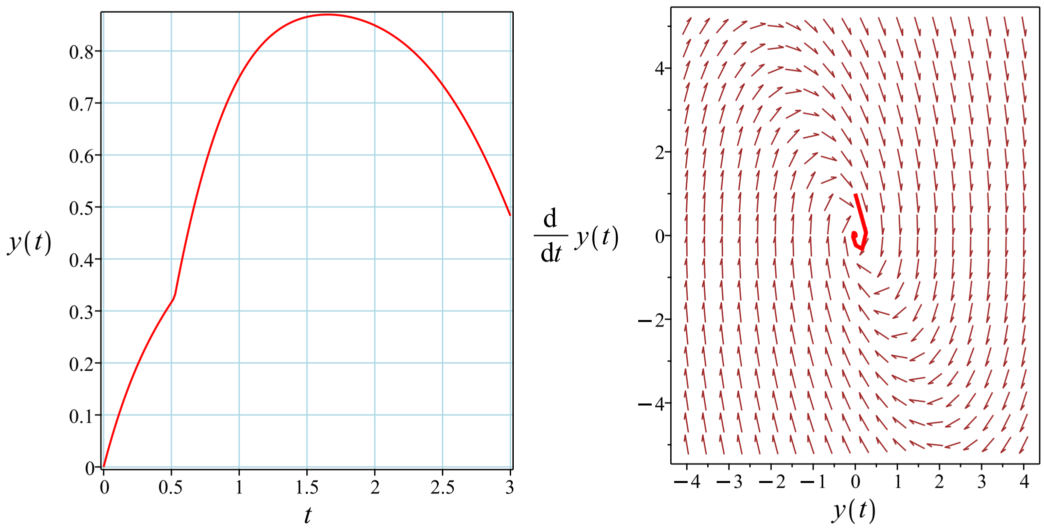
Simplifying the solution gives

$$
y = -\frac{\text{Heaviside}\left(t-\frac{\pi}{6}\right)\left(\cos(t)^2\sqrt{3} - \sin(t)\cos(t) - \frac{\sqrt{3}}{2}\right) e^{-t+\frac{\pi}{6}}}{2} + \frac{\left(4\cos(t)^2 + 3\sin(t)\cos(t) - 2\right) e^{-t}}{5} - \frac{2\cos(t)}{5} + \frac{4\sin(t)}{5}
$$

Summary

The solution(s) found are the following

$$
y = -\frac{\text{Heaviside}\left(t-\frac{\pi}{6}\right)\left(\cos(t)^2\sqrt{3} - \sin(t)\cos(t) - \frac{\sqrt{3}}{2}\right) e^{-t+\frac{\pi}{6}}}{2} + \frac{\left(4\cos(t)^2 + 3\sin(t)\cos(t) - 2\right) e^{-t}}{5} - \frac{2\cos(t)}{5} + \frac{4\sin(t)}{5} \tag{1}
$$



(a) Solution plot

(b) Slope field plot

<u>Verification of solutions</u>

$$y = -\frac{\text{Heaviside}\left(t-\frac{\pi}{6}\right)\left(\cos\left(t\right)^{2}\sqrt{3}-\sin\left(t\right)\cos\left(t\right)-\frac{\sqrt{3}}{2}\right)\mathrm{e}^{-t+\frac{\pi}{6}}}{2} + \frac{\left(4\cos\left(t\right)^{2}+3\sin\left(t\right)\cos\left(t\right)-2\right)\mathrm{e}^{-t}}{5} - \frac{2\cos\left(t\right)}{5} + \frac{4\sin\left(t\right)}{5}$$

Verified OK.

15.13.2 Maple step by step solution

Let's solve

$$\left[y''+2y'+5y=4\sin\left(t\right)+\mathit{Dirac}\left(t-\frac{\pi}{6}\right), y(0)=0, y'\Big|_{\{t=0\}}=1\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^{2}+2r+5=0$

- Use quadratic formula to solve for r

  $r=\frac{(-2)\pm\left(\sqrt{-16}\right)}{2}$

- Roots of the characteristic polynomial

$r = (-1 - 2\,\mathrm{I}, -1 + 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t) = \mathrm{e}^{-t} \cos(2t)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \mathrm{e}^{-t} \sin(2t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-t} \cos(2t) + c_2 \mathrm{e}^{-t} \sin(2t) + y_p(t)$

- ☐ Find a particular solution $y_p(t)$ of the ODE
  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t) \left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t) \left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = 4 \sin(t) + \mathit{Dirac}\left(t - \frac{\pi}{6}\right)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(t), y_2(t)) = \begin{bmatrix} \mathrm{e}^{-t} \cos(2t) & \mathrm{e}^{-t} \sin(2t) \\ -\mathrm{e}^{-t} \cos(2t) - 2\,\mathrm{e}^{-t} \sin(2t) & -\mathrm{e}^{-t} \sin(2t) + 2\,\mathrm{e}^{-t} \cos(2t) \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(t), y_2(t)) = 2\,\mathrm{e}^{-2t}$

  - Substitute functions into equation for $y_p(t)$

    $y_p(t) = \frac{\mathrm{e}^{-t} \left(-\cos(2t) \left(\int \left(16 \sin(t)^2 \cos(t) \mathrm{e}^{t} + \sqrt{3}\, \mathrm{e}^{\frac{\pi}{6}}\, \mathit{Dirac}\left(t - \frac{\pi}{6}\right)\right) dt\right) + \sin(2t) \left(\int \left(8 \cos(2t) \sin(t) \mathrm{e}^{t} + \mathrm{e}^{\frac{\pi}{6}}\, \mathit{Dirac}\left(t - \frac{\pi}{6}\right)\right) dt\right)\right)}{4}$

  - Compute integrals

    $y_p(t) = -\frac{\mathit{Heaviside}\left(t - \frac{\pi}{6}\right) \left(\cos(t)^2 \sqrt{3} - \sin(t) \cos(t) - \frac{\sqrt{3}}{2}\right) \mathrm{e}^{-t + \frac{\pi}{6}}}{2} - \frac{2 \cos(t)}{5} + \frac{4 \sin(t)}{5}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-t} \cos(2t) + c_2 \mathrm{e}^{-t} \sin(2t) - \frac{\mathit{Heaviside}\left(t - \frac{\pi}{6}\right) \left(\cos(t)^2 \sqrt{3} - \sin(t) \cos(t) - \frac{\sqrt{3}}{2}\right) \mathrm{e}^{-t + \frac{\pi}{6}}}{2} - \frac{2 \cos(t)}{5} + \frac{4 \sin(t)}{5}$

- ☐ Check validity of solution $y = c_1 \mathrm{e}^{-t} \cos(2t) + c_2 \mathrm{e}^{-t} \sin(2t) - \frac{\mathit{Heaviside}\left(t - \frac{\pi}{6}\right) \left(\cos(t)^2 \sqrt{3} - \sin(t) \cos(t) - \frac{\sqrt{3}}{2}\right.}{2}$
  - Use initial condition $y(0) = 0$

    $0 = c_1 - \frac{2}{5}$

  - Compute derivative of the solution

$$y' = -c_1 e^{-t} \cos(2t) - 2c_1 e^{-t} \sin(2t) - c_2 e^{-t} \sin(2t) + 2c_2 e^{-t} \cos(2t) - \frac{Dirac\left(t-\frac{\pi}{6}\right)\left(\cos(t)^2\sqrt{3}-\sin(t)\cos\right.}{2}$$

- Use the initial condition $y'\Big|_{\{t=0\}} = 1$

  $1 = -c_1 + \frac{4}{5} + 2c_2$

- Solve for c_1 and c_2

  $\left\{c_1 = \frac{2}{5}, c_2 = \frac{3}{10}\right\}$

- Substitute constant values into general solution and simplify

  $$y = -\frac{Heaviside\left(t-\frac{\pi}{6}\right)\left(\cos(t)^2\sqrt{3}-\sin(t)\cos(t)-\frac{\sqrt{3}}{2}\right)e^{-t+\frac{\pi}{6}}}{2} + \frac{\left(4\cos(t)^2+3\sin(t)\cos(t)-2\right)e^{-t}}{5} - \frac{2\cos(t)}{5} + \frac{4\sin(t)}{5}$$

- Solution to the IVP

  $$y = -\frac{Heaviside\left(t-\frac{\pi}{6}\right)\left(\cos(t)^2\sqrt{3}-\sin(t)\cos(t)-\frac{\sqrt{3}}{2}\right)e^{-t+\frac{\pi}{6}}}{2} + \frac{\left(4\cos(t)^2+3\sin(t)\cos(t)-2\right)e^{-t}}{5} - \frac{2\cos(t)}{5} + \frac{4\sin(t)}{5}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 3.406 (sec). Leaf size: 56

```
dsolve([diff(y(t),t$2)+2*diff(y(t),t)+5*y(t)=4*sin(t)+Dirac(t-Pi/6),y(0) = 0, D(y)(0) = 1],y
```

$$y(t) = -\frac{\text{Heaviside}\left(t-\frac{\pi}{6}\right)\left(\sqrt{3}\cos(t)^2 - \cos(t)\sin(t) - \frac{\sqrt{3}}{2}\right)e^{-t+\frac{\pi}{6}}}{2} + \frac{\left(4\cos(t)^2 + 3\cos(t)\sin(t) - 2\right)e^{-t}}{5} - \frac{2\cos(t)}{5} + \frac{4\sin(t)}{5}$$

✓ Solution by Mathematica

Time used: 0.644 (sec). Leaf size: 75

```
DSolve[{y''[t]+2*y'[t]+5*y[t]==4*Sin[t]+DiracDelta[t-Pi/6],{y[0]==0,y'[0]==1}},y[t],t,Includ
```

$$y(t) \to \frac{1}{20}e^{-t}\left(-5e^{\pi/6}\theta(6t-\pi)\left(\sqrt{3}\cos(2t)-\sin(2t)\right)+16e^{t}\sin(t)+6\sin(2t)\right.$$
$$\left.-8e^{t}\cos(t)+8\cos(2t)\right)$$

16 Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.2. page 739

16.1 problem Problem 1

Internal problem ID [2897]
Internal file name [`OUTPUT/2389_Sunday_June_05_2022_03_04_16_AM_27712915/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.2. page 739
Problem number: Problem 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using

Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0, y_0, y_0'} + \frac{x^3}{3!} f'|_{x_0, y_0, y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left. \frac{d^n f}{dx^n} \right|_{x_0, y_0, y_0'}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (610) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (611) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= y' \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= y
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= y(0) \\
F_1 &= y'(0) \\
F_2 &= y(0) \\
F_3 &= y'(0) \\
F_4 &= y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \frac{1}{720}x^6\right) y(0) + \left(x + \frac{1}{6}x^3 + \frac{1}{120}x^5\right) y'(0) + O(x^6)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard

power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} a_n x^n \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \sum_{n=0}^{\infty} (-a_n x^n) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \sum_{n=0}^{\infty} (-a_n x^n) = 0 \tag{3}$$

For $0 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) - a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{a_n}{(n+2)(n+1)} \tag{5}$$

For $n = 0$ the recurrence equation gives

$$2a_2 - a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_2 = \frac{a_0}{2}$$

For $n = 1$ the recurrence equation gives

$$6a_3 - a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{a_1}{6}$$

For $n = 2$ the recurrence equation gives

$$12a_4 - a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{24}$$

For $n = 3$ the recurrence equation gives

$$20a_5 - a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_1}{120}$$

For $n = 4$ the recurrence equation gives

$$30a_6 - a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0}{720}$$

For $n = 5$ the recurrence equation gives

$$42a_7 - a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{a_1}{5040}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \frac{1}{2} a_0 x^2 + \frac{1}{6} a_1 x^3 + \frac{1}{24} a_0 x^4 + \frac{1}{120} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + \frac{1}{2}x^2 + \frac{1}{24}x^4\right) a_0 + \left(x + \frac{1}{6}x^3 + \frac{1}{120}x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 + \frac{1}{2}x^2 + \frac{1}{24}x^4\right) c_1 + \left(x + \frac{1}{6}x^3 + \frac{1}{120}x^5\right) c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = \left(1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \frac{1}{720}x^6\right) y(0) + \left(x + \frac{1}{6}x^3 + \frac{1}{120}x^5\right) y'(0) + O(x^6) \tag{1}$$

$$y = \left(1 + \frac{1}{2}x^2 + \frac{1}{24}x^4\right) c_1 + \left(x + \frac{1}{6}x^3 + \frac{1}{120}x^5\right) c_2 + O(x^6) \tag{2}$$

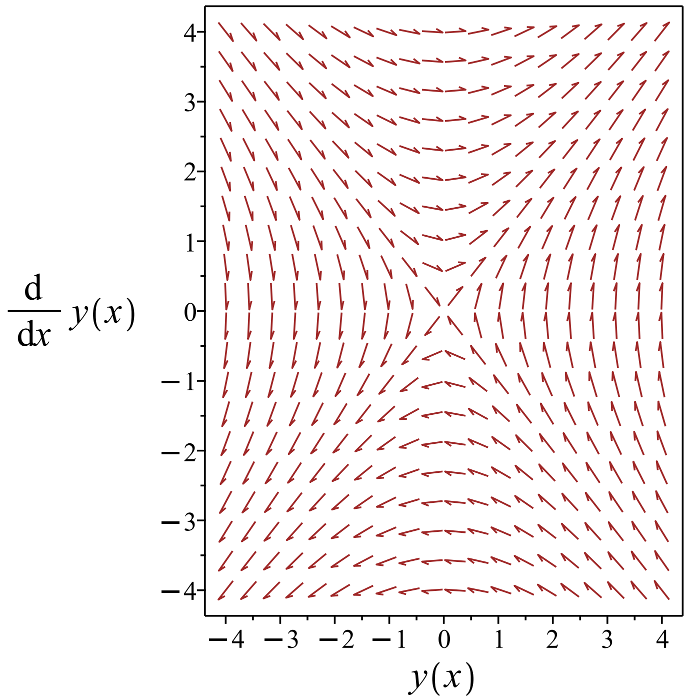


Figure 396: Slope field plot

Verification of solutions

$$y = \left(1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \frac{1}{720}x^6\right) y(0) + \left(x + \frac{1}{6}x^3 + \frac{1}{120}x^5\right) y'(0) + O(x^6)$$

Verified OK.

$$y = \left(1 + \frac{1}{2}x^2 + \frac{1}{24}x^4\right) c_1 + \left(x + \frac{1}{6}x^3 + \frac{1}{120}x^5\right) c_2 + O(x^6)$$

Verified OK.

16.1.1 Maple step by step solution

Let's solve

$y'' = y$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - y = 0$

- Characteristic polynomial of ODE

  $r^2 - 1 = 0$
- Factor the characteristic polynomial

  $(r-1)(r+1) = 0$
- Roots of the characteristic polynomial

  $r = (-1, 1)$
- 1st solution of the ODE

  $y_1(x) = e^{-x}$
- 2nd solution of the ODE

  $y_2(x) = e^{x}$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$
- Substitute in solutions

  $y = c_1 e^{-x} + c_2 e^{x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
Order:=6;
dsolve(diff(y(x),x$2)-y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 + \frac{1}{2}x^2 + \frac{1}{24}x^4\right) y(0) + \left(x + \frac{1}{6}x^3 + \frac{1}{120}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 42

```
AsymptoticDSolveValue[y''[x]-y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{x^5}{120}+\frac{x^3}{6}+x\right)+c_1\left(\frac{x^4}{24}+\frac{x^2}{2}+1\right)$$

16.2 problem Problem 2

Internal problem ID [2898]
Internal file name [`OUTPUT/2390_Sunday_June_05_2022_03_04_17_AM_4644177/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.2. page 739
Problem number: Problem 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

`[_erf]`

$$\boxed{y'' + 2xy' + 4y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + xy'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\ &= y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y'_0} \end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (613) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (614) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -2xy' - 4y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= 4y'x^2 + 8yx - 6y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= -8y'x^3 - 16x^2y + 28xy' + 32y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \left(16x^4 - 96x^2 + 60\right) y' + \left(32x^3 - 144x\right) y \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \left(-32x^5 + 288x^3 - 456x\right) y' - 64y\left(x^4 - \frac{15}{2}x^2 + 6\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -4y(0) \\
F_1 &= -6y'(0) \\
F_2 &= 32y(0) \\
F_3 &= 60y'(0) \\
F_4 &= -384y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 - 2x^2 + \frac{4}{3}x^4 - \frac{8}{15}x^6\right) y(0) + \left(x - x^3 + \frac{1}{2}x^5\right) y'(0) + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard

power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$\begin{aligned} y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\ y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} \end{aligned}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = -2x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - 4\left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} 2n\, x^n a_n\right) + \left(\sum_{n=0}^{\infty} 4 a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \left(\sum_{n=1}^{\infty} 2n\, x^n a_n\right) + \left(\sum_{n=0}^{\infty} 4 a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + 4a_0 = 0$$

$$a_2 = -2a_0$$

For $1 \leq n$, the recurrence equation is

$$(n+2)\,a_{n+2}(n+1) + 2na_n + 4a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{2a_n}{n+1} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 + 6a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -a_1$$

For $n = 2$ the recurrence equation gives

$$12a_4 + 8a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{4a_0}{3}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + 10a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_1}{2}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + 12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{8a_0}{15}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + 14a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{a_1}{6}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - 2a_0 x^2 - a_1 x^3 + \frac{4}{3} a_0 x^4 + \frac{1}{2} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - 2x^2 + \frac{4}{3}x^4\right) a_0 + \left(x - x^3 + \frac{1}{2}x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - 2x^2 + \frac{4}{3}x^4\right) c_1 + \left(x - x^3 + \frac{1}{2}x^5\right) c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = \left(1 - 2x^2 + \frac{4}{3}x^4 - \frac{8}{15}x^6\right) y(0) + \left(x - x^3 + \frac{1}{2}x^5\right) y'(0) + O(x^6) \tag{1}$$

$$y = \left(1 - 2x^2 + \frac{4}{3}x^4\right) c_1 + \left(x - x^3 + \frac{1}{2}x^5\right) c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1 - 2x^2 + \frac{4}{3}x^4 - \frac{8}{15}x^6\right) y(0) + \left(x - x^3 + \frac{1}{2}x^5\right) y'(0) + O(x^6)$$

Verified OK.

$$y = \left(1 - 2x^2 + \frac{4}{3}x^4\right) c_1 + \left(x - x^3 + \frac{1}{2}x^5\right) c_2 + O(x^6)$$

Verified OK.

16.2.1 Maple step by step solution

Let's solve

$y'' = -2xy' - 4y$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + 2xy' + 4y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ◦ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert y'' to series expansion

  $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- >k+2$

  $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} (a_{k+2}(k+2)\,(k+1) + 2a_k(k+2))\, x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+2)\,(ka_{k+2} + 2a_k + a_{k+2}) = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{2a_k}{k+1}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
Order:=6;
dsolve(diff(y(x),x$2)+2*x*diff(y(x),x)+4*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - 2x^2 + \frac{4}{3}x^4\right) y(0) + \left(x - x^3 + \frac{1}{2}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 38

```
AsymptoticDSolveValue[y''[x]+2*x*y'[x]+4*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{x^5}{2} - x^3 + x\right) + c_1\left(\frac{4x^4}{3} - 2x^2 + 1\right)$$

16.3 problem Problem 3

Internal problem ID [2899]
Internal file name [`OUTPUT/2391_Sunday_June_05_2022_03_04_19_AM_59879883/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.2. page 739
Problem number: Problem 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{y'' - 2xy' - 2y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using

Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2}\, f\big|_{x_0,y_0,y_0'} + \frac{x^3}{3!}\, f'\big|_{x_0,y_0,y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0,y_0,y_0'}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (616) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (617) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\ \ \vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= 2xy' + 2y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= 4y'x^2 + 4yx + 4y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= 8y'x^3 + 8x^2y + 20xy' + 12y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \left(16x^4 + 72x^2 + 32\right) y' + \left(16x^3 + 56x\right) y \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \left(32x^5 + 224x^3 + 264x\right) y' + 32\left(x^4 + 6x^2 + \frac{15}{4}\right) y
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 2y(0) \\
F_1 &= 4y'(0) \\
F_2 &= 12y(0) \\
F_3 &= 32y'(0) \\
F_4 &= 120y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 + x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6\right) y(0) + \left(x + \frac{2}{3}x^3 + \frac{4}{15}x^5\right) y'(0) + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard

power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = 2x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + 2\left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \sum_{n=1}^{\infty} (-2n\, x^n a_n) + \sum_{n=0}^{\infty} (-2a_n x^n) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \sum_{n=1}^{\infty} (-2n\, x^n a_n) + \sum_{n=0}^{\infty} (-2a_n x^n) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 - 2a_0 = 0$$

$$a_2 = a_0$$

For $1 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) - 2na_n - 2a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{2a_n}{n+2} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 - 4a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{2a_1}{3}$$

For $n = 2$ the recurrence equation gives

$$12a_4 - 6a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{2}$$

For $n = 3$ the recurrence equation gives

$$20a_5 - 8a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{4a_1}{15}$$

For $n = 4$ the recurrence equation gives

$$30a_6 - 10a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0}{6}$$

For $n = 5$ the recurrence equation gives

$$42a_7 - 12a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{8a_1}{105}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1x + a_0x^2 + \frac{2}{3}a_1x^3 + \frac{1}{2}a_0x^4 + \frac{4}{15}a_1x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + x^2 + \frac{1}{2}x^4\right) a_0 + \left(x + \frac{2}{3}x^3 + \frac{4}{15}x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 + x^2 + \frac{1}{2}x^4\right) c_1 + \left(x + \frac{2}{3}x^3 + \frac{4}{15}x^5\right) c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = \left(1 + x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6\right) y(0) + \left(x + \frac{2}{3}x^3 + \frac{4}{15}x^5\right) y'(0) + O(x^6) \tag{1}$$

$$y = \left(1 + x^2 + \frac{1}{2}x^4\right) c_1 + \left(x + \frac{2}{3}x^3 + \frac{4}{15}x^5\right) c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1 + x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6\right) y(0) + \left(x + \frac{2}{3}x^3 + \frac{4}{15}x^5\right) y'(0) + O(x^6)$$

Verified OK.

$$y = \left(1 + x^2 + \frac{1}{2}x^4\right) c_1 + \left(x + \frac{2}{3}x^3 + \frac{4}{15}x^5\right) c_2 + O(x^6)$$

Verified OK.

16.3.1 Maple step by step solution

Let's solve

$y'' = 2xy' + 2y$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - 2xy' - 2y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  - Convert y'' to series expansion

    $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - Shift index using $k- >k+2$

    $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - 2a_k(k+1)\right) x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+1)\left(a_{k+2}(k+2) - 2a_k\right) = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{2a_k}{k+2}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 32

```
Order:=6;
dsolve(diff(y(x),x$2)-2*x*diff(y(x),x)-2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 + x^2 + \frac{1}{2}x^4\right) y(0) + \left(x + \frac{2}{3}x^3 + \frac{4}{15}x^5\right) D(y)\,(0) + O\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 38

```
AsymptoticDSolveValue[y''[x]-2*x*y'[x]-2*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{4x^5}{15} + \frac{2x^3}{3} + x\right) + c_1\left(\frac{x^4}{2} + x^2 + 1\right)$$

16.4 problem Problem 4

Internal problem ID [2900]
Internal file name [`OUTPUT/2392_Sunday_June_05_2022_03_04_21_AM_28661475/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.2. page 739
Problem number: Problem 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{y'' - y'x^2 - 2yx = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using

Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0,y_0,y_0'} + \frac{x^3}{3!} f'|_{x_0,y_0,y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0,y_0,y_0'}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (619) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (620) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{align}
F_0 &= f(x,y,y') \tag{4}\\
F_1 &= \frac{df}{dx} \nonumber\\
&= \frac{dF_0}{dx} \nonumber\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \nonumber\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{5}\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0 \nonumber\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right) \nonumber\\
&= \frac{d}{dx}(F_1) \nonumber\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y'' \nonumber\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0 \nonumber\\
&\vdots \nonumber\\
F_n &= \frac{d}{dx}(F_{n-1}) \nonumber\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y'' \nonumber\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \tag{6}
\end{align}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= y'x^2 + 2yx \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \left(x^4 + 4x\right) y' + \left(2x^3 + 2\right) y \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \left(x^6 + 10x^3 + 6\right) y' + 2x^2 y\left(x^3 + 7\right) \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= x\left(\left(x^7 + 18x^4 + 50x\right) y' + 2y\left(x^6 + 15x^3 + 20\right)\right) \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \left(x^{10} + 28x^7 + 170x^4 + 140x\right) y' + 2y\left(x^9 + 25x^6 + 110x^3 + 20\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and $y'(0)=y'(0)$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= 2y(0) \\
F_2 &= 6y'(0) \\
F_3 &= 0 \\
F_4 &= 40y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 + \frac{1}{3}x^3 + \frac{1}{18}x^6\right) y(0) + \left(x + \frac{1}{4}x^4\right) y'(0) + O\left(x^6\right)
$$

Since the expansion point $x=0$ is an ordinary, we can also solve this using standard

power series Let the solution be represented as power series of the form

$$y=\sum_{n=0}^{\infty} a_n x^n$$

Then

$$y'=\sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y''=\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}=\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x^2+2\left(\sum_{n=0}^{\infty} a_n x^n\right) x \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right)+\sum_{n=1}^{\infty}\left(-n\, x^{1+n} a_n\right)+\sum_{n=0}^{\infty}\left(-2x^{1+n} a_n\right)=0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)\, a_{n+2}(1+n)\, x^n$$
$$\sum_{n=1}^{\infty}\left(-n\, x^{1+n} a_n\right)=\sum_{n=2}^{\infty}\left(-(n-1)\, a_{n-1} x^n\right)$$
$$\sum_{n=0}^{\infty}\left(-2x^{1+n} a_n\right)=\sum_{n=1}^{\infty}\left(-2a_{n-1} x^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty}(n+2)\, a_{n+2}(1+n)\, x^n\right)+\sum_{n=2}^{\infty}\left(-(n-1)\, a_{n-1} x^n\right)+\sum_{n=1}^{\infty}\left(-2a_{n-1} x^n\right)=0 \tag{3}$$

$n = 1$ gives

$$6a_3 - 2a_0 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = \frac{a_0}{3}$$

For $2 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(1+n) - (n-1)\, a_{n-1} - 2a_{n-1} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{a_{n-1}}{n+2} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$12a_4 - 3a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_1}{4}$$

For $n = 3$ the recurrence equation gives

$$20a_5 - 4a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = 0$$

For $n = 4$ the recurrence equation gives

$$30a_6 - 5a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0}{18}$$

For $n = 5$ the recurrence equation gives

$$42a_7 - 6a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{a_1}{28}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \frac{1}{3} a_0 x^3 + \frac{1}{4} a_1 x^4 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + \frac{x^3}{3}\right) a_0 + \left(x + \frac{1}{4} x^4\right) a_1 + O\left(x^6\right) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 + \frac{x^3}{3}\right) c_1 + \left(x + \frac{1}{4} x^4\right) c_2 + O\left(x^6\right)$$

Summary

The solution(s) found are the following

$$y = \left(1 + \frac{1}{3} x^3 + \frac{1}{18} x^6\right) y(0) + \left(x + \frac{1}{4} x^4\right) y'(0) + O\left(x^6\right) \tag{1}$$

$$y = \left(1 + \frac{x^3}{3}\right) c_1 + \left(x + \frac{1}{4} x^4\right) c_2 + O\left(x^6\right) \tag{2}$$

Verification of solutions

$$y = \left(1 + \frac{1}{3} x^3 + \frac{1}{18} x^6\right) y(0) + \left(x + \frac{1}{4} x^4\right) y'(0) + O\left(x^6\right)$$

Verified OK.

$$y = \left(1 + \frac{x^3}{3}\right) c_1 + \left(x + \frac{1}{4} x^4\right) c_2 + O\left(x^6\right)$$

Verified OK.

16.4.1 Maple step by step solution

Let's solve

$y'' = y'x^2 + 2yx$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - y'x^2 - 2yx = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x \cdot y$ to series expansion

  $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

  ◦ Shift index using $k- >k-1$

  $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$

  ◦ Convert $x^2 \cdot y'$ to series expansion

  $x^2 \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^{k+1}$

  ◦ Shift index using $k- >k-1$

  $x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1}(k-1)\, x^k$

  ◦ Convert y'' to series expansion

  $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- >k+2$

  $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite ODE with series expansions

  $2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - a_{k-1}(k+1)\right) x^k\right) = 0$

- Each term must be 0

  $2a_2 = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+1)\left(a_{k+2}(k+2) - a_{k-1}\right) = 0$

- Shift index using $k- > k+1$

  $(k+2)\left(a_{k+3}(k+3) - a_k\right) = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{a_k}{k+3}, 2a_2 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
Order:=6;
dsolve(diff(y(x),x$2)-x^2*diff(y(x),x)-2*x*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 + \frac{x^3}{3}\right) y(0) + \left(x + \frac{1}{4}x^4\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 28

```
AsymptoticDSolveValue[y''[x]-x^2*y'[x]-2*x*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{x^4}{4} + x\right) + c_1\left(\frac{x^3}{3} + 1\right)$$

16.5 problem Problem 5

Internal problem ID [2901]
Internal file name [`OUTPUT/2393_Sunday_June_05_2022_03_04_23_AM_61835031/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.2. page 739
Problem number: Problem 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_airy", "second_order_bessel_ode", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{y'' + yx = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using

Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0,y_0,y_0'} + \frac{x^3}{3!} f'|_{x_0,y_0,y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0,y_0,y_0'}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (622) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (623) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -yx \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= -xy' - y \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= -2y' + x^2 y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= x(xy' + 4y) \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= -yx^3 + 6xy' + 4y
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= -y(0) \\
F_2 &= -2y'(0) \\
F_3 &= 0 \\
F_4 &= 4y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 - \frac{1}{6}x^3 + \frac{1}{180}x^6\right) y(0) + \left(x - \frac{1}{12}x^4\right) y'(0) + O(x^6)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard

power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = -\left(\sum_{n=0}^{\infty} a_n x^n\right) x \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=0}^{\infty} x^{1+n} a_n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n$$
$$\sum_{n=0}^{\infty} x^{1+n} a_n = \sum_{n=1}^{\infty} a_{n-1} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^n\right) = 0 \tag{3}$$

For $1 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(1+n) + a_{n-1} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_{n-1}}{(n+2)(1+n)} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 + a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{a_0}{6}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{a_1}{12}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = 0$$

For $n = 4$ the recurrence equation gives

$$30a_6 + a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0}{180}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{a_1}{504}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{1}{6} a_0 x^3 - \frac{1}{12} a_1 x^4 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{x^3}{6}\right) a_0 + \left(x - \frac{1}{12} x^4\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{x^3}{6}\right) c_1 + \left(x - \frac{1}{12} x^4\right) c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = \left(1 - \frac{1}{6} x^3 + \frac{1}{180} x^6\right) y(0) + \left(x - \frac{1}{12} x^4\right) y'(0) + O(x^6) \tag{1}$$

$$y = \left(1 - \frac{x^3}{6}\right) c_1 + \left(x - \frac{1}{12} x^4\right) c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1 - \frac{1}{6} x^3 + \frac{1}{180} x^6\right) y(0) + \left(x - \frac{1}{12} x^4\right) y'(0) + O(x^6)$$

Verified OK.

$$y = \left(1 - \frac{x^3}{6}\right) c_1 + \left(x - \frac{1}{12} x^4\right) c_2 + O(x^6)$$

Verified OK.

16.5.1 Maple step by step solution

Let's solve

$y'' = -yx$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + yx = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x \cdot y$ to series expansion

  $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

  ◦ Shift index using $k- >k-1$

  $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$

  ◦ Convert y'' to series expansion

  $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- >k+2$

  $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite ODE with series expansions

  $2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_{k-1}\right) x^k\right) = 0$

- Each term must be 0

  $2a_2 = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2)\, a_{k+2} + a_{k-1} = 0$

- Shift index using $k- >k+1$

$$\left((k+1)^2+3k+5\right)a_{k+3}+a_k=0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+3}=-\frac{a_k}{k^2+5k+6}, 2a_2=0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 24

```
Order:=6;
dsolve(diff(y(x),x$2)+x*y(x)=0,y(x),type='series',x=0);
```

$$y(x)=\left(1-\frac{x^3}{6}\right)y(0)+\left(x-\frac{1}{12}x^4\right)D(y)(0)+O\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 28

```
AsymptoticDSolveValue[y''[x]+x*y[x]==0,y[x],{x,0,5}]
```

$$y(x)\to c_2\left(x-\frac{x^4}{12}\right)+c_1\left(1-\frac{x^3}{6}\right)$$

16.6 problem Problem 6

Internal problem ID [2902]
Internal file name [`OUTPUT/2394_Sunday_June_05_2022_03_04_25_AM_72312463/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.2. page 739
Problem number: Problem 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + xy' + 3y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + xy'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\
&= y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (625)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (626)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -xy' - 3y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= y'x^2 + 3yx - 4y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= -y'x^3 - 3x^2y + 9xy' + 15y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \left(x^4 - 15x^2 + 24\right) y' + 3xy\left(x^2 - 11\right) \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \left(-x^5 + 22x^3 - 87x\right) y' - 3y\left(x^4 - 18x^2 + 35\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -3y(0) \\
F_1 &= -4y'(0) \\
F_2 &= 15y(0) \\
F_3 &= 24y'(0) \\
F_4 &= -105y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 - \frac{3}{2}x^2 + \frac{5}{8}x^4 - \frac{7}{48}x^6\right) y(0) + \left(x - \frac{2}{3}x^3 + \frac{1}{5}x^5\right) y'(0) + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard

power series Let the solution be represented as power series of the form

$$y=\sum_{n=0}^{\infty} a_n x^n$$

Then

$$y'=\sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y''=\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}=-x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right)-3\left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right)+\left(\sum_{n=1}^{\infty} n\, x^n a_n\right)+\left(\sum_{n=0}^{\infty} 3a_n x^n\right)=0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}=\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right)+\left(\sum_{n=1}^{\infty} n\, x^n a_n\right)+\left(\sum_{n=0}^{\infty} 3a_n x^n\right)=0 \tag{3}$$

$n=0$ gives

$$2a_2+3a_0=0$$

$$a_2=-\frac{3a_0}{2}$$

For $1 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) + n a_n + 3a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_n(n+3)}{(n+2)\,(n+1)} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 + 4a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{2a_1}{3}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + 5a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{5a_0}{8}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + 6a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_1}{5}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + 7a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{7a_0}{48}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + 8a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{4a_1}{105}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{3}{2} a_0 x^2 - \frac{2}{3} a_1 x^3 + \frac{5}{8} a_0 x^4 + \frac{1}{5} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{3}{2}x^2 + \frac{5}{8}x^4\right) a_0 + \left(x - \frac{2}{3}x^3 + \frac{1}{5}x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{3}{2}x^2 + \frac{5}{8}x^4\right) c_1 + \left(x - \frac{2}{3}x^3 + \frac{1}{5}x^5\right) c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = \left(1 - \frac{3}{2}x^2 + \frac{5}{8}x^4 - \frac{7}{48}x^6\right) y(0) + \left(x - \frac{2}{3}x^3 + \frac{1}{5}x^5\right) y'(0) + O(x^6) \tag{1}$$

$$y = \left(1 - \frac{3}{2}x^2 + \frac{5}{8}x^4\right) c_1 + \left(x - \frac{2}{3}x^3 + \frac{1}{5}x^5\right) c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1 - \frac{3}{2}x^2 + \frac{5}{8}x^4 - \frac{7}{48}x^6\right) y(0) + \left(x - \frac{2}{3}x^3 + \frac{1}{5}x^5\right) y'(0) + O(x^6)$$

Verified OK.

$$y = \left(1 - \frac{3}{2}x^2 + \frac{5}{8}x^4\right) c_1 + \left(x - \frac{2}{3}x^3 + \frac{1}{5}x^5\right) c_2 + O(x^6)$$

Verified OK.

16.6.1 Maple step by step solution

Let's solve

$y'' = -xy' - 3y$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + xy' + 3y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k \, x^k$

  - Convert y'' to series expansion

    $y'' = \sum_{k=2}^{\infty} a_k k(k-1) \, x^{k-2}$

  - Shift index using $k- >k+2$

    $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1) \, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} (a_{k+2}(k+2)(k+1) + a_k(k+3)) \, x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2) \, a_{k+2} + a_k(k+3) = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k(k+3)}{k^2+3k+2}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form could result into a too large expression - returning special functi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
Order:=6;
dsolve(diff(y(x),x$2)+x*diff(y(x),x)+3*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{3}{2}x^2 + \frac{5}{8}x^4\right) y(0) + \left(x - \frac{2}{3}x^3 + \frac{1}{5}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 42

```
AsymptoticDSolveValue[y''[x]+x*y'[x]+3*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{x^5}{5} - \frac{2x^3}{3} + x\right) + c_1\left(\frac{5x^4}{8} - \frac{3x^2}{2} + 1\right)$$

16.7 problem Problem 7

Internal problem ID [2903]
Internal file name [`OUTPUT/2395_Sunday_June_05_2022_03_04_27_AM_94002059/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.2. page 739
Problem number: Problem 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - y'x^2 - 3yx = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + xy'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (628) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (629) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{align}
F_0 &= f(x, y, y') \tag{4}\\
F_1 &= \frac{df}{dx} \\
&= \frac{dF_0}{dx} \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \tag{5}\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right) \\
&= \frac{d}{dx}(F_1) \\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y'' \\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0 \\
&\vdots \\
F_n &= \frac{d}{dx}(F_{n-1}) \\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y'' \\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \tag{6}
\end{align}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n\big|_{x_0, y_0, y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= y'x^2 + 3yx \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \left(x^4 + 5x\right) y' + \left(3x^3 + 3\right) y \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \left(x^6 + 12x^3 + 8\right) y' + 3\left(x^5 + 8x^2\right) y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= x\left(\left(x^7 + 21x^4 + 68x\right) y' + 3y\left(x^6 + 17x^3 + 24\right)\right) \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \left(x^{10} + 32x^7 + 224x^4 + 208x\right) y' + 3y\left(x^3 + 6\right)\left(x^6 + 22x^3 + 4\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= 3y(0) \\
F_2 &= 8y'(0) \\
F_3 &= 0 \\
F_4 &= 72y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 + \frac{1}{2}x^3 + \frac{1}{10}x^6\right) y(0) + \left(x + \frac{1}{3}x^4\right) y'(0) + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard

power series Let the solution be represented as power series of the form

$$y=\sum_{n=0}^{\infty}a_nx^n$$

Then

$$y'=\sum_{n=1}^{\infty}na_nx^{n-1}$$
$$y''=\sum_{n=2}^{\infty}n(n-1)\,a_nx^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty}n(n-1)\,a_nx^{n-2}=\left(\sum_{n=1}^{\infty}na_nx^{n-1}\right)x^2+3\left(\sum_{n=0}^{\infty}a_nx^n\right)x \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty}n(n-1)\,a_nx^{n-2}\right)+\sum_{n=1}^{\infty}\left(-n\,x^{1+n}a_n\right)+\sum_{n=0}^{\infty}\left(-3x^{1+n}a_n\right)=0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty}n(n-1)\,a_nx^{n-2}=\sum_{n=0}^{\infty}(n+2)\,a_{n+2}(1+n)\,x^n$$
$$\sum_{n=1}^{\infty}\left(-n\,x^{1+n}a_n\right)=\sum_{n=2}^{\infty}\left(-(n-1)\,a_{n-1}x^n\right)$$
$$\sum_{n=0}^{\infty}\left(-3x^{1+n}a_n\right)=\sum_{n=1}^{\infty}\left(-3a_{n-1}x^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty}(n+2)\,a_{n+2}(1+n)\,x^n\right)+\sum_{n=2}^{\infty}\left(-(n-1)\,a_{n-1}x^n\right)+\sum_{n=1}^{\infty}\left(-3a_{n-1}x^n\right)=0 \tag{3}$$

$n = 1$ gives

$$6a_3 - 3a_0 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = \frac{a_0}{2}$$

For $2 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(1+n) - (n-1)\, a_{n-1} - 3a_{n-1} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{a_{n-1}}{1+n} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$12a_4 - 4a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_1}{3}$$

For $n = 3$ the recurrence equation gives

$$20a_5 - 5a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = 0$$

For $n = 4$ the recurrence equation gives

$$30a_6 - 6a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0}{10}$$

For $n = 5$ the recurrence equation gives

$$42a_7 - 7a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{a_1}{18}$$

And so on. Therefore the solution is

$$\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^n \\
&= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots
\end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \frac{1}{2} a_0 x^3 + \frac{1}{3} a_1 x^4 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + \frac{x^3}{2}\right) a_0 + \left(x + \frac{1}{3} x^4\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 + \frac{x^3}{2}\right) c_1 + \left(x + \frac{1}{3} x^4\right) c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = \left(1 + \frac{1}{2} x^3 + \frac{1}{10} x^6\right) y(0) + \left(x + \frac{1}{3} x^4\right) y'(0) + O(x^6) \tag{1}$$

$$y = \left(1 + \frac{x^3}{2}\right) c_1 + \left(x + \frac{1}{3} x^4\right) c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1 + \frac{1}{2} x^3 + \frac{1}{10} x^6\right) y(0) + \left(x + \frac{1}{3} x^4\right) y'(0) + O(x^6)$$

Verified OK.

$$y = \left(1 + \frac{x^3}{2}\right) c_1 + \left(x + \frac{1}{3} x^4\right) c_2 + O(x^6)$$

Verified OK.

16.7.1 Maple step by step solution

Let's solve

$y'' = y'x^2 + 3yx$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - y'x^2 - 3yx = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

  ◦ Shift index using $k- >k-1$

    $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$

  ◦ Convert $x^2 \cdot y'$ to series expansion

    $x^2 \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^{k+1}$

  ◦ Shift index using $k- >k-1$

    $x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1}(k-1)\, x^k$

  ◦ Convert y'' to series expansion

    $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- >k+2$

    $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\, x^k$

  Rewrite ODE with series expansions

  $2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)(k+1) - a_{k-1}(k+2)\right) x^k\right) = 0$

- Each term must be 0

  $2a_2 = 0$
- Each term in the series must be 0, giving the recursion relation

  $(k+2)(ka_{k+2} - a_{k-1} + a_{k+2}) = 0$
- Shift index using $k- >k+1$

  $(k+3)((k+1)a_{k+3} - a_k + a_{k+3}) = 0$
- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{a_k}{k+2}, 2a_2 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
Order:=6;
dsolve(diff(y(x),x$2)-x^2*diff(y(x),x)-3*x*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 + \frac{x^3}{2}\right) y(0) + \left(x + \frac{1}{3}x^4\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 28

```
AsymptoticDSolveValue[y''[x]-x^2*y'[x]-3*x*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{x^4}{3}+x\right)+c_1\left(\frac{x^3}{2}+1\right)$$

16.8 problem Problem 8

Internal problem ID [2904]
Internal file name [`OUTPUT/2396_Sunday_June_05_2022_03_04_29_AM_63670586/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.2. page 739
Problem number: Problem 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 2y'x^2 + 2yx = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + xy'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\ &= y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y'_0} \end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (631) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (632) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -2y'x^2 - 2yx \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= 4y'x^4 + 4yx^3 - 6xy' - 2y \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \left(-8x^6 + 32x^3 - 8\right) y' - 8x^2 y\left(x^3 - 3\right) \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= 16\left(\left(x^7 - \frac{15}{2}x^4 + \frac{17}{2}x\right) y' + y\left(x^6 - \frac{13}{2}x^3 + 4\right)\right) x \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \left(-32x^{10} + 384x^7 - 976x^4 + 336x\right) y' - 32\left(x^9 - 11x^6 + \frac{43}{2}x^3 - 2\right) y
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and $y'(0)=y'(0)$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= -2y(0) \\
F_2 &= -8y'(0) \\
F_3 &= 0 \\
F_4 &= 64y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 - \frac{1}{3}x^3 + \frac{4}{45}x^6\right) y(0) + \left(x - \frac{1}{3}x^4\right) y'(0) + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = -2\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x^2 - 2\left(\sum_{n=0}^{\infty} a_n x^n\right) x \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} 2n\, x^{1+n} a_n\right) + \left(\sum_{n=0}^{\infty} 2x^{1+n} a_n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n$$

$$\sum_{n=1}^{\infty} 2n\, x^{1+n} a_n = \sum_{n=2}^{\infty} 2(n-1)\, a_{n-1} x^n$$

$$\sum_{n=0}^{\infty} 2x^{1+n} a_n = \sum_{n=1}^{\infty} 2a_{n-1} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n\right) + \left(\sum_{n=2}^{\infty} 2(n-1)\, a_{n-1} x^n\right) + \left(\sum_{n=1}^{\infty} 2a_{n-1} x^n\right) = 0 \tag{3}$$

$n = 1$ gives

$$6a_3 + 2a_0 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{a_0}{3}$$

For $2 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(1+n) + 2(n-1)\, a_{n-1} + 2a_{n-1} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{2a_{n-1}n}{(n+2)\,(1+n)} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + 4a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{a_1}{3}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + 6a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = 0$$

For $n = 4$ the recurrence equation gives

$$30a_6 + 8a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{4a_0}{45}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + 10a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{5a_1}{63}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{1}{3} a_0 x^3 - \frac{1}{3} a_1 x^4 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{x^3}{3}\right) a_0 + \left(x - \frac{1}{3}x^4\right) a_1 + O\left(x^6\right) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{x^3}{3}\right) c_1 + \left(x - \frac{1}{3}x^4\right) c_2 + O\left(x^6\right)$$

Summary

The solution(s) found are the following

$$y = \left(1 - \frac{1}{3}x^3 + \frac{4}{45}x^6\right) y(0) + \left(x - \frac{1}{3}x^4\right) y'(0) + O\left(x^6\right) \tag{1}$$

$$y = \left(1 - \frac{x^3}{3}\right) c_1 + \left(x - \frac{1}{3}x^4\right) c_2 + O\left(x^6\right) \tag{2}$$

Verification of solutions

$$y = \left(1 - \frac{1}{3}x^3 + \frac{4}{45}x^6\right) y(0) + \left(x - \frac{1}{3}x^4\right) y'(0) + O\left(x^6\right)$$

Verified OK.

$$y = \left(1 - \frac{x^3}{3}\right) c_1 + \left(x - \frac{1}{3}x^4\right) c_2 + O\left(x^6\right)$$

Verified OK.

16.8.1 Maple step by step solution

Let's solve

$y'' = -2y'x^2 - 2yx$

- Highest derivative means the order of the ODE is 2

y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' + 2y'x^2 + 2yx = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x \cdot y$ to series expansion

  $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

  ◦ Shift index using $k- >k-1$

  $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$

  ◦ Convert $x^2 \cdot y'$ to series expansion

  $x^2 \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^{k+1}$

  ◦ Shift index using $k- >k-1$

  $x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1}(k-1)\, x^k$

  ◦ Convert y'' to series expansion

  $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- >k+2$

  $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

Rewrite ODE with series expansions

$2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + 2a_{k-1}k\right) x^k\right) = 0$

- Each term must be 0

  $2a_2 = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2)\, a_{k+2} + 2a_{k-1}k = 0$

- Shift index using $k- >k+1$

  $\left((k+1)^2 + 3k + 5\right) a_{k+3} + 2a_k(k+1) = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = -\frac{2a_k(k+1)}{k^2+5k+6}, 2a_2 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
Order:=6;
dsolve(diff(y(x),x$2)+2*x^2*diff(y(x),x)+2*x*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{x^3}{3}\right) y(0) + \left(x - \frac{1}{3}x^4\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 28

```
AsymptoticDSolveValue[y''[x]+2*x^2*y'[x]+2*x*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2 \left(x - \frac{x^4}{3}\right) + c_1 \left(1 - \frac{x^3}{3}\right)$$

16.9 problem Problem 9

Internal problem ID [2905]
Internal file name [`OUTPUT/2397_Sunday_June_05_2022_03_04_31_AM_78629810/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.2. page 739
Problem number: Problem 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{(x^2-3)\,y''-3xy'-5y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y_0'$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots\\
&= y_0+xy_0'+\frac{x^2}{2}\,f|_{x_0,y_0,y_0'}+\frac{x^3}{3!}\,f'|_{x_0,y_0,y_0'}+\cdots\\
&= y_0+xy_0'+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}\,\frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y_0'}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{634}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{635}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{align*}
F_0 &= f(x,y,y') \tag{4}\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{5}\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \tag{6}
\end{align*}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= \frac{3xy' + 5y}{x^2 - 3} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{11y'x^2 + 5yx - 24y'}{(x^2 - 3)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{16y'x^3 + 40x^2y - 57xy' - 135y}{(x^2 - 3)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{(40x^4 - 285x^2 + 576)\, y' + (-80x^3 + 285x)\, y}{(x^2 - 3)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{(-120x^5 + 900x^3 - 2025x)\, y' + 600\left(x^4 - \frac{9}{2}x^2 + \frac{27}{8}\right) y}{(x^2 - 3)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -\frac{5y(0)}{3} \\
F_1 &= -\frac{8y'(0)}{3} \\
F_2 &= 5y(0) \\
F_3 &= \frac{64y'(0)}{9} \\
F_4 &= -\frac{25y(0)}{3}
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$y = \left(1 - \frac{5}{6}x^2 + \frac{5}{24}x^4 - \frac{5}{432}x^6\right) y(0) + \left(x - \frac{4}{9}x^3 + \frac{8}{135}x^5\right) y'(0) + O(x^6)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(x^2 - 3)\, y'' - 3xy' - 5y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$(x^2 - 3)\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) - 3x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - 5\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \sum_{n=2}^{\infty} \left(-3n(n-1)\, a_n x^{n-2}\right) + \sum_{n=1}^{\infty} \left(-3n a_n x^n\right) + \sum_{n=0}^{\infty} \left(-5a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} \left(-3n(n-1)\, a_n x^{n-2}\right) = \sum_{n=0}^{\infty} \left(-3(n+2)\, a_{n+2}(n+1)\, x^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$
\left(\sum_{n=2}^{\infty} x^{n} a_{n} n(n-1)\right)+\sum_{n=0}^{\infty}\left(-3(n+2) a_{n+2}(n+1) x^{n}\right) \\
+\sum_{n=1}^{\infty}\left(-3 n a_{n} x^{n}\right)+\sum_{n=0}^{\infty}\left(-5 a_{n} x^{n}\right)=0 \tag{3}
$$

$n=0$ gives

$$
-6 a_{2}-5 a_{0}=0
$$

$$
a_{2}=-\frac{5 a_{0}}{6}
$$

$n=1$ gives

$$
-18 a_{3}-8 a_{1}=0
$$

Which after substituting earlier equations, simplifies to

$$
a_{3}=-\frac{4 a_{1}}{9}
$$

For $2 \leq n$, the recurrence equation is

$$
n a_{n}(n-1)-3(n+2) a_{n+2}(n+1)-3 n a_{n}-5 a_{n}=0 \tag{4}
$$

Solving for a_{n+2}, gives

$$
a_{n+2}=\frac{(n-5) a_{n}}{3 n+6} \tag{5}
$$

For $n=2$ the recurrence equation gives

$$
-9 a_{2}-36 a_{4}=0
$$

Which after substituting the earlier terms found becomes

$$
a_{4}=\frac{5 a_{0}}{24}
$$

For $n = 3$ the recurrence equation gives

$$-8a_3 - 60a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{8a_1}{135}$$

For $n = 4$ the recurrence equation gives

$$-5a_4 - 90a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{5a_0}{432}$$

For $n = 5$ the recurrence equation gives

$$-126a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = 0$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{5}{6} a_0 x^2 - \frac{4}{9} a_1 x^3 + \frac{5}{24} a_0 x^4 + \frac{8}{135} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{5}{6} x^2 + \frac{5}{24} x^4\right) a_0 + \left(x - \frac{4}{9} x^3 + \frac{8}{135} x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{5}{6}x^2 + \frac{5}{24}x^4\right)c_1 + \left(x - \frac{4}{9}x^3 + \frac{8}{135}x^5\right)c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = \left(1 - \frac{5}{6}x^2 + \frac{5}{24}x^4 - \frac{5}{432}x^6\right)y(0) + \left(x - \frac{4}{9}x^3 + \frac{8}{135}x^5\right)y'(0) + O(x^6) \tag{1}$$

$$y = \left(1 - \frac{5}{6}x^2 + \frac{5}{24}x^4\right)c_1 + \left(x - \frac{4}{9}x^3 + \frac{8}{135}x^5\right)c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1 - \frac{5}{6}x^2 + \frac{5}{24}x^4 - \frac{5}{432}x^6\right)y(0) + \left(x - \frac{4}{9}x^3 + \frac{8}{135}x^5\right)y'(0) + O(x^6)$$

Verified OK.

$$y = \left(1 - \frac{5}{6}x^2 + \frac{5}{24}x^4\right)c_1 + \left(x - \frac{4}{9}x^3 + \frac{8}{135}x^5\right)c_2 + O(x^6)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
Order:=6;
dsolve((x^2-3)*diff(y(x),x$2)-3*x*diff(y(x),x)-5*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{5}{6}x^2 + \frac{5}{24}x^4\right)y(0) + \left(x - \frac{4}{9}x^3 + \frac{8}{135}x^5\right)D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 42

```
AsymptoticDSolveValue[(x^2-3)*y''[x]-3*x*y'[x]-5*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \rightarrow c_2\left(\frac{8x^5}{135} - \frac{4x^3}{9} + x\right) + c_1\left(\frac{5x^4}{24} - \frac{5x^2}{6} + 1\right)$$

16.10 problem Problem 10

Internal problem ID [2906]
Internal file name [`OUTPUT/2398_Sunday_June_05_2022_03_04_33_AM_14543886/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.2. page 739
Problem number: Problem 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second order series method. Ordinary point", "second_order_change_of_variable_on_y_method_1", "linear_second_order_ode_solved_by_an_integrating_factor", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{(x^2+1)\, y''+4xy'+2y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using

Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} \left. f \right|_{x_0, y_0, y_0'} + \frac{x^3}{3!} \left. f' \right|_{x_0, y_0, y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left. \frac{d^n f}{dx^n} \right|_{x_0, y_0, y_0'}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (637) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (638) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{align*}
F_0 &= f(x,y,y') \tag{4}\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{5}\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \tag{6}
\end{align*}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{2(2xy' + y)}{x^2 + 1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{18y'x^2 + 12yx - 6y'}{(x^2 + 1)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{-96y'x^3 - 72x^2y + 96xy' + 24y}{(x^2 + 1)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{120(5x^4 - 10x^2 + 1)\, y' + 480(x^3 - x)\, y}{(x^2 + 1)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{(-4320x^5 + 14400x^3 - 4320x)\, y' - 3600y\left(x^4 - 2x^2 + \frac{1}{5}\right)}{(x^2 + 1)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -2y(0) \\
F_1 &= -6y'(0) \\
F_2 &= 24y(0) \\
F_3 &= 120y'(0) \\
F_4 &= -720y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(-x^6 + x^4 - x^2 + 1\right) y(0) + \left(x^5 - x^3 + x\right) y'(0) + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(x^2 + 1) y'' + 4xy' + 2y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$\begin{aligned} y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\ y'' &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \end{aligned}$$

Substituting the above back into the ode gives

$$(x^2 + 1) \left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \right) + 4x \left(\sum_{n=1}^{\infty} n a_n x^{n-1} \right) + 2 \left(\sum_{n=0}^{\infty} a_n x^n \right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} x^n a_n n(n-1) \right) + \left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \right) + \left(\sum_{n=1}^{\infty} 4n a_n x^n \right) + \left(\sum_{n=0}^{\infty} 2a_n x^n \right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\begin{aligned} \left(\sum_{n=2}^{\infty} x^n a_n n(n-1) \right) + \left(\sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n \right) \\ + \left(\sum_{n=1}^{\infty} 4n a_n x^n \right) + \left(\sum_{n=0}^{\infty} 2a_n x^n \right) = 0 \end{aligned} \tag{3}$$

$n = 0$ gives

$$2a_2 + 2a_0 = 0$$

$$a_2 = -a_0$$

$n = 1$ gives

$$6a_3 + 6a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -a_1$$

For $2 \leq n$, the recurrence equation is

$$na_n(n-1) + (n+2)\, a_{n+2}(n+1) + 4na_n + 2a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -a_n \tag{5}$$

For $n = 2$ the recurrence equation gives

$$12a_2 + 12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = a_0$$

For $n = 3$ the recurrence equation gives

$$20a_3 + 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = a_1$$

For $n = 4$ the recurrence equation gives

$$30a_4 + 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -a_0$$

For $n = 5$ the recurrence equation gives

$$42a_5 + 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -a_1$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_1 x^5 + a_0 x^4 - a_1 x^3 - a_0 x^2 + a_1 x + a_0 + \dots$$

Collecting terms, the solution becomes

$$y = (x^4 - x^2 + 1) a_0 + (x^5 - x^3 + x) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = (x^4 - x^2 + 1) c_1 + (x^5 - x^3 + x) c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = (-x^6 + x^4 - x^2 + 1) y(0) + (x^5 - x^3 + x) y'(0) + O(x^6) \tag{1}$$

$$y = (x^4 - x^2 + 1) c_1 + (x^5 - x^3 + x) c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$y = (-x^6 + x^4 - x^2 + 1) y(0) + (x^5 - x^3 + x) y'(0) + O(x^6)$$

Verified OK.

$$y = (x^4 - x^2 + 1) c_1 + (x^5 - x^3 + x) c_2 + O(x^6)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 30

```
Order:=6;
dsolve((1+x^2)*diff(y(x),x$2)+4*x*diff(y(x),x)+2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = (x^4 - x^2 + 1)\, y(0) + (x^5 - x^3 + x)\, D(y)\,(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 30

```
AsymptoticDSolveValue[(1+x^2)*y''[x]+4*x*y'[x]+2*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2(x^5 - x^3 + x) + c_1(x^4 - x^2 + 1)$$

16.11 problem Problem 11

Internal problem ID [2907]
Internal file name [`OUTPUT/2399_Sunday_June_05_2022_03_04_35_AM_57249669/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.2. page 739
Problem number: Problem 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

`[_Gegenbauer]`

$$\boxed{\left(-4x^2+1\right) y''-20xy'-16y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y_0'$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots \\
&= y_0+xy_0'+\frac{x^2}{2}\,f\big|_{x_0,y_0,y_0'}+\frac{x^3}{3!}\,f'\big|_{x_0,y_0,y_0'}+\cdots \\
&= y_0+xy_0'+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}\,\frac{d^nf}{dx^n}\bigg|_{x_0,y_0,y_0'}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{640}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{641}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{4(5xy'+4y)}{4x^2-1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0 \\
&= \frac{416y'x^2+448yx+36y'}{\left(4x^2-1\right)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y}y' + \frac{\partial F_1}{\partial y'}F_1 \\
&= -\frac{16(616y'x^3+752x^2y+161xy'+64y)}{\left(4x^2-1\right)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y}y' + \frac{\partial F_2}{\partial y'}F_2 \\
&= \frac{\left(267264x^4+140544x^2+3600\right)y' + \left(350208x^3+89856x\right)y}{\left(4x^2-1\right)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y}y' + \frac{\partial F_3}{\partial y'}F_3 \\
&= \frac{\left(-8220672x^5-7243776x^3-558144x\right)y' - 11280384y\left(x^4+\frac{631}{1224}x^2+\frac{2}{153}\right)}{\left(4x^2-1\right)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and $y'(0)=y'(0)$ gives

$$
\begin{aligned}
F_0 &= 16y(0) \\
F_1 &= 36y'(0) \\
F_2 &= 1024y(0) \\
F_3 &= 3600y'(0) \\
F_4 &= 147456y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1+8x^2+\frac{128}{3}x^4+\frac{1024}{5}x^6\right)y(0) + \left(30x^5+6x^3+x\right)y'(0) + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(-4x^2 + 1) y'' - 20xy' - 16y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the above back into the ode gives

$$(-4x^2 + 1) \left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) - 20x \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - 16 \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \quad (1)$$

Which simplifies to

$$\sum_{n=2}^{\infty} (-4x^n a_n n(n-1)) + \left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \sum_{n=1}^{\infty} (-20n a_n x^n) + \sum_{n=0}^{\infty} (-16 a_n x^n) = 0 \quad (2)$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\sum_{n=2}^{\infty} (-4x^n a_n n(n-1)) + \left(\sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n\right) + \sum_{n=1}^{\infty} (-20n a_n x^n) + \sum_{n=0}^{\infty} (-16 a_n x^n) = 0 \quad (3)$$

$n = 0$ gives

$$2a_2 - 16a_0 = 0$$

$$a_2 = 8a_0$$

$n = 1$ gives

$$6a_3 - 36a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = 6a_1$$

For $2 \leq n$, the recurrence equation is

$$-4na_n(n-1) + (n+2)\, a_{n+2}(n+1) - 20na_n - 16a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{4(n+2)\, a_n}{n+1} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$-64a_2 + 12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{128a_0}{3}$$

For $n = 3$ the recurrence equation gives

$$-100a_3 + 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = 30a_1$$

For $n = 4$ the recurrence equation gives

$$-144a_4 + 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{1024a_0}{5}$$

For $n = 5$ the recurrence equation gives

$$-196a_5 + 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = 140a_1$$

And so on. Therefore the solution is

$$\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^n \\
&= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots
\end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + 8a_0 x^2 + 6a_1 x^3 + \frac{128}{3} a_0 x^4 + 30a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + 8x^2 + \frac{128}{3} x^4\right) a_0 + \left(30x^5 + 6x^3 + x\right) a_1 + O\left(x^6\right) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 + 8x^2 + \frac{128}{3} x^4\right) c_1 + \left(30x^5 + 6x^3 + x\right) c_2 + O\left(x^6\right)$$

<u>Summary</u>
The solution(s) found are the following

$$y = \left(1 + 8x^2 + \frac{128}{3} x^4 + \frac{1024}{5} x^6\right) y(0) + \left(30x^5 + 6x^3 + x\right) y'(0) + O\left(x^6\right) \tag{1}$$

$$y = \left(1 + 8x^2 + \frac{128}{3} x^4\right) c_1 + \left(30x^5 + 6x^3 + x\right) c_2 + O\left(x^6\right) \tag{2}$$

Verification of solutions

$$y = \left(1+8x^2+\frac{128}{3}x^4+\frac{1024}{5}x^6\right)y(0)+\left(30x^5+6x^3+x\right)y'(0)+O(x^6)$$

Verified OK.

$$y = \left(1+8x^2+\frac{128}{3}x^4\right)c_1+\left(30x^5+6x^3+x\right)c_2+O(x^6)$$

Verified OK.

16.11.1 Maple step by step solution

Let's solve

$(-4x^2+1)\,y''-20xy'-16y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{20xy'}{4x^2-1}-\frac{16y}{4x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{20xy'}{4x^2-1}+\frac{16y}{4x^2-1}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{20x}{4x^2-1}, P_3(x)=\frac{16}{4x^2-1}\right]$

  ◦ $\left(x+\frac{1}{2}\right)\cdot P_2(x)$ is analytic at $x=-\frac{1}{2}$

    $\left(\left(x+\frac{1}{2}\right)\cdot P_2(x)\right)\Big|_{x=-\frac{1}{2}}=\frac{5}{2}$

  ◦ $\left(x+\frac{1}{2}\right)^2\cdot P_3(x)$ is analytic at $x=-\frac{1}{2}$

    $\left(\left(x+\frac{1}{2}\right)^2\cdot P_3(x)\right)\Big|_{x=-\frac{1}{2}}=0$

  ◦ $x=-\frac{1}{2}$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-\frac{1}{2}$

- Multiply by denominators

$y''(4x^2-1)+20xy'+16y=0$

- Change variables using $x=u-\frac{1}{2}$ so that the regular singular point is at $u=0$

$(4u^2-4u)\left(\frac{d^2}{du^2}y(u)\right)+(20u-10)\left(\frac{d}{du}y(u)\right)+16y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  - Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-2+m}$

  - Shift index using $k->k+2-m$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $-2a_0r(3+2r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-2a_{k+1}(k+1+r)(2k+5+2r)+4a_k(k+r+2)^2\right)u^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$-2r(3+2r)=0$

- Values of r that satisfy the indicial equation

$r\in\left\{0,-\frac{3}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

$4a_k(k+r+2)^2-4\left(k+\frac{5}{2}+r\right)a_{k+1}(k+1+r)=0$

- Recursion relation that defines series solution to ODE

$a_{k+1}=\frac{2a_k(k+r+2)^2}{(2k+5+2r)(k+1+r)}$

- Recursion relation for $r=0$

$$a_{k+1} = \frac{2a_k(k+2)^2}{(2k+5)(k+1)}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{2a_k(k+2)^2}{(2k+5)(k+1)}\right]$$

- Revert the change of variables $u = x + \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k \left(x + \frac{1}{2}\right)^k, a_{k+1} = \frac{2a_k(k+2)^2}{(2k+5)(k+1)}\right]$$

- Recursion relation for $r = -\frac{3}{2}$

$$a_{k+1} = \frac{2a_k\left(k+\frac{1}{2}\right)^2}{(2k+2)\left(k-\frac{1}{2}\right)}$$

- Solution for $r = -\frac{3}{2}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{3}{2}}, a_{k+1} = \frac{2a_k\left(k+\frac{1}{2}\right)^2}{(2k+2)\left(k-\frac{1}{2}\right)}\right]$$

- Revert the change of variables $u = x + \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k \left(x + \frac{1}{2}\right)^{k-\frac{3}{2}}, a_{k+1} = \frac{2a_k\left(k+\frac{1}{2}\right)^2}{(2k+2)\left(k-\frac{1}{2}\right)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k \left(x + \frac{1}{2}\right)^k\right) + \left(\sum_{k=0}^{\infty} b_k \left(x + \frac{1}{2}\right)^{k-\frac{3}{2}}\right), a_{1+k} = \frac{2a_k(2+k)^2}{(2k+5)(1+k)}, b_{1+k} = \frac{2b_k\left(k+\frac{1}{2}\right)^2}{(2+2k)\left(k-\frac{1}{2}\right)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
Order:=6;
dsolve((1-4*x^2)*diff(y(x),x$2)-20*x*diff(y(x),x)-16*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 + 8x^2 + \frac{128}{3}x^4\right) y(0) + \left(30x^5 + 6x^3 + x\right) D(y)(0) + O\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 36

```
AsymptoticDSolveValue[(1-4*x^2)*y''[x]-20*x*y'[x]-16*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(30x^5 + 6x^3 + x\right) + c_1\left(\frac{128x^4}{3} + 8x^2 + 1\right)$$

16.12 problem Problem 12

Internal problem ID [2908]
Internal file name [`OUTPUT/2400_Sunday_June_05_2022_03_04_36_AM_38874925/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.2. page 739
Problem number: Problem 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[_Gegenbauer]
```

$$\boxed{(x^2-1)\,y''-6xy'+12y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots \\
&= y_0+xy'_0+\frac{x^2}{2}\,f|_{x_0,y_0,y'_0}+\frac{x^3}{3!}\,f'|_{x_0,y_0,y'_0}+\cdots \\
&= y_0+xy'_0+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}\,\frac{d^nf}{dx^n}\bigg|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{643}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{644}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{align*}
F_0 &= f(x,y,y') \tag{4}\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{5}\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \tag{6}
\end{align*}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= \frac{6xy' - 12y}{x^2 - 1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{18y'x^2 - 48yx + 6y'}{\left(x^2 - 1\right)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{24y'x^3 - 72x^2y + 24xy' - 24y}{\left(x^2 - 1\right)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= 0 \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= 0
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 12y(0) \\
F_1 &= 6y'(0) \\
F_2 &= 24y(0) \\
F_3 &= 0 \\
F_4 &= 0
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(x^4 + 6x^2 + 1\right) y(0) + \left(x^3 + x\right) y'(0) + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$\left(x^2 - 1\right) y'' - 6xy' + 12y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) \left(x^2 - 1\right) - 6x \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + 12 \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \sum_{n=2}^{\infty} \left(-n(n-1) a_n x^{n-2}\right) + \sum_{n=1}^{\infty} \left(-6n a_n x^n\right) + \left(\sum_{n=0}^{\infty} 12 a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} \left(-n(n-1) a_n x^{n-2}\right) = \sum_{n=0}^{\infty} \left(-(n+2) a_{n+2}(n+1) x^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\begin{aligned} \left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \sum_{n=0}^{\infty} \left(-(n+2) a_{n+2}(n+1) x^n\right) \\ + \sum_{n=1}^{\infty} \left(-6n a_n x^n\right) + \left(\sum_{n=0}^{\infty} 12 a_n x^n\right) = 0 \end{aligned} \tag{3}$$

$n = 0$ gives

$$-2a_2 + 12a_0 = 0$$

$$a_2 = 6a_0$$

$n = 1$ gives

$$-6a_3 + 6a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = a_1$$

For $2 \leq n$, the recurrence equation is

$$na_n(n-1) - (n+2)\, a_{n+2}(n+1) - 6na_n + 12a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{a_n(n^2 - 7n + 12)}{(n+2)\,(n+1)} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$2a_2 - 12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = a_0$$

For $n = 3$ the recurrence equation gives

$$-20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = 0$$

For $n = 4$ the recurrence equation gives

$$-30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = 0$$

For $n = 5$ the recurrence equation gives

$$2a_5 - 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = 0$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 x^4 + a_1 x^3 + 6 a_0 x^2 + a_1 x + a_0 + \dots$$

Collecting terms, the solution becomes

$$y = (x^4 + 6x^2 + 1) a_0 + (x^3 + x) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = (x^4 + 6x^2 + 1) c_1 + (x^3 + x) c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = (x^4 + 6x^2 + 1) y(0) + (x^3 + x) y'(0) + O(x^6) \tag{1}$$
$$y = (x^4 + 6x^2 + 1) c_1 + (x^3 + x) c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$y = (x^4 + 6x^2 + 1) y(0) + (x^3 + x) y'(0) + O(x^6)$$

Verified OK.

$$y = (x^4 + 6x^2 + 1) c_1 + (x^3 + x) c_2 + O(x^6)$$

Verified OK.

16.12.1 Maple step by step solution

Let's solve

$y''(x^2-1)-6xy'+12y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{6xy'}{x^2-1}-\frac{12y}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{6xy'}{x^2-1}+\frac{12y}{x^2-1}=0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x)=-\frac{6x}{x^2-1}, P_3(x)=\frac{12}{x^2-1}\right]$

- ◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1}=-3$

- ◦ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1}=0$

- ◦ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $y''(x^2-1)-6xy'+12y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right)+(-6u+6)\left(\frac{d}{du}y(u)\right)+12y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$-2a_0 r(-4+r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2a_{k+1}(k+1+r)(k+r-3) + a_k(k+r-3)(k+r-4)\right) u^{k+r}\right)$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-2r(-4+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 4\}$$

- Each term in the series must be 0, giving the recursion relation

$$((-2k-2r-2)\, a_{k+1} + a_k(k+r-4))(k+r-3) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k(k+r-4)}{2(k+1+r)}$$

- Recursion relation for $r = 0$; series terminates at $k = 4$

$$a_{k+1} = \frac{a_k(k-4)}{2(k+1)}$$

- Apply recursion relation for $k = 0$

$$a_1 = -2a_0$$

- Apply recursion relation for $k = 1$

$$a_2 = -\frac{3a_1}{4}$$

- Express in terms of a_0

$$a_2 = \frac{3a_0}{2}$$

- Apply recursion relation for $k = 2$

$a_3 = -\frac{a_2}{3}$

- Express in terms of a_0

$a_3 = -\frac{a_0}{2}$

- Apply recursion relation for $k = 3$

$a_4 = -\frac{a_3}{8}$

- Express in terms of a_0

$a_4 = \frac{a_0}{16}$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

$y(u) = a_0 \cdot \left(1 - 2u + \frac{3}{2}u^2 - \frac{1}{2}u^3 + \frac{1}{16}u^4\right)$

- Revert the change of variables $u = x + 1$

$\left[y = \frac{a_0 (x-1)^4}{16}\right]$

- Recursion relation for $r = 4$

$a_{k+1} = \frac{a_k k}{2(k+5)}$

- Solution for $r = 4$

$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+4}, a_{k+1} = \frac{a_k k}{2(k+5)}\right]$

- Revert the change of variables $u = x + 1$

$\left[y = \sum_{k=0}^{\infty} a_k (x+1)^{k+4}, a_{k+1} = \frac{a_k k}{2(k+5)}\right]$

- Combine solutions and rename parameters

$\left[y = \frac{a_0 (x-1)^4}{16} + \left(\sum_{k=0}^{\infty} b_k (x+1)^{k+4}\right), b_{1+k} = \frac{b_k k}{2(k+5)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 25

```
Order:=6;
dsolve((x^2-1)*diff(y(x),x$2)-6*x*diff(y(x),x)+12*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = (x^4 + 6x^2 + 1)\, y(0) + (x^3 + x)\, D(y)\,(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 25

```
AsymptoticDSolveValue[(x^2-1)*y''[x]-6*x*y'[x]+12*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2(x^3 + x) + c_1(x^4 + 6x^2 + 1)$$

16.13 problem Problem 13

Internal problem ID [2909]
Internal file name [`OUTPUT/2401_Sunday_June_05_2022_03_04_38_AM_85255342/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.2. page 739
Problem number: Problem 13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_airy", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 2y' + 4yx = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using

Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0,y_0,y_0'} + \frac{x^3}{3!} f'|_{x_0,y_0,y_0'} + \cdots \\ &= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0,y_0,y_0'} \end{aligned}$$

But

$$\begin{aligned} \frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \qquad (1) \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' \qquad (646) \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \qquad (647) \\ \frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\ &= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f \qquad (2) \\ \frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\ &= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f \qquad (3) \\ &\vdots \end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -2y' - 4yx \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= (-4x + 4)\, y' + (8x - 4)\, y \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= 16(x - 1)\, y' + 16\left(x^2 - x + \frac{1}{2}\right) y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \left(16x^2 - 48x + 56\right) y' - 64y\left(x^2 - \frac{3}{2}x + \frac{1}{4}\right) \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \left(-96x^2 + 224x - 176\right) y' - 64\left(x^3 - 3x^2 + \frac{11}{2}x - \frac{3}{2}\right) y
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -2y'(0) \\
F_1 &= 4y'(0) - 4y(0) \\
F_2 &= -16y'(0) + 8y(0) \\
F_3 &= 56y'(0) - 16y(0) \\
F_4 &= -176y'(0) + 96y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y = &\left(1 - \frac{2}{3}x^3 + \frac{1}{3}x^4 - \frac{2}{15}x^5 + \frac{2}{15}x^6\right) y(0) \\
&+ \left(x - x^2 + \frac{2}{3}x^3 - \frac{2}{3}x^4 + \frac{7}{15}x^5 - \frac{11}{45}x^6\right) y'(0) + O\left(x^6\right)
\end{aligned}
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$\begin{aligned} y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\ y'' &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \end{aligned}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = -2\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - 4\left(\sum_{n=0}^{\infty} a_n x^n\right) x \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} 2n a_n x^{n-1}\right) + \left(\sum_{n=0}^{\infty} 4x^{1+n} a_n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\begin{aligned} \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} &= \sum_{n=0}^{\infty} (n+2) a_{n+2}(1+n) x^n \\ \sum_{n=1}^{\infty} 2n a_n x^{n-1} &= \sum_{n=0}^{\infty} 2(1+n) a_{1+n} x^n \\ \sum_{n=0}^{\infty} 4x^{1+n} a_n &= \sum_{n=1}^{\infty} 4a_{n-1} x^n \end{aligned}$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2) a_{n+2}(1+n) x^n\right) + \left(\sum_{n=0}^{\infty} 2(1+n) a_{1+n} x^n\right) + \left(\sum_{n=1}^{\infty} 4a_{n-1} x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + 2a_1 = 0$$

$$a_2 = -a_1$$

For $1 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(1+n) + 2(1+n)\, a_{1+n} + 4a_{n-1} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -\frac{2(na_{1+n} + a_{1+n} + 2a_{n-1})}{(n+2)\,(1+n)} \\ &= -\frac{2a_{1+n}}{n+2} - \frac{4a_{n-1}}{(n+2)\,(1+n)} \end{aligned} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 + 4a_2 + 4a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{2a_1}{3} - \frac{2a_0}{3}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + 6a_3 + 4a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{2a_1}{3} + \frac{a_0}{3}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + 8a_4 + 4a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{7a_1}{15} - \frac{2a_0}{15}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + 10a_5 + 4a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{11a_1}{45} + \frac{2a_0}{15}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + 12a_6 + 4a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{2a_1}{15} - \frac{22a_0}{315}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - a_1 x^2 + \left(\frac{2a_1}{3} - \frac{2a_0}{3}\right) x^3 + \left(-\frac{2a_1}{3} + \frac{a_0}{3}\right) x^4 + \left(\frac{7a_1}{15} - \frac{2a_0}{15}\right) x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{2}{3}x^3 + \frac{1}{3}x^4 - \frac{2}{15}x^5\right) a_0 + \left(x - x^2 + \frac{2}{3}x^3 - \frac{2}{3}x^4 + \frac{7}{15}x^5\right) a_1 + O(x^6) \quad (3)$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{2}{3}x^3 + \frac{1}{3}x^4 - \frac{2}{15}x^5\right) c_1 + \left(x - x^2 + \frac{2}{3}x^3 - \frac{2}{3}x^4 + \frac{7}{15}x^5\right) c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$
\begin{align*}
y &= \left(1-\frac{2}{3}x^{3}+\frac{1}{3}x^{4}-\frac{2}{15}x^{5}+\frac{2}{15}x^{6}\right) y(0) \\
&+ \left(x-x^{2}+\frac{2}{3}x^{3}-\frac{2}{3}x^{4}+\frac{7}{15}x^{5}-\frac{11}{45}x^{6}\right) y'(0)+O\left(x^{6}\right) \tag{1}
\end{align*}
$$

$$
y = \left(1-\frac{2}{3}x^{3}+\frac{1}{3}x^{4}-\frac{2}{15}x^{5}\right) c_{1}+\left(x-x^{2}+\frac{2}{3}x^{3}-\frac{2}{3}x^{4}+\frac{7}{15}x^{5}\right) c_{2}+O\left(x^{6}\right) \tag{2}
$$

Verification of solutions

$$
\begin{align*}
y &= \left(1-\frac{2}{3}x^{3}+\frac{1}{3}x^{4}-\frac{2}{15}x^{5}+\frac{2}{15}x^{6}\right) y(0) \\
&+ \left(x-x^{2}+\frac{2}{3}x^{3}-\frac{2}{3}x^{4}+\frac{7}{15}x^{5}-\frac{11}{45}x^{6}\right) y'(0)+O\left(x^{6}\right)
\end{align*}
$$

Verified OK.

$$
y = \left(1-\frac{2}{3}x^{3}+\frac{1}{3}x^{4}-\frac{2}{15}x^{5}\right) c_{1}+\left(x-x^{2}+\frac{2}{3}x^{3}-\frac{2}{3}x^{4}+\frac{7}{15}x^{5}\right) c_{2}+O\left(x^{6}\right)
$$

Verified OK.

16.13.1 Maple step by step solution

Let's solve

$y'' = -2y' - 4yx$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + 2y' + 4yx = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ○ Convert $x \cdot y$ to series expansion

  $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

- Shift index using $k->k-1$

  $$x \cdot y=\sum_{k=1}^{\infty} a_{k-1} x^{k}$$

- Convert y' to series expansion

  $$y'=\sum_{k=1}^{\infty} a_{k} k\, x^{k-1}$$

- Shift index using $k->k+1$

  $$y'=\sum_{k=0}^{\infty} a_{k+1}(k+1)\, x^{k}$$

- Convert y'' to series expansion

  $$y''=\sum_{k=2}^{\infty} a_{k} k(k-1)\, x^{k-2}$$

- Shift index using $k->k+2$

  $$y''=\sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^{k}$$

  Rewrite ODE with series expansions

  $$2a_{2}+2a_{1}+\left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1)+2a_{k+1}(k+1)+4a_{k-1}\right) x^{k}\right)=0$$

- Each term must be 0

  $$2a_{2}+2a_{1}=0$$

- Each term in the series must be 0, giving the recursion relation

  $$\left(k^{2}+3k+2\right) a_{k+2}+2a_{k+1}k+4a_{k-1}+2a_{k+1}=0$$

- Shift index using $k->k+1$

  $$\left((k+1)^{2}+3k+5\right) a_{k+3}+2a_{k+2}(k+1)+4a_{k}+2a_{k+2}=0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y=\sum_{k=0}^{\infty} a_{k} x^{k}, a_{k+3}=-\frac{2(k a_{k+2}+2a_{k}+2a_{k+2})}{k^{2}+5k+6}, 2a_{2}+2a_{1}=0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 49

```
Order:=6;
dsolve(diff(y(x),x$2)+2*diff(y(x),x)+4*x*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1-\frac{2}{3}x^3+\frac{1}{3}x^4-\frac{2}{15}x^5\right)y(0)+\left(x-x^2+\frac{2}{3}x^3-\frac{2}{3}x^4+\frac{7}{15}x^5\right)D(y)(0)+O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 61

```
AsymptoticDSolveValue[y''[x]+2*y'[x]+4*x*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(-\frac{2x^5}{15}+\frac{x^4}{3}-\frac{2x^3}{3}+1\right)+c_2\left(\frac{7x^5}{15}-\frac{2x^4}{3}+\frac{2x^3}{3}-x^2+x\right)$$

16.14 problem Problem 14

Internal problem ID [2910]
Internal file name [`OUTPUT/2402_Sunday_June_05_2022_03_04_41_AM_11973721/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.2. page 739
Problem number: Problem 14.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + xy' + (x+2)\, y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + xy'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\ &= y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y'_0} \end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (649) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (650) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') && (4) \\
F_1 &= \frac{df}{dx} \\
&= \frac{dF_0}{dx} \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (5) \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right) \\
&= \frac{d}{dx}(F_1) \\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y'' \\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0 \\
&\vdots \\
F_n &= \frac{d}{dx}(F_{n-1}) \\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y'' \\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -xy' - yx - 2y\\
F_1 &= \frac{dF_0}{dx}\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
&= \left(x^2 - x - 3\right) y' + y\left(x^2 + 2x - 1\right)\\
F_2 &= \frac{dF_1}{dx}\\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1\\
&= \left(-x^3 + 2x^2 + 7x - 2\right) y' - y\left(x^3 + x^2 - 7x - 8\right)\\
F_3 &= \frac{dF_2}{dx}\\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2\\
&= \left(x^4 - 3x^3 - 11x^2 + 13x + 15\right) y' + y\left(x^4 - 14x^2 - 14x + 11\right)\\
F_4 &= \frac{dF_3}{dx}\\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3\\
&= \left(-x^5 + 4x^4 + 15x^3 - 36x^2 - 51x + 24\right) y' - y\left(x^5 - x^4 - 21x^3 - 9x^2 + 69x + 44\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and $y'(0)=y'(0)$ gives

$$
\begin{aligned}
F_0 &= -2y(0)\\
F_1 &= -3y'(0) - y(0)\\
F_2 &= -2y'(0) + 8y(0)\\
F_3 &= 15y'(0) + 11y(0)\\
F_4 &= 24y'(0) - 44y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y = &\left(1 - x^2 - \frac{1}{6}x^3 + \frac{1}{3}x^4 + \frac{11}{120}x^5 - \frac{11}{180}x^6\right) y(0)\\
&+ \left(x - \frac{1}{2}x^3 - \frac{1}{12}x^4 + \frac{1}{8}x^5 + \frac{1}{30}x^6\right) y'(0) + O\left(x^6\right)
\end{aligned}
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = -x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - \left(\sum_{n=0}^{\infty} a_n x^n\right) x - 2\left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} n\, x^n a_n\right) + \left(\sum_{n=0}^{\infty} x^{1+n} a_n\right) + \left(\sum_{n=0}^{\infty} 2a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n$$
$$\sum_{n=0}^{\infty} x^{1+n} a_n = \sum_{n=1}^{\infty} a_{n-1} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n\right) + \left(\sum_{n=1}^{\infty} n\, x^n a_n\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^n\right) + \left(\sum_{n=0}^{\infty} 2a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + 2a_0 = 0$$

$$a_2 = -a_0$$

For $1 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(1+n) + n a_n + a_{n-1} + 2a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -\frac{n a_n + 2a_n + a_{n-1}}{(n+2)\,(1+n)} \\ &= -\frac{a_n}{1+n} - \frac{a_{n-1}}{(n+2)\,(1+n)} \end{aligned} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 + 3a_1 + a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{a_1}{2} - \frac{a_0}{6}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + 4a_2 + a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{3} - \frac{a_1}{12}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + 5a_3 + a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_1}{8} + \frac{11a_0}{120}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + 6a_4 + a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{11a_0}{180} + \frac{a_1}{30}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + 7a_5 + a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{19a_1}{1008} - \frac{13a_0}{560}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1x - a_0x^2 + \left(-\frac{a_1}{2} - \frac{a_0}{6}\right)x^3 + \left(\frac{a_0}{3} - \frac{a_1}{12}\right)x^4 + \left(\frac{a_1}{8} + \frac{11a_0}{120}\right)x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - x^2 - \frac{1}{6}x^3 + \frac{1}{3}x^4 + \frac{11}{120}x^5\right)a_0 + \left(x - \frac{1}{2}x^3 - \frac{1}{12}x^4 + \frac{1}{8}x^5\right)a_1 + O(x^6) \quad (3)$$

At $x = 0$ the solution above becomes

$$y = \left(1 - x^2 - \frac{1}{6}x^3 + \frac{1}{3}x^4 + \frac{11}{120}x^5\right)c_1 + \left(x - \frac{1}{2}x^3 - \frac{1}{12}x^4 + \frac{1}{8}x^5\right)c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = &\left(1-x^{2}-\frac{1}{6}x^{3}+\frac{1}{3}x^{4}+\frac{11}{120}x^{5}-\frac{11}{180}x^{6}\right) y(0) \\
&+\left(x-\frac{1}{2}x^{3}-\frac{1}{12}x^{4}+\frac{1}{8}x^{5}+\frac{1}{30}x^{6}\right) y'(0)+O\left(x^{6}\right)
\end{aligned} \tag{1}
$$

$$
y = \left(1-x^{2}-\frac{1}{6}x^{3}+\frac{1}{3}x^{4}+\frac{11}{120}x^{5}\right) c_{1}+\left(x-\frac{1}{2}x^{3}-\frac{1}{12}x^{4}+\frac{1}{8}x^{5}\right) c_{2}+O\left(x^{6}\right) \tag{2}
$$

Verification of solutions

$$
\begin{aligned}
y = &\left(1-x^{2}-\frac{1}{6}x^{3}+\frac{1}{3}x^{4}+\frac{11}{120}x^{5}-\frac{11}{180}x^{6}\right) y(0) \\
&+\left(x-\frac{1}{2}x^{3}-\frac{1}{12}x^{4}+\frac{1}{8}x^{5}+\frac{1}{30}x^{6}\right) y'(0)+O\left(x^{6}\right)
\end{aligned}
$$

Verified OK.

$$
y = \left(1-x^{2}-\frac{1}{6}x^{3}+\frac{1}{3}x^{4}+\frac{11}{120}x^{5}\right) c_{1}+\left(x-\frac{1}{2}x^{3}-\frac{1}{12}x^{4}+\frac{1}{8}x^{5}\right) c_{2}+O\left(x^{6}\right)
$$

Verified OK.

16.14.1 Maple step by step solution

Let's solve

$y'' = -xy' - yx - 2y$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = (-2-x)\, y - xy'$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + xy' + (x+2)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

$$x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$$

- Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$$

- Convert $x \cdot y'$ to series expansion

$$x \cdot y' = \sum_{k=0}^{\infty} a_k k \, x^k$$

- Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1) \, x^{k-2}$$

- Shift index using $k- >k + 2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1) \, x^k$$

Rewrite ODE with series expansions

$$2a_2 + 2a_0 + \left(\sum_{k=1}^{\infty} (a_{k+2}(k+2)(k+1) + a_k(k+2) + a_{k-1}) \, x^k\right) = 0$$

- Each term must be 0

$$2a_2 + 2a_0 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$(k^2 + 3k + 2) \, a_{k+2} + a_k k + 2a_k + a_{k-1} = 0$$

- Shift index using $k- >k + 1$

$$\left((k+1)^2 + 3k + 5\right) a_{k+3} + a_{k+1}(k+1) + 2a_{k+1} + a_k = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = -\frac{k a_{k+1} + a_k + 3a_{k+1}}{k^2 + 5k + 6}, 2a_2 + 2a_0 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form could result into a too large expression - returning special functi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 49

```
Order:=6;
dsolve(diff(y(x),x$2)+x*diff(y(x),x)+(2+x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x)=\left(1-x^{2}-\frac{1}{6}x^{3}+\frac{1}{3}x^{4}+\frac{11}{120}x^{5}\right)y(0)+\left(x-\frac{1}{2}x^{3}-\frac{1}{12}x^{4}+\frac{1}{8}x^{5}\right)D(y)(0)+O\left(x^{6}\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 61

```
AsymptoticDSolveValue[y''[x]+x*y'[x]+(2+x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{x^5}{8}-\frac{x^4}{12}-\frac{x^3}{2}+x\right)+c_1\left(\frac{11x^5}{120}+\frac{x^4}{3}-\frac{x^3}{6}-x^2+1\right)$$

16.15 problem Problem 15

Internal problem ID [2911]
Internal file name [`OUTPUT/2403_Sunday_June_05_2022_03_04_43_AM_47532475/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.2. page 739
Problem number: Problem 15.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode_form_A", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - e^{x} y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \left. \frac{d^n f}{dx^n} \right|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (652)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (653)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= \mathrm{e}^x y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \mathrm{e}^x (y' + y) \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \mathrm{e}^x (\mathrm{e}^x y + 2y' + y) \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= (4y + y')\,\mathrm{e}^{2x} + \mathrm{e}^x (y + 3y') \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= (11y + 6y')\,\mathrm{e}^{2x} + y\,\mathrm{e}^{3x} + \mathrm{e}^x (y + 4y')
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and $y'(0)=y'(0)$ gives

$$
\begin{aligned}
F_0 &= y(0) \\
F_1 &= y'(0) + y(0) \\
F_2 &= 2y(0) + 2y'(0) \\
F_3 &= 5y(0) + 4y'(0) \\
F_4 &= 13y(0) + 10y'(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y = {} & \left(1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{24}x^5 + \frac{13}{720}x^6\right) y(0) \\
& + \left(x + \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{30}x^5 + \frac{1}{72}x^6\right) y'(0) + O\left(x^6\right)
\end{aligned}
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}
\end{aligned}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \mathrm{e}^{x} \left(\sum_{n=0}^{\infty} a_n x^n \right) \tag{1}$$

Expanding $-\mathrm{e}^{x}$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned}
-\mathrm{e}^{x} &= -1 - x - \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{24}x^4 - \frac{1}{120}x^5 - \frac{1}{720}x^6 + \dots \\
&= -1 - x - \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{24}x^4 - \frac{1}{120}x^5 - \frac{1}{720}x^6
\end{aligned}$$

Hence the ODE in Eq (1) becomes

$$\begin{aligned}
&\left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \right) \\
&+ \left(-1 - x - \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{24}x^4 - \frac{1}{120}x^5 - \frac{1}{720}x^6 \right) \left(\sum_{n=0}^{\infty} a_n x^n \right) = 0
\end{aligned}$$

Expanding the second term in (1) gives

$$\begin{aligned}
&\left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \right) + -1 \cdot \left(\sum_{n=0}^{\infty} a_n x^n \right) - x \cdot \left(\sum_{n=0}^{\infty} a_n x^n \right) - \frac{x^2}{2} \cdot \left(\sum_{n=0}^{\infty} a_n x^n \right) \\
&- \frac{x^3}{6} \cdot \left(\sum_{n=0}^{\infty} a_n x^n \right) - \frac{x^4}{24} \cdot \left(\sum_{n=0}^{\infty} a_n x^n \right) - \frac{x^5}{120} \cdot \left(\sum_{n=0}^{\infty} a_n x^n \right) - \frac{x^6}{720} \cdot \left(\sum_{n=0}^{\infty} a_n x^n \right) = 0
\end{aligned}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right)+\sum_{n=0}^{\infty}\left(-a_n x^n\right)+\sum_{n=0}^{\infty}\left(-x^{1+n} a_n\right) \\
&\quad+\sum_{n=0}^{\infty}\left(-\frac{x^{n+2} a_n}{2}\right)+\sum_{n=0}^{\infty}\left(-\frac{x^{n+3} a_n}{6}\right)+\sum_{n=0}^{\infty}\left(-\frac{x^{n+4} a_n}{24}\right) \\
&\quad+\sum_{n=0}^{\infty}\left(-\frac{x^{n+5} a_n}{120}\right)+\sum_{n=0}^{\infty}\left(-\frac{x^{n+6} a_n}{720}\right)=0
\end{aligned} \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} &= \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n \\
\sum_{n=0}^{\infty} \left(-x^{1+n} a_n\right) &= \sum_{n=1}^{\infty} \left(-a_{n-1} x^n\right) \\
\sum_{n=0}^{\infty} \left(-\frac{x^{n+2} a_n}{2}\right) &= \sum_{n=2}^{\infty} \left(-\frac{a_{n-2} x^n}{2}\right) \\
\sum_{n=0}^{\infty} \left(-\frac{x^{n+3} a_n}{6}\right) &= \sum_{n=3}^{\infty} \left(-\frac{a_{n-3} x^n}{6}\right) \\
\sum_{n=0}^{\infty} \left(-\frac{x^{n+4} a_n}{24}\right) &= \sum_{n=4}^{\infty} \left(-\frac{a_{n-4} x^n}{24}\right) \\
\sum_{n=0}^{\infty} \left(-\frac{x^{n+5} a_n}{120}\right) &= \sum_{n=5}^{\infty} \left(-\frac{a_{n-5} x^n}{120}\right) \\
\sum_{n=0}^{\infty} \left(-\frac{x^{n+6} a_n}{720}\right) &= \sum_{n=6}^{\infty} \left(-\frac{a_{n-6} x^n}{720}\right)
\end{aligned}
$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty}(n+2)\,a_{n+2}(1+n)\,x^{n}\right)+\sum_{n=0}^{\infty}(-a_{n}x^{n})+\sum_{n=1}^{\infty}(-a_{n-1}x^{n}) \\ +\sum_{n=2}^{\infty}\left(-\frac{a_{n-2}x^{n}}{2}\right)+\sum_{n=3}^{\infty}\left(-\frac{a_{n-3}x^{n}}{6}\right)+\sum_{n=4}^{\infty}\left(-\frac{a_{n-4}x^{n}}{24}\right) \\ +\sum_{n=5}^{\infty}\left(-\frac{a_{n-5}x^{n}}{120}\right)+\sum_{n=6}^{\infty}\left(-\frac{a_{n-6}x^{n}}{720}\right)=0 \tag{3}$$

$n=0$ gives

$$2a_2 - a_0 = 0$$

$$a_2 = \frac{a_0}{2}$$

$n=1$ gives

$$6a_3 - a_1 - a_0 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = \frac{a_0}{6} + \frac{a_1}{6}$$

$n=2$ gives

$$12a_4 - a_2 - a_1 - \frac{a_0}{2} = 0$$

Which after substituting earlier equations, simplifies to

$$a_4 = \frac{a_0}{12} + \frac{a_1}{12}$$

$n=3$ gives

$$20a_5 - a_3 - a_2 - \frac{a_1}{2} - \frac{a_0}{6} = 0$$

Which after substituting earlier equations, simplifies to

$$a_5 = \frac{a_0}{24} + \frac{a_1}{30}$$

$n = 4$ gives

$$30a_6 - a_4 - a_3 - \frac{a_2}{2} - \frac{a_1}{6} - \frac{a_0}{24} = 0$$

Which after substituting earlier equations, simplifies to

$$a_6 = \frac{13a_0}{720} + \frac{a_1}{72}$$

$n = 5$ gives

$$42a_7 - a_5 - a_4 - \frac{a_3}{2} - \frac{a_2}{6} - \frac{a_1}{24} - \frac{a_0}{120} = 0$$

Which after substituting earlier equations, simplifies to

$$a_7 = \frac{a_0}{140} + \frac{29a_1}{5040}$$

For $6 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(1+n) - a_n - a_{n-1} - \frac{a_{n-2}}{2} - \frac{a_{n-3}}{6} - \frac{a_{n-4}}{24} - \frac{a_{n-5}}{120} - \frac{a_{n-6}}{720} = 0 \qquad (4)$$

Solving for a_{n+2}, gives

$$\begin{aligned}
a_{n+2} &= \frac{720a_n + 720a_{n-1} + 360a_{n-2} + 120a_{n-3} + 30a_{n-4} + 6a_{n-5} + a_{n-6}}{720\,(n+2)\,(1+n)} \\
(5) \quad &= \frac{a_n}{(n+2)\,(1+n)} + \frac{a_{n-6}}{720\,(n+2)\,(1+n)} + \frac{a_{n-5}}{120\,(n+2)\,(1+n)} \\
&\quad + \frac{a_{n-4}}{24\,(n+2)\,(1+n)} + \frac{a_{n-3}}{6\,(n+2)\,(1+n)} + \frac{a_{n-2}}{2\,(n+2)\,(1+n)} + \frac{a_{n-1}}{(n+2)\,(1+n)}
\end{aligned}$$

And so on. Therefore the solution is

$$\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^n \\
&= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots
\end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1x + \frac{a_0x^2}{2} + \left(\frac{a_0}{6} + \frac{a_1}{6}\right)x^3 + \left(\frac{a_0}{12} + \frac{a_1}{12}\right)x^4 + \left(\frac{a_0}{24} + \frac{a_1}{30}\right)x^5 + \dots$$

Collecting terms, the solution becomes

$$y=\left(1+\frac{1}{2}x^{2}+\frac{1}{6}x^{3}+\frac{1}{12}x^{4}+\frac{1}{24}x^{5}\right)a_{0}+\left(x+\frac{1}{6}x^{3}+\frac{1}{12}x^{4}+\frac{1}{30}x^{5}\right)a_{1}+O\left(x^{6}\right) \tag{3}$$

At $x=0$ the solution above becomes

$$y=\left(1+\frac{1}{2}x^{2}+\frac{1}{6}x^{3}+\frac{1}{12}x^{4}+\frac{1}{24}x^{5}\right)c_{1}+\left(x+\frac{1}{6}x^{3}+\frac{1}{12}x^{4}+\frac{1}{30}x^{5}\right)c_{2}+O\left(x^{6}\right)$$

Summary

The solution(s) found are the following

$$\begin{aligned}y=&\left(1+\frac{1}{2}x^{2}+\frac{1}{6}x^{3}+\frac{1}{12}x^{4}+\frac{1}{24}x^{5}+\frac{13}{720}x^{6}\right)y(0)\\&+\left(x+\frac{1}{6}x^{3}+\frac{1}{12}x^{4}+\frac{1}{30}x^{5}+\frac{1}{72}x^{6}\right)y'(0)+O\left(x^{6}\right)\end{aligned} \tag{1}$$

$$y=\left(1+\frac{1}{2}x^{2}+\frac{1}{6}x^{3}+\frac{1}{12}x^{4}+\frac{1}{24}x^{5}\right)c_{1}+\left(x+\frac{1}{6}x^{3}+\frac{1}{12}x^{4}+\frac{1}{30}x^{5}\right)c_{2}+O\left(x^{6}\right) \tag{2}$$

Verification of solutions

$$\begin{aligned}y=&\left(1+\frac{1}{2}x^{2}+\frac{1}{6}x^{3}+\frac{1}{12}x^{4}+\frac{1}{24}x^{5}+\frac{13}{720}x^{6}\right)y(0)\\&+\left(x+\frac{1}{6}x^{3}+\frac{1}{12}x^{4}+\frac{1}{30}x^{5}+\frac{1}{72}x^{6}\right)y'(0)+O\left(x^{6}\right)\end{aligned}$$

Verified OK.

$$y=\left(1+\frac{1}{2}x^{2}+\frac{1}{6}x^{3}+\frac{1}{12}x^{4}+\frac{1}{24}x^{5}\right)c_{1}+\left(x+\frac{1}{6}x^{3}+\frac{1}{12}x^{4}+\frac{1}{30}x^{5}\right)c_{2}+O\left(x^{6}\right)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a quadrature
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
   Change of variables used:
      [x = ln(t)]
   Linear ODE actually solved:
      -u(t)+diff(u(t),t)+t*diff(diff(u(t),t),t) = 0
<- change of variables successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 49

```
Order:=6;
dsolve(diff(y(x),x$2)-exp(x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{12}x^4+\frac{1}{24}x^5\right)y(0)+\left(x+\frac{1}{6}x^3+\frac{1}{12}x^4+\frac{1}{30}x^5\right)D(y)(0)+O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 63

```
AsymptoticDSolveValue[y''[x]-Exp[x]*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{x^5}{30}+\frac{x^4}{12}+\frac{x^3}{6}+x\right)+c_1\left(\frac{x^5}{24}+\frac{x^4}{12}+\frac{x^3}{6}+\frac{x^2}{2}+1\right)$$

16.16 problem Problem 17

Internal problem ID [2912]
Internal file name [`OUTPUT/2404_Sunday_June_05_2022_03_04_45_AM_57572708/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.2. page 739
Problem number: Problem 17.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{xy'' - (x-1)\,y' - yx = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + (1-x)\,y' - yx = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{x-1}{x}$$
$$q(x) = -1$$

Table 416: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{x-1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -1$	
singularity	type

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + (1-x)\,y' - yx = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) x + (1-x)\left(\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\right) - \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) x = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)\, x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n\right) = \sum_{n=2}^{\infty} \left(-a_{n-2} x^{n+r-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)\, x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \sum_{n=2}^{\infty} \left(-a_{n-2} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) + (n+r) a_n x^{n+r-1} = 0$$

When $n = 0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) + r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) + r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^{-1+r} r^2 = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^{-1+r}r^2 = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = \frac{r}{(1+r)^2}$$

For $2 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) - a_{n-1}(n+r-1) + a_n(n+r) - a_{n-2} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{n a_{n-1} + r a_{n-1} + a_{n-2} - a_{n-1}}{n^2 + 2nr + r^2} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = \frac{(n-1)\,a_{n-1} + a_{n-2}}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r}{(1+r)^2}$	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{2r+1}{(1+r)\,(r+2)^2}$$

Which for the root $r = 0$ becomes

$$a_2 = \frac{1}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r}{(1+r)^2}$	0
a_2	$\frac{2r+1}{(1+r)(r+2)^2}$	$\frac{1}{4}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{3r^2 + 5r + 1}{(1+r)^2\,(r+2)\,(r+3)^2}$$

Which for the root $r = 0$ becomes

$$a_3 = \frac{1}{18}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r}{(1+r)^2}$	0
a_2	$\frac{2r+1}{(1+r)(r+2)^2}$	$\frac{1}{4}$
a_3	$\frac{3r^2+5r+1}{(1+r)^2(r+2)(r+3)^2}$	$\frac{1}{18}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{5r^3 + 20r^2 + 21r + 5}{(r+2)^2 (1+r)^2 (r+3) (r+4)^2}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{5}{192}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r}{(1+r)^2}$	0
a_2	$\frac{2r+1}{(1+r)(r+2)^2}$	$\frac{1}{4}$
a_3	$\frac{3r^2+5r+1}{(1+r)^2(r+2)(r+3)^2}$	$\frac{1}{18}$
a_4	$\frac{5r^3+20r^2+21r+5}{(r+2)^2(1+r)^2(r+3)(r+4)^2}$	$\frac{5}{192}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{8r^4 + 58r^3 + 136r^2 + 114r + 23}{(r+3)^2 (1+r)^2 (r+2)^2 (r+4) (r+5)^2}$$

Which for the root $r = 0$ becomes

$$a_5 = \frac{23}{3600}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r}{(1+r)^2}$	0
a_2	$\frac{2r+1}{(1+r)(r+2)^2}$	$\frac{1}{4}$
a_3	$\frac{3r^2+5r+1}{(1+r)^2(r+2)(r+3)^2}$	$\frac{1}{18}$
a_4	$\frac{5r^3+20r^2+21r+5}{(r+2)^2(1+r)^2(r+3)(r+4)^2}$	$\frac{5}{192}$
a_5	$\frac{8r^4+58r^3+136r^2+114r+23}{(r+3)^2(1+r)^2(r+2)^2(r+4)(r+5)^2}$	$\frac{23}{3600}$

Using the above table, then the first solution $y_1(x)$ becomes

$$
\begin{aligned}
y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots \\
&= 1 + \frac{x^2}{4} + \frac{x^3}{18} + \frac{5x^4}{192} + \frac{23x^5}{3600} + O(x^6)
\end{aligned}
$$

Now the second solution is found. The second solution is given by

$$
y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r} \right)
$$

Where b_n is found using

$$
b_n = \frac{d}{dr} a_{n,r}
$$

And the above is then evaluated at $r = 0$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	$b_n(r =$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	$\frac{r}{(1+r)^2}$	0	$\frac{1-r}{(1+r)^3}$	1
b_2	$\frac{2r+1}{(1+r)(r+2)^2}$	$\frac{1}{4}$	$\frac{-4r^2-5r}{(r+2)^3(1+r)^2}$	0
b_3	$\frac{3r^2+5r+1}{(1+r)^2(r+2)(r+3)^2}$	$\frac{1}{18}$	$\frac{-9r^4-44r^3-66r^2-24r+11}{(r+3)^3(1+r)^3(r+2)^2}$	$\frac{11}{108}$
b_4	$\frac{5r^3+20r^2+21r+5}{(r+2)^2(1+r)^2(r+3)(r+4)^2}$	$\frac{5}{192}$	$\frac{-20r^6-215r^5-891r^4-1768r^3-1656r^2-552r+44}{(r+4)^3(r+2)^3(1+r)^3(r+3)^2}$	$\frac{11}{1152}$
b_5	$\frac{8r^4+58r^3+136r^2+114r+23}{(r+3)^2(1+r)^2(r+2)^2(r+4)(r+5)^2}$	$\frac{23}{3600}$	$\frac{-40r^8-740r^7-5696r^6-23580r^5-56475r^4-77715r^3-55823r^2-14473r+1766}{(r+5)^3(r+3)^3(1+r)^3(r+2)^3(r+4)^2}$	$\frac{883}{216000}$

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x) \ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\
&= \left(1 + \frac{x^2}{4} + \frac{x^3}{18} + \frac{5x^4}{192} + \frac{23x^5}{3600} + O(x^6)\right) \ln(x) + x + \frac{11x^3}{108} + \frac{11x^4}{1152} + \frac{883x^5}{216000} + O(x^6)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 \left(1 + \frac{x^2}{4} + \frac{x^3}{18} + \frac{5x^4}{192} + \frac{23x^5}{3600} + O(x^6)\right) \\
&\quad + c_2 \left(\left(1 + \frac{x^2}{4} + \frac{x^3}{18} + \frac{5x^4}{192} + \frac{23x^5}{3600} + O(x^6)\right) \ln(x) + x + \frac{11x^3}{108} + \frac{11x^4}{1152} \right. \\
&\qquad \left. + \frac{883x^5}{216000} + O(x^6)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 \left(1 + \frac{x^2}{4} + \frac{x^3}{18} + \frac{5x^4}{192} + \frac{23x^5}{3600} + O(x^6)\right) \\
&\quad + c_2 \left(\left(1 + \frac{x^2}{4} + \frac{x^3}{18} + \frac{5x^4}{192} + \frac{23x^5}{3600} + O(x^6)\right) \ln(x) + x + \frac{11x^3}{108} + \frac{11x^4}{1152} + \frac{883x^5}{216000} \right. \\
&\qquad \left. + O(x^6)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 \left(1 + \frac{x^2}{4} + \frac{x^3}{18} + \frac{5x^4}{192} + \frac{23x^5}{3600} + O(x^6)\right) \\
&\quad + c_2 \left(\left(1 + \frac{x^2}{4} + \frac{x^3}{18} + \frac{5x^4}{192} + \frac{23x^5}{3600} + O(x^6)\right) \ln(x) + x + \frac{11x^3}{108} + \frac{11x^4}{1152} \right. \\
&\qquad \left. + \frac{883x^5}{216000} + O(x^6)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1 \left(1 + \frac{x^2}{4} + \frac{x^3}{18} + \frac{5x^4}{192} + \frac{23x^5}{3600} + O(x^6)\right) \\
&\quad + c_2 \left(\left(1 + \frac{x^2}{4} + \frac{x^3}{18} + \frac{5x^4}{192} + \frac{23x^5}{3600} + O(x^6)\right) \ln(x) + x + \frac{11x^3}{108} + \frac{11x^4}{1152} + \frac{883x^5}{216000} \right. \\
&\qquad \left. + O(x^6)\right)
\end{aligned}
$$

Verified OK.

16.16.1 Maple step by step solution

Let's solve

$y''x + (1-x)\,y' - yx = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(x-1)y'}{x} + y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(x-1)y'}{x} - y = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -\frac{x-1}{x}, P_3(x) = -1\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y''x + (1-x)\,y' - yx = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x \cdot y$ to series expansion

  $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$

  ◦ Shift index using $k- >k-1$

$$x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

$$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k->k+1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r^2 x^{-1+r} + \left(a_1(1+r)^2 - a_0 r\right) x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)^2 - a_k(k+r) - a_{k-1}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = 0$$

- Each term must be 0

$$a_1(1+r)^2 - a_0 r = 0$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k+1}(k+1)^2 - a_k k - a_{k-1} = 0$$

- Shift index using $k->k+1$

$$a_{k+2}(k+2)^2 - a_{k+1}(k+1) - a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = \frac{k a_{k+1} + a_k + a_{k+1}}{(k+2)^2}$$

- Recursion relation for $r = 0$

$$a_{k+2} = \frac{k a_{k+1} + a_k + a_{k+1}}{(k+2)^2}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{k a_{k+1} + a_k + a_{k+1}}{(k+2)^2}, a_1 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Kummer successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 53

```
Order:=6;
dsolve(x*diff(y(x),x$2)-(x-1)*diff(y(x),x)-x*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = (c_2 \ln(x) + c_1)\left(1 + \frac{1}{4}x^2 + \frac{1}{18}x^3 + \frac{5}{192}x^4 + \frac{23}{3600}x^5 + \mathrm{O}\left(x^6\right)\right) + \left(x + \frac{11}{108}x^3 + \frac{11}{1152}x^4 + \frac{883}{216000}x^5 + \mathrm{O}\left(x^6\right)\right) c_2$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 96

```
AsymptoticDSolveValue[x*y''[x]-(x-1)*y'[x]-x*y[x]==0,y[x],{x,0,5}]
```

$$
\begin{aligned}
y(x) \to\; & c_1\left(\frac{23x^5}{3600}+\frac{5x^4}{192}+\frac{x^3}{18}+\frac{x^2}{4}+1\right) \\
& + c_2\left(\frac{883x^5}{216000}+\frac{11x^4}{1152}+\frac{11x^3}{108}+\left(\frac{23x^5}{3600}+\frac{5x^4}{192}+\frac{x^3}{18}+\frac{x^2}{4}+1\right)\log(x)+x\right)
\end{aligned}
$$

16.17 problem Problem 18

Internal problem ID [2913]
Internal file name [`OUTPUT/2405_Sunday_June_05_2022_03_04_49_AM_13349022/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.2. page 739
Problem number: Problem 18.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(2x^2+1)\,y''+7xy'+2y=0}$$

With initial conditions

$$[y(0)=0, y'(0)=1]$$

With the expansion point for the power series method at $x=0$.

16.17.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y''+p(x)y'+q(x)y=F$$

Where here

$$\begin{aligned} p(x) &= \frac{7x}{2x^2+1} \\ q(x) &= \frac{2}{2x^2+1} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + \frac{7xy'}{2x^2+1} + \frac{2y}{2x^2+1} = 0$$

The domain of $p(x) = \frac{7x}{2x^2+1}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \frac{2}{2x^2+1}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (656)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (657)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} F_n|_{x_0,y_0,y_0'} \qquad (7)$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{7xy' + 2y}{2x^2+1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{59y'x^2 + 22yx - 9y'}{(2x^2+1)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{-605y'x^3 - 250x^2y + 275xy' + 40y}{(2x^2+1)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{(7365x^4 - 6660x^2 + 315)\, y' + (3210x^3 - 1530x)\, y}{(2x^2+1)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{(-104055x^5 + 156150x^3 - 22095x)\, y' + (-46830x^4 + 44370x^2 - 2160)\, y}{(2x^2+1)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=0$ and $y'(0)=1$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= -9 \\
F_2 &= 0 \\
F_3 &= 315 \\
F_4 &= 0
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = x - \frac{3x^3}{2} + \frac{21x^5}{8} + O(x^6)
$$

$$
y = x-\frac{3 x^{3}}{2}+\frac{21 x^{5}}{8}+O\left(x^{6}\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$
\left(2 x^{2}+1\right) y''+7 x y'+2 y = 0
$$

Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_{n} x^{n}
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_{n} x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n \left(n-1\right) a_{n} x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\left(2 x^{2}+1\right) \left(\sum_{n=2}^{\infty} n \left(n-1\right) a_{n} x^{n-2}\right)+7 x \left(\sum_{n=1}^{\infty} n a_{n} x^{n-1}\right)+2\left(\sum_{n=0}^{\infty} a_{n} x^{n}\right) = 0 \tag{1}
$$

Which simplifies to

$$
\left(\sum_{n=2}^{\infty} 2 x^{n} a_{n} n \left(n-1\right)\right)+\left(\sum_{n=2}^{\infty} n \left(n-1\right) a_{n} x^{n-2}\right)+\left(\sum_{n=1}^{\infty} 7 n a_{n} x^{n}\right)+\left(\sum_{n=0}^{\infty} 2 a_{n} x^{n}\right) = 0 \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^{n} and adjusting the power and the corresponding index gives

$$
\sum_{n=2}^{\infty} n \left(n-1\right) a_{n} x^{n-2} = \sum_{n=0}^{\infty} \left(n+2\right) a_{n+2} \left(n+1\right) x^{n}
$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$
\begin{aligned}
&\left(\sum_{n=2}^{\infty} 2 x^{n} a_{n} n \left(n-1\right)\right)+\left(\sum_{n=0}^{\infty} \left(n+2\right) a_{n+2} \left(n+1\right) x^{n}\right) \\
&+\left(\sum_{n=1}^{\infty} 7 n a_{n} x^{n}\right)+\left(\sum_{n=0}^{\infty} 2 a_{n} x^{n}\right) = 0
\end{aligned} \tag{3}
$$

$n = 0$ gives

$$2a_2 + 2a_0 = 0$$

$$a_2 = -a_0$$

$n = 1$ gives

$$6a_3 + 9a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{3a_1}{2}$$

For $2 \leq n$, the recurrence equation is

$$2na_n(n-1) + (n+2)\, a_{n+2}(n+1) + 7na_n + 2a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{(2n+1)\, a_n}{n+1} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$20a_2 + 12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{5a_0}{3}$$

For $n = 3$ the recurrence equation gives

$$35a_3 + 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{21a_1}{8}$$

For $n = 4$ the recurrence equation gives

$$54a_4 + 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -3a_0$$

For $n = 5$ the recurrence equation gives

$$77a_5 + 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{77a_1}{16}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1x - a_0x^2 - \frac{3}{2}a_1x^3 + \frac{5}{3}a_0x^4 + \frac{21}{8}a_1x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - x^2 + \frac{5}{3}x^4\right)a_0 + \left(x - \frac{3}{2}x^3 + \frac{21}{8}x^5\right)a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - x^2 + \frac{5}{3}x^4\right)c_1 + \left(x - \frac{3}{2}x^3 + \frac{21}{8}x^5\right)c_2 + O(x^6)$$

$$y = x - \frac{3x^3}{2} + \frac{21x^5}{8} + O(x^6)$$

Summary
The solution(s) found are the following

$$y = x - \frac{3x^3}{2} + \frac{21x^5}{8} + O(x^6) \tag{1}$$
$$y = x - \frac{3x^3}{2} + \frac{21x^5}{8} + O(x^6) \tag{2}$$

Verification of solutions

$$y = x - \frac{3x^3}{2} + \frac{21x^5}{8} + O(x^6)$$

Verified OK.

$$y = x - \frac{3x^3}{2} + \frac{21x^5}{8} + O(x^6)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Legendre successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
Order:=6;
dsolve([(1+2*x^2)*diff(y(x),x$2)+7*x*diff(y(x),x)+2*y(x)=0,y(0) = 0, D(y)(0) = 1],y(x),type=
```

$$y(x) = x - \frac{3}{2}x^3 + \frac{21}{8}x^5 + \mathrm{O}\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 19

```
AsymptoticDSolveValue[{(1+2*x^2)*y''[x]+7*x*y'[x]+2*y[x]==0,{y[0]==0,y'[0]==1}},y[x],{x,0,5}
```

$$y(x) \to \frac{21x^5}{8} - \frac{3x^3}{2} + x$$

16.18 problem Problem 19

Internal problem ID [2914]
Internal file name [`OUTPUT/2406_Sunday_June_05_2022_03_04_53_AM_66127838/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.2. page 739
Problem number: Problem 19.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[_Lienard]
```

$$\boxed{4y'' + xy' + 4y = 0}$$

With initial conditions

$$[y(0) = 1, y'(0) = 0]$$

With the expansion point for the power series method at $x = 0$.

16.18.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= \frac{x}{4} \\ q(x) &= 1 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + \frac{xy'}{4} + y = 0$$

The domain of $p(x) = \frac{x}{4}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + xy'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\
&= y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (659)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (660)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{align}
F_0 &= f(x,y,y') \tag{4}\\
F_1 &= \frac{df}{dx} \nonumber\\
&= \frac{dF_0}{dx} \nonumber\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \nonumber\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{5}\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0 \nonumber\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right) \nonumber\\
&= \frac{d}{dx}(F_1) \nonumber\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y'' \nonumber\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0 \nonumber\\
&\vdots \nonumber\\
F_n &= \frac{d}{dx}(F_{n-1}) \nonumber\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y'' \nonumber\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \tag{6}
\end{align}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n\big|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{xy'}{4} - y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{y' x^2}{16} + \frac{yx}{4} - \frac{5y'}{4} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{(-x^3 + 44x)\, y'}{64} + \frac{(-4x^2 + 96)\, y}{64} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{(x^4 - 72x^2 + 560)\, y'}{256} + \frac{xy(x^2 - 52)}{64} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{(-x^5 + 104x^3 - 1968x)\, y'}{1024} - \frac{y(x^4 - 84x^2 + 768)}{256}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = 1$ and $y'(0) = 0$ gives

$$
\begin{aligned}
F_0 &= -1 \\
F_1 &= 0 \\
F_2 &= \frac{3}{2} \\
F_3 &= 0 \\
F_4 &= -3
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = 1 - \frac{x^2}{2} + \frac{x^4}{16} - \frac{x^6}{240} + O(x^6)
$$

$$
y = 1 - \frac{x^2}{2} + \frac{x^4}{16} - \frac{x^6}{240} + O(x^6)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = -\frac{x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right)}{4} - \left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}
$$

Which simplifies to

$$
\left(\sum_{n=2}^{\infty} 4n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} n\, x^n a_n\right) + \left(\sum_{n=0}^{\infty} 4 a_n x^n\right) = 0 \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$
\sum_{n=2}^{\infty} 4n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} 4(n+2)\, a_{n+2}(n+1)\, x^n
$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$
\left(\sum_{n=0}^{\infty} 4(n+2)\, a_{n+2}(n+1)\, x^n\right) + \left(\sum_{n=1}^{\infty} n\, x^n a_n\right) + \left(\sum_{n=0}^{\infty} 4 a_n x^n\right) = 0 \tag{3}
$$

$n = 0$ gives

$$8a_2 + 4a_0 = 0$$

$$a_2 = -\frac{a_0}{2}$$

For $1 \leq n$, the recurrence equation is

$$4(n+2)\, a_{n+2}(n+1) + na_n + 4a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_n(n+4)}{4\,(n+2)\,(n+1)} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$24a_3 + 5a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{5a_1}{24}$$

For $n = 2$ the recurrence equation gives

$$48a_4 + 6a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{16}$$

For $n = 3$ the recurrence equation gives

$$80a_5 + 7a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{7a_1}{384}$$

For $n = 4$ the recurrence equation gives

$$120a_6 + 8a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{a_0}{240}$$

For $n = 5$ the recurrence equation gives

$$168a_7 + 9a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{a_1}{1024}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{1}{2} a_0 x^2 - \frac{5}{24} a_1 x^3 + \frac{1}{16} a_0 x^4 + \frac{7}{384} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{16}x^4\right) a_0 + \left(x - \frac{5}{24}x^3 + \frac{7}{384}x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{16}x^4\right) c_1 + \left(x - \frac{5}{24}x^3 + \frac{7}{384}x^5\right) c_2 + O(x^6)$$

$$y = 1 - \frac{x^2}{2} + \frac{x^4}{16} + O(x^6)$$

Summary

The solution(s) found are the following

$$y = 1 - \frac{x^2}{2} + \frac{x^4}{16} - \frac{x^6}{240} + O(x^6) \tag{1}$$

$$y = 1 - \frac{x^2}{2} + \frac{x^4}{16} + O(x^6) \tag{2}$$

Verification of solutions

$$y = 1 - \frac{x^2}{2} + \frac{x^4}{16} - \frac{x^6}{240} + O(x^6)$$

Verified OK.

$$y = 1 - \frac{x^2}{2} + \frac{x^4}{16} + O(x^6)$$

Verified OK.

16.18.2 Maple step by step solution

Let's solve

$\left[y'' = -\frac{xy'}{4} - y, y(0) = 1, y'\Big|_{\{x=0\}} = 0\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{xy'}{4} + y = 0$

- Multiply by denominators

  $4y'' + xy' + 4y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ∘ Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ∘ Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k- >k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$$

Rewrite DE with series expansions

$$\sum_{k=0}^{\infty} \left(4a_{k+2}(k+2)\,(k+1) + a_k(k+4)\right) x^k = 0$$

- Each term in the series must be 0, giving the recursion relation

$$4(k^2+3k+2)\, a_{k+2} + a_k(k+4) = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k(k+4)}{4(k^2+3k+2)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric solution.
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form could result into a too large expression - returning special functi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
Order:=6;
dsolve([4*diff(y(x),x$2)+x*diff(y(x),x)+4*y(x)=0,y(0) = 1, D(y)(0) = 0],y(x),type='series',x
```

$$y(x) = 1 - \frac{1}{2}x^2 + \frac{1}{16}x^4 + \mathrm{O}\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 19

```
AsymptoticDSolveValue[{4*y''[x]+x*y'[x]+4*y[x]==0,{y[0]==1,y'[0]==0}},y[x],{x,0,5}]
```

$$y(x) \to \frac{x^4}{16} - \frac{x^2}{2} + 1$$

16.19 problem Problem 20

Internal problem ID [2915]
Internal file name [`OUTPUT/2407_Sunday_June_05_2022_03_04_56_AM_36856607/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.2. page 739
Problem number: Problem 20.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 2y'x^2 + yx = 2\cos(x)}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y_0'$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + xy_0' + \frac{x^2}{2} f|_{x_0,y_0,y_0'} + \frac{x^3}{3!} f'|_{x_0,y_0,y_0'} + \cdots \\
&= y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y_0'}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{662}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{663}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -2y'x^2 - yx + 2\cos(x) \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \left(4x^4 - 5x\right) y' + 2yx^3 - 4x^2 \cos(x) - y - 2\sin(x) \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \left(-8x^6 + 28x^3 - 6\right) y' + \left(8x^4 - 18x - 2\right) \cos(x) - 4\left(\left(x^3 - \frac{11}{4}\right) y - \sin(x)\right) x^2 \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \left(16x^8 - 108x^5 + 107x^2\right) y' + \left(-16x^6 + 88x^3 + 4x^2 - 30\right) \cos(x) + \left(-8x^4 + 26x + 2\right) \sin(x) + 8\left(x^6\right. \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \left(-32x^{10} + 352x^7 - 802x^4 + 242x\right) y' + \left(32x^8 - 312x^5 - 8x^4 + 478x^2 + 34x + 2\right) \cos(x) + \left(16x^6 -\right.
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 2 \\
F_1 &= -y(0) \\
F_2 &= -2 - 6y'(0) \\
F_3 &= -30 \\
F_4 &= 2 + 28y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 - \frac{1}{6}x^3 + \frac{7}{180}x^6\right) y(0) + \left(x - \frac{1}{4}x^4\right) y'(0) + x^2 - \frac{x^4}{12} - \frac{x^5}{4} + \frac{x^6}{360} + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
\end{aligned}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = -2\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x^2 - \left(\sum_{n=0}^{\infty} a_n x^n\right) x + 2\cos(x) \tag{1}$$

Expanding $2\cos(x)$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned}
2\cos(x) &= 2 - x^2 + \frac{1}{12}x^4 + \dots \\
&= 2 - x^2 + \frac{1}{12}x^4
\end{aligned}$$

Hence the ODE in Eq (1) becomes

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + 2\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x^2 + \left(\sum_{n=0}^{\infty} a_n x^n\right) x = 2 - x^2 + \frac{1}{12}x^4$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} 2n\, x^{1+n} a_n\right) + \left(\sum_{n=0}^{\infty} x^{1+n} a_n\right) = 2 - x^2 + \frac{1}{12}x^4 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n$$

$$\sum_{n=1}^{\infty} 2n\, x^{1+n} a_n = \sum_{n=2}^{\infty} 2(n-1)\, a_{n-1} x^n$$

$$\sum_{n=0}^{\infty} x^{1+n} a_n = \sum_{n=1}^{\infty} a_{n-1} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n\right) + \left(\sum_{n=2}^{\infty} 2(n-1)\, a_{n-1} x^n\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^n\right) = 2 - x^2 + \frac{1}{12} x^4 \tag{3}$$

$n = 0$ gives

$$(2a_2)\, 1 = 2$$
$$2a_2 = 2$$

Or

$$a_2 = 1$$

$n = 1$ gives

$$6a_3 + a_0 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{a_0}{6}$$

For $2 \leq n$, the recurrence equation is

$$((n+2)\, a_{n+2}(1+n) + 2(n-1)\, a_{n-1} + a_{n-1})\, x^n = 2 - x^2 + \frac{1}{12} x^4 \tag{4}$$

For $n = 2$ the recurrence equation gives

$$\begin{aligned}(12a_4 + 3a_1) x^2 &= -x^2 \\ 12a_4 + 3a_1 &= -1\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{1}{12} - \frac{a_1}{4}$$

For $n = 3$ the recurrence equation gives

$$\begin{aligned}(20a_5 + 5a_2) x^3 &= 0 \\ 20a_5 + 5a_2 &= 0\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{1}{4}$$

For $n = 4$ the recurrence equation gives

$$\begin{aligned}(30a_6 + 7a_3) x^4 &= \frac{x^4}{12} \\ 30a_6 + 7a_3 &= \frac{1}{12}\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{1}{360} + \frac{7a_0}{180}$$

For $n = 5$ the recurrence equation gives

$$\begin{aligned}(42a_7 + 9a_4) x^5 &= 0 \\ 42a_7 + 9a_4 &= 0\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{1}{56} + \frac{3a_1}{56}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + x^2 - \frac{a_0 x^3}{6} + \left(-\frac{1}{12} - \frac{a_1}{4}\right) x^4 - \frac{x^5}{4} + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{x^3}{6}\right) a_0 + \left(x - \frac{1}{4} x^4\right) a_1 + x^2 - \frac{x^4}{12} - \frac{x^5}{4} + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{x^3}{6}\right) c_1 + \left(x - \frac{1}{4} x^4\right) c_2 + x^2 - \frac{x^4}{12} - \frac{x^5}{4} + O(x^6)$$

Summary

The solution(s) found are the following

$$y = \left(1 - \frac{1}{6} x^3 + \frac{7}{180} x^6\right) y(0) + \left(x - \frac{1}{4} x^4\right) y'(0) + x^2 - \frac{x^4}{12} - \frac{x^5}{4} + \frac{x^6}{360} + O(x^6) \tag{1}$$

$$y = \left(1 - \frac{x^3}{6}\right) c_1 + \left(x - \frac{1}{4} x^4\right) c_2 + x^2 - \frac{x^4}{12} - \frac{x^5}{4} + O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1 - \frac{1}{6} x^3 + \frac{7}{180} x^6\right) y(0) + \left(x - \frac{1}{4} x^4\right) y'(0) + x^2 - \frac{x^4}{12} - \frac{x^5}{4} + \frac{x^6}{360} + O(x^6)$$

Verified OK.

$$y = \left(1 - \frac{x^3}{6}\right) c_1 + \left(x - \frac{1}{4} x^4\right) c_2 + x^2 - \frac{x^4}{12} - \frac{x^5}{4} + O(x^6)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      <- Kummer successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 30

```
Order:=6;
dsolve(diff(y(x),x$2)+2*x^2*diff(y(x),x)+x*y(x)=2*cos(x),y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{x^3}{6}\right) y(0) + \left(x - \frac{1}{4}x^4\right) D(y)(0) + x^2 - \frac{x^4}{12} - \frac{x^5}{4} + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 45

```
AsymptoticDSolveValue[y''[x]+2*x^2*y'[x]+x*y[x]==2*Cos[x],y[x],{x,0,5}]
```

$$y(x) \to -\frac{x^5}{4} - \frac{x^4}{12} + c_2\left(x - \frac{x^4}{4}\right) + c_1\left(1 - \frac{x^3}{6}\right) + x^2$$

16.20 problem Problem 21

Internal problem ID [2916]
Internal file name [`OUTPUT/2408_Sunday_June_05_2022_03_04_58_AM_50285274/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.2. page 739
Problem number: Problem 21.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + xy' - 4y = 6\,\mathrm{e}^{x}}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + xy'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\
&= y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (665) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (666) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n |_{x_0, y_0, y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -xy' + 4y + 6\,\mathrm{e}^{x} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= y'x^2 - 6x\,\mathrm{e}^{x} - 4yx + 6\,\mathrm{e}^{x} + 3y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \left(-x^3 - 5x\right) y' + 6\left(x^2 - x + 3\right) \mathrm{e}^{x} + 4\left(x^2 + 2\right) y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \left(x^4 + 6x^2 + 3\right) y' + 6\left(-x^3 + x^2 - 4x + 2\right) \mathrm{e}^{x} - 4xy\left(x^2 + 3\right) \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \left(-x^5 - 6x^3 - 3x\right) y' + 6\left(x^4 - x^3 + 4x^2 - 2x + 1\right) \mathrm{e}^{x} + 4x^2 y\left(x^2 + 3\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 6 + 4y(0) \\
F_1 &= 6 + 3y'(0) \\
F_2 &= 18 + 8y(0) \\
F_3 &= 12 + 3y'(0) \\
F_4 &= 6
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 + 2x^2 + \frac{1}{3}x^4\right) y(0) + \left(x + \frac{1}{2}x^3 + \frac{1}{40}x^5\right) y'(0) + 3x^2 + x^3 + \frac{3x^4}{4} + \frac{x^5}{10} + \frac{x^6}{120} + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard

power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = -x \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + 4 \left(\sum_{n=0}^{\infty} a_n x^n\right) + 6\,\mathrm{e}^{x} \tag{1}$$

Expanding $6\,\mathrm{e}^{x}$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned} 6\,\mathrm{e}^{x} &= 6 + 6x + 3x^2 + x^3 + \frac{1}{4}x^4 + \frac{1}{20}x^5 + \dots \\ &= 6 + 6x + 3x^2 + x^3 + \frac{1}{4}x^4 + \frac{1}{20}x^5 \end{aligned}$$

Hence the ODE in Eq (1) becomes

$$\begin{aligned} &\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + x \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - 4 \left(\sum_{n=0}^{\infty} a_n x^n\right) \\ &= 6 + 6x + 3x^2 + x^3 + \frac{1}{4}x^4 + \frac{1}{20}x^5 \end{aligned}$$

Which simplifies to

$$\begin{aligned} &\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} n\, x^n a_n\right) + \sum_{n=0}^{\infty} (-4 a_n x^n) \\ &= 6 + 6x + 3x^2 + x^3 + \frac{1}{4}x^4 + \frac{1}{20}x^5 \end{aligned} \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^{n}\right) + \left(\sum_{n=1}^{\infty} n\, x^{n} a_{n}\right) + \sum_{n=0}^{\infty} \left(-4 a_{n} x^{n}\right) = 6 + 6x + 3x^{2} + x^{3} + \frac{1}{4}x^{4} + \frac{1}{20}x^{5} \tag{3}$$

$n = 0$ gives

$$\left(2a_{2} - 4a_{0}\right) x^{0} = 6$$
$$2a_{2} - 4a_{0} = 6$$

$$a_{2} = 2a_{0} + 3$$

For $1 \leq n$, the recurrence equation is

$$\left((n+2)\, a_{n+2}(n+1) + n a_{n} - 4a_{n}\right) x^{n} = 6 + 6x + 3x^{2} + x^{3} + \frac{1}{4}x^{4} + \frac{1}{20}x^{5} \tag{4}$$

For $n = 1$ the recurrence equation gives

$$\left(6a_{3} - 3a_{1}\right) x = 6x$$
$$6a_{3} - 3a_{1} = 6$$

Which after substituting the earlier terms found becomes

$$a_{3} = 1 + \frac{a_{1}}{2}$$

For $n = 2$ the recurrence equation gives

$$\left(12a_{4} - 2a_{2}\right) x^{2} = 3x^{2}$$
$$12a_{4} - 2a_{2} = 3$$

Which after substituting the earlier terms found becomes

$$a_{4} = \frac{3}{4} + \frac{a_{0}}{3}$$

For $n = 3$ the recurrence equation gives

$$
\begin{aligned}
(20a_5 - a_3)\, x^3 &= x^3 \\
20a_5 - a_3 &= 1
\end{aligned}
$$

Which after substituting the earlier terms found becomes

$$
a_5 = \frac{1}{10} + \frac{a_1}{40}
$$

For $n = 4$ the recurrence equation gives

$$
\begin{aligned}
(30a_6)\, x^4 &= \frac{x^4}{4} \\
30a_6 &= \frac{1}{4}
\end{aligned}
$$

Which after substituting the earlier terms found becomes

$$
a_6 = \frac{1}{120}
$$

For $n = 5$ the recurrence equation gives

$$
\begin{aligned}
(42a_7 + a_5)\, x^5 &= \frac{x^5}{20} \\
42a_7 + a_5 &= \frac{1}{20}
\end{aligned}
$$

Which after substituting the earlier terms found becomes

$$
a_7 = -\frac{1}{840} - \frac{a_1}{1680}
$$

And so on. Therefore the solution is

$$
\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^n \\
&= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots
\end{aligned}
$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + (2a_0 + 3) x^2 + \left(1 + \frac{a_1}{2}\right) x^3 + \left(\frac{3}{4} + \frac{a_0}{3}\right) x^4 + \left(\frac{1}{10} + \frac{a_1}{40}\right) x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + 2x^2 + \frac{1}{3}x^4\right) a_0 + \left(x + \frac{1}{2}x^3 + \frac{1}{40}x^5\right) a_1 + 3x^2 + x^3 + \frac{3x^4}{4} + \frac{x^5}{10} + O\left(x^6\right) \quad (3)$$

At $x = 0$ the solution above becomes

$$y = \left(1 + 2x^2 + \frac{1}{3}x^4\right) c_1 + \left(x + \frac{1}{2}x^3 + \frac{1}{40}x^5\right) c_2 + 3x^2 + x^3 + \frac{3x^4}{4} + \frac{x^5}{10} + O\left(x^6\right)$$

Summary

The solution(s) found are the following

$$y = \left(1+2x^2+\frac{1}{3}x^4\right) y(0) + \left(x+\frac{1}{2}x^3+\frac{1}{40}x^5\right) y'(0) + 3x^2 + x^3 + \frac{3x^4}{4} + \frac{x^5}{10} + \frac{x^6}{120} + O\left(x^6\right) \quad (1)$$

$$y = \left(1 + 2x^2 + \frac{1}{3}x^4\right) c_1 + \left(x + \frac{1}{2}x^3 + \frac{1}{40}x^5\right) c_2 + 3x^2 + x^3 + \frac{3x^4}{4} + \frac{x^5}{10} + O\left(x^6\right) \quad (2)$$

Verification of solutions

$$y = \left(1+2x^2+\frac{1}{3}x^4\right) y(0) + \left(x+\frac{1}{2}x^3+\frac{1}{40}x^5\right) y'(0) + 3x^2 + x^3 + \frac{3x^4}{4} + \frac{x^5}{10} + \frac{x^6}{120} + O\left(x^6\right)$$

Verified OK.

$$y = \left(1 + 2x^2 + \frac{1}{3}x^4\right) c_1 + \left(x + \frac{1}{2}x^3 + \frac{1}{40}x^5\right) c_2 + 3x^2 + x^3 + \frac{3x^4}{4} + \frac{x^5}{10} + O\left(x^6\right)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Group is reducible, not completely reducible
      Solution has integrals. Trying a special function solution free of integrals...
      -> Trying a solution in terms of special functions:
         -> Bessel
         -> elliptic
         -> Legendre
         <- Kummer successful
      <- special function solution successful
         -> Trying to convert hypergeometric functions to elementary form...
         <- elementary form is not straightforward to achieve - returning special function s
      <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 42

```
Order:=6;
dsolve(diff(y(x),x$2)+x*diff(y(x),x)-4*y(x)=6*exp(x),y(x),type='series',x=0);
```

$$y(x) = \left(1+2x^2+\frac{1}{3}x^4\right)y(0)+\left(x+\frac{1}{2}x^3+\frac{1}{40}x^5\right)D(y)(0)+3x^2+x^3+\frac{3x^4}{4}+\frac{x^5}{10}+O(x^6)$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 62

```
AsymptoticDSolveValue[y''[x]+x*y'[x]-4*y[x]==6*Exp[x],y[x],{x,0,5}]
```

$$y(x) \to \frac{x^5}{10} + \frac{3x^4}{4} + x^3 + 3x^2 + c_2\left(\frac{x^5}{40} + \frac{x^3}{2} + x\right) + c_1\left(\frac{x^4}{3} + 2x^2 + 1\right)$$

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17.1 problem 1

Internal problem ID [2917]
Internal file name [`OUTPUT/2409_Sunday_June_05_2022_03_05_02_AM_46963906/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.4. page 758
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + \frac{y'}{1-x} + yx = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + xy'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\ &= y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y'_0} \end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{668}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{669}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{align*}
F_0 &= f(x, y, y') \tag{4}\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \tag{5}\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \tag{6}
\end{align*}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{x^2 y - yx - y'}{x-1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{-y' x^2 - 2yx + xy' + y}{x-1} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{\left(-4x^2+6x-2\right) y' + y(x^4-2x^3+x^2+1)}{\left(x-1\right)^2} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{\left(x^5-3x^4+3x^3-5x^2+9x-5\right) y' + 6\left(x^4-\frac{8}{3}x^3+\frac{7}{3}x^2-\frac{2}{3}x-\frac{1}{3}\right) y}{\left(x-1\right)^3} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{3(3x^5-11x^4+15x^3-9x^2+x+1)\, y' - y(x^7-4x^6+6x^5-14x^4+38x^3-48x^2+25x-10)}{\left(x-1\right)^4}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and $y'(0)=y'(0)$ gives

$$
\begin{aligned}
F_0 &= -y'(0) \\
F_1 &= -y(0) \\
F_2 &= y(0) - 2y'(0) \\
F_3 &= 2y(0) + 5y'(0) \\
F_4 &= 10y(0) + 3y'(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y = &\left(1-\frac{1}{6}x^3+\frac{1}{24}x^4+\frac{1}{60}x^5+\frac{1}{72}x^6\right) y(0) \\
&+\left(x-\frac{1}{2}x^2-\frac{1}{12}x^4+\frac{1}{24}x^5+\frac{1}{240}x^6\right) y'(0)+O(x^6)
\end{aligned}
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$
y''(x-1) - y' + x(x-1)\, y = 0
$$

Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right)(x-1) - \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + x(x-1)\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}
$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=2}^{\infty} n\, x^{n-1} a_n (n-1)\right) + \sum_{n=2}^{\infty} \left(-n(n-1)\, a_n x^{n-2}\right) \\
&+ \sum_{n=1}^{\infty} \left(-n a_n x^{n-1}\right) + \left(\sum_{n=0}^{\infty} x^{n+2} a_n\right) + \sum_{n=0}^{\infty} \left(-x^{1+n} a_n\right) = 0
\end{aligned} \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n\, x^{n-1} a_n (n-1) = \sum_{n=1}^{\infty} (1+n)\, a_{1+n} n\, x^n$$

$$\sum_{n=2}^{\infty} \left(-n(n-1)\, a_n x^{n-2}\right) = \sum_{n=0}^{\infty} \left(-(n+2)\, a_{n+2}(1+n)\, x^n\right)$$

$$\sum_{n=1}^{\infty} \left(-n a_n x^{n-1}\right) = \sum_{n=0}^{\infty} \left(-(1+n)\, a_{1+n} x^n\right)$$

$$\sum_{n=0}^{\infty} x^{n+2} a_n = \sum_{n=2}^{\infty} a_{n-2} x^n$$

$$\sum_{n=0}^{\infty} \left(-x^{1+n} a_n\right) = \sum_{n=1}^{\infty} \left(-a_{n-1} x^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=1}^{\infty} (1+n)\, a_{1+n} n\, x^n\right) + \sum_{n=0}^{\infty} \left(-(n+2)\, a_{n+2}(1+n)\, x^n\right) + \sum_{n=0}^{\infty} \left(-(1+n)\, a_{1+n} x^n\right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^n\right) + \sum_{n=1}^{\infty} \left(-a_{n-1} x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$-2a_2 - a_1 = 0$$

$$a_2 = -\frac{a_1}{2}$$

$n = 1$ gives

$$-6a_3 - a_0 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{a_0}{6}$$

For $2 \leq n$, the recurrence equation is

$$(1+n)\, a_{1+n} n - (n+2)\, a_{n+2}(1+n) - (1+n)\, a_{1+n} + a_{n-2} - a_{n-1} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= \frac{n^2 a_{1+n} - a_{1+n} + a_{n-2} - a_{n-1}}{(n+2)\,(1+n)} \\ &= \frac{(n^2-1)\, a_{1+n}}{(n+2)\,(1+n)} + \frac{a_{n-2}}{(n+2)\,(1+n)} - \frac{a_{n-1}}{(n+2)\,(1+n)} \end{aligned} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$3a_3 - 12a_4 + a_0 - a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{24} - \frac{a_1}{12}$$

For $n = 3$ the recurrence equation gives

$$8a_4 - 20a_5 + a_1 - a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_0}{60} + \frac{a_1}{24}$$

For $n = 4$ the recurrence equation gives

$$15a_5 - 30a_6 + a_2 - a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0}{72} + \frac{a_1}{240}$$

For $n = 5$ the recurrence equation gives

$$24a_6 - 42a_7 + a_3 - a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{a_0}{336} + \frac{11a_1}{2520}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{a_1 x^2}{2} - \frac{a_0 x^3}{6} + \left(\frac{a_0}{24} - \frac{a_1}{12}\right) x^4 + \left(\frac{a_0}{60} + \frac{a_1}{24}\right) x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{60}x^5\right) a_0 + \left(x - \frac{1}{2}x^2 - \frac{1}{12}x^4 + \frac{1}{24}x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{60}x^5\right) c_1 + \left(x - \frac{1}{2}x^2 - \frac{1}{12}x^4 + \frac{1}{24}x^5\right) c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = {} & \left(1 - \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{60}x^5 + \frac{1}{72}x^6\right) y(0) \\ & + \left(x - \frac{1}{2}x^2 - \frac{1}{12}x^4 + \frac{1}{24}x^5 + \frac{1}{240}x^6\right) y'(0) + O(x^6) \end{aligned} \tag{1}$$

$$y = \left(1 - \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{60}x^5\right) c_1 + \left(x - \frac{1}{2}x^2 - \frac{1}{12}x^4 + \frac{1}{24}x^5\right) c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$
\begin{aligned}
y = {} & \left(1 - \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{60}x^5 + \frac{1}{72}x^6\right) y(0) \\
& + \left(x - \frac{1}{2}x^2 - \frac{1}{12}x^4 + \frac{1}{24}x^5 + \frac{1}{240}x^6\right) y'(0) + O(x^6)
\end{aligned}
$$

Verified OK.

$$
y = \left(1 - \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{60}x^5\right) c_1 + \left(x - \frac{1}{2}x^2 - \frac{1}{12}x^4 + \frac{1}{24}x^5\right) c_2 + O(x^6)
$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying 2nd order exact linear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      trying 2nd order exact linear
      trying symmetries linear in x and y(x)
      trying to convert to a linear ODE with constant coefficients
      trying to convert to an ODE of Bessel type
   trying to convert to an ODE of Bessel type
   -> trying reduction of order to Riccati
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 44

```
Order:=6;
dsolve(diff(y(x),x$2)+1/(1-x)*diff(y(x),x)+x*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{60}x^5\right) y(0) + \left(x - \frac{1}{2}x^2 - \frac{1}{12}x^4 + \frac{1}{24}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 56

```
AsymptoticDSolveValue[y''[x]+1/(1-x)*y'[x]+x*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{x^5}{60} + \frac{x^4}{24} - \frac{x^3}{6} + 1\right) + c_2\left(\frac{x^5}{24} - \frac{x^4}{12} - \frac{x^2}{2} + x\right)$$

17.2 problem 3

Internal problem ID [2918]
Internal file name [`OUTPUT/2410_Sunday_June_05_2022_03_05_05_AM_52069117/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.4. page 758
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Complex roots"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + \frac{xy'}{(-x^2+1)^2} + y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + \frac{xy'}{(x^2-1)^2} + y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$\begin{aligned}p(x) &= \frac{1}{x(x^2-1)^2}\\ q(x) &= \frac{1}{x^2}\end{aligned}$$

Table 419: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x(x^2-1)^2}$	
singularity	type
$x = -1$	"irregular"
$x = 0$	"regular"
$x = 1$	"irregular"

$q(x) = \frac{1}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty]$

Irregular singular points : $[-1, 1]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$y''x^2(x^2-1)^2 + xy' + y(x^2-1)^2 = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) x^2(x^2-1)^2 + x\left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right)(x^2-1)^2 = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r+4} a_n(n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-2x^{n+r+2} a_n(n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)\right) + \left(\sum_{n=0}^{\infty} x^{n+r+4} a_n\right) + \sum_{n=0}^{\infty} \left(-2x^{n+r+2} a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r+4} a_n(n+r)(n+r-1) = \sum_{n=4}^{\infty} a_{n-4}(n-4+r)(n-5+r)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} \left(-2x^{n+r+2} a_n(n+r)(n+r-1)\right) = \sum_{n=2}^{\infty} \left(-2a_{n-2}(n+r-2)(n-3+r)\, x^{n+r}\right)$$

$$\sum_{n=0}^{\infty} x^{n+r+4} a_n = \sum_{n=4}^{\infty} a_{n-4} x^{n+r}$$

$$\sum_{n=0}^{\infty} \left(-2x^{n+r+2} a_n\right) = \sum_{n=2}^{\infty} \left(-2a_{n-2} x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=4}^{\infty} a_{n-4}(n-4+r)(n-5+r)\, x^{n+r}\right) + \sum_{n=2}^{\infty} \left(-2a_{n-2}(n+r-2)(n-3+r)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)\right) + \left(\sum_{n=4}^{\infty} a_{n-4} x^{n+r}\right) + \sum_{n=2}^{\infty} \left(-2a_{n-2} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$x^{n+r}a_n(n+r)(n+r-1) + x^{n+r}a_n(n+r) + a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$x^r a_0 r(-1+r) + x^r a_0 r + a_0 x^r = 0$$

Or

$$(x^r r(-1+r) + x^r r + x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2+1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 + 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= i \\ r_2 &= -i \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(r^2+1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the roots are complex conjugates, then two linearly independent solutions can be constructed using

$$\begin{aligned} y_1(x) &= x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+i} \\ y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n-i} \end{aligned}$$

$y_1(x)$ is found first. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

Substituting $n = 2$ in Eq. (2B) gives

$$a_2 = \frac{2r^2 - 2r + 2}{r^2 + 4r + 5}$$

Substituting $n = 3$ in Eq. (2B) gives

$$a_3 = 0$$

For $4 \leq n$ the recursive equation is

$$\begin{aligned} &a_{n-4}(n - 4 + r)(n - 5 + r) - 2a_{n-2}(n + r - 2)(n - 3 + r) \\ &+ a_n(n + r)(n + r - 1) + a_n(n + r) + a_{n-4} - 2a_{n-2} + a_n = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{n^2 a_{n-4} - 2n^2 a_{n-2} + 2nra_{n-4} - 4nra_{n-2} + r^2 a_{n-4} - 2r^2 a_{n-2} - 9na_{n-4} + 10na_{n-2} - 9ra_{n-4} + 10r}{n^2 + 2nr + r^2 + 1} \tag{4}$$

Which for the root $r = i$ becomes

$$a_n = \frac{(-a_{n-4} + 2a_{n-2})\, n^2 + ((9 - 2i)\, a_{n-4} + (-10 + 4i)\, a_{n-2})\, n + (-20 + 9i)\, a_{n-4} + (12 - 10i)\, a_{n-2}}{n\,(2i + n)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = i$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{2r^2-2r+2}{r^2+4r+5}$	$-\frac{1}{4} - \frac{i}{4}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{3r^4 + 5r^3 + 2r^2 + r + 7}{(r^2 + 4r + 5)(r^2 + 8r + 17)}$$

Which for the root $r = i$ becomes

$$a_4 = -\frac{1}{80} - \frac{7i}{80}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{2r^2-2r+2}{r^2+4r+5}$	$-\frac{1}{4} - \frac{i}{4}$
a_3	0	0
a_4	$\frac{3r^4+5r^3+2r^2+r+7}{(r^2+4r+5)(r^2+8r+17)}$	$-\frac{1}{80} - \frac{7i}{80}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{2r^2-2r+2}{r^2+4r+5}$	$-\frac{1}{4} - \frac{i}{4}$
a_3	0	0
a_4	$\frac{3r^4+5r^3+2r^2+r+7}{(r^2+4r+5)(r^2+8r+17)}$	$-\frac{1}{80} - \frac{7i}{80}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}
y_1(x) &= x^i \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= x^i \left(1 + \left(-\frac{1}{4} - \frac{i}{4}\right) x^2 + \left(-\frac{1}{80} - \frac{7i}{80}\right) x^4 + O(x^6)\right)
\end{aligned}$$

The second solution $y_2(x)$ is found by taking the complex conjugate of $y_1(x)$ which gives

$$y_2(x) = x^{-i} \left(1 + \left(-\frac{1}{4} + \frac{i}{4}\right) x^2 + \left(-\frac{1}{80} + \frac{7i}{80}\right) x^4 + O(x^6)\right)$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^{i} \left(1+\left(-\frac{1}{4}-\frac{i}{4}\right) x^{2}+\left(-\frac{1}{80}-\frac{7 i}{80}\right) x^{4}+O\left(x^{6}\right)\right) \\
&+ c_2 x^{-i} \left(1+\left(-\frac{1}{4}+\frac{i}{4}\right) x^{2}+\left(-\frac{1}{80}+\frac{7 i}{80}\right) x^{4}+O\left(x^{6}\right)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^{i} \left(1+\left(-\frac{1}{4}-\frac{i}{4}\right) x^{2}+\left(-\frac{1}{80}-\frac{7 i}{80}\right) x^{4}+O\left(x^{6}\right)\right) \\
&+ c_2 x^{-i} \left(1+\left(-\frac{1}{4}+\frac{i}{4}\right) x^{2}+\left(-\frac{1}{80}+\frac{7 i}{80}\right) x^{4}+O\left(x^{6}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = c_1 x^{i} &\left(1+\left(-\frac{1}{4}-\frac{i}{4}\right) x^{2}+\left(-\frac{1}{80}-\frac{7 i}{80}\right) x^{4}+O\left(x^{6}\right)\right) \\
&+ c_2 x^{-i} \left(1+\left(-\frac{1}{4}+\frac{i}{4}\right) x^{2}+\left(-\frac{1}{80}+\frac{7 i}{80}\right) x^{4}+O\left(x^{6}\right)\right)
\end{aligned}
\tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = c_1 x^{i} &\left(1+\left(-\frac{1}{4}-\frac{i}{4}\right) x^{2}+\left(-\frac{1}{80}-\frac{7 i}{80}\right) x^{4}+O\left(x^{6}\right)\right) \\
&+ c_2 x^{-i} \left(1+\left(-\frac{1}{4}+\frac{i}{4}\right) x^{2}+\left(-\frac{1}{80}+\frac{7 i}{80}\right) x^{4}+O\left(x^{6}\right)\right)
\end{aligned}
$$

Verified OK.

17.2.1 Maple step by step solution

Let's solve

$y''x^2(x^2-1)^2 + xy' + y(x^2-1)^2 = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y'}{x(x^2-1)^2} - \frac{y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x(x^2-1)^2} + \frac{y}{x^2} = 0$

☐ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{1}{x(x^2-1)^2}, P_3(x) = \frac{1}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = 1$

  - $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $y''x^2(x^2-1)^2 + xy' + y(x^2-1)^2 = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..4$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
- Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$
- Convert $x^m \cdot y''$ to series expansion for $m = 2..6$

  $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$
- Shift index using $k->k+2-m$

  $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m} (k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(r^2+1)\, x^r + a_1(r^2+2r+2)\, x^{1+r} + \left((r^2+4r+5)\, a_2 - 2a_0(r^2-r+1)\right) x^{2+r} + \left(a_3(r^2+6r+\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r^2+1=0$
- Values of r that satisfy the indicial equation

  $r \in \{-\mathrm{I}, \mathrm{I}\}$
- The coefficients of each power of x must be 0

  $[a_1(r^2+2r+2)=0, (r^2+4r+5)\, a_2 - 2a_0(r^2-r+1)=0, a_3(r^2+6r+10) - 2a_1(r^2+r+1)$
- Solve for the dependent coefficient(s)

  $\left\{a_1 = 0, a_2 = \frac{2a_0(r^2-r+1)}{r^2+4r+5}, a_3 = 0\right\}$
- Each term in the series must be 0, giving the recursion relation

  $(a_k + a_{k-4} - 2a_{k-2})\, k^2 + \left((2a_k + 2a_{k-4} - 4a_{k-2})\, r - 9a_{k-4} + 10a_{k-2}\right) k + (a_k + a_{k-4} - 2a_{k-2})\, r^2$
- Shift index using $k->k+4$

  $(a_{k+4} + a_k - 2a_{k+2})\,(k+4)^2 + \left((2a_{k+4} + 2a_k - 4a_{k+2})\, r - 9a_k + 10a_{k+2}\right)(k+4) + (a_{k+4} + a_k -$
- Recursion relation that defines series solution to ODE

  $a_{k+4} = -\frac{k^2 a_k - 2k^2 a_{k+2} + 2kr a_k - 4kr a_{k+2} + r^2 a_k - 2r^2 a_{k+2} - k a_k - 6k a_{k+2} - r a_k - 6r a_{k+2} + a_k - 6a_{k+2}}{k^2+2kr+r^2+8k+8r+17}$
- Recursion relation for $r = -\mathrm{I}$

  $a_{k+4} = -\frac{k^2 a_k - 2k^2 a_{k+2} - 2\,\mathrm{I}k a_k + 4\,\mathrm{I}k a_{k+2} - 4a_{k+2} - k a_k - 6k a_{k+2} + \mathrm{I}a_k + 6\,\mathrm{I}a_{k+2}}{k^2 - 2\,\mathrm{I}k + 16 - 8\,\mathrm{I} + 8k}$

- Solution for $r = -\mathrm{I}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\mathrm{I}}, a_{k+4} = -\frac{k^2 a_k - 2k^2 a_{k+2} - 2\,\mathrm{I} k a_k + 4\,\mathrm{I} k a_{k+2} - 4 a_{k+2} - k a_k - 6 k a_{k+2} + \mathrm{I} a_k + 6\,\mathrm{I} a_{k+2}}{k^2 - 2\,\mathrm{I} k + 16 - 8\,\mathrm{I} + 8k}, a_1 = 0, a_2 = \left(-\frac{1}{4}\right.\right.$$

- Recursion relation for $r = \mathrm{I}$

$$a_{k+4} = -\frac{k^2 a_k - 2k^2 a_{k+2} + 2\,\mathrm{I} k a_k - 4\,\mathrm{I} k a_{k+2} - 4 a_{k+2} - k a_k - 6 k a_{k+2} - \mathrm{I} a_k - 6\,\mathrm{I} a_{k+2}}{k^2 + 2\,\mathrm{I} k + 16 + 8\,\mathrm{I} + 8k}$$

- Solution for $r = \mathrm{I}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\mathrm{I}}, a_{k+4} = -\frac{k^2 a_k - 2k^2 a_{k+2} + 2\,\mathrm{I} k a_k - 4\,\mathrm{I} k a_{k+2} - 4 a_{k+2} - k a_k - 6 k a_{k+2} - \mathrm{I} a_k - 6\,\mathrm{I} a_{k+2}}{k^2 + 2\,\mathrm{I} k + 16 + 8\,\mathrm{I} + 8k}, a_1 = 0, a_2 = \left(-\frac{1}{4}\right.\right.$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\mathrm{I}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\mathrm{I}}\right), a_{k+4} = -\frac{k^2 a_k - 2k^2 a_{2+k} - 2\,\mathrm{I} k a_k + 4\,\mathrm{I} k a_{2+k} - 4 a_{2+k} - k a_k - 6 k a_{2+k} + \mathrm{I} a_k + 6\,\mathrm{I} a_2}{k^2 - 2\,\mathrm{I} k + 16 - 8\,\mathrm{I} + 8k}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
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trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
<- Heun successful: received ODE is equivalent to the  HeunC  ODE, case  a = 0, e <> 0, c <>
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 45

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+x/(1-x^2)^2*diff(y(x),x)+y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^{-i}\left(1+\left(-\frac{1}{4}+\frac{i}{4}\right)x^2+\left(-\frac{1}{80}+\frac{7i}{80}\right)x^4+\mathrm{O}\left(x^6\right)\right) + c_2 x^{i}\left(1+\left(-\frac{1}{4}-\frac{i}{4}\right)x^2+\left(-\frac{1}{80}-\frac{7i}{80}\right)x^4+\mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 70

```
AsymptoticDSolveValue[x^2*y''[x]+x/(1-x^2)^2*y'[x]+y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to \left(\frac{1}{80}+\frac{3i}{80}\right)c_2 x^{-i}\left((2+i)x^4+(4+8i)x^2+(8-24i)\right) - \left(\frac{3}{80}+\frac{i}{80}\right)c_1 x^{i}\left((1+2i)x^4+(8+4i)x^2-(24-8i)\right)$$

17.3 problem 4

Internal problem ID [2919]
Internal file name [`OUTPUT/2411_Sunday_June_05_2022_03_06_51_AM_76483647/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.4. page 758
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(x-2)^2 y''+(x-2)\,\mathrm{e}^{x} y'+\frac{4y}{x}=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(x^2-4x+4\right) y''+\left(x\,\mathrm{e}^{x}-2\,\mathrm{e}^{x}\right) y'+\frac{4y}{x}=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$\begin{aligned}p(x)&=\frac{\mathrm{e}^{x}}{x-2}\\ q(x)&=\frac{4}{(x-2)^2\,x}\end{aligned}$$

Table 421: Table $p(x), q(x)$ singularites.

$p(x) = \frac{e^x}{x-2}$	
singularity	type
$x = 2$	"regular"
$x = \infty$	"regular"

$q(x) = \frac{4}{(x-2)^2 x}$	
singularity	type
$x = 0$	"regular"
$x = 2$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[2, \infty, 0]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x(x^2 - 4x + 4)\, y'' + e^x x(x-2)\, y' + 4y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x(x^2 - 4x + 4) \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + e^x x(x-2) \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + 4\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Expanding $(x-2)\,\mathrm{e}^{x} x$ as Taylor series around $x=0$ and keeping only the first 6 terms gives

$$\begin{aligned}
(x-2)\,\mathrm{e}^{x} x &= -2x - x^{2} + \frac{1}{6}x^{4} + \frac{1}{12}x^{5} + \frac{1}{40}x^{6} + \dots \\
&= -2x - x^{2} + \frac{1}{6}x^{4} + \frac{1}{12}x^{5} + \frac{1}{40}x^{6}
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
&\left(\sum_{n=0}^{\infty} x^{1+n+r} a_{n}(n+r)\,(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-4x^{n+r} a_{n}(n+r)\,(n+r-1)\right) \\
&\quad + \left(\sum_{n=0}^{\infty} 4x^{n+r-1} a_{n}(n+r)\,(n+r-1)\right) \\
&\quad + \left(\sum_{n=0}^{\infty} \frac{x^{n+r+5} a_{n}(n+r)}{40}\right) + \left(\sum_{n=0}^{\infty} \frac{x^{n+r+4} a_{n}(n+r)}{12}\right) \\
&\quad + \left(\sum_{n=0}^{\infty} \frac{x^{n+r+3} a_{n}(n+r)}{6}\right) + \sum_{n=0}^{\infty} \left(-x^{1+n+r} a_{n}(n+r)\right) \\
&\quad + \sum_{n=0}^{\infty} \left(-2x^{n+r} a_{n}(n+r)\right) + \left(\sum_{n=0}^{\infty} 4a_{n} x^{n+r}\right) = 0
\end{aligned} \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{1+n+r} a_{n}(n+r)\,(n+r-1) = \sum_{n=2}^{\infty} a_{n-2}(n+r-2)\,(n-3+r)\,x^{n+r-1}$$

$$\sum_{n=0}^{\infty} \left(-4x^{n+r} a_{n}(n+r)\,(n+r-1)\right) = \sum_{n=1}^{\infty} \left(-4a_{n-1}(n+r-1)\,(n+r-2)\,x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty} \frac{x^{n+r+5} a_{n}(n+r)}{40} = \sum_{n=6}^{\infty} \frac{a_{n-6}(n+r-6)\,x^{n+r-1}}{40}$$

$$\sum_{n=0}^{\infty} \frac{x^{n+r+4} a_{n}(n+r)}{12} = \sum_{n=5}^{\infty} \frac{a_{n-5}(n+r-5)\,x^{n+r-1}}{12}$$

$$\sum_{n=0}^{\infty} \frac{x^{n+r+3} a_{n}(n+r)}{6} = \sum_{n=4}^{\infty} \frac{a_{n-4}(n+r-4)\,x^{n+r-1}}{6}$$

$$\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n (n+r)\right) = \sum_{n=2}^{\infty} \left(-a_{n-2}(n+r-2)\, x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty} \left(-2x^{n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-2a_{n-1}(n+r-1)\, x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty} 4a_n x^{n+r} = \sum_{n=1}^{\infty} 4a_{n-1} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n + r - 1$.

$$\begin{aligned}
&\left(\sum_{n=2}^{\infty} a_{n-2}(n+r-2)\,(n-3+r)\, x^{n+r-1}\right) \\
&+ \sum_{n=1}^{\infty} \left(-4a_{n-1}(n+r-1)\,(n+r-2)\, x^{n+r-1}\right) \\
&\quad + \left(\sum_{n=0}^{\infty} 4x^{n+r-1} a_n (n+r)\,(n+r-1)\right) \\
&\quad + \left(\sum_{n=6}^{\infty} \frac{a_{n-6}(n+r-6)\, x^{n+r-1}}{40}\right) + \left(\sum_{n=5}^{\infty} \frac{a_{n-5}(n+r-5)\, x^{n+r-1}}{12}\right) \\
&\quad + \left(\sum_{n=4}^{\infty} \frac{a_{n-4}(n+r-4)\, x^{n+r-1}}{6}\right) + \sum_{n=2}^{\infty} \left(-a_{n-2}(n+r-2)\, x^{n+r-1}\right) \\
&\quad + \sum_{n=1}^{\infty} \left(-2a_{n-1}(n+r-1)\, x^{n+r-1}\right) + \left(\sum_{n=1}^{\infty} 4a_{n-1} x^{n+r-1}\right) = 0
\end{aligned} \tag{2B}$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$4x^{n+r-1} a_n (n+r)\,(n+r-1) = 0$$

When $n = 0$ the above becomes

$$4x^{-1+r} a_0 r(-1+r) = 0$$

Or

$$4x^{-1+r} a_0 r(-1+r) = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$4x^{-1+r} r(-1+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$4r(-1+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= 1 \\ r_2 &= 0 \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$4x^{-1+r}r(-1+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= C y_1(x) \ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= x\left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{1+n} \\ y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = \frac{2r^2 - r - 2}{2r(1+r)}$$

Substituting $n = 2$ in Eq. (2B) gives

$$a_2 = \frac{3r^4 + 5r^3 - 7r^2 - 5r + 2}{4r\left(1+r\right)^2\left(2+r\right)}$$

Substituting $n = 3$ in Eq. (2B) gives

$$a_3 = \frac{2r^6 + 13r^5 + 18r^4 - 13r^3 - 28r^2 - 7r + 2}{4r\left(1+r\right)^2\left(2+r\right)^2\left(3+r\right)}$$

Substituting $n = 4$ in Eq. (2B) gives

$$a_4 = \frac{15r^8 + 208r^7 + 984r^6 + 1798r^5 + 363r^4 - 2576r^3 - 2562r^2 - 486r + 156}{48r\left(1+r\right)^2\left(2+r\right)^2\left(3+r\right)^2\left(4+r\right)}$$

Substituting $n = 5$ in Eq. (2B) gives

$$a_5 = \frac{18r^{10} + 425r^9 + 3902r^8 + 17872r^7 + 41868r^6 + 38587r^5 - 27540r^4 - 88804r^3 - 61280r^2 - 7068r +}{96r\left(1+r\right)^2\left(2+r\right)^2\left(3+r\right)^2\left(4+r\right)^2\left(5+r\right)}$$

For $6 \leq n$ the recursive equation is

$$\begin{aligned}
&a_{n-2}(n+r-2)\left(n-3+r\right) - 4a_{n-1}(n+r-1)\left(n+r-2\right) \\
&+ 4a_n(n+r)\left(n+r-1\right) + \frac{a_{n-6}(n+r-6)}{40} + \frac{a_{n-5}(n+r-5)}{12} \\
&+ \frac{a_{n-4}(n+r-4)}{6} - a_{n-2}(n+r-2) - 2a_{n-1}(n+r-1) + 4a_{n-1} = 0
\end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{120n^2a_{n-2} - 480n^2a_{n-1} + 240nra_{n-2} - 960nra_{n-1} + 120r^2a_{n-2} - 480r^2a_{n-1} + 3na_{n-6} + 10na_{n-5}}{} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = \frac{\left(-120a_{n-2} + 480a_{n-1}\right)n^2 + \left(-3a_{n-6} - 10a_{n-5} - 20a_{n-4} + 480a_{n-2} - 240a_{n-1}\right)n + 15a_{n-6} + 40a_{n-}}{480n\left(1+n\right)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r^2-r-2}{2r(1+r)}$	$-\frac{1}{4}$
a_2	$\frac{3r^4+5r^3-7r^2-5r+2}{4r(1+r)^2(2+r)}$	$-\frac{1}{24}$
a_3	$\frac{2r^6+13r^5+18r^4-13r^3-28r^2-7r+2}{4r(1+r)^2(2+r)^2(3+r)}$	$-\frac{13}{576}$
a_4	$\frac{15r^8+208r^7+984r^6+1798r^5+363r^4-2576r^3-2562r^2-486r+156}{48r(1+r)^2(2+r)^2(3+r)^2(4+r)}$	$-\frac{35}{2304}$
a_5	$\frac{18r^{10}+425r^9+3902r^8+17872r^7+41868r^6+38587r^5-27540r^4-88804r^3-61280r^2-7068r+4200}{96r(1+r)^2(2+r)^2(3+r)^2(4+r)^2(5+r)}$	$-\frac{1297}{138240}$

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= x\left(1 - \frac{x}{4} - \frac{x^2}{24} - \frac{13x^3}{576} - \frac{35x^4}{2304} - \frac{1297x^5}{138240} + O\left(x^6\right)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Let

$$
r_1 - r_2 = N
$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$
\begin{aligned}
a_N &= a_1 \\
&= \frac{2r^2 - r - 2}{2r(1+r)}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\lim_{r\to r_2} \frac{2r^2 - r - 2}{2r(1+r)} &= \lim_{r\to 0} \frac{2r^2 - r - 2}{2r(1+r)} \\
&= \textit{undefined}
\end{aligned}
$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$
y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)
$$

Therefore

$$
\begin{aligned}
\frac{d}{dx}y_2(x) &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\
&= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) \\
\frac{d^2}{dx^2}y_2(x) &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\
&= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right)
\end{aligned}
$$

Substituting these back into the given ode $x(x^2-4x+4)\,y'' + \mathrm{e}^x x(x-2)\,y' + 4y = 0$ gives

$$
\begin{aligned}
&x\left(x^2-4x+4\right)\left(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2}\right. \\
&\left. + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) \\
&+ \mathrm{e}^x x(x-2)\left(Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right) \\
&+ 4Cy_1(x)\ln(x) + 4\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned}
$$

Which can be written as

$$
\begin{aligned}
&\left(\left(x\left(x^2-4x+4\right)y_1''(x) + \mathrm{e}^x(x-2)\,xy_1'(x) + 4y_1(x)\right)\ln(x)\right. \\
&\left. + x\left(x^2-4x+4\right)\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right) + \mathrm{e}^x(x-2)\,y_1(x)\right)C \\
&+ x\left(x^2-4x+4\right)\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) \\
&+ \mathrm{e}^x(x-2)\,x\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right) + 4\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned} \tag{7}
$$

But since $y_1(x)$ is a solution to the ode, then

$$x\left(x^2-4x+4\right)y_1''(x)+\mathrm{e}^x(x-2)\,xy_1'(x)+4y_1(x)=0$$

Eq (7) simplifes to

$$\begin{aligned}&\left(x\left(x^2-4x+4\right)\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)+\mathrm{e}^x(x-2)\,y_1(x)\right)C\\&+x\left(x^2-4x+4\right)\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)\\&+\mathrm{e}^x(x-2)\,x\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)+4\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0\end{aligned}\tag{8}$$

Substituting $y_1=\sum_{n=0}^{\infty}a_n x^{n+r_1}$ into the above gives

$$\begin{aligned}&\frac{\left(2x(x-2)^2\left(\sum_{n=0}^{\infty}x^{-1+n+r_1}a_n(n+r_1)\right)+(x-2)\left(x\,\mathrm{e}^x-x+2\right)\left(\sum_{n=0}^{\infty}a_n x^{n+r_1}\right)\right)C}{x}\\&+\frac{x^2(x-2)^2\left(\sum_{n=0}^{\infty}x^{-2+n+r_2}b_n(n+r_2)(-1+n+r_2)\right)+\mathrm{e}^x x^2(x-2)\left(\sum_{n=0}^{\infty}x^{-1+n+r_2}b_n(n+r_2)\right)+4\left(\sum_{n=0}^{\infty}b_n\right.}{x}\\&=0\end{aligned}\tag{9}$$

Since $r_1=1$ and $r_2=0$ then the above becomes

$$\begin{aligned}&\frac{\left(2x(x-2)^2\left(\sum_{n=0}^{\infty}x^n a_n(1+n)\right)+(x-2)\left(x\,\mathrm{e}^x-x+2\right)\left(\sum_{n=0}^{\infty}a_n x^{1+n}\right)\right)C}{x}\\&+\frac{x^2(x-2)^2\left(\sum_{n=0}^{\infty}x^{n-2}b_n n(n-1)\right)+\mathrm{e}^x x^2(x-2)\left(\sum_{n=0}^{\infty}x^{n-1}b_n n\right)+4\left(\sum_{n=0}^{\infty}b_n x^n\right)x}{x}=0\end{aligned}\tag{10}$$

Expanding $C\,\mathrm{e}^x x^2$ as Taylor series around $x=0$ and keeping only the first 6 terms gives

$$\begin{aligned}C\,\mathrm{e}^x x^2&=C\,x^2+C\,x^3+\frac{1}{2}C\,x^4+\frac{1}{6}C\,x^5+\frac{1}{24}C\,x^6+\dots\\&=C\,x^2+C\,x^3+\frac{1}{2}C\,x^4+\frac{1}{6}C\,x^5+\frac{1}{24}C\,x^6\end{aligned}$$

Expanding $-2C\,e^{x}x$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned}-2C\,e^{x}x &= -2Cx - 2C\,x^{2} - C\,x^{3} - \frac{1}{3}C\,x^{4} - \frac{1}{12}C\,x^{5} - \frac{1}{60}C\,x^{6} + \dots \\ &= -2Cx - 2C\,x^{2} - C\,x^{3} - \frac{1}{3}C\,x^{4} - \frac{1}{12}C\,x^{5} - \frac{1}{60}C\,x^{6}\end{aligned}$$

Expanding $x\,e^{x}$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned}x\,e^{x} &= x + x^{2} + \frac{1}{2}x^{3} + \frac{1}{6}x^{4} + \frac{1}{24}x^{5} + \frac{1}{120}x^{6} + \dots \\ &= x + x^{2} + \frac{1}{2}x^{3} + \frac{1}{6}x^{4} + \frac{1}{24}x^{5} + \frac{1}{120}x^{6}\end{aligned}$$

Expanding $-2\,e^{x}$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned}-2\,e^{x} &= -2 - 2x - x^{2} - \frac{1}{3}x^{3} - \frac{1}{12}x^{4} - \frac{1}{60}x^{5} - \frac{1}{360}x^{6} + \dots \\ &= -2 - 2x - x^{2} - \frac{1}{3}x^{3} - \frac{1}{12}x^{4} - \frac{1}{60}x^{5} - \frac{1}{360}x^{6}\end{aligned}$$

Which simplifies to

$$\begin{aligned}
&\left(\sum_{n=0}^{\infty} C\,x^{n+3} a_n\right)+\left(\sum_{n=0}^{\infty} n\,x^{1+n} b_n\right)+\left(\sum_{n=0}^{\infty} \frac{C\,x^{n+4} a_n}{2}\right)\\
&+\left(\sum_{n=0}^{\infty} n\,x^{n+2} b_n\right)+\sum_{n=0}^{\infty} \left(-2C\,x^{1+n} a_n\right)+\sum_{n=0}^{\infty} \left(-4a_n x^n C\right)\\
&\quad+\sum_{n=0}^{\infty} \left(-\frac{n\,x^{n+6} b_n}{360}\right)+\left(\sum_{n=0}^{\infty} 2C\,x^{n+2} a_n(1+n)\right)\\
&\quad+\left(\sum_{n=0}^{\infty} \frac{n\,x^{n+5} b_n}{24}\right)+\left(\sum_{n=0}^{\infty} \frac{n\,x^{n+6} b_n}{120}\right)+\left(\sum_{n=0}^{\infty} n\,x^{1+n} b_n(n-1)\right)\\
&\quad+\left(\sum_{n=0}^{\infty} \frac{C\,x^{n+5} a_n}{6}\right)+\sum_{n=0}^{\infty} \left(-2C\,x^{n+2} a_n\right)+\sum_{n=0}^{\infty} \left(-\frac{C\,x^{n+5} a_n}{12}\right)\\
&\quad+\sum_{n=0}^{\infty} \left(-\frac{C\,x^{n+6} a_n}{60}\right)+\left(\sum_{n=0}^{\infty} \frac{n\,x^{n+3} b_n}{2}\right)+\sum_{n=0}^{\infty} \left(-\frac{n\,x^{n+4} b_n}{12}\right)\\
&\quad+\sum_{n=0}^{\infty} \left(-4x^n b_n n(n-1)\right)+\sum_{n=0}^{\infty} \left(-\frac{n\,x^{n+5} b_n}{60}\right)\\
&\quad+\sum_{n=0}^{\infty} \left(-C\,x^{n+2} a_n\right)+\sum_{n=0}^{\infty} \left(-2x^n b_n n\right)+\left(\sum_{n=0}^{\infty} C\,x^{n+2} a_n\right)\\
&\quad+\sum_{n=0}^{\infty} \left(-\frac{n\,x^{n+3} b_n}{3}\right)+\left(\sum_{n=0}^{\infty} 4C\,x^{1+n} a_n\right)\\
&\quad+\sum_{n=0}^{\infty} \left(-8C\,x^{1+n} a_n(1+n)\right)+\sum_{n=0}^{\infty} \left(-\frac{C\,x^{n+4} a_n}{3}\right)\\
&\quad+\sum_{n=0}^{\infty} \left(-C\,x^{n+3} a_n\right)+\left(\sum_{n=0}^{\infty} 4n\,x^{n-1} b_n(n-1)\right)\\
&\quad+\left(\sum_{n=0}^{\infty} \frac{C\,x^{n+6} a_n}{24}\right)+\sum_{n=0}^{\infty} \left(-2n\,x^{1+n} b_n\right)+\sum_{n=0}^{\infty} \left(-n\,x^{n+2} b_n\right)\\
&\quad+\left(\sum_{n=0}^{\infty} 4b_n x^n\right)+\left(\sum_{n=0}^{\infty} \frac{n\,x^{n+4} b_n}{6}\right)+\left(\sum_{n=0}^{\infty} 8C\,x^n a_n(1+n)\right)=0
\end{aligned}\tag{2A}$$

The next step is to make all powers of x be $n-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2C\, x^{n+2} a_n (1+n) = \sum_{n=3}^{\infty} 2C a_{n-3} (n-2)\, x^{n-1}$$

$$\sum_{n=0}^{\infty} \left(-8C\, x^{1+n} a_n (1+n)\right) = \sum_{n=2}^{\infty} \left(-8C a_{n-2} (n-1)\, x^{n-1}\right)$$

$$\sum_{n=0}^{\infty} 8C\, x^{n} a_n (1+n) = \sum_{n=1}^{\infty} 8C a_{n-1} n\, x^{n-1}$$

$$\sum_{n=0}^{\infty} C\, x^{n+2} a_n = \sum_{n=3}^{\infty} C a_{n-3} x^{n-1}$$

$$\sum_{n=0}^{\infty} C\, x^{n+3} a_n = \sum_{n=4}^{\infty} C a_{n-4} x^{n-1}$$

$$\sum_{n=0}^{\infty} \frac{C\, x^{n+4} a_n}{2} = \sum_{n=5}^{\infty} \frac{C a_{n-5} x^{n-1}}{2}$$

$$\sum_{n=0}^{\infty} \frac{C\, x^{n+5} a_n}{6} = \sum_{n=6}^{\infty} \frac{C a_{n-6} x^{n-1}}{6}$$

$$\sum_{n=0}^{\infty} \frac{C\, x^{n+6} a_n}{24} = \sum_{n=7}^{\infty} \frac{C a_{n-7} x^{n-1}}{24}$$

$$\sum_{n=0}^{\infty} \left(-C\, x^{n+2} a_n\right) = \sum_{n=3}^{\infty} \left(-C a_{n-3} x^{n-1}\right)$$

$$\sum_{n=0}^{\infty} \left(-2C\, x^{1+n} a_n\right) = \sum_{n=2}^{\infty} \left(-2C a_{n-2} x^{n-1}\right)$$

$$\sum_{n=0}^{\infty} \left(-2C\, x^{n+2} a_n\right) = \sum_{n=3}^{\infty} \left(-2C a_{n-3} x^{n-1}\right)$$

$$\sum_{n=0}^{\infty} \left(-C\, x^{n+3} a_n\right) = \sum_{n=4}^{\infty} \left(-C a_{n-4} x^{n-1}\right)$$

$$\sum_{n=0}^{\infty} \left(-\frac{C\, x^{n+4} a_n}{3}\right) = \sum_{n=5}^{\infty} \left(-\frac{C a_{n-5} x^{n-1}}{3}\right)$$

$$\sum_{n=0}^{\infty} \left(-\frac{C\, x^{n+5} a_n}{12}\right) = \sum_{n=6}^{\infty} \left(-\frac{C a_{n-6} x^{n-1}}{12}\right)$$

$$\sum_{n=0}^{\infty} \left(-\frac{C\, x^{n+6} a_n}{60}\right) = \sum_{n=7}^{\infty} \left(-\frac{C a_{n-7} x^{n-1}}{60}\right)$$

$$\sum_{n=0}^{\infty} 4C\, x^{1+n} a_n = \sum_{n=2}^{\infty} 4C a_{n-2} x^{n-1}$$

$$\sum_{n=0}^{\infty} (-4 a_n x^n C) = \sum_{n=1}^{\infty} \left(-4C a_{n-1} x^{n-1}\right)$$

$$\sum_{n=0}^{\infty} n\, x^{1+n} b_n (n-1) = \sum_{n=2}^{\infty} (n-2)\, b_{n-2} (n-3)\, x^{n-1}$$

$$\sum_{n=0}^{\infty} (-4 x^n b_n n (n-1)) = \sum_{n=1}^{\infty} \left(-4(n-1)\, b_{n-1} (n-2)\, x^{n-1}\right)$$

$$\sum_{n=0}^{\infty} n\, x^{1+n} b_n = \sum_{n=2}^{\infty} (n-2)\, b_{n-2} x^{n-1}$$

$$\sum_{n=0}^{\infty} n\, x^{n+2} b_n = \sum_{n=3}^{\infty} (n-3)\, b_{n-3} x^{n-1}$$

$$\sum_{n=0}^{\infty} \frac{n\, x^{n+3} b_n}{2} = \sum_{n=4}^{\infty} \frac{(n-4)\, b_{n-4} x^{n-1}}{2}$$

$$\sum_{n=0}^{\infty} \frac{n\, x^{n+4} b_n}{6} = \sum_{n=5}^{\infty} \frac{(n-5)\, b_{n-5} x^{n-1}}{6}$$

$$\sum_{n=0}^{\infty} \frac{n\, x^{n+5} b_n}{24} = \sum_{n=6}^{\infty} \frac{(n-6)\, b_{n-6} x^{n-1}}{24}$$

$$\sum_{n=0}^{\infty} \frac{n\, x^{n+6} b_n}{120} = \sum_{n=7}^{\infty} \frac{(n-7)\, b_{n-7} x^{n-1}}{120}$$

$$\sum_{n=0}^{\infty} (-2 x^n b_n n) = \sum_{n=1}^{\infty} \left(-2(n-1)\, b_{n-1} x^{n-1}\right)$$

$$\sum_{n=0}^{\infty} \left(-2n\, x^{1+n} b_n\right) = \sum_{n=2}^{\infty} \left(-2(n-2)\, b_{n-2} x^{n-1}\right)$$

$$\sum_{n=0}^{\infty} \left(-n\, x^{n+2} b_n\right) = \sum_{n=3}^{\infty} \left(-(n-3)\, b_{n-3} x^{n-1}\right)$$

$$\sum_{n=0}^{\infty} \left(-\frac{n\, x^{n+3} b_n}{3}\right) = \sum_{n=4}^{\infty} \left(-\frac{(n-4)\, b_{n-4} x^{n-1}}{3}\right)$$

$$\sum_{n=0}^{\infty} \left(-\frac{n\, x^{n+4} b_n}{12}\right) = \sum_{n=5}^{\infty} \left(-\frac{(n-5)\, b_{n-5} x^{n-1}}{12}\right)$$

$$\sum_{n=0}^{\infty} \left(-\frac{n\, x^{n+5} b_n}{60}\right) = \sum_{n=6}^{\infty} \left(-\frac{(n-6)\, b_{n-6} x^{n-1}}{60}\right)$$

$$\sum_{n=0}^{\infty} \left(-\frac{n\, x^{n+6} b_n}{360}\right) = \sum_{n=7}^{\infty} \left(-\frac{(n-7)\, b_{n-7} x^{n-1}}{360}\right)$$

$$\sum_{n=0}^{\infty} 4 b_n x^n = \sum_{n=1}^{\infty} 4 b_{n-1} x^{n-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-1$.

$$\text{Expression too large to display} \tag{2B}$$

For $n = 0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0 = 1$. For $n = N$, where $N = 1$ which is the difference between the two roots, we are free to choose $b_1 = 0$. Hence for $n = 1$, Eq (2B) gives

$$4C + 4 = 0$$

Which is solved for C. Solving for C gives

$$C = -1$$

For $n = 2$, Eq (2B) gives

$$(-6a_0 + 12a_1)\, C + 2b_1 + 8b_2 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$9 + 8b_2 = 0$$

Solving the above for b_2 gives

$$b_2 = -\frac{9}{8}$$

For $n = 3$, Eq (2B) gives

$$(-14a_1 + 20a_2) C - b_1 - 8b_2 + 24b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{19}{3} + 24b_3 = 0$$

Solving the above for b_3 gives

$$b_3 = -\frac{19}{72}$$

For $n = 4$, Eq (2B) gives

$$(2a_1 - 22a_2 + 28a_3) C - 26b_3 + 48b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{1019}{144} + 48b_4 = 0$$

Solving the above for b_4 gives

$$b_4 = -\frac{1019}{6912}$$

For $n = 5$, Eq (2B) gives

$$\frac{(a_0 + 24a_2 - 180a_3 + 216a_4) C}{6} + \frac{b_1}{6} + 3b_3 - 52b_4 + 80b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{5827}{864} + 80b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = -\frac{5827}{69120}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -1$ and all b_n, then the second solution becomes

$$y_2(x) = (-1) \left(x \left(1 - \frac{x}{4} - \frac{x^2}{24} - \frac{13x^3}{576} - \frac{35x^4}{2304} - \frac{1297x^5}{138240} + O(x^6)\right)\right) \ln(x)$$
$$+ 1 - \frac{9x^2}{8} - \frac{19x^3}{72} - \frac{1019x^4}{6912} - \frac{5827x^5}{69120} + O(x^6)$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x\left(1-\frac{x}{4}-\frac{x^2}{24}-\frac{13x^3}{576}-\frac{35x^4}{2304}-\frac{1297x^5}{138240}+O\left(x^6\right)\right) \\
&\quad + c_2\left((-1)\left(x\left(1-\frac{x}{4}-\frac{x^2}{24}-\frac{13x^3}{576}-\frac{35x^4}{2304}-\frac{1297x^5}{138240}+O\left(x^6\right)\right)\right)\ln\left(x\right)+1\right. \\
&\quad \left.-\frac{9x^2}{8}-\frac{19x^3}{72}-\frac{1019x^4}{6912}-\frac{5827x^5}{69120}+O\left(x^6\right)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x\left(1-\frac{x}{4}-\frac{x^2}{24}-\frac{13x^3}{576}-\frac{35x^4}{2304}-\frac{1297x^5}{138240}+O\left(x^6\right)\right) \\
&\quad + c_2\left(-x\left(1-\frac{x}{4}-\frac{x^2}{24}-\frac{13x^3}{576}-\frac{35x^4}{2304}-\frac{1297x^5}{138240}+O\left(x^6\right)\right)\ln\left(x\right)+1-\frac{9x^2}{8}\right. \\
&\quad \left.-\frac{19x^3}{72}-\frac{1019x^4}{6912}-\frac{5827x^5}{69120}+O\left(x^6\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x\left(1-\frac{x}{4}-\frac{x^2}{24}-\frac{13x^3}{576}-\frac{35x^4}{2304}-\frac{1297x^5}{138240}+O\left(x^6\right)\right) \\
&\quad + c_2\left(-x\left(1-\frac{x}{4}-\frac{x^2}{24}-\frac{13x^3}{576}-\frac{35x^4}{2304}-\frac{1297x^5}{138240}+O\left(x^6\right)\right)\ln\left(x\right)+1-\frac{9x^2}{8}\right. \\
&\quad \left.-\frac{19x^3}{72}-\frac{1019x^4}{6912}-\frac{5827x^5}{69120}+O\left(x^6\right)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1 x\left(1-\frac{x}{4}-\frac{x^2}{24}-\frac{13x^3}{576}-\frac{35x^4}{2304}-\frac{1297x^5}{138240}+O\left(x^6\right)\right) \\
&\quad + c_2\left(-x\left(1-\frac{x}{4}-\frac{x^2}{24}-\frac{13x^3}{576}-\frac{35x^4}{2304}-\frac{1297x^5}{138240}+O\left(x^6\right)\right)\ln\left(x\right)+1-\frac{9x^2}{8}\right. \\
&\quad \left.-\frac{19x^3}{72}-\frac{1019x^4}{6912}-\frac{5827x^5}{69120}+O\left(x^6\right)\right)
\end{aligned}
$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
      trying a symmetry of the form [xi=0, eta=F(x)]
      trying 2nd order exact linear
      trying symmetries linear in x and y(x)
      trying to convert to a linear ODE with constant coefficients
<- unable to find a useful change of variables
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying differential order: 2; exact nonlinear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
   -> Trying changes of variables to rationalize or make the ODE simpler
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
   <- unable to find a useful change of variables
      trying a symmetry of the form [xi=0, eta=F(x)]
   trying to convert to an ODE of Bessel type
   -> trying reduction of order to Riccati
      trying Riccati sub-methods:
         trying Riccati_symmetries
         -> trying a symmetry pattern of the form [F(x)*G(y), 0]
         -> trying a symmetry pattern of the form [0, F(x)*G(y)]
         -> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
--- Trying Lie symmetry methods, 2nd order ---
`, `-> Computing symmetries using: way = 3`[0, y]
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 60

```
Order:=6;
dsolve((x-2)^2*diff(y(x),x$2)+(x-2)*exp(x)*diff(y(x),x)+4/x*y(x)=0,y(x),type='series',x=0);
```

$$
\begin{aligned}
y(x) = {} & c_1 x \left(1 - \frac{1}{4}x - \frac{1}{24}x^2 - \frac{13}{576}x^3 - \frac{35}{2304}x^4 - \frac{1297}{138240}x^5 + \mathrm{O}\left(x^6\right)\right) \\
& + c_2 \left(\ln\left(x\right)\left(-x + \frac{1}{4}x^2 + \frac{1}{24}x^3 + \frac{13}{576}x^4 + \frac{35}{2304}x^5 + \mathrm{O}\left(x^6\right)\right) \right. \\
& \left. + \left(1 + \frac{1}{2}x - \frac{5}{4}x^2 - \frac{41}{144}x^3 - \frac{1097}{6912}x^4 - \frac{397}{4320}x^5 + \mathrm{O}\left(x^6\right)\right)\right)
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.063 (sec). Leaf size: 87

```
AsymptoticDSolveValue[(x-2)^2*y''[x]+(x-2)*Exp[x]*y'[x]+4/x*y[x]==0,y[x],{x,0,5}]
```

$$
\begin{aligned}
y(x) \rightarrow {} & c_1 \left(\frac{1}{576}x\left(13x^3 + 24x^2 + 144x - 576\right)\log(x) \right. \\
& \left. + \frac{-1097x^4 - 1968x^3 - 8640x^2 + 3456x + 6912}{6912}\right) \\
& + c_2 \left(-\frac{35x^5}{2304} - \frac{13x^4}{576} - \frac{x^3}{24} - \frac{x^2}{4} + x\right)
\end{aligned}
$$

17.4 problem 5

Internal problem ID [2920]
Internal file name [`OUTPUT/2412_Sunday_June_05_2022_03_06_58_AM_35030435/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.4. page 758
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Irregular singular point"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\boxed{y'' + \frac{2y'}{x(x-3)} - \frac{y}{x^3(x+3)} = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$y'' + \frac{2y'}{x(x-3)} - \frac{y}{x^3(x+3)} = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$\begin{aligned} p(x) &= \frac{2}{x(x-3)} \\ q(x) &= -\frac{1}{x^3(x+3)} \end{aligned}$$

Table 422: Table $p(x), q(x)$ singularites.

$p(x) = \frac{2}{x(x-3)}$	
singularity	type
$x = 0$	"regular"
$x = 3$	"regular"

$q(x) = -\frac{1}{x^3(x+3)}$	
singularity	type
$x = -3$	"regular"
$x = 0$	"irregular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[3, -3, \infty]$

Irregular singular points : $[0]$

Since $x = 0$ is not an ordinary point, then we will now check if it is a regular singular point. Unable to solve since $x = 0$ is not regular singular point. Terminating.

Verification of solutions N/A

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying 2nd order exact linear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      trying 2nd order exact linear
      trying symmetries linear in x and y(x)
      trying to convert to a linear ODE with constant coefficients
      trying to convert to an ODE of Bessel type
   trying to convert to an ODE of Bessel type
   -> trying reduction of order to Riccati
```

✗ Solution by Maple

```
Order:=6;
dsolve(diff(y(x),x$2)+2/(x*(x-3))*diff(y(x),x)-1/(x^3*(x+3))*y(x)=0,y(x),type='series',x=0);
```

No solution found

✓ Solution by Mathematica

Time used: 0.223 (sec). Leaf size: 258

```
AsymptoticDSolveValue[y''[x]+2/(x*(x-3))*y'[x]-1/(x^3*(x+3))*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 e^{-\frac{2}{\sqrt{3}\sqrt{x}}}\left(\frac{1087999600339049453 9x^{9/2}}{605967246346420224 0\sqrt{3}} + \frac{647134806104 17x^{7/2}}{328758271672320\sqrt{3}} + \frac{287821451x^{5/2}}{3397386240\sqrt{3}} + \frac{19817x^{3/2}}{73728\sqrt{3}} - \frac{4894564486149401320457x^5}{1246561192484064460800} - \frac{116612812982297797x^4}{378729528966512640} - \frac{22160647459x^3}{587068342272} + \frac{463507x^2}{42467328} + \frac{587x}{4608} + \frac{25\sqrt{x}}{16\sqrt{3}} + 1\right) x^{13/12} + c_2 e^{\frac{2}{\sqrt{3}\sqrt{x}}}\left(-\frac{10879996003390494539x^{9/2}}{6059672463464202240\sqrt{3}} - \frac{64713480610417x^{7/2}}{328758271672320\sqrt{3}} - \frac{287821451x^{5/2}}{3397386240\sqrt{3}} - \frac{19817x}{73728}\right.$$

17.5 problem 6

Internal problem ID [2921]
Internal file name [`OUTPUT/2413_Sunday_June_05_2022_03_07_03_AM_688095/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.4. page 758
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + x(1-x)\, y' - 7y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + \left(-x^2 + x\right) y' - 7y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{x-1}{x}$$
$$q(x) = -\frac{7}{x^2}$$

Table 423: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{x-1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{7}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + \left(-x^2 + x\right) y' - 7y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + \left(-x^2 + x\right)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) - 7\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)(n+r-1)\right)+\sum_{n=0}^{\infty}\left(-x^{1+n+r} a_n(n+r)\right) \\
&\quad+\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)\right)+\sum_{n=0}^{\infty}\left(-7 a_n x^{n+r}\right)=0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\sum_{n=0}^{\infty}\left(-x^{1+n+r} a_n(n+r)\right)=\sum_{n=1}^{\infty}\left(-a_{n-1}(n+r-1) x^{n+r}\right)
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)(n+r-1)\right)+\sum_{n=1}^{\infty}\left(-a_{n-1}(n+r-1) x^{n+r}\right) \\
&\quad+\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)\right)+\sum_{n=0}^{\infty}\left(-7 a_n x^{n+r}\right)=0
\end{aligned} \tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
x^{n+r} a_n(n+r)(n+r-1)+x^{n+r} a_n(n+r)-7 a_n x^{n+r}=0
$$

When $n=0$ the above becomes

$$
x^r a_0 r(-1+r)+x^r a_0 r-7 a_0 x^r=0
$$

Or

$$
\left(x^r r(-1+r)+x^r r-7 x^r\right) a_0=0
$$

Since $a_0 \neq 0$ then the above simplifies to

$$
\left(r^2-7\right) x^r=0
$$

Since the above is true for all x then the indicial equation becomes

$$
r^2-7=0
$$

Solving for r gives the roots of the indicial equation as

$$
\begin{aligned}
r_1 &= \sqrt{7} \\
r_2 &= -\sqrt{7}
\end{aligned}
$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$
\left(r^2 - 7\right) x^r = 0
$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 2\sqrt{7}$ is not an integer, then we can construct two linearly independent solutions

$$
\begin{aligned}
y_1(x) &= x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right) \\
y_2(x) &= x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+\sqrt{7}} \\
y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n-\sqrt{7}}
\end{aligned}
$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$
a_n(n+r)(n+r-1) - a_{n-1}(n+r-1) + a_n(n+r) - 7a_n = 0 \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$
a_n = \frac{a_{n-1}(n+r-1)}{n^2 + 2nr + r^2 - 7} \tag{4}
$$

Which for the root $r = \sqrt{7}$ becomes

$$
a_n = \frac{a_{n-1}\left(n+\sqrt{7}-1\right)}{n\left(2\sqrt{7}+n\right)} \tag{5}
$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \sqrt{7}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{r}{r^2 + 2r - 6}$$

Which for the root $r = \sqrt{7}$ becomes

$$a_1 = \frac{\sqrt{7}}{1 + 2\sqrt{7}}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r}{r^2+2r-6}$	$\frac{\sqrt{7}}{1+2\sqrt{7}}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{r(r + 1)}{(r^2 + 2r - 6)(r^2 + 4r - 3)}$$

Which for the root $r = \sqrt{7}$ becomes

$$a_2 = \frac{\sqrt{7}}{4 + 8\sqrt{7}}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r}{r^2+2r-6}$	$\frac{\sqrt{7}}{1+2\sqrt{7}}$
a_2	$\frac{r(r+1)}{(r^2+2r-6)(r^2+4r-3)}$	$\frac{\sqrt{7}}{4+8\sqrt{7}}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{r(r + 1)(2 + r)}{(r^2 + 2r - 6)(r^2 + 4r - 3)(r^2 + 6r + 2)}$$

Which for the root $r = \sqrt{7}$ becomes

$$a_3 = \frac{(2+\sqrt{7})\sqrt{7}}{372+96\sqrt{7}}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r}{r^2+2r-6}$	$\frac{\sqrt{7}}{1+2\sqrt{7}}$
a_2	$\frac{r(r+1)}{(r^2+2r-6)(r^2+4r-3)}$	$\frac{\sqrt{7}}{4+8\sqrt{7}}$
a_3	$\frac{r(r+1)(2+r)}{(r^2+2r-6)(r^2+4r-3)(r^2+6r+2)}$	$\frac{(2+\sqrt{7})\sqrt{7}}{372+96\sqrt{7}}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{r(r+1)(2+r)(3+r)}{(r^2+2r-6)(r^2+4r-3)(r^2+6r+2)(r^2+8r+9)}$$

Which for the root $r = \sqrt{7}$ becomes

$$a_4 = \frac{\sqrt{7}(3+\sqrt{7})}{2976+768\sqrt{7}}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r}{r^2+2r-6}$	$\frac{\sqrt{7}}{1+2\sqrt{7}}$
a_2	$\frac{r(r+1)}{(r^2+2r-6)(r^2+4r-3)}$	$\frac{\sqrt{7}}{4+8\sqrt{7}}$
a_3	$\frac{r(r+1)(2+r)}{(r^2+2r-6)(r^2+4r-3)(r^2+6r+2)}$	$\frac{(2+\sqrt{7})\sqrt{7}}{372+96\sqrt{7}}$
a_4	$\frac{r(r+1)(2+r)(3+r)}{(r^2+2r-6)(r^2+4r-3)(r^2+6r+2)(r^2+8r+9)}$	$\frac{\sqrt{7}(3+\sqrt{7})}{2976+768\sqrt{7}}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{r(r+1)(2+r)(3+r)(4+r)}{(r^2+2r-6)(r^2+4r-3)(r^2+6r+2)(r^2+8r+9)(r^2+10r+18)}$$

Which for the root $r = \sqrt{7}$ becomes

$$a_5 = \frac{\sqrt{7}\,(3+\sqrt{7})\,(4+\sqrt{7})}{48960\sqrt{7}+128160}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r}{r^2+2r-6}$	$\frac{\sqrt{7}}{1+2\sqrt{7}}$
a_2	$\frac{r(r+1)}{(r^2+2r-6)(r^2+4r-3)}$	$\frac{\sqrt{7}}{4+8\sqrt{7}}$
a_3	$\frac{r(r+1)(2+r)}{(r^2+2r-6)(r^2+4r-3)(r^2+6r+2)}$	$\frac{(2+\sqrt{7})\sqrt{7}}{372+96\sqrt{7}}$
a_4	$\frac{r(r+1)(2+r)(3+r)}{(r^2+2r-6)(r^2+4r-3)(r^2+6r+2)(r^2+8r+9)}$	$\frac{\sqrt{7}\,(3+\sqrt{7})}{2976+768\sqrt{7}}$
a_5	$\frac{r(r+1)(2+r)(3+r)(4+r)}{(r^2+2r-6)(r^2+4r-3)(r^2+6r+2)(r^2+8r+9)(r^2+10r+18)}$	$\frac{\sqrt{7}\,(3+\sqrt{7})\,(4+\sqrt{7})}{48960\sqrt{7}+128160}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\sqrt{7}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^{\sqrt{7}}\left(1 + \frac{\sqrt{7}\,x}{1+2\sqrt{7}} + \frac{\sqrt{7}\,x^2}{4+8\sqrt{7}} + \frac{(2+\sqrt{7})\,\sqrt{7}\,x^3}{372+96\sqrt{7}} + \frac{\sqrt{7}\,(3+\sqrt{7})\,x^4}{2976+768\sqrt{7}} + \frac{\sqrt{7}\,(3+\sqrt{7})\,(4+\sqrt{7})\,x^5}{48960\sqrt{7}+128160}\right. \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$b_n(n+r)\,(n+r-1) - b_{n-1}(n+r-1) + b_n(n+r) - 7b_n = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = \frac{b_{n-1}(n+r-1)}{n^2+2nr+r^2-7} \tag{4}$$

Which for the root $r = -\sqrt{7}$ becomes

$$b_n = \frac{b_{n-1}(-n+\sqrt{7}+1)}{n\,(2\sqrt{7}-n)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\sqrt{7}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{r}{r^2 + 2r - 6}$$

Which for the root $r = -\sqrt{7}$ becomes

$$b_1 = \frac{\sqrt{7}}{-1 + 2\sqrt{7}}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{r}{r^2+2r-6}$	$\frac{\sqrt{7}}{-1+2\sqrt{7}}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{r(r+1)}{(r^2 + 2r - 6)(r^2 + 4r - 3)}$$

Which for the root $r = -\sqrt{7}$ becomes

$$b_2 = \frac{\sqrt{7}}{-4 + 8\sqrt{7}}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{r}{r^2+2r-6}$	$\frac{\sqrt{7}}{-1+2\sqrt{7}}$
b_2	$\frac{r(r+1)}{(r^2+2r-6)(r^2+4r-3)}$	$\frac{\sqrt{7}}{-4+8\sqrt{7}}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{r(r+1)(2+r)}{(r^2+2r-6)(r^2+4r-3)(r^2+6r+2)}$$

Which for the root $r = -\sqrt{7}$ becomes

$$b_3 = \frac{\sqrt{7}\left(-2+\sqrt{7}\right)}{372-96\sqrt{7}}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{r}{r^2+2r-6}$	$\frac{\sqrt{7}}{-1+2\sqrt{7}}$
b_2	$\frac{r(r+1)}{(r^2+2r-6)(r^2+4r-3)}$	$\frac{\sqrt{7}}{-4+8\sqrt{7}}$
b_3	$\frac{r(r+1)(2+r)}{(r^2+2r-6)(r^2+4r-3)(r^2+6r+2)}$	$\frac{\sqrt{7}\left(-2+\sqrt{7}\right)}{372-96\sqrt{7}}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{r(r+1)(2+r)(3+r)}{(r^2+2r-6)(r^2+4r-3)(r^2+6r+2)(r^2+8r+9)}$$

Which for the root $r = -\sqrt{7}$ becomes

$$b_4 = \frac{\sqrt{7}\left(-3+\sqrt{7}\right)}{2976-768\sqrt{7}}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{r}{r^2+2r-6}$	$\frac{\sqrt{7}}{-1+2\sqrt{7}}$
b_2	$\frac{r(r+1)}{(r^2+2r-6)(r^2+4r-3)}$	$\frac{\sqrt{7}}{-4+8\sqrt{7}}$
b_3	$\frac{r(r+1)(2+r)}{(r^2+2r-6)(r^2+4r-3)(r^2+6r+2)}$	$\frac{\sqrt{7}\left(-2+\sqrt{7}\right)}{372-96\sqrt{7}}$
b_4	$\frac{r(r+1)(2+r)(3+r)}{(r^2+2r-6)(r^2+4r-3)(r^2+6r+2)(r^2+8r+9)}$	$\frac{\sqrt{7}\left(-3+\sqrt{7}\right)}{2976-768\sqrt{7}}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{r(r+1)(2+r)(3+r)(4+r)}{(r^2+2r-6)(r^2+4r-3)(r^2+6r+2)(r^2+8r+9)(r^2+10r+18)}$$

Which for the root $r = -\sqrt{7}$ becomes

$$b_5 = \frac{\sqrt{7}\left(-3+\sqrt{7}\right)\left(-4+\sqrt{7}\right)}{48960\sqrt{7}-128160}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{r}{r^2+2r-6}$	$\frac{\sqrt{7}}{-1+2\sqrt{7}}$
b_2	$\frac{r(r+1)}{(r^2+2r-6)(r^2+4r-3)}$	$\frac{\sqrt{7}}{-4+8\sqrt{7}}$
b_3	$\frac{r(r+1)(2+r)}{(r^2+2r-6)(r^2+4r-3)(r^2+6r+2)}$	$\frac{\sqrt{7}\left(-2+\sqrt{7}\right)}{372-96\sqrt{7}}$
b_4	$\frac{r(r+1)(2+r)(3+r)}{(r^2+2r-6)(r^2+4r-3)(r^2+6r+2)(r^2+8r+9)}$	$\frac{\sqrt{7}\left(-3+\sqrt{7}\right)}{2976-768\sqrt{7}}$
b_5	$\frac{r(r+1)(2+r)(3+r)(4+r)}{(r^2+2r-6)(r^2+4r-3)(r^2+6r+2)(r^2+8r+9)(r^2+10r+18)}$	$\frac{\sqrt{7}\left(-3+\sqrt{7}\right)\left(-4+\sqrt{7}\right)}{48960\sqrt{7}-128160}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x^{\sqrt{7}}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= x^{-\sqrt{7}}\left(1 + \frac{\sqrt{7}\,x}{-1+2\sqrt{7}} + \frac{\sqrt{7}\,x^2}{-4+8\sqrt{7}} + \frac{\sqrt{7}\left(-2+\sqrt{7}\right)x^3}{372-96\sqrt{7}} + \frac{\sqrt{7}\left(-3+\sqrt{7}\right)x^4}{2976-768\sqrt{7}} + \frac{\sqrt{7}\left(-3+\sqrt{7}\right)(\cdots}{48960\sqrt{7}-\cdots}\right. \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^{\sqrt{7}}\left(1 + \frac{\sqrt{7}\,x}{1+2\sqrt{7}} + \frac{\sqrt{7}\,x^2}{4+8\sqrt{7}} + \frac{\left(2+\sqrt{7}\right)\sqrt{7}\,x^3}{372+96\sqrt{7}} + \frac{\sqrt{7}\left(3+\sqrt{7}\right)x^4}{2976+768\sqrt{7}}\right. \\ &\qquad \left. + \frac{\sqrt{7}\left(3+\sqrt{7}\right)\left(4+\sqrt{7}\right)x^5}{48960\sqrt{7}+128160} + O\left(x^6\right)\right) \\ &\quad + c_2 x^{-\sqrt{7}}\left(1 + \frac{\sqrt{7}\,x}{-1+2\sqrt{7}} + \frac{\sqrt{7}\,x^2}{-4+8\sqrt{7}} + \frac{\sqrt{7}\left(-2+\sqrt{7}\right)x^3}{372-96\sqrt{7}} + \frac{\sqrt{7}\left(-3+\sqrt{7}\right)x^4}{2976-768\sqrt{7}}\right. \\ &\qquad \left. + \frac{\sqrt{7}\left(-3+\sqrt{7}\right)\left(-4+\sqrt{7}\right)x^5}{48960\sqrt{7}-128160} + O\left(x^6\right)\right) \end{aligned}$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^{\sqrt{7}} \left(1+\frac{\sqrt{7}\, x}{1+2\sqrt{7}}+\frac{\sqrt{7}\, x^{2}}{4+8\sqrt{7}}+\frac{\left(2+\sqrt{7}\right) \sqrt{7}\, x^{3}}{372+96\sqrt{7}}+\frac{\sqrt{7}\, \left(3+\sqrt{7}\right) x^{4}}{2976+768\sqrt{7}} \right. \\
&\quad \left. +\frac{\sqrt{7}\, \left(3+\sqrt{7}\right) \left(4+\sqrt{7}\right) x^{5}}{48960\sqrt{7}+128160}+O\left(x^{6}\right)\right)+c_2 x^{-\sqrt{7}} \left(1+\frac{\sqrt{7}\, x}{-1+2\sqrt{7}}+\frac{\sqrt{7}\, x^{2}}{-4+8\sqrt{7}} \right. \\
&\quad \left. +\frac{\sqrt{7}\, \left(-2+\sqrt{7}\right) x^{3}}{372-96\sqrt{7}}+\frac{\sqrt{7}\, \left(-3+\sqrt{7}\right) x^{4}}{2976-768\sqrt{7}}+\frac{\sqrt{7}\, \left(-3+\sqrt{7}\right) \left(-4+\sqrt{7}\right) x^{5}}{48960\sqrt{7}-128160}+O\left(x^{6}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = c_1 x^{\sqrt{7}} &\left(1+\frac{\sqrt{7}\, x}{1+2\sqrt{7}}+\frac{\sqrt{7}\, x^{2}}{4+8\sqrt{7}}+\frac{\left(2+\sqrt{7}\right) \sqrt{7}\, x^{3}}{372+96\sqrt{7}}+\frac{\sqrt{7}\, \left(3+\sqrt{7}\right) x^{4}}{2976+768\sqrt{7}} \right. \\
&\left. +\frac{\sqrt{7}\, \left(3+\sqrt{7}\right) \left(4+\sqrt{7}\right) x^{5}}{48960\sqrt{7}+128160}+O\left(x^{6}\right)\right) \\
+ c_2 x^{-\sqrt{7}} &\left(1+\frac{\sqrt{7}\, x}{-1+2\sqrt{7}}+\frac{\sqrt{7}\, x^{2}}{-4+8\sqrt{7}}+\frac{\sqrt{7}\, \left(-2+\sqrt{7}\right) x^{3}}{372-96\sqrt{7}} \right. \\
&\left. +\frac{\sqrt{7}\, \left(-3+\sqrt{7}\right) x^{4}}{2976-768\sqrt{7}}+\frac{\sqrt{7}\, \left(-3+\sqrt{7}\right) \left(-4+\sqrt{7}\right) x^{5}}{48960\sqrt{7}-128160}+O\left(x^{6}\right)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1 x^{\sqrt{7}} \left(1+\frac{\sqrt{7}\, x}{1+2\sqrt{7}}+\frac{\sqrt{7}\, x^{2}}{4+8\sqrt{7}}+\frac{\left(2+\sqrt{7}\right) \sqrt{7}\, x^{3}}{372+96\sqrt{7}}+\frac{\sqrt{7}\, \left(3+\sqrt{7}\right) x^{4}}{2976+768\sqrt{7}} \right. \\
&\quad \left. +\frac{\sqrt{7}\, \left(3+\sqrt{7}\right) \left(4+\sqrt{7}\right) x^{5}}{48960\sqrt{7}+128160}+O\left(x^{6}\right)\right)+c_2 x^{-\sqrt{7}} \left(1+\frac{\sqrt{7}\, x}{-1+2\sqrt{7}}+\frac{\sqrt{7}\, x^{2}}{-4+8\sqrt{7}} \right. \\
&\quad \left. +\frac{\sqrt{7}\, \left(-2+\sqrt{7}\right) x^{3}}{372-96\sqrt{7}}+\frac{\sqrt{7}\, \left(-3+\sqrt{7}\right) x^{4}}{2976-768\sqrt{7}}+\frac{\sqrt{7}\, \left(-3+\sqrt{7}\right) \left(-4+\sqrt{7}\right) x^{5}}{48960\sqrt{7}-128160}+O\left(x^{6}\right)\right)
\end{aligned}
$$

Verified OK.

17.5.1 Maple step by step solution

Let's solve

$x^2y'' + (-x^2 + x)\, y' - 7y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{7y}{x^2} + \frac{(x-1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(x-1)y'}{x} - \frac{7y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -\frac{x-1}{x}, P_3(x) = -\frac{7}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = -7$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2y'' - x(x-1)\, y' - 7y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

  ◦ Shift index using $k- >k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(r^2-7)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k^2+2kr+r^2-7) - a_{k-1}(k+r-1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$r^2-7=0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{\sqrt{7}, -\sqrt{7}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$a_k(k^2+2kr+r^2-7) - a_{k-1}(k+r-1) = 0$$

- Shift index using $k- >k+1$

  $$a_{k+1}\left((k+1)^2+2(k+1)\, r+r^2-7\right) - a_k(k+r) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = \frac{a_k(k+r)}{k^2+2kr+r^2+2k+2r-6}$$

- Recursion relation for $r=\sqrt{7}$

  $$a_{k+1} = \frac{a_k\left(k+\sqrt{7}\right)}{k^2+2k\sqrt{7}+1+2k+2\sqrt{7}}$$

- Solution for $r=\sqrt{7}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\sqrt{7}}, a_{k+1} = \frac{a_k\left(k+\sqrt{7}\right)}{k^2+2k\sqrt{7}+1+2k+2\sqrt{7}}\right]$$

- Recursion relation for $r=-\sqrt{7}$

  $$a_{k+1} = \frac{a_k\left(k-\sqrt{7}\right)}{k^2-2k\sqrt{7}+1+2k-2\sqrt{7}}$$

- Solution for $r=-\sqrt{7}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\sqrt{7}}, a_{k+1} = \frac{a_k\left(k-\sqrt{7}\right)}{k^2-2k\sqrt{7}+1+2k-2\sqrt{7}}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k+\sqrt{7}}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k-\sqrt{7}}\right), a_{1+k}=\frac{a_k\left(k+\sqrt{7}\right)}{k^2+2k\sqrt{7}+1+2k+2\sqrt{7}}, b_{1+k}=\frac{b_k\left(k-\sqrt{7}\right)}{k^2-2k\sqrt{7}+1+2k-2\sqrt{7}}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   <- Kummer successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 478

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+x*(1-x)*diff(y(x),x)-7*y(x)=0,y(x),type='series',x=0);
```

$$y(x)=c_1 x^{-\sqrt{7}}\left(1+\frac{\sqrt{7}}{-1+2\sqrt{7}}x+\frac{\sqrt{7}}{-4+8\sqrt{7}}x^2+\frac{\sqrt{7}\left(\sqrt{7}-2\right)}{372-96\sqrt{7}}x^3+\frac{\sqrt{7}\left(\sqrt{7}-3\right)}{2976-768\sqrt{7}}x^4\right.$$
$$\left.+\frac{\left(\sqrt{7}-4\right)\left(\sqrt{7}-3\right)\sqrt{7}}{48960\sqrt{7}-128160}x^5+\mathrm{O}\left(x^6\right)\right)+c_2 x^{\sqrt{7}}\left(1+\frac{\sqrt{7}}{1+2\sqrt{7}}x+\frac{\sqrt{7}}{4+8\sqrt{7}}x^2\right.$$
$$\left.+\frac{\sqrt{7}\left(\sqrt{7}+2\right)}{372+96\sqrt{7}}x^3+\frac{\left(\sqrt{7}+3\right)\sqrt{7}}{2976+768\sqrt{7}}x^4+\frac{\left(\sqrt{7}+4\right)\left(\sqrt{7}+3\right)\sqrt{7}}{48960\sqrt{7}+128160}x^5+\mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 1066

```
AsymptoticDSolveValue[x^2*y''[x]+x*(1-x)*y'[x]-7*y[x]==0,y[x],{x,0,5}]
```

$$
\begin{aligned}
y(x) \\
\rightarrow \Bigg(& \frac{\sqrt{7}(1+\sqrt{7})\,(2+\sqrt{7})\,(3+\sqrt{7})\,(4+\sqrt{7}}{(-6+\sqrt{7}+\sqrt{7}\,(1+\sqrt{7}))\,(-5+\sqrt{7}+(1+\sqrt{7})\,(2+\sqrt{7}))\,(-4+\sqrt{7}+(2+\sqrt{7})\,(3+\sqrt{7}))\,(-3} \\
& + \frac{\sqrt{7}(1+\sqrt{7})\,(2+\sqrt{7})\,(3+\sqrt{7})\,x^4}{(-6+\sqrt{7}+\sqrt{7}\,(1+\sqrt{7}))\,(-5+\sqrt{7}+(1+\sqrt{7})\,(2+\sqrt{7}))\,(-4+\sqrt{7}+(2+\sqrt{7})\,(3+\sqrt{7}))\,(-3} \\
& + \frac{\sqrt{7}(1+\sqrt{7})\,(2+\sqrt{7})\,x^3}{(-6+\sqrt{7}+\sqrt{7}\,(1+\sqrt{7}))\,(-5+\sqrt{7}+(1+\sqrt{7})\,(2+\sqrt{7}))\,(-4+\sqrt{7}+(2+\sqrt{7})\,(3+\sqrt{7}))} \\
& + \frac{\sqrt{7}(1+\sqrt{7})\,x^2}{(-6+\sqrt{7}+\sqrt{7}\,(1+\sqrt{7}))\,(-5+\sqrt{7}+(1+\sqrt{7})\,(2+\sqrt{7}))} \\
& + \frac{\sqrt{7}x}{-6+\sqrt{7}+\sqrt{7}\,(1+\sqrt{7})} + 1 \Bigg) c_1 x^{\sqrt{7}} \\
+ \Bigg(& -\frac{\sqrt{7}(1-\sqrt{7})\,(2-\sqrt{7})\,(3-\sqrt{7})\,(4-}{(-6-\sqrt{7}-\sqrt{7}\,(1-\sqrt{7}))\,(-5-\sqrt{7}+(1-\sqrt{7})\,(2-\sqrt{7}))\,(-4-\sqrt{7}+(2-\sqrt{7})\,(3-\sqrt{7}))\,(} \\
& - \frac{\sqrt{7}(1-\sqrt{7})\,(2-\sqrt{7})\,(3-\sqrt{7})\,x^4}{(-6-\sqrt{7}-\sqrt{7}\,(1-\sqrt{7}))\,(-5-\sqrt{7}+(1-\sqrt{7})\,(2-\sqrt{7}))\,(-4-\sqrt{7}+(2-\sqrt{7})\,(3-\sqrt{7}))\,(-3} \\
& - \frac{\sqrt{7}(1-\sqrt{7})\,(2-\sqrt{7})\,x^3}{(-6-\sqrt{7}-\sqrt{7}\,(1-\sqrt{7}))\,(-5-\sqrt{7}+(1-\sqrt{7})\,(2-\sqrt{7}))\,(-4-\sqrt{7}+(2-\sqrt{7})\,(3-\sqrt{7}))} \\
& - \frac{\sqrt{7}(1-\sqrt{7})\,x^2}{(-6-\sqrt{7}-\sqrt{7}\,(1-\sqrt{7}))\,(-5-\sqrt{7}+(1-\sqrt{7})\,(2-\sqrt{7}))} \\
& - \frac{\sqrt{7}x}{-6-\sqrt{7}-\sqrt{7}\,(1-\sqrt{7})} + 1 \Bigg) c_2 x^{-\sqrt{7}}
\end{aligned}
$$

17.6 problem 7

Internal problem ID [2922]
Internal file name [`OUTPUT/2414_Sunday_June_05_2022_03_07_14_AM_72399094/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.4. page 758
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{4x^2y'' + y'x\,\mathrm{e}^x - y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$4x^2y'' + y'x\,\mathrm{e}^x - y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$\begin{aligned} p(x) &= \frac{\mathrm{e}^x}{4x} \\ q(x) &= -\frac{1}{4x^2} \end{aligned}$$

Table 425: Table $p(x), q(x)$ singularites.

$p(x) = \frac{e^x}{4x}$	
singularity	type
$x = 0$	"regular"
$x = \infty$	"regular"

$q(x) = -\frac{1}{4x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$4x^2y'' + y'x\,e^x - y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$4x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) x\,e^x - \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Expanding $x\,\mathrm{e}^{x}$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$
\begin{aligned}
x\,\mathrm{e}^{x} &= x + x^{2} + \frac{1}{2}x^{3} + \frac{1}{6}x^{4} + \frac{1}{24}x^{5} + \frac{1}{120}x^{6} + \dots \\
&= x + x^{2} + \frac{1}{2}x^{3} + \frac{1}{6}x^{4} + \frac{1}{24}x^{5} + \frac{1}{120}x^{6}
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty} \frac{x^{n+r+5} a_n (n+r)}{120}\right) \\
&+ \left(\sum_{n=0}^{\infty} \frac{x^{n+r+4} a_n (n+r)}{24}\right) + \left(\sum_{n=0}^{\infty} \frac{x^{n+r+3} a_n (n+r)}{6}\right) \\
&+ \left(\sum_{n=0}^{\infty} \frac{x^{n+r+2} a_n (n+r)}{2}\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)\right) \\
&+ \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0
\end{aligned}
\tag{2A}
$$

The next step is to make all powers of x be $n + r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} \frac{x^{n+r+5} a_n (n+r)}{120} &= \sum_{n=5}^{\infty} \frac{a_{n-5} (n-5+r)\, x^{n+r}}{120} \\
\sum_{n=0}^{\infty} \frac{x^{n+r+4} a_n (n+r)}{24} &= \sum_{n=4}^{\infty} \frac{a_{n-4} (n-4+r)\, x^{n+r}}{24} \\
\sum_{n=0}^{\infty} \frac{x^{n+r+3} a_n (n+r)}{6} &= \sum_{n=3}^{\infty} \frac{a_{n-3} (n-3+r)\, x^{n+r}}{6} \\
\sum_{n=0}^{\infty} \frac{x^{n+r+2} a_n (n+r)}{2} &= \sum_{n=2}^{\infty} \frac{a_{n-2} (n+r-2)\, x^{n+r}}{2} \\
\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r) &= \sum_{n=1}^{\infty} a_{n-1} (n+r-1)\, x^{n+r}
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 4x^{n+r} a_n(n+r)\,(n+r-1)\right) + \left(\sum_{n=5}^{\infty} \frac{a_{n-5}(n-5+r)\,x^{n+r}}{120}\right) + \left(\sum_{n=4}^{\infty} \frac{a_{n-4}(n-4+r)\,x^{n+r}}{24}\right) + \left(\sum_{n=3}^{\infty} \frac{a_{n-3}(n-3+r)\,x^{n+r}}{6}\right) + \left(\sum_{n=2}^{\infty} \frac{a_{n-2}(n+r-2)\,x^{n+r}}{2}\right) + \left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1)\,x^{n+r}\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$4x^{n+r} a_n(n+r)\,(n+r-1) + x^{n+r} a_n(n+r) - a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$4x^r a_0 r(-1+r) + x^r a_0 r - a_0 x^r = 0$$

Or

$$\left(4x^r r(-1+r) + x^r r - x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(4r^2 - 3r - 1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$4r^2 - 3r - 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = -\frac{1}{4}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(4r^2 - 3r - 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{5}{4}$ is not an integer, then we can construct two linearly independent solutions

$$
\begin{aligned}
y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right) \\
y_2(x) &= x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^{1+n} \\
y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n-\frac{1}{4}}
\end{aligned}
$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$
a_1 = -\frac{1}{4r + 5}
$$

Substituting $n = 2$ in Eq. (2B) gives

$$
a_2 = \frac{-4r^2 - 3r + 2}{32r^3 + 144r^2 + 202r + 90}
$$

Substituting $n = 3$ in Eq. (2B) gives

$$
a_3 = \frac{-16r^4 - 48r^3 - 17r^2 + 33r + 15}{384r^5 + 3744r^4 + 13992r^3 + 25038r^2 + 21426r + 7020}
$$

Substituting $n = 4$ in Eq. (2B) gives

$$
a_4 = \frac{-64r^6 - 400r^5 - 572r^4 + 773r^3 + 2385r^2 + 1538r + 132}{24 \left(64r^5 + 624r^4 + 2332r^3 + 4173r^2 + 3571r + 1170\right) \left(4r^2 + 29r + 51\right)}
$$

For $5 \leq n$ the recursive equation is

$$
\begin{aligned}
&4a_n(n + r)(n + r - 1) + \frac{a_{n-5}(n - 5 + r)}{120} + \frac{a_{n-4}(n - 4 + r)}{24} \\
&+ \frac{a_{n-3}(n - 3 + r)}{6} + \frac{a_{n-2}(n + r - 2)}{2} + a_{n-1}(n + r - 1) + a_n(n + r) - a_n = 0
\end{aligned} \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{na_{n-5} + 5na_{n-4} + 20na_{n-3} + 60na_{n-2} + 120na_{n-1} + ra_{n-5} + 5ra_{n-4} + 20ra_{n-3} + 60ra_{n-2} + 120r}{120\left(4n^2 + 8nr + 4r^2 - 3n - 3r - 1\right)} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = \frac{\left(-a_{n-5} - 5a_{n-4} - 20a_{n-3} - 60a_{n-2} - 120a_{n-1}\right)n + 4a_{n-5} + 15a_{n-4} + 40a_{n-3} + 60a_{n-2}}{480n^2 + 600n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{4r+5}$	$-\frac{1}{9}$
a_2	$\frac{-4r^2-3r+2}{32r^3+144r^2+202r+90}$	$-\frac{5}{468}$
a_3	$\frac{-16r^4-48r^3-17r^2+33r+15}{384r^5+3744r^4+13992r^3+25038r^2+21426r+7020}$	$-\frac{11}{23868}$
a_4	$\frac{-64r^6-400r^5-572r^4+773r^3+2385r^2+1538r+132}{24(64r^5+624r^4+2332r^3+4173r^2+3571r+1170)(4r^2+29r+51)}$	$\frac{79}{501228}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-256r^8 - 2432r^7 - 3808r^6 + 33576r^5 + 168107r^4 + 308308r^3 + 239784r^2 + 45745r - 18960}{120\left(4r^2 + 37r + 84\right)\left(64r^5 + 624r^4 + 2332r^3 + 4173r^2 + 3571r + 1170\right)\left(4r^2 + 29r + 51\right)}$$

Which for the root $r = 1$ becomes

$$a_5 = \frac{16043}{313267500}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{4r+5}$	$-\frac{1}{9}$
a_2	$\frac{-4r^2-3r+2}{32r^3+144r^2+202r+90}$	$-\frac{5}{468}$
a_3	$\frac{-16r^4-48r^3-17r^2+33r+15}{384r^5+3744r^4+13992r^3+25038r^2+21426r+7020}$	$-\frac{11}{23868}$
a_4	$\frac{-64r^6-400r^5-572r^4+773r^3+2385r^2+1538r+132}{24(64r^5+624r^4+2332r^3+4173r^2+3571r+1170)(4r^2+29r+51)}$	$\frac{79}{501228}$
a_5	$\frac{-256r^8-2432r^7-3808r^6+33576r^5+168107r^4+308308r^3+239784r^2+45745r-18960}{120(4r^2+37r+84)(64r^5+624r^4+2332r^3+4173r^2+3571r+1170)(4r^2+29r+51)}$	$\frac{16043}{313267500}$

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= x\left(1 - \frac{x}{9} - \frac{5x^2}{468} - \frac{11x^3}{23868} + \frac{79x^4}{501228} + \frac{16043x^5}{313267500} + O\left(x^6\right)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$
b_1 = -\frac{1}{4r+5}
$$

Substituting $n = 2$ in Eq. (2B) gives

$$
b_2 = \frac{-4r^2 - 3r + 2}{32r^3 + 144r^2 + 202r + 90}
$$

Substituting $n = 3$ in Eq. (2B) gives

$$
b_3 = \frac{-16r^4 - 48r^3 - 17r^2 + 33r + 15}{384r^5 + 3744r^4 + 13992r^3 + 25038r^2 + 21426r + 7020}
$$

Substituting $n = 4$ in Eq. (2B) gives

$$
b_4 = \frac{-64r^6 - 400r^5 - 572r^4 + 773r^3 + 2385r^2 + 1538r + 132}{24\left(64r^5 + 624r^4 + 2332r^3 + 4173r^2 + 3571r + 1170\right)\left(4r^2 + 29r + 51\right)}
$$

For $5 \leq n$ the recursive equation is

$$
\begin{aligned}
&4b_n(n+r)(n+r-1) + \frac{b_{n-5}(n-5+r)}{120} + \frac{b_{n-4}(n-4+r)}{24} \\
&+ \frac{b_{n-3}(n-3+r)}{6} + \frac{b_{n-2}(n+r-2)}{2} + b_{n-1}(n+r-1) + b_n(n+r) - b_n = 0
\end{aligned}
\tag{3}
$$

Solving for b_n from recursive equation (4) gives

$$
b_n = -\frac{nb_{n-5} + 5nb_{n-4} + 20nb_{n-3} + 60nb_{n-2} + 120nb_{n-1} + rb_{n-5} + 5rb_{n-4} + 20rb_{n-3} + 60rb_{n-2} + 120rb_{n}}{120\left(4n^2 + 8nr + 4r^2 - 3n - 3r - 1\right)}
\tag{4}
$$

Which for the root $r = -\frac{1}{4}$ becomes

$$
b_n = \frac{\left(-4b_{n-5} - 20b_{n-4} - 80b_{n-3} - 240b_{n-2} - 480b_{n-1}\right)n + 21b_{n-5} + 85b_{n-4} + 260b_{n-3} + 540b_{n-2} + 600b_{n}}{1920n^2 - 2400n}
\tag{5}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\frac{1}{4}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{4r+5}$	$-\frac{1}{4}$
b_2	$\frac{-4r^2-3r+2}{32r^3+144r^2+202r+90}$	$\frac{5}{96}$
b_3	$\frac{-16r^4-48r^3-17r^2+33r+15}{384r^5+3744r^4+13992r^3+25038r^2+21426r+7020}$	$\frac{17}{8064}$
b_4	$\frac{-64r^6-400r^5-572r^4+773r^3+2385r^2+1538r+132}{24(64r^5+624r^4+2332r^3+4173r^2+3571r+1170)(4r^2+29r+51)}$	$-\frac{313}{1419264}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{-256r^8 - 2432r^7 - 3808r^6 + 33576r^5 + 168107r^4 + 308308r^3 + 239784r^2 + 45745r - 18960}{120\,(4r^2 + 37r + 84)\,(64r^5 + 624r^4 + 2332r^3 + 4173r^2 + 3571r + 1170)\,(4r^2 + 29r + 51)}$$

Which for the root $r = -\frac{1}{4}$ becomes

$$b_5 = -\frac{69703}{709632000}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{4r+5}$	$-\frac{1}{4}$
b_2	$\frac{-4r^2-3r+2}{32r^3+144r^2+202r+90}$	$\frac{5}{96}$
b_3	$\frac{-16r^4-48r^3-17r^2+33r+15}{384r^5+3744r^4+13992r^3+25038r^2+21426r+7020}$	$\frac{17}{8064}$
b_4	$\frac{-64r^6-400r^5-572r^4+773r^3+2385r^2+1538r+132}{24(64r^5+624r^4+2332r^3+4173r^2+3571r+1170)(4r^2+29r+51)}$	$-\frac{313}{1419264}$
b_5	$\frac{-256r^8-2432r^7-3808r^6+33576r^5+168107r^4+308308r^3+239784r^2+45745r-18960}{120(4r^2+37r+84)(64r^5+624r^4+2332r^3+4173r^2+3571r+1170)(4r^2+29r+51)}$	$-\frac{69703}{709632000}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= \frac{1 - \frac{x}{4} + \frac{5x^2}{96} + \frac{17x^3}{8064} - \frac{313x^4}{1419264} - \frac{69703x^5}{709632000} + O(x^6)}{x^{\frac{1}{4}}} \end{aligned}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x \left(1-\frac{x}{9}-\frac{5 x^{2}}{468}-\frac{11 x^{3}}{23868}+\frac{79 x^{4}}{501228}+\frac{16043 x^{5}}{313267500}+O\left(x^{6}\right)\right) \\
&\quad + \frac{c_2 \left(1-\frac{x}{4}+\frac{5 x^{2}}{96}+\frac{17 x^{3}}{8064}-\frac{313 x^{4}}{1419264}-\frac{69703 x^{5}}{709632000}+O\left(x^{6}\right)\right)}{x^{\frac{1}{4}}}
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x \left(1-\frac{x}{9}-\frac{5 x^{2}}{468}-\frac{11 x^{3}}{23868}+\frac{79 x^{4}}{501228}+\frac{16043 x^{5}}{313267500}+O\left(x^{6}\right)\right) \\
&\quad + \frac{c_2 \left(1-\frac{x}{4}+\frac{5 x^{2}}{96}+\frac{17 x^{3}}{8064}-\frac{313 x^{4}}{1419264}-\frac{69703 x^{5}}{709632000}+O\left(x^{6}\right)\right)}{x^{\frac{1}{4}}}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x \left(1-\frac{x}{9}-\frac{5 x^{2}}{468}-\frac{11 x^{3}}{23868}+\frac{79 x^{4}}{501228}+\frac{16043 x^{5}}{313267500}+O\left(x^{6}\right)\right) \\
&\quad + \frac{c_2 \left(1-\frac{x}{4}+\frac{5 x^{2}}{96}+\frac{17 x^{3}}{8064}-\frac{313 x^{4}}{1419264}-\frac{69703 x^{5}}{709632000}+O\left(x^{6}\right)\right)}{x^{\frac{1}{4}}}
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1 x \left(1-\frac{x}{9}-\frac{5 x^{2}}{468}-\frac{11 x^{3}}{23868}+\frac{79 x^{4}}{501228}+\frac{16043 x^{5}}{313267500}+O\left(x^{6}\right)\right) \\
&\quad + \frac{c_2 \left(1-\frac{x}{4}+\frac{5 x^{2}}{96}+\frac{17 x^{3}}{8064}-\frac{313 x^{4}}{1419264}-\frac{69703 x^{5}}{709632000}+O\left(x^{6}\right)\right)}{x^{\frac{1}{4}}}
\end{aligned}
$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
      trying a symmetry of the form [xi=0, eta=F(x)]
      trying 2nd order exact linear
      trying symmetries linear in x and y(x)
      trying to convert to a linear ODE with constant coefficients
<- unable to find a useful change of variables
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying differential order: 2; exact nonlinear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
   -> Trying changes of variables to rationalize or make the ODE simpler
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
   <- unable to find a useful change of variables
      trying a symmetry of the form [xi=0, eta=F(x)]
   trying to convert to an ODE of Bessel type
   -> trying reduction of order to Riccati
      trying Riccati sub-methods:
         trying Riccati_symmetries
         -> trying a symmetry pattern of the form [F(x)*G(y), 0]
         -> trying a symmetry pattern of the form [0, F(x)*G(y)]
         -> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
--- Trying Lie symmetry methods, 2nd order ---
`, `-> Computing symmetries using: way = 3`[0, y]
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 45

```
Order:=6;
dsolve(4*x^2*diff(y(x),x$2)+x*exp(x)*diff(y(x),x)-y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_1\left(1-\frac{1}{4}x+\frac{5}{96}x^2+\frac{17}{8064}x^3-\frac{313}{1419264}x^4-\frac{69703}{709632000}x^5+\mathrm{O}\left(x^6\right)\right)}{x^{\frac{1}{4}}} + c_2 x\left(1-\frac{1}{9}x-\frac{5}{468}x^2-\frac{11}{23868}x^3+\frac{79}{501228}x^4+\frac{16043}{313267500}x^5+\mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 86

```
AsymptoticDSolveValue[4*x^2*y''[x]+x*Exp[x]*y'[x]-y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 x\left(\frac{16043x^5}{313267500}+\frac{79x^4}{501228}-\frac{11x^3}{23868}-\frac{5x^2}{468}-\frac{x}{9}+1\right) + \frac{c_2\left(-\frac{69703x^5}{709632000}-\frac{313x^4}{1419264}+\frac{17x^3}{8064}+\frac{5x^2}{96}-\frac{x}{4}+1\right)}{\sqrt[4]{x}}$$

17.7 problem 8

Internal problem ID [2923]
Internal file name [`OUTPUT/2415_Sunday_June_05_2022_03_07_18_AM_7303244/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.4. page 758
Problem number: 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4xy'' - xy' + 2y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$4xy'' - xy' + 2y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{1}{4}$$
$$q(x) = \frac{1}{2x}$$

Table 426: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{1}{4}$	
singularity	type

$q(x) = \frac{1}{2x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$4xy'' - xy' + 2y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$4\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x - x\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + 2\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 4x^{n+r-1} a_n (n+r)\,(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 2a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-a_{n-1} (n+r-1)\, x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty} 2 a_n x^{n+r} = \sum_{n=1}^{\infty} 2 a_{n-1} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\begin{aligned} &\left(\sum_{n=0}^{\infty} 4 x^{n+r-1} a_n (n+r)\,(n+r-1)\right) \\ &+ \sum_{n=1}^{\infty} \left(-a_{n-1} (n+r-1)\, x^{n+r-1}\right) + \left(\sum_{n=1}^{\infty} 2 a_{n-1} x^{n+r-1}\right) = 0 \end{aligned} \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$4 x^{n+r-1} a_n (n+r)\,(n+r-1) = 0$$

When $n=0$ the above becomes

$$4 x^{-1+r} a_0 r(-1+r) = 0$$

Or

$$4 x^{-1+r} a_0 r(-1+r) = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$4 x^{-1+r} r(-1+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$4 r(-1+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$4x^{-1+r}r(-1+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$4a_n(n+r)(n+r-1) - a_{n-1}(n+r-1) + 2a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}(n+r-3)}{4(n+r)(n+r-1)} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = \frac{a_{n-1}(n-2)}{4(n+1)n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-2 + r}{4 (1 + r) r}$$

Which for the root $r = 1$ becomes

$$a_1 = -\frac{1}{8}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2+r}{4(1+r)r}$	$-\frac{1}{8}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{(-1 + r) (-2 + r)}{16 (1 + r)^2 r (2 + r)}$$

Which for the root $r = 1$ becomes

$$a_2 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2+r}{4(1+r)r}$	$-\frac{1}{8}$
a_2	$\frac{(-1+r)(-2+r)}{16(1+r)^2r(2+r)}$	0

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{(-1 + r) (-2 + r)}{64 (1 + r)^2 (2 + r)^2 (3 + r)}$$

Which for the root $r = 1$ becomes

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2+r}{4(1+r)r}$	$-\frac{1}{8}$
a_2	$\frac{(-1+r)(-2+r)}{16(1+r)^2r(2+r)}$	0
a_3	$\frac{(-1+r)(-2+r)}{64(1+r)^2(2+r)^2(3+r)}$	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{(-1+r)(-2+r)}{256(1+r)(2+r)^2(3+r)^2(4+r)}$$

Which for the root $r = 1$ becomes

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2+r}{4(1+r)r}$	$-\frac{1}{8}$
a_2	$\frac{(-1+r)(-2+r)}{16(1+r)^2r(2+r)}$	0
a_3	$\frac{(-1+r)(-2+r)}{64(1+r)^2(2+r)^2(3+r)}$	0
a_4	$\frac{(-1+r)(-2+r)}{256(1+r)(2+r)^2(3+r)^2(4+r)}$	0

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{(-1+r)(-2+r)}{1024(1+r)(2+r)(3+r)^2(4+r)^2(5+r)}$$

Which for the root $r = 1$ becomes

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2+r}{4(1+r)r}$	$-\frac{1}{8}$
a_2	$\frac{(-1+r)(-2+r)}{16(1+r)^2r(2+r)}$	0
a_3	$\frac{(-1+r)(-2+r)}{64(1+r)^2(2+r)^2(3+r)}$	0
a_4	$\frac{(-1+r)(-2+r)}{256(1+r)(2+r)^2(3+r)^2(4+r)}$	0
a_5	$\frac{(-1+r)(-2+r)}{1024(1+r)(2+r)(3+r)^2(4+r)^2(5+r)}$	0

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots\right) \\
&= x\left(1 - \frac{x}{8} + O(x^6)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Let

$$
r_1 - r_2 = N
$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$
\begin{aligned}
a_N &= a_1 \\
&= \frac{-2+r}{4(1+r)r}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\lim_{r \to r_2} \frac{-2+r}{4(1+r)r} &= \lim_{r \to 0} \frac{-2+r}{4(1+r)r} \\
&= \textit{undefined}
\end{aligned}
$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$
y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)
$$

Therefore

$$
\begin{aligned}
\frac{d}{dx}y_2(x) &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\
&= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right) \\
\frac{d^2}{dx^2}y_2(x) &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\
&= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right)
\end{aligned}
$$

Substituting these back into the given ode $4xy'' - xy' + 2y = 0$ gives

$$
\begin{aligned}
&4\Bigg(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\Bigg) x \\
&\quad - x\left(Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right) \\
&\quad + 2Cy_1(x)\ln(x) + 2\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned}
$$

Which can be written as

$$
\begin{aligned}
&\left(\left(-y_1'(x)\,x + 4y_1''(x)\,x + 2y_1(x)\right)\ln(x) + 4\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right)x - y_1(x)\right)C \\
&\quad + 4\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x \\
&\quad - x\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right) + 2\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned} \tag{7}
$$

But since $y_1(x)$ is a solution to the ode, then

$$-y_1'(x)\, x + 4y_1''(x)\, x + 2y_1(x) = 0$$

Eq (7) simplifes to

$$\begin{aligned} &\left(4\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right)x - y_1(x)\right)C \\ &+ 4\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x \\ &- x\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right) + 2\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0 \end{aligned} \tag{8}$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\begin{aligned} &\frac{\left(8\left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n (n+r_1)\right)x - (x+4)\left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right)\right)C}{x} \\ &+ \frac{-\left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right)x^2 + 4\left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right)x^2 + 2\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)x}{x} \\ &= 0 \end{aligned} \tag{9}$$

Since $r_1 = 1$ and $r_2 = 0$ then the above becomes

$$\begin{aligned} &\frac{\left(8\left(\sum_{n=0}^{\infty} x^n a_n (n+1)\right)x - (x+4)\left(\sum_{n=0}^{\infty} a_n x^{n+1}\right)\right)C}{x} \\ &+ \frac{-\left(\sum_{n=0}^{\infty} x^{n-1} b_n n\right)x^2 + 4\left(\sum_{n=0}^{\infty} x^{n-2} b_n n(n-1)\right)x^2 + 2\left(\sum_{n=0}^{\infty} b_n x^n\right)x}{x} = 0 \end{aligned} \tag{10}$$

Which simplifies to

$$\begin{aligned} &\left(\sum_{n=0}^{\infty} 8C\, x^n a_n (n+1)\right) + \sum_{n=0}^{\infty}\left(-C\, x^{n+1} a_n\right) + \sum_{n=0}^{\infty}\left(-4C a_n x^n\right) \\ &+ \sum_{n=0}^{\infty}\left(-x^n b_n n\right) + \left(\sum_{n=0}^{\infty} 4n\, x^{n-1} b_n (n-1)\right) + \left(\sum_{n=0}^{\infty} 2b_n x^n\right) = 0 \end{aligned} \tag{2A}$$

The next step is to make all powers of x be $n-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-1} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} 8C\,x^{n} a_{n} (n+1) &= \sum_{n=1}^{\infty} 8C a_{n-1} n\,x^{n-1} \\
\sum_{n=0}^{\infty} \left(-C\,x^{n+1} a_{n}\right) &= \sum_{n=2}^{\infty} \left(-C a_{n-2} x^{n-1}\right) \\
\sum_{n=0}^{\infty} \left(-4C a_{n} x^{n}\right) &= \sum_{n=1}^{\infty} \left(-4C a_{n-1} x^{n-1}\right) \\
\sum_{n=0}^{\infty} \left(-x^{n} b_{n} n\right) &= \sum_{n=1}^{\infty} \left(-(n-1)\, b_{n-1} x^{n-1}\right) \\
\sum_{n=0}^{\infty} 2 b_{n} x^{n} &= \sum_{n=1}^{\infty} 2 b_{n-1} x^{n-1}
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-1$.

$$
\begin{aligned}
&\left(\sum_{n=1}^{\infty} 8C a_{n-1} n\,x^{n-1}\right) + \sum_{n=2}^{\infty} \left(-C a_{n-2} x^{n-1}\right) \\
&+ \sum_{n=1}^{\infty} \left(-4C a_{n-1} x^{n-1}\right) + \sum_{n=1}^{\infty} \left(-(n-1)\, b_{n-1} x^{n-1}\right) \\
&+ \left(\sum_{n=0}^{\infty} 4n\,x^{n-1} b_{n} (n-1)\right) + \left(\sum_{n=1}^{\infty} 2 b_{n-1} x^{n-1}\right) = 0
\end{aligned} \tag{2B}
$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0 = 1$. For $n = N$, where $N = 1$ which is the difference between the two roots, we are free to choose $b_1 = 0$. Hence for $n = 1$, Eq (2B) gives

$$
4C + 2 = 0
$$

Which is solved for C. Solving for C gives

$$
C = -\frac{1}{2}
$$

For $n = 2$, Eq (2B) gives

$$(-a_0 + 12a_1) C + b_1 + 8b_2 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{5}{4} + 8b_2 = 0$$

Solving the above for b_2 gives

$$b_2 = -\frac{5}{32}$$

For $n = 3$, Eq (2B) gives

$$(-a_1 + 20a_2) C + 24b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-\frac{1}{16} + 24b_3 = 0$$

Solving the above for b_3 gives

$$b_3 = \frac{1}{384}$$

For $n = 4$, Eq (2B) gives

$$(-a_2 + 28a_3) C - b_3 + 48b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-\frac{1}{384} + 48b_4 = 0$$

Solving the above for b_4 gives

$$b_4 = \frac{1}{18432}$$

For $n = 5$, Eq (2B) gives

$$(-a_3 + 36a_4) C - 2b_4 + 80b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-\frac{1}{9216} + 80b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = \frac{1}{737280}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -\frac{1}{2}$ and all b_n, then the second solution becomes

$$y_2(x) = -\frac{1}{2}\left(x\left(1-\frac{x}{8}+O(x^6)\right)\right)\ln(x) + 1 - \frac{5x^2}{32} + \frac{x^3}{384} + \frac{x^4}{18432} + \frac{x^5}{737280} + O(x^6)$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x\left(1-\frac{x}{8}+O(x^6)\right) \\ &\quad + c_2\left(-\frac{1}{2}\left(x\left(1-\frac{x}{8}+O(x^6)\right)\right)\ln(x) + 1 - \frac{5x^2}{32} + \frac{x^3}{384} + \frac{x^4}{18432} + \frac{x^5}{737280} + O(x^6)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x\left(1-\frac{x}{8}+O(x^6)\right) \\ &\quad + c_2\left(-\frac{x\left(1-\frac{x}{8}+O(x^6)\right)\ln(x)}{2} + 1 - \frac{5x^2}{32} + \frac{x^3}{384} + \frac{x^4}{18432} + \frac{x^5}{737280} + O(x^6)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y &= c_1 x\left(1-\frac{x}{8}+O(x^6)\right) \\ &\quad + c_2\left(-\frac{x\left(1-\frac{x}{8}+O(x^6)\right)\ln(x)}{2} + 1 - \frac{5x^2}{32} + \frac{x^3}{384} + \frac{x^4}{18432} + \frac{x^5}{737280} + O(x^6)\right) \end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned} y &= c_1 x\left(1-\frac{x}{8}+O(x^6)\right) \\ &\quad + c_2\left(-\frac{x\left(1-\frac{x}{8}+O(x^6)\right)\ln(x)}{2} + 1 - \frac{5x^2}{32} + \frac{x^3}{384} + \frac{x^4}{18432} + \frac{x^5}{737280} + O(x^6)\right) \end{aligned}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 46

```
Order:=6;
dsolve(4*x*diff(y(x),x$2)-x*diff(y(x),x)+2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \ln(x)\left(-\frac{1}{2}x + \frac{1}{16}x^2 + \mathrm{O}\left(x^6\right)\right)c_2 + c_1 x\left(1 - \frac{1}{8}x + \mathrm{O}\left(x^6\right)\right) + \left(1 + \frac{1}{4}x - \frac{3}{16}x^2 + \frac{1}{384}x^3 + \frac{1}{18432}x^4 + \frac{1}{737280}x^5 + \mathrm{O}\left(x^6\right)\right)c_2$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 52

```
AsymptoticDSolveValue[4*x*y''[x]-x*y'[x]+2*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(x - \frac{x^2}{8}\right) + c_1\left(\frac{x^4 + 48x^3 - 4608x^2 + 13824x + 18432}{18432} + \frac{1}{16}(x-8)x\log(x)\right)$$

17.8 problem 9

Internal problem ID [2924]
Internal file name [`OUTPUT/2416_Sunday_June_05_2022_03_07_24_AM_191559/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.4. page 758
Problem number: 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Complex roots"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - x\cos(x)\, y' + 5y\,\mathrm{e}^{2x} = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' - x\cos(x)\, y' + 5y\,\mathrm{e}^{2x} = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{\cos(x)}{x}$$
$$q(x) = \frac{5\,\mathrm{e}^{2x}}{x^2}$$

Table 427: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{\cos(x)}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{5\,e^{2x}}{x^2}$	
singularity	type
$x = 0$	"regular"
$x = \infty$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2 y'' - x\cos(x)\, y' + 5y\,e^{2x} = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) - x\cos(x)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + 5\,e^{2x}\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Expanding $-\cos(x)\,x$ as Taylor series around $x=0$ and keeping only the first 6 terms gives

$$
\begin{aligned}
-\cos(x)\,x &= -x + \frac{1}{2}x^3 - \frac{1}{24}x^5 + \frac{1}{720}x^7 + \dots \\
&= -x + \frac{1}{2}x^3 - \frac{1}{24}x^5 + \frac{1}{720}x^7
\end{aligned}
$$

Expanding $5\,\mathrm{e}^{2x}$ as Taylor series around $x=0$ and keeping only the first 6 terms gives

$$
\begin{aligned}
5\,\mathrm{e}^{2x} &= 5 + 10x + 10x^2 + \frac{20}{3}x^3 + \frac{10}{3}x^4 + \frac{4}{3}x^5 + \frac{4}{9}x^6 + \dots \\
&= 5 + 10x + 10x^2 + \frac{20}{3}x^3 + \frac{10}{3}x^4 + \frac{4}{3}x^5 + \frac{4}{9}x^6
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} \frac{x^{n+r+6} a_n (n+r)}{720}\right) \\
&+ \sum_{n=0}^{\infty} \left(-\frac{x^{n+r+4} a_n (n+r)}{24}\right) + \left(\sum_{n=0}^{\infty} \frac{x^{n+r+2} a_n (n+r)}{2}\right) \\
&\quad + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 5 a_n x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 10 x^{1+n+r} a_n\right) \\
&\quad + \left(\sum_{n=0}^{\infty} 10 x^{n+r+2} a_n\right) + \left(\sum_{n=0}^{\infty} \frac{20 x^{n+r+3} a_n}{3}\right) + \left(\sum_{n=0}^{\infty} \frac{10 x^{n+r+4} a_n}{3}\right) \\
&\quad + \left(\sum_{n=0}^{\infty} \frac{4 x^{n+r+5} a_n}{3}\right) + \left(\sum_{n=0}^{\infty} \frac{4 x^{n+r+6} a_n}{9}\right) = 0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\sum_{n=0}^{\infty} \frac{x^{n+r+6} a_n (n+r)}{720} = \sum_{n=6}^{\infty} \frac{a_{n-6} (n-6+r)\, x^{n+r}}{720}
$$

$$
\sum_{n=0}^{\infty} \left(-\frac{x^{n+r+4} a_n (n+r)}{24}\right) = \sum_{n=4}^{\infty} \left(-\frac{a_{n-4} (n-4+r)\, x^{n+r}}{24}\right)
$$

$$
\sum_{n=0}^{\infty} \frac{x^{n+r+2} a_n (n+r)}{2} = \sum_{n=2}^{\infty} \frac{a_{n-2} (n+r-2)\, x^{n+r}}{2}
$$

$$\sum_{n=0}^{\infty} 10x^{1+n+r} a_n = \sum_{n=1}^{\infty} 10a_{n-1} x^{n+r}$$

$$\sum_{n=0}^{\infty} 10x^{n+r+2} a_n = \sum_{n=2}^{\infty} 10a_{n-2} x^{n+r}$$

$$\sum_{n=0}^{\infty} \frac{20x^{n+r+3} a_n}{3} = \sum_{n=3}^{\infty} \frac{20a_{n-3} x^{n+r}}{3}$$

$$\sum_{n=0}^{\infty} \frac{10x^{n+r+4} a_n}{3} = \sum_{n=4}^{\infty} \frac{10a_{n-4} x^{n+r}}{3}$$

$$\sum_{n=0}^{\infty} \frac{4x^{n+r+5} a_n}{3} = \sum_{n=5}^{\infty} \frac{4a_{n-5} x^{n+r}}{3}$$

$$\sum_{n=0}^{\infty} \frac{4x^{n+r+6} a_n}{9} = \sum_{n=6}^{\infty} \frac{4a_{n-6} x^{n+r}}{9}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\begin{aligned}
&\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=6}^{\infty} \frac{a_{n-6}(n-6+r)\, x^{n+r}}{720}\right) \\
&+ \sum_{n=4}^{\infty} \left(-\frac{a_{n-4}(n-4+r)\, x^{n+r}}{24}\right) + \left(\sum_{n=2}^{\infty} \frac{a_{n-2}(n+r-2)\, x^{n+r}}{2}\right) \\
&+ \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 5a_n x^{n+r}\right) + \left(\sum_{n=1}^{\infty} 10a_{n-1} x^{n+r}\right) \\
&+ \left(\sum_{n=2}^{\infty} 10a_{n-2} x^{n+r}\right) + \left(\sum_{n=3}^{\infty} \frac{20a_{n-3} x^{n+r}}{3}\right) + \left(\sum_{n=4}^{\infty} \frac{10a_{n-4} x^{n+r}}{3}\right) \\
&+ \left(\sum_{n=5}^{\infty} \frac{4a_{n-5} x^{n+r}}{3}\right) + \left(\sum_{n=6}^{\infty} \frac{4a_{n-6} x^{n+r}}{9}\right) = 0
\end{aligned} \tag{2B}$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) - x^{n+r} a_n (n+r) + 5a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$x^r a_0 r(-1+r) - x^r a_0 r + 5a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r)-x^r r+5x^r\right)a_0=0$$

Since $a_0\neq 0$ then the above simplifies to

$$\left(r^2-2r+5\right)x^r=0$$

Since the above is true for all x then the indicial equation becomes

$$r^2-2r+5=0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned}r_1&=1+2i\\ r_2&=1-2i\end{aligned}$$

Since $a_0\neq 0$ then the indicial equation becomes

$$\left(r^2-2r+5\right)x^r=0$$

Solving for r gives the roots of the indicial equation as Since the roots are complex conjugates, then two linearly independent solutions can be constructed using

$$\begin{aligned}y_1(x)&=x^{r_1}\left(\sum_{n=0}^{\infty}a_n x^n\right)\\ y_2(x)&=x^{r_2}\left(\sum_{n=0}^{\infty}b_n x^n\right)\end{aligned}$$

Or

$$\begin{aligned}y_1(x)&=\sum_{n=0}^{\infty}a_n x^{n+1+2i}\\ y_2(x)&=\sum_{n=0}^{\infty}b_n x^{n+1-2i}\end{aligned}$$

$y_1(x)$ is found first. Eq (2B) derived above is now used to find all a_n coefficients. The case $n=0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0=1$. Substituting $n=1$ in Eq. (2B) gives

$$a_1=-\frac{10}{r^2+4}$$

Substituting $n=2$ in Eq. (2B) gives

$$a_2=\frac{-r^3-20r^2-4r+120}{2\left(r^2+4\right)\left(r^2+2r+5\right)}$$

Substituting $n = 3$ in Eq. (2B) gives

$$a_3 = \frac{-\frac{10}{3}r^3 + 155r^2 + \frac{605}{3}r - \frac{20}{3}r^4 - \frac{625}{3}}{(r^2+4)(r^2+2r+5)(r^2+4r+8)}$$

Substituting $n = 4$ in Eq. (2B) gives

$$a_4 = \frac{r^7 - 68r^6 - 179r^5 + 4980r^4 + 18204r^3 - 8672r^2 - 67056r - 25520}{24(r^2+4)(r^2+2r+5)(r^2+4r+8)(r^2+6r+13)}$$

Substituting $n = 5$ in Eq. (2B) gives

$$a_5 = \frac{-16r^8 - 122r^7 + 1871r^6 + 16327r^5 + 12206r^4 - 150483r^3 - 334731r^2 - 69372r + 174270}{12(r^2+4)(r^2+2r+5)(r^2+4r+8)(r^2+6r+13)(r^2+8r+20)}$$

For $6 \leq n$ the recursive equation is

$$\begin{aligned} a_n(n+r)(n+r-1) + \frac{a_{n-6}(n-6+r)}{720} - \frac{a_{n-4}(n-4+r)}{24} + \frac{a_{n-2}(n+r-2)}{2} \\ - a_n(n+r) + 5a_n + 10a_{n-1} + 10a_{n-2} + \frac{20a_{n-3}}{3} + \frac{10a_{n-4}}{3} + \frac{4a_{n-5}}{3} + \frac{4a_{n-6}}{9} = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{na_{n-6} - 30na_{n-4} + 360na_{n-2} + ra_{n-6} - 30ra_{n-4} + 360ra_{n-2} + 314a_{n-6} + 960a_{n-5} + 2520a_{n-4} +}{720(n^2 + 2nr + r^2 - 2n - 2r + 5)} \tag{4}$$

Which for the root $r = 1 + 2i$ becomes

$$a_n = \frac{(-a_{n-6} + 30a_{n-4} - 360a_{n-2})\, n + (-315-2i)\, a_{n-6} + (-2490+60i)\, a_{n-4} + (-6840-720i)\, a_{n-2} -}{720n(n+4i)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1 + 2i$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{10}{r^2+4}$	$-\frac{10}{17} + \frac{40i}{17}$
a_2	$\frac{-r^3-20r^2-4r+120}{2(r^2+4)(r^2+2r+5)}$	$-\frac{365}{136} - \frac{13i}{17}$
a_3	$\frac{-\frac{10}{3}r^3+155r^2+\frac{605}{3}r-\frac{20}{3}r^4-\frac{625}{3}}{(r^2+4)(r^2+2r+5)(r^2+4r+8)}$	$\frac{223}{1020} - \frac{1723i}{765}$
a_4	$\frac{r^7-68r^6-179r^5+4980r^4+18204r^3-8672r^2-67056r-25520}{24(r^2+4)(r^2+2r+5)(r^2+4r+8)(r^2+6r+13)}$	$\frac{114911}{78336} - \frac{24835i}{78336}$
a_5	$\frac{-16r^8-122r^7+1871r^6+16327r^5+12206r^4-150483r^3-334731r^2-69372r+174270}{12(r^2+4)(r^2+2r+5)(r^2+4r+8)(r^2+6r+13)(r^2+8r+20)}$	$\frac{4041077}{8029440} + \frac{1112267i}{1605888}$

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x^{1+2i}(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots) \\
&= x^{1+2i}\left(1 + \left(-\frac{10}{17} + \frac{40i}{17}\right)x + \left(-\frac{365}{136} - \frac{13i}{17}\right)x^2 + \left(\frac{223}{1020} - \frac{1723i}{765}\right)x^3 \right. \\
&\left. + \left(\frac{114911}{78336} - \frac{24835i}{78336}\right)x^4 + \left(\frac{4041077}{8029440} + \frac{1112267i}{1605888}\right)x^5 + O(x^6)\right)
\end{aligned}
$$

The second solution $y_2(x)$ is found by taking the complex conjugate of $y_1(x)$ which gives

$$
\begin{aligned}
y_2(x) = x^{1-2i}&\left(1 + \left(-\frac{10}{17} - \frac{40i}{17}\right)x + \left(-\frac{365}{136} + \frac{13i}{17}\right)x^2 + \left(\frac{223}{1020} + \frac{1723i}{765}\right)x^3 \right. \\
&\left. + \left(\frac{114911}{78336} + \frac{24835i}{78336}\right)x^4 + \left(\frac{4041077}{8029440} - \frac{1112267i}{1605888}\right)x^5 + O(x^6)\right)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^{1+2i}\left(1 + \left(-\frac{10}{17} + \frac{40i}{17}\right)x + \left(-\frac{365}{136} - \frac{13i}{17}\right)x^2 + \left(\frac{223}{1020} - \frac{1723i}{765}\right)x^3 \right. \\
&\left. + \left(\frac{114911}{78336} - \frac{24835i}{78336}\right)x^4 + \left(\frac{4041077}{8029440} + \frac{1112267i}{1605888}\right)x^5 + O(x^6)\right) \\
&+ c_2 x^{1-2i}\left(1 + \left(-\frac{10}{17} - \frac{40i}{17}\right)x + \left(-\frac{365}{136} + \frac{13i}{17}\right)x^2 + \left(\frac{223}{1020} + \frac{1723i}{765}\right)x^3 \right. \\
&\left. + \left(\frac{114911}{78336} + \frac{24835i}{78336}\right)x^4 + \left(\frac{4041077}{8029440} - \frac{1112267i}{1605888}\right)x^5 + O(x^6)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^{1+2i}\left(1 + \left(-\frac{10}{17} + \frac{40i}{17}\right)x + \left(-\frac{365}{136} - \frac{13i}{17}\right)x^2 + \left(\frac{223}{1020} - \frac{1723i}{765}\right)x^3 \right. \\
&\left. + \left(\frac{114911}{78336} - \frac{24835i}{78336}\right)x^4 + \left(\frac{4041077}{8029440} + \frac{1112267i}{1605888}\right)x^5 + O(x^6)\right) \\
&+ c_2 x^{1-2i}\left(1 + \left(-\frac{10}{17} - \frac{40i}{17}\right)x + \left(-\frac{365}{136} + \frac{13i}{17}\right)x^2 + \left(\frac{223}{1020} + \frac{1723i}{765}\right)x^3 \right. \\
&\left. + \left(\frac{114911}{78336} + \frac{24835i}{78336}\right)x^4 + \left(\frac{4041077}{8029440} - \frac{1112267i}{1605888}\right)x^5 + O(x^6)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & c_1 x^{1+2i} \left(1 + \left(-\frac{10}{17} + \frac{40i}{17}\right) x + \left(-\frac{365}{136} - \frac{13i}{17}\right) x^2 + \left(\frac{223}{1020} - \frac{1723i}{765}\right) x^3 \right. \\
& \left. + \left(\frac{114911}{78336} - \frac{24835i}{78336}\right) x^4 + \left(\frac{4041077}{8029440} + \frac{1112267i}{1605888}\right) x^5 + O\left(x^6\right)\right) \\
& + c_2 x^{1-2i} \left(1 + \left(-\frac{10}{17} - \frac{40i}{17}\right) x + \left(-\frac{365}{136} + \frac{13i}{17}\right) x^2 + \left(\frac{223}{1020} + \frac{1723i}{765}\right) x^3 \right. \\
& \left. + \left(\frac{114911}{78336} + \frac{24835i}{78336}\right) x^4 + \left(\frac{4041077}{8029440} - \frac{1112267i}{1605888}\right) x^5 + O\left(x^6\right)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = {} & c_1 x^{1+2i} \left(1 + \left(-\frac{10}{17} + \frac{40i}{17}\right) x + \left(-\frac{365}{136} - \frac{13i}{17}\right) x^2 + \left(\frac{223}{1020} - \frac{1723i}{765}\right) x^3 \right. \\
& \left. + \left(\frac{114911}{78336} - \frac{24835i}{78336}\right) x^4 + \left(\frac{4041077}{8029440} + \frac{1112267i}{1605888}\right) x^5 + O\left(x^6\right)\right) \\
& + c_2 x^{1-2i} \left(1 + \left(-\frac{10}{17} - \frac{40i}{17}\right) x + \left(-\frac{365}{136} + \frac{13i}{17}\right) x^2 + \left(\frac{223}{1020} + \frac{1723i}{765}\right) x^3 \right. \\
& \left. + \left(\frac{114911}{78336} + \frac{24835i}{78336}\right) x^4 + \left(\frac{4041077}{8029440} - \frac{1112267i}{1605888}\right) x^5 + O\left(x^6\right)\right)
\end{aligned}
$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
      trying a symmetry of the form [xi=0, eta=F(x)]
      trying 2nd order exact linear
      trying symmetries linear in x and y(x)
      trying to convert to a linear ODE with constant coefficients
      -> trying with_periodic_functions in the coefficients
         --- Trying Lie symmetry methods, 2nd order ---
         `, `-> Computing symmetries using: way = 5
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
      trying a symmetry of the form [xi=0, eta=F(x)]
      trying 2nd order exact linear
      trying symmetries linear in x and y(x)
      trying to convert to a linear ODE with constant coefficients
      -> trying with_periodic_functions in the coefficients
         --- Trying Lie symmetry methods, 2nd order ---
         `, `-> Computing symmetries using: way = 5`[0, u]
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 71

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)-x*cos(x)*diff(y(x),x)+5*exp(2*x)*y(x)=0,y(x),type='series',x=0);
```

$$
\begin{aligned}
y(x) = {} & c_1 x^{1-2i}\left(1+\left(-\frac{10}{17}-\frac{40i}{17}\right)x+\left(-\frac{365}{136}+\frac{13i}{17}\right)x^2+\left(\frac{223}{1020}+\frac{1723i}{765}\right)x^3\right.\\
& \left.+\left(\frac{114911}{78336}+\frac{24835i}{78336}\right)x^4+\left(\frac{4041077}{8029440}-\frac{1112267i}{1605888}\right)x^5+\mathrm{O}\left(x^6\right)\right)\\
& + c_2 x^{1+2i}\left(1+\left(-\frac{10}{17}+\frac{40i}{17}\right)x+\left(-\frac{365}{136}-\frac{13i}{17}\right)x^2+\left(\frac{223}{1020}-\frac{1723i}{765}\right)x^3\right.\\
& \left.+\left(\frac{114911}{78336}-\frac{24835i}{78336}\right)x^4+\left(\frac{4041077}{8029440}+\frac{1112267i}{1605888}\right)x^5+\mathrm{O}\left(x^6\right)\right)
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 94

```
AsymptoticDSolveValue[x^2*y''[x]-x*Cos[x]*y'[x]+5*Exp[2*x]*y[x]==0,y[x],{x,0,5}]
```

$$
\begin{aligned}
y(x) \to {} & \left(\frac{11}{391680}+\frac{7i}{391680}\right)c_1\left((32064-31693i)x^4-(30784+60608i)x^3\right.\\
& \left.-(80352-23904i)x^2+(23040+69120i)x+(25344-16128i)\right)x^{1+2i}\\
& +\left(\frac{7}{391680}+\frac{11i}{391680}\right)c_2\left((31693-32064i)x^4+(60608+30784i)x^3\right.\\
& \left.-(23904-80352i)x^2-(69120+23040i)x+(16128-25344i)\right)x^{1-2i}
\end{aligned}
$$

17.9 problem 10

Internal problem ID [2925]
Internal file name [OUTPUT/2417_Sunday_June_05_2022_03_08_42_AM_17819108/index.tex]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.4. page 758
Problem number: 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{4x^2y'' + 3xy' + yx = 0}$$

With the expansion point for the power series method at $x = 0$.

The ODE is

$$4x^2y'' + 3xy' + yx = 0$$

Or

$$x(4xy'' + 3y' + y) = 0$$

For $x \neq 0$ the above simplifies to

$$4xy'' + 3y' + y = 0$$

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$4x^2y'' + 3xy' + yx = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$
\begin{aligned}
p(x) &= \frac{3}{4x} \\
q(x) &= \frac{1}{4x}
\end{aligned}
$$

Table 428: Table $p(x), q(x)$ singularites.

$p(x) = \frac{3}{4x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{1}{4x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$
4x^2y'' + 3xy' + yx = 0
$$

Let the solution be represented as Frobenius power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^{n+r}
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} \\
y'' &= \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\begin{aligned}
&4x^2\left(\sum_{n=0}^{\infty}(n+r)(n+r-1)\,a_n x^{n+r-2}\right) \\
&+3x\left(\sum_{n=0}^{\infty}(n+r)\,a_n x^{n+r-1}\right)+\left(\sum_{n=0}^{\infty}a_n x^{n+r}\right)x=0
\end{aligned} \tag{1}
$$

Which simplifies to

$$
\left(\sum_{n=0}^{\infty}4x^{n+r}a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty}3x^{n+r}a_n(n+r)\right)+\left(\sum_{n=0}^{\infty}x^{1+n+r}a_n\right)=0 \tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\sum_{n=0}^{\infty}x^{1+n+r}a_n=\sum_{n=1}^{\infty}a_{n-1}x^{n+r}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\left(\sum_{n=0}^{\infty}4x^{n+r}a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty}3x^{n+r}a_n(n+r)\right)+\left(\sum_{n=1}^{\infty}a_{n-1}x^{n+r}\right)=0 \tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
4x^{n+r}a_n(n+r)(n+r-1)+3x^{n+r}a_n(n+r)=0
$$

When $n=0$ the above becomes

$$
4x^r a_0 r(-1+r)+3x^r a_0 r=0
$$

Or

$$
(4x^r r(-1+r)+3x^r r)\,a_0=0
$$

Since $a_0\neq 0$ then the above simplifies to

$$
x^r r(-1+4r)=0
$$

Since the above is true for all x then the indicial equation becomes

$$4r^2 - r = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{4}$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^r r(-1 + 4r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{1}{4}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{4}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^n$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$4a_n(n + r)(n + r - 1) + 3a_n(n + r) + a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}}{4n^2 + 8nr + 4r^2 - n - r} \tag{4}$$

Which for the root $r = \frac{1}{4}$ becomes

$$a_n = -\frac{a_{n-1}}{4n^2 + n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{4}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{1}{4r^2 + 7r + 3}$$

Which for the root $r = \frac{1}{4}$ becomes

$$a_1 = -\frac{1}{5}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{4r^2+7r+3}$	$-\frac{1}{5}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{16r^4 + 88r^3 + 173r^2 + 143r + 42}$$

Which for the root $r = \frac{1}{4}$ becomes

$$a_2 = \frac{1}{90}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{4r^2+7r+3}$	$-\frac{1}{5}$
a_2	$\frac{1}{16r^4+88r^3+173r^2+143r+42}$	$\frac{1}{90}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{1}{64r^6 + 720r^5 + 3244r^4 + 7455r^3 + 9166r^2 + 5685r + 1386}$$

Which for the root $r = \frac{1}{4}$ becomes

$$a_3 = -\frac{1}{3510}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{4r^2+7r+3}$	$-\frac{1}{5}$
a_2	$\frac{1}{16r^4+88r^3+173r^2+143r+42}$	$\frac{1}{90}$
a_3	$-\frac{1}{64r^6+720r^5+3244r^4+7455r^3+9166r^2+5685r+1386}$	$-\frac{1}{3510}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{256r^8 + 4864r^7 + 39136r^6 + 173584r^5 + 462409r^4 + 754186r^3 + 731739r^2 + 384066r + 83160}$$

Which for the root $r = \frac{1}{4}$ becomes

$$a_4 = \frac{1}{238680}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{4r^2+7r+3}$	$-\frac{1}{5}$
a_2	$\frac{1}{16r^4+88r^3+173r^2+143r+42}$	$\frac{1}{90}$
a_3	$-\frac{1}{64r^6+720r^5+3244r^4+7455r^3+9166r^2+5685r+1386}$	$-\frac{1}{3510}$
a_4	$\frac{1}{256r^8+4864r^7+39136r^6+173584r^5+462409r^4+754186r^3+731739r^2+384066r+83160}$	$\frac{1}{238680}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{1}{(256r^8 + 4864r^7 + 39136r^6 + 173584r^5 + 462409r^4 + 754186r^3 + 731739r^2 + 384066r + 83160)\,(4$$

Which for the root $r = \frac{1}{4}$ becomes

$$a_5 = -\frac{1}{25061400}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{4r^2+7r+3}$	$-\frac{1}{5}$
a_2	$\frac{1}{16r^4+88r^3+173r^2+143r+42}$	$\frac{1}{90}$
a_3	$-\frac{1}{64r^6+720r^5+3244r^4+7455r^3+9166r^2+5685r+1386}$	$-\frac{1}{3510}$
a_4	$\frac{1}{256r^8+4864r^7+39136r^6+173584r^5+462409r^4+754186r^3+731739r^2+384066r+83160}$	$\frac{1}{238680}$
a_5	$-\frac{1}{(256r^8+4864r^7+39136r^6+173584r^5+462409r^4+754186r^3+731739r^2+384066r+83160)(4r^2+39r+95)}$	$-\frac{1}{25061400}$

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x^{\frac{1}{4}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= x^{\frac{1}{4}}\left(1 - \frac{x}{5} + \frac{x^2}{90} - \frac{x^3}{3510} + \frac{x^4}{238680} - \frac{x^5}{25061400} + O(x^6)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$
4b_n(n+r)(n+r-1) + 3b_n(n+r) + b_{n-1} = 0 \tag{3}
$$

Solving for b_n from recursive equation (4) gives

$$
b_n = -\frac{b_{n-1}}{4n^2 + 8nr + 4r^2 - n - r} \tag{4}
$$

Which for the root $r = 0$ becomes

$$
b_n = -\frac{b_{n-1}}{n(4n-1)} \tag{5}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = -\frac{1}{4r^2 + 7r + 3}$$

Which for the root $r = 0$ becomes

$$b_1 = -\frac{1}{3}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{4r^2+7r+3}$	$-\frac{1}{3}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1}{16r^4 + 88r^3 + 173r^2 + 143r + 42}$$

Which for the root $r = 0$ becomes

$$b_2 = \frac{1}{42}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{4r^2+7r+3}$	$-\frac{1}{3}$
b_2	$\frac{1}{16r^4+88r^3+173r^2+143r+42}$	$\frac{1}{42}$

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{1}{64r^6 + 720r^5 + 3244r^4 + 7455r^3 + 9166r^2 + 5685r + 1386}$$

Which for the root $r = 0$ becomes

$$b_3 = -\frac{1}{1386}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{4r^2+7r+3}$	$-\frac{1}{3}$
b_2	$\frac{1}{16r^4+88r^3+173r^2+143r+42}$	$\frac{1}{42}$
b_3	$-\frac{1}{64r^6+720r^5+3244r^4+7455r^3+9166r^2+5685r+1386}$	$-\frac{1}{1386}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{256r^8 + 4864r^7 + 39136r^6 + 173584r^5 + 462409r^4 + 754186r^3 + 731739r^2 + 384066r + 83160}$$

Which for the root $r = 0$ becomes

$$b_4 = \frac{1}{83160}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{4r^2+7r+3}$	$-\frac{1}{3}$
b_2	$\frac{1}{16r^4+88r^3+173r^2+143r+42}$	$\frac{1}{42}$
b_3	$-\frac{1}{64r^6+720r^5+3244r^4+7455r^3+9166r^2+5685r+1386}$	$-\frac{1}{1386}$
b_4	$\frac{1}{256r^8+4864r^7+39136r^6+173584r^5+462409r^4+754186r^3+731739r^2+384066r+83160}$	$\frac{1}{83160}$

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{1}{(256r^8 + 4864r^7 + 39136r^6 + 173584r^5 + 462409r^4 + 754186r^3 + 731739r^2 + 384066r + 83160) (4$$

Which for the root $r = 0$ becomes

$$b_5 = -\frac{1}{7900200}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{4r^2+7r+3}$	$-\frac{1}{3}$
b_2	$\frac{1}{16r^4+88r^3+173r^2+143r+42}$	$\frac{1}{42}$
b_3	$-\frac{1}{64r^6+720r^5+3244r^4+7455r^3+9166r^2+5685r+1386}$	$-\frac{1}{1386}$
b_4	$\frac{1}{256r^8+4864r^7+39136r^6+173584r^5+462409r^4+754186r^3+731739r^2+384066r+83160}$	$\frac{1}{83160}$
b_5	$-\frac{1}{(256r^8+4864r^7+39136r^6+173584r^5+462409r^4+754186r^3+731739r^2+384066r+83160)(4r^2+39r+95)}$	$-\frac{1}{7900200}$

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\
&= 1 - \frac{x}{3} + \frac{x^2}{42} - \frac{x^3}{1386} + \frac{x^4}{83160} - \frac{x^5}{7900200} + O(x^6)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^{\frac{1}{4}} \left(1 - \frac{x}{5} + \frac{x^2}{90} - \frac{x^3}{3510} + \frac{x^4}{238680} - \frac{x^5}{25061400} + O(x^6)\right) \\
&\quad + c_2 \left(1 - \frac{x}{3} + \frac{x^2}{42} - \frac{x^3}{1386} + \frac{x^4}{83160} - \frac{x^5}{7900200} + O(x^6)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^{\frac{1}{4}} \left(1 - \frac{x}{5} + \frac{x^2}{90} - \frac{x^3}{3510} + \frac{x^4}{238680} - \frac{x^5}{25061400} + O(x^6)\right) \\
&\quad + c_2 \left(1 - \frac{x}{3} + \frac{x^2}{42} - \frac{x^3}{1386} + \frac{x^4}{83160} - \frac{x^5}{7900200} + O(x^6)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = c_1 x^{\frac{1}{4}} &\left(1 - \frac{x}{5} + \frac{x^2}{90} - \frac{x^3}{3510} + \frac{x^4}{238680} - \frac{x^5}{25061400} + O(x^6)\right) \\
&+ c_2 \left(1 - \frac{x}{3} + \frac{x^2}{42} - \frac{x^3}{1386} + \frac{x^4}{83160} - \frac{x^5}{7900200} + O(x^6)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$y = c_1 x^{\frac{1}{4}} \left(1 - \frac{x}{5} + \frac{x^2}{90} - \frac{x^3}{3510} + \frac{x^4}{238680} - \frac{x^5}{25061400} + O(x^6)\right) \\ + c_2 \left(1 - \frac{x}{3} + \frac{x^2}{42} - \frac{x^3}{1386} + \frac{x^4}{83160} - \frac{x^5}{7900200} + O(x^6)\right)$$

Verified OK.

17.9.1 Maple step by step solution

Let's solve

$4x^2y'' + 3xy' + yx = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y}{4x} - \frac{3y'}{4x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{3y'}{4x} + \frac{y}{4x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{3}{4x}, P_3(x) = \frac{1}{4x}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = \frac{3}{4}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4y''x + 3y' + y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert y' to series expansion

$$y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$$

- Shift index using $k- >k+1$

$$y' = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)\, x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

$$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k- >k+1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)\,(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-1+4r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(4k+3+4r) + a_k\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(-1+4r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, \frac{1}{4}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$4\left(k+\frac{3}{4}+r\right)(k+1+r)\, a_{k+1} + a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{a_k}{(4k+3+4r)(k+1+r)}$$

- Recursion relation for $r = 0$

$$a_{k+1} = -\frac{a_k}{(4k+3)(k+1)}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{a_k}{(4k+3)(k+1)}\right]$$

- Recursion relation for $r = \frac{1}{4}$

$$a_{k+1} = -\frac{a_k}{(4k+4)\left(k+\frac{5}{4}\right)}$$

- Solution for $r = \frac{1}{4}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}, a_{k+1} = -\frac{a_k}{(4k+4)\left(k+\frac{5}{4}\right)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{4}}\right), a_{1+k} = -\frac{a_k}{(4k+3)(1+k)}, b_{1+k} = -\frac{b_k}{(4k+4)\left(k+\frac{5}{4}\right)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 44

```
Order:=6;
dsolve(4*x^2*diff(y(x),x$2)+3*x*diff(y(x),x)+x*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^{\frac{1}{4}}\left(1 - \frac{1}{5}x + \frac{1}{90}x^2 - \frac{1}{3510}x^3 + \frac{1}{238680}x^4 - \frac{1}{25061400}x^5 + \mathrm{O}\left(x^6\right)\right) + c_2\left(1 - \frac{1}{3}x + \frac{1}{42}x^2 - \frac{1}{1386}x^3 + \frac{1}{83160}x^4 - \frac{1}{7900200}x^5 + \mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 85

```
AsymptoticDSolveValue[4*x^2*y''[x]+3*x*y'[x]+x*y[x]==0,y[x],{x,0,5}]
```

$$
\begin{aligned}
y(x) \to & c_1 \sqrt[4]{x}\left(-\frac{x^5}{25061400}+\frac{x^4}{238680}-\frac{x^3}{3510}+\frac{x^2}{90}-\frac{x}{5}+1\right) \\
& + c_2\left(-\frac{x^5}{7900200}+\frac{x^4}{83160}-\frac{x^3}{1386}+\frac{x^2}{42}-\frac{x}{3}+1\right)
\end{aligned}
$$

17.10 problem 11

Internal problem ID [2926]
Internal file name [`OUTPUT/2418_Sunday_June_05_2022_03_08_47_AM_57705214/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.4. page 758
Problem number: 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{6x^2y'' + x(1+18x)\,y' + (1+12x)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$6x^2y'' + \left(18x^2 + x\right)y' + (1+12x)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1+18x}{6x}$$
$$q(x) = \frac{1+12x}{6x^2}$$

Table 430: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1+18x}{6x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{1+12x}{6x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$6x^2y'' + (18x^2 + x)\, y' + (1 + 12x)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$6x^2 \left(\sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2}\right) + (18x^2 + x) \left(\sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}\right) + (1 + 12x) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} 6x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 18x^{1+n+r} a_n (n+r)\right) \\
&+ \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 12x^{1+n+r} a_n\right) = 0
\end{aligned}
\tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} 18x^{1+n+r} a_n (n+r) &= \sum_{n=1}^{\infty} 18a_{n-1}(n+r-1)\, x^{n+r} \\
\sum_{n=0}^{\infty} 12x^{1+n+r} a_n &= \sum_{n=1}^{\infty} 12a_{n-1} x^{n+r}
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} 6x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} 18a_{n-1}(n+r-1)\, x^{n+r}\right) \\
&+ \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) + \left(\sum_{n=1}^{\infty} 12a_{n-1} x^{n+r}\right) = 0
\end{aligned}
\tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$6x^{n+r} a_n (n+r)(n+r-1) + x^{n+r} a_n (n+r) + a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$6x^r a_0 r(-1+r) + x^r a_0 r + a_0 x^r = 0$$

Or

$$\left(6x^r r(-1+r) + x^r r + x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(6r^2 - 5r + 1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$6r^2 - 5r + 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= \frac{1}{2} \\ r_2 &= \frac{1}{3} \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(6r^2 - 5r + 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{1}{6}$ is not an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}} \\ y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{3}} \end{aligned}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$6a_n(n+r)(n+r-1) + 18a_{n-1}(n+r-1) + a_n(n+r) + a_n + 12a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{6a_{n-1}}{2n + 2r - 1} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = -\frac{3a_{n-1}}{n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{6}{1 + 2r}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_1 = -3$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{6}{1+2r}$	-3

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{36}{4r^2 + 8r + 3}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = \frac{9}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{6}{1+2r}$	-3
a_2	$\frac{36}{4r^2+8r+3}$	$\frac{9}{2}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{216}{8r^3 + 36r^2 + 46r + 15}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_3 = -\frac{9}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{6}{1+2r}$	-3
a_2	$\frac{36}{4r^2+8r+3}$	$\frac{9}{2}$
a_3	$-\frac{216}{8r^3+36r^2+46r+15}$	$-\frac{9}{2}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1296}{16r^4 + 128r^3 + 344r^2 + 352r + 105}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = \frac{27}{8}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{6}{1+2r}$	-3
a_2	$\frac{36}{4r^2+8r+3}$	$\frac{9}{2}$
a_3	$-\frac{216}{8r^3+36r^2+46r+15}$	$-\frac{9}{2}$
a_4	$\frac{1296}{16r^4+128r^3+344r^2+352r+105}$	$\frac{27}{8}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{7776}{32r^5 + 400r^4 + 1840r^3 + 3800r^2 + 3378r + 945}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_5 = -\frac{81}{40}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{6}{1+2r}$	-3
a_2	$\frac{36}{4r^2+8r+3}$	$\frac{9}{2}$
a_3	$-\frac{216}{8r^3+36r^2+46r+15}$	$-\frac{9}{2}$
a_4	$\frac{1296}{16r^4+128r^3+344r^2+352r+105}$	$\frac{27}{8}$
a_5	$-\frac{7776}{32r^5+400r^4+1840r^3+3800r^2+3378r+945}$	$-\frac{81}{40}$

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= \sqrt{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= \sqrt{x}\left(1 - 3x + \frac{9x^2}{2} - \frac{9x^3}{2} + \frac{27x^4}{8} - \frac{81x^5}{40} + O(x^6)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$
6b_n(n+r)(n+r-1) + 18b_{n-1}(n+r-1) + b_n(n+r) + b_n + 12b_{n-1} = 0 \tag{3}
$$

Solving for b_n from recursive equation (4) gives

$$
b_n = -\frac{6b_{n-1}}{2n+2r-1} \tag{4}
$$

Which for the root $r = \frac{1}{3}$ becomes

$$
b_n = -\frac{18b_{n-1}}{6n-1} \tag{5}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{1}{3}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$
b_1 = -\frac{6}{1+2r}
$$

Which for the root $r = \frac{1}{3}$ becomes

$$b_1 = -\frac{18}{5}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{6}{1+2r}$	$-\frac{18}{5}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{36}{4r^2 + 8r + 3}$$

Which for the root $r = \frac{1}{3}$ becomes

$$b_2 = \frac{324}{55}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{6}{1+2r}$	$-\frac{18}{5}$
b_2	$\frac{36}{4r^2+8r+3}$	$\frac{324}{55}$

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{216}{8r^3 + 36r^2 + 46r + 15}$$

Which for the root $r = \frac{1}{3}$ becomes

$$b_3 = -\frac{5832}{935}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{6}{1+2r}$	$-\frac{18}{5}$
b_2	$\frac{36}{4r^2+8r+3}$	$\frac{324}{55}$
b_3	$-\frac{216}{8r^3+36r^2+46r+15}$	$-\frac{5832}{935}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1296}{16r^4 + 128r^3 + 344r^2 + 352r + 105}$$

Which for the root $r = \frac{1}{3}$ becomes

$$b_4 = \frac{104976}{21505}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{6}{1+2r}$	$-\frac{18}{5}$
b_2	$\frac{36}{4r^2+8r+3}$	$\frac{324}{55}$
b_3	$-\frac{216}{8r^3+36r^2+46r+15}$	$-\frac{5832}{935}$
b_4	$\frac{1296}{16r^4+128r^3+344r^2+352r+105}$	$\frac{104976}{21505}$

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{7776}{32r^5 + 400r^4 + 1840r^3 + 3800r^2 + 3378r + 945}$$

Which for the root $r = \frac{1}{3}$ becomes

$$b_5 = -\frac{1889568}{623645}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{6}{1+2r}$	$-\frac{18}{5}$
b_2	$\frac{36}{4r^2+8r+3}$	$\frac{324}{55}$
b_3	$-\frac{216}{8r^3+36r^2+46r+15}$	$-\frac{5832}{935}$
b_4	$\frac{1296}{16r^4+128r^3+344r^2+352r+105}$	$\frac{104976}{21505}$
b_5	$-\frac{7776}{32r^5+400r^4+1840r^3+3800r^2+3378r+945}$	$-\frac{1889568}{623645}$

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= \sqrt{x}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\
&= x^{\frac{1}{3}}\left(1 - \frac{18x}{5} + \frac{324x^2}{55} - \frac{5832x^3}{935} + \frac{104976x^4}{21505} - \frac{1889568x^5}{623645} + O(x^6)\right)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1\sqrt{x}\left(1 - 3x + \frac{9x^2}{2} - \frac{9x^3}{2} + \frac{27x^4}{8} - \frac{81x^5}{40} + O(x^6)\right) \\
&\quad + c_2 x^{\frac{1}{3}}\left(1 - \frac{18x}{5} + \frac{324x^2}{55} - \frac{5832x^3}{935} + \frac{104976x^4}{21505} - \frac{1889568x^5}{623645} + O(x^6)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1\sqrt{x}\left(1 - 3x + \frac{9x^2}{2} - \frac{9x^3}{2} + \frac{27x^4}{8} - \frac{81x^5}{40} + O(x^6)\right) \\
&\quad + c_2 x^{\frac{1}{3}}\left(1 - \frac{18x}{5} + \frac{324x^2}{55} - \frac{5832x^3}{935} + \frac{104976x^4}{21505} - \frac{1889568x^5}{623645} + O(x^6)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1\sqrt{x}\left(1 - 3x + \frac{9x^2}{2} - \frac{9x^3}{2} + \frac{27x^4}{8} - \frac{81x^5}{40} + O(x^6)\right) \\
&\quad + c_2 x^{\frac{1}{3}}\left(1 - \frac{18x}{5} + \frac{324x^2}{55} - \frac{5832x^3}{935} + \frac{104976x^4}{21505} - \frac{1889568x^5}{623645} + O(x^6)\right)
\end{aligned}
\tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1\sqrt{x}\left(1 - 3x + \frac{9x^2}{2} - \frac{9x^3}{2} + \frac{27x^4}{8} - \frac{81x^5}{40} + O(x^6)\right) \\
&\quad + c_2 x^{\frac{1}{3}}\left(1 - \frac{18x}{5} + \frac{324x^2}{55} - \frac{5832x^3}{935} + \frac{104976x^4}{21505} - \frac{1889568x^5}{623645} + O(x^6)\right)
\end{aligned}
$$

Verified OK.

17.10.1 Maple step by step solution

Let's solve

$6x^2y'' + (18x^2 + x)\, y' + (1 + 12x)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(1+12x)y}{6x^2} - \frac{(1+18x)y'}{6x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(1+18x)y'}{6x} + \frac{(1+12x)y}{6x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{1+18x}{6x}, P_3(x) = \frac{1+12x}{6x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = \frac{1}{6}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = \frac{1}{6}$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $6x^2y'' + x(1 + 18x)\, y' + (1 + 12x)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ◦ Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- ◦ Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- ◦ Shift index using $k- > k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- ◦ Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1+3r)\,(-1+2r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(3k+3r-1)\,(2k+2r-1) + 6a_{k-1}(3k+3r-1)\right) x^{k+r}\right) =$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-1+3r)\,(-1+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{\frac{1}{2}, \frac{1}{3}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$6\left(\left(k+r-\frac{1}{2}\right) a_k + 3a_{k-1}\right)\left(k+r-\frac{1}{3}\right) = 0$$

- Shift index using $k- > k+1$

$$6\left(\left(k+\frac{1}{2}+r\right) a_{k+1} + 3a_k\right)\left(k+\frac{2}{3}+r\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{6a_k}{2k+1+2r}$$

- Recursion relation for $r = \frac{1}{2}$

$$a_{k+1} = -\frac{6a_k}{2k+2}$$

- Solution for $r = \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = -\frac{6a_k}{2k+2}\right]$$

- Recursion relation for $r = \frac{1}{3}$

$$a_{k+1} = -\frac{6a_k}{2k+\frac{5}{3}}$$

- Solution for $r = \frac{1}{3}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+1} = -\frac{6a_k}{2k+\frac{5}{3}}\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), a_{1+k} = -\frac{6a_k}{2+2k}, b_{1+k} = -\frac{6b_k}{2k+\frac{5}{3}}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 47

```
Order:=6;
dsolve(6*x^2*diff(y(x),x$2)+x*(1+18*x)*diff(y(x),x)+(1+12*x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^{\frac{1}{3}} \left(1 - \frac{18}{5}x + \frac{324}{55}x^2 - \frac{5832}{935}x^3 + \frac{104976}{21505}x^4 - \frac{1889568}{623645}x^5 + \mathrm{O}\left(x^6\right)\right) + c_2\sqrt{x}\left(1 - 3x + \frac{9}{2}x^2 - \frac{9}{2}x^3 + \frac{27}{8}x^4 - \frac{81}{40}x^5 + \mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 88

```
AsymptoticDSolveValue[6*x^2*y''[x]+x*(1+18*x)*y'[x]+(1+12*x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\sqrt{x}\left(-\frac{81x^5}{40} + \frac{27x^4}{8} - \frac{9x^3}{2} + \frac{9x^2}{2} - 3x + 1\right) + c_2\sqrt[3]{x}\left(-\frac{1889568x^5}{623645} + \frac{104976x^4}{21505} - \frac{5832x^3}{935} + \frac{324x^2}{55} - \frac{18x}{5} + 1\right)$$

17.11 problem 12

Internal problem ID [2927]
Internal file name [`OUTPUT/2419_Sunday_June_05_2022_03_08_51_AM_30665821/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.4. page 758
Problem number: 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2 y'' + x y' - (x+2)\, y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2 y'' + x y' + (-2 - x)\, y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x) y' + q(x) y = 0$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = -\frac{x+2}{x^2}$$

Table 432: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{x+2}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2 y'' + x y' + (-2 - x)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + x \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + (-2 - x) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)\,(n+r-1)\right)+\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)\right) \\
&+\sum_{n=0}^{\infty}\left(-2 a_n x^{n+r}\right)+\sum_{n=0}^{\infty}\left(-x^{1+n+r} a_n\right)=0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\sum_{n=0}^{\infty}\left(-x^{1+n+r} a_n\right)=\sum_{n=1}^{\infty}\left(-a_{n-1} x^{n+r}\right)
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)\,(n+r-1)\right)+\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)\right) \\
&+\sum_{n=0}^{\infty}\left(-2 a_n x^{n+r}\right)+\sum_{n=1}^{\infty}\left(-a_{n-1} x^{n+r}\right)=0
\end{aligned} \tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
x^{n+r} a_n(n+r)\,(n+r-1)+x^{n+r} a_n(n+r)-2 a_n x^{n+r}=0
$$

When $n=0$ the above becomes

$$
x^r a_0 r(-1+r)+x^r a_0 r-2 a_0 x^r=0
$$

Or

$$
\left(x^r r(-1+r)+x^r r-2 x^r\right) a_0=0
$$

Since $a_0 \neq 0$ then the above simplifies to

$$
\left(r^2-2\right) x^r=0
$$

Since the above is true for all x then the indicial equation becomes

$$
r^2-2=0
$$

Solving for r gives the roots of the indicial equation as

$$
\begin{aligned}
r_1 &= \sqrt{2} \\
r_2 &= -\sqrt{2}
\end{aligned}
$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$
(r^2 - 2) x^r = 0
$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 2\sqrt{2}$ is not an integer, then we can construct two linearly independent solutions

$$
\begin{aligned}
y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right) \\
y_2(x) &= x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+\sqrt{2}} \\
y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n-\sqrt{2}}
\end{aligned}
$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$
a_n(n+r)(n+r-1) + a_n(n+r) - 2a_n - a_{n-1} = 0 \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$
a_n = \frac{a_{n-1}}{n^2 + 2nr + r^2 - 2} \tag{4}
$$

Which for the root $r = \sqrt{2}$ becomes

$$
a_n = \frac{a_{n-1}}{n(2\sqrt{2} + n)} \tag{5}
$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \sqrt{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{1}{r^2 + 2r - 1}$$

Which for the root $r = \sqrt{2}$ becomes

$$a_1 = \frac{1}{1 + 2\sqrt{2}}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{r^2+2r-1}$	$\frac{1}{1+2\sqrt{2}}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{(r^2 + 2r - 1)(r^2 + 4r + 2)}$$

Which for the root $r = \sqrt{2}$ becomes

$$a_2 = \frac{1}{20 + 12\sqrt{2}}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{r^2+2r-1}$	$\frac{1}{1+2\sqrt{2}}$
a_2	$\frac{1}{(r^2+2r-1)(r^2+4r+2)}$	$\frac{1}{20+12\sqrt{2}}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{1}{(r^2 + 2r - 1)(r^2 + 4r + 2)(r^2 + 6r + 7)}$$

Which for the root $r = \sqrt{2}$ becomes

$$a_3 = \frac{1}{228\sqrt{2} + 324}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{r^2+2r-1}$	$\frac{1}{1+2\sqrt{2}}$
a_2	$\frac{1}{(r^2+2r-1)(r^2+4r+2)}$	$\frac{1}{20+12\sqrt{2}}$
a_3	$\frac{1}{(r^2+2r-1)(r^2+4r+2)(r^2+6r+7)}$	$\frac{1}{228\sqrt{2}+324}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(r^2 + 2r - 1)(r^2 + 4r + 2)(r^2 + 6r + 7)(r^2 + 8r + 14)}$$

Which for the root $r = \sqrt{2}$ becomes

$$a_4 = \frac{1}{8832 + 6240\sqrt{2}}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{r^2+2r-1}$	$\frac{1}{1+2\sqrt{2}}$
a_2	$\frac{1}{(r^2+2r-1)(r^2+4r+2)}$	$\frac{1}{20+12\sqrt{2}}$
a_3	$\frac{1}{(r^2+2r-1)(r^2+4r+2)(r^2+6r+7)}$	$\frac{1}{228\sqrt{2}+324}$
a_4	$\frac{1}{(r^2+2r-1)(r^2+4r+2)(r^2+6r+7)(r^2+8r+14)}$	$\frac{1}{8832+6240\sqrt{2}}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{1}{(r^2 + 2r - 1)(r^2 + 4r + 2)(r^2 + 6r + 7)(r^2 + 8r + 14)(r^2 + 10r + 23)}$$

Which for the root $r = \sqrt{2}$ becomes

$$a_5 = \frac{1}{244320\sqrt{2} + 345600}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{r^2+2r-1}$	$\frac{1}{1+2\sqrt{2}}$
a_2	$\frac{1}{(r^2+2r-1)(r^2+4r+2)}$	$\frac{1}{20+12\sqrt{2}}$
a_3	$\frac{1}{(r^2+2r-1)(r^2+4r+2)(r^2+6r+7)}$	$\frac{1}{228\sqrt{2}+324}$
a_4	$\frac{1}{(r^2+2r-1)(r^2+4r+2)(r^2+6r+7)(r^2+8r+14)}$	$\frac{1}{8832+6240\sqrt{2}}$
a_5	$\frac{1}{(r^2+2r-1)(r^2+4r+2)(r^2+6r+7)(r^2+8r+14)(r^2+10r+23)}$	$\frac{1}{244320\sqrt{2}+345600}$

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x^{\sqrt{2}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= x^{\sqrt{2}}\left(1 + \frac{x}{1+2\sqrt{2}} + \frac{x^2}{20+12\sqrt{2}} + \frac{x^3}{228\sqrt{2}+324} + \frac{x^4}{8832+6240\sqrt{2}} + \frac{x^5}{244320\sqrt{2}+345600} + O(
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$
b_n(n+r)(n+r-1) + b_n(n+r) - 2b_n - b_{n-1} = 0 \tag{3}
$$

Solving for b_n from recursive equation (4) gives

$$
b_n = \frac{b_{n-1}}{n^2 + 2nr + r^2 - 2} \tag{4}
$$

Which for the root $r = -\sqrt{2}$ becomes

$$
b_n = \frac{b_{n-1}}{n\left(-2\sqrt{2} + n\right)} \tag{5}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\sqrt{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{1}{r^2 + 2r - 1}$$

Which for the root $r = -\sqrt{2}$ becomes

$$b_1 = \frac{1}{1 - 2\sqrt{2}}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{r^2+2r-1}$	$\frac{1}{1-2\sqrt{2}}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1}{(r^2 + 2r - 1)(r^2 + 4r + 2)}$$

Which for the root $r = -\sqrt{2}$ becomes

$$b_2 = \frac{1}{20 - 12\sqrt{2}}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{r^2+2r-1}$	$\frac{1}{1-2\sqrt{2}}$
b_2	$\frac{1}{(r^2+2r-1)(r^2+4r+2)}$	$\frac{1}{20-12\sqrt{2}}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{1}{(r^2 + 2r - 1)(r^2 + 4r + 2)(r^2 + 6r + 7)}$$

Which for the root $r = -\sqrt{2}$ becomes

$$b_3 = -\frac{1}{228\sqrt{2} - 324}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{r^2+2r-1}$	$\frac{1}{1-2\sqrt{2}}$
b_2	$\frac{1}{(r^2+2r-1)(r^2+4r+2)}$	$\frac{1}{20-12\sqrt{2}}$
b_3	$\frac{1}{(r^2+2r-1)(r^2+4r+2)(r^2+6r+7)}$	$-\frac{1}{228\sqrt{2}-324}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{(r^2 + 2r - 1)(r^2 + 4r + 2)(r^2 + 6r + 7)(r^2 + 8r + 14)}$$

Which for the root $r = -\sqrt{2}$ becomes

$$b_4 = \frac{1}{8832 - 6240\sqrt{2}}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{r^2+2r-1}$	$\frac{1}{1-2\sqrt{2}}$
b_2	$\frac{1}{(r^2+2r-1)(r^2+4r+2)}$	$\frac{1}{20-12\sqrt{2}}$
b_3	$\frac{1}{(r^2+2r-1)(r^2+4r+2)(r^2+6r+7)}$	$-\frac{1}{228\sqrt{2}-324}$
b_4	$\frac{1}{(r^2+2r-1)(r^2+4r+2)(r^2+6r+7)(r^2+8r+14)}$	$\frac{1}{8832-6240\sqrt{2}}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{1}{(r^2 + 2r - 1)(r^2 + 4r + 2)(r^2 + 6r + 7)(r^2 + 8r + 14)(r^2 + 10r + 23)}$$

Which for the root $r = -\sqrt{2}$ becomes

$$b_5 = -\frac{1}{480\left(-1 + 2\sqrt{2}\right)\left(\sqrt{2} - 1\right)\left(2\sqrt{2} - 3\right)\left(\sqrt{2} - 2\right)\left(-5 + 2\sqrt{2}\right)}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{r^2+2r-1}$	$\frac{1}{1-2\sqrt{2}}$
b_2	$\frac{1}{(r^2+2r-1)(r^2+4r+2)}$	$\frac{1}{20-12\sqrt{2}}$
b_3	$\frac{1}{(r^2+2r-1)(r^2+4r+2)(r^2+6r+7)}$	$-\frac{1}{228\sqrt{2}-324}$
b_4	$\frac{1}{(r^2+2r-1)(r^2+4r+2)(r^2+6r+7)(r^2+8r+14)}$	$\frac{1}{8832-6240\sqrt{2}}$
b_5	$\frac{1}{(r^2+2r-1)(r^2+4r+2)(r^2+6r+7)(r^2+8r+14)(r^2+10r+23)}$	$-\frac{1}{480\left(-1+2\sqrt{2}\right)\left(\sqrt{2}-1\right)\left(2\sqrt{2}-3\right)\left(\sqrt{2}-2\right)\left(-5+2\sqrt{2}\right)}$

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= x^{\sqrt{2}}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\
&= x^{-\sqrt{2}}\left(1 + \frac{x}{1-2\sqrt{2}} + \frac{x^2}{20-12\sqrt{2}} - \frac{x^3}{228\sqrt{2}-324} + \frac{x^4}{8832-6240\sqrt{2}} - \frac{}{480\left(-1+2\sqrt{2}\right)\left(\sqrt{2}-\right.}\right.
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^{\sqrt{2}}\left(1 + \frac{x}{1+2\sqrt{2}} + \frac{x^2}{20+12\sqrt{2}} + \frac{x^3}{228\sqrt{2}+324} + \frac{x^4}{8832+6240\sqrt{2}}\right. \\
&\qquad \left. + \frac{x^5}{244320\sqrt{2}+345600} + O\left(x^6\right)\right) \\
&\quad + c_2 x^{-\sqrt{2}}\left(1 + \frac{x}{1-2\sqrt{2}} + \frac{x^2}{20-12\sqrt{2}} - \frac{x^3}{228\sqrt{2}-324} + \frac{x^4}{8832-6240\sqrt{2}}\right. \\
&\qquad \left. - \frac{x^5}{480\left(-1+2\sqrt{2}\right)\left(\sqrt{2}-1\right)\left(2\sqrt{2}-3\right)\left(\sqrt{2}-2\right)\left(-5+2\sqrt{2}\right)} + O\left(x^6\right)\right)
\end{aligned}
$$

Hence the final solution is

$$
y = y_h
$$

$$
\begin{aligned}
&= c_1 x^{\sqrt{2}} \left(1 + \frac{x}{1+2\sqrt{2}} + \frac{x^2}{20+12\sqrt{2}} + \frac{x^3}{228\sqrt{2}+324} + \frac{x^4}{8832+6240\sqrt{2}} \right. \\
&\qquad \left. + \frac{x^5}{244320\sqrt{2}+345600} + O(x^6)\right) \\
&\quad + c_2 x^{-\sqrt{2}} \left(1 + \frac{x}{1-2\sqrt{2}} + \frac{x^2}{20-12\sqrt{2}} - \frac{x^3}{228\sqrt{2}-324} + \frac{x^4}{8832-6240\sqrt{2}} \right. \\
&\qquad \left. - \frac{x^5}{480\left(-1+2\sqrt{2}\right)\left(\sqrt{2}-1\right)\left(2\sqrt{2}-3\right)\left(\sqrt{2}-2\right)\left(-5+2\sqrt{2}\right)} + O(x^6)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x^{\sqrt{2}} \left(1 + \frac{x}{1+2\sqrt{2}} + \frac{x^2}{20+12\sqrt{2}} + \frac{x^3}{228\sqrt{2}+324} + \frac{x^4}{8832+6240\sqrt{2}} \right. \\
&\qquad \left. + \frac{x^5}{244320\sqrt{2}+345600} + O(x^6)\right) \\
&\quad + c_2 x^{-\sqrt{2}} \left(1 + \frac{x}{1-2\sqrt{2}} + \frac{x^2}{20-12\sqrt{2}} - \frac{x^3}{228\sqrt{2}-324} + \frac{x^4}{8832-6240\sqrt{2}} \right. \\
&\qquad \left. - \frac{x^5}{480\left(-1+2\sqrt{2}\right)\left(\sqrt{2}-1\right)\left(2\sqrt{2}-3\right)\left(\sqrt{2}-2\right)\left(-5+2\sqrt{2}\right)} + O(x^6)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1 x^{\sqrt{2}} \left(1 + \frac{x}{1+2\sqrt{2}} + \frac{x^2}{20+12\sqrt{2}} + \frac{x^3}{228\sqrt{2}+324} + \frac{x^4}{8832+6240\sqrt{2}} \right. \\
&\qquad \left. + \frac{x^5}{244320\sqrt{2}+345600} + O(x^6)\right) \\
&\quad + c_2 x^{-\sqrt{2}} \left(1 + \frac{x}{1-2\sqrt{2}} + \frac{x^2}{20-12\sqrt{2}} - \frac{x^3}{228\sqrt{2}-324} + \frac{x^4}{8832-6240\sqrt{2}} \right. \\
&\qquad \left. - \frac{x^5}{480\left(-1+2\sqrt{2}\right)\left(\sqrt{2}-1\right)\left(2\sqrt{2}-3\right)\left(\sqrt{2}-2\right)\left(-5+2\sqrt{2}\right)} + O(x^6)\right)
\end{aligned}
$$

Verified OK.

17.11.1 Maple step by step solution

Let's solve

$x^2y'' + xy' + (-2 - x)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y'}{x} + \frac{(x+2)y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} - \frac{(x+2)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{1}{x}, P_3(x) = -\frac{x+2}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = -2$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2y'' + xy' + (-2 - x)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ◦ Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x \cdot y'$ to series expansion

  $$x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(r^2-2)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k^2+2kr+r^2-2) - a_{k-1}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$r^2 - 2 = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{\sqrt{2}, -\sqrt{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$a_k(k^2+2kr+r^2-2) - a_{k-1} = 0$$

- Shift index using $k- >k+1$

  $$a_{k+1}\left((k+1)^2 + 2(k+1)\, r + r^2 - 2\right) - a_k = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = \frac{a_k}{k^2+2kr+r^2+2k+2r-1}$$

- Recursion relation for $r = \sqrt{2}$

  $$a_{k+1} = \frac{a_k}{k^2+2k\sqrt{2}+1+2k+2\sqrt{2}}$$

- Solution for $r = \sqrt{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\sqrt{2}}, a_{k+1} = \frac{a_k}{k^2+2k\sqrt{2}+1+2k+2\sqrt{2}}\right]$$

- Recursion relation for $r = -\sqrt{2}$

  $$a_{k+1} = \frac{a_k}{k^2-2k\sqrt{2}+1+2k-2\sqrt{2}}$$

- Solution for $r = -\sqrt{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\sqrt{2}}, a_{k+1} = \frac{a_k}{k^2-2k\sqrt{2}+1+2k-2\sqrt{2}}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+\sqrt{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-\sqrt{2}}\right), a_{1+k} = \frac{a_k}{k^2+2k\sqrt{2}+1+2k+2\sqrt{2}}, b_{1+k} = \frac{b_k}{k^2-2k\sqrt{2}+1+2k-2\sqrt{2}}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 321

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)-(2+x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^{-\sqrt{2}} \left(1 + \frac{1}{1-2\sqrt{2}} x + \frac{1}{20-12\sqrt{2}} x^2 - \frac{1}{228\sqrt{2}-324} x^3 + \frac{1}{8832-6240\sqrt{2}} x^4 - \frac{1}{480} \frac{1}{(-1+2\sqrt{2})(\sqrt{2}-1)(-3+2\sqrt{2})(\sqrt{2}-2)(-5+2\sqrt{2})} x^5 + \mathrm{O}\left(x^6\right)\right) + c_2 x^{\sqrt{2}} \left(1 + \frac{1}{1+2\sqrt{2}} x + \frac{1}{20+12\sqrt{2}} x^2 + \frac{1}{228\sqrt{2}+324} x^3 + \frac{1}{8832+6240\sqrt{2}} x^4 + \frac{1}{244320\sqrt{2}+345600} x^5 + \mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 843

```
AsymptoticDSolveValue[x^2*y''[x]+x*y'[x]-(2+x)*y[x]==0,y[x],{x,0,5}]
```

$$
\begin{aligned}
y(x) \\
\rightarrow \Bigg(& \frac{x^5}{\left(-1+\sqrt{2}+\sqrt{2}\left(1+\sqrt{2}\right)\right)\left(\sqrt{2}+\left(1+\sqrt{2}\right)\left(2+\sqrt{2}\right)\right)\left(1+\sqrt{2}+\left(2+\sqrt{2}\right)\left(3+\sqrt{2}\right)\right)\left(2+\sqrt{2}+(3\right.} \\
& + \frac{x^4}{\left(-1+\sqrt{2}+\sqrt{2}\left(1+\sqrt{2}\right)\right)\left(\sqrt{2}+\left(1+\sqrt{2}\right)\left(2+\sqrt{2}\right)\right)\left(1+\sqrt{2}+\left(2+\sqrt{2}\right)\left(3+\sqrt{2}\right)\right)\left(2+\sqrt{2}+(3\right.} \\
& + \frac{x^3}{\left(-1+\sqrt{2}+\sqrt{2}\left(1+\sqrt{2}\right)\right)\left(\sqrt{2}+\left(1+\sqrt{2}\right)\left(2+\sqrt{2}\right)\right)\left(1+\sqrt{2}+\left(2+\sqrt{2}\right)\left(3+\sqrt{2}\right)\right)} \\
& + \frac{x^2}{\left(-1+\sqrt{2}+\sqrt{2}\left(1+\sqrt{2}\right)\right)\left(\sqrt{2}+\left(1+\sqrt{2}\right)\left(2+\sqrt{2}\right)\right)} \\
& + \frac{x}{-1+\sqrt{2}+\sqrt{2}\left(1+\sqrt{2}\right)}+1\Bigg) c_1 x^{\sqrt{2}} \\
+ \Bigg(& \frac{x^5}{\left(-1-\sqrt{2}-\sqrt{2}\left(1-\sqrt{2}\right)\right)\left(-\sqrt{2}+\left(1-\sqrt{2}\right)\left(2-\sqrt{2}\right)\right)\left(1-\sqrt{2}+\left(2-\sqrt{2}\right)\left(3-\sqrt{2}\right)\right)\left(2-\sqrt{2}\right.} \\
& + \frac{x^4}{\left(-1-\sqrt{2}-\sqrt{2}\left(1-\sqrt{2}\right)\right)\left(-\sqrt{2}+\left(1-\sqrt{2}\right)\left(2-\sqrt{2}\right)\right)\left(1-\sqrt{2}+\left(2-\sqrt{2}\right)\left(3-\sqrt{2}\right)\right)\left(2-\sqrt{2}+\right.} \\
& + \frac{x^3}{\left(-1-\sqrt{2}-\sqrt{2}\left(1-\sqrt{2}\right)\right)\left(-\sqrt{2}+\left(1-\sqrt{2}\right)\left(2-\sqrt{2}\right)\right)\left(1-\sqrt{2}+\left(2-\sqrt{2}\right)\left(3-\sqrt{2}\right)\right)} \\
& + \frac{x^2}{\left(-1-\sqrt{2}-\sqrt{2}\left(1-\sqrt{2}\right)\right)\left(-\sqrt{2}+\left(1-\sqrt{2}\right)\left(2-\sqrt{2}\right)\right)} \\
& + \frac{x}{-1-\sqrt{2}-\sqrt{2}\left(1-\sqrt{2}\right)}+1\Bigg) c_2 x^{-\sqrt{2}}
\end{aligned}
$$

17.12 problem 13

Internal problem ID [2928]
Internal file name [`OUTPUT/2420_Sunday_June_05_2022_03_08_55_AM_69503755/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.4. page 758
Problem number: 13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2xy'' + y' - 2yx = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2xy'' + y' - 2yx = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{2x}$$
$$q(x) = -1$$

Table 434: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -1$	
singularity	type

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2xy'' + y' - 2yx = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x + \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) - 2\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) x = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r-1} a_n (n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-2x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-2x^{1+n+r} a_n\right) = \sum_{n=2}^{\infty} \left(-2a_{n-2} x^{n+r-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} 2x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \sum_{n=2}^{\infty} \left(-2a_{n-2} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r-1} a_n (n+r)(n+r-1) + (n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$2x^{-1+r} a_0 r(-1+r) + r a_0 x^{-1+r} = 0$$

Or

$$\left(2x^{-1+r} r(-1+r) + r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}(-1+2r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 - r = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{2}$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\, x^{-1+r}(-1+2r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$
\begin{aligned}
y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right) \\
y_2(x) &= x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}} \\
y_2(x) &= \sum_{n=0}^{\infty} b_n x^n
\end{aligned}
$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$
a_1 = 0
$$

For $2 \leq n$ the recursive equation is

$$
2a_n(n+r)(n+r-1) + a_n(n+r) - 2a_{n-2} = 0 \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$
a_n = \frac{2a_{n-2}}{2n^2 + 4nr + 2r^2 - n - r} \tag{4}
$$

Which for the root $r = \frac{1}{2}$ becomes

$$
a_n = \frac{2a_{n-2}}{2n^2 + n} \tag{5}
$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{2}{2r^2 + 7r + 6}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = \frac{1}{5}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{2}{2r^2+7r+6}$	$\frac{1}{5}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{2}{2r^2+7r+6}$	$\frac{1}{5}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{4}{(2r^2 + 7r + 6)(2r^2 + 15r + 28)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = \frac{1}{90}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{2}{2r^2+7r+6}$	$\frac{1}{5}$
a_3	0	0
a_4	$\frac{4}{(2r^2+7r+6)(2r^2+15r+28)}$	$\frac{1}{90}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{2}{2r^2+7r+6}$	$\frac{1}{5}$
a_3	0	0
a_4	$\frac{4}{(2r^2+7r+6)(2r^2+15r+28)}$	$\frac{1}{90}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}y_1(x) &= \sqrt{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= \sqrt{x}\left(1 + \frac{x^2}{5} + \frac{x^4}{90} + O(x^6)\right)\end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$2b_n(n+r)(n+r-1) + (n+r)b_n - 2b_{n-2} = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = \frac{2b_{n-2}}{2n^2 + 4nr + 2r^2 - n - r} \tag{4}$$

Which for the root $r = 0$ becomes

$$b_n = \frac{2b_{n-2}}{n\,(2n-1)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{2}{2r^2 + 7r + 6}$$

Which for the root $r = 0$ becomes

$$b_2 = \frac{1}{3}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{2}{2r^2+7r+6}$	$\frac{1}{3}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{2}{2r^2+7r+6}$	$\frac{1}{3}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{4}{(2r^2 + 7r + 6)(2r^2 + 15r + 28)}$$

Which for the root $r = 0$ becomes

$$b_4 = \frac{1}{42}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{2}{2r^2+7r+6}$	$\frac{1}{3}$
b_3	0	0
b_4	$\frac{4}{(2r^2+7r+6)(2r^2+15r+28)}$	$\frac{1}{42}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{2}{2r^2+7r+6}$	$\frac{1}{3}$
b_3	0	0
b_4	$\frac{4}{(2r^2+7r+6)(2r^2+15r+28)}$	$\frac{1}{42}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\ &= 1 + \frac{x^2}{3} + \frac{x^4}{42} + O(x^6) \end{aligned}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 \sqrt{x}\left(1+\frac{x^2}{5}+\frac{x^4}{90}+O(x^6)\right)+c_2\left(1+\frac{x^2}{3}+\frac{x^4}{42}+O(x^6)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 \sqrt{x}\left(1+\frac{x^2}{5}+\frac{x^4}{90}+O(x^6)\right)+c_2\left(1+\frac{x^2}{3}+\frac{x^4}{42}+O(x^6)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \sqrt{x}\left(1+\frac{x^2}{5}+\frac{x^4}{90}+O(x^6)\right)+c_2\left(1+\frac{x^2}{3}+\frac{x^4}{42}+O(x^6)\right) \tag{1}
$$

Verification of solutions

$$
y = c_1 \sqrt{x}\left(1+\frac{x^2}{5}+\frac{x^4}{90}+O(x^6)\right)+c_2\left(1+\frac{x^2}{3}+\frac{x^4}{42}+O(x^6)\right)
$$

Verified OK.

17.12.1 Maple step by step solution

Let's solve

$2y''x + y' - 2yx = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y'}{2x} + y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{2x} - y = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

$\left[P_2(x) = \frac{1}{2x}, P_3(x) = -1\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = \frac{1}{2}$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2y''x + y' - 2yx = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x \cdot y$ to series expansion

  $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$

- Shift index using $k- >k - 1$

  $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$

- Convert y' to series expansion

  $y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1}$

- Shift index using $k- >k + 1$

  $y' = \sum_{k=-1}^{\infty} a_{k+1} (k + r + 1)\, x^{k+r}$

- Convert $x \cdot y''$ to series expansion

  $x \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r-1}$

- Shift index using $k- >k + 1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-1+2r)\, x^{-1+r} + a_1(1+r)(1+2r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)(2k+1+2r) - 2a_{k-1}\right) x^{k+r}\right)$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(-1+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, \frac{1}{2}\right\}$$

- Each term must be 0

$$a_1(1+r)(1+2r) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$2\left(k+\frac{1}{2}+r\right)(k+r+1)\, a_{k+1} - 2a_{k-1} = 0$$

- Shift index using $k- >k+1$

$$2\left(k+\frac{3}{2}+r\right)(k+2+r)\, a_{k+2} - 2a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = \frac{2a_k}{(2k+3+2r)(k+2+r)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = \frac{2a_k}{(2k+3)(k+2)}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{2a_k}{(2k+3)(k+2)}, a_1 = 0\right]$$

- Recursion relation for $r = \frac{1}{2}$

$$a_{k+2} = \frac{2a_k}{(2k+4)\left(k+\frac{5}{2}\right)}$$

- Solution for $r = \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = \frac{2a_k}{(2k+4)\left(k+\frac{5}{2}\right)}, 3a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{2+k} = \frac{2a_k}{(2k+3)(2+k)}, a_1 = 0, b_{2+k} = \frac{2b_k}{(2k+4)\left(k+\frac{5}{2}\right)}, 3b_1 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 32

```
Order:=6;
dsolve(2*x*diff(y(x),x$2)+diff(y(x),x)-2*x*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1\sqrt{x}\left(1+\frac{1}{5}x^2+\frac{1}{90}x^4+\mathrm{O}\left(x^6\right)\right)+c_2\left(1+\frac{1}{3}x^2+\frac{1}{42}x^4+\mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 47

```
AsymptoticDSolveValue[2*x*y''[x]+y'[x]-2*x*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\sqrt{x}\left(\frac{x^4}{90}+\frac{x^2}{5}+1\right)+c_2\left(\frac{x^4}{42}+\frac{x^2}{3}+1\right)$$

17.13 problem 14

Internal problem ID [2929]
Internal file name [OUTPUT/2421_Sunday_June_05_2022_03_08_58_AM_82109773/index.tex]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.4. page 758
Problem number: 14.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{3x^2y'' - x(x+8)\,y' + 6y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$3x^2y'' + \left(-x^2 - 8x\right)y' + 6y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{x+8}{3x}$$
$$q(x) = \frac{2}{x^2}$$

Table 436: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{x+8}{3x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{2}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$3x^2y'' + \left(-x^2 - 8x\right) y' + 6y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$3x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + \left(-x^2 - 8x\right) \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + 6\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 3x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-8x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 6a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)\, x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 3x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)\, x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-8x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 6a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$3x^{n+r} a_n (n+r)(n+r-1) - 8x^{n+r} a_n (n+r) + 6a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$3x^r a_0 r(-1+r) - 8x^r a_0 r + 6a_0 x^r = 0$$

Or

$$\left(3x^r r(-1+r) - 8x^r r + 6x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(3r^2 - 11r + 6\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$3r^2 - 11r + 6 = 0$$

Solving for r gives the roots of the indicial equation as

$$
\begin{aligned}
r_1 &= 3 \\
r_2 &= \frac{2}{3}
\end{aligned}
$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$
(3r^2 - 11r + 6)\, x^r = 0
$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{7}{3}$ is not an integer, then we can construct two linearly independent solutions

$$
\begin{aligned}
y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right) \\
y_2(x) &= x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+3} \\
y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+\frac{2}{3}}
\end{aligned}
$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$
3a_n(n+r)(n+r-1) - a_{n-1}(n+r-1) - 8a_n(n+r) + 6a_n = 0 \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$
a_n = \frac{a_{n-1}(n+r-1)}{3n^2 + 6nr + 3r^2 - 11n - 11r + 6} \tag{4}
$$

Which for the root $r = 3$ becomes

$$
a_n = \frac{a_{n-1}(n+2)}{n(3n+7)} \tag{5}
$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 3$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{r}{3r^2 - 5r - 2}$$

Which for the root $r = 3$ becomes

$$a_1 = \frac{3}{10}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r}{3r^2-5r-2}$	$\frac{3}{10}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{r(1+r)}{9r^4 - 12r^3 - 23r^2 + 18r + 8}$$

Which for the root $r = 3$ becomes

$$a_2 = \frac{3}{65}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r}{3r^2-5r-2}$	$\frac{3}{10}$
a_2	$\frac{r(1+r)}{9r^4-12r^3-23r^2+18r+8}$	$\frac{3}{65}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{(1+r)(2+r)}{27r^5 + 27r^4 - 153r^3 - 107r^2 + 150r + 56}$$

Which for the root $r = 3$ becomes

$$a_3 = \frac{1}{208}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r}{3r^2-5r-2}$	$\frac{3}{10}$
a_2	$\frac{r(1+r)}{9r^4-12r^3-23r^2+18r+8}$	$\frac{3}{65}$
a_3	$\frac{(1+r)(2+r)}{27r^5+27r^4-153r^3-107r^2+150r+56}$	$\frac{1}{208}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{(2+r)(3+r)}{81r^6 + 351r^5 - 189r^4 - 1851r^3 - 620r^2 + 1668r + 560}$$

Which for the root $r = 3$ becomes

$$a_4 = \frac{3}{7904}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r}{3r^2-5r-2}$	$\frac{3}{10}$
a_2	$\frac{r(1+r)}{9r^4-12r^3-23r^2+18r+8}$	$\frac{3}{65}$
a_3	$\frac{(1+r)(2+r)}{27r^5+27r^4-153r^3-107r^2+150r+56}$	$\frac{1}{208}$
a_4	$\frac{(2+r)(3+r)}{81r^6+351r^5-189r^4-1851r^3-620r^2+1668r+560}$	$\frac{3}{7904}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{(4+r)(3+r)}{243r^7 + 2106r^6 + 3996r^5 - 8010r^4 - 25923r^3 - 3056r^2 + 23364r + 7280}$$

Which for the root $r = 3$ becomes

$$a_5 = \frac{21}{869440}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r}{3r^2-5r-2}$	$\frac{3}{10}$
a_2	$\frac{r(1+r)}{9r^4-12r^3-23r^2+18r+8}$	$\frac{3}{65}$
a_3	$\frac{(1+r)(2+r)}{27r^5+27r^4-153r^3-107r^2+150r+56}$	$\frac{1}{208}$
a_4	$\frac{(2+r)(3+r)}{81r^6+351r^5-189r^4-1851r^3-620r^2+1668r+560}$	$\frac{3}{7904}$
a_5	$\frac{(4+r)(3+r)}{243r^7+2106r^6+3996r^5-8010r^4-25923r^3-3056r^2+23364r+7280}$	$\frac{21}{869440}$

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x^3\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots\right) \\
&= x^3\left(1 + \frac{3x}{10} + \frac{3x^2}{65} + \frac{x^3}{208} + \frac{3x^4}{7904} + \frac{21x^5}{869440} + O(x^6)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$
3b_n(n+r)(n+r-1) - b_{n-1}(n+r-1) - 8b_n(n+r) + 6b_n = 0 \tag{3}
$$

Solving for b_n from recursive equation (4) gives

$$
b_n = \frac{b_{n-1}(n+r-1)}{3n^2 + 6nr + 3r^2 - 11n - 11r + 6} \tag{4}
$$

Which for the root $r = \frac{2}{3}$ becomes

$$
b_n = \frac{3nb_{n-1} - b_{n-1}}{9n^2 - 21n} \tag{5}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{2}{3}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$
b_1 = \frac{r}{3r^2 - 5r - 2}
$$

Which for the root $r = \frac{2}{3}$ becomes

$$b_1 = -\frac{1}{6}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{r}{3r^2-5r-2}$	$-\frac{1}{6}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{r(1+r)}{9r^4 - 12r^3 - 23r^2 + 18r + 8}$$

Which for the root $r = \frac{2}{3}$ becomes

$$b_2 = \frac{5}{36}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{r}{3r^2-5r-2}$	$-\frac{1}{6}$
b_2	$\frac{r(1+r)}{9r^4-12r^3-23r^2+18r+8}$	$\frac{5}{36}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{(1+r)(2+r)}{27r^5 + 27r^4 - 153r^3 - 107r^2 + 150r + 56}$$

Which for the root $r = \frac{2}{3}$ becomes

$$b_3 = \frac{5}{81}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{r}{3r^2-5r-2}$	$-\frac{1}{6}$
b_2	$\frac{r(1+r)}{9r^4-12r^3-23r^2+18r+8}$	$\frac{5}{36}$
b_3	$\frac{(1+r)(2+r)}{27r^5+27r^4-153r^3-107r^2+150r+56}$	$\frac{5}{81}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{(2+r)(3+r)}{81r^6 + 351r^5 - 189r^4 - 1851r^3 - 620r^2 + 1668r + 560}$$

Which for the root $r = \frac{2}{3}$ becomes

$$b_4 = \frac{11}{972}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{r}{3r^2-5r-2}$	$-\frac{1}{6}$
b_2	$\frac{r(1+r)}{9r^4-12r^3-23r^2+18r+8}$	$\frac{5}{36}$
b_3	$\frac{(1+r)(2+r)}{27r^5+27r^4-153r^3-107r^2+150r+56}$	$\frac{5}{81}$
b_4	$\frac{(2+r)(3+r)}{81r^6+351r^5-189r^4-1851r^3-620r^2+1668r+560}$	$\frac{11}{972}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{(4+r)(3+r)}{243r^7 + 2106r^6 + 3996r^5 - 8010r^4 - 25923r^3 - 3056r^2 + 23364r + 7280}$$

Which for the root $r = \frac{2}{3}$ becomes

$$b_5 = \frac{77}{58320}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{r}{3r^2-5r-2}$	$-\frac{1}{6}$
b_2	$\frac{r(1+r)}{9r^4-12r^3-23r^2+18r+8}$	$\frac{5}{36}$
b_3	$\frac{(1+r)(2+r)}{27r^5+27r^4-153r^3-107r^2+150r+56}$	$\frac{5}{81}$
b_4	$\frac{(2+r)(3+r)}{81r^6+351r^5-189r^4-1851r^3-620r^2+1668r+560}$	$\frac{11}{972}$
b_5	$\frac{(4+r)(3+r)}{243r^7+2106r^6+3996r^5-8010r^4-25923r^3-3056r^2+23364r+7280}$	$\frac{77}{58320}$

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= x^3 \left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \right) \\
&= x^{\frac{2}{3}} \left(1 - \frac{x}{6} + \frac{5x^2}{36} + \frac{5x^3}{81} + \frac{11x^4}{972} + \frac{77x^5}{58320} + O(x^6)\right)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^3 \left(1 + \frac{3x}{10} + \frac{3x^2}{65} + \frac{x^3}{208} + \frac{3x^4}{7904} + \frac{21x^5}{869440} + O(x^6)\right) \\
&\quad + c_2 x^{\frac{2}{3}} \left(1 - \frac{x}{6} + \frac{5x^2}{36} + \frac{5x^3}{81} + \frac{11x^4}{972} + \frac{77x^5}{58320} + O(x^6)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^3 \left(1 + \frac{3x}{10} + \frac{3x^2}{65} + \frac{x^3}{208} + \frac{3x^4}{7904} + \frac{21x^5}{869440} + O(x^6)\right) \\
&\quad + c_2 x^{\frac{2}{3}} \left(1 - \frac{x}{6} + \frac{5x^2}{36} + \frac{5x^3}{81} + \frac{11x^4}{972} + \frac{77x^5}{58320} + O(x^6)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x^3 \left(1 + \frac{3x}{10} + \frac{3x^2}{65} + \frac{x^3}{208} + \frac{3x^4}{7904} + \frac{21x^5}{869440} + O(x^6)\right) \\
&\quad + c_2 x^{\frac{2}{3}} \left(1 - \frac{x}{6} + \frac{5x^2}{36} + \frac{5x^3}{81} + \frac{11x^4}{972} + \frac{77x^5}{58320} + O(x^6)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1 x^3 \left(1 + \frac{3x}{10} + \frac{3x^2}{65} + \frac{x^3}{208} + \frac{3x^4}{7904} + \frac{21x^5}{869440} + O(x^6)\right) \\
&\quad + c_2 x^{\frac{2}{3}} \left(1 - \frac{x}{6} + \frac{5x^2}{36} + \frac{5x^3}{81} + \frac{11x^4}{972} + \frac{77x^5}{58320} + O(x^6)\right)
\end{aligned}
$$

Verified OK.

17.13.1 Maple step by step solution

Let's solve

$3x^2y'' + (-x^2 - 8x)\,y' + 6y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2y}{x^2} + \frac{(x+8)y'}{3x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(x+8)y'}{3x} + \frac{2y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -\frac{x+8}{3x}, P_3(x) = \frac{2}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = -\frac{8}{3}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = 2$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $3x^2y'' - x(x+8)\,y' + 6y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  ◦ Shift index using $k- >k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-2+3r)\,(-3+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(3k+3r-2)\,(k+r-3) - a_{k-1}(k+r-1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-2+3r)\,(-3+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{3, \frac{2}{3}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$3\left(k+r-\frac{2}{3}\right)(k+r-3)\, a_k - a_{k-1}(k+r-1) = 0$$

- Shift index using $k- >k+1$

$$3\left(k+\frac{1}{3}+r\right)(k-2+r)\, a_{k+1} - a_k(k+r) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k(k+r)}{(3k+1+3r)(k-2+r)}$$

- Recursion relation for $r=3$

$$a_{k+1} = \frac{a_k(k+3)}{(3k+10)(k+1)}$$

- Solution for $r=3$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+1} = \frac{a_k(k+3)}{(3k+10)(k+1)}\right]$$

- Recursion relation for $r=\frac{2}{3}$

$$a_{k+1} = \frac{a_k\left(k+\frac{2}{3}\right)}{(3k+3)\left(k-\frac{4}{3}\right)}$$

- Solution for $r=\frac{2}{3}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{2}{3}}, a_{k+1} = \frac{a_k\left(k+\frac{2}{3}\right)}{(3k+3)\left(k-\frac{4}{3}\right)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+3}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{2}{3}}\right), a_{1+k} = \frac{a_k(k+3)}{(3k+10)(1+k)}, b_{1+k} = \frac{b_k\left(k+\frac{2}{3}\right)}{(3k+3)\left(k-\frac{4}{3}\right)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric solution.
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 47

```
Order:=6;
dsolve(3*x^2*diff(y(x),x$2)-x*(x+8)*diff(y(x),x)+6*y(x)=0,y(x),type='series',x=0);
```

$$
\begin{aligned}
y(x) = {} & c_1 x^{\frac{2}{3}} \left(1 - \frac{1}{6}x + \frac{5}{36}x^2 + \frac{5}{81}x^3 + \frac{11}{972}x^4 + \frac{77}{58320}x^5 + \mathrm{O}\left(x^6\right)\right) \\
& + c_2 x^3 \left(1 + \frac{3}{10}x + \frac{3}{65}x^2 + \frac{1}{208}x^3 + \frac{3}{7904}x^4 + \frac{21}{869440}x^5 + \mathrm{O}\left(x^6\right)\right)
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 88

```
AsymptoticDSolveValue[3*x^2*y''[x]-x*(x+8)*y'[x]+6*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{21x^5}{869440}+\frac{3x^4}{7904}+\frac{x^3}{208}+\frac{3x^2}{65}+\frac{3x}{10}+1\right)x^3 + c_2\left(\frac{77x^5}{58320}+\frac{11x^4}{972}+\frac{5x^3}{81}+\frac{5x^2}{36}-\frac{x}{6}+1\right)x^{2/3}$$

17.14 problem 15

Internal problem ID [2930]
Internal file name [`OUTPUT/2422_Sunday_June_05_2022_03_09_02_AM_8091972/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.4. page 758
Problem number: 15.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2x^2y'' - x(1+2x)\,y' + 2(-1+4x)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2x^2y'' + \left(-2x^2 - x\right)y' + (8x-2)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{1+2x}{2x}$$
$$q(x) = \frac{-1+4x}{x^2}$$

Table 438: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{1+2x}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{-1+4x}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2x^2y'' + (-2x^2 - x)\, y' + (8x - 2)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + (-2x^2 - x) \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + (8x-2) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-2x^{1+n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 8x^{1+n+r} a_n\right) + \sum_{n=0}^{\infty} \left(-2a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-2x^{1+n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-2a_{n-1}(n+r-1)\, x^{n+r}\right)$$

$$\sum_{n=0}^{\infty} 8x^{1+n+r} a_n = \sum_{n=1}^{\infty} 8a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-2a_{n-1}(n+r-1)\, x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=1}^{\infty} 8a_{n-1} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-2a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$2x^{n+r} a_n (n+r)(n+r-1) - x^{n+r} a_n (n+r) - 2a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$2x^r a_0 r(-1+r) - x^r a_0 r - 2a_0 x^r = 0$$

Or

$$\left(2x^r r(-1+r) - x^r r - 2x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(2r^2 - 3r - 2\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 - 3r - 2 = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= 2 \\ r_2 &= -\frac{1}{2} \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(2r^2 - 3r - 2\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{5}{2}$ is not an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+2} \\ y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n-\frac{1}{2}} \end{aligned}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$2a_n(n+r)(n+r-1) - 2a_{n-1}(n+r-1) - a_n(n+r) + 8a_{n-1} - 2a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{2a_{n-1}(n+r-5)}{2n^2 + 4nr + 2r^2 - 3n - 3r - 2} \tag{4}$$

Which for the root $r = 2$ becomes

$$a_n = \frac{2a_{n-1}(n-3)}{n(2n+5)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 2$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-8 + 2r}{2r^2 + r - 3}$$

Which for the root $r = 2$ becomes

$$a_1 = -\frac{4}{7}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-8+2r}{2r^2+r-3}$	$-\frac{4}{7}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{4r^2 - 28r + 48}{4r^4 + 12r^3 - r^2 - 15r}$$

Which for the root $r = 2$ becomes

$$a_2 = \frac{4}{63}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-8+2r}{2r^2+r-3}$	$-\frac{4}{7}$
a_2	$\frac{4r^2-28r+48}{4r^4+12r^3-r^2-15r}$	$\frac{4}{63}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{8(-3 + r)(-4 + r)(-2 + r)}{8r^6 + 60r^5 + 134r^4 + 45r^3 - 142r^2 - 105r}$$

Which for the root $r = 2$ becomes

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-8+2r}{2r^2+r-3}$	$-\frac{4}{7}$
a_2	$\frac{4r^2-28r+48}{4r^4+12r^3-r^2-15r}$	$\frac{4}{63}$
a_3	$\frac{8(-3+r)(-4+r)(-2+r)}{8r^6+60r^5+134r^4+45r^3-142r^2-105r}$	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{16(-3+r)(-4+r)(-2+r)}{(8r^4+68r^3+202r^2+247r+105)\, r\,(2r^2+13r+18)}$$

Which for the root $r = 2$ becomes

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-8+2r}{2r^2+r-3}$	$-\frac{4}{7}$
a_2	$\frac{4r^2-28r+48}{4r^4+12r^3-r^2-15r}$	$\frac{4}{63}$
a_3	$\frac{8(-3+r)(-4+r)(-2+r)}{8r^6+60r^5+134r^4+45r^3-142r^2-105r}$	0
a_4	$\frac{16(-3+r)(-4+r)(-2+r)}{(8r^4+68r^3+202r^2+247r+105)r(2r^2+13r+18)}$	0

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{32(-3+r)(-4+r)(-2+r)}{(8r^4+68r^3+202r^2+247r+105)(2r^2+13r+18)(2r^2+17r+33)}$$

Which for the root $r = 2$ becomes

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-8+2r}{2r^2+r-3}$	$-\frac{4}{7}$
a_2	$\frac{4r^2-28r+48}{4r^4+12r^3-r^2-15r}$	$\frac{4}{63}$
a_3	$\frac{8(-3+r)(-4+r)(-2+r)}{8r^6+60r^5+134r^4+45r^3-142r^2-105r}$	0
a_4	$\frac{16(-3+r)(-4+r)(-2+r)}{(8r^4+68r^3+202r^2+247r+105)r(2r^2+13r+18)}$	0
a_5	$\frac{32(-3+r)(-4+r)(-2+r)}{(8r^4+68r^3+202r^2+247r+105)(2r^2+13r+18)(2r^2+17r+33)}$	0

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x^2\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= x^2\left(1 - \frac{4x}{7} + \frac{4x^2}{63} + O(x^6)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$
2b_n(n+r)(n+r-1) - 2b_{n-1}(n+r-1) - b_n(n+r) + 8b_{n-1} - 2b_n = 0 \tag{3}
$$

Solving for b_n from recursive equation (4) gives

$$
b_n = \frac{2b_{n-1}(n+r-5)}{2n^2+4nr+2r^2-3n-3r-2} \tag{4}
$$

Which for the root $r = -\frac{1}{2}$ becomes

$$
b_n = \frac{b_{n-1}(2n-11)}{n(2n-5)} \tag{5}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$
b_1 = \frac{-8+2r}{2r^2+r-3}
$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_1 = 3$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-8+2r}{2r^2+r-3}$	3

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{4r^2 - 28r + 48}{4r^4 + 12r^3 - r^2 - 15r}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_2 = \frac{21}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-8+2r}{2r^2+r-3}$	3
b_2	$\frac{4r^2-28r+48}{4r^4+12r^3-r^2-15r}$	$\frac{21}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{8(-3+r)(-4+r)(-2+r)}{8r^6 + 60r^5 + 134r^4 + 45r^3 - 142r^2 - 105r}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_3 = -\frac{35}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-8+2r}{2r^2+r-3}$	3
b_2	$\frac{4r^2-28r+48}{4r^4+12r^3-r^2-15r}$	$\frac{21}{2}$
b_3	$\frac{8(-3+r)(-4+r)(-2+r)}{8r^6+60r^5+134r^4+45r^3-142r^2-105r}$	$-\frac{35}{2}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{16(-3+r)(-4+r)(-2+r)}{(8r^4+68r^3+202r^2+247r+105)\, r\, (2r^2+13r+18)}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_4 = \frac{35}{8}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-8+2r}{2r^2+r-3}$	3
b_2	$\frac{4r^2-28r+48}{4r^4+12r^3-r^2-15r}$	$\frac{21}{2}$
b_3	$\frac{8(-3+r)(-4+r)(-2+r)}{8r^6+60r^5+134r^4+45r^3-142r^2-105r}$	$-\frac{35}{2}$
b_4	$\frac{16(-3+r)(-4+r)(-2+r)}{(8r^4+68r^3+202r^2+247r+105)r(2r^2+13r+18)}$	$\frac{35}{8}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{32(-3+r)(-4+r)(-2+r)}{(8r^4+68r^3+202r^2+247r+105)(2r^2+13r+18)(2r^2+17r+33)}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_5 = -\frac{7}{40}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-8+2r}{2r^2+r-3}$	3
b_2	$\frac{4r^2-28r+48}{4r^4+12r^3-r^2-15r}$	$\frac{21}{2}$
b_3	$\frac{8(-3+r)(-4+r)(-2+r)}{8r^6+60r^5+134r^4+45r^3-142r^2-105r}$	$-\frac{35}{2}$
b_4	$\frac{16(-3+r)(-4+r)(-2+r)}{(8r^4+68r^3+202r^2+247r+105)r(2r^2+13r+18)}$	$\frac{35}{8}$
b_5	$\frac{32(-3+r)(-4+r)(-2+r)}{(8r^4+68r^3+202r^2+247r+105)(2r^2+13r+18)(2r^2+17r+33)}$	$-\frac{7}{40}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned}y_2(x) &= x^2\left(b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 \dots\right)\\ &= \frac{1 + 3x + \frac{21x^2}{2} - \frac{35x^3}{2} + \frac{35x^4}{8} - \frac{7x^5}{40} + O(x^6)}{\sqrt{x}}\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}y_h(x) &= c_1y_1(x) + c_2y_2(x)\\ &= c_1x^2\left(1 - \frac{4x}{7} + \frac{4x^2}{63} + O(x^6)\right) + \frac{c_2\left(1 + 3x + \frac{21x^2}{2} - \frac{35x^3}{2} + \frac{35x^4}{8} - \frac{7x^5}{40} + O(x^6)\right)}{\sqrt{x}}\end{aligned}$$

Hence the final solution is

$$\begin{aligned}y &= y_h\\ &= c_1x^2\left(1 - \frac{4x}{7} + \frac{4x^2}{63} + O(x^6)\right) + \frac{c_2\left(1 + 3x + \frac{21x^2}{2} - \frac{35x^3}{2} + \frac{35x^4}{8} - \frac{7x^5}{40} + O(x^6)\right)}{\sqrt{x}}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1x^2\left(1 - \frac{4x}{7} + \frac{4x^2}{63} + O(x^6)\right) + \frac{c_2\left(1 + 3x + \frac{21x^2}{2} - \frac{35x^3}{2} + \frac{35x^4}{8} - \frac{7x^5}{40} + O(x^6)\right)}{\sqrt{x}} \tag{1}$$

Verification of solutions

$$y = c_1x^2\left(1 - \frac{4x}{7} + \frac{4x^2}{63} + O(x^6)\right) + \frac{c_2\left(1 + 3x + \frac{21x^2}{2} - \frac{35x^3}{2} + \frac{35x^4}{8} - \frac{7x^5}{40} + O(x^6)\right)}{\sqrt{x}}$$

Verified OK.

17.14.1 Maple step by step solution

Let's solve

$2x^2y'' + (-2x^2 - x)\,y' + (8x - 2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$$y'' = -\frac{(-1+4x)y}{x^2} + \frac{(1+2x)y'}{2x}$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$y'' - \frac{(1+2x)y'}{2x} + \frac{(-1+4x)y}{x^2} = 0$$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

$$\left[P_2(x) = -\frac{1+2x}{2x}, P_3(x) = \frac{-1+4x}{x^2}\right]$$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

$$(x \cdot P_2(x))\bigg|_{x=0} = -\frac{1}{2}$$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$(x^2 \cdot P_3(x))\bigg|_{x=0} = -1$$

  - $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$$x_0 = 0$$

- Multiply by denominators

$$2x^2y'' - x(1+2x)\,y' + (8x-2)\,y = 0$$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

  - Shift index using $k- >k-m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

  - Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

  - Shift index using $k- >k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+2r)\,(-2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r+1)\,(k+r-2) - 2a_{k-1}(k-5+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+2r)\,(-2+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{2, -\frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$2\left(k+\frac{1}{2}+r\right)(k+r-2)\, a_k - 2a_{k-1}(k-5+r) = 0$$

- Shift index using $k- >k+1$

$$2\left(k+\frac{3}{2}+r\right)(k+r-1)\, a_{k+1} - 2a_k(k+r-4) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{2a_k(k+r-4)}{(2k+3+2r)(k+r-1)}$$

- Recursion relation for $r=2$; series terminates at $k=2$

$$a_{k+1} = \frac{2a_k(k-2)}{(2k+7)(k+1)}$$

- Apply recursion relation for $k=0$

$$a_1 = -\frac{4a_0}{7}$$

- Apply recursion relation for $k=1$

$$a_2 = -\frac{a_1}{9}$$

- Express in terms of a_0

$$a_2 = \frac{4a_0}{63}$$

- Terminating series solution of the ODE for $r=2$. Use reduction of order to find the second li

$$y = a_0 \cdot \left(1 - \frac{4}{7}x + \frac{4}{63}x^2\right)$$

- Recursion relation for $r=-\frac{1}{2}$

$$a_{k+1} = \frac{2a_k\left(k-\frac{9}{2}\right)}{(2k+2)\left(k-\frac{3}{2}\right)}$$

- Solution for $r = -\frac{1}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+1} = \frac{2a_k \left(k-\frac{9}{2}\right)}{(2k+2)\left(k-\frac{3}{2}\right)}\right]$$

- Combine solutions and rename parameters

  $$\left[y = a_0 \cdot \left(1 - \frac{4}{7}x + \frac{4}{63}x^2\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-\frac{1}{2}}\right), b_{1+k} = \frac{2b_k \left(k-\frac{9}{2}\right)}{(2+2k)\left(k-\frac{3}{2}\right)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric solution.
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 41

```
Order:=6;
dsolve(2*x^2*diff(y(x),x$2)-x*(1+2*x)*diff(y(x),x)+2*(4*x-1)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_1\left(1+3x+\frac{21}{2}x^2-\frac{35}{2}x^3+\frac{35}{8}x^4-\frac{7}{40}x^5+\mathrm{O}\left(x^6\right)\right)}{\sqrt{x}} + c_2x^2\left(1-\frac{4}{7}x+\frac{4}{63}x^2+\mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 65

```
AsymptoticDSolveValue[2*x^2*y''[x]-x*(1+2*x)*y'[x]+2*(4*x-1)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{4x^2}{63}-\frac{4x}{7}+1\right)x^2+\frac{c_2\left(-\frac{7x^5}{40}+\frac{35x^4}{8}-\frac{35x^3}{2}+\frac{21x^2}{2}+3x+1\right)}{\sqrt{x}}$$

17.15 problem 16

Internal problem ID [2931]
Internal file name [`OUTPUT/2423_Sunday_June_05_2022_03_09_07_AM_220765/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.4. page 758
Problem number: 16.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + x(1-x)\,y' - (x+5)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + \left(-x^2 + x\right)y' + (-x-5)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{x-1}{x}$$
$$q(x) = -\frac{x+5}{x^2}$$

Table 440: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{x-1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{x+5}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + \left(-x^2 + x\right) y' + (-x - 5)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned} & x^2 \left(\sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2}\right) \\ & + \left(-x^2 + x\right) \left(\sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}\right) + (-x - 5) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \end{aligned} \tag{1}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n (n+r)\right) \\
&+ \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n\right) + \sum_{n=0}^{\infty} \left(-5 a_n x^{n+r}\right) = 0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n (n+r)\right) &= \sum_{n=1}^{\infty} \left(-a_{n-1} (n+r-1)\, x^{n+r}\right) \\
\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n\right) &= \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r}\right)
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-a_{n-1} (n+r-1)\, x^{n+r}\right) \\
&+ \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-5 a_n x^{n+r}\right) = 0
\end{aligned} \tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
x^{n+r} a_n (n+r)(n+r-1) + x^{n+r} a_n (n+r) - 5 a_n x^{n+r} = 0
$$

When $n=0$ the above becomes

$$
x^r a_0 r(-1+r) + x^r a_0 r - 5 a_0 x^r = 0
$$

Or

$$
\left(x^r r(-1+r) + x^r r - 5 x^r\right) a_0 = 0
$$

Since $a_0 \neq 0$ then the above simplifies to

$$
\left(r^2 - 5\right) x^r = 0
$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - 5 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \sqrt{5}$$
$$r_2 = -\sqrt{5}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(r^2 - 5)\, x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 2\sqrt{5}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\sqrt{5}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-\sqrt{5}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) - a_{n-1}(n+r-1) + a_n(n+r) - a_{n-1} - 5a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}(n+r)}{n^2 + 2nr + r^2 - 5} \tag{4}$$

Which for the root $r = \sqrt{5}$ becomes

$$a_n = \frac{a_{n-1}(n+\sqrt{5})}{n\,(2\sqrt{5}+n)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \sqrt{5}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{1+r}{r^2+2r-4}$$

Which for the root $r = \sqrt{5}$ becomes

$$a_1 = \frac{\sqrt{5}+1}{1+2\sqrt{5}}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1+r}{r^2+2r-4}$	$\frac{\sqrt{5}+1}{1+2\sqrt{5}}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{(1+r)(2+r)}{(r^2+2r-4)(r^2+4r-1)}$$

Which for the root $r = \sqrt{5}$ becomes

$$a_2 = \frac{2+\sqrt{5}}{4+8\sqrt{5}}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1+r}{r^2+2r-4}$	$\frac{\sqrt{5}+1}{1+2\sqrt{5}}$
a_2	$\frac{(1+r)(2+r)}{(r^2+2r-4)(r^2+4r-1)}$	$\frac{2+\sqrt{5}}{4+8\sqrt{5}}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{(1+r)(2+r)(3+r)}{(r^2+2r-4)(r^2+4r-1)(r^2+6r+4)}$$

Which for the root $r = \sqrt{5}$ becomes

$$a_3 = \frac{(2+\sqrt{5})(3+\sqrt{5})}{276+96\sqrt{5}}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1+r}{r^2+2r-4}$	$\frac{\sqrt{5}+1}{1+2\sqrt{5}}$
a_2	$\frac{(1+r)(2+r)}{(r^2+2r-4)(r^2+4r-1)}$	$\frac{2+\sqrt{5}}{4+8\sqrt{5}}$
a_3	$\frac{(1+r)(2+r)(3+r)}{(r^2+2r-4)(r^2+4r-1)(r^2+6r+4)}$	$\frac{(2+\sqrt{5})(3+\sqrt{5})}{276+96\sqrt{5}}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{(1+r)(2+r)(3+r)(4+r)}{(r^2+2r-4)(r^2+4r-1)(r^2+6r+4)(r^2+8r+11)}$$

Which for the root $r = \sqrt{5}$ becomes

$$a_4 = \frac{(4+\sqrt{5})(3+\sqrt{5})}{2208+768\sqrt{5}}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1+r}{r^2+2r-4}$	$\frac{\sqrt{5}+1}{1+2\sqrt{5}}$
a_2	$\frac{(1+r)(2+r)}{(r^2+2r-4)(r^2+4r-1)}$	$\frac{2+\sqrt{5}}{4+8\sqrt{5}}$
a_3	$\frac{(1+r)(2+r)(3+r)}{(r^2+2r-4)(r^2+4r-1)(r^2+6r+4)}$	$\frac{(2+\sqrt{5})(3+\sqrt{5})}{276+96\sqrt{5}}$
a_4	$\frac{(1+r)(2+r)(3+r)(4+r)}{(r^2+2r-4)(r^2+4r-1)(r^2+6r+4)(r^2+8r+11)}$	$\frac{(4+\sqrt{5})(3+\sqrt{5})}{2208+768\sqrt{5}}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{(1+r)(2+r)(3+r)(4+r)(5+r)}{(r^2+2r-4)(r^2+4r-1)(r^2+6r+4)(r^2+8r+11)(r^2+10r+20)}$$

Which for the root $r = \sqrt{5}$ becomes

$$a_5 = \frac{(3+\sqrt{5})(4+\sqrt{5})(5+\sqrt{5})}{41280\sqrt{5}+93600}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1+r}{r^2+2r-4}$	$\frac{\sqrt{5}+1}{1+2\sqrt{5}}$
a_2	$\frac{(1+r)(2+r)}{(r^2+2r-4)(r^2+4r-1)}$	$\frac{2+\sqrt{5}}{4+8\sqrt{5}}$
a_3	$\frac{(1+r)(2+r)(3+r)}{(r^2+2r-4)(r^2+4r-1)(r^2+6r+4)}$	$\frac{(2+\sqrt{5})(3+\sqrt{5})}{276+96\sqrt{5}}$
a_4	$\frac{(1+r)(2+r)(3+r)(4+r)}{(r^2+2r-4)(r^2+4r-1)(r^2+6r+4)(r^2+8r+11)}$	$\frac{(4+\sqrt{5})(3+\sqrt{5})}{2208+768\sqrt{5}}$
a_5	$\frac{(1+r)(2+r)(3+r)(4+r)(5+r)}{(r^2+2r-4)(r^2+4r-1)(r^2+6r+4)(r^2+8r+11)(r^2+10r+20)}$	$\frac{(3+\sqrt{5})(4+\sqrt{5})(5+\sqrt{5})}{41280\sqrt{5}+93600}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\sqrt{5}}(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots) \\ &= x^{\sqrt{5}}\left(1 + \frac{(\sqrt{5}+1)x}{1+2\sqrt{5}} + \frac{(2+\sqrt{5})x^2}{4+8\sqrt{5}} + \frac{(2+\sqrt{5})(3+\sqrt{5})x^3}{276+96\sqrt{5}} + \frac{(4+\sqrt{5})(3+\sqrt{5})x^4}{2208+768\sqrt{5}} + \frac{(3+\sqrt{5}\right.\end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$b_n(n+r)(n+r-1) - b_{n-1}(n+r-1) + b_n(n+r) - b_{n-1} - 5b_n = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = \frac{b_{n-1}(n+r)}{n^2+2nr+r^2-5} \tag{4}$$

Which for the root $r = -\sqrt{5}$ becomes

$$b_n = \frac{b_{n-1}\left(-n+\sqrt{5}\right)}{n\left(2\sqrt{5}-n\right)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\sqrt{5}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{1+r}{r^2+2r-4}$$

Which for the root $r = -\sqrt{5}$ becomes

$$b_1 = \frac{\sqrt{5}-1}{-1+2\sqrt{5}}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1+r}{r^2+2r-4}$	$\frac{\sqrt{5}-1}{-1+2\sqrt{5}}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{(1+r)(2+r)}{(r^2+2r-4)(r^2+4r-1)}$$

Which for the root $r = -\sqrt{5}$ becomes

$$b_2 = \frac{-2+\sqrt{5}}{-4+8\sqrt{5}}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1+r}{r^2+2r-4}$	$\frac{\sqrt{5}-1}{-1+2\sqrt{5}}$
b_2	$\frac{(1+r)(2+r)}{(r^2+2r-4)(r^2+4r-1)}$	$\frac{-2+\sqrt{5}}{-4+8\sqrt{5}}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{(1+r)(2+r)(3+r)}{(r^2+2r-4)(r^2+4r-1)(r^2+6r+4)}$$

Which for the root $r = -\sqrt{5}$ becomes

$$b_3 = \frac{(\sqrt{5}-3)(-2+\sqrt{5})}{276-96\sqrt{5}}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1+r}{r^2+2r-4}$	$\frac{\sqrt{5}-1}{-1+2\sqrt{5}}$
b_2	$\frac{(1+r)(2+r)}{(r^2+2r-4)(r^2+4r-1)}$	$\frac{-2+\sqrt{5}}{-4+8\sqrt{5}}$
b_3	$\frac{(1+r)(2+r)(3+r)}{(r^2+2r-4)(r^2+4r-1)(r^2+6r+4)}$	$\frac{(\sqrt{5}-3)(-2+\sqrt{5})}{276-96\sqrt{5}}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{(1+r)(2+r)(3+r)(4+r)}{(r^2+2r-4)(r^2+4r-1)(r^2+6r+4)(r^2+8r+11)}$$

Which for the root $r = -\sqrt{5}$ becomes

$$b_4 = \frac{(\sqrt{5}-3)(-4+\sqrt{5})}{2208-768\sqrt{5}}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1+r}{r^2+2r-4}$	$\frac{\sqrt{5}-1}{-1+2\sqrt{5}}$
b_2	$\frac{(1+r)(2+r)}{(r^2+2r-4)(r^2+4r-1)}$	$\frac{-2+\sqrt{5}}{-4+8\sqrt{5}}$
b_3	$\frac{(1+r)(2+r)(3+r)}{(r^2+2r-4)(r^2+4r-1)(r^2+6r+4)}$	$\frac{(\sqrt{5}-3)(-2+\sqrt{5})}{276-96\sqrt{5}}$
b_4	$\frac{(1+r)(2+r)(3+r)(4+r)}{(r^2+2r-4)(r^2+4r-1)(r^2+6r+4)(r^2+8r+11)}$	$\frac{(\sqrt{5}-3)(-4+\sqrt{5})}{2208-768\sqrt{5}}$

For $n=5$, using the above recursive equation gives

$$b_5 = \frac{(1+r)(2+r)(3+r)(4+r)(5+r)}{(r^2+2r-4)(r^2+4r-1)(r^2+6r+4)(r^2+8r+11)(r^2+10r+20)}$$

Which for the root $r=-\sqrt{5}$ becomes

$$b_5 = \frac{(\sqrt{5}-3)(-4+\sqrt{5})(-5+\sqrt{5})}{41280\sqrt{5}-93600}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1+r}{r^2+2r-4}$	$\frac{\sqrt{5}-1}{-1+2\sqrt{5}}$
b_2	$\frac{(1+r)(2+r)}{(r^2+2r-4)(r^2+4r-1)}$	$\frac{-2+\sqrt{5}}{-4+8\sqrt{5}}$
b_3	$\frac{(1+r)(2+r)(3+r)}{(r^2+2r-4)(r^2+4r-1)(r^2+6r+4)}$	$\frac{(\sqrt{5}-3)(-2+\sqrt{5})}{276-96\sqrt{5}}$
b_4	$\frac{(1+r)(2+r)(3+r)(4+r)}{(r^2+2r-4)(r^2+4r-1)(r^2+6r+4)(r^2+8r+11)}$	$\frac{(\sqrt{5}-3)(-4+\sqrt{5})}{2208-768\sqrt{5}}$
b_5	$\frac{(1+r)(2+r)(3+r)(4+r)(5+r)}{(r^2+2r-4)(r^2+4r-1)(r^2+6r+4)(r^2+8r+11)(r^2+10r+20)}$	$\frac{(\sqrt{5}-3)(-4+\sqrt{5})(-5+\sqrt{5})}{41280\sqrt{5}-93600}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned}
y_2(x) &= x^{\sqrt{5}}\left(b_0+b_1x+b_2x^2+b_3x^3+b_4x^4+b_5x^5+b_6x^6\dots\right) \\
&= x^{-\sqrt{5}}\left(1+\frac{(\sqrt{5}-1)\,x}{-1+2\sqrt{5}}+\frac{(-2+\sqrt{5})\,x^2}{-4+8\sqrt{5}}+\frac{(\sqrt{5}-3)(-2+\sqrt{5})\,x^3}{276-96\sqrt{5}}+\frac{(\sqrt{5}-3)(-4+\sqrt{5})\,x^4}{2208-768\sqrt{5}}+\right.
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1y_1(x)+c_2y_2(x) \\
&= c_1x^{\sqrt{5}}\left(1+\frac{(\sqrt{5}+1)\,x}{1+2\sqrt{5}}+\frac{(2+\sqrt{5})\,x^2}{4+8\sqrt{5}}+\frac{(2+\sqrt{5})(3+\sqrt{5})\,x^3}{276+96\sqrt{5}}\right. \\
&\qquad\left.+\frac{(4+\sqrt{5})(3+\sqrt{5})\,x^4}{2208+768\sqrt{5}}+\frac{(3+\sqrt{5})(4+\sqrt{5})(5+\sqrt{5})\,x^5}{41280\sqrt{5}+93600}+O(x^6)\right) \\
&\quad+c_2x^{-\sqrt{5}}\left(1+\frac{(\sqrt{5}-1)\,x}{-1+2\sqrt{5}}+\frac{(-2+\sqrt{5})\,x^2}{-4+8\sqrt{5}}+\frac{(\sqrt{5}-3)(-2+\sqrt{5})\,x^3}{276-96\sqrt{5}}\right. \\
&\qquad\left.+\frac{(\sqrt{5}-3)(-4+\sqrt{5})\,x^4}{2208-768\sqrt{5}}+\frac{(\sqrt{5}-3)(-4+\sqrt{5})(-5+\sqrt{5})\,x^5}{41280\sqrt{5}-93600}+O(x^6)\right)
\end{aligned}$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^{\sqrt{5}} \left(1+\frac{\left(\sqrt{5}+1\right) x}{1+2\sqrt{5}}+\frac{\left(2+\sqrt{5}\right) x^{2}}{4+8\sqrt{5}}+\frac{\left(2+\sqrt{5}\right) \left(3+\sqrt{5}\right) x^{3}}{276+96\sqrt{5}}\right. \\
&\quad \left.+\frac{\left(4+\sqrt{5}\right) \left(3+\sqrt{5}\right) x^{4}}{2208+768\sqrt{5}}+\frac{\left(3+\sqrt{5}\right) \left(4+\sqrt{5}\right) \left(5+\sqrt{5}\right) x^{5}}{41280\sqrt{5}+93600}+O\left(x^{6}\right)\right) \\
&\quad + c_2 x^{-\sqrt{5}} \left(1+\frac{\left(\sqrt{5}-1\right) x}{-1+2\sqrt{5}}+\frac{\left(-2+\sqrt{5}\right) x^{2}}{-4+8\sqrt{5}}+\frac{\left(\sqrt{5}-3\right) \left(-2+\sqrt{5}\right) x^{3}}{276-96\sqrt{5}}\right. \\
&\quad \left.+\frac{\left(\sqrt{5}-3\right) \left(-4+\sqrt{5}\right) x^{4}}{2208-768\sqrt{5}}+\frac{\left(\sqrt{5}-3\right) \left(-4+\sqrt{5}\right) \left(-5+\sqrt{5}\right) x^{5}}{41280\sqrt{5}-93600}+O\left(x^{6}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x^{\sqrt{5}} \left(1+\frac{\left(\sqrt{5}+1\right) x}{1+2\sqrt{5}}+\frac{\left(2+\sqrt{5}\right) x^{2}}{4+8\sqrt{5}}+\frac{\left(2+\sqrt{5}\right) \left(3+\sqrt{5}\right) x^{3}}{276+96\sqrt{5}}\right. \\
&\quad \left.+\frac{\left(4+\sqrt{5}\right) \left(3+\sqrt{5}\right) x^{4}}{2208+768\sqrt{5}}+\frac{\left(3+\sqrt{5}\right) \left(4+\sqrt{5}\right) \left(5+\sqrt{5}\right) x^{5}}{41280\sqrt{5}+93600}+O\left(x^{6}\right)\right) \\
&\quad + c_2 x^{-\sqrt{5}} \left(1+\frac{\left(\sqrt{5}-1\right) x}{-1+2\sqrt{5}}+\frac{\left(-2+\sqrt{5}\right) x^{2}}{-4+8\sqrt{5}}+\frac{\left(\sqrt{5}-3\right) \left(-2+\sqrt{5}\right) x^{3}}{276-96\sqrt{5}}\right. \\
&\quad \left.+\frac{\left(\sqrt{5}-3\right) \left(-4+\sqrt{5}\right) x^{4}}{2208-768\sqrt{5}}+\frac{\left(\sqrt{5}-3\right) \left(-4+\sqrt{5}\right) \left(-5+\sqrt{5}\right) x^{5}}{41280\sqrt{5}-93600}+O\left(x^{6}\right)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1 x^{\sqrt{5}} \left(1+\frac{\left(\sqrt{5}+1\right) x}{1+2\sqrt{5}}+\frac{\left(2+\sqrt{5}\right) x^{2}}{4+8\sqrt{5}}+\frac{\left(2+\sqrt{5}\right) \left(3+\sqrt{5}\right) x^{3}}{276+96\sqrt{5}}\right. \\
&\quad \left.+\frac{\left(4+\sqrt{5}\right) \left(3+\sqrt{5}\right) x^{4}}{2208+768\sqrt{5}}+\frac{\left(3+\sqrt{5}\right) \left(4+\sqrt{5}\right) \left(5+\sqrt{5}\right) x^{5}}{41280\sqrt{5}+93600}+O\left(x^{6}\right)\right) \\
&\quad + c_2 x^{-\sqrt{5}} \left(1+\frac{\left(\sqrt{5}-1\right) x}{-1+2\sqrt{5}}+\frac{\left(-2+\sqrt{5}\right) x^{2}}{-4+8\sqrt{5}}+\frac{\left(\sqrt{5}-3\right) \left(-2+\sqrt{5}\right) x^{3}}{276-96\sqrt{5}}\right. \\
&\quad \left.+\frac{\left(\sqrt{5}-3\right) \left(-4+\sqrt{5}\right) x^{4}}{2208-768\sqrt{5}}+\frac{\left(\sqrt{5}-3\right) \left(-4+\sqrt{5}\right) \left(-5+\sqrt{5}\right) x^{5}}{41280\sqrt{5}-93600}+O\left(x^{6}\right)\right)
\end{aligned}
$$

Verified OK.

17.15.1 Maple step by step solution

Let's solve

$x^2 y'' + (-x^2 + x) y' + (-x - 5) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(x+5)y}{x^2} + \frac{(x-1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(x-1)y'}{x} - \frac{(x+5)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{x-1}{x}, P_3(x) = -\frac{x+5}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -5$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2 y'' - x(x-1) y' + (-x - 5) y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k- > k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(r^2-5)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k^2+2kr+r^2-5) - a_{k-1}(k+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r^2 - 5 = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{\sqrt{5}, -\sqrt{5}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$a_k(k^2+2kr+r^2-5) - a_{k-1}(k+r) = 0$$

- Shift index using $k- > k+1$

$$a_{k+1}\left((k+1)^2 + 2(k+1)\, r + r^2 - 5\right) - a_k(k+r+1) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k(k+r+1)}{k^2+2kr+r^2+2k+2r-4}$$

- Recursion relation for $r = \sqrt{5}$

$$a_{k+1} = \frac{a_k\left(k+\sqrt{5}+1\right)}{k^2+2k\sqrt{5}+1+2k+2\sqrt{5}}$$

- Solution for $r = \sqrt{5}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\sqrt{5}}, a_{k+1} = \frac{a_k\left(k+\sqrt{5}+1\right)}{k^2+2k\sqrt{5}+1+2k+2\sqrt{5}}\right]$$

- Recursion relation for $r = -\sqrt{5}$

$$a_{k+1} = \frac{a_k\left(k-\sqrt{5}+1\right)}{k^2-2k\sqrt{5}+1+2k-2\sqrt{5}}$$

- Solution for $r = -\sqrt{5}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\sqrt{5}}, a_{k+1} = \frac{a_k \left(k-\sqrt{5}+1\right)}{k^2-2k\sqrt{5}+1+2k-2\sqrt{5}}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+\sqrt{5}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-\sqrt{5}}\right), a_{1+k} = \frac{a_k \left(k+\sqrt{5}+1\right)}{k^2+2k\sqrt{5}+1+2k+2\sqrt{5}}, b_{1+k} = \frac{b_k \left(k-\sqrt{5}+1\right)}{k^2-2k\sqrt{5}+1+2k-2\sqrt{5}}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   <- Kummer successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 503

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+x*(1-x)*diff(y(x),x)-(5+x)*y(x)=0,y(x),type='series',x=0);
```

$$
\begin{aligned}
y(x) = {} & c_1 x^{-\sqrt{5}} \left(1+\frac{\sqrt{5}-1}{-1+2\sqrt{5}}x+\frac{-2+\sqrt{5}}{-4+8\sqrt{5}}x^2+\frac{\left(-2+\sqrt{5}\right)\left(\sqrt{5}-3\right)}{276-96\sqrt{5}}x^3\right. \\
& \left.+\frac{\left(\sqrt{5}-3\right)\left(\sqrt{5}-4\right)}{2208-768\sqrt{5}}x^4+\frac{\left(\sqrt{5}-3\right)\left(\sqrt{5}-4\right)\left(-5+\sqrt{5}\right)}{41280\sqrt{5}-93600}x^5+\mathrm{O}\left(x^6\right)\right) \\
& + c_2 x^{\sqrt{5}} \left(1+\frac{\sqrt{5}+1}{1+2\sqrt{5}}x+\frac{\sqrt{5}+2}{4+8\sqrt{5}}x^2+\frac{\left(\sqrt{5}+2\right)\left(3+\sqrt{5}\right)}{276+96\sqrt{5}}x^3\right. \\
& \left.+\frac{\left(3+\sqrt{5}\right)\left(\sqrt{5}+4\right)}{2208+768\sqrt{5}}x^4+\frac{\left(3+\sqrt{5}\right)\left(\sqrt{5}+4\right)\left(5+\sqrt{5}\right)}{41280\sqrt{5}+93600}x^5+\mathrm{O}\left(x^6\right)\right)
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 1093

```
AsymptoticDSolveValue[x^2*y''[x]+x*(1-x)*y'[x]-(5+x)*y[x]==0,y[x],{x,0,5}]
```

$$
\begin{aligned}
y(x) \to \Bigg(& \frac{(-5-\sqrt{5})(-4-\sqrt{5})(-3-\sqrt{5})(-2-\sqrt{5})(1+}{(-4+\sqrt{5}+\sqrt{5}(1+\sqrt{5}))(-3+\sqrt{5}+(1+\sqrt{5})(2+\sqrt{5}))(-2+\sqrt{5}+(2+\sqrt{5})(3+\sqrt{5}))(-1} \\
& - \frac{(-4-\sqrt{5})(-3-\sqrt{5})(-2-\sqrt{5})(1+\sqrt{5})x^4}{(-4+\sqrt{5}+\sqrt{5}(1+\sqrt{5}))(-3+\sqrt{5}+(1+\sqrt{5})(2+\sqrt{5}))(-2+\sqrt{5}+(2+\sqrt{5})(3+\sqrt{5}))(-1} \\
& + \frac{(-3-\sqrt{5})(-2-\sqrt{5})(1+\sqrt{5})x^3}{(-4+\sqrt{5}+\sqrt{5}(1+\sqrt{5}))(-3+\sqrt{5}+(1+\sqrt{5})(2+\sqrt{5}))(-2+\sqrt{5}+(2+\sqrt{5})(3+\sqrt{5}))} \\
& - \frac{(-2-\sqrt{5})(1+\sqrt{5})x^2}{(-4+\sqrt{5}+\sqrt{5}(1+\sqrt{5}))(-3+\sqrt{5}+(1+\sqrt{5})(2+\sqrt{5}))} \\
& + \frac{(1+\sqrt{5})x}{-4+\sqrt{5}+\sqrt{5}(1+\sqrt{5})} + 1 \Bigg) c_1 x^{\sqrt{5}} \\
+ \Bigg(& \frac{(1-\sqrt{5})(-5+\sqrt{5})(-4+\sqrt{5})(-3+\sqrt{5})(-2}{(-4-\sqrt{5}-\sqrt{5}(1-\sqrt{5}))(-3-\sqrt{5}+(1-\sqrt{5})(2-\sqrt{5}))(-2-\sqrt{5}+(2-\sqrt{5})(3-\sqrt{5}))(-} \\
& - \frac{(1-\sqrt{5})(-4+\sqrt{5})(-3+\sqrt{5})(-2+\sqrt{5})x^4}{(-4-\sqrt{5}-\sqrt{5}(1-\sqrt{5}))(-3-\sqrt{5}+(1-\sqrt{5})(2-\sqrt{5}))(-2-\sqrt{5}+(2-\sqrt{5})(3-\sqrt{5}))(-1} \\
& + \frac{(1-\sqrt{5})(-3+\sqrt{5})(-2+\sqrt{5})x^3}{(-4-\sqrt{5}-\sqrt{5}(1-\sqrt{5}))(-3-\sqrt{5}+(1-\sqrt{5})(2-\sqrt{5}))(-2-\sqrt{5}+(2-\sqrt{5})(3-\sqrt{5}))} \\
& - \frac{(1-\sqrt{5})(-2+\sqrt{5})x^2}{(-4-\sqrt{5}-\sqrt{5}(1-\sqrt{5}))(-3-\sqrt{5}+(1-\sqrt{5})(2-\sqrt{5}))} \\
& + \frac{(1-\sqrt{5})x}{-4-\sqrt{5}-\sqrt{5}(1-\sqrt{5})} + 1 \Bigg) c_2 x^{-\sqrt{5}}
\end{aligned}
$$

17.16 problem 17

Internal problem ID [2932]
Internal file name [`OUTPUT/2424_Sunday_June_05_2022_03_09_11_AM_58770998/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.4. page 758
Problem number: 17.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{3x^2y'' + x(3x+7)\,y' + (6x+1)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$3x^2y'' + \left(3x^2 + 7x\right)y' + (6x+1)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{3x+7}{3x}$$
$$q(x) = \frac{6x+1}{3x^2}$$

Table 442: Table $p(x), q(x)$ singularites.

$p(x) = \frac{3x+7}{3x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{6x+1}{3x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$3x^2y'' + (3x^2 + 7x)\, y' + (6x + 1)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned} 3x^2 &\left(\sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2}\right) \\ &+ (3x^2 + 7x) \left(\sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}\right) + (6x + 1) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \end{aligned} \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 3x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 3x^{1+n+r} a_n (n+r)\right) \\ + \left(\sum_{n=0}^{\infty} 7x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 6x^{1+n+r} a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 3x^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} 3a_{n-1}(n+r-1)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} 6x^{1+n+r} a_n = \sum_{n=1}^{\infty} 6a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 3x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} 3a_{n-1}(n+r-1)\, x^{n+r}\right) \\ + \left(\sum_{n=0}^{\infty} 7x^{n+r} a_n (n+r)\right) + \left(\sum_{n=1}^{\infty} 6a_{n-1} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$3x^{n+r} a_n (n+r)(n+r-1) + 7x^{n+r} a_n (n+r) + a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$3x^r a_0 r(-1+r) + 7x^r a_0 r + a_0 x^r = 0$$

Or

$$\left(3x^r r(-1+r) + 7x^r r + x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(3r^2 + 4r + 1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$3r^2 + 4r + 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= -\frac{1}{3} \\ r_2 &= -1 \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(3r^2 + 4r + 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{2}{3}$ is not an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right) \\ y_2(x) &= x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n-\frac{1}{3}} \\ y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n-1} \end{aligned}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$3a_n(n + r)(n + r - 1) + 3a_{n-1}(n + r - 1) + 7a_n(n + r) + 6a_{n-1} + a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{3a_{n-1}}{3n + 3r + 1} \tag{4}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$a_n = -\frac{a_{n-1}}{n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = -\frac{1}{3}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{3}{4+3r}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$a_1 = -1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3}{4+3r}$	-1

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{9}{9r^2+33r+28}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$a_2 = \frac{1}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3}{4+3r}$	-1
a_2	$\frac{9}{9r^2+33r+28}$	$\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{27}{27r^3+189r^2+414r+280}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$a_3 = -\frac{1}{6}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3}{4+3r}$	-1
a_2	$\frac{9}{9r^2+33r+28}$	$\frac{1}{2}$
a_3	$-\frac{27}{27r^3+189r^2+414r+280}$	$-\frac{1}{6}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{81}{81r^4 + 918r^3 + 3699r^2 + 6222r + 3640}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$a_4 = \frac{1}{24}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3}{4+3r}$	-1
a_2	$\frac{9}{9r^2+33r+28}$	$\frac{1}{2}$
a_3	$-\frac{27}{27r^3+189r^2+414r+280}$	$-\frac{1}{6}$
a_4	$\frac{81}{81r^4+918r^3+3699r^2+6222r+3640}$	$\frac{1}{24}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{243}{243r^5 + 4050r^4 + 25785r^3 + 77850r^2 + 110472r + 58240}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$a_5 = -\frac{1}{120}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3}{4+3r}$	-1
a_2	$\frac{9}{9r^2+33r+28}$	$\frac{1}{2}$
a_3	$-\frac{27}{27r^3+189r^2+414r+280}$	$-\frac{1}{6}$
a_4	$\frac{81}{81r^4+918r^3+3699r^2+6222r+3640}$	$\frac{1}{24}$
a_5	$-\frac{243}{243r^5+4050r^4+25785r^3+77850r^2+110472r+58240}$	$-\frac{1}{120}$

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= \frac{1}{x^{\frac{1}{3}}}\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots\right) \\
&= \frac{1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + O(x^6)}{x^{\frac{1}{3}}}
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$
3b_n(n+r)(n+r-1) + 3b_{n-1}(n+r-1) + 7b_n(n+r) + 6b_{n-1} + b_n = 0 \tag{3}
$$

Solving for b_n from recursive equation (4) gives

$$
b_n = -\frac{3b_{n-1}}{3n+3r+1} \tag{4}
$$

Which for the root $r = -1$ becomes

$$
b_n = -\frac{3b_{n-1}}{3n-2} \tag{5}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = -\frac{3}{4 + 3r}$$

Which for the root $r = -1$ becomes

$$b_1 = -3$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{3}{4+3r}$	-3

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{9}{9r^2 + 33r + 28}$$

Which for the root $r = -1$ becomes

$$b_2 = \frac{9}{4}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{3}{4+3r}$	-3
b_2	$\frac{9}{9r^2+33r+28}$	$\frac{9}{4}$

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{27}{27r^3 + 189r^2 + 414r + 280}$$

Which for the root $r = -1$ becomes

$$b_3 = -\frac{27}{28}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{3}{4+3r}$	-3
b_2	$\frac{9}{9r^2+33r+28}$	$\frac{9}{4}$
b_3	$-\frac{27}{27r^3+189r^2+414r+280}$	$-\frac{27}{28}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{81}{81r^4 + 918r^3 + 3699r^2 + 6222r + 3640}$$

Which for the root $r = -1$ becomes

$$b_4 = \frac{81}{280}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{3}{4+3r}$	-3
b_2	$\frac{9}{9r^2+33r+28}$	$\frac{9}{4}$
b_3	$-\frac{27}{27r^3+189r^2+414r+280}$	$-\frac{27}{28}$
b_4	$\frac{81}{81r^4+918r^3+3699r^2+6222r+3640}$	$\frac{81}{280}$

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{243}{243r^5 + 4050r^4 + 25785r^3 + 77850r^2 + 110472r + 58240}$$

Which for the root $r = -1$ becomes

$$b_5 = -\frac{243}{3640}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{3}{4+3r}$	-3
b_2	$\frac{9}{9r^2+33r+28}$	$\frac{9}{4}$
b_3	$-\frac{27}{27r^3+189r^2+414r+280}$	$-\frac{27}{28}$
b_4	$\frac{81}{81r^4+918r^3+3699r^2+6222r+3640}$	$\frac{81}{280}$
b_5	$-\frac{243}{243r^5+4050r^4+25785r^3+77850r^2+110472r+58240}$	$-\frac{243}{3640}$

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= \frac{1}{x^{\frac{1}{3}}}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\
&= \frac{1 - 3x + \frac{9x^2}{4} - \frac{27x^3}{28} + \frac{81x^4}{280} - \frac{243x^5}{3640} + O(x^6)}{x}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= \frac{c_1\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + O(x^6)\right)}{x^{\frac{1}{3}}} \\
&\quad + \frac{c_2\left(1 - 3x + \frac{9x^2}{4} - \frac{27x^3}{28} + \frac{81x^4}{280} - \frac{243x^5}{3640} + O(x^6)\right)}{x}
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= \frac{c_1\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + O(x^6)\right)}{x^{\frac{1}{3}}} \\
&\quad + \frac{c_2\left(1 - 3x + \frac{9x^2}{4} - \frac{27x^3}{28} + \frac{81x^4}{280} - \frac{243x^5}{3640} + O(x^6)\right)}{x}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{align*}
y &= \frac{c_1 \left(1-x+\frac{x^2}{2}-\frac{x^3}{6}+\frac{x^4}{24}-\frac{x^5}{120}+O(x^6)\right)}{x^{\frac{1}{3}}} \\
&+ \frac{c_2 \left(1-3x+\frac{9x^2}{4}-\frac{27x^3}{28}+\frac{81x^4}{280}-\frac{243x^5}{3640}+O(x^6)\right)}{x} \tag{1}
\end{align*}
$$

Verification of solutions

$$
\begin{align*}
y &= \frac{c_1 \left(1-x+\frac{x^2}{2}-\frac{x^3}{6}+\frac{x^4}{24}-\frac{x^5}{120}+O(x^6)\right)}{x^{\frac{1}{3}}} \\
&+ \frac{c_2 \left(1-3x+\frac{9x^2}{4}-\frac{27x^3}{28}+\frac{81x^4}{280}-\frac{243x^5}{3640}+O(x^6)\right)}{x}
\end{align*}
$$

Verified OK.

17.16.1 Maple step by step solution

Let's solve

$3x^2y''+(3x^2+7x)\,y'+(6x+1)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(6x+1)y}{3x^2}-\frac{(3x+7)y'}{3x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(3x+7)y'}{3x}+\frac{(6x+1)y}{3x^2}=0$

□ Check to see if $x_0=0$ is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{3x+7}{3x}, P_3(x)=\frac{6x+1}{3x^2}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left(x\cdot P_2(x)\right)\bigg|_{x=0}=\frac{7}{3}$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = \frac{1}{3}$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $3x^2 y'' + x(3x+7)\, y' + (6x+1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+r)\,(1+3r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)\,(3k+3r+1) + 3a_{k-1}(k+r+1)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)\,(1+3r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-1, -\frac{1}{3}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $3\left(\left(k+r+\frac{1}{3}\right)a_k+a_{k-1}\right)(k+r+1)=0$

- Shift index using $k->k+1$

  $3\left(\left(k+\frac{4}{3}+r\right)a_{k+1}+a_k\right)(k+r+2)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=-\frac{3a_k}{3k+4+3r}$

- Recursion relation for $r=-1$

  $a_{k+1}=-\frac{3a_k}{3k+1}$

- Solution for $r=-1$

  $\left[y=\sum_{k=0}^{\infty}a_k x^{k-1}, a_{k+1}=-\frac{3a_k}{3k+1}\right]$

- Recursion relation for $r=-\frac{1}{3}$

  $a_{k+1}=-\frac{3a_k}{3k+3}$

- Solution for $r=-\frac{1}{3}$

  $\left[y=\sum_{k=0}^{\infty}a_k x^{k-\frac{1}{3}}, a_{k+1}=-\frac{3a_k}{3k+3}\right]$

- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty}a_k x^{k-1}\right)+\left(\sum_{k=0}^{\infty}b_k x^{k-\frac{1}{3}}\right), a_{1+k}=-\frac{3a_k}{3k+1}, b_{1+k}=-\frac{3b_k}{3k+3}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
   One independent solution has integrals. Trying a hypergeometric solution free of integral
   -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   -> Trying to convert hypergeometric functions to elementary form...
   <- elementary form is not straightforward to achieve - returning hypergeometric solution
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 47

```
Order:=6;
dsolve(3*x^2*diff(y(x),x$2)+x*(7+3*x)*diff(y(x),x)+(1+6*x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_1\left(1-3x+\frac{9}{4}x^2-\frac{27}{28}x^3+\frac{81}{280}x^4-\frac{243}{3640}x^5+\mathrm{O}\left(x^6\right)\right)}{x} + \frac{c_2\left(1-x+\frac{1}{2}x^2-\frac{1}{6}x^3+\frac{1}{24}x^4-\frac{1}{120}x^5+\mathrm{O}\left(x^6\right)\right)}{x^{\frac{1}{3}}}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 84

```
AsymptoticDSolveValue[3*x^2*y''[x]+x*(7+3*x)*y'[x]+(1+6*x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to \frac{c_1\left(-\frac{x^5}{120}+\frac{x^4}{24}-\frac{x^3}{6}+\frac{x^2}{2}-x+1\right)}{\sqrt[3]{x}} + \frac{c_2\left(-\frac{243x^5}{3640}+\frac{81x^4}{280}-\frac{27x^3}{28}+\frac{9x^2}{4}-3x+1\right)}{x}$$

17.17 problem 18

Internal problem ID [2933]
Internal file name [`OUTPUT/2425_Sunday_June_05_2022_03_09_15_AM_10981741/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.4. page 758
Problem number: 18.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Complex roots"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + xy' + (1-x)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + xy' + (1-x)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= -\frac{x-1}{x^2} \end{aligned}$$

Table 444: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{x-1}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2 y'' + x y' + (1 - x) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2} \right) + x \left(\sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1} \right) + (1 - x) \left(\sum_{n=0}^{\infty} a_n x^{n+r} \right) = 0 \tag{1}$$

Which simplifies to

$$
\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) \\
+ \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n\right) = 0 \tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n\right) = \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r}\right)
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) \\
+ \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) + \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r}\right) = 0 \tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
x^{n+r} a_n (n+r)(n+r-1) + x^{n+r} a_n (n+r) + a_n x^{n+r} = 0
$$

When $n=0$ the above becomes

$$
x^r a_0 r(-1+r) + x^r a_0 r + a_0 x^r = 0
$$

Or

$$
\left(x^r r(-1+r) + x^r r + x^r\right) a_0 = 0
$$

Since $a_0 \neq 0$ then the above simplifies to

$$
\left(r^2+1\right) x^r = 0
$$

Since the above is true for all x then the indicial equation becomes

$$
r^2 + 1 = 0
$$

Solving for r gives the roots of the indicial equation as

$$
\begin{aligned}
r_1 &= i \\
r_2 &= -i
\end{aligned}
$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$
(r^2 + 1)\, x^r = 0
$$

Solving for r gives the roots of the indicial equation as Since the roots are complex conjugates, then two linearly independent solutions can be constructed using

$$
\begin{aligned}
y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right) \\
y_2(x) &= x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+i} \\
y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n-i}
\end{aligned}
$$

$y_1(x)$ is found first. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$
a_n(n+r)(n+r-1) + a_n(n+r) + a_n - a_{n-1} = 0 \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$
a_n = \frac{a_{n-1}}{n^2 + 2nr + r^2 + 1} \tag{4}
$$

Which for the root $r = i$ becomes

$$
a_n = \frac{a_{n-1}}{n\,(2i + n)} \tag{5}
$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = i$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{1}{r^2 + 2r + 2}$$

Which for the root $r = i$ becomes

$$a_1 = \frac{1}{5} - \frac{2i}{5}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{r^2+2r+2}$	$\frac{1}{5} - \frac{2i}{5}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{(r^2 + 2r + 2)(r^2 + 4r + 5)}$$

Which for the root $r = i$ becomes

$$a_2 = -\frac{1}{40} - \frac{3i}{40}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{r^2+2r+2}$	$\frac{1}{5} - \frac{2i}{5}$
a_2	$\frac{1}{(r^2+2r+2)(r^2+4r+5)}$	$-\frac{1}{40} - \frac{3i}{40}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{1}{(r^2 + 2r + 2)(r^2 + 4r + 5)(r^2 + 6r + 10)}$$

Which for the root $r = i$ becomes

$$a_3 = -\frac{3}{520} - \frac{7i}{1560}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{r^2+2r+2}$	$\frac{1}{5} - \frac{2i}{5}$
a_2	$\frac{1}{(r^2+2r+2)(r^2+4r+5)}$	$-\frac{1}{40} - \frac{3i}{40}$
a_3	$\frac{1}{(r^2+2r+2)(r^2+4r+5)(r^2+6r+10)}$	$-\frac{3}{520} - \frac{7i}{1560}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(r^2+2r+2)(r^2+4r+5)(r^2+6r+10)(r^2+8r+17)}$$

Which for the root $r = i$ becomes

$$a_4 = -\frac{1}{2496} - \frac{i}{12480}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{r^2+2r+2}$	$\frac{1}{5} - \frac{2i}{5}$
a_2	$\frac{1}{(r^2+2r+2)(r^2+4r+5)}$	$-\frac{1}{40} - \frac{3i}{40}$
a_3	$\frac{1}{(r^2+2r+2)(r^2+4r+5)(r^2+6r+10)}$	$-\frac{3}{520} - \frac{7i}{1560}$
a_4	$\frac{1}{(r^2+2r+2)(r^2+4r+5)(r^2+6r+10)(r^2+8r+17)}$	$-\frac{1}{2496} - \frac{i}{12480}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{1}{(r^2+2r+2)(r^2+4r+5)(r^2+6r+10)(r^2+8r+17)(r^2+10r+26)}$$

Which for the root $r = i$ becomes

$$a_5 = -\frac{9}{603200} + \frac{i}{361920}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{r^2+2r+2}$	$\frac{1}{5}-\frac{2i}{5}$
a_2	$\frac{1}{(r^2+2r+2)(r^2+4r+5)}$	$-\frac{1}{40}-\frac{3i}{40}$
a_3	$\frac{1}{(r^2+2r+2)(r^2+4r+5)(r^2+6r+10)}$	$-\frac{3}{520}-\frac{7i}{1560}$
a_4	$\frac{1}{(r^2+2r+2)(r^2+4r+5)(r^2+6r+10)(r^2+8r+17)}$	$-\frac{1}{2496}-\frac{i}{12480}$
a_5	$\frac{1}{(r^2+2r+2)(r^2+4r+5)(r^2+6r+10)(r^2+8r+17)(r^2+10r+26)}$	$-\frac{9}{603200}+\frac{i}{361920}$

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x^i(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6\dots) \\
&= x^i\left(1+\left(\frac{1}{5}-\frac{2i}{5}\right)x+\left(-\frac{1}{40}-\frac{3i}{40}\right)x^2+\left(-\frac{3}{520}-\frac{7i}{1560}\right)x^3\right. \\
&\quad \left.+\left(-\frac{1}{2496}-\frac{i}{12480}\right)x^4+\left(-\frac{9}{603200}+\frac{i}{361920}\right)x^5+O(x^6)\right)
\end{aligned}
$$

The second solution $y_2(x)$ is found by taking the complex conjugate of $y_1(x)$ which gives

$$
\begin{aligned}
y_2(x) &= x^{-i}\left(1+\left(\frac{1}{5}+\frac{2i}{5}\right)x+\left(-\frac{1}{40}+\frac{3i}{40}\right)x^2+\left(-\frac{3}{520}+\frac{7i}{1560}\right)x^3\right. \\
&\quad \left.+\left(-\frac{1}{2496}+\frac{i}{12480}\right)x^4+\left(-\frac{9}{603200}-\frac{i}{361920}\right)x^5+O(x^6)\right)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1y_1(x) + c_2y_2(x) \\
&= c_1x^i\left(1+\left(\frac{1}{5}-\frac{2i}{5}\right)x+\left(-\frac{1}{40}-\frac{3i}{40}\right)x^2+\left(-\frac{3}{520}-\frac{7i}{1560}\right)x^3+\left(-\frac{1}{2496}-\frac{i}{12480}\right)x^4\right. \\
&\quad \left.+\left(-\frac{9}{603200}+\frac{i}{361920}\right)x^5+O(x^6)\right)+c_2x^{-i}\left(1+\left(\frac{1}{5}+\frac{2i}{5}\right)x\right. \\
&\quad +\left(-\frac{1}{40}+\frac{3i}{40}\right)x^2+\left(-\frac{3}{520}+\frac{7i}{1560}\right)x^3+\left(-\frac{1}{2496}+\frac{i}{12480}\right)x^4 \\
&\quad \left.+\left(-\frac{9}{603200}-\frac{i}{361920}\right)x^5+O(x^6)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^i \left(1+\left(\frac{1}{5}-\frac{2i}{5}\right) x+\left(-\frac{1}{40}-\frac{3i}{40}\right) x^2+\left(-\frac{3}{520}-\frac{7i}{1560}\right) x^3+\left(-\frac{1}{2496}-\frac{i}{12480}\right) x^4 \right. \\
&\quad \left. +\left(-\frac{9}{603200}+\frac{i}{361920}\right) x^5+O\left(x^6\right)\right)+c_2 x^{-i}\left(1+\left(\frac{1}{5}+\frac{2i}{5}\right) x+\left(-\frac{1}{40}+\frac{3i}{40}\right) x^2 \right. \\
&\quad \left. +\left(-\frac{3}{520}+\frac{7i}{1560}\right) x^3+\left(-\frac{1}{2496}+\frac{i}{12480}\right) x^4+\left(-\frac{9}{603200}-\frac{i}{361920}\right) x^5+O\left(x^6\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = c_1 x^i &\left(1+\left(\frac{1}{5}-\frac{2i}{5}\right) x+\left(-\frac{1}{40}-\frac{3i}{40}\right) x^2+\left(-\frac{3}{520}-\frac{7i}{1560}\right) x^3 \right. \\
&\left. +\left(-\frac{1}{2496}-\frac{i}{12480}\right) x^4+\left(-\frac{9}{603200}+\frac{i}{361920}\right) x^5+O\left(x^6\right)\right) \\
+ c_2 x^{-i} &\left(1+\left(\frac{1}{5}+\frac{2i}{5}\right) x+\left(-\frac{1}{40}+\frac{3i}{40}\right) x^2+\left(-\frac{3}{520}+\frac{7i}{1560}\right) x^3 \right. \\
&\left. +\left(-\frac{1}{2496}+\frac{i}{12480}\right) x^4+\left(-\frac{9}{603200}-\frac{i}{361920}\right) x^5+O\left(x^6\right)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1 x^i \left(1+\left(\frac{1}{5}-\frac{2i}{5}\right) x+\left(-\frac{1}{40}-\frac{3i}{40}\right) x^2+\left(-\frac{3}{520}-\frac{7i}{1560}\right) x^3+\left(-\frac{1}{2496}-\frac{i}{12480}\right) x^4 \right. \\
&\quad \left. +\left(-\frac{9}{603200}+\frac{i}{361920}\right) x^5+O\left(x^6\right)\right)+c_2 x^{-i}\left(1+\left(\frac{1}{5}+\frac{2i}{5}\right) x+\left(-\frac{1}{40}+\frac{3i}{40}\right) x^2 \right. \\
&\quad \left. +\left(-\frac{3}{520}+\frac{7i}{1560}\right) x^3+\left(-\frac{1}{2496}+\frac{i}{12480}\right) x^4+\left(-\frac{9}{603200}-\frac{i}{361920}\right) x^5+O\left(x^6\right)\right)
\end{aligned}
$$

Verified OK.

17.17.1 Maple step by step solution

Let's solve

$x^2 y'' + x y' + (1-x) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{y'}{x} + \frac{(x-1)y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' + \frac{y'}{x} - \frac{(x-1)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- ○ Define functions

$\left[P_2(x) = \frac{1}{x}, P_3(x) = -\frac{x-1}{x^2}\right]$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

$(x \cdot P_2(x))\bigg|_{x=0} = 1$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$(x^2 \cdot P_3(x))\bigg|_{x=0} = 1$

- ○ $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

$x^2y'' + xy' + (1-x)\,y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- ○ Shift index using $k- >k - m$

$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- ○ Convert $x \cdot y'$ to series expansion

$x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$

- ○ Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(r^2+1)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k^2+2kr+r^2+1) - a_{k-1}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$r^2+1=0$$

- Values of r that satisfy the indicial equation

  $$r \in \{-\mathrm{I}, \mathrm{I}\}$$

- Each term in the series must be 0, giving the recursion relation

  $$a_k(k^2+2kr+r^2+1) - a_{k-1} = 0$$

- Shift index using $k- >k+1$

  $$a_{k+1}\left((k+1)^2 + 2(k+1)\, r + r^2 + 1\right) - a_k = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = \frac{a_k}{k^2+2kr+r^2+2k+2r+2}$$

- Recursion relation for $r = -\mathrm{I}$

  $$a_{k+1} = \frac{a_k}{k^2-2\,\mathrm{I}k+1-2\,\mathrm{I}+2k}$$

- Solution for $r = -\mathrm{I}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\mathrm{I}}, a_{k+1} = \frac{a_k}{k^2-2\,\mathrm{I}k+1-2\,\mathrm{I}+2k}\right]$$

- Recursion relation for $r = \mathrm{I}$

  $$a_{k+1} = \frac{a_k}{k^2+2\,\mathrm{I}k+1+2\,\mathrm{I}+2k}$$

- Solution for $r = \mathrm{I}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\mathrm{I}}, a_{k+1} = \frac{a_k}{k^2+2\,\mathrm{I}k+1+2\,\mathrm{I}+2k}\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\mathrm{I}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\mathrm{I}}\right), a_{1+k} = \frac{a_k}{k^2-2\,\mathrm{I}k+1-2\,\mathrm{I}+2k}, b_{1+k} = \frac{b_k}{k^2+2\,\mathrm{I}k+1+2\,\mathrm{I}+2k}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 69

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+(1-x)*y(x)=0,y(x),type='series',x=0);
```

$$
\begin{aligned}
y(x) = {} & c_1 x^{-i}\left(1+\left(\frac{1}{5}+\frac{2i}{5}\right)x+\left(-\frac{1}{40}+\frac{3i}{40}\right)x^2+\left(-\frac{3}{520}+\frac{7i}{1560}\right)x^3\right. \\
& \left.+\left(-\frac{1}{2496}+\frac{i}{12480}\right)x^4+\left(-\frac{9}{603200}-\frac{i}{361920}\right)x^5+\mathrm{O}\left(x^6\right)\right) \\
& + c_2 x^{i}\left(1+\left(\frac{1}{5}-\frac{2i}{5}\right)x+\left(-\frac{1}{40}-\frac{3i}{40}\right)x^2+\left(-\frac{3}{520}-\frac{7i}{1560}\right)x^3\right. \\
& \left.+\left(-\frac{1}{2496}-\frac{i}{12480}\right)x^4+\left(-\frac{9}{603200}+\frac{i}{361920}\right)x^5+\mathrm{O}\left(x^6\right)\right)
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 90

```
AsymptoticDSolveValue[x^2*y''[x]+x*y'[x]+(1-x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to \left(\frac{1}{12480}+\frac{i}{2496}\right) c_2 x^{-i}\left(i x^4+(8+16 i) x^3+(168+96 i) x^2+(1056-288 i) x+(480-2400 i)\right)-\left(\frac{1}{2496}+\frac{i}{12480}\right) c_1 x^i\left(x^4+(16+8 i) x^3+(96+168 i) x^2-(288-1056 i) x-(2400-480 i)\right)$$

17.18 problem 19

Internal problem ID [2934]
Internal file name [`OUTPUT/2426_Sunday_June_05_2022_03_09_22_AM_60454255/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.4. page 758
Problem number: 19.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{3x^2y'' + x(3x^2+1)\,y' - 2yx = 0}$$

With the expansion point for the power series method at $x = 0$.

The ODE is

$$3x^2y'' + (3x^3+x)\,y' - 2yx = 0$$

Or

$$x(3y'x^2 + 3xy'' + y' - 2y) = 0$$

For $x \neq 0$ the above simplifies to

$$3y'x^2 + 3xy'' + y' - 2y = 0$$

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$3x^2y'' + (3x^3+x)\,y' - 2yx = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$
\begin{aligned}
p(x) &= \frac{3x^2+1}{3x} \\
q(x) &= -\frac{2}{3x}
\end{aligned}
$$

Table 446: Table $p(x), q(x)$ singularites.

$p(x) = \frac{3x^2+1}{3x}$	
singularity	type
$x = 0$	"regular"
$x = \infty$	"regular"
$x = -\infty$	"regular"

$q(x) = -\frac{2}{3x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty, -\infty]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$
3x^2y'' + (3x^3 + x)\, y' - 2yx = 0
$$

Let the solution be represented as Frobenius power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^{n+r}
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} \\
y'' &= \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\begin{aligned}
&3x^2\left(\sum_{n=0}^{\infty}(n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) \\
&+\left(3x^3+x\right)\left(\sum_{n=0}^{\infty}(n+r)\,a_n x^{n+r-1}\right)-2\left(\sum_{n=0}^{\infty}a_n x^{n+r}\right)x=0
\end{aligned} \tag{1}
$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty}3x^{n+r}a_n(n+r)\,(n+r-1)\right)+\left(\sum_{n=0}^{\infty}3x^{n+r+2}a_n(n+r)\right) \\
&+\left(\sum_{n=0}^{\infty}x^{n+r}a_n(n+r)\right)+\sum_{n=0}^{\infty}\left(-2x^{1+n+r}a_n\right)=0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty}3x^{n+r+2}a_n(n+r)&=\sum_{n=2}^{\infty}3a_{n-2}(n+r-2)\,x^{n+r} \\
\sum_{n=0}^{\infty}\left(-2x^{1+n+r}a_n\right)&=\sum_{n=1}^{\infty}\left(-2a_{n-1}x^{n+r}\right)
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty}3x^{n+r}a_n(n+r)\,(n+r-1)\right)+\left(\sum_{n=2}^{\infty}3a_{n-2}(n+r-2)\,x^{n+r}\right) \\
&+\left(\sum_{n=0}^{\infty}x^{n+r}a_n(n+r)\right)+\sum_{n=1}^{\infty}\left(-2a_{n-1}x^{n+r}\right)=0
\end{aligned} \tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
3x^{n+r}a_n(n+r)\,(n+r-1)+x^{n+r}a_n(n+r)=0
$$

When $n = 0$ the above becomes

$$3x^r a_0 r(-1+r) + x^r a_0 r = 0$$

Or

$$(3x^r r(-1+r) + x^r r) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^r r(-2+3r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$3r^2 - 2r = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{2}{3}$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^r r(-2+3r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{2}{3}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{2}{3}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^n$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = \frac{2}{3r^2 + 4r + 1}$$

For $2 \leq n$ the recursive equation is

$$3a_n(n+r)(n+r-1) + 3a_{n-2}(n+r-2) + a_n(n+r) - 2a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{3na_{n-2} + 3ra_{n-2} - 6a_{n-2} - 2a_{n-1}}{3n^2 + 6nr + 3r^2 - 2n - 2r} \tag{4}$$

Which for the root $r = \frac{2}{3}$ becomes

$$a_n = \frac{-3na_{n-2} + 4a_{n-2} + 2a_{n-1}}{n(3n+2)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{2}{3}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2}{3r^2+4r+1}$	$\frac{2}{5}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{-9r^3 - 12r^2 - 3r + 4}{9r^4 + 42r^3 + 67r^2 + 42r + 8}$$

Which for the root $r = \frac{2}{3}$ becomes

$$a_2 = -\frac{3}{40}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2}{3r^2+4r+1}$	$\frac{2}{5}$
a_2	$\frac{-9r^3-12r^2-3r+4}{9r^4+42r^3+67r^2+42r+8}$	$-\frac{3}{40}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-36r^3 - 102r^2 - 114r - 40}{27r^6 + 270r^5 + 1062r^4 + 2080r^3 + 2103r^2 + 1010r + 168}$$

Which for the root $r = \frac{2}{3}$ becomes

$$a_3 = -\frac{43}{660}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2}{3r^2+4r+1}$	$\frac{2}{5}$
a_2	$\frac{-9r^3-12r^2-3r+4}{9r^4+42r^3+67r^2+42r+8}$	$-\frac{3}{40}$
a_3	$\frac{-36r^3-102r^2-114r-40}{27r^6+270r^5+1062r^4+2080r^3+2103r^2+1010r+168}$	$-\frac{43}{660}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{81r^6 + 702r^5 + 2250r^4 + 3132r^3 + 1521r^2 - 486r - 584}{81r^8 + 1404r^7 + 10206r^6 + 40404r^5 + 94549r^4 + 132496r^3 + 106844r^2 + 44096r + 6720}$$

Which for the root $r = \frac{2}{3}$ becomes

$$a_4 = \frac{31}{3696}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2}{3r^2+4r+1}$	$\frac{2}{5}$
a_2	$\frac{-9r^3-12r^2-3r+4}{9r^4+42r^3+67r^2+42r+8}$	$-\frac{3}{40}$
a_3	$\frac{-36r^3-102r^2-114r-40}{27r^6+270r^5+1062r^4+2080r^3+2103r^2+1010r+168}$	$-\frac{43}{660}$
a_4	$\frac{81r^6+702r^5+2250r^4+3132r^3+1521r^2-486r-584}{81r^8+1404r^7+10206r^6+40404r^5+94549r^4+132496r^3+106844r^2+44096r+6720}$	$\frac{31}{3696}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{486r^6 + 5670r^5 + 26460r^4 + 62622r^3 + 79734r^2 + 52788r + 13232}{(81r^8 + 1404r^7 + 10206r^6 + 40404r^5 + 94549r^4 + 132496r^3 + 106844r^2 + 44096r + 6720)(3r^2 + 28}$$

Which for the root $r = \frac{2}{3}$ becomes

$$a_5 = \frac{2259}{261800}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2}{3r^2+4r+1}$	$\frac{2}{5}$
a_2	$\frac{-9r^3-12r^2-3r+4}{9r^4+42r^3+67r^2+42r+8}$	$-\frac{3}{40}$
a_3	$\frac{-36r^3-102r^2-114r-40}{27r^6+270r^5+1062r^4+2080r^3+2103r^2+1010r+168}$	$-\frac{43}{660}$
a_4	$\frac{81r^6+702r^5+2250r^4+3132r^3+1521r^2-486r-584}{81r^8+1404r^7+10206r^6+40404r^5+94549r^4+132496r^3+106844r^2+44096r+6720}$	$\frac{31}{3696}$
a_5	$\frac{486r^6+5670r^5+26460r^4+62622r^3+79734r^2+52788r+13232}{(81r^8+1404r^7+10206r^6+40404r^5+94549r^4+132496r^3+106844r^2+44096r+6720)(3r^2+28r+65)}$	$\frac{2259}{261800}$

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x^{\frac{2}{3}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= x^{\frac{2}{3}}\left(1 + \frac{2x}{5} - \frac{3x^2}{40} - \frac{43x^3}{660} + \frac{31x^4}{3696} + \frac{2259x^5}{261800} + O(x^6)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$
b_1 = \frac{2}{3r^2 + 4r + 1}
$$

For $2 \leq n$ the recursive equation is

$$
3b_n(n + r)(n + r - 1) + 3b_{n-2}(n + r - 2) + b_n(n + r) - 2b_{n-1} = 0 \tag{3}
$$

Solving for b_n from recursive equation (4) gives

$$
b_n = -\frac{3nb_{n-2} + 3rb_{n-2} - 6b_{n-2} - 2b_{n-1}}{3n^2 + 6nr + 3r^2 - 2n - 2r} \tag{4}
$$

Which for the root $r = 0$ becomes

$$
b_n = \frac{-3nb_{n-2} + 6b_{n-2} + 2b_{n-1}}{n(3n - 2)} \tag{5}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2}{3r^2+4r+1}$	2

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{-9r^3 - 12r^2 - 3r + 4}{9r^4 + 42r^3 + 67r^2 + 42r + 8}$$

Which for the root $r = 0$ becomes

$$b_2 = \frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2}{3r^2+4r+1}$	2
b_2	$\frac{-9r^3-12r^2-3r+4}{9r^4+42r^3+67r^2+42r+8}$	$\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{-36r^3 - 102r^2 - 114r - 40}{27r^6 + 270r^5 + 1062r^4 + 2080r^3 + 2103r^2 + 1010r + 168}$$

Which for the root $r = 0$ becomes

$$b_3 = -\frac{5}{21}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2}{3r^2+4r+1}$	2
b_2	$\frac{-9r^3-12r^2-3r+4}{9r^4+42r^3+67r^2+42r+8}$	$\frac{1}{2}$
b_3	$\frac{-36r^3-102r^2-114r-40}{27r^6+270r^5+1062r^4+2080r^3+2103r^2+1010r+168}$	$-\frac{5}{21}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{81r^6 + 702r^5 + 2250r^4 + 3132r^3 + 1521r^2 - 486r - 584}{81r^8 + 1404r^7 + 10206r^6 + 40404r^5 + 94549r^4 + 132496r^3 + 106844r^2 + 44096r + 6720}$$

Which for the root $r = 0$ becomes

$$b_4 = -\frac{73}{840}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2}{3r^2+4r+1}$	2
b_2	$\frac{-9r^3-12r^2-3r+4}{9r^4+42r^3+67r^2+42r+8}$	$\frac{1}{2}$
b_3	$\frac{-36r^3-102r^2-114r-40}{27r^6+270r^5+1062r^4+2080r^3+2103r^2+1010r+168}$	$-\frac{5}{21}$
b_4	$\frac{81r^6+702r^5+2250r^4+3132r^3+1521r^2-486r-584}{81r^8+1404r^7+10206r^6+40404r^5+94549r^4+132496r^3+106844r^2+44096r+6720}$	$-\frac{73}{840}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{486r^6 + 5670r^5 + 26460r^4 + 62622r^3 + 79734r^2 + 52788r + 13232}{(81r^8 + 1404r^7 + 10206r^6 + 40404r^5 + 94549r^4 + 132496r^3 + 106844r^2 + 44096r + 6720)\,(3r^2 + 28r}$$

Which for the root $r = 0$ becomes

$$b_5 = \frac{827}{27300}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2}{3r^2+4r+1}$	2
b_2	$\frac{-9r^3-12r^2-3r+4}{9r^4+42r^3+67r^2+42r+8}$	$\frac{1}{2}$
b_3	$\frac{-36r^3-102r^2-114r-40}{27r^6+270r^5+1062r^4+2080r^3+2103r^2+1010r+168}$	$-\frac{5}{21}$
b_4	$\frac{81r^6+702r^5+2250r^4+3132r^3+1521r^2-486r-584}{81r^8+1404r^7+10206r^6+40404r^5+94549r^4+132496r^3+106844r^2+44096r+6720}$	$-\frac{73}{840}$
b_5	$\frac{486r^6+5670r^5+26460r^4+62622r^3+79734r^2+52788r+13232}{(81r^8+1404r^7+10206r^6+40404r^5+94549r^4+132496r^3+106844r^2+44096r+6720)(3r^2+28r+65)}$	$\frac{827}{27300}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\ &= 1 + 2x + \frac{x^2}{2} - \frac{5x^3}{21} - \frac{73x^4}{840} + \frac{827x^5}{27300} + O(x^6) \end{aligned}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^{\frac{2}{3}} \left(1 + \frac{2x}{5} - \frac{3x^2}{40} - \frac{43x^3}{660} + \frac{31x^4}{3696} + \frac{2259x^5}{261800} + O(x^6)\right) \\
&\quad + c_2 \left(1 + 2x + \frac{x^2}{2} - \frac{5x^3}{21} - \frac{73x^4}{840} + \frac{827x^5}{27300} + O(x^6)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^{\frac{2}{3}} \left(1 + \frac{2x}{5} - \frac{3x^2}{40} - \frac{43x^3}{660} + \frac{31x^4}{3696} + \frac{2259x^5}{261800} + O(x^6)\right) \\
&\quad + c_2 \left(1 + 2x + \frac{x^2}{2} - \frac{5x^3}{21} - \frac{73x^4}{840} + \frac{827x^5}{27300} + O(x^6)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = c_1 x^{\frac{2}{3}} &\left(1 + \frac{2x}{5} - \frac{3x^2}{40} - \frac{43x^3}{660} + \frac{31x^4}{3696} + \frac{2259x^5}{261800} + O(x^6)\right) \\
&+ c_2 \left(1 + 2x + \frac{x^2}{2} - \frac{5x^3}{21} - \frac{73x^4}{840} + \frac{827x^5}{27300} + O(x^6)\right)
\end{aligned}
\tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = c_1 x^{\frac{2}{3}} &\left(1 + \frac{2x}{5} - \frac{3x^2}{40} - \frac{43x^3}{660} + \frac{31x^4}{3696} + \frac{2259x^5}{261800} + O(x^6)\right) \\
&+ c_2 \left(1 + 2x + \frac{x^2}{2} - \frac{5x^3}{21} - \frac{73x^4}{840} + \frac{827x^5}{27300} + O(x^6)\right)
\end{aligned}
$$

Verified OK.

17.18.1 Maple step by step solution

Let's solve

$3x^2y'' + (3x^3 + x)\,y' - 2yx = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{2y}{3x} - \frac{(3x^2+1)y'}{3x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(3x^2+1)y'}{3x} - \frac{2y}{3x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{3x^2+1}{3x}, P_3(x) = -\frac{2}{3x}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = \frac{1}{3}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $3y''x + (3x^2 + 1)\,y' - 2y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y'$ to series expansion for $m = 0..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k + r)\,x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

  $$x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k->k+1$

  $$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(-2+3r)\, x^{-1+r} + (a_1(1+r)\,(1+3r) - 2a_0)\, x^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+1+r)\,(3k+1+3r) - 2a_k$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$r(-2+3r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{0, \frac{2}{3}\right\}$$

- Each term must be 0

  $$a_1(1+r)\,(1+3r) - 2a_0 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$a_{k+1}(k+1+r)\,(3k+1+3r) - 2a_k + 3a_{k-1}(k+r-1) = 0$$

- Shift index using $k->k+1$

  $$a_{k+2}(k+2+r)\,(3k+4+3r) - 2a_{k+1} + 3a_k(k+r) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = -\frac{3ka_k+3ra_k-2a_{k+1}}{(k+2+r)(3k+4+3r)}$$

- Recursion relation for $r = 0$

  $$a_{k+2} = -\frac{3ka_k-2a_{k+1}}{(k+2)(3k+4)}$$

- Solution for $r = 0$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{3ka_k-2a_{k+1}}{(k+2)(3k+4)}, a_1 - 2a_0 = 0\right]$$

- Recursion relation for $r = \frac{2}{3}$

  $$a_{k+2} = -\frac{3ka_k+2a_k-2a_{k+1}}{\left(k+\frac{8}{3}\right)(3k+6)}$$

- Solution for $r = \frac{2}{3}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{2}{3}}, a_{k+2} = -\frac{3k a_k + 2a_k - 2a_{k+1}}{\left(k+\frac{8}{3}\right)(3k+6)}, 5a_1 - 2a_0 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{2}{3}}\right), a_{2+k} = -\frac{3k a_k - 2a_{1+k}}{(2+k)(3k+4)}, a_1 - 2a_0 = 0, b_{2+k} = -\frac{3k b_k + 2b_k - 2b_{1+k}}{\left(k+\frac{8}{3}\right)(3k+6)}, 5b_1\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
<- Heun successful: received ODE is equivalent to the  HeunB  ODE, case  c = 0 `
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 44

```
Order:=6;
dsolve(3*x^2*diff(y(x),x$2)+x*(1+3*x^2)*diff(y(x),x)-2*x*y(x)=0,y(x),type='series',x=0);
```

$$
\begin{aligned}
y(x) = {} & c_1 x^{\frac{2}{3}} \left(1+\frac{2}{5}x-\frac{3}{40}x^2-\frac{43}{660}x^3+\frac{31}{3696}x^4+\frac{2259}{261800}x^5+\mathrm{O}\left(x^6\right)\right) \\
& + c_2 \left(1+2x+\frac{1}{2}x^2-\frac{5}{21}x^3-\frac{73}{840}x^4+\frac{827}{27300}x^5+\mathrm{O}\left(x^6\right)\right)
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 83

```
AsymptoticDSolveValue[3*x^2*y''[x]+x*(1+3*x^2)*y'[x]-2*x*y[x]==0,y[x],{x,0,5}]
```

$$
\begin{aligned}
y(x) \to {} & c_2 \left(\frac{827x^5}{27300}-\frac{73x^4}{840}-\frac{5x^3}{21}+\frac{x^2}{2}+2x+1\right) \\
& + c_1 x^{2/3} \left(\frac{2259x^5}{261800}+\frac{31x^4}{3696}-\frac{43x^3}{660}-\frac{3x^2}{40}+\frac{2x}{5}+1\right)
\end{aligned}
$$

17.19 problem 20

Internal problem ID [2935]
Internal file name [OUTPUT/2427_Sunday_June_05_2022_03_09_27_AM_23375353/index.tex]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.4. page 758
Problem number: 20.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2y'' - 4y'x^2 + (1+2x)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$4x^2y'' - 4y'x^2 + (1+2x)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$\begin{aligned} p(x) &= -1 \\ q(x) &= \frac{1+2x}{4x^2} \end{aligned}$$

Table 448: Table $p(x), q(x)$ singularites.

$p(x) = -1$	
singularity	type

$q(x) = \frac{1+2x}{4x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$4x^2y'' - 4y'x^2 + (1 + 2x)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$4x^2 \left(\sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2}\right) - 4\left(\sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}\right) x^2 + (1 + 2x) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-4x^{1+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 2x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-4x^{1+n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-4a_{n-1}(n+r-1)\, x^{n+r}\right)$$

$$\sum_{n=0}^{\infty} 2x^{1+n+r} a_n = \sum_{n=1}^{\infty} 2a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-4a_{n-1}(n+r-1)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) + \left(\sum_{n=1}^{\infty} 2a_{n-1} x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$4x^{n+r} a_n (n+r)(n+r-1) + a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$4x^r a_0 r(-1+r) + a_0 x^r = 0$$

Or

$$\left(4x^r r(-1+r) + x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^r (-1+2r)^2 = 0$$

Since the above is true for all x then the indicial equation becomes

$$(-1+2r)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{2}$$
$$r_2 = \frac{1}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^r(-1+2r)^2 = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = \frac{1}{2}$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}$$
$$y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+\frac{1}{2}}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$4a_n(n+r)(n+r-1) - 4a_{n-1}(n+r-1) + a_n + 2a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{2a_{n-1}(2n+2r-3)}{4n^2+8nr+4r^2-4n-4r+1} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = \frac{a_{n-1}(n-1)}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-2+4r}{(2r+1)^2}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_1 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2+4r}{(2r+1)^2}$	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{8r-4}{(2r+1)(3+2r)^2}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2+4r}{(2r+1)^2}$	0
a_2	$\frac{8r-4}{(2r+1)(3+2r)^2}$	0

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-8+16r}{(3+2r)(2r+1)(5+2r)^2}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2+4r}{(2r+1)^2}$	0
a_2	$\frac{8r-4}{(2r+1)(3+2r)^2}$	0
a_3	$\frac{-8+16r}{(3+2r)(2r+1)(5+2r)^2}$	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{-16+32r}{(3+2r)(5+2r)(2r+1)(7+2r)^2}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2+4r}{(2r+1)^2}$	0
a_2	$\frac{8r-4}{(2r+1)(3+2r)^2}$	0
a_3	$\frac{-8+16r}{(3+2r)(2r+1)(5+2r)^2}$	0
a_4	$\frac{-16+32r}{(3+2r)(5+2r)(2r+1)(7+2r)^2}$	0

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-32 + 64r}{(2r + 1)(5 + 2r)(3 + 2r)(7 + 2r)(9 + 2r)^2}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2+4r}{(2r+1)^2}$	0
a_2	$\frac{8r-4}{(2r+1)(3+2r)^2}$	0
a_3	$\frac{-8+16r}{(3+2r)(2r+1)(5+2r)^2}$	0
a_4	$\frac{-16+32r}{(3+2r)(5+2r)(2r+1)(7+2r)^2}$	0
a_5	$\frac{-32+64r}{(2r+1)(5+2r)(3+2r)(7+2r)(9+2r)^2}$	0

Using the above table, then the first solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= \sqrt{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= \sqrt{x}\left(1 + O(x^6)\right) \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = \frac{1}{2}$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr}a_{n,r}$	$b_n(r = \frac{1}{2})$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	$\frac{-2+4r}{(2r+1)^2}$	0	$\frac{-8r+12}{(2r+1)^3}$	1
b_2	$\frac{8r-4}{(2r+1)(3+2r)^2}$	0	$\frac{-64r^2+32r+64}{(2r+1)^2(3+2r)^3}$	$\frac{1}{4}$
b_3	$\frac{-8+16r}{(3+2r)(2r+1)(5+2r)^2}$	0	$\frac{-384r^3-576r^2+608r+656}{(3+2r)^2(2r+1)^2(5+2r)^3}$	$\frac{1}{18}$
b_4	$\frac{-16+32r}{(3+2r)(5+2r)(2r+1)(7+2r)^2}$	0	$-\frac{256(8r^4+36r^3+26r^2-45r-37)}{(3+2r)^2(5+2r)^2(2r+1)^2(7+2r)^3}$	$\frac{1}{96}$
b_5	$\frac{-32+64r}{(2r+1)(5+2r)(3+2r)(7+2r)(9+2r)^2}$	0	$-\frac{64(160r^5+1360r^4+3440r^3+1400r^2-3942r-2739)}{(2r+1)^2(5+2r)^2(3+2r)^2(7+2r)^2(9+2r)^3}$	$\frac{1}{600}$

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6\dots \\
&= \sqrt{x}\,(1+O(x^6))\ln(x) + \sqrt{x}\left(x + \frac{x^2}{4} + \frac{x^3}{18} + \frac{x^4}{96} + \frac{x^5}{600} + O(x^6)\right)
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1y_1(x) + c_2y_2(x) \\
&= c_1\sqrt{x}\,(1+O(x^6)) + c_2\left(\sqrt{x}\,(1+O(x^6))\ln(x) + \sqrt{x}\left(x + \frac{x^2}{4} + \frac{x^3}{18} + \frac{x^4}{96} + \frac{x^5}{600} \right.\right. \\
&\qquad \left.\left. + O(x^6)\right)\right)
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= c_1\sqrt{x}\,(1+O(x^6)) + c_2\left(\sqrt{x}\,(1+O(x^6))\ln(x) + \sqrt{x}\left(x + \frac{x^2}{4} + \frac{x^3}{18} + \frac{x^4}{96} + \frac{x^5}{600} + O(x^6)\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}
y &= c_1\sqrt{x}\,(1+O(x^6)) \\
&\quad + c_2\left(\sqrt{x}\,(1+O(x^6))\ln(x) + \sqrt{x}\left(x + \frac{x^2}{4} + \frac{x^3}{18} + \frac{x^4}{96} + \frac{x^5}{600} + O(x^6)\right)\right)
\end{aligned} \tag{1}$$

Verification of solutions

$$y = c_1\sqrt{x}\,(1+O(x^6)) + c_2\left(\sqrt{x}\,(1+O(x^6))\ln(x) + \sqrt{x}\left(x+\frac{x^2}{4}+\frac{x^3}{18}+\frac{x^4}{96}+\frac{x^5}{600}+O(x^6)\right)\right)$$

Verified OK.

17.19.1 Maple step by step solution

Let's solve

$4x^2y'' - 4y'x^2 + (1+2x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = y' - \frac{(1+2x)y}{4x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - y' + \frac{(1+2x)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -1, P_3(x) = \frac{1+2x}{4x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = 0$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = \frac{1}{4}$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2y'' - 4y'x^2 + (1+2x)\,y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- > k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^2 \cdot y'$ to series expansion

$$x^2 \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r+1}$$

- Shift index using $k- > k - 1$

$$x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1} (k-1+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k-1+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 (-1+2r)^2 x^r + \left(\sum_{k=1}^{\infty} \left(a_k (2k+2r-1)^2 - 2a_{k-1} (2k-3+2r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-1+2r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = \frac{1}{2}$$

- Each term in the series must be 0, giving the recursion relation

$$a_k (2k+2r-1)^2 + (-4k+6-4r)\, a_{k-1} = 0$$

- Shift index using $k- > k + 1$

$$a_{k+1} (2k+1+2r)^2 + a_k (-4k-4r+2) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{2a_k (2k+2r-1)}{(2k+1+2r)^2}$$

- Recursion relation for $r = \frac{1}{2}$

$a_{k+1} = \frac{4a_k k}{(2k+2)^2}$

- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = \frac{4a_k k}{(2k+2)^2}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 49

```
Order:=6;
dsolve(4*x^2*diff(y(x),x$2)-4*x^2*diff(y(x),x)+(1+2*x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \sqrt{x}\left(\left(x+\frac{1}{4}x^2+\frac{1}{18}x^3+\frac{1}{96}x^4+\frac{1}{600}x^5+O\left(x^6\right)\right)c_2+\left(c_2\ln\left(x\right)+c_1\right)\left(1+O\left(x^6\right)\right)\right)$$

✓ Solution by Mathematica
Time used: 0.004 (sec). Leaf size: 60

```
AsymptoticDSolveValue[4*x^2*y''[x]-4*x^2*y'[x]+(1+2*x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\sqrt{x}\left(\frac{x^5}{600}+\frac{x^4}{96}+\frac{x^3}{18}+\frac{x^2}{4}+x\right)+\sqrt{x}\log(x)\right)+c_1\sqrt{x}$$

17.20 problem 21

Internal problem ID [2936]
Internal file name [`OUTPUT/2428_Sunday_June_05_2022_03_09_30_AM_49471608/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.4. page 758
Problem number: 21.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2 y'' + x(3-2x)\, y' + (1-2x)\, y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2 y'' + \left(-2x^2 + 3x\right) y' + (1-2x)\, y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{-3+2x}{x}$$
$$q(x) = -\frac{2x-1}{x^2}$$

Table 450: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{-3+2x}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{2x-1}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + \left(-2x^2 + 3x\right) y' + (1 - 2x) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2} \right) + \left(-2x^2 + 3x\right) \left(\sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1} \right) + (1 - 2x) \left(\sum_{n=0}^{\infty} a_n x^{n+r} \right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-2x^{1+n+r} a_n (n+r)\right) \\ + \left(\sum_{n=0}^{\infty} 3x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-2x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-2x^{1+n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-2a_{n-1}(n+r-1)\, x^{n+r}\right)$$

$$\sum_{n=0}^{\infty} \left(-2x^{1+n+r} a_n\right) = \sum_{n=1}^{\infty} \left(-2a_{n-1} x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-2a_{n-1}(n+r-1)\, x^{n+r}\right) \\ + \left(\sum_{n=0}^{\infty} 3x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) + \sum_{n=1}^{\infty} \left(-2a_{n-1} x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) + 3x^{n+r} a_n (n+r) + a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) + 3x^r a_0 r + a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) + 3x^r r + x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(r+1)^2 x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(r+1)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = -1$$
$$r_2 = -1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(r+1)^2 x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = -1$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n-1}$$
$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n-1}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) - 2a_{n-1}(n+r-1) + 3a_n(n+r) + a_n - 2a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{2a_{n-1}(n+r)}{n^2+2nr+r^2+2n+2r+1} \tag{4}$$

Which for the root $r=-1$ becomes

$$a_n = \frac{2a_{n-1}(n-1)}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r=-1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n=1$, using the above recursive equation gives

$$a_1 = \frac{2r+2}{(r+2)^2}$$

Which for the root $r=-1$ becomes

$$a_1 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r+2}{(r+2)^2}$	0

For $n=2$, using the above recursive equation gives

$$a_2 = \frac{4r+4}{(r+2)(r+3)^2}$$

Which for the root $r=-1$ becomes

$$a_2 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r+2}{(r+2)^2}$	0
a_2	$\frac{4r+4}{(r+2)(r+3)^2}$	0

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{8r+8}{(r+2)(r+3)(r+4)^2}$$

Which for the root $r = -1$ becomes

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r+2}{(r+2)^2}$	0
a_2	$\frac{4r+4}{(r+2)(r+3)^2}$	0
a_3	$\frac{8r+8}{(r+2)(r+3)(r+4)^2}$	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{16r+16}{(r+2)(r+3)(r+4)(5+r)^2}$$

Which for the root $r = -1$ becomes

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r+2}{(r+2)^2}$	0
a_2	$\frac{4r+4}{(r+2)(r+3)^2}$	0
a_3	$\frac{8r+8}{(r+2)(r+3)(r+4)^2}$	0
a_4	$\frac{16r+16}{(r+2)(r+3)(r+4)(5+r)^2}$	0

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{32r + 32}{(r + 2)(r + 3)(r + 4)(5 + r)(6 + r)^2}$$

Which for the root $r = -1$ becomes

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r+2}{(r+2)^2}$	0
a_2	$\frac{4r+4}{(r+2)(r+3)^2}$	0
a_3	$\frac{8r+8}{(r+2)(r+3)(r+4)^2}$	0
a_4	$\frac{16r+16}{(r+2)(r+3)(r+4)(5+r)^2}$	0
a_5	$\frac{32r+32}{(r+2)(r+3)(r+4)(5+r)(6+r)^2}$	0

Using the above table, then the first solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= \frac{1}{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= \frac{1 + O(x^6)}{x} \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = -1$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr}a_{n,r}$	$b_n(r=-1)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	$\frac{2r+2}{(r+2)^2}$	0	$-\frac{2r}{(r+2)^3}$	2
b_2	$\frac{4r+4}{(r+2)(r+3)^2}$	0	$\frac{-8r^2-20r-4}{(r+2)^2(r+3)^3}$	1
b_3	$\frac{8r+8}{(r+2)(r+3)(r+4)^2}$	0	$\frac{-24r^3-144r^2-232r-64}{(r+2)^2(r+3)^2(r+4)^3}$	$\frac{4}{9}$
b_4	$\frac{16r+16}{(r+2)(r+3)(r+4)(5+r)^2}$	0	$-\frac{32(2r^4+21r^3+74r^2+96r+29)}{(r+2)^2(r+3)^2(r+4)^2(5+r)^3}$	$\frac{1}{6}$
b_5	$\frac{32r+32}{(r+2)(r+3)(r+4)(5+r)(6+r)^2}$	0	$-\frac{32(5r^5+80r^4+475r^3+1270r^2+1434r+444)}{(r+2)^2(r+3)^2(r+4)^2(5+r)^2(6+r)^3}$	$\frac{4}{75}$

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\
&= \frac{(1+O(x^6))\ln(x)}{x} + \frac{x^2 + 2x + \frac{4x^3}{9} + \frac{x^4}{6} + \frac{4x^5}{75} + O(x^6)}{x}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= \frac{c_1(1+O(x^6))}{x} + c_2\left(\frac{(1+O(x^6))\ln(x)}{x} + \frac{x^2 + 2x + \frac{4x^3}{9} + \frac{x^4}{6} + \frac{4x^5}{75} + O(x^6)}{x}\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= \frac{c_1(1+O(x^6))}{x} + c_2\left(\frac{(1+O(x^6))\ln(x)}{x} + \frac{x^2 + 2x + \frac{4x^3}{9} + \frac{x^4}{6} + \frac{4x^5}{75} + O(x^6)}{x}\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = \frac{c_1(1+O(x^6))}{x} + c_2\left(\frac{(1+O(x^6))\ln(x)}{x} + \frac{x^2 + 2x + \frac{4x^3}{9} + \frac{x^4}{6} + \frac{4x^5}{75} + O(x^6)}{x}\right) \quad (1)
$$

Verification of solutions

$$y = \frac{c_1(1+O(x^6))}{x} + c_2\left(\frac{(1+O(x^6))\ln(x)}{x} + \frac{x^2+2x+\frac{4x^3}{9}+\frac{x^4}{6}+\frac{4x^5}{75}+O(x^6)}{x}\right)$$

Verified OK.

17.20.1 Maple step by step solution

Let's solve

$x^2y''+(-2x^2+3x)\,y'+(1-2x)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{(2x-1)y}{x^2}+\frac{(-3+2x)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{(-3+2x)y'}{x}-\frac{(2x-1)y}{x^2}=0$

□ Check to see if $x_0=0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=-\frac{-3+2x}{x}, P_3(x)=-\frac{2x-1}{x^2}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=3$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2\cdot P_3(x))\right|_{x=0}=1$

  ◦ $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x^2y''-x(-3+2x)\,y'+(1-2x)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_kx^{k+r}$

- Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k->k-m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$
  - Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 (1+r)^2 x^r + \left(\sum_{k=1}^{\infty} \left(a_k (k+r+1)^2 - 2a_{k-1}(k+r)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)^2 = 0$
- Values of r that satisfy the indicial equation

  $r = -1$
- Each term in the series must be 0, giving the recursion relation

  $a_k (k+r+1)^2 - 2a_{k-1}(k+r) = 0$
- Shift index using $k->k+1$

  $a_{k+1} (k+2+r)^2 - 2a_k (k+r+1) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{2a_k (k+r+1)}{(k+2+r)^2}$
- Recursion relation for $r = -1$

  $a_{k+1} = \frac{2a_k k}{(k+1)^2}$

- Solution for $r = -1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+1} = \frac{2a_k k}{(k+1)^2}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 49

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+x*(3-2*x)*diff(y(x),x)+(1-2*x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{\left(2x + x^2 + \frac{4}{9}x^3 + \frac{1}{6}x^4 + \frac{4}{75}x^5 + \mathrm{O}\left(x^6\right)\right)c_2 + \left(c_2 \ln\left(x\right) + c_1\right)\left(1 + \mathrm{O}\left(x^6\right)\right)}{x}$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 52

```
AsymptoticDSolveValue[x^2*y''[x]+x*(3-2*x)*y'[x]+(1-2*x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{\frac{4x^5}{75} + \frac{x^4}{6} + \frac{4x^3}{9} + x^2 + 2x}{x} + \frac{\log(x)}{x}\right) + \frac{c_1}{x}$$

18 Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.5. page 771

18.1 problem Example 11.5.2 page 763

Internal problem ID [2937]
Internal file name [`OUTPUT/2429_Sunday_June_05_2022_03_09_33_AM_65018095/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.5. page 771
Problem number: Example 11.5.2 page 763.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - x(x+3)\,y' + (-x+4)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + \left(-x^2 - 3x\right)y' + (-x+4)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{x+3}{x}$$
$$q(x) = -\frac{x-4}{x^2}$$

Table 452: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{x+3}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{x-4}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + \left(-x^2 - 3x\right) y' + (-x + 4)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned} &x^2 \left(\sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2}\right) \\ &+ \left(-x^2 - 3x\right) \left(\sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}\right) + (-x + 4) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \end{aligned} \tag{1}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n (n+r)\right) \\
&+ \sum_{n=0}^{\infty} \left(-3x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n\right) + \left(\sum_{n=0}^{\infty} 4a_n x^{n+r}\right) = 0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)\, x^{n+r}\right)
$$

$$
\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n\right) = \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r}\right)
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)\, x^{n+r}\right) \\
&+ \sum_{n=0}^{\infty} \left(-3x^{n+r} a_n (n+r)\right) + \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 4a_n x^{n+r}\right) = 0
\end{aligned} \tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
x^{n+r} a_n (n+r)(n+r-1) - 3x^{n+r} a_n (n+r) + 4a_n x^{n+r} = 0
$$

When $n=0$ the above becomes

$$
x^r a_0 r(-1+r) - 3x^r a_0 r + 4a_0 x^r = 0
$$

Or

$$
\left(x^r r(-1+r) - 3x^r r + 4x^r\right) a_0 = 0
$$

Since $a_0 \neq 0$ then the above simplifies to

$$
(r-2)^2 x^r = 0
$$

Since the above is true for all x then the indicial equation becomes

$$(r-2)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 2$$
$$r_2 = 2$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(r-2)^2 x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = 2$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+2}$$
$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+2}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) - a_{n-1}(n+r-1) - 3a_n(n+r) - a_{n-1} + 4a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}(n+r)}{n^2 + 2nr + r^2 - 4n - 4r + 4} \tag{4}$$

Which for the root $r = 2$ becomes

$$a_n = \frac{a_{n-1}(n+2)}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 2$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{1+r}{(-1+r)^2}$$

Which for the root $r = 2$ becomes

$$a_1 = 3$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1+r}{(-1+r)^2}$	3

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{(1+r)(2+r)}{(-1+r)^2 r^2}$$

Which for the root $r = 2$ becomes

$$a_2 = 3$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1+r}{(-1+r)^2}$	3
a_2	$\frac{(1+r)(2+r)}{(-1+r)^2 r^2}$	3

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{(2+r)(3+r)}{(1+r)(-1+r)^2 r^2}$$

Which for the root $r = 2$ becomes

$$a_3 = \frac{5}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1+r}{(-1+r)^2}$	3
a_2	$\frac{(1+r)(2+r)}{(-1+r)^2 r^2}$	3
a_3	$\frac{(2+r)(3+r)}{(1+r)(-1+r)^2 r^2}$	$\frac{5}{3}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{(3+r)(4+r)}{(1+r)(2+r)(-1+r)^2 r^2}$$

Which for the root $r = 2$ becomes

$$a_4 = \frac{5}{8}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1+r}{(-1+r)^2}$	3
a_2	$\frac{(1+r)(2+r)}{(-1+r)^2 r^2}$	3
a_3	$\frac{(2+r)(3+r)}{(1+r)(-1+r)^2 r^2}$	$\frac{5}{3}$
a_4	$\frac{(3+r)(4+r)}{(1+r)(2+r)(-1+r)^2 r^2}$	$\frac{5}{8}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{(4+r)(5+r)}{(1+r)(2+r)(-1+r)^2 r^2 (3+r)}$$

Which for the root $r = 2$ becomes

$$a_5 = \frac{7}{40}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1+r}{(-1+r)^2}$	3
a_2	$\frac{(1+r)(2+r)}{(-1+r)^2 r^2}$	3
a_3	$\frac{(2+r)(3+r)}{(1+r)(-1+r)^2 r^2}$	$\frac{5}{3}$
a_4	$\frac{(3+r)(4+r)}{(1+r)(2+r)(-1+r)^2 r^2}$	$\frac{5}{8}$
a_5	$\frac{(4+r)(5+r)}{(1+r)(2+r)(-1+r)^2 r^2(3+r)}$	$\frac{7}{40}$

Using the above table, then the first solution $y_1(x)$ is

$$\begin{aligned}
y_1(x) &= x^2\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= x^2\left(3x^2 + 3x + 1 + \frac{5x^3}{3} + \frac{5x^4}{8} + \frac{7x^5}{40} + O(x^6)\right)
\end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = 2$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr}a_{n,r}$	$b_n(r=2)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	$\frac{1+r}{(-1+r)^2}$	3	$\frac{-3-r}{(-1+r)^3}$	-5
b_2	$\frac{(1+r)(2+r)}{(-1+r)^2 r^2}$	3	$\frac{-2r^3-9r^2-5r+4}{(-1+r)^3 r^3}$	$-\frac{29}{4}$
b_3	$\frac{(2+r)(3+r)}{(1+r)(-1+r)^2 r^2}$	$\frac{5}{3}$	$\frac{-3r^4-21r^3-35r^2-r+12}{(1+r)^2(-1+r)^3 r^3}$	$-\frac{173}{36}$
b_4	$\frac{(3+r)(4+r)}{(1+r)(2+r)(-1+r)^2 r^2}$	$\frac{5}{8}$	$\frac{-4r^5-42r^4-136r^3-132r^2+26r+48}{(1+r)^2(2+r)^2(-1+r)^3 r^3}$	$-\frac{193}{96}$
b_5	$\frac{(4+r)(5+r)}{(1+r)(2+r)(-1+r)^2 r^2(3+r)}$	$\frac{7}{40}$	$\frac{-5r^6-75r^5-395r^4-855r^3-584r^2+234r+240}{(1+r)^2(2+r)^2(-1+r)^3 r^3(3+r)^2}$	$-\frac{1459}{2400}$

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\
&= x^2\left(3x^2 + 3x + 1 + \frac{5x^3}{3} + \frac{5x^4}{8} + \frac{7x^5}{40} + O(x^6)\right)\ln(x) \\
&\quad + x^2\left(-5x - \frac{29x^2}{4} - \frac{173x^3}{36} - \frac{193x^4}{96} - \frac{1459x^5}{2400} + O(x^6)\right)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^2\left(3x^2 + 3x + 1 + \frac{5x^3}{3} + \frac{5x^4}{8} + \frac{7x^5}{40} + O(x^6)\right) \\
&\quad + c_2\left(x^2\left(3x^2 + 3x + 1 + \frac{5x^3}{3} + \frac{5x^4}{8} + \frac{7x^5}{40} + O(x^6)\right)\ln(x)\right. \\
&\quad\left. + x^2\left(-5x - \frac{29x^2}{4} - \frac{173x^3}{36} - \frac{193x^4}{96} - \frac{1459x^5}{2400} + O(x^6)\right)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^2\left(3x^2 + 3x + 1 + \frac{5x^3}{3} + \frac{5x^4}{8} + \frac{7x^5}{40} + O(x^6)\right) \\
&\quad + c_2\left(x^2\left(3x^2 + 3x + 1 + \frac{5x^3}{3} + \frac{5x^4}{8} + \frac{7x^5}{40} + O(x^6)\right)\ln(x)\right. \\
&\quad\left. + x^2\left(-5x - \frac{29x^2}{4} - \frac{173x^3}{36} - \frac{193x^4}{96} - \frac{1459x^5}{2400} + O(x^6)\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & c_1 x^2 \left(3x^2+3x+1+\frac{5x^3}{3}+\frac{5x^4}{8}+\frac{7x^5}{40}+O\left(x^6\right)\right) \\
& + c_2 \left(x^2 \left(3x^2+3x+1+\frac{5x^3}{3}+\frac{5x^4}{8}+\frac{7x^5}{40}+O\left(x^6\right)\right)\ln\left(x\right)\right. \\
& \left. + x^2 \left(-5x-\frac{29x^2}{4}-\frac{173x^3}{36}-\frac{193x^4}{96}-\frac{1459x^5}{2400}+O\left(x^6\right)\right)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = {} & c_1 x^2 \left(3x^2+3x+1+\frac{5x^3}{3}+\frac{5x^4}{8}+\frac{7x^5}{40}+O\left(x^6\right)\right) \\
& + c_2 \left(x^2 \left(3x^2+3x+1+\frac{5x^3}{3}+\frac{5x^4}{8}+\frac{7x^5}{40}+O\left(x^6\right)\right)\ln\left(x\right)\right. \\
& \left. + x^2 \left(-5x-\frac{29x^2}{4}-\frac{173x^3}{36}-\frac{193x^4}{96}-\frac{1459x^5}{2400}+O\left(x^6\right)\right)\right)
\end{aligned}
$$

Verified OK.

18.1.1 Maple step by step solution

Let's solve

$x^2 y'' + \left(-x^2-3x\right) y' + \left(-x+4\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(x-4)y}{x^2} + \frac{(x+3)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(x+3)y'}{x} - \frac{(x-4)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{x+3}{x}, P_3(x) = -\frac{x-4}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -3$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 4$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2 y'' - x(x+3)\,y' + (-x+4)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

  - Shift index using $k- >k + 1 - m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

  - Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-2+r)^2\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k (k+r-2)^2 - a_{k-1}(k+r)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-2+r)^2 = 0$

- Values of r that satisfy the indicial equation

$r = 2$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r-2)^2 - a_{k-1}(k+r) = 0$

- Shift index using $k- >k+1$

  $a_{k+1}(k+r-1)^2 - a_k(k+r+1) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r+1)}{(k+r-1)^2}$

- Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_k(k+3)}{(k+1)^2}$

- Solution for $r = 2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = \frac{a_k(k+3)}{(k+1)^2}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 69

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)-x*(3+x)*diff(y(x),x)+(4-x)*y(x)=0,y(x),type='series',x=0);
```

$$
\begin{aligned}
y(x) = \Bigg(\bigg(&(-5)\,x - \frac{29}{4}x^2 - \frac{173}{36}x^3 - \frac{193}{96}x^4 - \frac{1459}{2400}x^5 + \mathrm{O}\left(x^6\right)\bigg)\, c_2 \\
&+ \left(1 + 3x + 3x^2 + \frac{5}{3}x^3 + \frac{5}{8}x^4 + \frac{7}{40}x^5 + \mathrm{O}\left(x^6\right)\right)\left(c_2 \ln\left(x\right) + c_1\right)\Bigg)\, x^2
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 118

```
AsymptoticDSolveValue[x^2*y''[x]-x*(3+x)*y'[x]+(4-x)*y[x]==0,y[x],{x,0,5}]
```

$$
\begin{aligned}
y(x) \rightarrow\ & c_1\left(\frac{7x^5}{40} + \frac{5x^4}{8} + \frac{5x^3}{3} + 3x^2 + 3x + 1\right)x^2 \\
&+ c_2\Bigg(\left(-\frac{1459x^5}{2400} - \frac{193x^4}{96} - \frac{173x^3}{36} - \frac{29x^2}{4} - 5x\right)x^2 \\
&+ \left(\frac{7x^5}{40} + \frac{5x^4}{8} + \frac{5x^3}{3} + 3x^2 + 3x + 1\right)x^2 \log(x)\Bigg)
\end{aligned}
$$

18.2 problem Example 11.5.4 page 765

Internal problem ID [2938]
Internal file name [`OUTPUT/2430_Sunday_June_05_2022_03_09_37_AM_63942638/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.5. page 771
Problem number: Example 11.5.4 page 765.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + x(-x+3)\, y' + y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + \left(-x^2 + 3x\right) y' + y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{x-3}{x}$$
$$q(x) = \frac{1}{x^2}$$

Table 454: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{x-3}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{1}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2 y'' + \left(-x^2 + 3x\right) y' + y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned} & x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) \\ & + \left(-x^2 + 3x\right) \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \end{aligned} \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 3x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)\, x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 3x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) + 3x^{n+r} a_n (n+r) + a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) + 3x^r a_0 r + a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) + 3x^r r + x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(r+1)^2 x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(r+1)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = -1$$
$$r_2 = -1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(r+1)^2 x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = -1$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n-1}$$
$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n-1}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) - a_{n-1}(n+r-1) + 3a_n(n+r) + a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}(n+r-1)}{n^2 + 2nr + r^2 + 2n + 2r + 1} \tag{4}$$

Which for the root $r = -1$ becomes

$$a_n = \frac{a_{n-1}(n-2)}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{r}{(r+2)^2}$$

Which for the root $r = -1$ becomes

$$a_1 = -1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r}{(r+2)^2}$	-1

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{r(r+1)}{(r+2)^2 (r+3)^2}$$

Which for the root $r = -1$ becomes

$$a_2 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r}{(r+2)^2}$	-1
a_2	$\frac{r(r+1)}{(r+2)^2(r+3)^2}$	0

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{r(r+1)}{(r+2)(r+3)^2(r+4)^2}$$

Which for the root $r = -1$ becomes

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r}{(r+2)^2}$	-1
a_2	$\frac{r(r+1)}{(r+2)^2(r+3)^2}$	0
a_3	$\frac{r(r+1)}{(r+2)(r+3)^2(r+4)^2}$	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{r(r+1)}{(r+2)(r+3)(r+4)^2(5+r)^2}$$

Which for the root $r = -1$ becomes

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r}{(r+2)^2}$	-1
a_2	$\frac{r(r+1)}{(r+2)^2(r+3)^2}$	0
a_3	$\frac{r(r+1)}{(r+2)(r+3)^2(r+4)^2}$	0
a_4	$\frac{r(r+1)}{(r+2)(r+3)(r+4)^2(5+r)^2}$	0

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{r(r+1)}{(r+2)(r+3)(r+4)(5+r)^2(6+r)^2}$$

Which for the root $r = -1$ becomes

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r}{(r+2)^2}$	-1
a_2	$\frac{r(r+1)}{(r+2)^2(r+3)^2}$	0
a_3	$\frac{r(r+1)}{(r+2)(r+3)^2(r+4)^2}$	0
a_4	$\frac{r(r+1)}{(r+2)(r+3)(r+4)^2(5+r)^2}$	0
a_5	$\frac{r(r+1)}{(r+2)(r+3)(r+4)(5+r)^2(6+r)^2}$	0

Using the above table, then the first solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= \frac{1}{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= \frac{1 - x + O(x^6)}{x} \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = -1$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr}a_{n,r}$	$b_n(r=-1)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	$\frac{r}{(r+2)^2}$	-1	$\frac{-r+2}{(r+2)^3}$	3
b_2	$\frac{r(r+1)}{(r+2)^2(r+3)^2}$	0	$\frac{-2r^3-3r^2+7r+6}{(r+2)^3(r+3)^3}$	$-\frac{1}{4}$
b_3	$\frac{r(r+1)}{(r+2)(r+3)^2(r+4)^2}$	0	$\frac{-3r^4-15r^3-8r^2+34r+24}{(r+2)^2(r+3)^3(r+4)^3}$	$-\frac{1}{36}$
b_4	$\frac{r(r+1)}{(r+2)(r+3)(r+4)^2(5+r)^2}$	0	$\frac{-4r^5-38r^4-104r^3-28r^2+186r+120}{(r+2)^2(r+3)^2(r+4)^3(5+r)^3}$	$-\frac{1}{288}$
b_5	$\frac{r(r+1)}{(r+2)(r+3)(r+4)(5+r)^2(6+r)^2}$	0	$\frac{-5r^6-75r^5-395r^4-795r^3-134r^2+1176r+720}{(r+2)^2(r+3)^2(r+4)^2(5+r)^3(6+r)^3}$	$-\frac{1}{2400}$

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 \dots \\
&= \frac{(1 - x + O(x^6))\ln(x)}{x} + \frac{3x - \frac{x^2}{4} - \frac{x^3}{36} - \frac{x^4}{288} - \frac{x^5}{2400} + O(x^6)}{x}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1y_1(x) + c_2y_2(x) \\
&= \frac{c_1(1 - x + O(x^6))}{x} + c_2\left(\frac{(1 - x + O(x^6))\ln(x)}{x} + \frac{3x - \frac{x^2}{4} - \frac{x^3}{36} - \frac{x^4}{288} - \frac{x^5}{2400} + O(x^6)}{x}\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= \frac{c_1(1 - x + O(x^6))}{x} + c_2\left(\frac{(1 - x + O(x^6))\ln(x)}{x} + \frac{3x - \frac{x^2}{4} - \frac{x^3}{36} - \frac{x^4}{288} - \frac{x^5}{2400} + O(x^6)}{x}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & \frac{c_1(1 - x + O(x^6))}{x} \\
& + c_2\left(\frac{(1 - x + O(x^6))\ln(x)}{x} + \frac{3x - \frac{x^2}{4} - \frac{x^3}{36} - \frac{x^4}{288} - \frac{x^5}{2400} + O(x^6)}{x}\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$y = \frac{c_1(1-x+O(x^6))}{x} + c_2\left(\frac{(1-x+O(x^6))\ln(x)}{x} + \frac{3x-\frac{x^2}{4}-\frac{x^3}{36}-\frac{x^4}{288}-\frac{x^5}{2400}+O(x^6)}{x}\right)$$

Verified OK.

18.2.1 Maple step by step solution

Let's solve

$x^2y'' + (-x^2+3x)\,y' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y}{x^2} + \frac{(x-3)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(x-3)y'}{x} + \frac{y}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -\frac{x-3}{x}, P_3(x) = \frac{1}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = 3$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = 1$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2y'' - x(x-3)\,y' + y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions
  - Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$
  - Shift index using $k- >k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$
  - Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+r)^2\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)^2 - a_{k-1}(k+r-1)\right) x^{k+r}\right) = 0$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)^2 = 0$
- Values of r that satisfy the indicial equation

  $r = -1$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+1)^2 - a_{k-1}(k+r-1) = 0$
- Shift index using $k- >k+1$

  $a_{k+1}(k+2+r)^2 - a_k(k+r) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r)}{(k+2+r)^2}$
- Recursion relation for $r = -1$; series terminates at $k = 1$

  $a_{k+1} = \frac{a_k(k-1)}{(k+1)^2}$
- Apply recursion relation for $k = 0$

  $a_1 = -a_0$
- Terminating series solution of the ODE for $r = -1$. Use reduction of order to find the second

  $y = a_0 \cdot (1-x)$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 53

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+x*(3-x)*diff(y(x),x)+y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{\left(3x - \frac{1}{4}x^2 - \frac{1}{36}x^3 - \frac{1}{288}x^4 - \frac{1}{2400}x^5 + \mathrm{O}\left(x^6\right)\right)c_2 + \left(c_2 \ln\left(x\right) + c_1\right)\left(1 - x + \mathrm{O}\left(x^6\right)\right)}{x}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 66

```
AsymptoticDSolveValue[x^2*y''[x]+x*(3-x)*y'[x]+y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{-\frac{x^5}{2400} - \frac{x^4}{288} - \frac{x^3}{36} - \frac{x^2}{4} + 3x}{x} + \frac{(1-x)\log(x)}{x}\right) + \frac{c_1(1-x)}{x}$$

18.3 problem Example 11.5.5 page 768

Internal problem ID [2939]
Internal file name [OUTPUT/2431_Sunday_June_05_2022_03_09_40_AM_25398422/index.tex]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.5. page 771
Problem number: Example 11.5.5 page 768.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + xy' - (x+4)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + xy' + (-x-4)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = -\frac{x+4}{x^2}$$

Table 456: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{x+4}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2 y'' + x y' + (-x - 4) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}\right) + x \left(\sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}\right) + (-x - 4) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) \tag{2A}$$
$$+ \sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n\right) + \sum_{n=0}^{\infty} \left(-4a_n x^{n+r}\right) = 0$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n\right) = \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) \tag{2B}$$
$$+ \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-4a_n x^{n+r}\right) = 0$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) + x^{n+r} a_n (n+r) - 4a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) + x^r a_0 r - 4a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) + x^r r - 4x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2 - 4\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - 4 = 0$$

Solving for r gives the roots of the indicial equation as

$$
\begin{aligned}
r_1 &= 2 \\
r_2 &= -2
\end{aligned}
$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$
\left(r^2 - 4\right) x^r = 0
$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 4$ is an integer, then we can construct two linearly independent solutions

$$
\begin{aligned}
y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right) \\
y_2(x) &= C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= x^2 \left(\sum_{n=0}^{\infty} a_n x^n\right) \\
y_2(x) &= C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x^2}
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+2} \\
y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-2}\right)
\end{aligned}
$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$
a_n(n + r)(n + r - 1) + a_n(n + r) - a_{n-1} - 4a_n = 0 \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$
a_n = \frac{a_{n-1}}{n^2 + 2nr + r^2 - 4} \tag{4}
$$

Which for the root $r = 2$ becomes

$$a_n = \frac{a_{n-1}}{n(n+4)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 2$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{1}{r^2 + 2r - 3}$$

Which for the root $r = 2$ becomes

$$a_1 = \frac{1}{5}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{r^2+2r-3}$	$\frac{1}{5}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{(r+3)(-1+r)r(r+4)}$$

Which for the root $r = 2$ becomes

$$a_2 = \frac{1}{60}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{r^2+2r-3}$	$\frac{1}{5}$
a_2	$\frac{1}{(r+3)(-1+r)r(r+4)}$	$\frac{1}{60}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{1}{(r+3)(-1+r)r(r+4)(5+r)(r+1)}$$

Which for the root $r = 2$ becomes

$$a_3 = \frac{1}{1260}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{r^2+2r-3}$	$\frac{1}{5}$
a_2	$\frac{1}{(r+3)(-1+r)r(r+4)}$	$\frac{1}{60}$
a_3	$\frac{1}{(r+3)(-1+r)r(r+4)(5+r)(r+1)}$	$\frac{1}{1260}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(r+3)r(r+4)(5+r)(r^2+8r+12)(r^2-1)}$$

Which for the root $r = 2$ becomes

$$a_4 = \frac{1}{40320}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{r^2+2r-3}$	$\frac{1}{5}$
a_2	$\frac{1}{(r+3)(-1+r)r(r+4)}$	$\frac{1}{60}$
a_3	$\frac{1}{(r+3)(-1+r)r(r+4)(5+r)(r+1)}$	$\frac{1}{1260}$
a_4	$\frac{1}{(r+3)r(r+4)(5+r)(r^2+8r+12)(r^2-1)}$	$\frac{1}{40320}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{1}{(r+3)^2 r(r+4)(5+r)(6+r)(r+2)(-1+r)(r+1)(r+7)}$$

Which for the root $r = 2$ becomes

$$a_5 = \frac{1}{1814400}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{r^2+2r-3}$	$\frac{1}{5}$
a_2	$\frac{1}{(r+3)(-1+r)r(r+4)}$	$\frac{1}{60}$
a_3	$\frac{1}{(r+3)(-1+r)r(r+4)(5+r)(r+1)}$	$\frac{1}{1260}$
a_4	$\frac{1}{(r+3)r(r+4)(5+r)(r^2+8r+12)(r^2-1)}$	$\frac{1}{40320}$
a_5	$\frac{1}{(r+3)^2r(r+4)(5+r)(6+r)(r+2)(-1+r)(r+1)(r+7)}$	$\frac{1}{1814400}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}
y_1(x) &= x^2\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots\right) \\
&= x^2\left(1 + \frac{x}{5} + \frac{x^2}{60} + \frac{x^3}{1260} + \frac{x^4}{40320} + \frac{x^5}{1814400} + O(x^6)\right)
\end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 4$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\rightarrow r_2} a_4(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned}
a_N &= a_4 \\
&= \frac{1}{(r+3)\, r\, (r+4)\, (5+r)\, (r^2+8r+12)\, (r^2-1)}
\end{aligned}$$

Therefore

$$\begin{aligned}
\lim_{r\rightarrow r_2} \frac{1}{(r+3)\, r\, (r+4)\, (5+r)\, (r^2+8r+12)\, (r^2-1)} &= \lim_{r\rightarrow -2} \frac{1}{(r+3)\, r\, (r+4)\, (5+r)\, (r^2+8r+12)\, (r^2-1)} \\
&= \mathit{undefined}
\end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$\begin{aligned}
\frac{d}{dx}y_2(x) &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\
&= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right) \\
\frac{d^2}{dx^2}y_2(x) &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\
&= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right)
\end{aligned}$$

Substituting these back into the given ode $x^2y'' + xy' + (-x-4)\,y = 0$ gives

$$\begin{aligned}
&x^2\left(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2}\right. \\
&\quad \left. + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) \\
&\quad + x\left(Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right) \\
&\quad + (-x-4)\left(Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)\right) = 0
\end{aligned}$$

Which can be written as

$$
\begin{aligned}
&\Bigg(\left(x^{2}y_1''(x)+y_1'(x)\,x+(-x-4)\,y_1(x)\right)\ln(x)+x^{2}\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^{2}}\right)\\
&+y_1(x)\Bigg)C+x^{2}\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^{2}}{x^{2}}-\frac{b_n x^{n+r_2}(n+r_2)}{x^{2}}\right)\right)\\
&+x\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)+(-x-4)\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0
\end{aligned}
\tag{7}
$$

But since $y_1(x)$ is a solution to the ode, then

$$
x^{2}y_1''(x)+y_1'(x)\,x+(-x-4)\,y_1(x)=0
$$

Eq (7) simplifes to

$$
\begin{aligned}
&\left(x^{2}\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^{2}}\right)+y_1(x)\right)C\\
&+x^{2}\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^{2}}{x^{2}}-\frac{b_n x^{n+r_2}(n+r_2)}{x^{2}}\right)\right)\\
&+x\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)+(-x-4)\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0
\end{aligned}
\tag{8}
$$

Substituting $y_1=\sum_{n=0}^{\infty}a_n x^{n+r_1}$ into the above gives

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty}x^{-2+n+r_2}b_n(n+r_2)(-1+n+r_2)\right)x^{2}+2x\left(\sum_{n=0}^{\infty}x^{-1+n+r_1}a_n(n+r_1)\right)C\\
&+\left(\sum_{n=0}^{\infty}x^{-1+n+r_2}b_n(n+r_2)\right)x-(x+4)\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0
\end{aligned}
\tag{9}
$$

Since $r_1=2$ and $r_2=-2$ then the above becomes

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty}x^{-4+n}b_n(n-2)(-3+n)\right)x^{2}+2x\left(\sum_{n=0}^{\infty}x^{1+n}a_n(n+2)\right)C\\
&+\left(\sum_{n=0}^{\infty}x^{-3+n}b_n(n-2)\right)x-(x+4)\left(\sum_{n=0}^{\infty}b_n x^{n-2}\right)=0
\end{aligned}
\tag{10}
$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} x^{n-2} b_n \left(n^2-5n+6\right)\right)+\left(\sum_{n=0}^{\infty} 2C\, x^{n+2} a_n (n+2)\right) \\
&+\left(\sum_{n=0}^{\infty} x^{n-2} b_n (n-2)\right)+\sum_{n=0}^{\infty}\left(-x^{n-1} b_n\right)+\sum_{n=0}^{\infty}\left(-4 b_n x^{n-2}\right)=0
\end{aligned}
\tag{2A}
$$

The next step is to make all powers of x be $n-2$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-2} and adjusting the power and the corresponding index gives

$$
\sum_{n=0}^{\infty} 2C\, x^{n+2} a_n (n+2)=\sum_{n=4}^{\infty} 2C a_{-4+n} (n-2)\, x^{n-2}
$$

$$
\sum_{n=0}^{\infty}\left(-x^{n-1} b_n\right)=\sum_{n=1}^{\infty}\left(-b_{n-1} x^{n-2}\right)
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-2$.

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} x^{n-2} b_n \left(n^2-5n+6\right)\right)+\left(\sum_{n=4}^{\infty} 2C a_{-4+n} (n-2)\, x^{n-2}\right) \\
&+\left(\sum_{n=0}^{\infty} x^{n-2} b_n (n-2)\right)+\sum_{n=1}^{\infty}\left(-b_{n-1} x^{n-2}\right)+\sum_{n=0}^{\infty}\left(-4 b_n x^{n-2}\right)=0
\end{aligned}
\tag{2B}
$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0=1$. For $n=1$, Eq (2B) gives

$$
-3b_1-b_0=0
$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$
-3b_1-1=0
$$

Solving the above for b_1 gives

$$
b_1=-\frac{1}{3}
$$

For $n=2$, Eq (2B) gives

$$
-b_1-4b_2=0
$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{1}{3} - 4b_2 = 0$$

Solving the above for b_2 gives

$$b_2 = \frac{1}{12}$$

For $n = 3$, Eq (2B) gives

$$-3b_3 - b_2 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-3b_3 - \frac{1}{12} = 0$$

Solving the above for b_3 gives

$$b_3 = -\frac{1}{36}$$

For $n = N$, where $N = 4$ which is the difference between the two roots, we are free to choose $b_4 = 0$. Hence for $n = 4$, Eq (2B) gives

$$4C + \frac{1}{36} = 0$$

Which is solved for C. Solving for C gives

$$C = -\frac{1}{144}$$

For $n = 5$, Eq (2B) gives

$$6Ca_1 - b_4 + 5b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$5b_5 - \frac{1}{120} = 0$$

Solving the above for b_5 gives

$$b_5 = \frac{1}{600}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -\frac{1}{144}$ and all b_n, then the second solution becomes

$$y_2(x) = -\frac{1}{144}\left(x^2\left(1+\frac{x}{5}+\frac{x^2}{60}+\frac{x^3}{1260}+\frac{x^4}{40320}+\frac{x^5}{1814400}+O(x^6)\right)\right)\ln(x) + \frac{1-\frac{x}{3}+\frac{x^2}{12}-\frac{x^3}{36}+\frac{x^5}{600}+O(x^6)}{x^2}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^2\left(1+\frac{x}{5}+\frac{x^2}{60}+\frac{x^3}{1260}+\frac{x^4}{40320}+\frac{x^5}{1814400}+O(x^6)\right) \\ &\quad + c_2\left(-\frac{1}{144}\left(x^2\left(1+\frac{x}{5}+\frac{x^2}{60}+\frac{x^3}{1260}+\frac{x^4}{40320}+\frac{x^5}{1814400}+O(x^6)\right)\right)\ln(x) \right. \\ &\quad \left. + \frac{1-\frac{x}{3}+\frac{x^2}{12}-\frac{x^3}{36}+\frac{x^5}{600}+O(x^6)}{x^2}\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x^2\left(1+\frac{x}{5}+\frac{x^2}{60}+\frac{x^3}{1260}+\frac{x^4}{40320}+\frac{x^5}{1814400}+O(x^6)\right) \\ &\quad + c_2\left(-\frac{x^2\left(1+\frac{x}{5}+\frac{x^2}{60}+\frac{x^3}{1260}+\frac{x^4}{40320}+\frac{x^5}{1814400}+O(x^6)\right)\ln(x)}{144} \right. \\ &\quad \left. + \frac{1-\frac{x}{3}+\frac{x^2}{12}-\frac{x^3}{36}+\frac{x^5}{600}+O(x^6)}{x^2}\right) \end{aligned}$$

<u>Summary</u>

The solution(s) found are the following

$$\begin{aligned} y &= c_1 x^2\left(1+\frac{x}{5}+\frac{x^2}{60}+\frac{x^3}{1260}+\frac{x^4}{40320}+\frac{x^5}{1814400}+O(x^6)\right) \\ &\quad + c_2\left(-\frac{x^2\left(1+\frac{x}{5}+\frac{x^2}{60}+\frac{x^3}{1260}+\frac{x^4}{40320}+\frac{x^5}{1814400}+O(x^6)\right)\ln(x)}{144} \right. \\ &\quad \left. + \frac{1-\frac{x}{3}+\frac{x^2}{12}-\frac{x^3}{36}+\frac{x^5}{600}+O(x^6)}{x^2}\right) \end{aligned} \tag{1}$$

Verification of solutions

$$y = c_1 x^2 \left(1 + \frac{x}{5} + \frac{x^2}{60} + \frac{x^3}{1260} + \frac{x^4}{40320} + \frac{x^5}{1814400} + O(x^6)\right) + c_2 \left(-\frac{x^2 \left(1 + \frac{x}{5} + \frac{x^2}{60} + \frac{x^3}{1260} + \frac{x^4}{40320} + \frac{x^5}{1814400} + O(x^6)\right) \ln(x)}{144} + \frac{1 - \frac{x}{3} + \frac{x^2}{12} - \frac{x^3}{36} + \frac{x^5}{600} + O(x^6)}{x^2}\right)$$

Verified OK.

18.3.1 Maple step by step solution

Let's solve

$x^2 y'' + x y' + (-x - 4) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y'}{x} + \frac{(x+4) y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} - \frac{(x+4) y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = -\frac{x+4}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -4$

  ◦ $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $x^2 y'' + x y' + (-x-4) y = 0$
- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k- > k-m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r) x^{k+r}$
  - Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(2+r)(-2+r) x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+2)(k+r-2) - a_{k-1}\right) x^{k+r}\right) = 0$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(2+r)(-2+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{-2, 2\}$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+2)(k+r-2) - a_{k-1} = 0$
- Shift index using $k- > k+1$

  $a_{k+1}(k+3+r)(k+r-1) - a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k}{(k+3+r)(k+r-1)}$
- Recursion relation for $r = -2$

$$a_{k+1} = \frac{a_k}{(k+1)(k-3)}$$

- Series not valid for $r = -2$, division by 0 in the recursion relation at $k = 3$

$$a_{k+1} = \frac{a_k}{(k+1)(k-3)}$$

- Recursion relation for $r = 2$

$$a_{k+1} = \frac{a_k}{(k+5)(k+1)}$$

- Solution for $r = 2$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = \frac{a_k}{(k+5)(k+1)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 61

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)-(4+x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_1 x^4 \left(1 + \frac{1}{5}x + \frac{1}{60}x^2 + \frac{1}{1260}x^3 + \frac{1}{40320}x^4 + \frac{1}{1814400}x^5 + \mathrm{O}\left(x^6\right)\right) + c_2 \left(\ln\left(x\right)\left(x^4 + \frac{1}{5}x^5 + \mathrm{O}\left(x^6\right)\right) + \left(-144 - \right.\right.}{x^2}$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 77

```
AsymptoticDSolveValue[x^2*y''[x]+x*y'[x]-(4+x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(\frac{x^4 - 16x^3 + 48x^2 - 192x + 576}{576x^2} - \frac{1}{144} x^2 \log(x) \right) + c_2 \left(\frac{x^6}{40320} + \frac{x^5}{1260} + \frac{x^4}{60} + \frac{x^3}{5} + x^2 \right)$$

18.4 problem (a)

Internal problem ID [2940]
Internal file name [`OUTPUT/2432_Sunday_June_05_2022_03_09_46_AM_34906402/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.5. page 771
Problem number: (a).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - \left(-x^2 + x\right) y' + \left(x^3 + 1\right) y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + \left(x^2 - x\right) y' + \left(x^3 + 1\right) y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{x - 1}{x}$$
$$q(x) = \frac{x^3 + 1}{x^2}$$

Table 458: Table $p(x), q(x)$ singularites.

$p(x) = \frac{x-1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{x^3+1}{x^2}$	
singularity	type
$x = 0$	"regular"
$x = \infty$	"regular"
$x = -\infty$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty, -\infty]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + (x^2 - x)\, y' + (x^3 + 1)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$
$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + (x^2 - x)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + (x^3+1)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)\right) \\
&+ \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{n+r+3} a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} a_{n-1}(n+r-1)\, x^{n+r}
$$

$$
\sum_{n=0}^{\infty} x^{n+r+3} a_n = \sum_{n=3}^{\infty} a_{n-3} x^{n+r}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1)\, x^{n+r}\right) \\
&+ \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=3}^{\infty} a_{n-3} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0
\end{aligned} \tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
x^{n+r} a_n (n+r)(n+r-1) - x^{n+r} a_n (n+r) + a_n x^{n+r} = 0
$$

When $n=0$ the above becomes

$$
x^r a_0 r(-1+r) - x^r a_0 r + a_0 x^r = 0
$$

Or

$$
\left(x^r r(-1+r) - x^r r + x^r\right) a_0 = 0
$$

Since $a_0 \neq 0$ then the above simplifies to

$$
(-1+r)^2 x^r = 0
$$

Since the above is true for all x then the indicial equation becomes

$$(-1+r)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = 1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(-1+r)^2 x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = 1$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{1+n}$$
$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{1+n}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = -\frac{1}{r}$$

Substituting $n = 2$ in Eq. (2B) gives

$$a_2 = \frac{1}{(1+r)\, r}$$

For $3 \leq n$ the recursive equation is

$$a_n(n+r)\,(n+r-1) + a_{n-1}(n+r-1) - a_n(n+r) + a_{n-3} + a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{n a_{n-1} + r a_{n-1} + a_{n-3} - a_{n-1}}{n^2 + 2nr + r^2 - 2n - 2r + 1} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = \frac{-n a_{n-1} - a_{n-3}}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r}$	-1
a_2	$\frac{1}{(1+r)r}$	$\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-r^2 - 2r - 2}{(1+r)\, r\, (2+r)^2}$$

Which for the root $r = 1$ becomes

$$a_3 = -\frac{5}{18}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r}$	-1
a_2	$\frac{1}{(1+r)r}$	$\frac{1}{2}$
a_3	$\frac{-r^2-2r-2}{(1+r)r(2+r)^2}$	$-\frac{5}{18}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{2r^3 + 10r^2 + 16r + 10}{(1+r)\, r\, (2+r)^2\, (r+3)^2}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{19}{144}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r}$	-1
a_2	$\frac{1}{(1+r)r}$	$\frac{1}{2}$
a_3	$\frac{-r^2-2r-2}{(1+r)r(2+r)^2}$	$-\frac{5}{18}$
a_4	$\frac{2r^3+10r^2+16r+10}{(1+r)r(2+r)^2(r+3)^2}$	$\frac{19}{144}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-3r^4 - 28r^3 - 93r^2 - 134r - 76}{(1+r)\, r\, (2+r)^2\, (r+3)^2\, (r+4)^2}$$

Which for the root $r = 1$ becomes

$$a_5 = -\frac{167}{3600}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r}$	-1
a_2	$\frac{1}{(1+r)r}$	$\frac{1}{2}$
a_3	$\frac{-r^2-2r-2}{(1+r)r(2+r)^2}$	$-\frac{5}{18}$
a_4	$\frac{2r^3+10r^2+16r+10}{(1+r)r(2+r)^2(r+3)^2}$	$\frac{19}{144}$
a_5	$\frac{-3r^4-28r^3-93r^2-134r-76}{(1+r)r(2+r)^2(r+3)^2(r+4)^2}$	$-\frac{167}{3600}$

Using the above table, then the first solution $y_1(x)$ is

$$y_1(x) = x\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots\right)$$
$$= x\left(1 - x + \frac{x^2}{2} - \frac{5x^3}{18} + \frac{19x^4}{144} - \frac{167x^5}{3600} + O(x^6)\right)$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr}a_{n,r}$$

And the above is then evaluated at $r = 1$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr}a_{n,r}$	$b_n(r=1)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	$-\frac{1}{r}$	-1	$\frac{1}{r^2}$	1
b_2	$\frac{1}{(1+r)r}$	$\frac{1}{2}$	$\frac{-2r-1}{(1+r)^2r^2}$	$-\frac{3}{4}$
b_3	$\frac{-r^2-2r-2}{(1+r)r(2+r)^2}$	$-\frac{5}{18}$	$\frac{2r^4+7r^3+14r^2+14r+4}{(1+r)^2r^2(2+r)^3}$	$\frac{41}{108}$
b_4	$\frac{2r^3+10r^2+16r+10}{(1+r)r(2+r)^2(r+3)^2}$	$\frac{19}{144}$	$\frac{-6r^6-54r^5-198r^4-390r^3-446r^2-270r-60}{(1+r)^2r^2(2+r)^3(r+3)^3}$	$-\frac{89}{432}$
b_5	$\frac{-3r^4-28r^3-93r^2-134r-76}{(1+r)r(2+r)^2(r+3)^2(r+4)^2}$	$-\frac{167}{3600}$	$\frac{12r^8+203r^7+1453r^6+5761r^5+13901r^4+20986r^3+19276r^2+9576r+1824}{(1+r)^2r^2(2+r)^3(r+3)^3(r+4)^3}$	$\frac{2281}{27000}$

The above table gives all values of b_n needed. Hence the second solution is

$$y_2(x) = y_1(x)\ln(x) + b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 \dots$$
$$= x\left(1 - x + \frac{x^2}{2} - \frac{5x^3}{18} + \frac{19x^4}{144} - \frac{167x^5}{3600} + O(x^6)\right)\ln(x)$$
$$+ x\left(x - \frac{3x^2}{4} + \frac{41x^3}{108} - \frac{89x^4}{432} + \frac{2281x^5}{27000} + O(x^6)\right)$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x \left(1 - x + \frac{x^2}{2} - \frac{5x^3}{18} + \frac{19x^4}{144} - \frac{167x^5}{3600} + O(x^6)\right) \\
&+ c_2 \left(x \left(1 - x + \frac{x^2}{2} - \frac{5x^3}{18} + \frac{19x^4}{144} - \frac{167x^5}{3600} + O(x^6)\right) \ln(x) \right. \\
&\left. + x \left(x - \frac{3x^2}{4} + \frac{41x^3}{108} - \frac{89x^4}{432} + \frac{2281x^5}{27000} + O(x^6)\right)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x \left(1 - x + \frac{x^2}{2} - \frac{5x^3}{18} + \frac{19x^4}{144} - \frac{167x^5}{3600} + O(x^6)\right) \\
&+ c_2 \left(x \left(1 - x + \frac{x^2}{2} - \frac{5x^3}{18} + \frac{19x^4}{144} - \frac{167x^5}{3600} + O(x^6)\right) \ln(x) \right. \\
&\left. + x \left(x - \frac{3x^2}{4} + \frac{41x^3}{108} - \frac{89x^4}{432} + \frac{2281x^5}{27000} + O(x^6)\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x \left(1 - x + \frac{x^2}{2} - \frac{5x^3}{18} + \frac{19x^4}{144} - \frac{167x^5}{3600} + O(x^6)\right) \\
&+ c_2 \left(x \left(1 - x + \frac{x^2}{2} - \frac{5x^3}{18} + \frac{19x^4}{144} - \frac{167x^5}{3600} + O(x^6)\right) \ln(x) \right. \\
&\left. + x \left(x - \frac{3x^2}{4} + \frac{41x^3}{108} - \frac{89x^4}{432} + \frac{2281x^5}{27000} + O(x^6)\right)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1 x \left(1 - x + \frac{x^2}{2} - \frac{5x^3}{18} + \frac{19x^4}{144} - \frac{167x^5}{3600} + O(x^6)\right) \\
&+ c_2 \left(x \left(1 - x + \frac{x^2}{2} - \frac{5x^3}{18} + \frac{19x^4}{144} - \frac{167x^5}{3600} + O(x^6)\right) \ln(x) \right. \\
&\left. + x \left(x - \frac{3x^2}{4} + \frac{41x^3}{108} - \frac{89x^4}{432} + \frac{2281x^5}{27000} + O(x^6)\right)\right)
\end{aligned}
$$

Verified OK.

18.4.1 Maple step by step solution

Let's solve

$x^2 y'' + (x^2 - x) y' + (x^3 + 1) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x^3+1)y}{x^2} - \frac{(x-1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(x-1)y'}{x} + \frac{(x^3+1)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{x-1}{x}, P_3(x) = \frac{x^3+1}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = -1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 1$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2 y'' + x(x - 1) y' + (x^3 + 1) y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..3$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^{m}\cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
- Convert $x^{m}\cdot y'$ to series expansion for $m=1..2$

  $x^{m}\cdot y'=\sum_{k=0}^{\infty} a_{k}(k+r)\, x^{k+r-1+m}$
- Shift index using $k->k+1-m$

  $x^{m}\cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$
- Convert $x^{2}\cdot y''$ to series expansion

  $x^{2}\cdot y''=\sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_{0}(-1+r)^{2} x^{r}+\left(a_{1}r^{2}+a_{0}r\right) x^{1+r}+\left(a_{2}(1+r)^{2}+a_{1}(1+r)\right) x^{2+r}+\left(\sum_{k=3}^{\infty}\left(a_{k}(k+r-1)^{2}+a_{k-}\right.\right.$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $(-1+r)^{2}=0$
- Values of r that satisfy the indicial equation

  $r=1$
- The coefficients of each power of x must be 0

  $\left[a_{1}r^{2}+a_{0}r=0, a_{2}(1+r)^{2}+a_{1}(1+r)=0\right]$
- Solve for the dependent coefficient(s)

  $\left\{a_{1}=-\frac{a_{0}}{r}, a_{2}=\frac{a_{0}}{r(1+r)}\right\}$
- Each term in the series must be 0, giving the recursion relation

  $a_{k}(k+r-1)^{2}+a_{k-1}(k+r-1)+a_{k-3}=0$
- Shift index using $k->k+3$

  $a_{k+3}(k+2+r)^{2}+a_{k+2}(k+2+r)+a_{k}=0$
- Recursion relation that defines series solution to ODE

  $a_{k+3}=-\frac{k a_{k+2}+r a_{k+2}+a_{k}+2a_{k+2}}{(k+2+r)^{2}}$
- Recursion relation for $r=1$

  $a_{k+3}=-\frac{k a_{k+2}+a_{k}+3a_{k+2}}{(k+3)^{2}}$

- Solution for $r = 1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+3} = -\frac{k a_{k+2} + a_k + 3a_{k+2}}{(k+3)^2}, a_1 = -a_0, a_2 = \frac{a_0}{2}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying differential order: 2; exact nonlinear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      trying 2nd order exact linear
      trying symmetries linear in x and y(x)
      trying to convert to a linear ODE with constant coefficients
      trying to convert to an ODE of Bessel type
   trying to convert to an ODE of Bessel type
   -> trying reduction of order to Riccati
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 63

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)-(x-x^2)*diff(y(x),x)+(1+x^3)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(\left(x - \frac{3}{4}x^2 + \frac{41}{108}x^3 - \frac{89}{432}x^4 + \frac{2281}{27000}x^5 + \mathrm{O}\left(x^6\right)\right)c_2 + \left(1 - x + \frac{1}{2}x^2 - \frac{5}{18}x^3 + \frac{19}{144}x^4 - \frac{167}{3600}x^5 + \mathrm{O}\left(x^6\right)\right)\left(c_2 \ln\left(x\right) + c_1\right)\right)x$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 114

```
AsymptoticDSolveValue[x^2*y''[x]-(x-x^2)*y'[x]+(1+x^3)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 x\left(-\frac{167x^5}{3600} + \frac{19x^4}{144} - \frac{5x^3}{18} + \frac{x^2}{2} - x + 1\right) + c_2\left(x\left(\frac{2281x^5}{27000} - \frac{89x^4}{432} + \frac{41x^3}{108} - \frac{3x^2}{4} + x\right) + x\left(-\frac{167x^5}{3600} + \frac{19x^4}{144} - \frac{5x^3}{18} + \frac{x^2}{2} - x + 1\right)\log(x)\right)$$

18.5 problem (b)

Internal problem ID [2941]
Internal file name [`OUTPUT/2433_Sunday_June_05_2022_03_09_50_AM_7140631/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.5. page 771
Problem number: (b).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - \left(-1+2\sqrt{5}\right)xy' + \left(\frac{19}{4} - 3x^2\right)y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + x\left(1-2\sqrt{5}\right)y' + \left(\frac{19}{4} - 3x^2\right)y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{-1+2\sqrt{5}}{x}$$
$$q(x) = -\frac{12x^2-19}{4x^2}$$

Table 460: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{-1+2\sqrt{5}}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{12x^2-19}{4x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + x\left(1 - 2\sqrt{5}\right) y' + \left(\frac{19}{4} - 3x^2\right) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned} &x^2\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) \\ &+ x\left(1 - 2\sqrt{5}\right) \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \left(\frac{19}{4} - 3x^2\right) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \end{aligned} \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-\left(-1+2\sqrt{5}\right) x^n x^r a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} \frac{19 a_n x^{n+r}}{4}\right) + \sum_{n=0}^{\infty} \left(-3 x^{n+r+2} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-3 x^{n+r+2} a_n\right) = \sum_{n=2}^{\infty} \left(-3 a_{n-2} x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-\left(-1+2\sqrt{5}\right) x^n x^r a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} \frac{19 a_n x^{n+r}}{4}\right) + \sum_{n=2}^{\infty} \left(-3 a_{n-2} x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) + x^{n+r} a_n (n+r)\left(1-2\sqrt{5}\right) + \frac{19 a_n x^{n+r}}{4} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) + x^r a_0 r\left(1-2\sqrt{5}\right) + \frac{19 a_0 x^r}{4} = 0$$

Or

$$\left(x^r r(-1+r) + x^r r\left(1-2\sqrt{5}\right) + \frac{19 x^r}{4}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2 - 2r\sqrt{5} + \frac{19}{4}\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - 2r\sqrt{5} + \frac{19}{4} = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{2} + \sqrt{5}$$
$$r_2 = -\frac{1}{2} + \sqrt{5}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(r^2 - 2r\sqrt{5} + \frac{19}{4}\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x^{\frac{1}{2}+\sqrt{5}} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{-\frac{1}{2}+\sqrt{5}} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}+\sqrt{5}}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-\frac{1}{2}+\sqrt{5}}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1)+a_n\left(1-2\sqrt{5}\right)(n+r)+\frac{19a_n}{4}-3a_{n-2}=0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n=-\frac{12a_{n-2}}{-19+8\sqrt{5}\,n+8r\sqrt{5}-4n^2-8nr-4r^2} \tag{4}$$

Which for the root $r=\frac{1}{2}+\sqrt{5}$ becomes

$$a_n=\frac{3a_{n-2}}{n(n+1)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r=\frac{1}{2}+\sqrt{5}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n=2$, using the above recursive equation gives

$$a_2=\frac{12}{(-8r-16)\sqrt{5}+4r^2+16r+35}$$

Which for the root $r=\frac{1}{2}+\sqrt{5}$ becomes

$$a_2=\frac{1}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{12}{(-8r-16)\sqrt{5}+4r^2+16r+35}$	$\frac{1}{2}$

For $n=3$, using the above recursive equation gives

$$a_3=0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{12}{(-8r-16)\sqrt{5}+4r^2+16r+35}$	$\frac{1}{2}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{144}{(-64r^3 - 576r^2 - 1968r - 2448)\sqrt{5} + 16r^4 + 192r^3 + 1304r^2 + 4368r + 5465}$$

Which for the root $r = \frac{1}{2} + \sqrt{5}$ becomes

$$a_4 = \frac{3}{40}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{12}{(-8r-16)\sqrt{5}+4r^2+16r+35}$	$\frac{1}{2}$
a_3	0	0
a_4	$\frac{144}{(-64r^3-576r^2-1968r-2448)\sqrt{5}+16r^4+192r^3+1304r^2+4368r+5465}$	$\frac{3}{40}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{12}{(-8r-16)\sqrt{5}+4r^2+16r+35}$	$\frac{1}{2}$
a_3	0	0
a_4	$\frac{144}{(-64r^3-576r^2-1968r-2448)\sqrt{5}+16r^4+192r^3+1304r^2+4368r+5465}$	$\frac{3}{40}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x^{\frac{1}{2}+\sqrt{5}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= x^{\frac{1}{2}+\sqrt{5}}\left(1 + \frac{x^2}{2} + \frac{3x^4}{40} + O\left(x^6\right)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Let

$$
r_1 - r_2 = N
$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$
\begin{aligned}
a_N &= a_1 \\
&= 0
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\lim_{r\to r_2} 0 &= \lim_{r\to -\frac{1}{2}+\sqrt{5}} 0 \\
&= 0
\end{aligned}
$$

The limit is 0. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$
\begin{aligned}
y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\
&= \sum_{n=0}^{\infty} b_n x^{n-\frac{1}{2}+\sqrt{5}}
\end{aligned}
$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq(3) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$b_n(n+r)(n+r-1) - 2b_n\left(-\frac{1}{2}+\sqrt{5}\right)(n+r) + \frac{19b_n}{4} - 3b_{n-2} = 0 \tag{4}$$

Which for for the root $r = -\frac{1}{2}+\sqrt{5}$ becomes

$$b_n\left(n-\frac{1}{2}+\sqrt{5}\right)\left(n-\frac{3}{2}+\sqrt{5}\right) - 2b_n\left(-\frac{1}{2}+\sqrt{5}\right)\left(n-\frac{1}{2}+\sqrt{5}\right) + \frac{19b_n}{4} - 3b_{n-2} = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = -\frac{12b_{n-2}}{-19+8\sqrt{5}\,n+8r\sqrt{5}-4n^2-8nr-4r^2} \tag{5}$$

Which for the root $r = -\frac{1}{2}+\sqrt{5}$ becomes

$$b_n = -\frac{12b_{n-2}}{-19+8\sqrt{5}\,n+8\left(-\frac{1}{2}+\sqrt{5}\right)\sqrt{5}-4n^2-8n\left(-\frac{1}{2}+\sqrt{5}\right)-4\left(-\frac{1}{2}+\sqrt{5}\right)^2} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\frac{1}{2}+\sqrt{5}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = -\frac{12}{-35+16\sqrt{5}+8r\sqrt{5}-16r-4r^2}$$

Which for the root $r = -\frac{1}{2}+\sqrt{5}$ becomes

$$b_2 = -\frac{12}{-27+8\left(-\frac{1}{2}+\sqrt{5}\right)\sqrt{5}-4\left(-\frac{1}{2}+\sqrt{5}\right)^2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{12}{(-8r-16)\sqrt{5}+4r^2+16r+35}$	$-\frac{12}{-27+8\left(-\frac{1}{2}+\sqrt{5}\right)\sqrt{5}-4\left(-\frac{1}{2}+\sqrt{5}\right)^2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{12}{(-8r-16)\sqrt{5}+4r^2+16r+35}$	$-\frac{12}{-27+8\left(-\frac{1}{2}+\sqrt{5}\right)\sqrt{5}-4\left(-\frac{1}{2}+\sqrt{5}\right)^2}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = -\frac{144}{-1304r^2 + 64\sqrt{5}\,r^3 - 16r^4 - 4368r + 576\sqrt{5}\,r^2 - 192r^3 - 5465 + 1968r\sqrt{5} + 2448\sqrt{5}}$$

Which for the root $r = -\frac{1}{2} + \sqrt{5}$ becomes

$$b_4 = -\frac{144}{-1304\left(-\frac{1}{2}+\sqrt{5}\right)^2 + 64\sqrt{5}\left(-\frac{1}{2}+\sqrt{5}\right)^3 - 16\left(-\frac{1}{2}+\sqrt{5}\right)^4 - 3281 - 1920\sqrt{5} + 576\sqrt{5}\left(-\frac{1}{2}+\sqrt{5}\right.}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{12}{(-8r-16)\sqrt{5}+4r^2+16r+35}$	$-\frac{12}{-27+8\left(-\frac{1}{2}+\sqrt{5}\right)\sqrt{5}-4\left(-\frac{1}{2}+\sqrt{5}\right)^2}$
b_3	0	0
b_4	$\frac{144}{(-64r^3-576r^2-1968r-2448)\sqrt{5}+16r^4+192r^3+1304r^2+4368r+5465}$	$-\frac{}{-1304\left(-\frac{1}{2}+\sqrt{5}\right)^2+64\sqrt{5}\left(-\frac{1}{2}+\sqrt{5}\right)^3-16\left(-\frac{1}{2}+\sqrt{5}\right)^4-328}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{12}{(-8r-16)\sqrt{5}+4r^2+16r+35}$	$-\frac{12}{-27+8\left(-\frac{1}{2}+\sqrt{5}\right)\sqrt{5}-4\left(-\frac{1}{2}+\sqrt{5}\right)^2}$
b_3	0	0
b_4	$\frac{144}{(-64r^3-576r^2-1968r-2448)\sqrt{5}+16r^4+192r^3+1304r^2+4368r+5465}$	$-\frac{}{-1304\left(-\frac{1}{2}+\sqrt{5}\right)^2+64\sqrt{5}\left(-\frac{1}{2}+\sqrt{5}\right)^3-16\left(-\frac{1}{2}+\sqrt{5}\right)^4-328}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned}
y_2(x) &= x^{\frac{1}{2}+\sqrt{5}}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\
&= x^{-\frac{1}{2}+\sqrt{5}}\left(1 - \frac{12x^2}{-27 + 8\left(-\frac{1}{2}+\sqrt{5}\right)\sqrt{5} - 4\left(-\frac{1}{2}+\sqrt{5}\right)^2} - \frac{}{-1304\left(-\frac{1}{2}+\sqrt{5}\right)^2 + 64\sqrt{5}\,\left(-\frac{1}{2}+\sqrt{5}\right.}\right.
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^{\frac{1}{2}+\sqrt{5}}\left(1 + \frac{x^2}{2} + \frac{3x^4}{40} + O\left(x^6\right)\right) \\
&\quad + c_2 x^{-\frac{1}{2}+\sqrt{5}}\left(1 - \frac{12x^2}{-27 + 8\left(-\frac{1}{2}+\sqrt{5}\right)\sqrt{5} - 4\left(-\frac{1}{2}+\sqrt{5}\right)^2}\right. \\
&\quad - \frac{144x^4}{-1304\left(-\frac{1}{2}+\sqrt{5}\right)^2 + 64\sqrt{5}\,\left(-\frac{1}{2}+\sqrt{5}\right)^3 - 16\left(-\frac{1}{2}+\sqrt{5}\right)^4 - 3281 - 1920\sqrt{5} + 576\sqrt{5}\,\left(-\frac{1}{2}+\right.} \\
&\quad \left. + O\left(x^6\right)\right)
\end{aligned}$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^{\frac{1}{2}+\sqrt{5}} \left(1+\frac{x^2}{2}+\frac{3x^4}{40}+O\left(x^6\right)\right) \\
&\quad + c_2 x^{-\frac{1}{2}+\sqrt{5}} \left(1-\frac{12x^2}{-27+8\left(-\frac{1}{2}+\sqrt{5}\right)\sqrt{5}-4\left(-\frac{1}{2}+\sqrt{5}\right)^2} \right. \\
&\quad - \frac{144x^4}{-1304\left(-\frac{1}{2}+\sqrt{5}\right)^2+64\sqrt{5}\,\left(-\frac{1}{2}+\sqrt{5}\right)^3-16\left(-\frac{1}{2}+\sqrt{5}\right)^4-3281-1920\sqrt{5}+576\sqrt{5}\,\left(-\frac{1}{2}+\sqrt{5}\right)} \\
&\quad \left. +O\left(x^6\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x^{\frac{1}{2}+\sqrt{5}} \left(1+\frac{x^2}{2}+\frac{3x^4}{40}+O\left(x^6\right)\right) \\
&\quad + c_2 x^{-\frac{1}{2}+\sqrt{5}} \left(1-\frac{12x^2}{-27+8\left(-\frac{1}{2}+\sqrt{5}\right)\sqrt{5}-4\left(-\frac{1}{2}+\sqrt{5}\right)^2} \right. \\
&\quad - \frac{144x^4}{-1304\left(-\frac{1}{2}+\sqrt{5}\right)^2+64\sqrt{5}\,\left(-\frac{1}{2}+\sqrt{5}\right)^3-16\left(-\frac{1}{2}+\sqrt{5}\right)^4-3281-1920\sqrt{5}+576\sqrt{5}\,\left(-\frac{1}{2}+\sqrt{5}\right)^2} \\
&\quad \left. +O\left(x^6\right)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1 x^{\frac{1}{2}+\sqrt{5}} \left(1+\frac{x^2}{2}+\frac{3x^4}{40}+O\left(x^6\right)\right) \\
&\quad + c_2 x^{-\frac{1}{2}+\sqrt{5}} \left(1-\frac{12x^2}{-27+8\left(-\frac{1}{2}+\sqrt{5}\right)\sqrt{5}-4\left(-\frac{1}{2}+\sqrt{5}\right)^2} \right. \\
&\quad - \frac{144x^4}{-1304\left(-\frac{1}{2}+\sqrt{5}\right)^2+64\sqrt{5}\,\left(-\frac{1}{2}+\sqrt{5}\right)^3-16\left(-\frac{1}{2}+\sqrt{5}\right)^4-3281-1920\sqrt{5}+576\sqrt{5}\,\left(-\frac{1}{2}+\sqrt{5}\right)^2} \\
&\quad \left. +O\left(x^6\right)\right)
\end{aligned}
$$

Verified OK.

18.5.1 Maple step by step solution

Let's solve

$x^2y'' + x\left(1 - 2\sqrt{5}\right) y' + \left(\frac{19}{4} - 3x^2\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(12x^2-19)y}{4x^2} + \frac{\left(-1+2\sqrt{5}\right)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{\left(-1+2\sqrt{5}\right)y'}{x} - \frac{(12x^2-19)y}{4x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- ○ Define functions

  $\left[P_2(x) = -\frac{-1+2\sqrt{5}}{x}, P_3(x) = -\frac{12x^2-19}{4x^2}\right]$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 1 - 2\sqrt{5}$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = \frac{19}{4}$

- ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2y'' - 4\left(-1 + 2\sqrt{5}\right) xy' + \left(-12x^2 + 19\right) y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r) x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r}$

  Rewrite ODE with series expansions

  $\left(1+2\sqrt{5}-2r\right)\left(-1+2\sqrt{5}-2r\right) a_0 x^r + \left(-1+2\sqrt{5}-2r\right)\left(-3+2\sqrt{5}-2r\right) a_1 x^{1+r} + \left(\sum_{k=2}^{\infty} \left(\left(-\right.\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $\left(1+2\sqrt{5}-2r\right)\left(-1+2\sqrt{5}-2r\right) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}+\sqrt{5}, \frac{1}{2}+\sqrt{5}\right\}$

- Each term must be 0

  $\left(-1+2\sqrt{5}-2r\right)\left(-3+2\sqrt{5}-2r\right) a_1 = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $-8a_k(k+r)\sqrt{5} + (4k^2+8kr+4r^2+19) a_k - 12a_{k-2} = 0$

- Shift index using $k->k+2$

  $-8a_{k+2}(k+2+r)\sqrt{5} + \left(4(k+2)^2 + 8(k+2) r + 4r^2 + 19\right) a_{k+2} - 12a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{12a_k}{-35+8k\sqrt{5}+8\sqrt{5}\, r-4k^2-8kr-4r^2+16\sqrt{5}-16k-16r}$

- Recursion relation for $r = -\frac{1}{2}+\sqrt{5}$

  $a_{k+2} = -\frac{12a_k}{-27+8k\sqrt{5}+8\left(-\frac{1}{2}+\sqrt{5}\right)\sqrt{5}-4k^2-8k\left(-\frac{1}{2}+\sqrt{5}\right)-4\left(-\frac{1}{2}+\sqrt{5}\right)^2-16k}$

- Solution for $r = -\frac{1}{2}+\sqrt{5}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}+\sqrt{5}}, a_{k+2} = -\frac{12a_k}{-27+8k\sqrt{5}+8\left(-\frac{1}{2}+\sqrt{5}\right)\sqrt{5}-4k^2-8k\left(-\frac{1}{2}+\sqrt{5}\right)-4\left(-\frac{1}{2}+\sqrt{5}\right)^2-16k}, a_1 = 0\right]$$

- Recursion relation for $r = \frac{1}{2} + \sqrt{5}$

$$a_{k+2} = -\frac{12a_k}{-43+8k\sqrt{5}+8\left(\frac{1}{2}+\sqrt{5}\right)\sqrt{5}-4k^2-8k\left(\frac{1}{2}+\sqrt{5}\right)-4\left(\frac{1}{2}+\sqrt{5}\right)^2-16k}$$

- Solution for $r = \frac{1}{2} + \sqrt{5}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}+\sqrt{5}}, a_{k+2} = -\frac{12a_k}{-43+8k\sqrt{5}+8\left(\frac{1}{2}+\sqrt{5}\right)\sqrt{5}-4k^2-8k\left(\frac{1}{2}+\sqrt{5}\right)-4\left(\frac{1}{2}+\sqrt{5}\right)^2-16k}, a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}+\sqrt{5}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}+\sqrt{5}}\right), a_{2+k} = -\frac{12a_k}{-27+8k\sqrt{5}+8\left(-\frac{1}{2}+\sqrt{5}\right)\sqrt{5}-4k^2-8k\left(-\frac{1}{2}+\sqrt{5}\right)-4\left(-\frac{1}{2}\right.}\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.046 (sec). Leaf size: 325

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)-(2*sqrt(5)-1)*x*diff(y(x),x)+(19/4-3*x^2)*y(x)=0,y(x),type='series
```

$$y(x) = x^{-\frac{1}{2}+\sqrt{5}}\left(\left(1+\frac{3}{2}x^2+\frac{3}{8}x^4+\mathrm{O}\left(x^6\right)\right)c_1 + c_2 x\left(\left(1+\frac{1}{2}x^2+\frac{3}{40}x^4+\mathrm{O}\left(x^6\right)\right)\ln(x)+\left(-\frac{5}{12}x^2-\frac{77}{800}x^4+\mathrm{O}\left(x^6\right)\right)\right)\right)$$

✓ Solution by Mathematica

Time used: 0.054 (sec). Leaf size: 94

```
AsymptoticDSolveValue[x^2*y''[x]-(2*Sqrt[5]-1)*x*y'[x]+(19/4-3*x^2)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{3}{8}x^{\frac{7}{2}+\sqrt{5}} + \frac{3}{2}x^{\frac{3}{2}+\sqrt{5}} + x^{\sqrt{5}-\frac{1}{2}}\right) + c_2\left(\frac{3}{40}x^{\frac{9}{2}+\sqrt{5}} + \frac{1}{2}x^{\frac{5}{2}+\sqrt{5}} + x^{\frac{1}{2}+\sqrt{5}}\right)$$

18.6 problem (c)

Internal problem ID [2942]
Internal file name [OUTPUT/2434_Sunday_June_05_2022_03_09_54_AM_68650022/index.tex]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.5. page 771
Problem number: (c).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Complex roots"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + \left(-2x^5 + 9x\right) y' + \left(10x^4 + 5x^2 + 25\right) y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + \left(-2x^5 + 9x\right) y' + \left(10x^4 + 5x^2 + 25\right) y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{2x^4 - 9}{x}$$
$$q(x) = \frac{10x^4 + 5x^2 + 25}{x^2}$$

Table 462: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{2x^4-9}{x}$	
singularity	type
$x = 0$	"regular"
$x = \infty$	"regular"
$x = -\infty$	"regular"

$q(x) = \frac{10x^4+5x^2+25}{x^2}$	
singularity	type
$x = 0$	"regular"
$x = \infty$	"regular"
$x = -\infty$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty, -\infty]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + (-2x^5 + 9x)\, y' + (10x^4 + 5x^2 + 25)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + (-2x^5 + 9x) \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + (10x^4 + 5x^2 + 25) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-2x^{n+r+4} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 9x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 10x^{n+r+4} a_n\right) + \left(\sum_{n=0}^{\infty} 5x^{n+r+2} a_n\right) + \left(\sum_{n=0}^{\infty} 25a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-2x^{n+r+4} a_n (n+r)\right) = \sum_{n=4}^{\infty} \left(-2a_{n-4}(n-4+r)\, x^{n+r}\right)$$

$$\sum_{n=0}^{\infty} 10x^{n+r+4} a_n = \sum_{n=4}^{\infty} 10a_{n-4} x^{n+r}$$

$$\sum_{n=0}^{\infty} 5x^{n+r+2} a_n = \sum_{n=2}^{\infty} 5a_{n-2} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=4}^{\infty} \left(-2a_{n-4}(n-4+r)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 9x^{n+r} a_n (n+r)\right) + \left(\sum_{n=4}^{\infty} 10a_{n-4} x^{n+r}\right) + \left(\sum_{n=2}^{\infty} 5a_{n-2} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 25a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) + 9x^{n+r} a_n (n+r) + 25a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$x^r a_0 r(-1+r) + 9x^r a_0 r + 25a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) + 9x^r r + 25x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2 + 8r + 25\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 + 8r + 25 = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= -4 + 3i \\ r_2 &= -4 - 3i \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(r^2 + 8r + 25\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the roots are complex conjugates, then two linearly independent solutions can be constructed using

$$\begin{aligned} y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n-4+3i} \\ y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n-4-3i} \end{aligned}$$

$y_1(x)$ is found first. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

Substituting $n = 2$ in Eq. (2B) gives

$$a_2 = -\frac{5}{r^2 + 12r + 45}$$

Substituting $n = 3$ in Eq. (2B) gives

$$a_3 = 0$$

For $4 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) - 2a_{n-4}(n-4+r) + 9a_n(n+r) + 10a_{n-4} + 5a_{n-2} + 25a_n = 0 \quad (3)$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{2na_{n-4} + 2ra_{n-4} - 18a_{n-4} - 5a_{n-2}}{n^2 + 2nr + r^2 + 8n + 8r + 25} \tag{4}$$

Which for the root $r = -4 + 3i$ becomes

$$a_n = \frac{(-26 + 6i + 2n)\, a_{n-4} - 5a_{n-2}}{n\,(n + 6i)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = -4 + 3i$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{5}{r^2+12r+45}$	$-\frac{1}{8} + \frac{3i}{8}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{2r^3 + 14r^2 - 30r - 425}{(r^2 + 12r + 45)(r^2 + 16r + 73)}$$

Which for the root $r = -4 + 3i$ becomes

$$a_4 = -\frac{179}{832} + \frac{483i}{832}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{5}{r^2+12r+45}$	$-\frac{1}{8}+\frac{3i}{8}$
a_3	0	0
a_4	$\frac{2r^3+14r^2-30r-425}{(r^2+12r+45)(r^2+16r+73)}$	$-\frac{179}{832}+\frac{483i}{832}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{5}{r^2+12r+45}$	$-\frac{1}{8}+\frac{3i}{8}$
a_3	0	0
a_4	$\frac{2r^3+14r^2-30r-425}{(r^2+12r+45)(r^2+16r+73)}$	$-\frac{179}{832}+\frac{483i}{832}$
a_5	0	0

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{-20r^3 - 200r^2 - 100r + 4315}{(r^2 + 12r + 45)(r^2 + 16r + 73)(r^2 + 20r + 109)}$$

Which for the root $r = -4 + 3i$ becomes

$$a_6 = -\frac{433}{3744} - \frac{3943i}{29952}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{5}{r^2+12r+45}$	$-\frac{1}{8}+\frac{3i}{8}$
a_3	0	0
a_4	$\frac{2r^3+14r^2-30r-425}{(r^2+12r+45)(r^2+16r+73)}$	$-\frac{179}{832}+\frac{483i}{832}$
a_5	0	0
a_6	$\frac{-20r^3-200r^2-100r+4315}{(r^2+12r+45)(r^2+16r+73)(r^2+20r+109)}$	$-\frac{433}{3744}-\frac{3943i}{29952}$

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x^{-4+3i}\left(a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+a_6x^6+a_7x^7\dots\right)\\
&= x^{-4+3i}\left(1+\left(-\frac{1}{8}+\frac{3i}{8}\right)x^2+\left(-\frac{179}{832}+\frac{483i}{832}\right)x^4+\left(-\frac{433}{3744}-\frac{3943i}{29952}\right)x^6+O\left(x^7\right)\right)
\end{aligned}
$$

The second solution $y_2(x)$ is found by taking the complex conjugate of $y_1(x)$ which gives

$$
y_2(x) = x^{-4-3i}\left(1+\left(-\frac{1}{8}-\frac{3i}{8}\right)x^2+\left(-\frac{179}{832}-\frac{483i}{832}\right)x^4+\left(-\frac{433}{3744}+\frac{3943i}{29952}\right)x^6+O\left(x^7\right)\right)
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1y_1(x)+c_2y_2(x)\\
&= c_1x^{-4+3i}\left(1+\left(-\frac{1}{8}+\frac{3i}{8}\right)x^2+\left(-\frac{179}{832}+\frac{483i}{832}\right)x^4+\left(-\frac{433}{3744}-\frac{3943i}{29952}\right)x^6+O\left(x^7\right)\right)\\
&\quad + c_2x^{-4-3i}\left(1+\left(-\frac{1}{8}-\frac{3i}{8}\right)x^2+\left(-\frac{179}{832}-\frac{483i}{832}\right)x^4+\left(-\frac{433}{3744}+\frac{3943i}{29952}\right)x^6\right.\\
&\quad \left.+O\left(x^7\right)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h\\
&= c_1x^{-4+3i}\left(1+\left(-\frac{1}{8}+\frac{3i}{8}\right)x^2+\left(-\frac{179}{832}+\frac{483i}{832}\right)x^4+\left(-\frac{433}{3744}-\frac{3943i}{29952}\right)x^6+O\left(x^7\right)\right)\\
&\quad +c_2x^{-4-3i}\left(1+\left(-\frac{1}{8}-\frac{3i}{8}\right)x^2+\left(-\frac{179}{832}-\frac{483i}{832}\right)x^4+\left(-\frac{433}{3744}+\frac{3943i}{29952}\right)x^6+O\left(x^7\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = c_1 x^{-4+3i}\left(1+\left(-\frac{1}{8}+\frac{3i}{8}\right)x^2+\left(-\frac{179}{832}+\frac{483i}{832}\right)x^4+\left(-\frac{433}{3744}-\frac{3943i}{29952}\right)x^6 + O\left(x^7\right)\right) + c_2 x^{-4-3i}\left(1+\left(-\frac{1}{8}-\frac{3i}{8}\right)x^2+\left(-\frac{179}{832}-\frac{483i}{832}\right)x^4 + \left(-\frac{433}{3744}+\frac{3943i}{29952}\right)x^6+O\left(x^7\right)\right) \tag{1}$$

Verification of solutions

$$y = c_1 x^{-4+3i}\left(1+\left(-\frac{1}{8}+\frac{3i}{8}\right)x^2+\left(-\frac{179}{832}+\frac{483i}{832}\right)x^4+\left(-\frac{433}{3744}-\frac{3943i}{29952}\right)x^6+O\left(x^7\right)\right) + c_2 x^{-4-3i}\left(1+\left(-\frac{1}{8}-\frac{3i}{8}\right)x^2+\left(-\frac{179}{832}-\frac{483i}{832}\right)x^4+\left(-\frac{433}{3744}+\frac{3943i}{29952}\right)x^6 + O\left(x^7\right)\right)$$

Verified OK.

18.6.1 Maple step by step solution

Let's solve

$x^2y''+\left(-2x^5+9x\right)y'+\left(10x^4+5x^2+25\right)y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{5(2x^4+x^2+5)y}{x^2}+\frac{(2x^4-9)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{(2x^4-9)y'}{x}+\frac{5(2x^4+x^2+5)y}{x^2}=0$

□ Check to see if $x_0=0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=-\frac{2x^4-9}{x}, P_3(x)=\frac{5(2x^4+x^2+5)}{x^2}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left(x\cdot P_2(x)\right)\Big|_{x=0}=9$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 25$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2 y'' - x(2x^4 - 9)\, y' + (10x^4 + 5x^2 + 25)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..4$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..5$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1+m}$

- Shift index using $k- >k + 1 - m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k + 1 - m + r)\, x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(r^2 + 8r + 25)\, x^r + a_1(r^2 + 10r + 34)\, x^{1+r} + \left((r^2 + 12r + 45)\, a_2 + 5a_0\right) x^{2+r} + \left(a_3(r^2 + 14r +\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r^2 + 8r + 25 = 0$

- Values of r that satisfy the indicial equation

$r \in \{-4-3\,\mathrm{I}, -4+3\,\mathrm{I}\}$

- The coefficients of each power of x must be 0

  $[a_1(r^2+10r+34)=0, (r^2+12r+45)\,a_2+5a_0=0, a_3(r^2+14r+58)+5a_1=0]$

- Solve for the dependent coefficient(s)

  $\left\{a_1=0, a_2=-\frac{5a_0}{r^2+12r+45}, a_3=0\right\}$

- Each term in the series must be 0, giving the recursion relation

  $(k^2+(2r+8)\,k+r^2+8r+25)\,a_k+5a_{k-2}-2a_{k-4}(k-9+r)=0$

- Shift index using $k->k+4$

  $\left((k+4)^2+(2r+8)\,(k+4)+r^2+8r+25\right)a_{k+4}+5a_{k+2}-2a_k(k+r-5)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+4}=\frac{2ka_k+2ra_k-10a_k-5a_{k+2}}{k^2+2kr+r^2+16k+16r+73}$

- Recursion relation for $r=-4-3\,\mathrm{I}$

  $a_{k+4}=\frac{2ka_k-(8+6\,\mathrm{I})a_k-10a_k-5a_{k+2}}{k^2-(8+6\,\mathrm{I})k+16-24\,\mathrm{I}+16k}$

- Solution for $r=-4-3\,\mathrm{I}$

  $\left[y=\sum_{k=0}^{\infty}a_kx^{k-4-3\,\mathrm{I}}, a_{k+4}=\frac{2ka_k-(8+6\,\mathrm{I})a_k-10a_k-5a_{k+2}}{k^2-(8+6\,\mathrm{I})k+16-24\,\mathrm{I}+16k}, a_1=0, a_2=\left(-\frac{1}{8}-\frac{3\,\mathrm{I}}{8}\right)a_0, a_3=0\right]$

- Recursion relation for $r=-4+3\,\mathrm{I}$

  $a_{k+4}=\frac{2ka_k+(-8+6\,\mathrm{I})a_k-10a_k-5a_{k+2}}{k^2+(-8+6\,\mathrm{I})k+16+24\,\mathrm{I}+16k}$

- Solution for $r=-4+3\,\mathrm{I}$

  $\left[y=\sum_{k=0}^{\infty}a_kx^{k-4+3\,\mathrm{I}}, a_{k+4}=\frac{2ka_k+(-8+6\,\mathrm{I})a_k-10a_k-5a_{k+2}}{k^2+(-8+6\,\mathrm{I})k+16+24\,\mathrm{I}+16k}, a_1=0, a_2=\left(-\frac{1}{8}+\frac{3\,\mathrm{I}}{8}\right)a_0, a_3=0\right]$

- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty}a_kx^{k-4-3\,\mathrm{I}}\right)+\left(\sum_{k=0}^{\infty}b_kx^{k-4+3\,\mathrm{I}}\right), a_{k+4}=\frac{2ka_k-(8+6\,\mathrm{I})a_k-10a_k-5a_{2+k}}{k^2-(8+6\,\mathrm{I})k+16-24\,\mathrm{I}+16k}, a_1=0, a_2=\left(-\frac{1}{8}-\right.\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
<- Heun successful: received ODE is equivalent to the  HeunB  ODE, case  c = 0 `
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 55

```
Order:=7;
dsolve(x^2*diff(y(x),x$2)+(9*x-2*x^5)*diff(y(x),x)+(25+5*x^2+10*x^4)*y(x)=0,y(x),type='serie
```

$$y(x) = c_1 x^{-4-3i}\left(1+\left(-\frac{1}{8}-\frac{3i}{8}\right)x^2+\left(-\frac{179}{832}-\frac{483i}{832}\right)x^4+\left(-\frac{433}{3744}+\frac{3943i}{29952}\right)x^6+\mathrm{O}\left(x^7\right)\right)+c_2 x^{-4+3i}\left(1+\left(-\frac{1}{8}+\frac{3i}{8}\right)x^2+\left(-\frac{179}{832}+\frac{483i}{832}\right)x^4+\left(-\frac{433}{3744}-\frac{3943i}{29952}\right)x^6+\mathrm{O}\left(x^7\right)\right)$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 70

```
AsymptoticDSolveValue[x^2*y''[x]+(9*x-2*x^5)*y'[x]+(25+5*x^2+10*x^4)*y[x]==0,y[x],{x,0,6}]
```

$$
\begin{aligned}
y(x) \to & \left(\frac{1}{832}+\frac{5i}{832}\right)c_1x^{-4+3i}\left((86+53i)x^4+(56+32i)x^2+(32-160i)\right) \\
& -\left(\frac{5}{832}+\frac{i}{832}\right)c_2x^{-4-3i}\left((53+86i)x^4+(32+56i)x^2-(160-32i)\right)
\end{aligned}
$$

18.7 problem (d)

Internal problem ID [2943]
Internal file name [`OUTPUT/2435_Sunday_June_05_2022_03_10_44_AM_75711485/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.5. page 771
Problem number: (d).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + \left(4x + \frac{1}{2}x^2 - \frac{1}{3}x^3\right)y' - \frac{7y}{4} = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + \left(4x + \frac{1}{2}x^2 - \frac{1}{3}x^3\right)y' - \frac{7y}{4} = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{2x^2 - 3x - 24}{6x}$$
$$q(x) = -\frac{7}{4x^2}$$

Table 464: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{2x^2-3x-24}{6x}$	
singularity	type
$x = 0$	"regular"
$x = \infty$	"regular"
$x = -\infty$	"regular"

$q(x) = -\frac{7}{4x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty, -\infty]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + \left(4x + \frac{1}{2}x^2 - \frac{1}{3}x^3\right) y' - \frac{7y}{4} = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) + \left(4x + \frac{1}{2}x^2 - \frac{1}{3}x^3\right) \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) - \frac{7\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right)}{4} = 0 \tag{1}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) \\
&+\sum_{n=0}^{\infty}\left(-\frac{x^{n+r+2} a_n (n+r)}{3}\right)+\left(\sum_{n=0}^{\infty} \frac{x^{1+n+r} a_n (n+r)}{2}\right) \\
&+\left(\sum_{n=0}^{\infty} 4 x^{n+r} a_n (n+r)\right)+\sum_{n=0}^{\infty}\left(-\frac{7 a_n x^{n+r}}{4}\right)=0
\end{aligned}
\tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty}\left(-\frac{x^{n+r+2} a_n (n+r)}{3}\right) &= \sum_{n=2}^{\infty}\left(-\frac{a_{n-2}(n+r-2)\, x^{n+r}}{3}\right) \\
\sum_{n=0}^{\infty} \frac{x^{1+n+r} a_n (n+r)}{2} &= \sum_{n=1}^{\infty} \frac{a_{n-1}(n+r-1)\, x^{n+r}}{2}
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right)+\sum_{n=2}^{\infty}\left(-\frac{a_{n-2}(n+r-2)\, x^{n+r}}{3}\right) \\
&+\left(\sum_{n=1}^{\infty} \frac{a_{n-1}(n+r-1)\, x^{n+r}}{2}\right) \\
&+\left(\sum_{n=0}^{\infty} 4 x^{n+r} a_n (n+r)\right)+\sum_{n=0}^{\infty}\left(-\frac{7 a_n x^{n+r}}{4}\right)=0
\end{aligned}
\tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
x^{n+r} a_n (n+r)(n+r-1)+4 x^{n+r} a_n (n+r)-\frac{7 a_n x^{n+r}}{4}=0
$$

When $n=0$ the above becomes

$$
x^r a_0 r(-1+r)+4 x^r a_0 r-\frac{7 a_0 x^r}{4}=0
$$

Or

$$\left(x^r r(-1+r)+4x^r r-\frac{7x^r}{4}\right)a_0=0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\frac{(4r^2+12r-7)\,x^r}{4}=0$$

Since the above is true for all x then the indicial equation becomes

$$r^2+3r-\frac{7}{4}=0$$

Solving for r gives the roots of the indicial equation as

$$r_1=\frac{1}{2}$$
$$r_2=-\frac{7}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\frac{(4r^2+12r-7)\,x^r}{4}=0$$

Solving for r gives the roots of the indicial equation as Since $r_1-r_2=4$ is an integer, then we can construct two linearly independent solutions

$$y_1(x)=x^{r_1}\left(\sum_{n=0}^{\infty}a_n x^n\right)$$
$$y_2(x)=Cy_1(x)\ln(x)+x^{r_2}\left(\sum_{n=0}^{\infty}b_n x^n\right)$$

Or

$$y_1(x)=\sqrt{x}\left(\sum_{n=0}^{\infty}a_n x^n\right)$$
$$y_2(x)=Cy_1(x)\ln(x)+\frac{\sum_{n=0}^{\infty}b_n x^n}{x^{\frac{7}{2}}}$$

Or

$$y_1(x)=\sum_{n=0}^{\infty}a_n x^{n+\frac{1}{2}}$$
$$y_2(x)=Cy_1(x)\ln(x)+\left(\sum_{n=0}^{\infty}b_n x^{n-\frac{7}{2}}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = -\frac{2r}{4r^2 + 20r + 9}$$

For $2 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) - \frac{a_{n-2}(n+r-2)}{3} + \frac{a_{n-1}(n+r-1)}{2} + 4a_n(n+r) - \frac{7a_n}{4} = 0 \quad (3)$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{\frac{4}{3}na_{n-2} - 2na_{n-1} + \frac{4}{3}ra_{n-2} - 2ra_{n-1} - \frac{8}{3}a_{n-2} + 2a_{n-1}}{4n^2 + 8nr + 4r^2 + 12n + 12r - 7} \quad (4)$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = \frac{(4a_{n-2} - 6a_{n-1})\, n - 6a_{n-2} + 3a_{n-1}}{12n\,(n+4)} \quad (5)$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2r}{4r^2+20r+9}$	$-\frac{1}{20}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{16r^3 + 92r^2 + 48r}{48r^4 + 576r^3 + 2184r^2 + 2736r + 891}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = \frac{49}{2880}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2r}{4r^2+20r+9}$	$-\frac{1}{20}$
a_2	$\frac{16r^3+92r^2+48r}{48r^4+576r^3+2184r^2+2736r+891}$	$\frac{49}{2880}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{8r(8r^3 + 63r^2 + 119r + 57)}{3\left(16r^4 + 192r^3 + 728r^2 + 912r + 297\right)\left(4r^2 + 36r + 65\right)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_3 = -\frac{533}{241920}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2r}{4r^2+20r+9}$	$-\frac{1}{20}$
a_2	$\frac{16r^3+92r^2+48r}{48r^4+576r^3+2184r^2+2736r+891}$	$\frac{49}{2880}$
a_3	$-\frac{8r(8r^3+63r^2+119r+57)}{3(16r^4+192r^3+728r^2+912r+297)(4r^2+36r+65)}$	$-\frac{533}{241920}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{16r(16r^5 + 292r^4 + 1869r^3 + 5123r^2 + 5876r + 2073)}{9\left(16r^4 + 192r^3 + 728r^2 + 912r + 297\right)\left(4r^2 + 36r + 65\right)\left(4r^2 + 44r + 105\right)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = \frac{277}{491520}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2r}{4r^2+20r+9}$	$-\frac{1}{20}$
a_2	$\frac{16r^3+92r^2+48r}{48r^4+576r^3+2184r^2+2736r+891}$	$\frac{49}{2880}$
a_3	$-\frac{8r(8r^3+63r^2+119r+57)}{3(16r^4+192r^3+728r^2+912r+297)(4r^2+36r+65)}$	$-\frac{533}{241920}$
a_4	$\frac{16r(16r^5+292r^4+1869r^3+5123r^2+5876r+2073)}{9(16r^4+192r^3+728r^2+912r+297)(4r^2+36r+65)(4r^2+44r+105)}$	$\frac{277}{491520}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{32r(48r^6 + 1056r^5 + 8937r^4 + 36942r^3 + 77608r^2 + 76571r + 26247)}{9\left(16r^4 + 192r^3 + 728r^2 + 912r + 297\right)\left(4r^2 + 36r + 65\right)\left(4r^2 + 44r + 105\right)\left(4r^2 + 52r + 153\right)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_5 = -\frac{203759}{2388787200}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2r}{4r^2+20r+9}$	$-\frac{1}{20}$
a_2	$\frac{16r^3+92r^2+48r}{48r^4+576r^3+2184r^2+2736r+891}$	$\frac{49}{2880}$
a_3	$-\frac{8r(8r^3+63r^2+119r+57)}{3(16r^4+192r^3+728r^2+912r+297)(4r^2+36r+65)}$	$-\frac{533}{241920}$
a_4	$\frac{16r(16r^5+292r^4+1869r^3+5123r^2+5876r+2073)}{9(16r^4+192r^3+728r^2+912r+297)(4r^2+36r+65)(4r^2+44r+105)}$	$\frac{277}{491520}$
a_5	$-\frac{32r(48r^6+1056r^5+8937r^4+36942r^3+77608r^2+76571r+26247)}{9(16r^4+192r^3+728r^2+912r+297)(4r^2+36r+65)(4r^2+44r+105)(4r^2+52r+153)}$	$-\frac{203759}{2388787200}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= \sqrt{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= \sqrt{x}\left(1 - \frac{x}{20} + \frac{49x^2}{2880} - \frac{533x^3}{241920} + \frac{277x^4}{491520} - \frac{203759x^5}{2388787200} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 4$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_4(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_4 \\ &= \frac{16r(16r^5 + 292r^4 + 1869r^3 + 5123r^2 + 5876r + 2073)}{9\left(16r^4 + 192r^3 + 728r^2 + 912r + 297\right)\left(4r^2 + 36r + 65\right)\left(4r^2 + 44r + 105\right)} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r \to r_2} \frac{16r(16r^5 + 292r^4 + 1869r^3 + 5123r^2 + 5876r + 2073)}{9\left(16r^4 + 192r^3 + 728r^2 + 912r + 297\right)\left(4r^2 + 36r + 65\right)\left(4r^2 + 44r + 105\right)} &= \lim_{r \to -\frac{7}{2}} \frac{16r(16r}{9\left(16r^4 + 192r^3 + \right.} \\ &= \textit{undefined} \end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$\frac{d}{dx}y_2(x) = Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right)$$

$$= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right)$$

$$\frac{d^2}{dx^2}y_2(x) = Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)$$

$$= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right)$$

Substituting these back into the given ode $x^2y'' + \left(4x + \frac{1}{2}x^2 - \frac{1}{3}x^3\right)y' - \frac{7y}{4} = 0$ gives

$$x^2\left(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) + \left(4x + \frac{1}{2}x^2 - \frac{1}{3}x^3\right)\left(Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right) - \frac{7Cy_1(x)\ln(x)}{4} - \frac{7\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)}{4} = 0$$

Which can be written as

$$\begin{aligned}&\Bigg(\left(x^2 y_1''(x) + \left(4x + \frac{1}{2}x^2 - \frac{1}{3}x^3\right) y_1'(x) - \frac{7 y_1(x)}{4}\right) \ln(x)\\&+ x^2\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right) + \frac{\left(4x + \frac{1}{2}x^2 - \frac{1}{3}x^3\right) y_1(x)}{x}\Bigg) C\\&+ x^2\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)\Bigg)\\&+ \left(4x + \frac{1}{2}x^2 - \frac{1}{3}x^3\right)\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right) - \frac{7\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)}{4} = 0\end{aligned} \tag{7}$$

But since $y_1(x)$ is a solution to the ode, then

$$x^2 y_1''(x) + \left(4x + \frac{1}{2}x^2 - \frac{1}{3}x^3\right) y_1'(x) - \frac{7 y_1(x)}{4} = 0$$

Eq (7) simplifes to

$$\begin{aligned}&\left(x^2\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right) + \frac{\left(4x + \frac{1}{2}x^2 - \frac{1}{3}x^3\right) y_1(x)}{x}\right) C\\&+ x^2\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)\Bigg)\\&+ \left(4x + \frac{1}{2}x^2 - \frac{1}{3}x^3\right)\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right) - \frac{7\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)}{4} = 0\end{aligned} \tag{8}$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\begin{aligned}&\left(2\left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n(n+r_1)\right) x - \frac{\left(x^2 - \frac{3}{2}x - 9\right)\left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right)}{3}\right) C\\&+ \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right) x^2\\&+ \frac{(-2x^3 + 3x^2 + 24x)\left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right)}{6} - \frac{7\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)}{4} = 0\end{aligned} \tag{9}$$

Since $r_1 = \frac{1}{2}$ and $r_2 = -\frac{7}{2}$ then the above becomes

$$
\begin{aligned}
&\left(2\left(\sum_{n=0}^{\infty} x^{-\frac{1}{2}+n} a_n \left(n+\frac{1}{2}\right)\right) x - \frac{\left(x^2-\frac{3}{2}x-9\right)\left(\sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}\right)}{3}\right) C \\
&+ \left(\sum_{n=0}^{\infty} x^{-\frac{11}{2}+n} b_n \left(n-\frac{7}{2}\right)\left(-\frac{9}{2}+n\right)\right) x^2 \\
&+ \frac{\left(-2x^3+3x^2+24x\right)\left(\sum_{n=0}^{\infty} x^{-\frac{9}{2}+n} b_n \left(n-\frac{7}{2}\right)\right)}{6} - \frac{7\left(\sum_{n=0}^{\infty} b_n x^{n-\frac{7}{2}}\right)}{4} = 0
\end{aligned}
\tag{10}
$$

Expanding $-\frac{C x^{\frac{5}{2}}}{3}$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$
\begin{aligned}
-\frac{C x^{\frac{5}{2}}}{3} &= -\frac{C x^{\frac{5}{2}}}{3} + \dots \\
&= -\frac{C x^{\frac{5}{2}}}{3}
\end{aligned}
$$

Expanding $\frac{C x^{\frac{3}{2}}}{2}$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$
\begin{aligned}
\frac{C x^{\frac{3}{2}}}{2} &= \frac{C x^{\frac{3}{2}}}{2} + \dots \\
&= \frac{C x^{\frac{3}{2}}}{2}
\end{aligned}
$$

Expanding $3C\sqrt{x}$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$
\begin{aligned}
3C\sqrt{x} &= 3C\sqrt{x} + \dots \\
&= 3C\sqrt{x}
\end{aligned}
$$

Expanding $-\frac{1}{6x^{\frac{3}{2}}}$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$
\begin{aligned}
-\frac{1}{6x^{\frac{3}{2}}} &= -\frac{1}{6x^{\frac{3}{2}}} + \dots \\
&= -\frac{1}{6x^{\frac{3}{2}}}
\end{aligned}
$$

Expanding $\frac{1}{4x^{\frac{5}{2}}}$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$
\begin{aligned}
\frac{1}{4x^{\frac{5}{2}}} &= \frac{1}{4x^{\frac{5}{2}}} + \dots \\
&= \frac{1}{4x^{\frac{5}{2}}}
\end{aligned}
$$

Expanding $\frac{2}{x^{\frac{7}{2}}}$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned}\frac{2}{x^{\frac{7}{2}}} &= \frac{2}{x^{\frac{7}{2}}} + \dots \\ &= \frac{2}{x^{\frac{7}{2}}}\end{aligned}$$

Which simplifies to

$$\begin{aligned}&\left(\sum_{n=0}^{\infty} C\, x^{n+\frac{1}{2}} a_n (2n+1)\right) + \sum_{n=0}^{\infty} \left(-\frac{C\, x^{n+\frac{5}{2}} a_n}{3}\right) + \left(\sum_{n=0}^{\infty} \frac{C\, x^{n+\frac{3}{2}} a_n}{2}\right) \\ &+ \left(\sum_{n=0}^{\infty} 3C\, x^{n+\frac{1}{2}} a_n\right) + \left(\sum_{n=0}^{\infty} \frac{x^{n-\frac{7}{2}} b_n (4n^2 - 32n + 63)}{4}\right) \\ &+ \sum_{n=0}^{\infty} \left(-\frac{x^{n-\frac{3}{2}} b_n (2n-7)}{6}\right) + \left(\sum_{n=0}^{\infty} \frac{x^{n-\frac{5}{2}} b_n (2n-7)}{4}\right) \\ &+ \left(\sum_{n=0}^{\infty} (4n-14)\, b_n x^{n-\frac{7}{2}}\right) + \sum_{n=0}^{\infty} \left(-\frac{7 b_n x^{n-\frac{7}{2}}}{4}\right) = 0\end{aligned} \tag{2A}$$

The next step is to make all powers of x be $n - \frac{7}{2}$ in each summation term. Going over each summation term above with power of x in it which is not already $x^{n-\frac{7}{2}}$ and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} C\, x^{n+\frac{1}{2}} a_n (2n+1) = \sum_{n=4}^{\infty} C a_{n-4} (2n-7)\, x^{n-\frac{7}{2}}$$

$$\sum_{n=0}^{\infty} \left(-\frac{C\, x^{n+\frac{5}{2}} a_n}{3}\right) = \sum_{n=6}^{\infty} \left(-\frac{C a_{n-6} x^{n-\frac{7}{2}}}{3}\right)$$

$$\sum_{n=0}^{\infty} \frac{C\, x^{n+\frac{3}{2}} a_n}{2} = \sum_{n=5}^{\infty} \frac{C a_{n-5} x^{n-\frac{7}{2}}}{2}$$

$$\sum_{n=0}^{\infty} 3C\, x^{n+\frac{1}{2}} a_n = \sum_{n=4}^{\infty} 3C a_{n-4} x^{n-\frac{7}{2}}$$

$$\sum_{n=0}^{\infty} \left(-\frac{x^{n-\frac{3}{2}} b_n (2n-7)}{6}\right) = \sum_{n=2}^{\infty} \left(-\frac{b_{n-2} (-11+2n)\, x^{n-\frac{7}{2}}}{6}\right)$$

$$\sum_{n=0}^{\infty} \frac{x^{n-\frac{5}{2}} b_n (2n-7)}{4} = \sum_{n=1}^{\infty} \frac{b_{n-1} (-9+2n)\, x^{n-\frac{7}{2}}}{4}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-\frac{7}{2}$.

$$\begin{aligned}
&\left(\sum_{n=4}^{\infty} C a_{n-4}(2n-7)\, x^{n-\frac{7}{2}}\right)+\sum_{n=6}^{\infty}\left(-\frac{C a_{n-6} x^{n-\frac{7}{2}}}{3}\right)+\left(\sum_{n=5}^{\infty} \frac{C a_{n-5} x^{n-\frac{7}{2}}}{2}\right) \\
&+\left(\sum_{n=4}^{\infty} 3 C a_{n-4} x^{n-\frac{7}{2}}\right)+\left(\sum_{n=0}^{\infty} \frac{x^{n-\frac{7}{2}} b_n\left(4n^2-32n+63\right)}{4}\right) \\
&+\sum_{n=2}^{\infty}\left(-\frac{b_{n-2}(-11+2n)\, x^{n-\frac{7}{2}}}{6}\right)+\left(\sum_{n=1}^{\infty} \frac{b_{n-1}(-9+2n)\, x^{n-\frac{7}{2}}}{4}\right) \\
&+\left(\sum_{n=0}^{\infty}(4n-14)\, b_n x^{n-\frac{7}{2}}\right)+\sum_{n=0}^{\infty}\left(-\frac{7 b_n x^{n-\frac{7}{2}}}{4}\right)=0
\end{aligned} \tag{2B}$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0=1$. For $n=1$, Eq (2B) gives

$$-3b_1-\frac{7b_0}{4}=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-3b_1-\frac{7}{4}=0$$

Solving the above for b_1 gives

$$b_1=-\frac{7}{12}$$

For $n=2$, Eq (2B) gives

$$-4b_2+\frac{7b_0}{6}-\frac{5b_1}{4}=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-4b_2+\frac{91}{48}=0$$

Solving the above for b_2 gives

$$b_2=\frac{91}{192}$$

For $n=3$, Eq (2B) gives

$$-3b_3+\frac{5b_1}{6}-\frac{3b_2}{4}=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-3b_3-\frac{1939}{2304}=0$$

Solving the above for b_3 gives

$$b_3 = -\frac{1939}{6912}$$

For $n = N$, where $N = 4$ which is the difference between the two roots, we are free to choose $b_4 = 0$. Hence for $n = 4$, Eq (2B) gives

$$4C + \frac{8491}{27648} = 0$$

Which is solved for C. Solving for C gives

$$C = -\frac{8491}{110592}$$

For $n = 5$, Eq (2B) gives

$$\frac{(a_0 + 12a_1)\,C}{2} + \frac{b_3}{6} + \frac{b_4}{4} + 5b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-\frac{103033}{1658880} + 5b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = \frac{103033}{8294400}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -\frac{8491}{110592}$ and all b_n, then the second solution becomes

$$y_2(x) = -\frac{8491}{110592}\left(\sqrt{x}\left(1 - \frac{x}{20} + \frac{49x^2}{2880} - \frac{533x^3}{241920} + \frac{277x^4}{491520} - \frac{203759x^5}{2388787200} + O\left(x^6\right)\right)\right)\ln(x) + \frac{1 - \frac{7x}{12} + \frac{91x^2}{192} - \frac{1939x^3}{6912} + \frac{103033x^5}{8294400} + O\left(x^6\right)}{x^{\frac{7}{2}}}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1\sqrt{x}\left(1 - \frac{x}{20} + \frac{49x^2}{2880} - \frac{533x^3}{241920} + \frac{277x^4}{491520} - \frac{203759x^5}{2388787200} + O\left(x^6\right)\right) \\ &\quad + c_2\left(-\frac{8491}{110592}\left(\sqrt{x}\left(1 - \frac{x}{20} + \frac{49x^2}{2880} - \frac{533x^3}{241920} + \frac{277x^4}{491520} - \frac{203759x^5}{2388787200} + O\left(x^6\right)\right)\right)\ln(x) \right. \\ &\quad \left. + \frac{1 - \frac{7x}{12} + \frac{91x^2}{192} - \frac{1939x^3}{6912} + \frac{103033x^5}{8294400} + O\left(x^6\right)}{x^{\frac{7}{2}}}\right) \end{aligned}$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1\sqrt{x}\left(1-\frac{x}{20}+\frac{49x^2}{2880}-\frac{533x^3}{241920}+\frac{277x^4}{491520}-\frac{203759x^5}{2388787200}+O(x^6)\right) \\
&\quad + c_2\left(-\frac{8491\sqrt{x}\left(1-\frac{x}{20}+\frac{49x^2}{2880}-\frac{533x^3}{241920}+\frac{277x^4}{491520}-\frac{203759x^5}{2388787200}+O(x^6)\right)\ln(x)}{110592}\right. \\
&\quad \left. + \frac{1-\frac{7x}{12}+\frac{91x^2}{192}-\frac{1939x^3}{6912}+\frac{103033x^5}{8294400}+O(x^6)}{x^{\frac{7}{2}}}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1\sqrt{x}\left(1-\frac{x}{20}+\frac{49x^2}{2880}-\frac{533x^3}{241920}+\frac{277x^4}{491520}-\frac{203759x^5}{2388787200}+O(x^6)\right) \\
&\quad + c_2\left(-\frac{8491\sqrt{x}\left(1-\frac{x}{20}+\frac{49x^2}{2880}-\frac{533x^3}{241920}+\frac{277x^4}{491520}-\frac{203759x^5}{2388787200}+O(x^6)\right)\ln(x)}{110592}\right. \\
&\quad \left. + \frac{1-\frac{7x}{12}+\frac{91x^2}{192}-\frac{1939x^3}{6912}+\frac{103033x^5}{8294400}+O(x^6)}{x^{\frac{7}{2}}}\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1\sqrt{x}\left(1-\frac{x}{20}+\frac{49x^2}{2880}-\frac{533x^3}{241920}+\frac{277x^4}{491520}-\frac{203759x^5}{2388787200}+O(x^6)\right) \\
&\quad + c_2\left(-\frac{8491\sqrt{x}\left(1-\frac{x}{20}+\frac{49x^2}{2880}-\frac{533x^3}{241920}+\frac{277x^4}{491520}-\frac{203759x^5}{2388787200}+O(x^6)\right)\ln(x)}{110592}\right. \\
&\quad \left. + \frac{1-\frac{7x}{12}+\frac{91x^2}{192}-\frac{1939x^3}{6912}+\frac{103033x^5}{8294400}+O(x^6)}{x^{\frac{7}{2}}}\right)
\end{aligned}
$$

Verified OK.

18.7.1 Maple step by step solution

Let's solve

$x^2 y'' + \left(4x + \frac{1}{2}x^2 - \frac{1}{3}x^3\right) y' - \frac{7y}{4} = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{7y}{4x^2} + \frac{(2x^2-3x-24)y'}{6x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(2x^2-3x-24)y'}{6x} - \frac{7y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{2x^2-3x-24}{6x}, P_3(x) = -\frac{7}{4x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 4$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -\frac{7}{4}$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $12x^2 y'' - 2x(2x^2 - 3x - 24) y' - 21y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r) x^{k+r-1+m}$

- Shift index using $k- > k+1-m$

  $$x^{m}\cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^{2}\cdot y''$ to series expansion

  $$x^{2}\cdot y'' = \sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$3a_{0}(7+2r)\,(-1+2r)\, x^{r} + \left(3a_{1}(9+2r)\,(1+2r)+6a_{0}r\right) x^{1+r} + \left(\sum_{k=2}^{\infty} (3a_{k}(2k+2r+7)\,(2k+2r$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$3(7+2r)\,(-1+2r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{-\frac{7}{2}, \frac{1}{2}\right\}$$

- Each term must be 0

  $$3a_{1}(9+2r)\,(1+2r)+6a_{0}r = 0$$

- Solve for the dependent coefficient(s)

  $$a_{1} = -\frac{2a_{0}r}{4r^{2}+20r+9}$$

- Each term in the series must be 0, giving the recursion relation

  $$3a_{k}(2k+2r+7)\,(2k+2r-1)+6a_{k-1}(k+r-1)-4a_{k-2}(k-2+r) = 0$$

- Shift index using $k- > k+2$

  $$3a_{k+2}(2k+11+2r)\,(2k+3+2r)+6a_{k+1}(k+1+r)-4a_{k}(k+r) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = \frac{2(2ka_{k}-3ka_{k+1}+2ra_{k}-3ra_{k+1}-3a_{k+1})}{3(2k+11+2r)(2k+3+2r)}$$

- Recursion relation for $r = -\frac{7}{2}$

  $$a_{k+2} = \frac{2\left(2ka_{k}-3ka_{k+1}-7a_{k}+\frac{15}{2}a_{k+1}\right)}{3(2k+4)(2k-4)}$$

- Series not valid for $r = -\frac{7}{2}$, division by 0 in the recursion relation at $k = 2$

  $$a_{k+2} = \frac{2\left(2ka_{k}-3ka_{k+1}-7a_{k}+\frac{15}{2}a_{k+1}\right)}{3(2k+4)(2k-4)}$$

- Recursion relation for $r = \frac{1}{2}$

  $$a_{k+2} = \frac{2\left(2ka_{k}-3ka_{k+1}+a_{k}-\frac{9}{2}a_{k+1}\right)}{3(2k+12)(2k+4)}$$

- Solution for $r = \frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2}=\frac{2\left(2k a_k-3k a_{k+1}+a_k-\frac{9}{2}a_{k+1}\right)}{3(2k+12)(2k+4)}, a_1=-\frac{a_0}{20}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
<- Heun successful: received ODE is equivalent to the  HeunB  ODE, case  c = 0 `
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 63

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+(4*x+1/2*x^2-1/3*x^3)*diff(y(x),x)-7/4*y(x)=0,y(x),type='series',x
```

$$y(x) = \frac{c_1 x^4\left(1-\frac{1}{20}x+\frac{49}{2880}x^2-\frac{533}{241920}x^3+\frac{277}{491520}x^4-\frac{203759}{2388787200}x^5+\mathrm{O}\left(x^6\right)\right)+c_2\left(\ln\left(x\right)\left(\frac{8491}{768}x^4-\frac{8491}{15360}x^5+\mathrm{O}\left(\right.\right.\right.}{x^{\frac{7}{2}}}$$

✓ Solution by Mathematica

Time used: 0.038 (sec). Leaf size: 93

```
AsymptoticDSolveValue[x^2*y''[x]+(4*x+1/2*x^2-1/3*x^3)*y'[x]-7/4*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2 \left(\frac{277x^{9/2}}{491520} - \frac{533x^{7/2}}{241920} + \frac{49x^{5/2}}{2880} - \frac{x^{3/2}}{20} + \sqrt{x} \right) + c_1 \left(\frac{65067x^4 - 124096x^3 + 209664x^2 - 258048x + 442368}{442368x^{7/2}} - \frac{8491\sqrt{x}\log(x)}{110592} \right)$$

18.8 problem (e)

Internal problem ID [2944]
Internal file name [OUTPUT/2436_Sunday_June_05_2022_03_10_52_AM_13049648/index.tex]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.5. page 771
Problem number: (e).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries], [_2nd_order, _linear, `
   _with_symmetry_[0,F(x)]`]]
```

$$\boxed{x^2y'' + y'x^2 + yx = 0}$$

With the expansion point for the power series method at $x = 0$.

The ODE is

$$x^2y'' + y'x^2 + yx = 0$$

Or

$$x(y''x + xy' + y) = 0$$

For $x \neq 0$ the above simplifies to

$$y''x + xy' + y = 0$$

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + y'x^2 + yx = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = 1$$
$$q(x) = \frac{1}{x}$$

Table 466: Table $p(x), q(x)$ singularites.

$p(x) = 1$	
singularity	type

$q(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + y'x^2 + yx = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$
$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2\left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) x^2 + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) x = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r) (n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n\right) = 0 \quad \text{(2A)}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} a_{n-1} (n+r-1) x^{n+r}$$

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n = \sum_{n=1}^{\infty} a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r) (n+r-1)\right) + \left(\sum_{n=1}^{\infty} a_{n-1} (n+r-1) x^{n+r}\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r}\right) = 0 \quad \text{(2B)}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r) (n+r-1) = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) = 0$$

Or

$$x^r a_0 r(-1+r) = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^r r(-1+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(-1+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$
\begin{aligned}
r_1 &= 1 \\
r_2 &= 0
\end{aligned}
$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$
x^r r(-1+r) = 0
$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$
\begin{aligned}
y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right) \\
y_2(x) &= C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= x \left(\sum_{n=0}^{\infty} a_n x^n\right) \\
y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^{1+n} \\
y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}
$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$
a_n(n+r)(n+r-1) + a_{n-1}(n+r-1) + a_{n-1} = 0 \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$
a_n = -\frac{a_{n-1}}{n+r-1} \tag{4}
$$

Which for the root $r = 1$ becomes

$$a_n = -\frac{a_{n-1}}{n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{1}{r}$$

Which for the root $r = 1$ becomes

$$a_1 = -1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r}$	-1

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{r(1+r)}$$

Which for the root $r = 1$ becomes

$$a_2 = \frac{1}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r}$	-1
a_2	$\frac{1}{r(1+r)}$	$\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{1}{r(1+r)(2+r)}$$

Which for the root $r = 1$ becomes

$$a_3 = -\frac{1}{6}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r}$	-1
a_2	$\frac{1}{r(1+r)}$	$\frac{1}{2}$
a_3	$-\frac{1}{r(1+r)(2+r)}$	$-\frac{1}{6}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{r(1+r)(2+r)(3+r)}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{1}{24}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r}$	-1
a_2	$\frac{1}{r(1+r)}$	$\frac{1}{2}$
a_3	$-\frac{1}{r(1+r)(2+r)}$	$-\frac{1}{6}$
a_4	$\frac{1}{r(1+r)(2+r)(3+r)}$	$\frac{1}{24}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{1}{r(1+r)(2+r)(3+r)(4+r)}$$

Which for the root $r = 1$ becomes

$$a_5 = -\frac{1}{120}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r}$	-1
a_2	$\frac{1}{r(1+r)}$	$\frac{1}{2}$
a_3	$-\frac{1}{r(1+r)(2+r)}$	$-\frac{1}{6}$
a_4	$\frac{1}{r(1+r)(2+r)(3+r)}$	$\frac{1}{24}$
a_5	$-\frac{1}{r(1+r)(2+r)(3+r)(4+r)}$	$-\frac{1}{120}$

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= x\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + O\left(x^6\right)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Let

$$
r_1 - r_2 = N
$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$
\begin{aligned}
a_N &= a_1 \\
&= -\frac{1}{r}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\lim_{r \to r_2} -\frac{1}{r} &= \lim_{r \to 0} -\frac{1}{r} \\
&= \textit{undefined}
\end{aligned}
$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$
y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)
$$

Therefore

$$
\begin{aligned}
\frac{d}{dx}y_2(x) &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\
&= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right) \\
\frac{d^2}{dx^2}y_2(x) &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\
&= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right)
\end{aligned}
$$

Substituting these back into the given ode $x^2y'' + y'x^2 + yx = 0$ gives

$$
\begin{aligned}
&x^2\Bigg(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\Bigg) \\
&\quad + \left(Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right)x^2 \\
&\quad + \left(Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)\right)x = 0
\end{aligned}
$$

Which can be written as

$$
\begin{aligned}
&\left(\left(y_1'(x)\,x^2 + x^2y_1''(x) + y_1(x)\,x\right)\ln(x) + x^2\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right) + y_1(x)\,x\right)C \\
&\quad + x^2\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) \\
&\quad + \left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)x^2 + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)x = 0
\end{aligned}
\tag{7}
$$

But since $y_1(x)$ is a solution to the ode, then

$$y_1'(x)\,x^2 + x^2 y_1''(x) + y_1(x)\,x = 0$$

Eq (7) simplifes to

$$\begin{aligned}&\left(x^2\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right) + y_1(x)\,x\right) C\\ &+ x^2\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)\\ &+ \left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)x^2 + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)x = 0\end{aligned} \tag{8}$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\begin{aligned}&\left(2\left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n(n+r_1)\right)x + (x-1)\left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right)\right) C\\ &+ \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)\,(-1+n+r_2)\right)x^2\\ &+ \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right)x^2 + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)x = 0\end{aligned} \tag{9}$$

Since $r_1 = 1$ and $r_2 = 0$ then the above becomes

$$\begin{aligned}&\left(2\left(\sum_{n=0}^{\infty} x^n a_n(1+n)\right)x + (x-1)\left(\sum_{n=0}^{\infty} a_n x^{1+n}\right)\right) C\\ &+ \left(\sum_{n=0}^{\infty} x^{-2+n} b_n n(n-1)\right)x^2 + \left(\sum_{n=0}^{\infty} x^{n-1} b_n n\right)x^2 + \left(\sum_{n=0}^{\infty} b_n x^n\right)x = 0\end{aligned} \tag{10}$$

Which simplifies to

$$\begin{aligned}&\left(\sum_{n=0}^{\infty} 2C\,x^{1+n} a_n(1+n)\right) + \left(\sum_{n=0}^{\infty} C\,x^{n+2} a_n\right) + \sum_{n=0}^{\infty}\left(-C\,x^{1+n} a_n\right)\\ &+ \left(\sum_{n=0}^{\infty} n\,x^n b_n(n-1)\right) + \left(\sum_{n=0}^{\infty} n\,x^{1+n} b_n\right) + \left(\sum_{n=0}^{\infty} x^{1+n} b_n\right) = 0\end{aligned} \tag{2A}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} 2C\, x^{1+n} a_n (1+n) &= \sum_{n=1}^{\infty} 2C a_{n-1} n\, x^n \\
\sum_{n=0}^{\infty} C\, x^{n+2} a_n &= \sum_{n=2}^{\infty} C a_{-2+n} x^n \\
\sum_{n=0}^{\infty} \left(-C\, x^{1+n} a_n\right) &= \sum_{n=1}^{\infty} \left(-C a_{n-1} x^n\right) \\
\sum_{n=0}^{\infty} n\, x^{1+n} b_n &= \sum_{n=1}^{\infty} (n-1)\, b_{n-1} x^n \\
\sum_{n=0}^{\infty} x^{1+n} b_n &= \sum_{n=1}^{\infty} b_{n-1} x^n
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to n.

$$
\begin{aligned}
&\left(\sum_{n=1}^{\infty} 2C a_{n-1} n\, x^n\right) + \left(\sum_{n=2}^{\infty} C a_{-2+n} x^n\right) + \sum_{n=1}^{\infty} \left(-C a_{n-1} x^n\right) \\
&\quad + \left(\sum_{n=0}^{\infty} n\, x^n b_n (n-1)\right) + \left(\sum_{n=1}^{\infty} (n-1)\, b_{n-1} x^n\right) + \left(\sum_{n=1}^{\infty} b_{n-1} x^n\right) = 0
\end{aligned} \tag{2B}
$$

For $n = 0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0 = 1$. For $n = N$, where $N = 1$ which is the difference between the two roots, we are free to choose $b_1 = 0$. Hence for $n = 1$, Eq (2B) gives

$$
C + 1 = 0
$$

Which is solved for C. Solving for C gives

$$
C = -1
$$

For $n = 2$, Eq (2B) gives

$$
(a_0 + 3a_1)\, C + 2b_1 + 2b_2 = 0
$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$2 + 2b_2 = 0$$

Solving the above for b_2 gives

$$b_2 = -1$$

For $n = 3$, Eq (2B) gives

$$(a_1 + 5a_2)\, C + 3b_2 + 6b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-\frac{9}{2} + 6b_3 = 0$$

Solving the above for b_3 gives

$$b_3 = \frac{3}{4}$$

For $n = 4$, Eq (2B) gives

$$(a_2 + 7a_3)\, C + 4b_3 + 12b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{11}{3} + 12b_4 = 0$$

Solving the above for b_4 gives

$$b_4 = -\frac{11}{36}$$

For $n = 5$, Eq (2B) gives

$$(a_3 + 9a_4)\, C + 5b_4 + 20b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-\frac{125}{72} + 20b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = \frac{25}{288}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -1$ and all b_n, then the second solution becomes

$$
\begin{aligned}
y_2(x) = (-1) \left(x\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + O(x^6)\right)\right) \ln(x) \\
+ 1 - x^2 + \frac{3x^3}{4} - \frac{11x^4}{36} + \frac{25x^5}{288} + O(x^6)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + O(x^6)\right) \\
&\quad + c_2\left((-1) \left(x\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + O(x^6)\right)\right) \ln(x) + 1 - x^2 + \frac{3x^3}{4} \right. \\
&\quad \left. - \frac{11x^4}{36} + \frac{25x^5}{288} + O(x^6)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + O(x^6)\right) \\
&\quad + c_2\left(-x\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + O(x^6)\right) \ln(x) + 1 - x^2 + \frac{3x^3}{4} - \frac{11x^4}{36} \right. \\
&\quad \left. + \frac{25x^5}{288} + O(x^6)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + O(x^6)\right) \\
&\quad + c_2\left(-x\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + O(x^6)\right) \ln(x) + 1 - x^2 + \frac{3x^3}{4} \right. \\
&\quad \left. - \frac{11x^4}{36} + \frac{25x^5}{288} + O(x^6)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = {} & c_1 x \left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + O(x^6)\right) \\
& + c_2 \left(-x \left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + O(x^6)\right) \ln(x) + 1 - x^2 + \frac{3x^3}{4} - \frac{11x^4}{36} \right. \\
& \left. + \frac{25x^5}{288} + O(x^6)\right)
\end{aligned}
$$

Verified OK.

18.8.1 Maple step by step solution

Let's solve

$x^2 y'' + y' x^2 + yx = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -y' - \frac{y}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + y' + \frac{y}{x} = 0$

- □ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = 1, P_3(x) = \frac{1}{x}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 0$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

$$y''x + xy' + y = 0$$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- ◦ Convert $x \cdot y'$ to series expansion

$$x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$$

- ◦ Convert $x \cdot y''$ to series expansion

$$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$$

- ◦ Shift index using $k- >k+1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-1+r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(k+r) + a_k(k+1+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(-1+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 1\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k+1+r)\,(a_{k+1}(k+r) + a_k) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{a_k}{k+r}$$

- Recursion relation for $r = 0$

$$a_{k+1} = -\frac{a_k}{k}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{a_k}{k}\right]$$

- Recursion relation for $r = 1$

$$a_{k+1} = -\frac{a_k}{k+1}$$

- Solution for $r = 1$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = -\frac{a_k}{k+1}\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{1+k}\right), a_{1+k} = -\frac{a_k}{k}, b_{1+k} = -\frac{b_k}{1+k}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 58

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+x^2*diff(y(x),x)+x*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x \left(1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{1}{120}x^5 + \mathrm{O}\left(x^6\right)\right) + c_2 \left(\ln\left(x\right)\left(-x + x^2 - \frac{1}{2}x^3 + \frac{1}{6}x^4 - \frac{1}{24}x^5 + \mathrm{O}\left(x^6\right)\right) + \left(1 - x + \frac{1}{4}x^3 - \frac{5}{36}x^4 + \frac{13}{288}x^5 + \mathrm{O}\left(x^6\right)\right)\right)$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 80

```
AsymptoticDSolveValue[x^2*y''[x]+x^2*y'[x]+x*y[x]==0,y[x],{x,0,5}]
```

$$
y(x) \to c_1 \left(\frac{1}{6} x \left(x^3 - 3x^2 + 6x - 6 \right) \log(x) + \frac{1}{36} \left(-11x^4 + 27x^3 - 36x^2 + 36 \right) \right) + c_2 \left(\frac{x^5}{24} - \frac{x^4}{6} + \frac{x^3}{2} - x^2 + x \right)
$$

18.9 problem 1

Internal problem ID [2945]
Internal file name [`OUTPUT/2437_Sunday_June_05_2022_03_10_58_AM_51214053/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.5. page 771
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + x(x-3)\,y' + (-x+4)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + \left(x^2 - 3x\right)y' + (-x+4)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{x-3}{x}$$
$$q(x) = -\frac{x-4}{x^2}$$

Table 468: Table $p(x), q(x)$ singularites.

$p(x) = \frac{x-3}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{x-4}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + (x^2 - 3x)\, y' + (-x + 4)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned} x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) \\ + (x^2 - 3x) \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + (-x+4) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \end{aligned} \tag{1}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)\right) \\
&+ \sum_{n=0}^{\infty} \left(-3x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n\right) + \left(\sum_{n=0}^{\infty} 4a_n x^{n+r}\right) = 0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r) &= \sum_{n=1}^{\infty} a_{n-1}(n+r-1)\, x^{n+r} \\
\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n\right) &= \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r}\right)
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1)\, x^{n+r}\right) \\
&+ \sum_{n=0}^{\infty} \left(-3x^{n+r} a_n (n+r)\right) + \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 4a_n x^{n+r}\right) = 0
\end{aligned} \tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
x^{n+r} a_n (n+r)(n+r-1) - 3x^{n+r} a_n (n+r) + 4a_n x^{n+r} = 0
$$

When $n=0$ the above becomes

$$
x^r a_0 r(-1+r) - 3x^r a_0 r + 4a_0 x^r = 0
$$

Or

$$
\left(x^r r(-1+r) - 3x^r r + 4x^r\right) a_0 = 0
$$

Since $a_0 \neq 0$ then the above simplifies to

$$
(r-2)^2 x^r = 0
$$

Since the above is true for all x then the indicial equation becomes

$$(r-2)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= 2 \\ r_2 &= 2 \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(r-2)^2 x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = 2$, Eqs (1A,1B) become

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+2} \\ y_2(x) &= y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+2}\right) \end{aligned}$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_{n-1}(n+r-1) - 3a_n(n+r) - a_{n-1} + 4a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}}{n+r-2} \tag{4}$$

Which for the root $r = 2$ becomes

$$a_n = -\frac{a_{n-1}}{n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 2$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{1}{-1+r}$$

Which for the root $r = 2$ becomes

$$a_1 = -1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{-1+r}$	-1

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{r(-1+r)}$$

Which for the root $r = 2$ becomes

$$a_2 = \frac{1}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{-1+r}$	-1
a_2	$\frac{1}{r(-1+r)}$	$\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{1}{r^3 - r}$$

Which for the root $r = 2$ becomes

$$a_3 = -\frac{1}{6}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{-1+r}$	-1
a_2	$\frac{1}{r(-1+r)}$	$\frac{1}{2}$
a_3	$-\frac{1}{r^3-r}$	$-\frac{1}{6}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(2+r)\,r\,(r^2-1)}$$

Which for the root $r = 2$ becomes

$$a_4 = \frac{1}{24}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{-1+r}$	-1
a_2	$\frac{1}{r(-1+r)}$	$\frac{1}{2}$
a_3	$-\frac{1}{r^3-r}$	$-\frac{1}{6}$
a_4	$\frac{1}{(2+r)r(r^2-1)}$	$\frac{1}{24}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{1}{r^5 + 5r^4 + 5r^3 - 5r^2 - 6r}$$

Which for the root $r = 2$ becomes

$$a_5 = -\frac{1}{120}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{-1+r}$	-1
a_2	$\frac{1}{r(-1+r)}$	$\frac{1}{2}$
a_3	$-\frac{1}{r^3-r}$	$-\frac{1}{6}$
a_4	$\frac{1}{(2+r)r(r^2-1)}$	$\frac{1}{24}$
a_5	$-\frac{1}{r^5+5r^4+5r^3-5r^2-6r}$	$-\frac{1}{120}$

Using the above table, then the first solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x^2\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots\right) \\
&= x^2\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + O(x^6)\right)
\end{aligned}
$$

Now the second solution is found. The second solution is given by

$$
y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)
$$

Where b_n is found using

$$
b_n = \frac{d}{dr} a_{n,r}
$$

And the above is then evaluated at $r = 2$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr}a_{n,r}$	$b_n(r=2)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	$-\frac{1}{-1+r}$	-1	$\frac{1}{(-1+r)^2}$	1
b_2	$\frac{1}{r(-1+r)}$	$\frac{1}{2}$	$\frac{1-2r}{r^2(-1+r)^2}$	$-\frac{3}{4}$
b_3	$-\frac{1}{r^3-r}$	$-\frac{1}{6}$	$\frac{3r^2-1}{(r^3-r)^2}$	$\frac{11}{36}$
b_4	$\frac{1}{(2+r)r(r^2-1)}$	$\frac{1}{24}$	$\frac{-4r^3-6r^2+2r+2}{(2+r)^2r^2(r^2-1)^2}$	$-\frac{25}{288}$
b_5	$-\frac{1}{r^5+5r^4+5r^3-5r^2-6r}$	$-\frac{1}{120}$	$\frac{5r^4+20r^3+15r^2-10r-6}{r^2(r^4+5r^3+5r^2-5r-6)^2}$	$\frac{137}{7200}$

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\
&= x^2\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + O(x^6)\right)\ln(x) \\
&\quad + x^2\left(x - \frac{3x^2}{4} + \frac{11x^3}{36} - \frac{25x^4}{288} + \frac{137x^5}{7200} + O(x^6)\right)
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^2\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + O(x^6)\right) \\
&\quad + c_2\left(x^2\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + O(x^6)\right)\ln(x)\right. \\
&\qquad \left. + x^2\left(x - \frac{3x^2}{4} + \frac{11x^3}{36} - \frac{25x^4}{288} + \frac{137x^5}{7200} + O(x^6)\right)\right)
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= c_1 x^2\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + O(x^6)\right) \\
&\quad + c_2\left(x^2\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + O(x^6)\right)\ln(x)\right. \\
&\qquad \left. + x^2\left(x - \frac{3x^2}{4} + \frac{11x^3}{36} - \frac{25x^4}{288} + \frac{137x^5}{7200} + O(x^6)\right)\right)
\end{aligned}$$

Summary
The solution(s) found are the following

$$\begin{aligned}
y &= c_1 x^2\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + O(x^6)\right) \\
&\quad + c_2\left(x^2\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + O(x^6)\right)\ln(x)\right. \\
&\qquad \left. + x^2\left(x - \frac{3x^2}{4} + \frac{11x^3}{36} - \frac{25x^4}{288} + \frac{137x^5}{7200} + O(x^6)\right)\right)
\end{aligned} \tag{1}$$

Verification of solutions

$$
\begin{aligned}
y = c_1 x^2 \left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + O(x^6)\right) \\
+ c_2 \left(x^2 \left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + O(x^6)\right) \ln (x) \right. \\
\left. + x^2 \left(x - \frac{3x^2}{4} + \frac{11x^3}{36} - \frac{25x^4}{288} + \frac{137x^5}{7200} + O(x^6)\right)\right)
\end{aligned}
$$

Verified OK.

18.9.1 Maple step by step solution

Let's solve

$x^2 y'' + (x^2 - 3x) y' + (-x + 4) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(x-4)y}{x^2} - \frac{(x-3)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(x-3)y'}{x} - \frac{(x-4)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{x-3}{x}, P_3(x) = -\frac{x-4}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -3$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 4$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

$$x^2y'' + x(x-3)\,y' + (-x+4)\,y = 0$$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- ∘ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- ∘ Shift index using $k- > k-m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- ∘ Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$$

- ∘ Shift index using $k- > k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$$

- ∘ Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-2+r)^2\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r-2)^2 + a_{k-1}(k+r-2)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-2+r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = 2$$

- Each term in the series must be 0, giving the recursion relation

$$(k+r-2)\,(a_k(k+r-2) + a_{k-1}) = 0$$

- Shift index using $k- > k+1$

$$(k+r-1)\,(a_{k+1}(k+r-1) + a_k) = 0$$

- Recursion relation that defines series solution to ODE

$a_{k+1} = -\frac{a_k}{k+r-1}$

- Recursion relation for $r = 2$

$a_{k+1} = -\frac{a_k}{k+1}$

- Solution for $r = 2$

$\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = -\frac{a_k}{k+1}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 69

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+x*(x-3)*diff(y(x),x)+(4-x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(\left(x - \frac{3}{4}x^2 + \frac{11}{36}x^3 - \frac{25}{288}x^4 + \frac{137}{7200}x^5 + \mathrm{O}\left(x^6\right)\right)c_2 + \left(1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{1}{120}x^5 + \mathrm{O}\left(x^6\right)\right)\left(c_2 \ln\left(x\right) + c_1\right)\right)x^2$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 120

```
AsymptoticDSolveValue[x^2*y''[x]+x*(x-3)*y'[x]+(4-x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(-\frac{x^5}{120}+\frac{x^4}{24}-\frac{x^3}{6}+\frac{x^2}{2}-x+1\right)x^2+c_2\left(\left(\frac{137x^5}{7200}-\frac{25x^4}{288}+\frac{11x^3}{36}-\frac{3x^2}{4}+x\right)x^2 + \left(-\frac{x^5}{120}+\frac{x^4}{24}-\frac{x^3}{6}+\frac{x^2}{2}-x+1\right)x^2\log(x)\right)$$

18.10 problem 2

Internal problem ID [2946]
Internal file name [`OUTPUT/2438_Sunday_June_05_2022_03_11_01_AM_67401820/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.5. page 771
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2y'' + 2y'x^2 + y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$4x^2y'' + 2y'x^2 + y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{2}$$
$$q(x) = \frac{1}{4x^2}$$

Table 470: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{2}$	
singularity	type

$q(x) = \frac{1}{4x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$4x^2y'' + 2y'x^2 + y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$4x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + 2\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) x^2 + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2x^{1+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2x^{1+n+r} a_n(n+r) = \sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)\, x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 4x^{n+r} a_n(n+r)\,(n+r-1)\right) + \left(\sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$4x^{n+r} a_n(n+r)\,(n+r-1) + a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$4x^r a_0 r(-1+r) + a_0 x^r = 0$$

Or

$$(4x^r r(-1+r) + x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^r (2r-1)^2 = 0$$

Since the above is true for all x then the indicial equation becomes

$$(2r-1)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{2}$$
$$r_2 = \frac{1}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^r (2r-1)^2 = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = \frac{1}{2}$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}$$
$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+\frac{1}{2}}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$4a_n(n+r)(n+r-1) + 2a_{n-1}(n+r-1) + a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{2a_{n-1}(n+r-1)}{4n^2 + 8nr + 4r^2 - 4n - 4r + 1} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = -\frac{a_{n-1}(2n-1)}{4n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{2r}{(2r+1)^2}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_1 = -\frac{1}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2r}{(2r+1)^2}$	$-\frac{1}{4}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{4r(1+r)}{(2r+1)^2 (3+2r)^2}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = \frac{3}{64}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2r}{(2r+1)^2}$	$-\frac{1}{4}$
a_2	$\frac{4r(1+r)}{(2r+1)^2(3+2r)^2}$	$\frac{3}{64}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{8r(1+r)(2+r)}{(2r+1)^2 (3+2r)^2 (2r+5)^2}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_3 = -\frac{5}{768}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2r}{(2r+1)^2}$	$-\frac{1}{4}$
a_2	$\frac{4r(1+r)}{(2r+1)^2(3+2r)^2}$	$\frac{3}{64}$
a_3	$-\frac{8r(1+r)(2+r)}{(2r+1)^2(3+2r)^2(2r+5)^2}$	$-\frac{5}{768}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{16r(1+r)(2+r)(3+r)}{(2r+1)^2(3+2r)^2(2r+5)^2(2r+7)^2}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = \frac{35}{49152}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2r}{(2r+1)^2}$	$-\frac{1}{4}$
a_2	$\frac{4r(1+r)}{(2r+1)^2(3+2r)^2}$	$\frac{3}{64}$
a_3	$-\frac{8r(1+r)(2+r)}{(2r+1)^2(3+2r)^2(2r+5)^2}$	$-\frac{5}{768}$
a_4	$\frac{16r(1+r)(2+r)(3+r)}{(2r+1)^2(3+2r)^2(2r+5)^2(2r+7)^2}$	$\frac{35}{49152}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{32r(1+r)(2+r)(3+r)(4+r)}{(2r+1)^2(3+2r)^2(2r+5)^2(2r+7)^2(2r+9)^2}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_5 = -\frac{21}{327680}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2r}{(2r+1)^2}$	$-\frac{1}{4}$
a_2	$\frac{4r(1+r)}{(2r+1)^2(3+2r)^2}$	$\frac{3}{64}$
a_3	$-\frac{8r(1+r)(2+r)}{(2r+1)^2(3+2r)^2(2r+5)^2}$	$-\frac{5}{768}$
a_4	$\frac{16r(1+r)(2+r)(3+r)}{(2r+1)^2(3+2r)^2(2r+5)^2(2r+7)^2}$	$\frac{35}{49152}$
a_5	$-\frac{32r(1+r)(2+r)(3+r)(4+r)}{(2r+1)^2(3+2r)^2(2r+5)^2(2r+7)^2(2r+9)^2}$	$-\frac{21}{327680}$

Using the above table, then the first solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= \sqrt{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= \sqrt{x}\left(1 - \frac{x}{4} + \frac{3x^2}{64} - \frac{5x^3}{768} + \frac{35x^4}{49152} - \frac{21x^5}{327680} + O(x^6)\right) \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = \frac{1}{2}$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr}a_{n,r}$
b_0	1	1	N/A since b_n starts from 1
b_1	$-\frac{2r}{(2r+1)^2}$	$-\frac{1}{4}$	$\frac{4r-2}{(2r+1)^3}$
b_2	$\frac{4r(1+r)}{(2r+1)^2(3+2r)^2}$	$\frac{3}{64}$	$\frac{-32r^3-48r^2-8r+12}{(3+2r)^3(2r+1)^3}$
b_3	$-\frac{8r(1+r)(2+r)}{(2r+1)^2(3+2r)^2(2r+5)^2}$	$-\frac{5}{768}$	$\frac{192r^5+1056r^4+2000r^3+1368r^2+16r-240}{(2r+5)^3(3+2r)^3(2r+1)^3}$
b_4	$\frac{16r(1+r)(2+r)(3+r)}{(2r+1)^2(3+2r)^2(2r+5)^2(2r+7)^2}$	$\frac{35}{49152}$	$-\frac{32(32r^7+368r^6+1680r^5+3832r^4+4438r^3+2151r^2-99r-315)}{(2r+1)^3(3+2r)^3(2r+5)^3(2r+7)^3}$
b_5	$-\frac{32r(1+r)(2+r)(3+r)(4+r)}{(2r+1)^2(3+2r)^2(2r+5)^2(2r+7)^2(2r+9)^2}$	$-\frac{21}{327680}$	$\frac{5120r^9+99840r^8+821760r^7+3705600r^6+9919296r^5+15869280r^4+1423264}{(2r+1)^3(3+2r)^3(2r+5)^3(2r+7)^3(2r+9)^3}$

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 \dots \\
&= \sqrt{x}\left(1 - \frac{x}{4} + \frac{3x^2}{64} - \frac{5x^3}{768} + \frac{35x^4}{49152} - \frac{21x^5}{327680} + O(x^6)\right)\ln(x) \\
&\quad + \sqrt{x}\left(-\frac{x^2}{64} + \frac{x^3}{256} - \frac{19x^4}{32768} + \frac{25x^5}{393216} + O(x^6)\right)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1y_1(x) + c_2y_2(x) \\
&= c_1\sqrt{x}\left(1 - \frac{x}{4} + \frac{3x^2}{64} - \frac{5x^3}{768} + \frac{35x^4}{49152} - \frac{21x^5}{327680} + O(x^6)\right) \\
&\quad + c_2\left(\sqrt{x}\left(1 - \frac{x}{4} + \frac{3x^2}{64} - \frac{5x^3}{768} + \frac{35x^4}{49152} - \frac{21x^5}{327680} + O(x^6)\right)\ln(x)\right. \\
&\qquad \left. + \sqrt{x}\left(-\frac{x^2}{64} + \frac{x^3}{256} - \frac{19x^4}{32768} + \frac{25x^5}{393216} + O(x^6)\right)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1\sqrt{x}\left(1 - \frac{x}{4} + \frac{3x^2}{64} - \frac{5x^3}{768} + \frac{35x^4}{49152} - \frac{21x^5}{327680} + O(x^6)\right) \\
&\quad + c_2\left(\sqrt{x}\left(1 - \frac{x}{4} + \frac{3x^2}{64} - \frac{5x^3}{768} + \frac{35x^4}{49152} - \frac{21x^5}{327680} + O(x^6)\right)\ln(x)\right. \\
&\qquad \left. + \sqrt{x}\left(-\frac{x^2}{64} + \frac{x^3}{256} - \frac{19x^4}{32768} + \frac{25x^5}{393216} + O(x^6)\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
\begin{aligned}
y = c_1\sqrt{x}\left(1-\frac{x}{4}+\frac{3x^2}{64}-\frac{5x^3}{768}+\frac{35x^4}{49152}-\frac{21x^5}{327680}+O(x^6)\right) \\
+c_2\left(\sqrt{x}\left(1-\frac{x}{4}+\frac{3x^2}{64}-\frac{5x^3}{768}+\frac{35x^4}{49152}-\frac{21x^5}{327680}+O(x^6)\right)\ln(x)\right. \\
\left.+\sqrt{x}\left(-\frac{x^2}{64}+\frac{x^3}{256}-\frac{19x^4}{32768}+\frac{25x^5}{393216}+O(x^6)\right)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = c_1\sqrt{x}\left(1-\frac{x}{4}+\frac{3x^2}{64}-\frac{5x^3}{768}+\frac{35x^4}{49152}-\frac{21x^5}{327680}+O(x^6)\right) \\
+c_2\left(\sqrt{x}\left(1-\frac{x}{4}+\frac{3x^2}{64}-\frac{5x^3}{768}+\frac{35x^4}{49152}-\frac{21x^5}{327680}+O(x^6)\right)\ln(x)\right. \\
\left.+\sqrt{x}\left(-\frac{x^2}{64}+\frac{x^3}{256}-\frac{19x^4}{32768}+\frac{25x^5}{393216}+O(x^6)\right)\right)
\end{aligned}
$$

Verified OK.

18.10.1 Maple step by step solution

Let's solve
$4x^2y''+2y'x^2+y=0$

- Highest derivative means the order of the ODE is 2
  y''
- Isolate 2nd derivative
  $y''=-\frac{y'}{2}-\frac{y}{4x^2}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear
  $y''+\frac{y'}{2}+\frac{y}{4x^2}=0$

□ Check to see if $x_0=0$ is a regular singular point
  - Define functions
    $\left[P_2(x)=\frac{1}{2},P_3(x)=\frac{1}{4x^2}\right]$
  - $x\cdot P_2(x)$ is analytic at $x=0$
    $\left(x\cdot P_2(x)\right)\Big|_{x=0}=0$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = \frac{1}{4}$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2y'' + 2y'x^2 + y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions

  - Convert $x^2 \cdot y'$ to series expansion

    $x^2 \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r+1}$

  - Shift index using $k- > k-1$

    $x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1}(k-1+r)\, x^{k+r}$

  - Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k-1+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+2r)^2\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r-1)^2 + 2a_{k-1}(k-1+r)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+2r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = \frac{1}{2}$

- Each term in the series must be 0, giving the recursion relation

  $a_k(2k+2r-1)^2 + 2a_{k-1}(k-1+r) = 0$

- Shift index using $k- > k+1$

  $a_{k+1}(2k+1+2r)^2 + 2a_k(k+r) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{2a_k(k+r)}{(2k+1+2r)^2}$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+1} = -\frac{2a_k\left(k+\frac{1}{2}\right)}{(2k+2)^2}$

- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = -\frac{2a_k\left(k+\frac{1}{2}\right)}{(2k+2)^2}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 67

```
Order:=6;
dsolve(4*x^2*diff(y(x),x$2)+2*x^2*diff(y(x),x)+y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \sqrt{x}\left((c_2 \ln(x) + c_1)\left(1 - \frac{1}{4}x + \frac{3}{64}x^2 - \frac{5}{768}x^3 + \frac{35}{49152}x^4 - \frac{21}{327680}x^5 + \mathrm{O}\left(x^6\right)\right) + \left(-\frac{1}{64}x^2 + \frac{1}{256}x^3 - \frac{19}{32768}x^4 + \frac{25}{393216}x^5 + \mathrm{O}\left(x^6\right)\right)c_2\right)$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 129

```
AsymptoticDSolveValue[4*x^2*y''[x]+2*x^2*y'[x]+y[x]==0,y[x],{x,0,5}]
```

$$
\begin{aligned}
y(x) \to\ & c_1\sqrt{x}\left(-\frac{21x^5}{327680}+\frac{35x^4}{49152}-\frac{5x^3}{768}+\frac{3x^2}{64}-\frac{x}{4}+1\right) \\
& + c_2\left(\sqrt{x}\left(\frac{25x^5}{393216}-\frac{19x^4}{32768}+\frac{x^3}{256}-\frac{x^2}{64}\right)\right. \\
& \left. + \sqrt{x}\left(-\frac{21x^5}{327680}+\frac{35x^4}{49152}-\frac{5x^3}{768}+\frac{3x^2}{64}-\frac{x}{4}+1\right)\log(x)\right)
\end{aligned}
$$

18.11 problem 3

Internal problem ID [2947]
Internal file name [`OUTPUT/2439_Sunday_June_05_2022_03_11_04_AM_72846286/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.5. page 771
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + x\cos(x)\,y' - 2\,\mathrm{e}^x y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + x\cos(x)\,y' - 2\,\mathrm{e}^x y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{\cos(x)}{x}$$
$$q(x) = -\frac{2\,\mathrm{e}^x}{x^2}$$

Table 472: Table $p(x), q(x)$ singularites.

$p(x) = \frac{\cos(x)}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{2\,e^x}{x^2}$	
singularity	type
$x = 0$	"regular"
$x = \infty$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2 y'' + x \cos(x)\, y' - 2\,e^x y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned} &x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) \\ &+ x\cos(x) \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) - 2\,e^x \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \end{aligned} \tag{1}$$

Expanding $\cos(x)x$ as Taylor series around $x=0$ and keeping only the first 6 terms gives

$$
\begin{aligned}
\cos(x)x &= x - \frac{1}{2}x^3 + \frac{1}{24}x^5 - \frac{1}{720}x^7 + \dots \\
&= x - \frac{1}{2}x^3 + \frac{1}{24}x^5 - \frac{1}{720}x^7
\end{aligned}
$$

Expanding $-2\,\mathrm{e}^x$ as Taylor series around $x=0$ and keeping only the first 6 terms gives

$$
\begin{aligned}
-2\,\mathrm{e}^x &= -2 - 2x - x^2 - \frac{1}{3}x^3 - \frac{1}{12}x^4 - \frac{1}{60}x^5 - \frac{1}{360}x^6 + \dots \\
&= -2 - 2x - x^2 - \frac{1}{3}x^3 - \frac{1}{12}x^4 - \frac{1}{60}x^5 - \frac{1}{360}x^6
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty}\left(-\frac{x^{n+r+6} a_n (n+r)}{720}\right) \\
&+ \left(\sum_{n=0}^{\infty} \frac{x^{n+r+4} a_n (n+r)}{24}\right) + \sum_{n=0}^{\infty}\left(-\frac{x^{n+r+2} a_n (n+r)}{2}\right) \\
&+ \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty}\left(-2 a_n x^{n+r}\right) + \sum_{n=0}^{\infty}\left(-2 x^{1+n+r} a_n\right) \\
&+ \sum_{n=0}^{\infty}\left(-x^{n+r+2} a_n\right) + \sum_{n=0}^{\infty}\left(-\frac{x^{n+r+3} a_n}{3}\right) + \sum_{n=0}^{\infty}\left(-\frac{x^{n+r+4} a_n}{12}\right) \\
&+ \sum_{n=0}^{\infty}\left(-\frac{x^{n+r+5} a_n}{60}\right) + \sum_{n=0}^{\infty}\left(-\frac{x^{n+r+6} a_n}{360}\right) = 0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty}\left(-\frac{x^{n+r+6} a_n (n+r)}{720}\right) &= \sum_{n=6}^{\infty}\left(-\frac{a_{n-6}(n-6+r)\,x^{n+r}}{720}\right) \\
\sum_{n=0}^{\infty} \frac{x^{n+r+4} a_n (n+r)}{24} &= \sum_{n=4}^{\infty} \frac{a_{n-4}(n-4+r)\,x^{n+r}}{24} \\
\sum_{n=0}^{\infty}\left(-\frac{x^{n+r+2} a_n (n+r)}{2}\right) &= \sum_{n=2}^{\infty}\left(-\frac{a_{n-2}(n+r-2)\,x^{n+r}}{2}\right)
\end{aligned}
$$

$$\sum_{n=0}^{\infty} \left(-2x^{1+n+r} a_n\right) = \sum_{n=1}^{\infty} \left(-2a_{n-1} x^{n+r}\right)$$

$$\sum_{n=0}^{\infty} \left(-x^{n+r+2} a_n\right) = \sum_{n=2}^{\infty} \left(-a_{n-2} x^{n+r}\right)$$

$$\sum_{n=0}^{\infty} \left(-\frac{x^{n+r+3} a_n}{3}\right) = \sum_{n=3}^{\infty} \left(-\frac{a_{n-3} x^{n+r}}{3}\right)$$

$$\sum_{n=0}^{\infty} \left(-\frac{x^{n+r+4} a_n}{12}\right) = \sum_{n=4}^{\infty} \left(-\frac{a_{n-4} x^{n+r}}{12}\right)$$

$$\sum_{n=0}^{\infty} \left(-\frac{x^{n+r+5} a_n}{60}\right) = \sum_{n=5}^{\infty} \left(-\frac{a_{n-5} x^{n+r}}{60}\right)$$

$$\sum_{n=0}^{\infty} \left(-\frac{x^{n+r+6} a_n}{360}\right) = \sum_{n=6}^{\infty} \left(-\frac{a_{n-6} x^{n+r}}{360}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\begin{aligned} &\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=6}^{\infty} \left(-\frac{a_{n-6}(n-6+r)\, x^{n+r}}{720}\right) \\ &+ \left(\sum_{n=4}^{\infty} \frac{a_{n-4}(n-4+r)\, x^{n+r}}{24}\right) + \sum_{n=2}^{\infty} \left(-\frac{a_{n-2}(n+r-2)\, x^{n+r}}{2}\right) \\ &+ \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-2a_n x^{n+r}\right) + \sum_{n=1}^{\infty} \left(-2a_{n-1} x^{n+r}\right) \\ &+ \sum_{n=2}^{\infty} \left(-a_{n-2} x^{n+r}\right) + \sum_{n=3}^{\infty} \left(-\frac{a_{n-3} x^{n+r}}{3}\right) + \sum_{n=4}^{\infty} \left(-\frac{a_{n-4} x^{n+r}}{12}\right) \\ &+ \sum_{n=5}^{\infty} \left(-\frac{a_{n-5} x^{n+r}}{60}\right) + \sum_{n=6}^{\infty} \left(-\frac{a_{n-6} x^{n+r}}{360}\right) = 0 \end{aligned} \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) + x^{n+r} a_n (n+r) - 2a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) + x^r a_0 r - 2a_0 x^r = 0$$

Or

$$(x^r r(-1+r) + x^r r - 2x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(r^2 - 2)\, x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - 2 = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= \sqrt{2} \\ r_2 &= -\sqrt{2} \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(r^2 - 2)\, x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 2\sqrt{2}$ is not an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right) \\ y_2(x) &= x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+\sqrt{2}} \\ y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n-\sqrt{2}} \end{aligned}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = \frac{2}{r^2 + 2r - 1}$$

Substituting $n = 2$ in Eq. (2B) gives

$$a_2 = \frac{r^3 + 4r^2 + 3r + 6}{2\,(r^2 + 2r - 1)\,(r^2 + 4r + 2)}$$

Substituting $n = 3$ in Eq. (2B) gives

$$a_3 = \frac{r^4 + 12r^3 + 42r^2 + 51r + 34}{3\left(r^2 + 2r - 1\right)\left(r^2 + 4r + 2\right)\left(r^2 + 6r + 7\right)}$$

Substituting $n = 4$ in Eq. (2B) gives

$$a_4 = \frac{-r^7 - 4r^6 + 56r^5 + 484r^4 + 1611r^3 + 2924r^2 + 3066r + 1748}{24\left(r^2 + 2r - 1\right)\left(r^2 + 4r + 2\right)\left(r^2 + 6r + 7\right)\left(r^2 + 8r + 14\right)}$$

Substituting $n = 5$ in Eq. (2B) gives

$$a_5 = \frac{r^8 + 30r^7 + 487r^6 + 4215r^5 + 20182r^4 + 55655r^3 + 89604r^2 + 82830r + 39204}{60\left(r^2 + 2r - 1\right)\left(r^2 + 4r + 2\right)\left(r^2 + 6r + 7\right)\left(r^2 + 8r + 14\right)\left(r^2 + 10r + 23\right)}$$

For $6 \leq n$ the recursive equation is

$$\begin{aligned} &a_n(n+r)\left(n+r-1\right) - \frac{a_{n-6}(n-6+r)}{720} + \frac{a_{n-4}(n-4+r)}{24} - \frac{a_{n-2}(n+r-2)}{2} \\ &+ a_n(n+r) - 2a_n - 2a_{n-1} - a_{n-2} - \frac{a_{n-3}}{3} - \frac{a_{n-4}}{12} - \frac{a_{n-5}}{60} - \frac{a_{n-6}}{360} = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{na_{n-6} - 30na_{n-4} + 360na_{n-2} + ra_{n-6} - 30ra_{n-4} + 360ra_{n-2} - 4a_{n-6} + 12a_{n-5} + 180a_{n-4} + 240a_{n-}}{720n^2 + 1440nr + 720r^2 - 1440} \tag{4}$$

Which for the root $r = \sqrt{2}$ becomes

$$a_n = \frac{\left(a_{n-6} - 30a_{n-4} + 360a_{n-2}\right)\sqrt{2} + \left(a_{n-6} - 30a_{n-4} + 360a_{n-2}\right)n - 4a_{n-6} + 12a_{n-5} + 180a_{n-4} + 240a_{}}{720n\left(2\sqrt{2} + n\right)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \sqrt{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2}{r^2+2r-1}$	$\frac{2}{1+2\sqrt{2}}$
a_2	$\frac{r^3+4r^2+3r+6}{2(r^2+2r-1)(r^2+4r+2)}$	$\frac{5\sqrt{2}+14}{40+24\sqrt{2}}$
a_3	$\frac{r^4+12r^3+42r^2+51r+34}{3(r^2+2r-1)(r^2+4r+2)(r^2+6r+7)}$	$\frac{122+75\sqrt{2}}{684\sqrt{2}+972}$
a_4	$\frac{-r^7-4r^6+56r^5+484r^4+1611r^3+2924r^2+3066r+1748}{24(r^2+2r-1)(r^2+4r+2)(r^2+6r+7)(r^2+8r+14)}$	$\frac{1626\sqrt{2}+2375}{52992+37440\sqrt{2}}$
a_5	$\frac{r^8+30r^7+487r^6+4215r^5+20182r^4+55655r^3+89604r^2+82830r+39204}{60(r^2+2r-1)(r^2+4r+2)(r^2+6r+7)(r^2+8r+14)(r^2+10r+23)}$	$\frac{75763+52810\sqrt{2}}{7200\left(1+2\sqrt{2}\right)\left(1+\sqrt{2}\right)\left(3+2\sqrt{2}\right)\left(2+\sqrt{2}\right)\left(5+2\sqrt{2}\right)}$

Using the above table, then the solution $y_1(x)$ is

$$y_1(x) = x^{\sqrt{2}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right)$$
$$= x^{\sqrt{2}}\left(1 + \frac{2x}{1+2\sqrt{2}} + \frac{\left(5\sqrt{2}+14\right)x^2}{40+24\sqrt{2}} + \frac{\left(122+75\sqrt{2}\right)x^3}{684\sqrt{2}+972} + \frac{\left(1626\sqrt{2}+2375\right)x^4}{52992+37440\sqrt{2}} + \frac{}{7200\left(1+2\sqrt{\right.}}\right.$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = \frac{2}{r^2 + 2r - 1}$$

Substituting $n = 2$ in Eq. (2B) gives

$$b_2 = \frac{r^3 + 4r^2 + 3r + 6}{2\left(r^2 + 2r - 1\right)\left(r^2 + 4r + 2\right)}$$

Substituting $n = 3$ in Eq. (2B) gives

$$b_3 = \frac{r^4 + 12r^3 + 42r^2 + 51r + 34}{3\left(r^2 + 2r - 1\right)\left(r^2 + 4r + 2\right)\left(r^2 + 6r + 7\right)}$$

Substituting $n = 4$ in Eq. (2B) gives

$$b_4 = \frac{-r^7 - 4r^6 + 56r^5 + 484r^4 + 1611r^3 + 2924r^2 + 3066r + 1748}{24\left(r^2 + 2r - 1\right)\left(r^2 + 4r + 2\right)\left(r^2 + 6r + 7\right)\left(r^2 + 8r + 14\right)}$$

Substituting $n = 5$ in Eq. (2B) gives

$$b_5 = \frac{r^8 + 30r^7 + 487r^6 + 4215r^5 + 20182r^4 + 55655r^3 + 89604r^2 + 82830r + 39204}{60\left(r^2 + 2r - 1\right)\left(r^2 + 4r + 2\right)\left(r^2 + 6r + 7\right)\left(r^2 + 8r + 14\right)\left(r^2 + 10r + 23\right)}$$

For $6 \leq n$ the recursive equation is

$$\begin{aligned}
&b_n(n+r)\left(n+r-1\right) - \frac{b_{n-6}(n-6+r)}{720} + \frac{b_{n-4}(n-4+r)}{24} - \frac{b_{n-2}(n+r-2)}{2} \\
&+ b_n(n+r) - 2b_n - 2b_{n-1} - b_{n-2} - \frac{b_{n-3}}{3} - \frac{b_{n-4}}{12} - \frac{b_{n-5}}{60} - \frac{b_{n-6}}{360} = 0
\end{aligned} \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = \frac{nb_{n-6} - 30nb_{n-4} + 360nb_{n-2} + rb_{n-6} - 30rb_{n-4} + 360rb_{n-2} - 4b_{n-6} + 12b_{n-5} + 180b_{n-4} + 240b_{n-3}}{720n^2 + 1440nr + 720r^2 - 1440} \tag{4}$$

Which for the root $r = -\sqrt{2}$ becomes

$$b_n = \frac{(-b_{n-6} + 30b_{n-4} - 360b_{n-2})\sqrt{2} + (b_{n-6} - 30b_{n-4} + 360b_{n-2})\,n - 4b_{n-6} + 12b_{n-5} + 180b_{n-4} + 240b_n}{720n\left(-2\sqrt{2} + n\right)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\sqrt{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2}{r^2+2r-1}$	$-\frac{2}{-1+2\sqrt{2}}$
b_2	$\frac{r^3+4r^2+3r+6}{2(r^2+2r-1)(r^2+4r+2)}$	$\frac{-5\sqrt{2}+14}{40-24\sqrt{2}}$
b_3	$\frac{r^4+12r^3+42r^2+51r+34}{3(r^2+2r-1)(r^2+4r+2)(r^2+6r+7)}$	$\frac{-122+75\sqrt{2}}{684\sqrt{2}-972}$
b_4	$\frac{-r^7-4r^6+56r^5+484r^4+1611r^3+2924r^2+3066r+1748}{24(r^2+2r-1)(r^2+4r+2)(r^2+6r+7)(r^2+8r+14)}$	$\frac{-1626\sqrt{2}+2375}{52992-37440\sqrt{2}}$
b_5	$\frac{r^8+30r^7+487r^6+4215r^5+20182r^4+55655r^3+89604r^2+82830r+39204}{60(r^2+2r-1)(r^2+4r+2)(r^2+6r+7)(r^2+8r+14)(r^2+10r+23)}$	$\frac{-75763+52810\sqrt{2}}{7200\left(-1+2\sqrt{2}\right)\left(\sqrt{2}-1\right)\left(2\sqrt{2}-3\right)\left(\sqrt{2}-2\right)\left(-5+2\sqrt{2}\right)}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x^{\sqrt{2}}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= x^{-\sqrt{2}}\left(1 - \frac{2x}{-1+2\sqrt{2}} + \frac{\left(-5\sqrt{2}+14\right)x^2}{40-24\sqrt{2}} + \frac{\left(-122+75\sqrt{2}\right)x^3}{684\sqrt{2}-972} + \frac{\left(-1626\sqrt{2}+2375\right)x^4}{52992-37440\sqrt{2}} + \frac{}{7200} \right. \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^{\sqrt{2}}\left(1 + \frac{2x}{1+2\sqrt{2}} + \frac{\left(5\sqrt{2}+14\right)x^2}{40+24\sqrt{2}} + \frac{\left(122+75\sqrt{2}\right)x^3}{684\sqrt{2}+972} + \frac{\left(1626\sqrt{2}+2375\right)x^4}{52992+37440\sqrt{2}} \right. \\ &\quad \left. + \frac{\left(75763+52810\sqrt{2}\right)x^5}{7200\left(1+2\sqrt{2}\right)\left(1+\sqrt{2}\right)\left(3+2\sqrt{2}\right)\left(2+\sqrt{2}\right)\left(5+2\sqrt{2}\right)} + O\left(x^6\right)\right) \\ &\quad + c_2 x^{-\sqrt{2}}\left(1 - \frac{2x}{-1+2\sqrt{2}} + \frac{\left(-5\sqrt{2}+14\right)x^2}{40-24\sqrt{2}} + \frac{\left(-122+75\sqrt{2}\right)x^3}{684\sqrt{2}-972} \right. \\ &\quad + \frac{\left(-1626\sqrt{2}+2375\right)x^4}{52992-37440\sqrt{2}} \\ &\quad \left. + \frac{\left(-75763+52810\sqrt{2}\right)x^5}{7200\left(-1+2\sqrt{2}\right)\left(\sqrt{2}-1\right)\left(2\sqrt{2}-3\right)\left(\sqrt{2}-2\right)\left(-5+2\sqrt{2}\right)} + O\left(x^6\right)\right) \end{aligned}$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^{\sqrt{2}} \left(1 + \frac{2x}{1+2\sqrt{2}} + \frac{\left(5\sqrt{2}+14\right)x^2}{40+24\sqrt{2}} + \frac{\left(122+75\sqrt{2}\right)x^3}{684\sqrt{2}+972} + \frac{\left(1626\sqrt{2}+2375\right)x^4}{52992+37440\sqrt{2}} \right. \\
&\quad \left. + \frac{\left(75763+52810\sqrt{2}\right)x^5}{7200\left(1+2\sqrt{2}\right)\left(1+\sqrt{2}\right)\left(3+2\sqrt{2}\right)\left(2+\sqrt{2}\right)\left(5+2\sqrt{2}\right)} + O\left(x^6\right)\right) + c_2 x^{-\sqrt{2}} \left(1 \right. \\
&\quad - \frac{2x}{-1+2\sqrt{2}} + \frac{\left(-5\sqrt{2}+14\right)x^2}{40-24\sqrt{2}} + \frac{\left(-122+75\sqrt{2}\right)x^3}{684\sqrt{2}-972} + \frac{\left(-1626\sqrt{2}+2375\right)x^4}{52992-37440\sqrt{2}} \\
&\quad \left. + \frac{\left(-75763+52810\sqrt{2}\right)x^5}{7200\left(-1+2\sqrt{2}\right)\left(\sqrt{2}-1\right)\left(2\sqrt{2}-3\right)\left(\sqrt{2}-2\right)\left(-5+2\sqrt{2}\right)} + O\left(x^6\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x^{\sqrt{2}} \left(1 + \frac{2x}{1+2\sqrt{2}} + \frac{\left(5\sqrt{2}+14\right)x^2}{40+24\sqrt{2}} + \frac{\left(122+75\sqrt{2}\right)x^3}{684\sqrt{2}+972} + \frac{\left(1626\sqrt{2}+2375\right)x^4}{52992+37440\sqrt{2}} \right. \\
&\quad \left. + \frac{\left(75763+52810\sqrt{2}\right)x^5}{7200\left(1+2\sqrt{2}\right)\left(1+\sqrt{2}\right)\left(3+2\sqrt{2}\right)\left(2+\sqrt{2}\right)\left(5+2\sqrt{2}\right)} + O\left(x^6\right)\right) + c_2 x^{-\sqrt{2}} \left(1 \right. \\
&\quad - \frac{2x}{-1+2\sqrt{2}} + \frac{\left(-5\sqrt{2}+14\right)x^2}{40-24\sqrt{2}} + \frac{\left(-122+75\sqrt{2}\right)x^3}{684\sqrt{2}-972} + \frac{\left(-1626\sqrt{2}+2375\right)x^4}{52992-37440\sqrt{2}} \\
&\quad \left. + \frac{\left(-75763+52810\sqrt{2}\right)x^5}{7200\left(-1+2\sqrt{2}\right)\left(\sqrt{2}-1\right)\left(2\sqrt{2}-3\right)\left(\sqrt{2}-2\right)\left(-5+2\sqrt{2}\right)} + O\left(x^6\right)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1 x^{\sqrt{2}} \left(1 + \frac{2x}{1+2\sqrt{2}} + \frac{\left(5\sqrt{2}+14\right)x^2}{40+24\sqrt{2}} + \frac{\left(122+75\sqrt{2}\right)x^3}{684\sqrt{2}+972} + \frac{\left(1626\sqrt{2}+2375\right)x^4}{52992+37440\sqrt{2}} \right. \\
&\quad \left. + \frac{\left(75763+52810\sqrt{2}\right)x^5}{7200\left(1+2\sqrt{2}\right)\left(1+\sqrt{2}\right)\left(3+2\sqrt{2}\right)\left(2+\sqrt{2}\right)\left(5+2\sqrt{2}\right)} + O\left(x^6\right)\right) + c_2 x^{-\sqrt{2}} \left(1 \right. \\
&\quad - \frac{2x}{-1+2\sqrt{2}} + \frac{\left(-5\sqrt{2}+14\right)x^2}{40-24\sqrt{2}} + \frac{\left(-122+75\sqrt{2}\right)x^3}{684\sqrt{2}-972} + \frac{\left(-1626\sqrt{2}+2375\right)x^4}{52992-37440\sqrt{2}} \\
&\quad \left. + \frac{\left(-75763+52810\sqrt{2}\right)x^5}{7200\left(-1+2\sqrt{2}\right)\left(\sqrt{2}-1\right)\left(2\sqrt{2}-3\right)\left(\sqrt{2}-2\right)\left(-5+2\sqrt{2}\right)} + O\left(x^6\right)\right)
\end{aligned}
$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
      trying a symmetry of the form [xi=0, eta=F(x)]
      trying 2nd order exact linear
      trying symmetries linear in x and y(x)
      trying to convert to a linear ODE with constant coefficients
      -> trying with_periodic_functions in the coefficients
         --- Trying Lie symmetry methods, 2nd order ---
         `, `-> Computing symmetries using: way = 5
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
      trying a symmetry of the form [xi=0, eta=F(x)]
      trying 2nd order exact linear
      trying symmetries linear in x and y(x)
      trying to convert to a linear ODE with constant coefficients
      -> trying with_periodic_functions in the coefficients
         --- Trying Lie symmetry methods, 2nd order ---
         `, `-> Computing symmetries using: way = 5`[0, u]
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 389

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+x*cos(x)*diff(y(x),x)-2*exp(x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^{-\sqrt{2}} \left(1 - 2\frac{1}{-1+2\sqrt{2}}x + \frac{-5\sqrt{2}+14}{40-24\sqrt{2}}x^2 + \frac{-122+75\sqrt{2}}{684\sqrt{2}-972}x^3 + \frac{-1626\sqrt{2}+2375}{52992-37440\sqrt{2}}x^4 + \frac{1}{7200}\frac{-75763+52810\sqrt{2}}{\left(-1+2\sqrt{2}\right)\left(\sqrt{2}-1\right)\left(-3+2\sqrt{2}\right)\left(\sqrt{2}-2\right)\left(-5+2\sqrt{2}\right)}x^5 + \mathrm{O}\left(x^6\right)\right) + c_2 x^{\sqrt{2}} \left(1+2\frac{1}{1+2\sqrt{2}}x + \frac{5\sqrt{2}+14}{40+24\sqrt{2}}x^2 + \frac{122+75\sqrt{2}}{684\sqrt{2}+972}x^3 + \frac{1626\sqrt{2}+2375}{52992+37440\sqrt{2}}x^4 + \frac{1}{7200}\frac{75763+52810\sqrt{2}}{\left(1+2\sqrt{2}\right)\left(1+\sqrt{2}\right)\left(3+2\sqrt{2}\right)\left(2+\sqrt{2}\right)\left(5+2\sqrt{2}\right)}x^5 + \mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 2210

```
AsymptoticDSolveValue[x^2*y''[x]+x*Cos[x]*y'[x]-2*Exp[x]*y[x]==0,y[x],{x,0,5}]
```

Too large to display

18.12 problem 4

Internal problem ID [2948]
Internal file name [`OUTPUT/2440_Sunday_June_05_2022_03_11_12_AM_44382120/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.5. page 771
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + y'x^2 - (x+2)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + y'x^2 + (-2-x)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$\begin{aligned}p(x) &= 1\\ q(x) &= -\frac{x+2}{x^2}\end{aligned}$$

Table 473: Table $p(x), q(x)$ singularites.

$p(x) = 1$	
singularity	type

$q(x) = -\frac{x+2}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2 y'' + y' x^2 + (-2 - x) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) x^2 + (-2 - x) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty} x^{1+n+r} a_n(n+r)\right) \\
&+\sum_{n=0}^{\infty}\left(-2 a_n x^{n+r}\right)+\sum_{n=0}^{\infty}\left(-x^{1+n+r} a_n\right)=0
\end{aligned}
\tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} x^{1+n+r} a_n(n+r) &= \sum_{n=1}^{\infty} a_{n-1}(n+r-1) x^{n+r} \\
\sum_{n=0}^{\infty}\left(-x^{1+n+r} a_n\right) &= \sum_{n=1}^{\infty}\left(-a_{n-1} x^{n+r}\right)
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)(n+r-1)\right)+\left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1) x^{n+r}\right) \\
&+\sum_{n=0}^{\infty}\left(-2 a_n x^{n+r}\right)+\sum_{n=1}^{\infty}\left(-a_{n-1} x^{n+r}\right)=0
\end{aligned}
\tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
x^{n+r} a_n(n+r)(n+r-1)-2 a_n x^{n+r}=0
$$

When $n=0$ the above becomes

$$
x^r a_0 r(-1+r)-2 a_0 x^r=0
$$

Or

$$
\left(x^r r(-1+r)-2 x^r\right) a_0=0
$$

Since $a_0 \neq 0$ then the above simplifies to

$$
\left(r^2-r-2\right) x^r=0
$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - r - 2 = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= 2 \\ r_2 &= -1 \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(r^2 - r - 2\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 3$ is an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= x^2 \left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x} \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+2} \\ y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-1}\right) \end{aligned}$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n + r)(n + r - 1) + a_{n-1}(n + r - 1) - 2a_n - a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}}{1+n+r} \tag{4}$$

Which for the root $r = 2$ becomes

$$a_n = -\frac{a_{n-1}}{3+n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 2$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{1}{2+r}$$

Which for the root $r = 2$ becomes

$$a_1 = -\frac{1}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2+r}$	$-\frac{1}{4}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{(2+r)(3+r)}$$

Which for the root $r = 2$ becomes

$$a_2 = \frac{1}{20}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2+r}$	$-\frac{1}{4}$
a_2	$\frac{1}{(2+r)(3+r)}$	$\frac{1}{20}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{1}{(2+r)(3+r)(4+r)}$$

Which for the root $r = 2$ becomes

$$a_3 = -\frac{1}{120}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2+r}$	$-\frac{1}{4}$
a_2	$\frac{1}{(2+r)(3+r)}$	$\frac{1}{20}$
a_3	$-\frac{1}{(2+r)(3+r)(4+r)}$	$-\frac{1}{120}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(3+r)(4+r)(2+r)(5+r)}$$

Which for the root $r = 2$ becomes

$$a_4 = \frac{1}{840}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2+r}$	$-\frac{1}{4}$
a_2	$\frac{1}{(2+r)(3+r)}$	$\frac{1}{20}$
a_3	$-\frac{1}{(2+r)(3+r)(4+r)}$	$-\frac{1}{120}$
a_4	$\frac{1}{(3+r)(4+r)(2+r)(5+r)}$	$\frac{1}{840}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{1}{(4+r)(2+r)(5+r)(3+r)(6+r)}$$

Which for the root $r = 2$ becomes

$$a_5 = -\frac{1}{6720}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2+r}$	$-\frac{1}{4}$
a_2	$\frac{1}{(2+r)(3+r)}$	$\frac{1}{20}$
a_3	$-\frac{1}{(2+r)(3+r)(4+r)}$	$-\frac{1}{120}$
a_4	$\frac{1}{(3+r)(4+r)(2+r)(5+r)}$	$\frac{1}{840}$
a_5	$-\frac{1}{(4+r)(2+r)(5+r)(3+r)(6+r)}$	$-\frac{1}{6720}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^2 (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots) \\ &= x^2 \left(1 - \frac{x}{4} + \frac{x^2}{20} - \frac{x^3}{120} + \frac{x^4}{840} - \frac{x^5}{6720} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 3$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_3(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_3 \\ &= -\frac{1}{(2+r)(3+r)(4+r)} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r \to r_2} -\frac{1}{(2+r)(3+r)(4+r)} &= \lim_{r \to -1} -\frac{1}{(2+r)(3+r)(4+r)} \\ &= -\frac{1}{6} \end{aligned}$$

The limit is $-\frac{1}{6}$. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$
\begin{aligned}
y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\
&= \sum_{n=0}^{\infty} b_n x^{n-1}
\end{aligned}
$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$
b_n(n+r)(n+r-1) + b_{n-1}(n+r-1) - 2b_n - b_{n-1} = 0 \tag{4}
$$

Which for for the root $r = -1$ becomes

$$
b_n(n-1)(n-2) + b_{n-1}(n-2) - 2b_n - b_{n-1} = 0 \tag{4A}
$$

Solving for b_n from the recursive equation (4) gives

$$
b_n = -\frac{b_{n-1}}{1+n+r} \tag{5}
$$

Which for the root $r = -1$ becomes

$$
b_n = -\frac{b_{n-1}}{n} \tag{6}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$
b_1 = -\frac{1}{2+r}
$$

Which for the root $r = -1$ becomes

$$
b_1 = -1
$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{2+r}$	-1

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1}{(2+r)(3+r)}$$

Which for the root $r = -1$ becomes

$$b_2 = \frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{2+r}$	-1
b_2	$\frac{1}{(2+r)(3+r)}$	$\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{1}{(2+r)(3+r)(4+r)}$$

Which for the root $r = -1$ becomes

$$b_3 = -\frac{1}{6}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{2+r}$	-1
b_2	$\frac{1}{(2+r)(3+r)}$	$\frac{1}{2}$
b_3	$-\frac{1}{(2+r)(3+r)(4+r)}$	$-\frac{1}{6}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{(3+r)(4+r)(2+r)(5+r)}$$

Which for the root $r = -1$ becomes

$$b_4 = \frac{1}{24}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{2+r}$	-1
b_2	$\frac{1}{(2+r)(3+r)}$	$\frac{1}{2}$
b_3	$-\frac{1}{(2+r)(3+r)(4+r)}$	$-\frac{1}{6}$
b_4	$\frac{1}{(3+r)(4+r)(2+r)(5+r)}$	$\frac{1}{24}$

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{1}{(4+r)(2+r)(5+r)(3+r)(6+r)}$$

Which for the root $r = -1$ becomes

$$b_5 = -\frac{1}{120}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{2+r}$	-1
b_2	$\frac{1}{(2+r)(3+r)}$	$\frac{1}{2}$
b_3	$-\frac{1}{(2+r)(3+r)(4+r)}$	$-\frac{1}{6}$
b_4	$\frac{1}{(3+r)(4+r)(2+r)(5+r)}$	$\frac{1}{24}$
b_5	$-\frac{1}{(4+r)(2+r)(5+r)(3+r)(6+r)}$	$-\frac{1}{120}$

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= x^2 \left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\
&= \frac{1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + O(x^6)}{x}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^2 \left(1 - \frac{x}{4} + \frac{x^2}{20} - \frac{x^3}{120} + \frac{x^4}{840} - \frac{x^5}{6720} + O(x^6)\right) \\
&+ \frac{c_2 \left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + O(x^6)\right)}{x}
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^2 \left(1 - \frac{x}{4} + \frac{x^2}{20} - \frac{x^3}{120} + \frac{x^4}{840} - \frac{x^5}{6720} + O(x^6)\right) \\
&+ \frac{c_2 \left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + O(x^6)\right)}{x}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x^2 \left(1 - \frac{x}{4} + \frac{x^2}{20} - \frac{x^3}{120} + \frac{x^4}{840} - \frac{x^5}{6720} + O(x^6)\right) \\
&+ \frac{c_2 \left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + O(x^6)\right)}{x}
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1 x^2 \left(1 - \frac{x}{4} + \frac{x^2}{20} - \frac{x^3}{120} + \frac{x^4}{840} - \frac{x^5}{6720} + O(x^6)\right) \\
&+ \frac{c_2 \left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + O(x^6)\right)}{x}
\end{aligned}
$$

Verified OK.

18.12.1 Maple step by step solution

Let's solve

$x^2y'' + y'x^2 + (-2-x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -y' + \frac{(x+2)y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + y' - \frac{(x+2)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = 1, P_3(x) = -\frac{x+2}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = 0$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -2$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2y'' + y'x^2 + (-2-x)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ◦ Shift index using $k- >k-m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^2 \cdot y'$ to series expansion

  $$x^2 \cdot y' = \sum_{k=0}^{\infty} a_k (k+r) \, x^{k+r+1}$$

- Shift index using $k- >k-1$

  $$x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1} (k-1+r) \, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r) \, (k-1+r) \, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 (1+r) \, (-2+r) \, x^r + \left(\sum_{k=1}^{\infty} \left(a_k (k+r+1) \, (k+r-2) + a_{k-1} (k+r-2)\right) x^{k+r} \right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(1+r) \, (-2+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{-1, 2\}$$

- Each term in the series must be 0, giving the recursion relation

  $$(k+r-2) \left(a_k (k+r+1) + a_{k-1}\right) = 0$$

- Shift index using $k- >k+1$

  $$(k-1+r) \left(a_{k+1} (k+2+r) + a_k\right) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = -\frac{a_k}{k+2+r}$$

- Recursion relation for $r = -1$

  $$a_{k+1} = -\frac{a_k}{k+1}$$

- Solution for $r = -1$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+1} = -\frac{a_k}{k+1}\right]$$

- Recursion relation for $r = 2$

  $$a_{k+1} = -\frac{a_k}{k+4}$$

- Solution for $r = 2$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1}=-\frac{a_k}{k+4}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-1}\right)+\left(\sum_{k=0}^{\infty} b_k x^{2+k}\right), a_{1+k}=-\frac{a_k}{1+k}, b_{1+k}=-\frac{b_k}{k+4}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 47

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+x^2*diff(y(x),x)-(2+x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x)=c_1 x^2\left(1-\frac{1}{4}x+\frac{1}{20}x^2-\frac{1}{120}x^3+\frac{1}{840}x^4-\frac{1}{6720}x^5+\mathrm{O}\left(x^6\right)\right)+\frac{c_2\left(12-12x+6x^2-2x^3+\frac{1}{2}x^4-\frac{1}{10}x^5+\mathrm{O}\left(x^6\right)\right)}{x}$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 66

```
AsymptoticDSolveValue[x^2*y''[x]+x^2*y'[x]-(2+x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{x^3}{24} - \frac{x^2}{6} + \frac{x}{2} + \frac{1}{x} - 1\right) + c_2\left(\frac{x^6}{840} - \frac{x^5}{120} + \frac{x^4}{20} - \frac{x^3}{4} + x^2\right)$$

18.13 problem 5

Internal problem ID [2949]
Internal file name [`OUTPUT/2441_Sunday_June_05_2022_03_11_16_AM_23810315/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.5. page 771
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + 2y'x^2 + \left(x - \frac{3}{4}\right)y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + 2y'x^2 + \left(x - \frac{3}{4}\right)y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$\begin{aligned} p(x) &= 2 \\ q(x) &= \frac{4x - 3}{4x^2} \end{aligned}$$

Table 475: Table $p(x), q(x)$ singularites.

$p(x) = 2$	
singularity	type

$q(x) = \frac{4x-3}{4x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + 2y'x^2 + \left(x - \frac{3}{4}\right) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned} &x^2\left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) \\ &+ 2\left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) x^2 + \left(x - \frac{3}{4}\right)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \end{aligned} \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2x^{1+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n\right) + \sum_{n=0}^{\infty} \left(-\frac{3a_n x^{n+r}}{4}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2x^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n = \sum_{n=1}^{\infty} a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)\, x^{n+r}\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-\frac{3a_n x^{n+r}}{4}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) - \frac{3a_n x^{n+r}}{4} = 0$$

When $n = 0$ the above becomes

$$x^r a_0 r(-1+r) - \frac{3a_0 x^r}{4} = 0$$

Or

$$\left(x^r r(-1+r) - \frac{3x^r}{4}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\frac{(4r^2 - 4r - 3)\, x^r}{4} = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - r - \frac{3}{4} = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{3}{2}$$
$$r_2 = -\frac{1}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\frac{(4r^2 - 4r - 3)\, x^r}{4} = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 2$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x^{\frac{3}{2}} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{\sqrt{x}}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{3}{2}}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-\frac{1}{2}}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + 2a_{n-1}(n+r-1) + a_{n-1} - \frac{3a_n}{4} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{4a_{n-1}(2n+2r-1)}{4n^2+8nr+4r^2-4n-4r-3} \tag{4}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_n = -\frac{2a_{n-1}(1+n)}{n(n+2)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{3}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-8r-4}{4r^2+4r-3}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_1 = -\frac{4}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-8r-4}{4r^2+4r-3}$	$-\frac{4}{3}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{16}{4r^2+8r-5}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_2 = 1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-8r-4}{4r^2+4r-3}$	$-\frac{4}{3}$
a_2	$\frac{16}{4r^2+8r-5}$	1

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{64}{8r^3 + 36r^2 + 22r - 21}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_3 = -\frac{8}{15}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-8r-4}{4r^2+4r-3}$	$-\frac{4}{3}$
a_2	$\frac{16}{4r^2+8r-5}$	1
a_3	$-\frac{64}{8r^3+36r^2+22r-21}$	$-\frac{8}{15}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{256}{(4r^2 + 28r + 45)(4r^2 + 4r - 3)}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_4 = \frac{2}{9}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-8r-4}{4r^2+4r-3}$	$-\frac{4}{3}$
a_2	$\frac{16}{4r^2+8r-5}$	1
a_3	$-\frac{64}{8r^3+36r^2+22r-21}$	$-\frac{8}{15}$
a_4	$\frac{256}{(4r^2+28r+45)(4r^2+4r-3)}$	$\frac{2}{9}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{1024}{(4r^2 + 36r + 77)(4r^2 + 4r - 3)(2r + 5)}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_5 = -\frac{8}{105}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-8r-4}{4r^2+4r-3}$	$-\frac{4}{3}$
a_2	$\frac{16}{4r^2+8r-5}$	1
a_3	$-\frac{64}{8r^3+36r^2+22r-21}$	$-\frac{8}{15}$
a_4	$\frac{256}{(4r^2+28r+45)(4r^2+4r-3)}$	$\frac{2}{9}$
a_5	$-\frac{1024}{(4r^2+36r+77)(4r^2+4r-3)(2r+5)}$	$-\frac{8}{105}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\frac{3}{2}}\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots\right) \\ &= x^{\frac{3}{2}}\left(1 - \frac{4x}{3} + x^2 - \frac{8x^3}{15} + \frac{2x^4}{9} - \frac{8x^5}{105} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 2$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_2(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_2 \\ &= \frac{16}{4r^2 + 8r - 5} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\to r_2} \frac{16}{4r^2 + 8r - 5} &= \lim_{r\to -\frac{1}{2}} \frac{16}{4r^2 + 8r - 5} \\ &= -2 \end{aligned}$$

The limit is -2. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\ &= \sum_{n=0}^{\infty} b_n x^{n-\frac{1}{2}} \end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$b_n(n + r)(n + r - 1) + 2b_{n-1}(n + r - 1) + b_{n-1} - \frac{3b_n}{4} = 0 \tag{4}$$

Which for for the root $r = -\frac{1}{2}$ becomes

$$b_n\left(n - \frac{1}{2}\right)\left(n - \frac{3}{2}\right) + 2b_{n-1}\left(n - \frac{3}{2}\right) + b_{n-1} - \frac{3b_n}{4} = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = -\frac{4b_{n-1}(2n + 2r - 1)}{4n^2 + 8nr + 4r^2 - 4n - 4r - 3} \tag{5}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_n = -\frac{4b_{n-1}(2n - 2)}{4n^2 - 8n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = -\frac{4(1+2r)}{4r^2+4r-3}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_1 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-8r-4}{4r^2+4r-3}$	0

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{16}{(2r+5)(2r-1)}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_2 = -2$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-8r-4}{4r^2+4r-3}$	0
b_2	$\frac{16}{4r^2+8r-5}$	-2

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{64}{(4r^2+20r+21)(2r-1)}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_3 = \frac{8}{3}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-8r-4}{4r^2+4r-3}$	0
b_2	$\frac{16}{4r^2+8r-5}$	-2
b_3	$-\frac{64}{8r^3+36r^2+22r-21}$	$\frac{8}{3}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{256}{(2r-1)(3+2r)(4r^2+28r+45)}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_4 = -2$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-8r-4}{4r^2+4r-3}$	0
b_2	$\frac{16}{4r^2+8r-5}$	-2
b_3	$-\frac{64}{8r^3+36r^2+22r-21}$	$\frac{8}{3}$
b_4	$\frac{256}{16r^4+128r^3+280r^2+96r-135}$	-2

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{1024}{(2r+5)(2r-1)(3+2r)(4r^2+36r+77)}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_5 = \frac{16}{15}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-8r-4}{4r^2+4r-3}$	0
b_2	$\frac{16}{4r^2+8r-5}$	-2
b_3	$-\frac{64}{8r^3+36r^2+22r-21}$	$\frac{8}{3}$
b_4	$\frac{256}{16r^4+128r^3+280r^2+96r-135}$	-2
b_5	$-\frac{1024}{(2r+5)(2r-1)(3+2r)(4r^2+36r+77)}$	$\frac{16}{15}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned}y_2(x) &= x^{\frac{3}{2}}\left(b_0+b_1x+b_2x^2+b_3x^3+b_4x^4+b_5x^5+b_6x^6\dots\right)\\ &= \frac{1-2x^2+\frac{8x^3}{3}-2x^4+\frac{16x^5}{15}+O(x^6)}{\sqrt{x}}\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}y_h(x) &= c_1y_1(x)+c_2y_2(x)\\ &= c_1x^{\frac{3}{2}}\left(1-\frac{4x}{3}+x^2-\frac{8x^3}{15}+\frac{2x^4}{9}-\frac{8x^5}{105}+O(x^6)\right)\\ &\quad+\frac{c_2\left(1-2x^2+\frac{8x^3}{3}-2x^4+\frac{16x^5}{15}+O(x^6)\right)}{\sqrt{x}}\end{aligned}$$

Hence the final solution is

$$\begin{aligned}y &= y_h\\ &= c_1x^{\frac{3}{2}}\left(1-\frac{4x}{3}+x^2-\frac{8x^3}{15}+\frac{2x^4}{9}-\frac{8x^5}{105}+O(x^6)\right)\\ &\quad+\frac{c_2\left(1-2x^2+\frac{8x^3}{3}-2x^4+\frac{16x^5}{15}+O(x^6)\right)}{\sqrt{x}}\end{aligned}$$

Summary
The solution(s) found are the following

$$
y = c_1 x^{\frac{3}{2}} \left(1 - \frac{4x}{3} + x^2 - \frac{8x^3}{15} + \frac{2x^4}{9} - \frac{8x^5}{105} + O(x^6)\right) + \frac{c_2 \left(1 - 2x^2 + \frac{8x^3}{3} - 2x^4 + \frac{16x^5}{15} + O(x^6)\right)}{\sqrt{x}} \tag{1}
$$

Verification of solutions

$$
y = c_1 x^{\frac{3}{2}} \left(1 - \frac{4x}{3} + x^2 - \frac{8x^3}{15} + \frac{2x^4}{9} - \frac{8x^5}{105} + O(x^6)\right) + \frac{c_2 \left(1 - 2x^2 + \frac{8x^3}{3} - 2x^4 + \frac{16x^5}{15} + O(x^6)\right)}{\sqrt{x}}
$$

Verified OK.

18.13.1 Maple step by step solution

Let's solve
$x^2 y'' + 2y' x^2 + \left(x - \frac{3}{4}\right) y = 0$

- Highest derivative means the order of the ODE is 2
  y''
- Isolate 2nd derivative
  $y'' = -\frac{y(4x-3)}{4x^2} - 2y'$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear
  $y'' + 2y' + \frac{y(4x-3)}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(x) = 2, P_3(x) = \frac{4x-3}{4x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$
    $\left.(x \cdot P_2(x))\right|_{x=0} = 0$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$
    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -\frac{3}{4}$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2y'' + 8y'x^2 + y(4x-3) = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k-m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^2 \cdot y'$ to series expansion

    $x^2 \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r+1}$

  - Shift index using $k- >k-1$

    $x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1}(k-1+r)\, x^{k+r}$

  - Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k-1+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)\,(-3+2r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r+1)\,(2k+2r-3) + 4a_{k-1}(2k-1+2r)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)\,(-3+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{3}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

$$4\left(k+r+\frac{1}{2}\right)\left(k+r-\frac{3}{2}\right)a_k+8\left(k-\frac{1}{2}+r\right)a_{k-1}=0$$

- Shift index using $k->k+1$

$$4\left(k+\frac{3}{2}+r\right)\left(k-\frac{1}{2}+r\right)a_{k+1}+8\left(k+r+\frac{1}{2}\right)a_k=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1}=-\frac{4(2k+2r+1)a_k}{(2k+3+2r)(2k-1+2r)}$$

- Recursion relation for $r=-\frac{1}{2}$

$$a_{k+1}=-\frac{8ka_k}{(2k+2)(2k-2)}$$

- Series not valid for $r=-\frac{1}{2}$, division by 0 in the recursion relation at $k=1$

$$a_{k+1}=-\frac{8ka_k}{(2k+2)(2k-2)}$$

- Recursion relation for $r=\frac{3}{2}$

$$a_{k+1}=-\frac{4(2k+4)a_k}{(2k+6)(2k+2)}$$

- Solution for $r=\frac{3}{2}$

$$\left[y=\sum_{k=0}^{\infty}a_k x^{k+\frac{3}{2}},a_{k+1}=-\frac{4(2k+4)a_k}{(2k+6)(2k+2)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 45

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+2*x^2*diff(y(x),x)+(x-3/4)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_1 x^2 \left(1 - \frac{4}{3}x + x^2 - \frac{8}{15}x^3 + \frac{2}{9}x^4 - \frac{8}{105}x^5 + \mathrm{O}\left(x^6\right)\right) + c_2\left(-2 + 4x^2 - \frac{16}{3}x^3 + 4x^4 - \frac{32}{15}x^5 + \mathrm{O}\left(x^6\right)\right)}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.033 (sec). Leaf size: 77

```
AsymptoticDSolveValue[x^2*y''[x]+2*x^2*y'[x]+(x-3/4)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \rightarrow c_1\left(-2x^{7/2} + \frac{8x^{5/2}}{3} - 2x^{3/2} + \frac{1}{\sqrt{x}}\right) + c_2\left(\frac{2x^{11/2}}{9} - \frac{8x^{9/2}}{15} + x^{7/2} - \frac{4x^{5/2}}{3} + x^{3/2}\right)$$

18.14 problem 6

Internal problem ID [2950]
Internal file name [OUTPUT/2442_Sunday_June_05_2022_03_11_19_AM_45498510/index.tex]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.5. page 771
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + xy' + (2x-1)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + xy' + (2x-1)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= \frac{2x-1}{x^2} \end{aligned}$$

Table 477: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{2x-1}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + xy' + (2x - 1)\,y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) + x\left(\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\right) + (2x-1)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 2x^{1+n+r} a_n\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2x^{1+n+r} a_n = \sum_{n=1}^{\infty} 2a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \left(\sum_{n=1}^{\infty} 2a_{n-1} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) + x^{n+r} a_n (n+r) - a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) + x^r a_0 r - a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) + x^r r - x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2-1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$
\begin{aligned}
r_1 &= 1\\
r_2 &= -1
\end{aligned}
$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$
(r^2 - 1)\, x^r = 0
$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 2$ is an integer, then we can construct two linearly independent solutions

$$
\begin{aligned}
y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)\\
y_2(x) &= C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= x \left(\sum_{n=0}^{\infty} a_n x^n\right)\\
y_2(x) &= C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x}
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^{1+n}\\
y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-1}\right)
\end{aligned}
$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$
a_n(n+r)(n+r-1) + a_n(n+r) + 2a_{n-1} - a_n = 0 \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$
a_n = -\frac{2a_{n-1}}{n^2 + 2nr + r^2 - 1} \tag{4}
$$

Which for the root $r = 1$ becomes

$$a_n = -\frac{2a_{n-1}}{n(n+2)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{2}{r(r+2)}$$

Which for the root $r = 1$ becomes

$$a_1 = -\frac{2}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{r(r+2)}$	$-\frac{2}{3}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{4}{r(r+2)(3+r)(r+1)}$$

Which for the root $r = 1$ becomes

$$a_2 = \frac{1}{6}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{r(r+2)}$	$-\frac{2}{3}$
a_2	$\frac{4}{r(r+2)(3+r)(r+1)}$	$\frac{1}{6}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{8}{r(r+2)^2(3+r)(r+1)(r+4)}$$

Which for the root $r = 1$ becomes

$$a_3 = -\frac{1}{45}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{r(r+2)}$	$-\frac{2}{3}$
a_2	$\frac{4}{r(r+2)(3+r)(r+1)}$	$\frac{1}{6}$
a_3	$-\frac{8}{r(r+2)^2(3+r)(r+1)(r+4)}$	$-\frac{1}{45}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{16}{r(r+2)^2(3+r)^2(r+1)(r+4)(5+r)}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{1}{540}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{r(r+2)}$	$-\frac{2}{3}$
a_2	$\frac{4}{r(r+2)(3+r)(r+1)}$	$\frac{1}{6}$
a_3	$-\frac{8}{r(r+2)^2(3+r)(r+1)(r+4)}$	$-\frac{1}{45}$
a_4	$\frac{16}{r(r+2)^2(3+r)^2(r+1)(r+4)(5+r)}$	$\frac{1}{540}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{32}{r(r+2)^2(3+r)^2(r+1)(r+4)^2(5+r)(6+r)}$$

Which for the root $r = 1$ becomes

$$a_5 = -\frac{1}{9450}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{r(r+2)}$	$-\frac{2}{3}$
a_2	$\frac{4}{r(r+2)(3+r)(r+1)}$	$\frac{1}{6}$
a_3	$-\frac{8}{r(r+2)^2(3+r)(r+1)(r+4)}$	$-\frac{1}{45}$
a_4	$\frac{16}{r(r+2)^2(3+r)^2(r+1)(r+4)(5+r)}$	$\frac{1}{540}$
a_5	$-\frac{32}{r(r+2)^2(3+r)^2(r+1)(r+4)^2(5+r)(6+r)}$	$-\frac{1}{9450}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}
y_1(x) &= x\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots\right) \\
&= x\left(1 - \frac{2x}{3} + \frac{x^2}{6} - \frac{x^3}{45} + \frac{x^4}{540} - \frac{x^5}{9450} + O(x^6)\right)
\end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 2$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_2(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned}
a_N &= a_2 \\
&= \frac{4}{r\left(r+2\right)\left(3+r\right)\left(r+1\right)}
\end{aligned}$$

Therefore

$$\begin{aligned}
\lim_{r\to r_2} \frac{4}{r\left(r+2\right)\left(3+r\right)\left(r+1\right)} &= \lim_{r\to -1} \frac{4}{r\left(r+2\right)\left(3+r\right)\left(r+1\right)} \\
&= \textit{undefined}
\end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$
y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)
$$

Therefore

$$
\begin{aligned}
\frac{d}{dx} y_2(x) &= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\
&= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) \\
\frac{d^2}{dx^2} y_2(x) &= C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\
&= C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right)
\end{aligned}
$$

Substituting these back into the given ode $x^2 y'' + x y' + (2x-1) y = 0$ gives

$$
\begin{aligned}
& x^2 \left(C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} \right. \\
& \left. + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) \\
& + x \left(C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right) \\
& + (2x-1) \left(C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)\right) = 0
\end{aligned}
$$

Which can be written as

$$\left(\left(\left(x^{2} y_{1}''(x)+y_{1}'(x)\, x+(2 x-1)\, y_{1}(x)\right) \ln (x)+x^{2}\left(\frac{2 y_{1}'(x)}{x}-\frac{y_{1}(x)}{x^{2}}\right)+y_{1}(x)\right) C \right.$$
$$+x^{2}\left(\sum_{n=0}^{\infty}\left(\frac{b_{n} x^{n+r_{2}}\left(n+r_{2}\right)^{2}}{x^{2}}-\frac{b_{n} x^{n+r_{2}}\left(n+r_{2}\right)}{x^{2}}\right)\right) \tag{7}$$
$$+x\left(\sum_{n=0}^{\infty} \frac{b_{n} x^{n+r_{2}}\left(n+r_{2}\right)}{x}\right)+(2 x-1)\left(\sum_{n=0}^{\infty} b_{n} x^{n+r_{2}}\right)=0$$

But since $y_1(x)$ is a solution to the ode, then

$$x^{2} y_{1}''(x)+y_{1}'(x)\, x+(2 x-1)\, y_{1}(x)=0$$

Eq (7) simplifes to

$$\left(x^{2}\left(\frac{2 y_{1}'(x)}{x}-\frac{y_{1}(x)}{x^{2}}\right)+y_{1}(x)\right) C$$
$$+x^{2}\left(\sum_{n=0}^{\infty}\left(\frac{b_{n} x^{n+r_{2}}\left(n+r_{2}\right)^{2}}{x^{2}}-\frac{b_{n} x^{n+r_{2}}\left(n+r_{2}\right)}{x^{2}}\right)\right) \tag{8}$$
$$+x\left(\sum_{n=0}^{\infty} \frac{b_{n} x^{n+r_{2}}\left(n+r_{2}\right)}{x}\right)+(2 x-1)\left(\sum_{n=0}^{\infty} b_{n} x^{n+r_{2}}\right)=0$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\left(\sum_{n=0}^{\infty} x^{-2+n+r_{2}} b_{n}\left(n+r_{2}\right)\left(-1+n+r_{2}\right)\right) x^{2}+2 x\left(\sum_{n=0}^{\infty} x^{-1+n+r_{1}} a_{n}\left(n+r_{1}\right)\right) C$$
$$+\left(\sum_{n=0}^{\infty} x^{-1+n+r_{2}} b_{n}\left(n+r_{2}\right)\right) x+(2 x-1)\left(\sum_{n=0}^{\infty} b_{n} x^{n+r_{2}}\right)=0 \tag{9}$$

Since $r_1 = 1$ and $r_2 = -1$ then the above becomes

$$\left(\sum_{n=0}^{\infty} x^{-3+n} b_{n}(n-1)(-2+n)\right) x^{2}+2 x\left(\sum_{n=0}^{\infty} x^{n} a_{n}(1+n)\right) C$$
$$+\left(\sum_{n=0}^{\infty} x^{-2+n} b_{n}(n-1)\right) x+(2 x-1)\left(\sum_{n=0}^{\infty} b_{n} x^{n-1}\right)=0 \tag{10}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n-1} b_n (n^2 - 3n + 2)\right) + \left(\sum_{n=0}^{\infty} 2C\, x^{1+n} a_n (1+n)\right) \\ + \left(\sum_{n=0}^{\infty} x^{n-1} b_n (n-1)\right) + \left(\sum_{n=0}^{\infty} 2b_n x^n\right) + \sum_{n=0}^{\infty} \left(-b_n x^{n-1}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2C\, x^{1+n} a_n (1+n) = \sum_{n=2}^{\infty} 2C a_{-2+n} (n-1)\, x^{n-1}$$

$$\sum_{n=0}^{\infty} 2b_n x^n = \sum_{n=1}^{\infty} 2b_{n-1} x^{n-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-1$.

$$\left(\sum_{n=0}^{\infty} x^{n-1} b_n (n^2 - 3n + 2)\right) + \left(\sum_{n=2}^{\infty} 2C a_{-2+n} (n-1)\, x^{n-1}\right) \\ + \left(\sum_{n=0}^{\infty} x^{n-1} b_n (n-1)\right) + \left(\sum_{n=1}^{\infty} 2b_{n-1} x^{n-1}\right) + \sum_{n=0}^{\infty} \left(-b_n x^{n-1}\right) = 0 \tag{2B}$$

For $n = 0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0 = 1$. For $n = 1$, Eq (2B) gives

$$2b_0 - b_1 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$2 - b_1 = 0$$

Solving the above for b_1 gives

$$b_1 = 2$$

For $n = N$, where $N = 2$ which is the difference between the two roots, we are free to choose $b_2 = 0$. Hence for $n = 2$, Eq (2B) gives

$$2C + 4 = 0$$

Which is solved for C. Solving for C gives

$$C = -2$$

For $n = 3$, Eq (2B) gives

$$4Ca_1 + 2b_2 + 3b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$3b_3 + \frac{16}{3} = 0$$

Solving the above for b_3 gives

$$b_3 = -\frac{16}{9}$$

For $n = 4$, Eq (2B) gives

$$6Ca_2 + 2b_3 + 8b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$8b_4 - \frac{50}{9} = 0$$

Solving the above for b_4 gives

$$b_4 = \frac{25}{36}$$

For $n = 5$, Eq (2B) gives

$$8Ca_3 + 2b_4 + 15b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$15b_5 + \frac{157}{90} = 0$$

Solving the above for b_5 gives

$$b_5 = -\frac{157}{1350}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -2$ and all b_n, then the second solution becomes

$$y_2(x) = (-2)\left(x\left(1 - \frac{2x}{3} + \frac{x^2}{6} - \frac{x^3}{45} + \frac{x^4}{540} - \frac{x^5}{9450} + O(x^6)\right)\right)\ln(x) + \frac{1 + 2x - \frac{16x^3}{9} + \frac{25x^4}{36} - \frac{157x^5}{1350} + O(x^6)}{x}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x \left(1 - \frac{2x}{3} + \frac{x^2}{6} - \frac{x^3}{45} + \frac{x^4}{540} - \frac{x^5}{9450} + O(x^6)\right) \\
&\quad + c_2 \left((-2) \left(x \left(1 - \frac{2x}{3} + \frac{x^2}{6} - \frac{x^3}{45} + \frac{x^4}{540} - \frac{x^5}{9450} + O(x^6)\right)\right) \ln(x) \right. \\
&\qquad \left. + \frac{1 + 2x - \frac{16x^3}{9} + \frac{25x^4}{36} - \frac{157x^5}{1350} + O(x^6)}{x}\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x \left(1 - \frac{2x}{3} + \frac{x^2}{6} - \frac{x^3}{45} + \frac{x^4}{540} - \frac{x^5}{9450} + O(x^6)\right) \\
&\quad + c_2 \left(-2x \left(1 - \frac{2x}{3} + \frac{x^2}{6} - \frac{x^3}{45} + \frac{x^4}{540} - \frac{x^5}{9450} + O(x^6)\right) \ln(x) \right. \\
&\qquad \left. + \frac{1 + 2x - \frac{16x^3}{9} + \frac{25x^4}{36} - \frac{157x^5}{1350} + O(x^6)}{x}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x \left(1 - \frac{2x}{3} + \frac{x^2}{6} - \frac{x^3}{45} + \frac{x^4}{540} - \frac{x^5}{9450} + O(x^6)\right) \\
&\quad + c_2 \left(-2x \left(1 - \frac{2x}{3} + \frac{x^2}{6} - \frac{x^3}{45} + \frac{x^4}{540} - \frac{x^5}{9450} + O(x^6)\right) \ln(x) \right. \\
&\qquad \left. + \frac{1 + 2x - \frac{16x^3}{9} + \frac{25x^4}{36} - \frac{157x^5}{1350} + O(x^6)}{x}\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = c_1 x \left(1-\frac{2x}{3}+\frac{x^2}{6}-\frac{x^3}{45}+\frac{x^4}{540}-\frac{x^5}{9450}+O(x^6)\right) \\
+ c_2 \left(-2x \left(1-\frac{2x}{3}+\frac{x^2}{6}-\frac{x^3}{45}+\frac{x^4}{540}-\frac{x^5}{9450}+O(x^6)\right)\ln(x) \right. \\
\left. + \frac{1+2x-\frac{16x^3}{9}+\frac{25x^4}{36}-\frac{157x^5}{1350}+O(x^6)}{x}\right)
\end{aligned}
$$

Verified OK.

18.14.1 Maple step by step solution

Let's solve

$x^2 y'' + x y' + (2x-1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y'}{x} - \frac{(2x-1)y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{(2x-1)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{2x-1}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = -1$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

$x^2y'' + xy' + (2x-1)\,y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- ◦ Shift index using $k- > k-m$

$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- ◦ Convert $x \cdot y'$ to series expansion

$x \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r}$

- ◦ Convert $x^2 \cdot y''$ to series expansion

$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

Rewrite ODE with series expansions

$a_0(1+r)\,(-1+r)\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)\,(k+r-1) + 2a_{k-1}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$(1+r)\,(-1+r) = 0$

- Values of r that satisfy the indicial equation

$r \in \{-1, 1\}$

- Each term in the series must be 0, giving the recursion relation

$a_k(k+r+1)\,(k+r-1) + 2a_{k-1} = 0$

- Shift index using $k- > k+1$

$a_{k+1}(k+2+r)\,(k+r) + 2a_k = 0$

- Recursion relation that defines series solution to ODE

$a_{k+1} = -\frac{2a_k}{(k+2+r)(k+r)}$

- Recursion relation for $r = -1$

$a_{k+1} = -\frac{2a_k}{(k+1)(k-1)}$

- Series not valid for $r = -1$, division by 0 in the recursion relation at $k = 1$

  $a_{k+1} = -\frac{2a_k}{(k+1)(k-1)}$

- Recursion relation for $r = 1$

  $a_{k+1} = -\frac{2a_k}{(k+3)(k+1)}$

- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = -\frac{2a_k}{(k+3)(k+1)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.046 (sec). Leaf size: 63

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+(2*x-1)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_1 x^2 \left(1 - \frac{2}{3}x + \frac{1}{6}x^2 - \frac{1}{45}x^3 + \frac{1}{540}x^4 - \frac{1}{9450}x^5 + \mathrm{O}\left(x^6\right)\right) + c_2 \left(\ln\left(x\right)\left(4x^2 - \frac{8}{3}x^3 + \frac{2}{3}x^4 - \frac{4}{45}x^5 + \mathrm{O}\left(x^6\right)\right) +}{x}$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 83

```
AsymptoticDSolveValue[x^2*y''[x]+x*y'[x]+(2*x-1)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{31x^4 - 88x^3 + 36x^2 + 72x + 36}{36x} - \frac{1}{3}x(x^2 - 4x + 6)\log(x)\right) + c_2\left(\frac{x^5}{540} - \frac{x^4}{45} + \frac{x^3}{6} - \frac{2x^2}{3} + x\right)$$

18.15 problem 7

Internal problem ID [2951]
Internal file name [`OUTPUT/2443_Sunday_June_05_2022_03_11_25_AM_44152525/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.5. page 771
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + y'x^3 - (x+2)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + y'x^3 + (-2-x)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = x$$
$$q(x) = -\frac{x+2}{x^2}$$

Table 479: Table $p(x), q(x)$ singularites.

$p(x) = x$	
singularity	type
$x = \infty$	"regular"
$x = -\infty$	"regular"

$q(x) = -\frac{x+2}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[\infty, -\infty, 0]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + y'x^3 + (-2 - x)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$
$$y'' = \sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2}\right) + \left(\sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}\right) x^3 + (-2 - x) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty} x^{2+n+r} a_n(n+r)\right) \\ +\sum_{n=0}^{\infty}\left(-2 a_n x^{n+r}\right)+\sum_{n=0}^{\infty}\left(-x^{1+n+r} a_n\right)=0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{2+n+r} a_n(n+r)=\sum_{n=2}^{\infty} a_{n-2}(n+r-2) x^{n+r}$$

$$\sum_{n=0}^{\infty}\left(-x^{1+n+r} a_n\right)=\sum_{n=1}^{\infty}\left(-a_{n-1} x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)(n+r-1)\right)+\left(\sum_{n=2}^{\infty} a_{n-2}(n+r-2) x^{n+r}\right) \\ +\sum_{n=0}^{\infty}\left(-2 a_n x^{n+r}\right)+\sum_{n=1}^{\infty}\left(-a_{n-1} x^{n+r}\right)=0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n(n+r)(n+r-1)-2 a_n x^{n+r}=0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r)-2 a_0 x^r=0$$

Or

$$\left(x^r r(-1+r)-2 x^r\right) a_0=0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2-r-2\right) x^r=0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - r - 2 = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= 2 \\ r_2 &= -1 \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(r^2 - r - 2\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 3$ is an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right) \\ y_2(x) &= C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= x^2 \left(\sum_{n=0}^{\infty} a_n x^n \right) \\ y_2(x) &= C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x} \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+2} \\ y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-1} \right) \end{aligned}$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = \frac{1}{r^2 + r - 2}$$

For $2 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1)+a_{n-2}(n+r-2)-2a_n-a_{n-1}=0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n=-\frac{n a_{n-2}+r a_{n-2}-2a_{n-2}-a_{n-1}}{n^2+2nr+r^2-n-r-2} \tag{4}$$

Which for the root $r=2$ becomes

$$a_n=\frac{-n a_{n-2}+a_{n-1}}{n(n+3)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r=2$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{r^2+r-2}$	$\frac{1}{4}$

For $n=2$, using the above recursive equation gives

$$a_2=\frac{-r^3-r^2+2r+1}{(r^2+r-2)\,r\,(r+3)}$$

Which for the root $r=2$ becomes

$$a_2=-\frac{7}{40}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{r^2+r-2}$	$\frac{1}{4}$
a_2	$\frac{-r^3-r^2+2r+1}{(r^2+r-2)r(r+3)}$	$-\frac{7}{40}$

For $n=3$, using the above recursive equation gives

$$a_3=\frac{-2r^3-5r^2-r+1}{r^6+9r^5+25r^4+15r^3-26r^2-24r}$$

Which for the root $r = 2$ becomes

$$a_3 = -\frac{37}{720}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{r^2+r-2}$	$\frac{1}{4}$
a_2	$\frac{-r^3-r^2+2r+1}{(r^2+r-2)r(r+3)}$	$-\frac{7}{40}$
a_3	$\frac{-2r^3-5r^2-r+1}{r^6+9r^5+25r^4+15r^3-26r^2-24r}$	$-\frac{37}{720}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{r^6 + 8r^5 + 19r^4 + 5r^3 - 32r^2 - 31r - 7}{(5 + r)(r + 2)^2 r(-1 + r)(r + 4)(r + 3)(1 + r)}$$

Which for the root $r = 2$ becomes

$$a_4 = \frac{467}{20160}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{r^2+r-2}$	$\frac{1}{4}$
a_2	$\frac{-r^3-r^2+2r+1}{(r^2+r-2)r(r+3)}$	$-\frac{7}{40}$
a_3	$\frac{-2r^3-5r^2-r+1}{r^6+9r^5+25r^4+15r^3-26r^2-24r}$	$-\frac{37}{720}$
a_4	$\frac{r^6+8r^5+19r^4+5r^3-32r^2-31r-7}{(5+r)(r+2)^2r(-1+r)(r+4)(r+3)(1+r)}$	$\frac{467}{20160}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{3r^6 + 33r^5 + 132r^4 + 229r^3 + 139r^2 - 32r - 37}{(6 + r)(r + 3)^2 (5 + r)(r + 2)^2 r(r + 4)(-1 + r)(1 + r)}$$

Which for the root $r = 2$ becomes

$$a_5 = \frac{5647}{806400}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{r^2+r-2}$	$\frac{1}{4}$
a_2	$\frac{-r^3-r^2+2r+1}{(r^2+r-2)r(r+3)}$	$-\frac{7}{40}$
a_3	$\frac{-2r^3-5r^2-r+1}{r^6+9r^5+25r^4+15r^3-26r^2-24r}$	$-\frac{37}{720}$
a_4	$\frac{r^6+8r^5+19r^4+5r^3-32r^2-31r-7}{(5+r)(r+2)^2r(-1+r)(r+4)(r+3)(1+r)}$	$\frac{467}{20160}$
a_5	$\frac{3r^6+33r^5+132r^4+229r^3+139r^2-32r-37}{(6+r)(r+3)^2(5+r)(r+2)^2r(r+4)(-1+r)(1+r)}$	$\frac{5647}{806400}$

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x^2\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots\right) \\
&= x^2\left(1 + \frac{x}{4} - \frac{7x^2}{40} - \frac{37x^3}{720} + \frac{467x^4}{20160} + \frac{5647x^5}{806400} + O(x^6)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Let

$$
r_1 - r_2 = N
$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 3$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_3(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$
\begin{aligned}
a_N &= a_3 \\
&= \frac{-2r^3 - 5r^2 - r + 1}{r^6 + 9r^5 + 25r^4 + 15r^3 - 26r^2 - 24r}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\lim_{r\to r_2} \frac{-2r^3 - 5r^2 - r + 1}{r^6 + 9r^5 + 25r^4 + 15r^3 - 26r^2 - 24r} &= \lim_{r\to -1} \frac{-2r^3 - 5r^2 - r + 1}{r^6 + 9r^5 + 25r^4 + 15r^3 - 26r^2 - 24r} \\
&= \textit{undefined}
\end{aligned}
$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$
y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)
$$

Therefore

$$
\begin{aligned}
\frac{d}{dx}y_2(x) &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\
&= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right) \\
\frac{d^2}{dx^2}y_2(x) &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\
&= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right)
\end{aligned}
$$

Substituting these back into the given ode $x^2y'' + y'x^3 + (-2-x)\,y = 0$ gives

$$
\begin{aligned}
& x^2\left(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2}\right. \\
&\left. + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) \\
& + \left(Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right)x^3 \\
& + (-2-x)\left(Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)\right) = 0
\end{aligned}
$$

Which can be written as

$$
\begin{aligned}
& \left(\left(x^2y_1''(x) + y_1'(x)\,x^3 + (-2-x)\,y_1(x)\right)\ln(x) + x^2\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right)\right. \\
&\left. + y_1(x)\,x^2\right)C + x^2\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) \\
& + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right)x^3 + (-2-x)\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned}
\tag{7}
$$

But since $y_1(x)$ is a solution to the ode, then

$$x^2 y_1''(x) + y_1'(x)\, x^3 + (-2 - x)\, y_1(x) = 0$$

Eq (7) simplifes to

$$\begin{aligned} &\left(x^2\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right) + y_1(x)\, x^2\right) C \\ &+ x^2\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) \\ &+ \left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right) x^3 + (-2 - x)\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0 \end{aligned} \tag{8}$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\begin{aligned} &\left(2\left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n(n+r_1)\right) x + \left(x^2 - 1\right)\left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right)\right) C \\ &+ \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)\,(-1+n+r_2)\right) x^2 \\ &+ \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right) x^3 - (x+2)\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0 \end{aligned} \tag{9}$$

Since $r_1 = 2$ and $r_2 = -1$ then the above becomes

$$\begin{aligned} &\left(2\left(\sum_{n=0}^{\infty} x^{1+n} a_n(n+2)\right) x + \left(x^2 - 1\right)\left(\sum_{n=0}^{\infty} a_n x^{n+2}\right)\right) C \\ &+ \left(\sum_{n=0}^{\infty} x^{-3+n} b_n(n-1)\,(n-2)\right) x^2 \\ &+ \left(\sum_{n=0}^{\infty} x^{n-2} b_n(n-1)\right) x^3 - (x+2)\left(\sum_{n=0}^{\infty} b_n x^{n-1}\right) = 0 \end{aligned} \tag{10}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} 2C\, x^{n+2} a_n (n+2)\right) + \left(\sum_{n=0}^{\infty} C\, x^{n+4} a_n\right) \\
&+ \sum_{n=0}^{\infty} \left(-C\, x^{n+2} a_n\right) + \left(\sum_{n=0}^{\infty} x^{n-1} b_n \left(n^2 - 3n + 2\right)\right) \\
&+ \left(\sum_{n=0}^{\infty} x^{1+n} b_n (n-1)\right) + \sum_{n=0}^{\infty} \left(-b_n x^n\right) + \sum_{n=0}^{\infty} \left(-2 b_n x^{n-1}\right) = 0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-1} and adjusting the power and the corresponding index gives

$$
\sum_{n=0}^{\infty} 2C\, x^{n+2} a_n (n+2) = \sum_{n=3}^{\infty} 2C a_{-3+n} (n-1)\, x^{n-1}
$$

$$
\sum_{n=0}^{\infty} C\, x^{n+4} a_n = \sum_{n=5}^{\infty} C a_{n-5} x^{n-1}
$$

$$
\sum_{n=0}^{\infty} \left(-C\, x^{n+2} a_n\right) = \sum_{n=3}^{\infty} \left(-C a_{-3+n} x^{n-1}\right)
$$

$$
\sum_{n=0}^{\infty} x^{1+n} b_n (n-1) = \sum_{n=2}^{\infty} b_{n-2} (-3+n)\, x^{n-1}
$$

$$
\sum_{n=0}^{\infty} \left(-b_n x^n\right) = \sum_{n=1}^{\infty} \left(-b_{n-1} x^{n-1}\right)
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-1$.

$$
\begin{aligned}
&\left(\sum_{n=3}^{\infty} 2C a_{-3+n} (n-1)\, x^{n-1}\right) + \left(\sum_{n=5}^{\infty} C a_{n-5} x^{n-1}\right) \\
&+ \sum_{n=3}^{\infty} \left(-C a_{-3+n} x^{n-1}\right) + \left(\sum_{n=0}^{\infty} x^{n-1} b_n \left(n^2 - 3n + 2\right)\right) \\
&+ \left(\sum_{n=2}^{\infty} b_{n-2} (-3+n)\, x^{n-1}\right) + \sum_{n=1}^{\infty} \left(-b_{n-1} x^{n-1}\right) + \sum_{n=0}^{\infty} \left(-2 b_n x^{n-1}\right) = 0
\end{aligned} \tag{2B}
$$

For $n = 0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0 = 1$. For $n = 1$, Eq (2B) gives

$$-b_0 - 2b_1 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-1 - 2b_1 = 0$$

Solving the above for b_1 gives

$$b_1 = -\frac{1}{2}$$

For $n = 2$, Eq (2B) gives

$$-b_0 - b_1 - 2b_2 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-\frac{1}{2} - 2b_2 = 0$$

Solving the above for b_2 gives

$$b_2 = -\frac{1}{4}$$

For $n = N$, where $N = 3$ which is the difference between the two roots, we are free to choose $b_3 = 0$. Hence for $n = 3$, Eq (2B) gives

$$3C + \frac{1}{4} = 0$$

Which is solved for C. Solving for C gives

$$C = -\frac{1}{12}$$

For $n = 4$, Eq (2B) gives

$$5Ca_1 + b_2 - b_3 + 4b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$4b_4 - \frac{17}{48} = 0$$

Solving the above for b_4 gives

$$b_4 = \frac{17}{192}$$

For $n = 5$, Eq (2B) gives

$$(a_0 + 7a_2)\,C + 2b_3 - b_4 + 10b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-\frac{67}{960}+10b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = \frac{67}{9600}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -\frac{1}{12}$ and all b_n, then the second solution becomes

$$\begin{aligned}
y_2(x) = &-\frac{1}{12}\left(x^2\left(1+\frac{x}{4}-\frac{7x^2}{40}-\frac{37x^3}{720}+\frac{467x^4}{20160}+\frac{5647x^5}{806400}+O(x^6)\right)\right)\ln(x) \\
&+\frac{1-\frac{x}{2}-\frac{x^2}{4}+\frac{17x^4}{192}+\frac{67x^5}{9600}+O(x^6)}{x}
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1y_1(x)+c_2y_2(x) \\
&= c_1x^2\left(1+\frac{x}{4}-\frac{7x^2}{40}-\frac{37x^3}{720}+\frac{467x^4}{20160}+\frac{5647x^5}{806400}+O(x^6)\right) \\
&\quad + c_2\left(-\frac{1}{12}\left(x^2\left(1+\frac{x}{4}-\frac{7x^2}{40}-\frac{37x^3}{720}+\frac{467x^4}{20160}+\frac{5647x^5}{806400}+O(x^6)\right)\right)\ln(x)\right. \\
&\quad\left. +\frac{1-\frac{x}{2}-\frac{x^2}{4}+\frac{17x^4}{192}+\frac{67x^5}{9600}+O(x^6)}{x}\right)
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= c_1x^2\left(1+\frac{x}{4}-\frac{7x^2}{40}-\frac{37x^3}{720}+\frac{467x^4}{20160}+\frac{5647x^5}{806400}+O(x^6)\right) \\
&\quad + c_2\left(-\frac{x^2\left(1+\frac{x}{4}-\frac{7x^2}{40}-\frac{37x^3}{720}+\frac{467x^4}{20160}+\frac{5647x^5}{806400}+O(x^6)\right)\ln(x)}{12}\right. \\
&\quad\left. +\frac{1-\frac{x}{2}-\frac{x^2}{4}+\frac{17x^4}{192}+\frac{67x^5}{9600}+O(x^6)}{x}\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = c_1 x^2 \left(1+\frac{x}{4}-\frac{7x^2}{40}-\frac{37x^3}{720}+\frac{467x^4}{20160}+\frac{5647x^5}{806400}+O(x^6)\right) \\
+ c_2 \left(-\frac{x^2\left(1+\frac{x}{4}-\frac{7x^2}{40}-\frac{37x^3}{720}+\frac{467x^4}{20160}+\frac{5647x^5}{806400}+O(x^6)\right)\ln(x)}{12}\right. \\
\left. + \frac{1-\frac{x}{2}-\frac{x^2}{4}+\frac{17x^4}{192}+\frac{67x^5}{9600}+O(x^6)}{x}\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = c_1 x^2 \left(1+\frac{x}{4}-\frac{7x^2}{40}-\frac{37x^3}{720}+\frac{467x^4}{20160}+\frac{5647x^5}{806400}+O(x^6)\right) \\
+ c_2 \left(-\frac{x^2\left(1+\frac{x}{4}-\frac{7x^2}{40}-\frac{37x^3}{720}+\frac{467x^4}{20160}+\frac{5647x^5}{806400}+O(x^6)\right)\ln(x)}{12}\right. \\
\left. + \frac{1-\frac{x}{2}-\frac{x^2}{4}+\frac{17x^4}{192}+\frac{67x^5}{9600}+O(x^6)}{x}\right)
\end{aligned}
$$

Verified OK.

18.15.1 Maple step by step solution

Let's solve

$x^2 y'' + y' x^3 + (-2-x)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -xy' + \frac{(x+2)y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + xy' - \frac{(x+2)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = x, P_3(x) = -\frac{x+2}{x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 0$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -2$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2y'' + y'x^3 + (-2 - x)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^3 \cdot y'$ to series expansion

  $x^3 \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r+2}$

- Shift index using $k- >k - 2$

  $x^3 \cdot y' = \sum_{k=2}^{\infty} a_{k-2}(k - 2 + r)\, x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1 + r)\,(-2 + r)\, x^r + (a_1(2 + r)\,(-1 + r) - a_0)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k + r + 1)\,(k - 2 + r) - a_{k-1}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)(-2+r)=0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 2\}$

- Each term must be 0

  $a_1(2+r)(-1+r)-a_0=0$

- Solve for the dependent coefficient(s)

  $a_1 = \frac{a_0}{r^2+r-2}$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+1)(k-2+r)+a_{k-2}k+a_{k-2}r-2a_{k-2}-a_{k-1}=0$

- Shift index using $k->k+2$

  $a_{k+2}(k+3+r)(k+r)+a_k(k+2)+ra_k-2a_k-a_{k+1}=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{ka_k+ra_k-a_{k+1}}{(k+3+r)(k+r)}$

- Recursion relation for $r=-1$

  $a_{k+2} = -\frac{ka_k-a_k-a_{k+1}}{(k+2)(k-1)}$

- Series not valid for $r=-1$, division by 0 in the recursion relation at $k=1$

  $a_{k+2} = -\frac{ka_k-a_k-a_{k+1}}{(k+2)(k-1)}$

- Recursion relation for $r=2$

  $a_{k+2} = -\frac{ka_k+2a_k-a_{k+1}}{(k+5)(k+2)}$

- Solution for $r=2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = -\frac{ka_k+2a_k-a_{k+1}}{(k+5)(k+2)}, a_1 = \frac{a_0}{4}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
<- Heun successful: received ODE is equivalent to the  HeunB  ODE, case  c = 0 `
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 65

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+x^3*diff(y(x),x)-(2+x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_1 x^3 \left(1 + \frac{1}{4}x - \frac{7}{40}x^2 - \frac{37}{720}x^3 + \frac{467}{20160}x^4 + \frac{5647}{806400}x^5 + \mathrm{O}\left(x^6\right)\right) + c_2 \left(\ln\left(x\right)\left(-x^3 - \frac{1}{4}x^4 + \frac{7}{40}x^5 + \mathrm{O}\left(x^6\right)\right) + \right.}{x}$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 82

```
AsymptoticDSolveValue[x^2*y''[x]+x^3*y'[x]-(2+x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{91x^4+160x^3-144x^2-288x+576}{576x}-\frac{1}{48}x^2(x+4)\log(x)\right)+c_2\left(\frac{467x^6}{20160}-\frac{37x^5}{720}-\frac{7x^4}{40}+\frac{x^3}{4}+x^2\right)$$

18.16 problem 8

Internal problem ID [2952]
Internal file name [`OUTPUT/2444_Sunday_June_05_2022_03_11_32_AM_19801593/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.5. page 771
Problem number: 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(x^2+1)\,y''+7y'x\,\mathrm{e}^x+9(1+\tan(x))\,y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(x^4+x^2)\,y''+7y'x\,\mathrm{e}^x+(9\tan(x)+9)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{7\,\mathrm{e}^x}{x\,(x^2+1)}$$
$$q(x)=\frac{9\tan(x)+9}{x^2\,(x^2+1)}$$

Table 481: Table $p(x), q(x)$ singularites.

$p(x) = \frac{7\,e^x}{x(x^2+1)}$	
singularity	type
$x = 0$	"regular"
$x = -i$	"regular"
$x = i$	"regular"
$x = \infty$	"regular"

$q(x) = \frac{9\tan(x)+9}{x^2(x^2+1)}$	
singularity	type
$x = 0$	"regular"
$x = -i$	"regular"
$x = i$	"regular"
$x = \frac{1}{2}\pi + Z\pi$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $\left[0, -i, i, \infty, \frac{1}{2}\pi + Z\pi\right]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2\left(x^2+1\right)y'' + 7y'x\,e^x + \left(9\tan\left(x\right)+9\right)y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned} &x^2\left(x^2+1\right)\left(\sum_{n=0}^{\infty}(n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) \\ &+ 7\left(\sum_{n=0}^{\infty}(n+r)\,a_n x^{n+r-1}\right)x\,e^x + \left(9\tan\left(x\right)+9\right)\left(\sum_{n=0}^{\infty}a_n x^{n+r}\right) = 0 \end{aligned} \tag{1}$$

Expanding $7x\,\mathrm{e}^{x}$ as Taylor series around $x = 0$ and keeping only the first 7 terms gives

$$
\begin{aligned}
7x\,\mathrm{e}^{x} &= 7x + 7x^{2} + \frac{7}{2}x^{3} + \frac{7}{6}x^{4} + \frac{7}{24}x^{5} + \frac{7}{120}x^{6} + \frac{7}{720}x^{7} + \dots \\
&= 7x + 7x^{2} + \frac{7}{2}x^{3} + \frac{7}{6}x^{4} + \frac{7}{24}x^{5} + \frac{7}{120}x^{6} + \frac{7}{720}x^{7}
\end{aligned}
$$

Expanding $9\tan(x) + 9$ as Taylor series around $x = 0$ and keeping only the first 7 terms gives

$$
\begin{aligned}
9\tan(x) + 9 &= 9 + 9x + 3x^{3} + \frac{6}{5}x^{5} + \frac{17}{35}x^{7} + \dots \\
&= 9 + 9x + 3x^{3} + \frac{6}{5}x^{5} + \frac{17}{35}x^{7}
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} x^{n+r+2} a_{n} (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_{n} (n+r)(n+r-1)\right) \\
&+ \left(\sum_{n=0}^{\infty} \frac{7x^{n+r+6} a_{n} (n+r)}{720}\right) + \left(\sum_{n=0}^{\infty} \frac{7x^{n+r+5} a_{n} (n+r)}{120}\right) \\
&+ \left(\sum_{n=0}^{\infty} \frac{7x^{n+r+4} a_{n} (n+r)}{24}\right) + \left(\sum_{n=0}^{\infty} \frac{7x^{n+r+3} a_{n} (n+r)}{6}\right) \\
&+ \left(\sum_{n=0}^{\infty} \frac{7x^{n+r+2} a_{n} (n+r)}{2}\right) + \left(\sum_{n=0}^{\infty} 7x^{1+n+r} a_{n} (n+r)\right) \\
&+ \left(\sum_{n=0}^{\infty} 7x^{n+r} a_{n} (n+r)\right) + \left(\sum_{n=0}^{\infty} 9a_{n} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 9x^{1+n+r} a_{n}\right) \\
&+ \left(\sum_{n=0}^{\infty} 3x^{n+r+3} a_{n}\right) + \left(\sum_{n=0}^{\infty} \frac{6x^{n+r+5} a_{n}}{5}\right) + \left(\sum_{n=0}^{\infty} \frac{17x^{n+r+7} a_{n}}{35}\right) = 0
\end{aligned}
\tag{2A}
$$

The next step is to make all powers of x be $n + r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\sum_{n=0}^{\infty} x^{n+r+2} a_{n} (n+r)(n+r-1) = \sum_{n=2}^{\infty} a_{n-2} (n+r-2)(n-3+r)\, x^{n+r}
$$

$$
\sum_{n=0}^{\infty} \frac{7x^{n+r+6} a_{n} (n+r)}{720} = \sum_{n=6}^{\infty} \frac{7a_{n-6} (n-6+r)\, x^{n+r}}{720}
$$

$$\sum_{n=0}^{\infty} \frac{7 x^{n+r+5} a_{n} \left(n+r\right)}{120} = \sum_{n=5}^{\infty} \frac{7 a_{n-5} \left(n-5+r\right) x^{n+r}}{120}$$

$$\sum_{n=0}^{\infty} \frac{7 x^{n+r+4} a_{n} \left(n+r\right)}{24} = \sum_{n=4}^{\infty} \frac{7 a_{n-4} \left(n-4+r\right) x^{n+r}}{24}$$

$$\sum_{n=0}^{\infty} \frac{7 x^{n+r+3} a_{n} \left(n+r\right)}{6} = \sum_{n=3}^{\infty} \frac{7 a_{n-3} \left(n-3+r\right) x^{n+r}}{6}$$

$$\sum_{n=0}^{\infty} \frac{7 x^{n+r+2} a_{n} \left(n+r\right)}{2} = \sum_{n=2}^{\infty} \frac{7 a_{n-2} \left(n+r-2\right) x^{n+r}}{2}$$

$$\sum_{n=0}^{\infty} 7 x^{1+n+r} a_{n} \left(n+r\right) = \sum_{n=1}^{\infty} 7 a_{n-1} \left(n+r-1\right) x^{n+r}$$

$$\sum_{n=0}^{\infty} 9 x^{1+n+r} a_{n} = \sum_{n=1}^{\infty} 9 a_{n-1} x^{n+r}$$

$$\sum_{n=0}^{\infty} 3 x^{n+r+3} a_{n} = \sum_{n=3}^{\infty} 3 a_{n-3} x^{n+r}$$

$$\sum_{n=0}^{\infty} \frac{6 x^{n+r+5} a_{n}}{5} = \sum_{n=5}^{\infty} \frac{6 a_{n-5} x^{n+r}}{5}$$

$$\sum_{n=0}^{\infty} \frac{17 x^{n+r+7} a_{n}}{35} = \sum_{n=7}^{\infty} \frac{17 a_{n-7} x^{n+r}}{35}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=2}^{\infty} a_{n-2}(n+r-2)\,(n-3+r)\,x^{n+r}\right)+\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)\,(n+r-1)\right)$$
$$+\left(\sum_{n=6}^{\infty} \frac{7a_{n-6}(n-6+r)\,x^{n+r}}{720}\right)+\left(\sum_{n=5}^{\infty} \frac{7a_{n-5}(n-5+r)\,x^{n+r}}{120}\right)$$
$$+\left(\sum_{n=4}^{\infty} \frac{7a_{n-4}(n-4+r)\,x^{n+r}}{24}\right)+\left(\sum_{n=3}^{\infty} \frac{7a_{n-3}(n-3+r)\,x^{n+r}}{6}\right) \tag{2B}$$
$$+\left(\sum_{n=2}^{\infty} \frac{7a_{n-2}(n+r-2)\,x^{n+r}}{2}\right)+\left(\sum_{n=1}^{\infty} 7a_{n-1}(n+r-1)\,x^{n+r}\right)$$
$$+\left(\sum_{n=0}^{\infty} 7x^{n+r} a_n(n+r)\right)+\left(\sum_{n=0}^{\infty} 9a_n x^{n+r}\right)+\left(\sum_{n=1}^{\infty} 9a_{n-1}x^{n+r}\right)$$
$$+\left(\sum_{n=3}^{\infty} 3a_{n-3}x^{n+r}\right)+\left(\sum_{n=5}^{\infty} \frac{6a_{n-5}x^{n+r}}{5}\right)+\left(\sum_{n=7}^{\infty} \frac{17a_{n-7}x^{n+r}}{35}\right)=0$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n(n+r)\,(n+r-1)+7x^{n+r} a_n(n+r)+9a_n x^{n+r}=0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r)+7x^r a_0 r+9a_0 x^r=0$$

Or

$$(x^r r(-1+r)+7x^r r+9x^r)\,a_0=0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(3+r)^2\,x^r=0$$

Since the above is true for all x then the indicial equation becomes

$$(3+r)^2=0$$

Solving for r gives the roots of the indicial equation as

$$r_1=-3$$
$$r_2=-3$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(3+r)^2 x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = -3$, Eqs (1A,1B) become

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n-3} \\ y_2(x) &= y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n-3}\right) \end{aligned}$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = \frac{-7r-9}{(r+4)^2}$$

Substituting $n = 2$ in Eq. (2B) gives

$$a_2 = \frac{-2r^4 - 21r^3 + 26r^2 + 270r + 288}{2(r+4)^2 (5+r)^2}$$

Substituting $n = 3$ in Eq. (2B) gives

$$a_3 = \frac{77r^5 + 1098r^4 + 3602r^3 - 1998r^2 - 22318r - 22347}{6(r+4)^2 (5+r)^2 (6+r)^2}$$

Substituting $n = 4$ in Eq. (2B) gives

$$a_4 = \frac{24r^8 + 689r^7 + 5132r^6 - 7139r^5 - 200328r^4 - 605868r^3 + 73708r^2 + 2422164r + 2371896}{24 (r+4)^2 (5+r)^2 (6+r)^2 (r+7)^2}$$

Substituting $n = 5$ in Eq. (2B) gives

$$a_5 = \frac{-2247r^9 - 79622r^8 - 1065025r^7 - 6575402r^6 - 16492594r^5 + 7514875r^4 + 98942906r^3 + 50216229}{120 (r+4)^2 (5+r)^2 (6+r)^2 (r+7)^2 (r+8)^2}$$

Substituting $n = 6$ in Eq. (2B) gives

$$a_6 = \frac{-720r^{12} - 39547r^{11} - 792562r^{10} - 6305240r^9 + 6536464r^8 + 490611417r^7 + 3743792400r^6 + 12951}{720 (r+4)^2 (5+r)^2 (6+r)^2}$$

For $7 \leq n$ the recursive equation is

$$\begin{aligned} &a_{n-2}(n+r-2)(n-3+r) + a_n(n+r)(n+r-1) \\ &+ \frac{7a_{n-6}(n-6+r)}{720} + \frac{7a_{n-5}(n-5+r)}{120} + \frac{7a_{n-4}(n-4+r)}{24} \\ &+ \frac{7a_{n-3}(n-3+r)}{6} + \frac{7a_{n-2}(n+r-2)}{2} + 7a_{n-1}(n+r-1) \\ &+ 7a_n(n+r) + 9a_n + 9a_{n-1} + 3a_{n-3} + \frac{6a_{n-5}}{5} + \frac{17a_{n-7}}{35} = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{5040n^2a_{n-2} + 10080nra_{n-2} + 5040r^2a_{n-2} + 49na_{n-6} + 294na_{n-5} + 1470na_{n-4} + 5880na_{n-3} - 75}{} \tag{4}$$

Which for the root $r = -3$ becomes

$$a_n = \frac{-5040n^2a_{n-2} + (-49a_{n-6} - 294a_{n-5} - 1470a_{n-4} - 5880a_{n-3} + 37800a_{n-2} - 35280a_{n-1})n - 2448a}{5040n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = -3$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$
a_0	1
a_1	$\frac{-7r-9}{(r+4)^2}$
a_2	$\frac{-2r^4-21r^3+26r^2+270r+288}{2(r+4)^2(5+r)^2}$
a_3	$\frac{77r^5+1098r^4+3602r^3-1998r^2-22318r-22347}{6(r+4)^2(5+r)^2(6+r)^2}$
a_4	$\frac{24r^8+689r^7+5132r^6-7139r^5-200328r^4-605868r^3+73708r^2+2422164r+2371896}{24(r+4)^2(5+r)^2(6+r)^2(r+7)^2}$
a_5	$\frac{-2247r^9-79622r^8-1065025r^7-6575402r^6-16492594r^5+7514875r^4+98942906r^3+50216229r^2-289883610r-327734910}{120(r+4)^2(5+r)^2(6+r)^2(r+7)^2(r+8)^2}$
a_6	$\frac{-720r^{12}-39547r^{11}-792562r^{10}-6305240r^9+6536464r^8+490611417r^7+3743792400r^6+12951080410r^5+18943551934r^4-1275211246r^3}{720(r+4)^2(5+r)^2(6+r)^2(r+7)^2(r+8)^2(r+9)^2}$

Using the above table, then the first solution $y_1(x)$ is

$$y_1(x) = \frac{1}{x^3}\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 \dots\right)$$
$$= \frac{1 + 12x + \frac{117x^2}{8} - \frac{67x^3}{36} + \frac{505x^4}{256} - \frac{262x^5}{125} + \frac{2443637x^6}{2304000} + O(x^7)}{x^3}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr}a_{n,r}$$

And the above is then evaluated at $r = -3$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$
b_0	1
b_1	$\frac{-7r-9}{(r+4)^2}$
b_2	$\frac{-2r^4-21r^3+26r^2+270r+288}{2(r+4)^2(5+r)^2}$
b_3	$\frac{77r^5+1098r^4+3602r^3-1998r^2-22318r-22347}{6(r+4)^2(5+r)^2(6+r)^2}$
b_4	$\frac{24r^8+689r^7+5132r^6-7139r^5-200328r^4-605868r^3+73708r^2+2422164r+2371896}{24(r+4)^2(5+r)^2(6+r)^2(r+7)^2}$
b_5	$\frac{-2247r^9-79622r^8-1065025r^7-6575402r^6-16492594r^5+7514875r^4+98942906r^3+50216229r^2-289883610r-327734910}{120(r+4)^2(5+r)^2(6+r)^2(r+7)^2(r+8)^2}$
b_6	$\frac{-720r^{12}-39547r^{11}-792562r^{10}-6305240r^9+6536464r^8+490611417r^7+3743792400r^6+12951080410r^5+18943551934r^4-1275211246r^3-}{720(r+4)^2(5+r)^2(6+r)^2(r+7)^2(r+8)^2(r+9)^2}$

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 \dots \\
&= \frac{\left(1 + 12x + \frac{117x^2}{8} - \frac{67x^3}{36} + \frac{505x^4}{256} - \frac{262x^5}{125} + \frac{2443637x^6}{2304000} + O(x^7)\right)\ln(x)}{x^3} \\
&\quad + \frac{-31x - \frac{147x^2}{2} + \frac{37x^3}{8} - \frac{44803x^4}{4608} + \frac{5057587x^5}{480000} - \frac{3797765581x^6}{622080000} + O(x^7)}{x^3}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= \frac{c_1\left(1 + 12x + \frac{117x^2}{8} - \frac{67x^3}{36} + \frac{505x^4}{256} - \frac{262x^5}{125} + \frac{2443637x^6}{2304000} + O(x^7)\right)}{x^3} \\
&\quad + c_2\left(\frac{\left(1 + 12x + \frac{117x^2}{8} - \frac{67x^3}{36} + \frac{505x^4}{256} - \frac{262x^5}{125} + \frac{2443637x^6}{2304000} + O(x^7)\right)\ln(x)}{x^3}\right. \\
&\quad \left. + \frac{-31x - \frac{147x^2}{2} + \frac{37x^3}{8} - \frac{44803x^4}{4608} + \frac{5057587x^5}{480000} - \frac{3797765581x^6}{622080000} + O(x^7)}{x^3}\right)
\end{aligned}
$$

Hence the final solution is

$$
y = y_h
$$

$$= \frac{c_1\left(1+12x+\frac{117x^2}{8}-\frac{67x^3}{36}+\frac{505x^4}{256}-\frac{262x^5}{125}+\frac{2443637x^6}{2304000}+O(x^7)\right)}{x^3} + c_2\left(\frac{\left(1+12x+\frac{117x^2}{8}-\frac{67x^3}{36}+\frac{505x^4}{256}-\frac{262x^5}{125}+\frac{2443637x^6}{2304000}+O(x^7)\right)\ln(x)}{x^3} + \frac{-31x-\frac{147x^2}{2}+\frac{37x^3}{8}-\frac{44803x^4}{4608}+\frac{5057587x^5}{480000}-\frac{3797765581x^6}{622080000}+O(x^7)}{x^3}\right)$$

Summary

The solution(s) found are the following

$$y = \frac{c_1\left(1+12x+\frac{117x^2}{8}-\frac{67x^3}{36}+\frac{505x^4}{256}-\frac{262x^5}{125}+\frac{2443637x^6}{2304000}+O(x^7)\right)}{x^3} + c_2\left(\frac{\left(1+12x+\frac{117x^2}{8}-\frac{67x^3}{36}+\frac{505x^4}{256}-\frac{262x^5}{125}+\frac{2443637x^6}{2304000}+O(x^7)\right)\ln(x)}{x^3} + \frac{-31x-\frac{147x^2}{2}+\frac{37x^3}{8}-\frac{44803x^4}{4608}+\frac{5057587x^5}{480000}-\frac{3797765581x^6}{622080000}+O(x^7)}{x^3}\right) \tag{1}$$

Verification of solutions

$$y = \frac{c_1\left(1+12x+\frac{117x^2}{8}-\frac{67x^3}{36}+\frac{505x^4}{256}-\frac{262x^5}{125}+\frac{2443637x^6}{2304000}+O(x^7)\right)}{x^3} + c_2\left(\frac{\left(1+12x+\frac{117x^2}{8}-\frac{67x^3}{36}+\frac{505x^4}{256}-\frac{262x^5}{125}+\frac{2443637x^6}{2304000}+O(x^7)\right)\ln(x)}{x^3} + \frac{-31x-\frac{147x^2}{2}+\frac{37x^3}{8}-\frac{44803x^4}{4608}+\frac{5057587x^5}{480000}-\frac{3797765581x^6}{622080000}+O(x^7)}{x^3}\right)$$

Verified OK.

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 75

```
Order:=7;
dsolve(x^2*(x^2+1)*diff(y(x),x$2)+7*x*exp(x)*diff(y(x),x)+9*(1+tan(x))*y(x)=0,y(x),type='ser
```

$$y(x) = \frac{(c_2 \ln(x) + c_1)\left(1 + 12x + \frac{117}{8}x^2 - \frac{67}{36}x^3 + \frac{505}{256}x^4 - \frac{262}{125}x^5 + \frac{2443637}{2304000}x^6 + \mathrm{O}(x^7)\right) + \left((-31)x - \frac{147}{2}x^2 + \frac{37}{8}\right.}{x^3}$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 143

```
AsymptoticDSolveValue[x^2*(x^2+1)*y''[x]+7*x*Exp[x]*y'[x]+9*(1+Tan[x])*y[x]==0,y[x],{x,0,6}]
```

$$y(x) \to \frac{c_1\left(\frac{2443637x^6}{2304000} - \frac{262x^5}{125} + \frac{505x^4}{256} - \frac{67x^3}{36} + \frac{117x^2}{8} + 12x + 1\right)}{x^3} + c_2\left(\frac{-\frac{3797765581x^6}{622080000} + \frac{5057587x^5}{480000} - \frac{44803x^4}{4608} + \frac{37x^3}{8} - \frac{147x^2}{2} - 31x}{x^3} + \frac{\left(\frac{2443637x^6}{2304000} - \frac{262x^5}{125} + \frac{505x^4}{256} - \frac{67x^3}{36} + \frac{117x^2}{8} + 12x + 1\right)\log(x)}{x^3}\right)$$

18.17 problem 11

Internal problem ID [2953]
Internal file name [OUTPUT/2445_Sunday_June_05_2022_03_13_28_AM_88092337/index.tex]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.5. page 771
Problem number: 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(x+1)\,y''+y'x^2-2y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(x^3+x^2\right)y''+y'x^2-2y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{1}{x+1}$$
$$q(x)=-\frac{2}{x^2\,(x+1)}$$

Table 482: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x+1}$	
singularity	type
$x = -1$	"regular"

$q(x) = -\frac{2}{x^2(x+1)}$	
singularity	type
$x = -1$	"regular"
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-1, 0, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2(x+1)\,y'' + y'x^2 - 2y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2(x+1)\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) + \left(\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\right) x^2 - 2\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-2 a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)(n+r-1) = \sum_{n=1}^{\infty} a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} a_{n-1}(n+r-1)\, x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1)\, x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-2 a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) - 2 a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) - 2 a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) - 2 x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2 - r - 2\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - r - 2 = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= 2 \\ r_2 &= -1 \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(r^2 - r - 2)\, x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 3$ is an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right) \\ y_2(x) &= C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= x^2 \left(\sum_{n=0}^{\infty} a_n x^n \right) \\ y_2(x) &= C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x} \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+2} \\ y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-1} \right) \end{aligned}$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_{n-1}(n + r - 1)(n + r - 2) + a_n(n + r)(n + r - 1) + a_{n-1}(n + r - 1) - 2a_n = 0 \quad (3)$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}(n^2 + 2nr + r^2 - 2n - 2r + 1)}{n^2 + 2nr + r^2 - n - r - 2} \tag{4}$$

Which for the root $r = 2$ becomes

$$a_n = -\frac{a_{n-1}(1+n)^2}{n(n+3)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 2$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{r^2}{r^2 + r - 2}$$

Which for the root $r = 2$ becomes

$$a_1 = -1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r^2}{r^2+r-2}$	-1

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{r(r+1)^2}{r^3 + 4r^2 + r - 6}$$

Which for the root $r = 2$ becomes

$$a_2 = \frac{9}{10}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r^2}{r^2+r-2}$	-1
a_2	$\frac{r(r+1)^2}{r^3+4r^2+r-6}$	$\frac{9}{10}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{(r+1)\, r(r+2)}{r^3 + 6r^2 + 5r - 12}$$

Which for the root $r = 2$ becomes

$$a_3 = -\frac{4}{5}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r^2}{r^2+r-2}$	-1
a_2	$\frac{r(r+1)^2}{r^3+4r^2+r-6}$	$\frac{9}{10}$
a_3	$-\frac{(r+1)r(r+2)}{r^3+6r^2+5r-12}$	$-\frac{4}{5}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{r(r+1)\,(r+3)}{r^3 + 8r^2 + 11r - 20}$$

Which for the root $r = 2$ becomes

$$a_4 = \frac{5}{7}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r^2}{r^2+r-2}$	-1
a_2	$\frac{r(r+1)^2}{r^3+4r^2+r-6}$	$\frac{9}{10}$
a_3	$-\frac{(r+1)r(r+2)}{r^3+6r^2+5r-12}$	$-\frac{4}{5}$
a_4	$\frac{r(r+1)(r+3)}{r^3+8r^2+11r-20}$	$\frac{5}{7}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{(r+1)\, r(r+4)}{r^3 + 10r^2 + 19r - 30}$$

Which for the root $r = 2$ becomes

$$a_5 = -\frac{9}{14}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r^2}{r^2+r-2}$	-1
a_2	$\frac{r(r+1)^2}{r^3+4r^2+r-6}$	$\frac{9}{10}$
a_3	$-\frac{(r+1)r(r+2)}{r^3+6r^2+5r-12}$	$-\frac{4}{5}$
a_4	$\frac{r(r+1)(r+3)}{r^3+8r^2+11r-20}$	$\frac{5}{7}$
a_5	$-\frac{(r+1)r(r+4)}{r^3+10r^2+19r-30}$	$-\frac{9}{14}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^2\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots\right) \\ &= x^2\left(1 - x + \frac{9x^2}{10} - \frac{4x^3}{5} + \frac{5x^4}{7} - \frac{9x^5}{14} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 3$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_3(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_3 \\ &= -\frac{(r+1)\, r(r+2)}{r^3 + 6r^2 + 5r - 12} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\to r_2} -\frac{(r+1)\, r(r+2)}{r^3 + 6r^2 + 5r - 12} &= \lim_{r\to -1} -\frac{(r+1)\, r(r+2)}{r^3 + 6r^2 + 5r - 12} \\ &= 0 \end{aligned}$$

The limit is 0. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$
\begin{aligned}
y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\
&= \sum_{n=0}^{\infty} b_n x^{n-1}
\end{aligned}
$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$
b_{n-1}(n+r-1)(n+r-2) + b_n(n+r)(n+r-1) + b_{n-1}(n+r-1) - 2b_n = 0 \quad (4)
$$

Which for for the root $r = -1$ becomes

$$
b_{n-1}(n-2)(n-3) + b_n(n-1)(n-2) + b_{n-1}(n-2) - 2b_n = 0 \qquad (4\text{A})
$$

Solving for b_n from the recursive equation (4) gives

$$
b_n = -\frac{b_{n-1}(n^2 + 2nr + r^2 - 2n - 2r + 1)}{n^2 + 2nr + r^2 - n - r - 2} \qquad (5)
$$

Which for the root $r = -1$ becomes

$$
b_n = -\frac{b_{n-1}(n^2 - 4n + 4)}{n^2 - 3n} \qquad (6)
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$
b_1 = -\frac{r^2}{r^2 + r - 2}
$$

Which for the root $r = -1$ becomes

$$
b_1 = \frac{1}{2}
$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{r^2}{r^2+r-2}$	$\frac{1}{2}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{r(r^2 + 2r + 1)}{(r^2 + r - 2)(r + 3)}$$

Which for the root $r = -1$ becomes

$$b_2 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{r^2}{r^2+r-2}$	$\frac{1}{2}$
b_2	$\frac{r(r+1)^2}{r^3+4r^2+r-6}$	0

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{(r + 1)\, r(r + 2)}{(r + 4)(r + 3)(-1 + r)}$$

Which for the root $r = -1$ becomes

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{r^2}{r^2+r-2}$	$\frac{1}{2}$
b_2	$\frac{r(r+1)^2}{r^3+4r^2+r-6}$	0
b_3	$-\frac{(r+1)r(r+2)}{(r+4)(r+3)(-1+r)}$	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{r(r+1)(r+3)}{(r+5)(-1+r)(r+4)}$$

Which for the root $r = -1$ becomes

$$b_4 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{r^2}{r^2+r-2}$	$\frac{1}{2}$
b_2	$\frac{r(r+1)^2}{r^3+4r^2+r-6}$	0
b_3	$-\frac{(r+1)r(r+2)}{(r+4)(r+3)(-1+r)}$	0
b_4	$\frac{r(r+1)(r+3)}{(r+5)(-1+r)(r+4)}$	0

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{(r+1)r(r+4)}{(r+6)(-1+r)(r+5)}$$

Which for the root $r = -1$ becomes

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{r^2}{r^2+r-2}$	$\frac{1}{2}$
b_2	$\frac{r(r+1)^2}{r^3+4r^2+r-6}$	0
b_3	$-\frac{(r+1)r(r+2)}{(r+4)(r+3)(-1+r)}$	0
b_4	$\frac{r(r+1)(r+3)}{(r+5)(-1+r)(r+4)}$	0
b_5	$-\frac{(r+1)r(r+4)}{(r+6)(-1+r)(r+5)}$	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x^2\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= \frac{1 + \frac{x}{2} + O(x^6)}{x} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^2\left(1 - x + \frac{9x^2}{10} - \frac{4x^3}{5} + \frac{5x^4}{7} - \frac{9x^5}{14} + O(x^6)\right) + \frac{c_2\left(1 + \frac{x}{2} + O(x^6)\right)}{x} \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x^2\left(1 - x + \frac{9x^2}{10} - \frac{4x^3}{5} + \frac{5x^4}{7} - \frac{9x^5}{14} + O(x^6)\right) + \frac{c_2\left(1 + \frac{x}{2} + O(x^6)\right)}{x} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x^2\left(1 - x + \frac{9x^2}{10} - \frac{4x^3}{5} + \frac{5x^4}{7} - \frac{9x^5}{14} + O(x^6)\right) + \frac{c_2\left(1 + \frac{x}{2} + O(x^6)\right)}{x} \tag{1}$$

Verification of solutions

$$y = c_1 x^2\left(1 - x + \frac{9x^2}{10} - \frac{4x^3}{5} + \frac{5x^4}{7} - \frac{9x^5}{14} + O(x^6)\right) + \frac{c_2\left(1 + \frac{x}{2} + O(x^6)\right)}{x}$$

Verified OK.

18.17.1 Maple step by step solution

Let's solve

$x^2(x+1)\, y'' + y' x^2 - 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y'}{x+1} + \frac{2y}{x^2(x+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x+1} - \frac{2y}{x^2(x+1)} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{1}{x+1}, P_3(x) = -\frac{2}{x^2(x+1)}\right]$

  - $(x+1) \cdot P_2(x)$ is analytic at $x = -1$

    $\left((x+1) \cdot P_2(x)\right)\bigg|_{x=-1} = 1$

  - $(x+1)^2 \cdot P_3(x)$ is analytic at $x = -1$

    $\left((x+1)^2 \cdot P_3(x)\right)\bigg|_{x=-1} = 0$

  - $x = -1$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = -1$

- Multiply by denominators

  $x^2(x+1)\, y'' + y'x^2 - 2y = 0$

- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

  $(u^3 - 2u^2 + u)\left(\frac{d^2}{du^2}y(u)\right) + (u^2 - 2u + 1)\left(\frac{d}{du}y(u)\right) - 2y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

  - Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

    $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $$u^{m}\cdot\left(\frac{d^{2}}{du^{2}}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$$

  Rewrite ODE with series expansions

  $$a_{0}r^{2}u^{-1+r}+\left(a_{1}(1+r)^{2}-2a_{0}(r^{2}+1)\right)u^{r}+\left(\sum_{k=1}^{\infty}\left(a_{k+1}(k+1+r)^{2}-2a_{k}(k^{2}+2kr+r^{2}+1)+\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$r^{2}=0$$

- Values of r that satisfy the indicial equation

  $$r=0$$

- Each term must be 0

  $$a_{1}(1+r)^{2}-2a_{0}(r^{2}+1)=0$$

- Each term in the series must be 0, giving the recursion relation

  $$a_{k+1}(k+1)^{2}-2a_{k}(k^{2}+1)+a_{k-1}(k-1)^{2}=0$$

- Shift index using $k->k+1$

  $$a_{k+2}(k+2)^{2}-2a_{k+1}\left((k+1)^{2}+1\right)+k^{2}a_{k}=0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2}=-\frac{k^{2}a_{k}-2k^{2}a_{k+1}-4ka_{k+1}-4a_{k+1}}{(k+2)^{2}}$$

- Recursion relation for $r=0$

  $$a_{k+2}=-\frac{k^{2}a_{k}-2k^{2}a_{k+1}-4ka_{k+1}-4a_{k+1}}{(k+2)^{2}}$$

- Solution for $r=0$

  $$\left[y(u)=\sum_{k=0}^{\infty}a_{k}u^{k},a_{k+2}=-\frac{k^{2}a_{k}-2k^{2}a_{k+1}-4ka_{k+1}-4a_{k+1}}{(k+2)^{2}},a_{1}-2a_{0}=0\right]$$

- Revert the change of variables $u=x+1$

  $$\left[y=\sum_{k=0}^{\infty}a_{k}(x+1)^{k},a_{k+2}=-\frac{k^{2}a_{k}-2k^{2}a_{k+1}-4ka_{k+1}-4a_{k+1}}{(k+2)^{2}},a_{1}-2a_{0}=0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.015 (sec). Leaf size: 39

```
Order:=6;
dsolve(x^2*(1+x)*diff(y(x),x$2)+x^2*diff(y(x),x)-2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^2 \left(1 - x + \frac{9}{10}x^2 - \frac{4}{5}x^3 + \frac{5}{7}x^4 - \frac{9}{14}x^5 + \mathrm{O}\left(x^6\right)\right) + \frac{c_2(12 + 6x + \mathrm{O}\left(x^6\right))}{x}$$

✓ Solution by Mathematica
Time used: 0.029 (sec). Leaf size: 47

```
AsymptoticDSolveValue[x^2*(1+x)*y''[x]+x^2*y'[x]-2*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2 \left(\frac{5x^6}{7} - \frac{4x^5}{5} + \frac{9x^4}{10} - x^3 + x^2\right) + c_1 \left(\frac{1}{x} + \frac{1}{2}\right)$$

18.18 problem 12

Internal problem ID [2954]
Internal file name [OUTPUT/2446_Sunday_June_05_2022_03_13_32_AM_90924725/index.tex]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.5. page 771
Problem number: 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + 3xy' + (1-x)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + 3xy' + (1-x)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$\begin{aligned}p(x) &= \frac{3}{x}\\ q(x) &= -\frac{x-1}{x^2}\end{aligned}$$

Table 484: Table $p(x), q(x)$ singularites.

$p(x) = \frac{3}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{x-1}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + 3xy' + (1 - x)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2}\right) + 3x \left(\sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}\right) + (1 - x) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 3x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n\right) = \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 3x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) + \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) + 3x^{n+r} a_n (n+r) + a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) + 3x^r a_0 r + a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) + 3x^r r + x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(r+1)^2 x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(r+1)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = -1$$
$$r_2 = -1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(r+1)^2 x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = -1$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n-1}$$
$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n-1}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + 3a_n(n+r) + a_n - a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}}{n^2 + 2nr + r^2 + 2n + 2r + 1} \tag{4}$$

Which for the root $r = -1$ becomes

$$a_n = \frac{a_{n-1}}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{1}{(r+2)^2}$$

Which for the root $r = -1$ becomes

$$a_1 = 1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{(r+2)^2}$	1

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{(r+2)^2 (3+r)^2}$$

Which for the root $r = -1$ becomes

$$a_2 = \frac{1}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{(r+2)^2}$	1
a_2	$\frac{1}{(r+2)^2(3+r)^2}$	$\frac{1}{4}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{1}{(r+2)^2 (3+r)^2 (r+4)^2}$$

Which for the root $r = -1$ becomes

$$a_3 = \frac{1}{36}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{(r+2)^2}$	1
a_2	$\frac{1}{(r+2)^2(3+r)^2}$	$\frac{1}{4}$
a_3	$\frac{1}{(r+2)^2(3+r)^2(r+4)^2}$	$\frac{1}{36}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(r+2)^2 (3+r)^2 (r+4)^2 (5+r)^2}$$

Which for the root $r = -1$ becomes

$$a_4 = \frac{1}{576}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{(r+2)^2}$	1
a_2	$\frac{1}{(r+2)^2(3+r)^2}$	$\frac{1}{4}$
a_3	$\frac{1}{(r+2)^2(3+r)^2(r+4)^2}$	$\frac{1}{36}$
a_4	$\frac{1}{(r+2)^2(3+r)^2(r+4)^2(5+r)^2}$	$\frac{1}{576}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{1}{(r+2)^2 (3+r)^2 (r+4)^2 (5+r)^2 (6+r)^2}$$

Which for the root $r = -1$ becomes

$$a_5 = \frac{1}{14400}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{(r+2)^2}$	1
a_2	$\frac{1}{(r+2)^2(3+r)^2}$	$\frac{1}{4}$
a_3	$\frac{1}{(r+2)^2(3+r)^2(r+4)^2}$	$\frac{1}{36}$
a_4	$\frac{1}{(r+2)^2(3+r)^2(r+4)^2(5+r)^2}$	$\frac{1}{576}$
a_5	$\frac{1}{(r+2)^2(3+r)^2(r+4)^2(5+r)^2(6+r)^2}$	$\frac{1}{14400}$

Using the above table, then the first solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= \frac{1}{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= \frac{x + 1 + \frac{x^2}{4} + \frac{x^3}{36} + \frac{x^4}{576} + \frac{x^5}{14400} + O(x^6)}{x} \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = -1$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr}a_{n,r}$	$b_n(r = -1)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	$\frac{1}{(r+2)^2}$	1	$-\frac{2}{(r+2)^3}$	-2
b_2	$\frac{1}{(r+2)^2(3+r)^2}$	$\frac{1}{4}$	$\frac{-10-4r}{(r+2)^3(3+r)^3}$	$-\frac{3}{4}$
b_3	$\frac{1}{(r+2)^2(3+r)^2(r+4)^2}$	$\frac{1}{36}$	$\frac{-6r^2-36r-52}{(r+2)^3(3+r)^3(r+4)^3}$	$-\frac{11}{108}$
b_4	$\frac{1}{(r+2)^2(3+r)^2(r+4)^2(5+r)^2}$	$\frac{1}{576}$	$\frac{-8r^3-84r^2-284r-308}{(r+2)^3(3+r)^3(r+4)^3(5+r)^3}$	$-\frac{25}{3456}$
b_5	$\frac{1}{(r+2)^2(3+r)^2(r+4)^2(5+r)^2(6+r)^2}$	$\frac{1}{14400}$	$\frac{-10r^4-160r^3-930r^2-2320r-2088}{(r+2)^3(3+r)^3(r+4)^3(5+r)^3(6+r)^3}$	$-\frac{137}{432000}$

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 \dots \\
&= \frac{\left(x + 1 + \frac{x^2}{4} + \frac{x^3}{36} + \frac{x^4}{576} + \frac{x^5}{14400} + O(x^6)\right)\ln(x)}{x} \\
&\quad + \frac{-2x - \frac{3x^2}{4} - \frac{11x^3}{108} - \frac{25x^4}{3456} - \frac{137x^5}{432000} + O(x^6)}{x}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1y_1(x) + c_2y_2(x) \\
&= \frac{c_1\left(x + 1 + \frac{x^2}{4} + \frac{x^3}{36} + \frac{x^4}{576} + \frac{x^5}{14400} + O(x^6)\right)}{x} \\
&\quad + c_2\left(\frac{\left(x + 1 + \frac{x^2}{4} + \frac{x^3}{36} + \frac{x^4}{576} + \frac{x^5}{14400} + O(x^6)\right)\ln(x)}{x}\right. \\
&\qquad \left. + \frac{-2x - \frac{3x^2}{4} - \frac{11x^3}{108} - \frac{25x^4}{3456} - \frac{137x^5}{432000} + O(x^6)}{x}\right)
\end{aligned}
$$

Hence the final solution is

$$
y = y_h
$$

$$
= \frac{c_1\left(x+1+\frac{x^2}{4}+\frac{x^3}{36}+\frac{x^4}{576}+\frac{x^5}{14400}+O(x^6)\right)}{x} + c_2\left(\frac{\left(x+1+\frac{x^2}{4}+\frac{x^3}{36}+\frac{x^4}{576}+\frac{x^5}{14400}+O(x^6)\right)\ln(x)}{x} + \frac{-2x-\frac{3x^2}{4}-\frac{11x^3}{108}-\frac{25x^4}{3456}-\frac{137x^5}{432000}+O(x^6)}{x}\right)
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = \frac{c_1\left(x+1+\frac{x^2}{4}+\frac{x^3}{36}+\frac{x^4}{576}+\frac{x^5}{14400}+O(x^6)\right)}{x} + c_2\left(\frac{\left(x+1+\frac{x^2}{4}+\frac{x^3}{36}+\frac{x^4}{576}+\frac{x^5}{14400}+O(x^6)\right)\ln(x)}{x} + \frac{-2x-\frac{3x^2}{4}-\frac{11x^3}{108}-\frac{25x^4}{3456}-\frac{137x^5}{432000}+O(x^6)}{x}\right) \tag{1}
$$

<u>Verification of solutions</u>

$$
y = \frac{c_1\left(x+1+\frac{x^2}{4}+\frac{x^3}{36}+\frac{x^4}{576}+\frac{x^5}{14400}+O(x^6)\right)}{x} + c_2\left(\frac{\left(x+1+\frac{x^2}{4}+\frac{x^3}{36}+\frac{x^4}{576}+\frac{x^5}{14400}+O(x^6)\right)\ln(x)}{x} + \frac{-2x-\frac{3x^2}{4}-\frac{11x^3}{108}-\frac{25x^4}{3456}-\frac{137x^5}{432000}+O(x^6)}{x}\right)
$$

Verified OK.

18.18.1 Maple step by step solution

Let's solve

$x^2y'' + 3xy' + (1-x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{3y'}{x} + \frac{(x-1)y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{3y'}{x} - \frac{(x-1)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{3}{x}, P_3(x) = -\frac{x-1}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 3$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 1$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2y'' + 3xy' + (1-x)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ◦ Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x \cdot y'$ to series expansion

  $$x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 (1+r)^2\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k (k+r+1)^2 - a_{k-1}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(1+r)^2 = 0$$

- Values of r that satisfy the indicial equation

  $$r = -1$$

- Each term in the series must be 0, giving the recursion relation

  $$a_k (k+r+1)^2 - a_{k-1} = 0$$

- Shift index using $k- > k+1$

  $$a_{k+1} (k+2+r)^2 - a_k = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = \frac{a_k}{(k+2+r)^2}$$

- Recursion relation for $r = -1$

  $$a_{k+1} = \frac{a_k}{(k+1)^2}$$

- Solution for $r = -1$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+1} = \frac{a_k}{(k+1)^2}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 69

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+3*x*diff(y(x),x)+(1-x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{(c_2 \ln(x) + c_1)\left(1 + x + \frac{1}{4}x^2 + \frac{1}{36}x^3 + \frac{1}{576}x^4 + \frac{1}{14400}x^5 + \mathrm{O}(x^6)\right) + \left((-2)x - \frac{3}{4}x^2 - \frac{11}{108}x^3 - \frac{25}{3456}x^4 - \cdots\right.}{x}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 118

```
AsymptoticDSolveValue[x^2*y''[x]+3*x*y'[x]+(1-x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to \frac{c_1\left(\frac{x^5}{14400} + \frac{x^4}{576} + \frac{x^3}{36} + \frac{x^2}{4} + x + 1\right)}{x} + c_2\left(\frac{-\frac{137x^5}{432000} - \frac{25x^4}{3456} - \frac{11x^3}{108} - \frac{3x^2}{4} - 2x}{x} + \frac{\left(\frac{x^5}{14400} + \frac{x^4}{576} + \frac{x^3}{36} + \frac{x^2}{4} + x + 1\right)\log(x)}{x}\right)$$

18.19 problem 13

Internal problem ID [2955]
Internal file name [`OUTPUT/2447_Sunday_June_05_2022_03_13_35_AM_18055943/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.5. page 771
Problem number: 13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{xy'' - y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' - y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = 0$$
$$q(x) = -\frac{1}{x}$$

Table 486: Table $p(x), q(x)$ singularites.

$p(x) = 0$	
singularity	type

$q(x) = -\frac{1}{x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' - y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\, (n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)\, (n+r-1)\, a_n x^{n+r-2}\right) x - \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)\, (n+r-1)\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) = 0$$

Or

$$x^{-1+r} a_0 r(-1+r) = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^{-1+r} r(-1+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(-1+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^{-1+r} r(-1+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right)$$

$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right)$$

Or

$$y_1(x) = x \left(\sum_{n=0}^{\infty} a_n x^n \right)$$

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n \right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$$

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n \right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) - a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}}{(n+r)(n+r-1)} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = \frac{a_{n-1}}{(n+1)\,n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{1}{(1+r)\,r}$$

Which for the root $r = 1$ becomes

$$a_1 = \frac{1}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{(1+r)r}$	$\frac{1}{2}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{(1+r)^2\,r\,(2+r)}$$

Which for the root $r = 1$ becomes

$$a_2 = \frac{1}{12}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{(1+r)r}$	$\frac{1}{2}$
a_2	$\frac{1}{(1+r)^2r(2+r)}$	$\frac{1}{12}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{1}{(1+r)^2\,r\,(2+r)^2\,(3+r)}$$

Which for the root $r = 1$ becomes

$$a_3 = \frac{1}{144}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{(1+r)r}$	$\frac{1}{2}$
a_2	$\frac{1}{(1+r)^2 r(2+r)}$	$\frac{1}{12}$
a_3	$\frac{1}{(1+r)^2 r(2+r)^2 (3+r)}$	$\frac{1}{144}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(1+r)^2 r (2+r)^2 (3+r)^2 (4+r)}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{1}{2880}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{(1+r)r}$	$\frac{1}{2}$
a_2	$\frac{1}{(1+r)^2 r(2+r)}$	$\frac{1}{12}$
a_3	$\frac{1}{(1+r)^2 r(2+r)^2 (3+r)}$	$\frac{1}{144}$
a_4	$\frac{1}{(1+r)^2 r(2+r)^2 (3+r)^2 (4+r)}$	$\frac{1}{2880}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{1}{(1+r)^2 r (2+r)^2 (3+r)^2 (4+r)^2 (5+r)}$$

Which for the root $r = 1$ becomes

$$a_5 = \frac{1}{86400}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{(1+r)r}$	$\frac{1}{2}$
a_2	$\frac{1}{(1+r)^2 r(2+r)}$	$\frac{1}{12}$
a_3	$\frac{1}{(1+r)^2 r(2+r)^2 (3+r)}$	$\frac{1}{144}$
a_4	$\frac{1}{(1+r)^2 r(2+r)^2 (3+r)^2 (4+r)}$	$\frac{1}{2880}$
a_5	$\frac{1}{(1+r)^2 r(2+r)^2 (3+r)^2 (4+r)^2 (5+r)}$	$\frac{1}{86400}$

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= x\left(1 + \frac{x}{2} + \frac{x^2}{12} + \frac{x^3}{144} + \frac{x^4}{2880} + \frac{x^5}{86400} + O(x^6)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Let

$$
r_1 - r_2 = N
$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\rightarrow r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$
\begin{aligned}
a_N &= a_1 \\
&= \frac{1}{(1+r)\,r}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\lim_{r\rightarrow r_2} \frac{1}{(1+r)\,r} &= \lim_{r\rightarrow 0} \frac{1}{(1+r)\,r} \\
&= \textit{undefined}
\end{aligned}
$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$
y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)
$$

Therefore

$$
\begin{aligned}
\frac{d}{dx}y_2(x) &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\
&= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right) \\
\frac{d^2}{dx^2}y_2(x) &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\
&= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right)
\end{aligned}
$$

Substituting these back into the given ode $xy'' - y = 0$ gives

$$
\begin{aligned}
&\left(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2}\right. \\
&\left.+ \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x \\
&- Cy_1(x)\ln(x) - \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned}
$$

Which can be written as

$$
\begin{aligned}
&\left(\left(y_1''(x)\,x - y_1(x)\right)\ln(x) + \left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right)x\right)C \\
&+ \left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x - \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned} \tag{7}
$$

But since $y_1(x)$ is a solution to the ode, then

$$
y_1''(x)\,x - y_1(x) = 0
$$

Eq (7) simplifes to

$$\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right) xC + \left(\sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2} (n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2} (n+r_2)}{x^2}\right)\right) x \\ - \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0 \tag{8}$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\frac{\left(2\left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n (n+r_1)\right) x - \left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right)\right) C}{x} \\ + \frac{\left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2) (-1+n+r_2)\right) x^2 - \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) x}{x} = 0 \tag{9}$$

Since $r_1 = 1$ and $r_2 = 0$ then the above becomes

$$\frac{\left(2\left(\sum_{n=0}^{\infty} x^n a_n (n+1)\right) x - \left(\sum_{n=0}^{\infty} a_n x^{n+1}\right)\right) C}{x} \\ + \frac{\left(\sum_{n=0}^{\infty} x^{-2+n} b_n n(n-1)\right) x^2 - \left(\sum_{n=0}^{\infty} b_n x^n\right) x}{x} = 0 \tag{10}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2C\, x^n a_n (n+1)\right) + \sum_{n=0}^{\infty} (-C\, x^n a_n) + \left(\sum_{n=0}^{\infty} n\, x^{n-1} b_n (n-1)\right) + \sum_{n=0}^{\infty} (-b_n x^n) = 0 \tag{2A}$$

The next step is to make all powers of x be $n-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2C\, x^n a_n (n+1) = \sum_{n=1}^{\infty} 2C a_{n-1} n\, x^{n-1}$$

$$\sum_{n=0}^{\infty} \left(-C\, x^n a_n\right) = \sum_{n=1}^{\infty} \left(-C a_{n-1} x^{n-1}\right)$$

$$\sum_{n=0}^{\infty} \left(-b_n x^n\right) = \sum_{n=1}^{\infty} \left(-b_{n-1} x^{n-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-1$.

$$\left(\sum_{n=1}^{\infty} 2C a_{n-1} n\, x^{n-1}\right) + \sum_{n=1}^{\infty} \left(-C a_{n-1} x^{n-1}\right) + \left(\sum_{n=0}^{\infty} n\, x^{n-1} b_n (n-1)\right) + \sum_{n=1}^{\infty} \left(-b_{n-1} x^{n-1}\right) = 0 \tag{2B}$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0 = 1$. For $n = N$, where $N = 1$ which is the difference between the two roots, we are free to choose $b_1 = 0$. Hence for $n = 1$, Eq (2B) gives

$$C - 1 = 0$$

Which is solved for C. Solving for C gives

$$C = 1$$

For $n = 2$, Eq (2B) gives

$$3Ca_1 - b_1 + 2b_2 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$2b_2 + \frac{3}{2} = 0$$

Solving the above for b_2 gives

$$b_2 = -\frac{3}{4}$$

For $n = 3$, Eq (2B) gives

$$5Ca_2 - b_2 + 6b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$6b_3 + \frac{7}{6} = 0$$

Solving the above for b_3 gives

$$b_3 = -\frac{7}{36}$$

For $n = 4$, Eq (2B) gives

$$7Ca_3 - b_3 + 12b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$12b_4 + \frac{35}{144} = 0$$

Solving the above for b_4 gives

$$b_4 = -\frac{35}{1728}$$

For $n = 5$, Eq (2B) gives

$$9Ca_4 - b_4 + 20b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$20b_5 + \frac{101}{4320} = 0$$

Solving the above for b_5 gives

$$b_5 = -\frac{101}{86400}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = 1$ and all b_n, then the second solution becomes

$$\begin{aligned}y_2(x) = 1\left(x\left(1 + \frac{x}{2} + \frac{x^2}{12} + \frac{x^3}{144} + \frac{x^4}{2880} + \frac{x^5}{86400} + O(x^6)\right)\right)\ln(x)\\ + 1 - \frac{3x^2}{4} - \frac{7x^3}{36} - \frac{35x^4}{1728} - \frac{101x^5}{86400} + O(x^6)\end{aligned}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x \left(1 + \frac{x}{2} + \frac{x^2}{12} + \frac{x^3}{144} + \frac{x^4}{2880} + \frac{x^5}{86400} + O(x^6)\right) \\
&\quad + c_2 \left(1 \left(x \left(1 + \frac{x}{2} + \frac{x^2}{12} + \frac{x^3}{144} + \frac{x^4}{2880} + \frac{x^5}{86400} + O(x^6)\right)\right) \ln(x) + 1 - \frac{3x^2}{4} \right. \\
&\qquad \left. - \frac{7x^3}{36} - \frac{35x^4}{1728} - \frac{101x^5}{86400} + O(x^6)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x \left(1 + \frac{x}{2} + \frac{x^2}{12} + \frac{x^3}{144} + \frac{x^4}{2880} + \frac{x^5}{86400} + O(x^6)\right) \\
&\quad + c_2 \left(x \left(1 + \frac{x}{2} + \frac{x^2}{12} + \frac{x^3}{144} + \frac{x^4}{2880} + \frac{x^5}{86400} + O(x^6)\right) \ln(x) + 1 - \frac{3x^2}{4} - \frac{7x^3}{36} \right. \\
&\qquad \left. - \frac{35x^4}{1728} - \frac{101x^5}{86400} + O(x^6)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x \left(1 + \frac{x}{2} + \frac{x^2}{12} + \frac{x^3}{144} + \frac{x^4}{2880} + \frac{x^5}{86400} + O(x^6)\right) \\
&\quad + c_2 \left(x \left(1 + \frac{x}{2} + \frac{x^2}{12} + \frac{x^3}{144} + \frac{x^4}{2880} + \frac{x^5}{86400} + O(x^6)\right) \ln(x) + 1 - \frac{3x^2}{4} - \frac{7x^3}{36} \right. \\
&\qquad \left. - \frac{35x^4}{1728} - \frac{101x^5}{86400} + O(x^6)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1 x \left(1 + \frac{x}{2} + \frac{x^2}{12} + \frac{x^3}{144} + \frac{x^4}{2880} + \frac{x^5}{86400} + O(x^6)\right) \\
&\quad + c_2 \left(x \left(1 + \frac{x}{2} + \frac{x^2}{12} + \frac{x^3}{144} + \frac{x^4}{2880} + \frac{x^5}{86400} + O(x^6)\right) \ln(x) + 1 - \frac{3x^2}{4} - \frac{7x^3}{36} \right. \\
&\qquad \left. - \frac{35x^4}{1728} - \frac{101x^5}{86400} + O(x^6)\right)
\end{aligned}
$$

Verified OK.

18.19.1 Maple step by step solution

Let's solve

$y''x - y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{y}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{y}{x} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- ○ Define functions

  $\left[P_2(x) = 0, P_3(x) = -\frac{1}{x}\right]$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = 0$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = 0$

- ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y''x - y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- ○ Convert $x \cdot y''$ to series expansion

  $x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r-1}$

- ○ Shift index using $k- >k+1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-1+r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)(k+r) - a_k\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-1+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{0, 1\}$
- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)(k+r) - a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k}{(k+1+r)(k+r)}$
- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k}{(k+1)k}$
- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = \frac{a_k}{(k+1)k}\right]$
- Recursion relation for $r = 1$

  $a_{k+1} = \frac{a_k}{(k+2)(k+1)}$
- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = \frac{a_k}{(k+2)(k+1)}\right]$
- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{1+k}\right), a_{1+k} = \frac{a_k}{(1+k)k}, b_{1+k} = \frac{b_k}{(2+k)(1+k)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 58

```
Order:=6;
dsolve(x*diff(y(x),x$2)-y(x)=0,y(x),type='series',x=0);
```

$$\begin{aligned} y(x) = c_1 x\left(1+\frac{1}{2}x+\frac{1}{12}x^2+\frac{1}{144}x^3+\frac{1}{2880}x^4+\frac{1}{86400}x^5+\mathrm{O}\left(x^6\right)\right) \\ + c_2\left(\ln\left(x\right)\left(x+\frac{1}{2}x^2+\frac{1}{12}x^3+\frac{1}{144}x^4+\frac{1}{2880}x^5+\mathrm{O}\left(x^6\right)\right)\right. \\ \left. + \left(1-\frac{3}{4}x^2-\frac{7}{36}x^3-\frac{35}{1728}x^4-\frac{101}{86400}x^5+\mathrm{O}\left(x^6\right)\right)\right) \end{aligned}$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 85

```
AsymptoticDSolveValue[x*y''[x]-y[x]==0,y[x],{x,0,5}]
```

$$\begin{aligned} y(x) \to c_1\left(\frac{1}{144}x\left(x^3+12x^2+72x+144\right)\log(x)\right. \\ \left. + \frac{-47x^4-480x^3-2160x^2-1728x+1728}{1728}\right) + c_2\left(\frac{x^5}{2880}+\frac{x^4}{144}+\frac{x^3}{12}+\frac{x^2}{2}+x\right) \end{aligned}$$

18.20 problem 14

Internal problem ID [2956]
Internal file name [`OUTPUT/2448_Sunday_June_05_2022_03_13_39_AM_73802209/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.5. page 771
Problem number: 14.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + x(x^2 + 6)\,y' + 6y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + (x^3 + 6x)\,y' + 6y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{x^2 + 6}{x}$$
$$q(x) = \frac{6}{x^2}$$

Table 488: Table $p(x), q(x)$ singularites.

$p(x) = \frac{x^2+6}{x}$	
singularity	type
$x = 0$	"regular"
$x = \infty$	"regular"
$x = -\infty$	"regular"

$q(x) = \frac{6}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty, -\infty]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2 y'' + (x^3 + 6x)\, y' + 6y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + (x^3 + 6x) \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + 6\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r+2} a_n (n+r)\right) \\ + \left(\sum_{n=0}^{\infty} 6x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 6a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n (n+r) = \sum_{n=2}^{\infty} a_{n-2}(n+r-2)\, x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=2}^{\infty} a_{n-2}(n+r-2)\, x^{n+r}\right) \\ + \left(\sum_{n=0}^{\infty} 6x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 6a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) + 6x^{n+r} a_n (n+r) + 6a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$x^r a_0 r(-1+r) + 6x^r a_0 r + 6a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) + 6x^r r + 6x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2 + 5r + 6\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 + 5r + 6 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = -2$$
$$r_2 = -3$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(r^2 + 5r + 6\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \frac{\sum_{n=0}^{\infty} a_n x^n}{x^2}$$
$$y_2(x) = C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x^3}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n-2}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-3}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_{n-2}(n+r-2) + 6a_n(n+r) + 6a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-2}(n+r-2)}{n^2+2nr+r^2+5n+5r+6} \tag{4}$$

Which for the root $r=-2$ becomes

$$a_n = -\frac{a_{n-2}(n-4)}{n\,(n+1)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r=-2$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n=2$, using the above recursive equation gives

$$a_2 = -\frac{r}{r^2+9r+20}$$

Which for the root $r=-2$ becomes

$$a_2 = \frac{1}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{r}{r^2+9r+20}$	$\frac{1}{3}$

For $n=3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{r}{r^2+9r+20}$	$\frac{1}{3}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{r(2+r)}{(5+r)(r+4)(r+7)(6+r)}$$

Which for the root $r = -2$ becomes

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{r}{r^2+9r+20}$	$\frac{1}{3}$
a_3	0	0
a_4	$\frac{r(2+r)}{(5+r)(r+4)(r+7)(6+r)}$	0

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{r}{r^2+9r+20}$	$\frac{1}{3}$
a_3	0	0
a_4	$\frac{r(2+r)}{(5+r)(r+4)(r+7)(6+r)}$	0
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= \frac{1}{x^2}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= \frac{1 + \frac{x^2}{3} + O(x^6)}{x^2} \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \rightarrow r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_1 \\ &= 0 \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r \rightarrow r_2} 0 &= \lim_{r \rightarrow -3} 0 \\ &= 0 \end{aligned}$$

The limit is 0. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\ &= \sum_{n=0}^{\infty} b_n x^{n-3} \end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq(3) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$b_n(n+r)(n+r-1) + b_{n-2}(n+r-2) + 6b_n(n+r) + 6b_n = 0 \tag{4}$$

Which for for the root $r = -3$ becomes

$$b_n(n-3)(n-4) + b_{n-2}(n-5) + 6b_n(n-3) + 6b_n = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = -\frac{b_{n-2}(n+r-2)}{n^2 + 2nr + r^2 + 5n + 5r + 6} \tag{5}$$

Which for the root $r = -3$ becomes

$$b_n = -\frac{b_{n-2}(n-5)}{n^2 - n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -3$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = -\frac{r}{r^2 + 9r + 20}$$

Which for the root $r = -3$ becomes

$$b_2 = \frac{3}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{r}{r^2+9r+20}$	$\frac{3}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{r}{r^2+9r+20}$	$\frac{3}{2}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{r(2+r)}{(r^2+9r+20)(r^2+13r+42)}$$

Which for the root $r = -3$ becomes

$$b_4 = \frac{1}{8}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{r}{r^2+9r+20}$	$\frac{3}{2}$
b_3	0	0
b_4	$\frac{r(2+r)}{(5+r)(r+4)(r+7)(6+r)}$	$\frac{1}{8}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{r}{r^2+9r+20}$	$\frac{3}{2}$
b_3	0	0
b_4	$\frac{r(2+r)}{(5+r)(r+4)(r+7)(6+r)}$	$\frac{1}{8}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= \frac{1}{x^2}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= \frac{1 + \frac{3x^2}{2} + \frac{x^4}{8} + O(x^6)}{x^3} \end{aligned}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= \frac{c_1\left(1+\frac{x^2}{3}+O(x^6)\right)}{x^2} + \frac{c_2\left(1+\frac{3x^2}{2}+\frac{x^4}{8}+O(x^6)\right)}{x^3}
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= \frac{c_1\left(1+\frac{x^2}{3}+O(x^6)\right)}{x^2} + \frac{c_2\left(1+\frac{3x^2}{2}+\frac{x^4}{8}+O(x^6)\right)}{x^3}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1\left(1+\frac{x^2}{3}+O(x^6)\right)}{x^2} + \frac{c_2\left(1+\frac{3x^2}{2}+\frac{x^4}{8}+O(x^6)\right)}{x^3} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1\left(1+\frac{x^2}{3}+O(x^6)\right)}{x^2} + \frac{c_2\left(1+\frac{3x^2}{2}+\frac{x^4}{8}+O(x^6)\right)}{x^3}
$$

Verified OK.

18.20.1 Maple step by step solution

Let's solve

$x^2y'' + (x^3+6x)\,y' + 6y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{6y}{x^2} - \frac{(x^2+6)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(x^2+6)y'}{x} + \frac{6y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- Define functions

  $\left[P_2(x)=\frac{x^2+6}{x}, P_3(x)=\frac{6}{x^2}\right]$
- $x \cdot P_2(x)$ is analytic at $x=0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0}=6$
- $x^2 \cdot P_3(x)$ is analytic at $x=0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0}=6$
- $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x^2y''+x(x^2+6)\,y'+6y=0$
- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y'$ to series expansion for $m=1..3$

  $x^m \cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$
- Shift index using $k->k+1-m$

  $x^m \cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$
- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(3+r)\,(2+r)\,x^r+a_1(4+r)\,(3+r)\,x^{1+r}+\left(\sum_{k=2}^{\infty} (a_k(k+r+3)\,(k+r+2)+a_{k-2}(k-2+r)\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(3+r)\,(2+r)=0$
- Values of r that satisfy the indicial equation

$r \in \{-3, -2\}$

- Each term must be 0

  $a_1(4+r)(3+r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+3)(k+r+2) + a_{k-2}(k-2+r) = 0$

- Shift index using $k- > k+2$

  $a_{k+2}(k+5+r)(k+4+r) + a_k(k+r) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k(k+r)}{(k+5+r)(k+4+r)}$

- Recursion relation for $r = -3$

  $a_{k+2} = -\frac{a_k(k-3)}{(k+2)(k+1)}$

- Solution for $r = -3$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-3}, a_{k+2} = -\frac{a_k(k-3)}{(k+2)(k+1)}, a_1 = 0\right]$

- Recursion relation for $r = -2$; series terminates at $k = 2$

  $a_{k+2} = -\frac{a_k(k-2)}{(k+3)(k+2)}$

- Solution for $r = -2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2} = -\frac{a_k(k-2)}{(k+3)(k+2)}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-3}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-2}\right), a_{2+k} = -\frac{a_k(k-3)}{(2+k)(1+k)}, a_1 = 0, b_{2+k} = -\frac{b_k(k-2)}{(k+3)(2+k)}, b_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric solution.
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 33

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+x*(6+x^2)*diff(y(x),x)+6*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_1\left(1+\frac{1}{3}x^2+\mathrm{O}\left(x^6\right)\right)x+c_2\left(1+\frac{3}{2}x^2+\frac{1}{8}x^4+\mathrm{O}\left(x^6\right)\right)}{x^3}$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 33

```
AsymptoticDSolveValue[x^2*y''[x]+x*(6+x^2)*y'[x]+6*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(\frac{1}{x^3} + \frac{x}{8} + \frac{3}{2x}\right) + c_2 \left(\frac{1}{x^2} + \frac{1}{3}\right)$$

18.21 problem 15

Internal problem ID [2957]
Internal file name [`OUTPUT/2449_Sunday_June_05_2022_03_13_42_AM_36455973/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.5. page 771
Problem number: 15.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + x(1-x)\,y' - y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + \left(-x^2 + x\right) y' - y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{x-1}{x}$$
$$q(x) = -\frac{1}{x^2}$$

Table 490: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{x-1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{1}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + (-x^2 + x)\, y' - y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + (-x^2 + x)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) - \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)\, x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) + x^{n+r} a_n (n+r) - a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) + x^r a_0 r - a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) + x^r r - x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2-1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= 1 \\ r_2 &= -1 \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(r^2 - 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 2$ is an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= x \left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x} \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{1+n} \\ y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-1}\right) \end{aligned}$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) - a_{n-1}(n+r-1) + a_n(n+r) - a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}}{1+n+r} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = \frac{a_{n-1}}{2+n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{1}{2+r}$$

Which for the root $r = 1$ becomes

$$a_1 = \frac{1}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2+r}$	$\frac{1}{3}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{(2+r)(3+r)}$$

Which for the root $r = 1$ becomes

$$a_2 = \frac{1}{12}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2+r}$	$\frac{1}{3}$
a_2	$\frac{1}{(2+r)(3+r)}$	$\frac{1}{12}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{1}{(3+r)(2+r)(4+r)}$$

Which for the root $r = 1$ becomes

$$a_3 = \frac{1}{60}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2+r}$	$\frac{1}{3}$
a_2	$\frac{1}{(2+r)(3+r)}$	$\frac{1}{12}$
a_3	$\frac{1}{(3+r)(2+r)(4+r)}$	$\frac{1}{60}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(2+r)(4+r)(5+r)(3+r)}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{1}{360}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2+r}$	$\frac{1}{3}$
a_2	$\frac{1}{(2+r)(3+r)}$	$\frac{1}{12}$
a_3	$\frac{1}{(3+r)(2+r)(4+r)}$	$\frac{1}{60}$
a_4	$\frac{1}{(2+r)(4+r)(5+r)(3+r)}$	$\frac{1}{360}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{1}{(2+r)(5+r)(3+r)(4+r)(6+r)}$$

Which for the root $r = 1$ becomes

$$a_5 = \frac{1}{2520}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2+r}$	$\frac{1}{3}$
a_2	$\frac{1}{(2+r)(3+r)}$	$\frac{1}{12}$
a_3	$\frac{1}{(3+r)(2+r)(4+r)}$	$\frac{1}{60}$
a_4	$\frac{1}{(2+r)(4+r)(5+r)(3+r)}$	$\frac{1}{360}$
a_5	$\frac{1}{(2+r)(5+r)(3+r)(4+r)(6+r)}$	$\frac{1}{2520}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}
y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \ldots\right) \\
&= x\left(1 + \frac{x}{3} + \frac{x^2}{12} + \frac{x^3}{60} + \frac{x^4}{360} + \frac{x^5}{2520} + O\left(x^6\right)\right)
\end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 2$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\rightarrow r_2} a_2(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned}
a_N &= a_2 \\
&= \frac{1}{(2+r)(3+r)}
\end{aligned}$$

Therefore

$$\begin{aligned}
\lim_{r\rightarrow r_2} \frac{1}{(2+r)(3+r)} &= \lim_{r\rightarrow -1} \frac{1}{(2+r)(3+r)} \\
&= \frac{1}{2}
\end{aligned}$$

The limit is $\frac{1}{2}$. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned}
y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\
&= \sum_{n=0}^{\infty} b_n x^{n-1}
\end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$b_n(n+r)(n+r-1) - b_{n-1}(n+r-1) + b_n(n+r) - b_n = 0 \tag{4}$$

Which for for the root $r = -1$ becomes

$$b_n(n-1)(n-2) - b_{n-1}(n-2) + b_n(n-1) - b_n = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = \frac{b_{n-1}}{1+n+r} \tag{5}$$

Which for the root $r = -1$ becomes

$$b_n = \frac{b_{n-1}}{n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{1}{2+r}$$

Which for the root $r = -1$ becomes

$$b_1 = 1$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{2+r}$	1

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1}{(2+r)(3+r)}$$

Which for the root $r = -1$ becomes

$$b_2 = \frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{2+r}$	1
b_2	$\frac{1}{(2+r)(3+r)}$	$\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{1}{(3+r)(2+r)(4+r)}$$

Which for the root $r = -1$ becomes

$$b_3 = \frac{1}{6}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{2+r}$	1
b_2	$\frac{1}{(2+r)(3+r)}$	$\frac{1}{2}$
b_3	$\frac{1}{(3+r)(2+r)(4+r)}$	$\frac{1}{6}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{(2+r)(4+r)(5+r)(3+r)}$$

Which for the root $r = -1$ becomes

$$b_4 = \frac{1}{24}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{2+r}$	1
b_2	$\frac{1}{(2+r)(3+r)}$	$\frac{1}{2}$
b_3	$\frac{1}{(3+r)(2+r)(4+r)}$	$\frac{1}{6}$
b_4	$\frac{1}{(2+r)(4+r)(5+r)(3+r)}$	$\frac{1}{24}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{1}{(2+r)(5+r)(3+r)(4+r)(6+r)}$$

Which for the root $r = -1$ becomes

$$b_5 = \frac{1}{120}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{2+r}$	1
b_2	$\frac{1}{(2+r)(3+r)}$	$\frac{1}{2}$
b_3	$\frac{1}{(3+r)(2+r)(4+r)}$	$\frac{1}{6}$
b_4	$\frac{1}{(2+r)(4+r)(5+r)(3+r)}$	$\frac{1}{24}$
b_5	$\frac{1}{(2+r)(5+r)(3+r)(4+r)(6+r)}$	$\frac{1}{120}$

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= x\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\
&= \frac{1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6)}{x}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x\left(1+\frac{x}{3}+\frac{x^2}{12}+\frac{x^3}{60}+\frac{x^4}{360}+\frac{x^5}{2520}+O\left(x^6\right)\right)+\frac{c_2\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6)\right)}{x}
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x\left(1+\frac{x}{3}+\frac{x^2}{12}+\frac{x^3}{60}+\frac{x^4}{360}+\frac{x^5}{2520}+O\left(x^6\right)\right)+\frac{c_2\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6)\right)}{x}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = c_1 x\left(1+\frac{x}{3}+\frac{x^2}{12}+\frac{x^3}{60}+\frac{x^4}{360}+\frac{x^5}{2520}+O\left(x^6\right)\right) \\
+ \frac{c_2\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6)\right)}{x}
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = c_1 x\left(1+\frac{x}{3}+\frac{x^2}{12}+\frac{x^3}{60}+\frac{x^4}{360}+\frac{x^5}{2520}+O\left(x^6\right)\right) \\
+ \frac{c_2\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6)\right)}{x}
\end{aligned}
$$

Verified OK.

18.21.1 Maple step by step solution

Let's solve

$x^2y'' + (-x^2 + x)\, y' - y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{y}{x^2} + \frac{(x-1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(x-1)y'}{x} - \frac{y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -\frac{x-1}{x}, P_3(x) = -\frac{1}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = -1$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2y'' - x(x-1)\, y' - y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

  ◦ Shift index using $k- >k + 1 - m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+r)\,(-1+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)\,(k+r-1) - a_{k-1}(k+r-1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+r)\,(-1+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{-1, 1\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k+r-1)\,(a_k(k+r+1) - a_{k-1}) = 0$$

- Shift index using $k- >k+1$

$$(k+r)\,(a_{k+1}(k+2+r) - a_k) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k}{k+2+r}$$

- Recursion relation for $r = -1$

$$a_{k+1} = \frac{a_k}{k+1}$$

- Solution for $r = -1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+1} = \frac{a_k}{k+1}\right]$$

- Recursion relation for $r = 1$

$$a_{k+1} = \frac{a_k}{k+3}$$

- Solution for $r = 1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = \frac{a_k}{k+3}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{1+k}\right), a_{1+k} = \frac{a_k}{1+k}, b_{1+k} = \frac{b_k}{k+3}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 45

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+x*(1-x)*diff(y(x),x)-y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x \left(1 + \frac{1}{3}x + \frac{1}{12}x^2 + \frac{1}{60}x^3 + \frac{1}{360}x^4 + \frac{1}{2520}x^5 + \mathrm{O}\left(x^6\right)\right) + \frac{c_2\left(-2 - 2x - x^2 - \frac{1}{3}x^3 - \frac{1}{12}x^4 - \frac{1}{60}x^5 + \mathrm{O}\left(x^6\right)\right)}{x}$$

✓ Solution by Mathematica

Time used: 0.018 (sec). Leaf size: 64

```
AsymptoticDSolveValue[x^2*y''[x]+x*(1-x)*y'[x]-y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(\frac{x^3}{24} + \frac{x^2}{6} + \frac{x}{2} + \frac{1}{x} + 1\right) + c_2 \left(\frac{x^5}{360} + \frac{x^4}{60} + \frac{x^3}{12} + \frac{x^2}{3} + x\right)$$

18.22 problem 16

Internal problem ID [2958]
Internal file name [`OUTPUT/2450_Sunday_June_05_2022_03_13_45_AM_15374412/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.5. page 771
Problem number: 16.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2y'' + (1-4x)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$4x^2y'' + (1-4x)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$\begin{aligned}p(x) &= 0\\ q(x) &= -\frac{-1+4x}{4x^2}\end{aligned}$$

Table 492: Table $p(x), q(x)$ singularites.

$p(x) = 0$	
singularity	type

$q(x) = -\frac{-1+4x}{4x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$4x^2y'' + (1 - 4x)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$4x^2 \left(\sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2} \right) + (1 - 4x) \left(\sum_{n=0}^{\infty} a_n x^{n+r} \right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n + r)\, (n + r - 1) \right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r} \right) + \sum_{n=0}^{\infty} \left(-4x^{1+n+r} a_n \right) = 0 \quad \text{(2A)}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-4x^{1+n+r} a_n\right) = \sum_{n=1}^{\infty} \left(-4a_{n-1} x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) + \sum_{n=1}^{\infty} \left(-4a_{n-1} x^{n+r}\right) = 0 \quad \text{(2B)}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$4x^{n+r} a_n (n+r)(n+r-1) + a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$4x^r a_0 r(-1+r) + a_0 x^r = 0$$

Or

$$\left(4x^r r(-1+r) + x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^r (2r-1)^2 = 0$$

Since the above is true for all x then the indicial equation becomes

$$(2r-1)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{2}$$
$$r_2 = \frac{1}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^r (2r-1)^2 = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = \frac{1}{2}$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}$$

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+\frac{1}{2}}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$4a_n(n+r)(n+r-1) + a_n - 4a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{4a_{n-1}}{4n^2 + 8nr + 4r^2 - 4n - 4r + 1} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = \frac{a_{n-1}}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{4}{(2r + 1)^2}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_1 = 1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{4}{(2r+1)^2}$	1

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{16}{(2r + 1)^2 (2r + 3)^2}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = \frac{1}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{4}{(2r+1)^2}$	1
a_2	$\frac{16}{(2r+1)^2(2r+3)^2}$	$\frac{1}{4}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{64}{(2r + 1)^2 (2r + 3)^2 (2r + 5)^2}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_3 = \frac{1}{36}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{4}{(2r+1)^2}$	1
a_2	$\frac{16}{(2r+1)^2(2r+3)^2}$	$\frac{1}{4}$
a_3	$\frac{64}{(2r+1)^2(2r+3)^2(2r+5)^2}$	$\frac{1}{36}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{256}{(2r+1)^2 (2r+3)^2 (2r+5)^2 (2r+7)^2}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = \frac{1}{576}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{4}{(2r+1)^2}$	1
a_2	$\frac{16}{(2r+1)^2(2r+3)^2}$	$\frac{1}{4}$
a_3	$\frac{64}{(2r+1)^2(2r+3)^2(2r+5)^2}$	$\frac{1}{36}$
a_4	$\frac{256}{(2r+1)^2(2r+3)^2(2r+5)^2(2r+7)^2}$	$\frac{1}{576}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{1024}{(2r+1)^2 (2r+3)^2 (2r+5)^2 (2r+7)^2 (2r+9)^2}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_5 = \frac{1}{14400}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{4}{(2r+1)^2}$	1
a_2	$\frac{16}{(2r+1)^2(2r+3)^2}$	$\frac{1}{4}$
a_3	$\frac{64}{(2r+1)^2(2r+3)^2(2r+5)^2}$	$\frac{1}{36}$
a_4	$\frac{256}{(2r+1)^2(2r+3)^2(2r+5)^2(2r+7)^2}$	$\frac{1}{576}$
a_5	$\frac{1024}{(2r+1)^2(2r+3)^2(2r+5)^2(2r+7)^2(2r+9)^2}$	$\frac{1}{14400}$

Using the above table, then the first solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= \sqrt{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= \sqrt{x}\left(x + 1 + \frac{x^2}{4} + \frac{x^3}{36} + \frac{x^4}{576} + \frac{x^5}{14400} + O\left(x^6\right)\right)
\end{aligned}
$$

Now the second solution is found. The second solution is given by

$$
y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)
$$

Where b_n is found using

$$
b_n = \frac{d}{dr} a_{n,r}
$$

And the above is then evaluated at $r = \frac{1}{2}$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	$b_n\left(r = \frac{1}{2}\right)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	$\frac{4}{(2r+1)^2}$	1	$-\frac{16}{(2r+1)^3}$	-2
b_2	$\frac{16}{(2r+1)^2(2r+3)^2}$	$\frac{1}{4}$	$\frac{-256r-256}{(2r+1)^3(2r+3)^3}$	$-\frac{3}{4}$
b_3	$\frac{64}{(2r+1)^2(2r+3)^2(2r+5)^2}$	$\frac{1}{36}$	$\frac{-3072r^2-9216r-5888}{(2r+1)^3(2r+3)^3(2r+5)^3}$	$-\frac{11}{108}$
b_4	$\frac{256}{(2r+1)^2(2r+3)^2(2r+5)^2(2r+7)^2}$	$\frac{1}{576}$	$-\frac{32768(2+r)\left(r^2+4r+\frac{11}{4}\right)}{(2r+1)^3(2r+3)^3(2r+5)^3(2r+7)^3}$	$-\frac{25}{3456}$
b_5	$\frac{1024}{(2r+1)^2(2r+3)^2(2r+5)^2(2r+7)^2(2r+9)^2}$	$\frac{1}{14400}$	$-\frac{4096\left(80r^4+800r^3+2760r^2+3800r+1689\right)}{(2r+1)^3(2r+3)^3(2r+5)^3(2r+7)^3(2r+9)^3}$	$-\frac{137}{432000}$

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 \ldots \\
&= \sqrt{x}\left(x+1+\frac{x^2}{4}+\frac{x^3}{36}+\frac{x^4}{576}+\frac{x^5}{14400}+O(x^6)\right)\ln(x) \\
&\quad + \sqrt{x}\left(-2x-\frac{3x^2}{4}-\frac{11x^3}{108}-\frac{25x^4}{3456}-\frac{137x^5}{432000}+O(x^6)\right)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1y_1(x) + c_2y_2(x) \\
&= c_1\sqrt{x}\left(x+1+\frac{x^2}{4}+\frac{x^3}{36}+\frac{x^4}{576}+\frac{x^5}{14400}+O(x^6)\right) \\
&\quad + c_2\left(\sqrt{x}\left(x+1+\frac{x^2}{4}+\frac{x^3}{36}+\frac{x^4}{576}+\frac{x^5}{14400}+O(x^6)\right)\ln(x)\right. \\
&\qquad \left.+ \sqrt{x}\left(-2x-\frac{3x^2}{4}-\frac{11x^3}{108}-\frac{25x^4}{3456}-\frac{137x^5}{432000}+O(x^6)\right)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1\sqrt{x}\left(x+1+\frac{x^2}{4}+\frac{x^3}{36}+\frac{x^4}{576}+\frac{x^5}{14400}+O(x^6)\right) \\
&\quad + c_2\left(\sqrt{x}\left(x+1+\frac{x^2}{4}+\frac{x^3}{36}+\frac{x^4}{576}+\frac{x^5}{14400}+O(x^6)\right)\ln(x)\right. \\
&\qquad \left.+ \sqrt{x}\left(-2x-\frac{3x^2}{4}-\frac{11x^3}{108}-\frac{25x^4}{3456}-\frac{137x^5}{432000}+O(x^6)\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1\sqrt{x}\left(x+1+\frac{x^2}{4}+\frac{x^3}{36}+\frac{x^4}{576}+\frac{x^5}{14400}+O(x^6)\right) \\
&\quad + c_2\left(\sqrt{x}\left(x+1+\frac{x^2}{4}+\frac{x^3}{36}+\frac{x^4}{576}+\frac{x^5}{14400}+O(x^6)\right)\ln(x)\right. \\
&\qquad \left.+ \sqrt{x}\left(-2x-\frac{3x^2}{4}-\frac{11x^3}{108}-\frac{25x^4}{3456}-\frac{137x^5}{432000}+O(x^6)\right)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$y = c_1\sqrt{x}\left(x+1+\frac{x^2}{4}+\frac{x^3}{36}+\frac{x^4}{576}+\frac{x^5}{14400}+O(x^6)\right) + c_2\left(\sqrt{x}\left(x+1+\frac{x^2}{4}+\frac{x^3}{36}+\frac{x^4}{576}+\frac{x^5}{14400}+O(x^6)\right)\ln(x) + \sqrt{x}\left(-2x-\frac{3x^2}{4}-\frac{11x^3}{108}-\frac{25x^4}{3456}-\frac{137x^5}{432000}+O(x^6)\right)\right)$$

Verified OK.

18.22.1 Maple step by step solution

Let's solve

$4x^2y''+(1-4x)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{(-1+4x)y}{4x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{(-1+4x)y}{4x^2}=0$

□ Check to see if $x_0=0$ is a regular singular point

  - Define functions

    $\left[P_2(x)=0, P_3(x)=-\frac{-1+4x}{4x^2}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=0$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2\cdot P_3(x))\right|_{x=0}=\frac{1}{4}$

  - $x=0$is a regular singular point

    Check to see if $x_0=0$ is a regular singular point

    $x_0=0$

- Multiply by denominators

$$4x^2y'' + (1-4x)\,y = 0$$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- ○ Shift index using $k- >k-m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- ○ Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\,x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1+2r)^2\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r-1)^2 - 4a_{k-1}\right)x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-1+2r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = \tfrac{1}{2}$$

- Each term in the series must be 0, giving the recursion relation

$$4\left(k+r-\tfrac{1}{2}\right)^2 a_k - 4a_{k-1} = 0$$

- Shift index using $k- >k+1$

$$4\left(k+\tfrac{1}{2}+r\right)^2 a_{k+1} - 4a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{4a_k}{(2k+1+2r)^2}$$

- Recursion relation for $r = \frac{1}{2}$

$$a_{k+1} = \frac{4a_k}{(2k+2)^2}$$

- Solution for $r = \frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1}=\frac{4a_k}{(2k+2)^2}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 69

```
Order:=6;
dsolve(4*x^2*diff(y(x),x$2)+(1-4*x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x)=\sqrt{x}\left((c_2\ln(x)+c_1)\left(1+x+\frac{1}{4}x^2+\frac{1}{36}x^3+\frac{1}{576}x^4+\frac{1}{14400}x^5+\mathrm{O}\left(x^6\right)\right)+\left((-2)\,x-\frac{3}{4}x^2-\frac{11}{108}x^3-\frac{25}{3456}x^4-\frac{137}{432000}x^5+\mathrm{O}\left(x^6\right)\right)c_2\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 124

```
AsymptoticDSolveValue[4*x^2*y''[x]+(1-4*x)*y[x]==0,y[x],{x,0,5}]
```

$$
\begin{aligned}
y(x) \to\; & c_1\sqrt{x}\left(\frac{x^5}{14400}+\frac{x^4}{576}+\frac{x^3}{36}+\frac{x^2}{4}+x+1\right)\\
&+c_2\left(\sqrt{x}\left(-\frac{137x^5}{432000}-\frac{25x^4}{3456}-\frac{11x^3}{108}-\frac{3x^2}{4}-2x\right)\right.\\
&\left.+\sqrt{x}\left(\frac{x^5}{14400}+\frac{x^4}{576}+\frac{x^3}{36}+\frac{x^2}{4}+x+1\right)\log(x)\right)
\end{aligned}
$$

18.23 problem 17

Internal problem ID [2959]
Internal file name [`OUTPUT/2451_Sunday_June_05_2022_03_13_48_AM_843029/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.5. page 771
Problem number: 17.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{xy'' + y' - 2y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + y' - 2y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = -\frac{2}{x}$$

Table 494: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{2}{x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + y' - 2y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) x + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) - 2\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-2a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-2a_n x^{n+r}\right) = \sum_{n=1}^{\infty} \left(-2a_{n-1} x^{n+r-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \sum_{n=1}^{\infty} \left(-2a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) + (n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) + r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) + r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^{-1+r} r^2 = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^{-1+r} r^2 = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_n(n+r) - 2a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{2a_{n-1}}{n^2 + 2nr + r^2} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = \frac{2a_{n-1}}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{2}{(r+1)^2}$$

Which for the root $r = 0$ becomes

$$a_1 = 2$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2}{(r+1)^2}$	2

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{4}{(r+1)^2 (r+2)^2}$$

Which for the root $r = 0$ becomes

$$a_2 = 1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2}{(r+1)^2}$	2
a_2	$\frac{4}{(r+1)^2(r+2)^2}$	1

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{8}{(r+1)^2 (r+2)^2 (3+r)^2}$$

Which for the root $r = 0$ becomes

$$a_3 = \frac{2}{9}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2}{(r+1)^2}$	2
a_2	$\frac{4}{(r+1)^2(r+2)^2}$	1
a_3	$\frac{8}{(r+1)^2(r+2)^2(3+r)^2}$	$\frac{2}{9}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{16}{(r+1)^2 (r+2)^2 (3+r)^2 (r+4)^2}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{1}{36}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2}{(r+1)^2}$	2
a_2	$\frac{4}{(r+1)^2(r+2)^2}$	1
a_3	$\frac{8}{(r+1)^2(r+2)^2(3+r)^2}$	$\frac{2}{9}$
a_4	$\frac{16}{(r+1)^2(r+2)^2(3+r)^2(r+4)^2}$	$\frac{1}{36}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{32}{(r+1)^2 (r+2)^2 (3+r)^2 (r+4)^2 (5+r)^2}$$

Which for the root $r = 0$ becomes

$$a_5 = \frac{1}{450}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2}{(r+1)^2}$	2
a_2	$\frac{4}{(r+1)^2(r+2)^2}$	1
a_3	$\frac{8}{(r+1)^2(r+2)^2(3+r)^2}$	$\frac{2}{9}$
a_4	$\frac{16}{(r+1)^2(r+2)^2(3+r)^2(r+4)^2}$	$\frac{1}{36}$
a_5	$\frac{32}{(r+1)^2(r+2)^2(3+r)^2(r+4)^2(5+r)^2}$	$\frac{1}{450}$

Using the above table, then the first solution $y_1(x)$ becomes

$$
\begin{aligned}
y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots \\
&= x^2 + 2x + 1 + \frac{2x^3}{9} + \frac{x^4}{36} + \frac{x^5}{450} + O(x^6)
\end{aligned}
$$

Now the second solution is found. The second solution is given by

$$
y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)
$$

Where b_n is found using

$$
b_n = \frac{d}{dr} a_{n,r}
$$

And the above is then evaluated at $r = 0$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	$b_n(r = 0)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	$\frac{2}{(r+1)^2}$	2	$-\frac{4}{(r+1)^3}$	-4
b_2	$\frac{4}{(r+1)^2 (r+2)^2}$	1	$\frac{-16r-24}{(r+1)^3 (r+2)^3}$	-3
b_3	$\frac{8}{(r+1)^2 (r+2)^2 (3+r)^2}$	$\frac{2}{9}$	$\frac{-48r^2-192r-176}{(r+1)^3 (r+2)^3 (3+r)^3}$	$-\frac{22}{27}$
b_4	$\frac{16}{(r+1)^2 (r+2)^2 (3+r)^2 (r+4)^2}$	$\frac{1}{36}$	$-\frac{64(2r^3+15r^2+35r+25)}{(r+1)^3 (r+2)^3 (3+r)^3 (r+4)^3}$	$-\frac{25}{216}$
b_5	$\frac{32}{(r+1)^2 (r+2)^2 (3+r)^2 (r+4)^2 (5+r)^2}$	$\frac{1}{450}$	$-\frac{64(5r^4+60r^3+255r^2+450r+274)}{(r+1)^3 (r+2)^3 (3+r)^3 (r+4)^3 (5+r)^3}$	$-\frac{137}{13500}$

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x) \ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\
&= \left(x^2+2x+1+\frac{2x^3}{9}+\frac{x^4}{36}+\frac{x^5}{450}+O(x^6)\right) \ln(x) - 3x^2 - 4x - \frac{22x^3}{27} - \frac{25x^4}{216} - \frac{137x^5}{13500} \\
&\quad + O(x^6)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 \left(x^2 + 2x + 1 + \frac{2x^3}{9} + \frac{x^4}{36} + \frac{x^5}{450} + O(x^6)\right) \\
&\quad + c_2 \left(\left(x^2 + 2x + 1 + \frac{2x^3}{9} + \frac{x^4}{36} + \frac{x^5}{450} + O(x^6)\right) \ln(x) - 3x^2 - 4x - \frac{22x^3}{27} \right. \\
&\quad \left. - \frac{25x^4}{216} - \frac{137x^5}{13500} + O(x^6)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 \left(x^2 + 2x + 1 + \frac{2x^3}{9} + \frac{x^4}{36} + \frac{x^5}{450} + O(x^6)\right) \\
&\quad + c_2 \left(\left(x^2 + 2x + 1 + \frac{2x^3}{9} + \frac{x^4}{36} + \frac{x^5}{450} + O(x^6)\right) \ln(x) - 3x^2 - 4x - \frac{22x^3}{27} - \frac{25x^4}{216} \right. \\
&\quad \left. - \frac{137x^5}{13500} + O(x^6)\right)
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 \left(x^2 + 2x + 1 + \frac{2x^3}{9} + \frac{x^4}{36} + \frac{x^5}{450} + O(x^6)\right) \\
&\quad + c_2 \left(\left(x^2 + 2x + 1 + \frac{2x^3}{9} + \frac{x^4}{36} + \frac{x^5}{450} + O(x^6)\right) \ln(x) - 3x^2 - 4x - \frac{22x^3}{27} \right. \\
&\quad \left. - \frac{25x^4}{216} - \frac{137x^5}{13500} + O(x^6)\right)
\end{aligned} \tag{1}
$$

<u>Verification of solutions</u>

$$
\begin{aligned}
y &= c_1 \left(x^2 + 2x + 1 + \frac{2x^3}{9} + \frac{x^4}{36} + \frac{x^5}{450} + O(x^6)\right) \\
&\quad + c_2 \left(\left(x^2 + 2x + 1 + \frac{2x^3}{9} + \frac{x^4}{36} + \frac{x^5}{450} + O(x^6)\right) \ln(x) - 3x^2 - 4x - \frac{22x^3}{27} - \frac{25x^4}{216} \right. \\
&\quad \left. - \frac{137x^5}{13500} + O(x^6)\right)
\end{aligned}
$$

Verified OK.

18.23.1 Maple step by step solution

Let's solve

$y''x + y' - 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{2y}{x} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} - \frac{2y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = -\frac{2}{x}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = 0$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $y''x + y' - 2y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert y' to series expansion

    $y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1}$

  - Shift index using $k- >k + 1$

$$y' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

$$x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k->k+1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r^2 x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)^2 - 2a_k\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = 0$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k+1}(k+1)^2 - 2a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{2a_k}{(k+1)^2}$$

- Recursion relation for $r = 0$

$$a_{k+1} = \frac{2a_k}{(k+1)^2}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = \frac{2a_k}{(k+1)^2}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 59

```
Order:=6;
dsolve(x*diff(y(x),x$2)+diff(y(x),x)-2*y(x)=0,y(x),type='series',x=0);
```

$$
\begin{aligned}
y(x) = {} & (c_2 \ln(x) + c_1)\left(1 + 2x + x^2 + \frac{2}{9}x^3 + \frac{1}{36}x^4 + \frac{1}{450}x^5 + \mathrm{O}\left(x^6\right)\right) \\
& + \left((-4)\,x - 3x^2 - \frac{22}{27}x^3 - \frac{25}{216}x^4 - \frac{137}{13500}x^5 + \mathrm{O}\left(x^6\right)\right) c_2
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 101

```
AsymptoticDSolveValue[x*y''[x]+y'[x]-2*y[x]==0,y[x],{x,0,5}]
```

$$
\begin{aligned}
y(x) \to {} & c_1\left(\frac{x^5}{450} + \frac{x^4}{36} + \frac{2x^3}{9} + x^2 + 2x + 1\right) \\
& + c_2\left(-\frac{137x^5}{13500} - \frac{25x^4}{216} - \frac{22x^3}{27} - 3x^2 + \left(\frac{x^5}{450} + \frac{x^4}{36} + \frac{2x^3}{9} + x^2 + 2x + 1\right)\log(x) - 4x\right)
\end{aligned}
$$

18.24 problem 18

Internal problem ID [2960]
Internal file name [`OUTPUT/2452_Sunday_June_05_2022_03_13_55_AM_63124368/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.5. page 771
Problem number: 18.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + xy' - y(x+1) = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + xy' + (-x-1)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = -\frac{x+1}{x^2}$$

Table 496: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{x+1}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2 y'' + x y' + (-x - 1)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned} &x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) \\ &+ x \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + (-x-1) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \end{aligned} \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) \tag{2A}$$
$$+\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n\right) = \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) \tag{2B}$$
$$+\sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) + x^{n+r} a_n (n+r) - a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) + x^r a_0 r - a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) + x^r r - x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2-1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = -1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(r^2 - 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 2$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{1+n}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-1}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_n(n+r) - a_{n-1} - a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}}{n^2 + 2nr + r^2 - 1} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = \frac{a_{n-1}}{n(n+2)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{1}{r(r+2)}$$

Which for the root $r = 1$ becomes

$$a_1 = \frac{1}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{r(r+2)}$	$\frac{1}{3}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{r(r+2)(3+r)(r+1)}$$

Which for the root $r = 1$ becomes

$$a_2 = \frac{1}{24}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{r(r+2)}$	$\frac{1}{3}$
a_2	$\frac{1}{r(r+2)(3+r)(r+1)}$	$\frac{1}{24}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{1}{r \left(r+2\right)^{2} \left(3+r\right) \left(r+1\right) \left(r+4\right)}$$

Which for the root $r = 1$ becomes

$$a_3 = \frac{1}{360}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{r(r+2)}$	$\frac{1}{3}$
a_2	$\frac{1}{r(r+2)(3+r)(r+1)}$	$\frac{1}{24}$
a_3	$\frac{1}{r(r+2)^2(3+r)(r+1)(r+4)}$	$\frac{1}{360}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{r \left(r+2\right)^{2} \left(3+r\right)^{2} \left(r+1\right) \left(r+4\right) \left(5+r\right)}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{1}{8640}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{r(r+2)}$	$\frac{1}{3}$
a_2	$\frac{1}{r(r+2)(3+r)(r+1)}$	$\frac{1}{24}$
a_3	$\frac{1}{r(r+2)^2(3+r)(r+1)(r+4)}$	$\frac{1}{360}$
a_4	$\frac{1}{r(r+2)^2(3+r)^2(r+1)(r+4)(5+r)}$	$\frac{1}{8640}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{1}{r \left(r+2\right)^{2} \left(3+r\right)^{2} \left(r+1\right) \left(r+4\right)^{2} \left(5+r\right) \left(6+r\right)}$$

Which for the root $r = 1$ becomes

$$a_5 = \frac{1}{302400}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{r(r+2)}$	$\frac{1}{3}$
a_2	$\frac{1}{r(r+2)(3+r)(r+1)}$	$\frac{1}{24}$
a_3	$\frac{1}{r(r+2)^2(3+r)(r+1)(r+4)}$	$\frac{1}{360}$
a_4	$\frac{1}{r(r+2)^2(3+r)^2(r+1)(r+4)(5+r)}$	$\frac{1}{8640}$
a_5	$\frac{1}{r(r+2)^2(3+r)^2(r+1)(r+4)^2(5+r)(6+r)}$	$\frac{1}{302400}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x\left(1 + \frac{x}{3} + \frac{x^2}{24} + \frac{x^3}{360} + \frac{x^4}{8640} + \frac{x^5}{302400} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 2$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_2(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_2 \\ &= \frac{1}{r(r+2)(3+r)(r+1)} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r \to r_2} \frac{1}{r(r+2)(3+r)(r+1)} &= \lim_{r \to -1} \frac{1}{r(r+2)(3+r)(r+1)} \\ &= \textit{undefined} \end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$\begin{aligned}
\frac{d}{dx}y_2(x) &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\
&= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right) \\
\frac{d^2}{dx^2}y_2(x) &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\
&= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right)
\end{aligned}$$

Substituting these back into the given ode $x^2y'' + xy' + (-x-1)\,y = 0$ gives

$$\begin{aligned}
&x^2\left(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2}\right. \\
&\left.\quad + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) \\
&\quad + x\left(Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right) \\
&\quad + (-x-1)\left(Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)\right) = 0
\end{aligned}$$

Which can be written as

$$\left(\left(\left(x^2 y_1''(x) + y_1'(x)\, x + (-x-1)\, y_1(x)\right) \ln(x) + x^2 \left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right) + y_1(x)\right) C + x^2 \left(\sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2} (n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2} (n+r_2)}{x^2}\right)\right) + x \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2} (n+r_2)}{x}\right) + (-x-1) \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0 \tag{7}$$

But since $y_1(x)$ is a solution to the ode, then

$$x^2 y_1''(x) + y_1'(x)\, x + (-x-1)\, y_1(x) = 0$$

Eq (7) simplifes to

$$\left(x^2 \left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right) + y_1(x)\right) C + x^2 \left(\sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2} (n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2} (n+r_2)}{x^2}\right)\right) + x \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2} (n+r_2)}{x}\right) + (-x-1) \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0 \tag{8}$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2) (-1+n+r_2)\right) x^2 + 2x \left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n (n+r_1)\right) C + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) x - (x+1) \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0 \tag{9}$$

Since $r_1 = 1$ and $r_2 = -1$ then the above becomes

$$\left(\sum_{n=0}^{\infty} x^{-3+n} b_n (n-1) (-2+n)\right) x^2 + 2x \left(\sum_{n=0}^{\infty} x^{n} a_n (1+n)\right) C + \left(\sum_{n=0}^{\infty} x^{-2+n} b_n (n-1)\right) x - (x+1) \left(\sum_{n=0}^{\infty} b_n x^{n-1}\right) = 0 \tag{10}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n-1} b_n \left(n^2-3n+2\right)\right)+\left(\sum_{n=0}^{\infty} 2C\, x^{1+n} a_n (1+n)\right) \tag{2A}$$
$$+\left(\sum_{n=0}^{\infty} x^{n-1} b_n (n-1)\right)+\sum_{n=0}^{\infty}\left(-b_n x^n\right)+\sum_{n=0}^{\infty}\left(-b_n x^{n-1}\right)=0$$

The next step is to make all powers of x be $n-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2C\, x^{1+n} a_n (1+n)=\sum_{n=2}^{\infty} 2C a_{-2+n} (n-1)\, x^{n-1}$$

$$\sum_{n=0}^{\infty}\left(-b_n x^n\right)=\sum_{n=1}^{\infty}\left(-b_{n-1} x^{n-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-1$.

$$\left(\sum_{n=0}^{\infty} x^{n-1} b_n \left(n^2-3n+2\right)\right)+\left(\sum_{n=2}^{\infty} 2C a_{-2+n} (n-1)\, x^{n-1}\right) \tag{2B}$$
$$+\left(\sum_{n=0}^{\infty} x^{n-1} b_n (n-1)\right)+\sum_{n=1}^{\infty}\left(-b_{n-1} x^{n-1}\right)+\sum_{n=0}^{\infty}\left(-b_n x^{n-1}\right)=0$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0=1$. For $n=1$, Eq (2B) gives

$$-b_0-b_1=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-1-b_1=0$$

Solving the above for b_1 gives

$$b_1=-1$$

For $n=N$, where $N=2$ which is the difference between the two roots, we are free to choose $b_2=0$. Hence for $n=2$, Eq (2B) gives

$$2C+1=0$$

Which is solved for C. Solving for C gives

$$C = -\frac{1}{2}$$

For $n = 3$, Eq (2B) gives

$$4Ca_1 - b_2 + 3b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$3b_3 - \frac{2}{3} = 0$$

Solving the above for b_3 gives

$$b_3 = \frac{2}{9}$$

For $n = 4$, Eq (2B) gives

$$6Ca_2 - b_3 + 8b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$8b_4 - \frac{25}{72} = 0$$

Solving the above for b_4 gives

$$b_4 = \frac{25}{576}$$

For $n = 5$, Eq (2B) gives

$$8Ca_3 - b_4 + 15b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$15b_5 - \frac{157}{2880} = 0$$

Solving the above for b_5 gives

$$b_5 = \frac{157}{43200}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -\frac{1}{2}$ and all b_n, then the second solution becomes

$$y_2(x) = -\frac{1}{2}\left(x\left(1 + \frac{x}{3} + \frac{x^2}{24} + \frac{x^3}{360} + \frac{x^4}{8640} + \frac{x^5}{302400} + O(x^6)\right)\right)\ln(x) + \frac{1 - x + \frac{2x^3}{9} + \frac{25x^4}{576} + \frac{157x^5}{43200} + O(x^6)}{x}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x \left(1+\frac{x}{3}+\frac{x^2}{24}+\frac{x^3}{360}+\frac{x^4}{8640}+\frac{x^5}{302400}+O\left(x^6\right)\right) \\
&\quad + c_2 \left(-\frac{1}{2}\left(x\left(1+\frac{x}{3}+\frac{x^2}{24}+\frac{x^3}{360}+\frac{x^4}{8640}+\frac{x^5}{302400}+O\left(x^6\right)\right)\right)\ln\left(x\right) \right. \\
&\quad \left. + \frac{1-x+\frac{2x^3}{9}+\frac{25x^4}{576}+\frac{157x^5}{43200}+O(x^6)}{x}\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x \left(1+\frac{x}{3}+\frac{x^2}{24}+\frac{x^3}{360}+\frac{x^4}{8640}+\frac{x^5}{302400}+O\left(x^6\right)\right) \\
&\quad + c_2 \left(-\frac{x\left(1+\frac{x}{3}+\frac{x^2}{24}+\frac{x^3}{360}+\frac{x^4}{8640}+\frac{x^5}{302400}+O(x^6)\right)\ln\left(x\right)}{2} \right. \\
&\quad \left. + \frac{1-x+\frac{2x^3}{9}+\frac{25x^4}{576}+\frac{157x^5}{43200}+O(x^6)}{x}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x \left(1+\frac{x}{3}+\frac{x^2}{24}+\frac{x^3}{360}+\frac{x^4}{8640}+\frac{x^5}{302400}+O\left(x^6\right)\right) \\
&\quad + c_2 \left(-\frac{x\left(1+\frac{x}{3}+\frac{x^2}{24}+\frac{x^3}{360}+\frac{x^4}{8640}+\frac{x^5}{302400}+O(x^6)\right)\ln\left(x\right)}{2} \right. \\
&\quad \left. + \frac{1-x+\frac{2x^3}{9}+\frac{25x^4}{576}+\frac{157x^5}{43200}+O(x^6)}{x}\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$y = c_1 x \left(1 + \frac{x}{3} + \frac{x^2}{24} + \frac{x^3}{360} + \frac{x^4}{8640} + \frac{x^5}{302400} + O(x^6)\right) + c_2 \left(-\frac{x\left(1 + \frac{x}{3} + \frac{x^2}{24} + \frac{x^3}{360} + \frac{x^4}{8640} + \frac{x^5}{302400} + O(x^6)\right) \ln(x)}{2} + \frac{1 - x + \frac{2x^3}{9} + \frac{25x^4}{576} + \frac{157x^5}{43200} + O(x^6)}{x}\right)$$

Verified OK.

18.24.1 Maple step by step solution

Let's solve

$x^2 y'' + x y' + (-x - 1) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y'}{x} + \frac{(x+1)y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} - \frac{(x+1)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = -\frac{x+1}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = -1$

  ◦ $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $x^2y''+xy'+(-x-1)\,y=0$
- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m\cdot y$ to series expansion for $m=0..1$

    $x^m\cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k->k-m$

    $x^m\cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x\cdot y'$ to series expansion

    $x\cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r}$
  - Convert $x^2\cdot y''$ to series expansion

    $x^2\cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+r)\,(-1+r)\,x^r+\left(\sum_{k=1}^{\infty} (a_k(k+r+1)\,(k+r-1)-a_{k-1})\,x^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)\,(-1+r)=0$
- Values of r that satisfy the indicial equation

  $r\in\{-1,1\}$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+1)\,(k+r-1)-a_{k-1}=0$
- Shift index using $k->k+1$

  $a_{k+1}(k+2+r)\,(k+r)-a_k=0$
- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{a_k}{(k+2+r)(k+r)}$
- Recursion relation for $r=-1$

$$a_{k+1} = \frac{a_k}{(k+1)(k-1)}$$

- Series not valid for $r = -1$, division by 0 in the recursion relation at $k = 1$

$$a_{k+1} = \frac{a_k}{(k+1)(k-1)}$$

- Recursion relation for $r = 1$

$$a_{k+1} = \frac{a_k}{(k+3)(k+1)}$$

- Solution for $r = 1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = \frac{a_k}{(k+3)(k+1)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 63

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)-(1+x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_1 x^2 \left(1 + \frac{1}{3}x + \frac{1}{24}x^2 + \frac{1}{360}x^3 + \frac{1}{8640}x^4 + \frac{1}{302400}x^5 + O\left(x^6\right)\right) + c_2 \left(\ln\left(x\right) \left(x^2 + \frac{1}{3}x^3 + \frac{1}{24}x^4 + \frac{1}{360}x^5 + O\left(x^6\right.\right.\right.}{x}$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 83

```
AsymptoticDSolveValue[x^2*y''[x]+x*y'[x]-(1+x)*y[x]==0,y[x],{x,0,5}]
```

$$
y(x) \to c_1 \left(\frac{31x^4 + 176x^3 + 144x^2 - 576x + 576}{576x} - \frac{1}{48} x(x^2 + 8x + 24) \log(x) \right) + c_2 \left(\frac{x^5}{8640} + \frac{x^4}{360} + \frac{x^3}{24} + \frac{x^2}{3} + x \right)
$$

18.25 problem 19

Internal problem ID [2961]
Internal file name [`OUTPUT/2453_Sunday_June_05_2022_03_13_59_AM_65157335/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.5. page 771
Problem number: 19.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - x(x+3)\,y' + 4y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + \left(-x^2 - 3x\right) y' + 4y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{x+3}{x}$$
$$q(x) = \frac{4}{x^2}$$

Table 498: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{x+3}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{4}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + \left(-x^2 - 3x\right) y' + 4y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$
$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + \left(-x^2 - 3x\right)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + 4\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-3x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 4a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)\, x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)\, x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-3x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 4a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) - 3x^{n+r} a_n (n+r) + 4a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) - 3x^r a_0 r + 4a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) - 3x^r r + 4x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(r-2)^2 x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(r-2)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 2$$
$$r_2 = 2$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(r-2)^2 x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = 2$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+2}$$
$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+2}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) - a_{n-1}(n+r-1) - 3a_n(n+r) + 4a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}(n+r-1)}{n^2 + 2nr + r^2 - 4n - 4r + 4} \tag{4}$$

Which for the root $r = 2$ becomes

$$a_n = \frac{a_{n-1}(1+n)}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 2$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{r}{(-1+r)^2}$$

Which for the root $r = 2$ becomes

$$a_1 = 2$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r}{(-1+r)^2}$	2

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1+r}{r(-1+r)^2}$$

Which for the root $r = 2$ becomes

$$a_2 = \frac{3}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r}{(-1+r)^2}$	2
a_2	$\frac{1+r}{r(-1+r)^2}$	$\frac{3}{2}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{2 + r}{r (1 + r) (-1 + r)^2}$$

Which for the root $r = 2$ becomes

$$a_3 = \frac{2}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r}{(-1+r)^2}$	2
a_2	$\frac{1+r}{r(-1+r)^2}$	$\frac{3}{2}$
a_3	$\frac{2+r}{r(1+r)(-1+r)^2}$	$\frac{2}{3}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{3 + r}{r (1 + r) (-1 + r)^2 (2 + r)}$$

Which for the root $r = 2$ becomes

$$a_4 = \frac{5}{24}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r}{(-1+r)^2}$	2
a_2	$\frac{1+r}{r(-1+r)^2}$	$\frac{3}{2}$
a_3	$\frac{2+r}{r(1+r)(-1+r)^2}$	$\frac{2}{3}$
a_4	$\frac{3+r}{r(1+r)(-1+r)^2(2+r)}$	$\frac{5}{24}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{4 + r}{r (1 + r) (-1 + r)^2 (2 + r) (3 + r)}$$

Which for the root $r = 2$ becomes

$$a_5 = \frac{1}{20}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r}{(-1+r)^2}$	2
a_2	$\frac{1+r}{r(-1+r)^2}$	$\frac{3}{2}$
a_3	$\frac{2+r}{r(1+r)(-1+r)^2}$	$\frac{2}{3}$
a_4	$\frac{3+r}{r(1+r)(-1+r)^2(2+r)}$	$\frac{5}{24}$
a_5	$\frac{4+r}{r(1+r)(-1+r)^2(2+r)(3+r)}$	$\frac{1}{20}$

Using the above table, then the first solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^2\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^2\left(1 + 2x + \frac{3x^2}{2} + \frac{2x^3}{3} + \frac{5x^4}{24} + \frac{x^5}{20} + O(x^6)\right) \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = 2$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr}a_{n,r}$	$b_n(r=2)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	$\frac{r}{(-1+r)^2}$	2	$\frac{-1-r}{(-1+r)^3}$	-3
b_2	$\frac{1+r}{r(-1+r)^2}$	$\frac{3}{2}$	$\frac{-2r^2-3r+1}{r^2(-1+r)^3}$	$-\frac{13}{4}$
b_3	$\frac{2+r}{r(1+r)(-1+r)^2}$	$\frac{2}{3}$	$\frac{-3r^3-9r^2-2r+2}{r^2(1+r)^2(-1+r)^3}$	$-\frac{31}{18}$
b_4	$\frac{3+r}{r(1+r)(-1+r)^2(2+r)}$	$\frac{5}{24}$	$\frac{-4r^4-22r^3-28r^2+6}{r^2(1+r)^2(-1+r)^3(2+r)^2}$	$-\frac{173}{288}$
b_5	$\frac{4+r}{r(1+r)(-1+r)^2(2+r)(3+r)}$	$\frac{1}{20}$	$\frac{-5r^5-45r^4-125r^3-105r^2+16r+24}{r^2(1+r)^2(-1+r)^3(2+r)^2(3+r)^2}$	$-\frac{187}{1200}$

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6\dots \\
&= x^2\left(1+2x+\frac{3x^2}{2}+\frac{2x^3}{3}+\frac{5x^4}{24}+\frac{x^5}{20}+O(x^6)\right)\ln(x) \\
&\quad + x^2\left(-3x-\frac{13x^2}{4}-\frac{31x^3}{18}-\frac{173x^4}{288}-\frac{187x^5}{1200}+O(x^6)\right)
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1y_1(x) + c_2y_2(x) \\
&= c_1x^2\left(1+2x+\frac{3x^2}{2}+\frac{2x^3}{3}+\frac{5x^4}{24}+\frac{x^5}{20}+O(x^6)\right) \\
&\quad + c_2\left(x^2\left(1+2x+\frac{3x^2}{2}+\frac{2x^3}{3}+\frac{5x^4}{24}+\frac{x^5}{20}+O(x^6)\right)\ln(x)\right. \\
&\quad \left. + x^2\left(-3x-\frac{13x^2}{4}-\frac{31x^3}{18}-\frac{173x^4}{288}-\frac{187x^5}{1200}+O(x^6)\right)\right)
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= c_1x^2\left(1+2x+\frac{3x^2}{2}+\frac{2x^3}{3}+\frac{5x^4}{24}+\frac{x^5}{20}+O(x^6)\right) \\
&\quad + c_2\left(x^2\left(1+2x+\frac{3x^2}{2}+\frac{2x^3}{3}+\frac{5x^4}{24}+\frac{x^5}{20}+O(x^6)\right)\ln(x)\right. \\
&\quad \left. + x^2\left(-3x-\frac{13x^2}{4}-\frac{31x^3}{18}-\frac{173x^4}{288}-\frac{187x^5}{1200}+O(x^6)\right)\right)
\end{aligned}$$

Summary
The solution(s) found are the following

$$
\begin{align*}
y &= c_1 x^2 \left(1+2x+\frac{3x^2}{2}+\frac{2x^3}{3}+\frac{5x^4}{24}+\frac{x^5}{20}+O(x^6)\right) \\
&+ c_2 \left(x^2 \left(1+2x+\frac{3x^2}{2}+\frac{2x^3}{3}+\frac{5x^4}{24}+\frac{x^5}{20}+O(x^6)\right)\ln(x) \right. \tag{1} \\
&\left. + x^2 \left(-3x-\frac{13x^2}{4}-\frac{31x^3}{18}-\frac{173x^4}{288}-\frac{187x^5}{1200}+O(x^6)\right)\right)
\end{align*}
$$

Verification of solutions

$$
\begin{align*}
y &= c_1 x^2 \left(1+2x+\frac{3x^2}{2}+\frac{2x^3}{3}+\frac{5x^4}{24}+\frac{x^5}{20}+O(x^6)\right) \\
&+ c_2 \left(x^2 \left(1+2x+\frac{3x^2}{2}+\frac{2x^3}{3}+\frac{5x^4}{24}+\frac{x^5}{20}+O(x^6)\right)\ln(x) \right. \\
&\left. + x^2 \left(-3x-\frac{13x^2}{4}-\frac{31x^3}{18}-\frac{173x^4}{288}-\frac{187x^5}{1200}+O(x^6)\right)\right)
\end{align*}
$$

Verified OK.

18.25.1 Maple step by step solution

Let's solve

$x^2 y'' + (-x^2 - 3x) y' + 4y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{4y}{x^2} + \frac{(x+3)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(x+3)y'}{x} + \frac{4y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{x+3}{x}, P_3(x) = \frac{4}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -3$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 4$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2 y'' - x(x+3)\, y' + 4y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions

  - Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

  - Shift index using $k- >k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

  - Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-2+r)^2 x^r + \left(\sum_{k=1}^{\infty} \left(a_k (k+r-2)^2 - a_{k-1}(k+r-1)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-2+r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = 2$

- Each term in the series must be 0, giving the recursion relation

  $a_k (k+r-2)^2 - a_{k-1}(k+r-1) = 0$

- Shift index using $k- >k+1$

  $a_{k+1}(k+r-1)^2 - a_k (k+r) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r)}{(k+r-1)^2}$

- Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_k(k+2)}{(k+1)^2}$

- Solution for $r = 2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = \frac{a_k(k+2)}{(k+1)^2}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 69

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)-x*(x+3)*diff(y(x),x)+4*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left((c_2 \ln(x) + c_1)\left(1 + 2x + \frac{3}{2}x^2 + \frac{2}{3}x^3 + \frac{5}{24}x^4 + \frac{1}{20}x^5 + \mathrm{O}\left(x^6\right)\right) + \left((-3)\,x - \frac{13}{4}x^2 - \frac{31}{18}x^3 - \frac{173}{288}x^4 - \frac{187}{1200}x^5 + \mathrm{O}\left(x^6\right)\right) c_2\right) x^2$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 122

```
AsymptoticDSolveValue[x^2*y''[x]-x*(x+3)*y'[x]+4*y[x]==0,y[x],{x,0,5}]
```

$$
\begin{aligned}
y(x) \to\ & c_1\left(\frac{x^5}{20}+\frac{5x^4}{24}+\frac{2x^3}{3}+\frac{3x^2}{2}+2x+1\right)x^2 \\
& + c_2\left(\left(-\frac{187x^5}{1200}-\frac{173x^4}{288}-\frac{31x^3}{18}-\frac{13x^2}{4}-3x\right)x^2\right. \\
& \left.+\left(\frac{x^5}{20}+\frac{5x^4}{24}+\frac{2x^3}{3}+\frac{3x^2}{2}+2x+1\right)x^2\log(x)\right)
\end{aligned}
$$

18.26 problem 20

Internal problem ID [2962]
Internal file name [OUTPUT/2454_Sunday_June_05_2022_03_14_02_AM_42909298/index.tex]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.5. page 771
Problem number: 20.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - y'x^2 - 2y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' - y'x^2 - 2y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -1$$
$$q(x) = -\frac{2}{x^2}$$

Table 500: Table $p(x), q(x)$ singularites.

$p(x) = -1$	
singularity	type

$q(x) = -\frac{2}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' - y'x^2 - 2y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) - \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) x^2 - 2\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\,(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-2a_n x^{n+r}\right) = 0 \quad \text{(2A)}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty}\left(-x^{1+n+r} a_n(n+r)\right)=\sum_{n=1}^{\infty}\left(-a_{n-1}(n+r-1)\, x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)\,(n+r-1)\right)+\sum_{n=1}^{\infty}\left(-a_{n-1}(n+r-1)\, x^{n+r}\right)+\sum_{n=0}^{\infty}\left(-2 a_n x^{n+r}\right)=0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n(n+r)\,(n+r-1)-2 a_n x^{n+r}=0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r)-2 a_0 x^r=0$$

Or

$$\left(x^r r(-1+r)-2 x^r\right) a_0=0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2-r-2\right) x^r=0$$

Since the above is true for all x then the indicial equation becomes

$$r^2-r-2=0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= 2 \\ r_2 &= -1 \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(r^2-r-2\right) x^r=0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 3$ is an integer, then we can construct two linearly independent solutions

$$
\begin{aligned}
y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right) \\
y_2(x) &= C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= x^2 \left(\sum_{n=0}^{\infty} a_n x^n\right) \\
y_2(x) &= C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x}
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+2} \\
y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-1}\right)
\end{aligned}
$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$
a_n(n+r)(n+r-1) - a_{n-1}(n+r-1) - 2a_n = 0 \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$
a_n = \frac{a_{n-1}(n+r-1)}{n^2 + 2nr + r^2 - n - r - 2} \tag{4}
$$

Which for the root $r = 2$ becomes

$$
a_n = \frac{a_{n-1}(1+n)}{n(n+3)} \tag{5}
$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 2$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{r}{r^2 + r - 2}$$

Which for the root $r = 2$ becomes

$$a_1 = \frac{1}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r}{r^2+r-2}$	$\frac{1}{2}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1 + r}{r^3 + 4r^2 + r - 6}$$

Which for the root $r = 2$ becomes

$$a_2 = \frac{3}{20}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r}{r^2+r-2}$	$\frac{1}{2}$
a_2	$\frac{1+r}{r^3+4r^2+r-6}$	$\frac{3}{20}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{1}{r^3 + 6r^2 + 5r - 12}$$

Which for the root $r = 2$ becomes

$$a_3 = \frac{1}{30}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r}{r^2+r-2}$	$\frac{1}{2}$
a_2	$\frac{1+r}{r^3+4r^2+r-6}$	$\frac{3}{20}$
a_3	$\frac{1}{r^3+6r^2+5r-12}$	$\frac{1}{30}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{r^4 + 10r^3 + 27r^2 + 2r - 40}$$

Which for the root $r = 2$ becomes

$$a_4 = \frac{1}{168}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r}{r^2+r-2}$	$\frac{1}{2}$
a_2	$\frac{1+r}{r^3+4r^2+r-6}$	$\frac{3}{20}$
a_3	$\frac{1}{r^3+6r^2+5r-12}$	$\frac{1}{30}$
a_4	$\frac{1}{r^4+10r^3+27r^2+2r-40}$	$\frac{1}{168}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{1}{r^5 + 15r^4 + 75r^3 + 125r^2 - 36r - 180}$$

Which for the root $r = 2$ becomes

$$a_5 = \frac{1}{1120}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r}{r^2+r-2}$	$\frac{1}{2}$
a_2	$\frac{1+r}{r^3+4r^2+r-6}$	$\frac{3}{20}$
a_3	$\frac{1}{r^3+6r^2+5r-12}$	$\frac{1}{30}$
a_4	$\frac{1}{r^4+10r^3+27r^2+2r-40}$	$\frac{1}{168}$
a_5	$\frac{1}{r^5+15r^4+75r^3+125r^2-36r-180}$	$\frac{1}{1120}$

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x^2\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots\right) \\
&= x^2\left(1 + \frac{x}{2} + \frac{3x^2}{20} + \frac{x^3}{30} + \frac{x^4}{168} + \frac{x^5}{1120} + O(x^6)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Let

$$
r_1 - r_2 = N
$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 3$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\rightarrow r_2} a_3(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$
\begin{aligned}
a_N &= a_3 \\
&= \frac{1}{r^3 + 6r^2 + 5r - 12}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\lim_{r\rightarrow r_2} \frac{1}{r^3 + 6r^2 + 5r - 12} &= \lim_{r\rightarrow -1} \frac{1}{r^3 + 6r^2 + 5r - 12} \\
&= -\frac{1}{12}
\end{aligned}
$$

The limit is $-\frac{1}{12}$. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$
\begin{aligned}
y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\
&= \sum_{n=0}^{\infty} b_n x^{n-1}
\end{aligned}
$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$b_n(n + r)(n + r - 1) - b_{n-1}(n + r - 1) - 2b_n = 0 \tag{4}$$

Which for for the root $r = -1$ becomes

$$b_n(n - 1)(n - 2) - b_{n-1}(n - 2) - 2b_n = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = \frac{b_{n-1}(n + r - 1)}{n^2 + 2nr + r^2 - n - r - 2} \tag{5}$$

Which for the root $r = -1$ becomes

$$b_n = \frac{b_{n-1}(n - 2)}{n^2 - 3n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{r}{r^2 + r - 2}$$

Which for the root $r = -1$ becomes

$$b_1 = \frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{r}{r^2+r-2}$	$\frac{1}{2}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1 + r}{(r^2 + r - 2)(r + 3)}$$

Which for the root $r = -1$ becomes

$$b_2 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{r}{r^2+r-2}$	$\frac{1}{2}$
b_2	$\frac{1+r}{r^3+4r^2+r-6}$	0

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{1}{(r + 4)(r + 3)(-1 + r)}$$

Which for the root $r = -1$ becomes

$$b_3 = -\frac{1}{12}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{r}{r^2+r-2}$	$\frac{1}{2}$
b_2	$\frac{1+r}{r^3+4r^2+r-6}$	0
b_3	$\frac{1}{(r+4)(r+3)(-1+r)}$	$-\frac{1}{12}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{(r + 4)(-1 + r)(r^2 + 7r + 10)}$$

Which for the root $r = -1$ becomes

$$b_4 = -\frac{1}{24}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{r}{r^2+r-2}$	$\frac{1}{2}$
b_2	$\frac{1+r}{r^3+4r^2+r-6}$	0
b_3	$\frac{1}{(r+4)(r+3)(-1+r)}$	$-\frac{1}{12}$
b_4	$\frac{1}{(5+r)(2+r)(r+4)(-1+r)}$	$-\frac{1}{24}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{1}{(-1+r)(r^2+7r+10)(r^2+9r+18)}$$

Which for the root $r = -1$ becomes

$$b_5 = -\frac{1}{80}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{r}{r^2+r-2}$	$\frac{1}{2}$
b_2	$\frac{1+r}{r^3+4r^2+r-6}$	0
b_3	$\frac{1}{(r+4)(r+3)(-1+r)}$	$-\frac{1}{12}$
b_4	$\frac{1}{(5+r)(2+r)(r+4)(-1+r)}$	$-\frac{1}{24}$
b_5	$\frac{1}{(6+r)(r+3)(5+r)(-1+r)(2+r)}$	$-\frac{1}{80}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned}
y_2(x) &= x^2 \left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\
&= \frac{1 + \frac{x}{2} - \frac{x^3}{12} - \frac{x^4}{24} - \frac{x^5}{80} + O(x^6)}{x}
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^2 \left(1 + \frac{x}{2} + \frac{3x^2}{20} + \frac{x^3}{30} + \frac{x^4}{168} + \frac{x^5}{1120} + O(x^6)\right) + \frac{c_2 \left(1 + \frac{x}{2} - \frac{x^3}{12} - \frac{x^4}{24} - \frac{x^5}{80} + O(x^6)\right)}{x}
\end{aligned}$$

Hence the final solution is

$$y = y_h$$

$$= c_1x^2\left(1+\frac{x}{2}+\frac{3x^2}{20}+\frac{x^3}{30}+\frac{x^4}{168}+\frac{x^5}{1120}+O(x^6)\right)+\frac{c_2\left(1+\frac{x}{2}-\frac{x^3}{12}-\frac{x^4}{24}-\frac{x^5}{80}+O(x^6)\right)}{x}$$

Summary

The solution(s) found are the following

$$y = c_1x^2\left(1+\frac{x}{2}+\frac{3x^2}{20}+\frac{x^3}{30}+\frac{x^4}{168}+\frac{x^5}{1120}+O(x^6)\right) + \frac{c_2\left(1+\frac{x}{2}-\frac{x^3}{12}-\frac{x^4}{24}-\frac{x^5}{80}+O(x^6)\right)}{x} \tag{1}$$

Verification of solutions

$$y = c_1x^2\left(1+\frac{x}{2}+\frac{3x^2}{20}+\frac{x^3}{30}+\frac{x^4}{168}+\frac{x^5}{1120}+O(x^6)\right)+\frac{c_2\left(1+\frac{x}{2}-\frac{x^3}{12}-\frac{x^4}{24}-\frac{x^5}{80}+O(x^6)\right)}{x}$$

Verified OK.

18.26.1 Maple step by step solution

Let's solve

$x^2y'' - y'x^2 - 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = y' + \frac{2y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - y' - \frac{2y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -1, P_3(x) = -\frac{2}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

$$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 0$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -2$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $$x^2y'' - y'x^2 - 2y = 0$$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x^2 \cdot y'$ to series expansion

  $$x^2 \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r+1}$$

- Shift index using $k- >k-1$

  $$x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1}(k-1+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k-1+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(1+r)\,(-2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)\,(k+r-2) - a_{k-1}(k-1+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(1+r)\,(-2+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{-1, 2\}$$

- Each term in the series must be 0, giving the recursion relation

  $$a_k(k+r+1)\,(k+r-2) - a_{k-1}(k-1+r) = 0$$

- Shift index using $k- >k+1$

  $a_{k+1}(k+2+r)\,(k-1+r)-a_{k}(k+r)=0$
- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{a_{k}(k+r)}{(k+2+r)(k-1+r)}$
- Recursion relation for $r=-1$; series terminates at $k=1$

  $a_{k+1}=\frac{a_{k}(k-1)}{(k+1)(k-2)}$
- Apply recursion relation for $k=0$

  $a_{1}=\frac{a_{0}}{2}$
- Terminating series solution of the ODE for $r=-1$. Use reduction of order to find the second

  $y=a_{0}\cdot\left(1+\frac{x}{2}\right)$
- Recursion relation for $r=2$

  $a_{k+1}=\frac{a_{k}(k+2)}{(k+4)(k+1)}$
- Solution for $r=2$

  $\left[y=\sum_{k=0}^{\infty}a_{k}x^{k+2}, a_{k+1}=\frac{a_{k}(k+2)}{(k+4)(k+1)}\right]$
- Combine solutions and rename parameters

  $\left[y=a_{0}\cdot\left(1+\frac{x}{2}\right)+\left(\sum_{k=0}^{\infty}b_{k}x^{2+k}\right), b_{1+k}=\frac{b_{k}(2+k)}{(k+4)(1+k)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 45

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)-x^2*diff(y(x),x)-2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^2 \left(1 + \frac{1}{2}x + \frac{3}{20}x^2 + \frac{1}{30}x^3 + \frac{1}{168}x^4 + \frac{1}{1120}x^5 + \mathrm{O}\left(x^6\right)\right) + \frac{c_2\left(12 + 6x - x^3 - \frac{1}{2}x^4 - \frac{3}{20}x^5 + \mathrm{O}\left(x^6\right)\right)}{x}$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 63

```
AsymptoticDSolveValue[x^2*y''[x]-x^2*y'[x]-2*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(-\frac{x^3}{24} - \frac{x^2}{12} + \frac{1}{x} + \frac{1}{2}\right) + c_2\left(\frac{x^6}{168} + \frac{x^5}{30} + \frac{3x^4}{20} + \frac{x^3}{2} + x^2\right)$$

18.27 problem 21

Internal problem ID [2963]
Internal file name [OUTPUT/2455_Sunday_June_05_2022_03_14_06_AM_92181284/index.tex]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.5. page 771
Problem number: 21.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - y'x^2 - (2+3x)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' - y'x^2 + (-3x-2)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$\begin{aligned} p(x) &= -1 \\ q(x) &= -\frac{2+3x}{x^2} \end{aligned}$$

Table 502: Table $p(x), q(x)$ singularites.

$p(x) = -1$	
singularity	type

$q(x) = -\frac{2+3x}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' - y'x^2 + (-3x - 2)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) - \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) x^2 + (-3x-2) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n (n+r)\right) \\
&+ \sum_{n=0}^{\infty} \left(-3x^{1+n+r} a_n\right) + \sum_{n=0}^{\infty} \left(-2a_n x^{n+r}\right) = 0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)\, x^{n+r}\right)
$$

$$
\sum_{n=0}^{\infty} \left(-3x^{1+n+r} a_n\right) = \sum_{n=1}^{\infty} \left(-3a_{n-1} x^{n+r}\right)
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)\, x^{n+r}\right) \\
&+ \sum_{n=1}^{\infty} \left(-3a_{n-1} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-2a_n x^{n+r}\right) = 0
\end{aligned} \tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
x^{n+r} a_n (n+r)(n+r-1) - 2a_n x^{n+r} = 0
$$

When $n=0$ the above becomes

$$
x^r a_0 r(-1+r) - 2a_0 x^r = 0
$$

Or

$$
\left(x^r r(-1+r) - 2x^r\right) a_0 = 0
$$

Since $a_0 \neq 0$ then the above simplifies to

$$
\left(r^2 - r - 2\right) x^r = 0
$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - r - 2 = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= 2 \\ r_2 &= -1 \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(r^2 - r - 2\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 3$ is an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= x^2 \left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x} \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+2} \\ y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-1}\right) \end{aligned}$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n + r)(n + r - 1) - a_{n-1}(n + r - 1) - 3a_{n-1} - 2a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}(n+r+2)}{n^2+2nr+r^2-n-r-2} \tag{4}$$

Which for the root $r = 2$ becomes

$$a_n = \frac{a_{n-1}(n+4)}{n\,(n+3)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 2$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{3+r}{r^2+r-2}$$

Which for the root $r = 2$ becomes

$$a_1 = \frac{5}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{3+r}{r^2+r-2}$	$\frac{5}{4}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{4+r}{r\,(r^2+r-2)}$$

Which for the root $r = 2$ becomes

$$a_2 = \frac{3}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{3+r}{r^2+r-2}$	$\frac{5}{4}$
a_2	$\frac{4+r}{r(r^2+r-2)}$	$\frac{3}{4}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{5+r}{(r+1)\, r\, (r^2+r-2)}$$

Which for the root $r = 2$ becomes

$$a_3 = \frac{7}{24}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{3+r}{r^2+r-2}$	$\frac{5}{4}$
a_2	$\frac{4+r}{r(r^2+r-2)}$	$\frac{3}{4}$
a_3	$\frac{5+r}{(r+1)r(r^2+r-2)}$	$\frac{7}{24}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{6+r}{(r+2)^2\, (r+1)\, r\, (-1+r)}$$

Which for the root $r = 2$ becomes

$$a_4 = \frac{1}{12}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{3+r}{r^2+r-2}$	$\frac{5}{4}$
a_2	$\frac{4+r}{r(r^2+r-2)}$	$\frac{3}{4}$
a_3	$\frac{5+r}{(r+1)r(r^2+r-2)}$	$\frac{7}{24}$
a_4	$\frac{6+r}{(r+2)^2(r+1)r(-1+r)}$	$\frac{1}{12}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{7+r}{(3+r)\,(r+2)^2\, r\, (r^2-1)}$$

Which for the root $r = 2$ becomes

$$a_5 = \frac{3}{160}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{3+r}{r^2+r-2}$	$\frac{5}{4}$
a_2	$\frac{4+r}{r(r^2+r-2)}$	$\frac{3}{4}$
a_3	$\frac{5+r}{(r+1)r(r^2+r-2)}$	$\frac{7}{24}$
a_4	$\frac{6+r}{(r+2)^2(r+1)r(-1+r)}$	$\frac{1}{12}$
a_5	$\frac{7+r}{(3+r)(r+2)^2r(r^2-1)}$	$\frac{3}{160}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}
y_1(x) &= x^2(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots) \\
&= x^2\left(1 + \frac{5x}{4} + \frac{3x^2}{4} + \frac{7x^3}{24} + \frac{x^4}{12} + \frac{3x^5}{160} + O(x^6)\right)
\end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 3$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_3(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned}
a_N &= a_3 \\
&= \frac{5+r}{(r+1)\,r\,(r^2+r-2)}
\end{aligned}$$

Therefore

$$\begin{aligned}
\lim_{r\to r_2} \frac{5+r}{(r+1)\,r\,(r^2+r-2)} &= \lim_{r\to -1} \frac{5+r}{(r+1)\,r\,(r^2+r-2)} \\
&= \textit{undefined}
\end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$\begin{aligned}
\frac{d}{dx}y_2(x) &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\
&= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right) \\
\frac{d^2}{dx^2}y_2(x) &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\
&= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right)
\end{aligned}$$

Substituting these back into the given ode $x^2y'' - y'x^2 + (-3x-2)\,y = 0$ gives

$$\begin{aligned}
&x^2\left(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2}\right. \\
&\quad \left. + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) \\
&\quad - \left(Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right)x^2 \\
&\quad + (-3x-2)\left(Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)\right) = 0
\end{aligned}$$

Which can be written as

$$\left(\left(\left(x^2 y_1''(x) - y_1'(x)\, x^2 + (-3x-2)\, y_1(x)\right)\ln(x) + x^2\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right)\right.\right.$$
$$\left.\left. - y_1(x)\, x\right) C + x^2\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)\right. \tag{7}$$
$$- \left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)x^2 + (-3x-2)\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0$$

But since $y_1(x)$ is a solution to the ode, then

$$x^2 y_1''(x) - y_1'(x)\, x^2 + (-3x-2)\, y_1(x) = 0$$

Eq (7) simplifes to

$$\left(x^2\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right) - y_1(x)\, x\right) C$$
$$+ x^2\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) \tag{8}$$
$$- \left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)x^2 + (-3x-2)\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\left(2\left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n (n+r_1)\right)x - (x+1)\left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right)\right) C$$
$$+ \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right)x^2 \tag{9}$$
$$- \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right)x^2 + (-3x-2)\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0$$

Since $r_1 = 2$ and $r_2 = -1$ then the above becomes

$$\begin{aligned}
&\left(2\left(\sum_{n=0}^{\infty} x^{1+n} a_n (n+2)\right) x - (x+1)\left(\sum_{n=0}^{\infty} a_n x^{n+2}\right)\right) C \\
&+ \left(\sum_{n=0}^{\infty} x^{-3+n} b_n (n-1)(-2+n)\right) x^2 \\
&- \left(\sum_{n=0}^{\infty} x^{-2+n} b_n (n-1)\right) x^2 + (-3x-2)\left(\sum_{n=0}^{\infty} b_n x^{n-1}\right) = 0
\end{aligned} \tag{10}$$

Which simplifies to

$$\begin{aligned}
&\left(\sum_{n=0}^{\infty} 2C\, x^{n+2} a_n (n+2)\right) + \sum_{n=0}^{\infty} \left(-C\, x^{n+3} a_n\right) \\
&+ \sum_{n=0}^{\infty} \left(-C\, x^{n+2} a_n\right) + \left(\sum_{n=0}^{\infty} x^{n-1} b_n \left(n^2 - 3n + 2\right)\right) \\
&+ \sum_{n=0}^{\infty} \left(-x^n b_n (n-1)\right) + \sum_{n=0}^{\infty} \left(-3 b_n x^n\right) + \sum_{n=0}^{\infty} \left(-2 b_n x^{n-1}\right) = 0
\end{aligned} \tag{2A}$$

The next step is to make all powers of x be $n-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2C\, x^{n+2} a_n (n+2) = \sum_{n=3}^{\infty} 2C a_{-3+n} (n-1)\, x^{n-1}$$

$$\sum_{n=0}^{\infty} \left(-C\, x^{n+3} a_n\right) = \sum_{n=4}^{\infty} \left(-C a_{n-4} x^{n-1}\right)$$

$$\sum_{n=0}^{\infty} \left(-C\, x^{n+2} a_n\right) = \sum_{n=3}^{\infty} \left(-C a_{-3+n} x^{n-1}\right)$$

$$\sum_{n=0}^{\infty} \left(-x^n b_n (n-1)\right) = \sum_{n=1}^{\infty} \left(-b_{n-1} (-2+n)\, x^{n-1}\right)$$

$$\sum_{n=0}^{\infty} \left(-3 b_n x^n\right) = \sum_{n=1}^{\infty} \left(-3 b_{n-1} x^{n-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-1$.

$$\left(\sum_{n=3}^{\infty} 2Ca_{-3+n}(n-1)\, x^{n-1}\right)+\sum_{n=4}^{\infty}\left(-Ca_{n-4}x^{n-1}\right)+\sum_{n=3}^{\infty}\left(-Ca_{-3+n}x^{n-1}\right) + \left(\sum_{n=0}^{\infty} x^{n-1} b_n\left(n^2-3n+2\right)\right)+\sum_{n=1}^{\infty}\left(-b_{n-1}(-2+n)\, x^{n-1}\right) + \sum_{n=1}^{\infty}\left(-3b_{n-1}x^{n-1}\right)+\sum_{n=0}^{\infty}\left(-2b_n x^{n-1}\right)=0 \tag{2B}$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0=1$. For $n=1$, Eq (2B) gives

$$-2b_0-2b_1=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-2-2b_1=0$$

Solving the above for b_1 gives

$$b_1=-1$$

For $n=2$, Eq (2B) gives

$$-3b_1-2b_2=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$3-2b_2=0$$

Solving the above for b_2 gives

$$b_2=\frac{3}{2}$$

For $n=N$, where $N=3$ which is the difference between the two roots, we are free to choose $b_3=0$. Hence for $n=3$, Eq (2B) gives

$$3C-6=0$$

Which is solved for C. Solving for C gives

$$C=2$$

For $n=4$, Eq (2B) gives

$$(-a_0+5a_1)\, C-5b_3+4b_4=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{21}{2}+4b_4=0$$

Solving the above for b_4 gives

$$b_4=-\frac{21}{8}$$

For $n=5$, Eq (2B) gives

$$(-a_1+7a_2)\,C-6b_4+10b_5=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{95}{4}+10b_5=0$$

Solving the above for b_5 gives

$$b_5=-\frac{19}{8}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x)=Cy_1(x)\ln(x)+\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C=2$ and all b_n, then the second solution becomes

$$\begin{aligned}y_2(x)=&\,2\left(x^2\left(1+\frac{5x}{4}+\frac{3x^2}{4}+\frac{7x^3}{24}+\frac{x^4}{12}+\frac{3x^5}{160}+O\left(x^6\right)\right)\right)\ln(x)\\&+\frac{1-x+\frac{3x^2}{2}-\frac{21x^4}{8}-\frac{19x^5}{8}+O(x^6)}{x}\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}y_h(x)&=c_1y_1(x)+c_2y_2(x)\\&=c_1x^2\left(1+\frac{5x}{4}+\frac{3x^2}{4}+\frac{7x^3}{24}+\frac{x^4}{12}+\frac{3x^5}{160}+O\left(x^6\right)\right)\\&\quad+c_2\left(2\left(x^2\left(1+\frac{5x}{4}+\frac{3x^2}{4}+\frac{7x^3}{24}+\frac{x^4}{12}+\frac{3x^5}{160}+O\left(x^6\right)\right)\right)\ln(x)\right.\\&\qquad\left.+\frac{1-x+\frac{3x^2}{2}-\frac{21x^4}{8}-\frac{19x^5}{8}+O(x^6)}{x}\right)\end{aligned}$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^2 \left(1+\frac{5x}{4}+\frac{3x^2}{4}+\frac{7x^3}{24}+\frac{x^4}{12}+\frac{3x^5}{160}+O(x^6)\right) \\
&\quad + c_2 \left(2x^2 \left(1+\frac{5x}{4}+\frac{3x^2}{4}+\frac{7x^3}{24}+\frac{x^4}{12}+\frac{3x^5}{160}+O(x^6)\right)\ln(x)\right. \\
&\quad \left. + \frac{1-x+\frac{3x^2}{2}-\frac{21x^4}{8}-\frac{19x^5}{8}+O(x^6)}{x}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x^2 \left(1+\frac{5x}{4}+\frac{3x^2}{4}+\frac{7x^3}{24}+\frac{x^4}{12}+\frac{3x^5}{160}+O(x^6)\right) \\
&\quad + c_2 \left(2x^2 \left(1+\frac{5x}{4}+\frac{3x^2}{4}+\frac{7x^3}{24}+\frac{x^4}{12}+\frac{3x^5}{160}+O(x^6)\right)\ln(x)\right. \\
&\quad \left. + \frac{1-x+\frac{3x^2}{2}-\frac{21x^4}{8}-\frac{19x^5}{8}+O(x^6)}{x}\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1 x^2 \left(1+\frac{5x}{4}+\frac{3x^2}{4}+\frac{7x^3}{24}+\frac{x^4}{12}+\frac{3x^5}{160}+O(x^6)\right) \\
&\quad + c_2 \left(2x^2 \left(1+\frac{5x}{4}+\frac{3x^2}{4}+\frac{7x^3}{24}+\frac{x^4}{12}+\frac{3x^5}{160}+O(x^6)\right)\ln(x)\right. \\
&\quad \left. + \frac{1-x+\frac{3x^2}{2}-\frac{21x^4}{8}-\frac{19x^5}{8}+O(x^6)}{x}\right)
\end{aligned}
$$

Verified OK.

18.27.1 Maple step by step solution

Let's solve

$x^2y'' - y'x^2 + (-3x-2)\,y = 0$

- Highest derivative means the order of the ODE is 2

y''

- Isolate 2nd derivative

$y'' = \frac{(2+3x)y}{x^2} + y'$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' - y' - \frac{(2+3x)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -1, P_3(x) = -\frac{2+3x}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = 0$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = -2$

  ◦ $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

$x^2y'' - y'x^2 + (-3x-2)\,y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ◦ Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^2 \cdot y'$ to series expansion

  $$x^2 \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r+1}$$

- Shift index using $k->k-1$

  $$x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1}(k-1+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k-1+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(1+r)\,(-2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)\,(k+r-2) - a_{k-1}(k+2+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(1+r)\,(-2+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{-1, 2\}$$

- Each term in the series must be 0, giving the recursion relation

  $$a_k(k+r+1)\,(k+r-2) - a_{k-1}(k+2+r) = 0$$

- Shift index using $k->k+1$

  $$a_{k+1}(k+2+r)\,(k-1+r) - a_k(k+r+3) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = \frac{a_k(k+r+3)}{(k+2+r)(k-1+r)}$$

- Recursion relation for $r=-1$

  $$a_{k+1} = \frac{a_k(k+2)}{(k+1)(k-2)}$$

- Series not valid for $r=-1$, division by 0 in the recursion relation at $k=2$

  $$a_{k+1} = \frac{a_k(k+2)}{(k+1)(k-2)}$$

- Recursion relation for $r=2$

  $$a_{k+1} = \frac{a_k(k+5)}{(k+4)(k+1)}$$

- Solution for $r=2$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1}=\frac{a_k(k+5)}{(k+4)(k+1)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 65

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)-x^2*diff(y(x),x)-(3*x+2)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_1 x^3\left(1+\frac{5}{4}x+\frac{3}{4}x^2+\frac{7}{24}x^3+\frac{1}{12}x^4+\frac{3}{160}x^5+\mathrm{O}\left(x^6\right)\right)+c_2\left(\ln\left(x\right)\left(24x^3+30x^4+18x^5+\mathrm{O}\left(x^6\right)\right)+(12-}{x}$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 84

```
AsymptoticDSolveValue[x^2*y''[x]-x^2*y'[x]-(3*x+2)*y[x]==0,y[x],{x,0,5}]
```

$$y(x)\to c_1\left(\frac{1}{2}x^2(5x+4)\log(x)-\frac{3x^4-6x^3-6x^2+4x-4}{4x}\right)+c_2\left(\frac{x^6}{12}+\frac{7x^5}{24}+\frac{3x^4}{4}+\frac{5x^3}{4}+x^2\right)$$

18.28 problem 22

Internal problem ID [2964]
Internal file name [`OUTPUT/2456_Sunday_June_05_2022_03_14_10_AM_15449460/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.5. page 771
Problem number: 22.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + x(5-x)\,y' + 4y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + \left(-x^2 + 5x\right) y' + 4y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{x-5}{x}$$
$$q(x) = \frac{4}{x^2}$$

Table 504: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{x-5}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{4}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + \left(-x^2 + 5x\right) y' + 4y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$
$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + \left(-x^2 + 5x\right)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + 4\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 5 x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 4 a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-a_{n-1} (n+r-1) x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-a_{n-1} (n+r-1) x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 5 x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 4 a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) + 5 x^{n+r} a_n (n+r) + 4 a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$x^r a_0 r(-1+r) + 5 x^r a_0 r + 4 a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) + 5 x^r r + 4 x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(r+2)^2 x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(r+2)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = -2$$
$$r_2 = -2$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(r+2)^2 x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = -2$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n-2}$$
$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n-2}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) - a_{n-1}(n+r-1) + 5a_n(n+r) + 4a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}(n+r-1)}{n^2 + 2nr + r^2 + 4n + 4r + 4} \tag{4}$$

Which for the root $r = -2$ becomes

$$a_n = \frac{a_{n-1}(n-3)}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = -2$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{r}{(3+r)^2}$$

Which for the root $r = -2$ becomes

$$a_1 = -2$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r}{(3+r)^2}$	-2

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{r(1+r)}{(3+r)^2 (r+4)^2}$$

Which for the root $r = -2$ becomes

$$a_2 = \frac{1}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r}{(3+r)^2}$	-2
a_2	$\frac{r(1+r)}{(3+r)^2(r+4)^2}$	$\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{r(1+r)(r+2)}{(3+r)^2(r+4)^2(r+5)^2}$$

Which for the root $r = -2$ becomes

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r}{(3+r)^2}$	-2
a_2	$\frac{r(1+r)}{(3+r)^2(r+4)^2}$	$\frac{1}{2}$
a_3	$\frac{r(1+r)(r+2)}{(3+r)^2(r+4)^2(r+5)^2}$	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{r(1+r)(r+2)}{(3+r)(r+4)^2(r+5)^2(6+r)^2}$$

Which for the root $r = -2$ becomes

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r}{(3+r)^2}$	-2
a_2	$\frac{r(1+r)}{(3+r)^2(r+4)^2}$	$\frac{1}{2}$
a_3	$\frac{r(1+r)(r+2)}{(3+r)^2(r+4)^2(r+5)^2}$	0
a_4	$\frac{r(1+r)(r+2)}{(3+r)(r+4)^2(r+5)^2(6+r)^2}$	0

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{r(1+r)(r+2)}{(3+r)(r+4)(r+5)^2(6+r)^2(r+7)^2}$$

Which for the root $r = -2$ becomes

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r}{(3+r)^2}$	-2
a_2	$\frac{r(1+r)}{(3+r)^2(r+4)^2}$	$\frac{1}{2}$
a_3	$\frac{r(1+r)(r+2)}{(3+r)^2(r+4)^2(r+5)^2}$	0
a_4	$\frac{r(1+r)(r+2)}{(3+r)(r+4)^2(r+5)^2(6+r)^2}$	0
a_5	$\frac{r(1+r)(r+2)}{(3+r)(r+4)(r+5)^2(6+r)^2(r+7)^2}$	0

Using the above table, then the first solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= \frac{1}{x^2}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= \frac{1 - 2x + \frac{x^2}{2} + O(x^6)}{x^2} \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = -2$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr}a_{n,r}$	$b_n(r=-2)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	$\frac{r}{(3+r)^2}$	-2	$\frac{3-r}{(3+r)^3}$	5
b_2	$\frac{r(1+r)}{(3+r)^2(r+4)^2}$	$\frac{1}{2}$	$\frac{-2r^3-3r^2+17r+12}{(3+r)^3(r+4)^3}$	$-\frac{9}{4}$
b_3	$\frac{r(1+r)(r+2)}{(3+r)^2(r+4)^2(r+5)^2}$	0	$\frac{-3r^5-24r^4-35r^3+108r^2+266r+120}{(3+r)^3(r+4)^3(r+5)^3}$	$\frac{1}{18}$
b_4	$\frac{r(1+r)(r+2)}{(3+r)(r+4)^2(r+5)^2(6+r)^2}$	0	$\frac{-4r^6-54r^5-228r^4-180r^3+874r^2+1716r+720}{(3+r)^2(r+4)^3(r+5)^3(6+r)^3}$	$\frac{1}{288}$
b_5	$\frac{r(1+r)(r+2)}{(3+r)(r+4)(r+5)^2(6+r)^2(r+7)^2}$	0	$\frac{-5r^7-100r^6-730r^5-2190r^4-1089r^3+7258r^2+12552r+5040}{(3+r)^2(r+4)^2(r+5)^3(6+r)^3(r+7)^3}$	$\frac{1}{3600}$

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 \dots \\
&= \frac{\left(1-2x+\frac{x^2}{2}+O(x^6)\right)\ln(x)}{x^2} + \frac{5x-\frac{9x^2}{4}+\frac{x^3}{18}+\frac{x^4}{288}+\frac{x^5}{3600}+O(x^6)}{x^2}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1y_1(x) + c_2y_2(x) \\
&= \frac{c_1\left(1-2x+\frac{x^2}{2}+O(x^6)\right)}{x^2} \\
&\quad + c_2\left(\frac{\left(1-2x+\frac{x^2}{2}+O(x^6)\right)\ln(x)}{x^2} + \frac{5x-\frac{9x^2}{4}+\frac{x^3}{18}+\frac{x^4}{288}+\frac{x^5}{3600}+O(x^6)}{x^2}\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= \frac{c_1\left(1-2x+\frac{x^2}{2}+O(x^6)\right)}{x^2} \\
&\quad + c_2\left(\frac{\left(1-2x+\frac{x^2}{2}+O(x^6)\right)\ln(x)}{x^2} + \frac{5x-\frac{9x^2}{4}+\frac{x^3}{18}+\frac{x^4}{288}+\frac{x^5}{3600}+O(x^6)}{x^2}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & \frac{c_1\left(1-2x+\frac{x^2}{2}+O(x^6)\right)}{x^2} \\
& + c_2\left(\frac{\left(1-2x+\frac{x^2}{2}+O(x^6)\right)\ln(x)}{x^2}+\frac{5x-\frac{9x^2}{4}+\frac{x^3}{18}+\frac{x^4}{288}+\frac{x^5}{3600}+O(x^6)}{x^2}\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = {} & \frac{c_1\left(1-2x+\frac{x^2}{2}+O(x^6)\right)}{x^2} \\
& + c_2\left(\frac{\left(1-2x+\frac{x^2}{2}+O(x^6)\right)\ln(x)}{x^2}+\frac{5x-\frac{9x^2}{4}+\frac{x^3}{18}+\frac{x^4}{288}+\frac{x^5}{3600}+O(x^6)}{x^2}\right)
\end{aligned}
$$

Verified OK.

18.28.1 Maple step by step solution

Let's solve

$x^2y''+(-x^2+5x)\,y'+4y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{4y}{x^2}+\frac{(x-5)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{(x-5)y'}{x}+\frac{4y}{x^2}=0$

□ Check to see if $x_0=0$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x)=-\frac{x-5}{x}, P_3(x)=\frac{4}{x^2}\right]$

- ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=5$

- ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 4$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2 y'' - x(x-5)\, y' + 4y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- Rewrite ODE with series expansions

  - Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

  - Shift index using $k- > k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

  - Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(2+r)^2 x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+2)^2 - a_{k-1}(k+r-1)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(2+r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = -2$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+2)^2 - a_{k-1}(k+r-1) = 0$

- Shift index using $k- > k+1$

  $a_{k+1}(k+3+r)^2 - a_k(k+r) = 0$

- Recursion relation that defines series solution to ODE

$a_{k+1} = \frac{a_k(k+r)}{(k+3+r)^2}$

- Recursion relation for $r = -2$; series terminates at $k = 2$

  $a_{k+1} = \frac{a_k(k-2)}{(k+1)^2}$

- Apply recursion relation for $k = 0$

  $a_1 = -2a_0$

- Apply recursion relation for $k = 1$

  $a_2 = -\frac{a_1}{4}$

- Express in terms of a_0

  $a_2 = \frac{a_0}{2}$

- Terminating series solution of the ODE for $r = -2$. Use reduction of order to find the second

  $y = a_0 \cdot \left(1 - 2x + \frac{1}{2}x^2\right)$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 57

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+x*(5-x)*diff(y(x),x)+4*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{(c_2 \ln(x) + c_1)\left(1 - 2x + \frac{1}{2}x^2 + \mathrm{O}(x^6)\right) + \left(5x - \frac{9}{4}x^2 + \frac{1}{18}x^3 + \frac{1}{288}x^4 + \frac{1}{3600}x^5 + \mathrm{O}(x^6)\right)c_2}{x^2}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 80

```
AsymptoticDSolveValue[x^2*y''[x]+x*(5-x)*y'[x]+4*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to \frac{c_1\left(\frac{x^2}{2} - 2x + 1\right)}{x^2} + c_2\left(\frac{\left(\frac{x^2}{2} - 2x + 1\right)\log(x)}{x^2} + \frac{\frac{x^5}{3600} + \frac{x^4}{288} + \frac{x^3}{18} - \frac{9x^2}{4} + 5x}{x^2}\right)$$

18.29 problem 23

Internal problem ID [2965]
Internal file name [`OUTPUT/2457_Sunday_June_05_2022_03_14_13_AM_71883718/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.5. page 771
Problem number: 23.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2y'' + 4x(1-x)\,y' + (2x-9)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$4x^2y'' + \left(-4x^2 + 4x\right)y' + (2x-9)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{x-1}{x}$$
$$q(x) = \frac{2x-9}{4x^2}$$

Table 506: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{x-1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{2x-9}{4x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$4x^2y'' + \left(-4x^2 + 4x\right) y' + (2x - 9)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$4x^2 \left(\sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2} \right) + \left(-4x^2 + 4x\right) \left(\sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1} \right) + (2x - 9) \left(\sum_{n=0}^{\infty} a_n x^{n+r} \right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-4x^{1+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 2x^{1+n+r} a_n\right) + \sum_{n=0}^{\infty} \left(-9a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-4x^{1+n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-4a_{n-1}(n+r-1)\, x^{n+r}\right)$$

$$\sum_{n=0}^{\infty} 2x^{1+n+r} a_n = \sum_{n=1}^{\infty} 2a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-4a_{n-1}(n+r-1)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)\right) + \left(\sum_{n=1}^{\infty} 2a_{n-1} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-9a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$4x^{n+r} a_n (n+r)(n+r-1) + 4x^{n+r} a_n (n+r) - 9a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$4x^r a_0 r(-1+r) + 4x^r a_0 r - 9a_0 x^r = 0$$

Or

$$\left(4x^r r(-1+r) + 4x^r r - 9x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(4r^2 - 9\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$4r^2 - 9 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{3}{2}$$
$$r_2 = -\frac{3}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(4r^2 - 9\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 3$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x^{\frac{3}{2}} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x^{\frac{3}{2}}}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{3}{2}}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-\frac{3}{2}}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$4a_n(n+r)(n+r-1) - 4a_{n-1}(n+r-1) + 4a_n(n+r) + 2a_{n-1} - 9a_n = 0 \quad (3)$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{2a_{n-1}}{2n + 2r + 3} \tag{4}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_n = \frac{a_{n-1}}{n + 3} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{3}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{2}{5 + 2r}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_1 = \frac{1}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2}{5+2r}$	$\frac{1}{4}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{4}{4r^2 + 24r + 35}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_2 = \frac{1}{20}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2}{5+2r}$	$\frac{1}{4}$
a_2	$\frac{4}{4r^2+24r+35}$	$\frac{1}{20}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{8}{8r^3 + 84r^2 + 286r + 315}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_3 = \frac{1}{120}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2}{5+2r}$	$\frac{1}{4}$
a_2	$\frac{4}{4r^2+24r+35}$	$\frac{1}{20}$
a_3	$\frac{8}{8r^3+84r^2+286r+315}$	$\frac{1}{120}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{16}{(4r^2 + 32r + 63)(5 + 2r)(11 + 2r)}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_4 = \frac{1}{840}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2}{5+2r}$	$\frac{1}{4}$
a_2	$\frac{4}{4r^2+24r+35}$	$\frac{1}{20}$
a_3	$\frac{8}{8r^3+84r^2+286r+315}$	$\frac{1}{120}$
a_4	$\frac{16}{(4r^2+32r+63)(5+2r)(11+2r)}$	$\frac{1}{840}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{32}{(11 + 2r)(5 + 2r)(9 + 2r)(13 + 2r)(7 + 2r)}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_5 = \frac{1}{6720}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2}{5+2r}$	$\frac{1}{4}$
a_2	$\frac{4}{4r^2+24r+35}$	$\frac{1}{20}$
a_3	$\frac{8}{8r^3+84r^2+286r+315}$	$\frac{1}{120}$
a_4	$\frac{16}{(4r^2+32r+63)(5+2r)(11+2r)}$	$\frac{1}{840}$
a_5	$\frac{32}{(11+2r)(5+2r)(9+2r)(13+2r)(7+2r)}$	$\frac{1}{6720}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}
y_1(x) &= x^{\frac{3}{2}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= x^{\frac{3}{2}}\left(1 + \frac{x}{4} + \frac{x^2}{20} + \frac{x^3}{120} + \frac{x^4}{840} + \frac{x^5}{6720} + O(x^6)\right)
\end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 3$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_3(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned}
a_N &= a_3 \\
&= \frac{8}{8r^3 + 84r^2 + 286r + 315}
\end{aligned}$$

Therefore

$$\begin{aligned}
\lim_{r \to r_2} \frac{8}{8r^3 + 84r^2 + 286r + 315} &= \lim_{r \to -\frac{3}{2}} \frac{8}{8r^3 + 84r^2 + 286r + 315} \\
&= \frac{1}{6}
\end{aligned}$$

The limit is $\frac{1}{6}$. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\ &= \sum_{n=0}^{\infty} b_n x^{n-\frac{3}{2}} \end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$4b_n(n+r)(n+r-1) - 4b_{n-1}(n+r-1) + 4b_n(n+r) + 2b_{n-1} - 9b_n = 0 \tag{4}$$

Which for for the root $r = -\frac{3}{2}$ becomes

$$4b_n\left(n-\frac{3}{2}\right)\left(n-\frac{5}{2}\right) - 4b_{n-1}\left(n-\frac{5}{2}\right) + 4b_n\left(n-\frac{3}{2}\right) + 2b_{n-1} - 9b_n = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = \frac{2b_{n-1}}{2n+2r+3} \tag{5}$$

Which for the root $r = -\frac{3}{2}$ becomes

$$b_n = \frac{b_{n-1}}{n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\frac{3}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{2}{5+2r}$$

Which for the root $r = -\frac{3}{2}$ becomes

$$b_1 = 1$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2}{5+2r}$	1

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{4}{(5+2r)(7+2r)}$$

Which for the root $r = -\frac{3}{2}$ becomes

$$b_2 = \frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2}{5+2r}$	1
b_2	$\frac{4}{4r^2+24r+35}$	$\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{8}{(5+2r)(7+2r)(9+2r)}$$

Which for the root $r = -\frac{3}{2}$ becomes

$$b_3 = \frac{1}{6}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2}{5+2r}$	1
b_2	$\frac{4}{4r^2+24r+35}$	$\frac{1}{2}$
b_3	$\frac{8}{8r^3+84r^2+286r+315}$	$\frac{1}{6}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{16}{(7 + 2r)(9 + 2r)(5 + 2r)(11 + 2r)}$$

Which for the root $r = -\frac{3}{2}$ becomes

$$b_4 = \frac{1}{24}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2}{5+2r}$	1
b_2	$\frac{4}{4r^2+24r+35}$	$\frac{1}{2}$
b_3	$\frac{8}{8r^3+84r^2+286r+315}$	$\frac{1}{6}$
b_4	$\frac{16}{16r^4+256r^3+1496r^2+3776r+3465}$	$\frac{1}{24}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{32}{(11 + 2r)(5 + 2r)(9 + 2r)(13 + 2r)(7 + 2r)}$$

Which for the root $r = -\frac{3}{2}$ becomes

$$b_5 = \frac{1}{120}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2}{5+2r}$	1
b_2	$\frac{4}{4r^2+24r+35}$	$\frac{1}{2}$
b_3	$\frac{8}{8r^3+84r^2+286r+315}$	$\frac{1}{6}$
b_4	$\frac{16}{16r^4+256r^3+1496r^2+3776r+3465}$	$\frac{1}{24}$
b_5	$\frac{32}{(11+2r)(5+2r)(9+2r)(13+2r)(7+2r)}$	$\frac{1}{120}$

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= x^{\frac{3}{2}}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\
&= \frac{1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6)}{x^{\frac{3}{2}}}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^{\frac{3}{2}}\left(1 + \frac{x}{4} + \frac{x^2}{20} + \frac{x^3}{120} + \frac{x^4}{840} + \frac{x^5}{6720} + O(x^6)\right) \\
&\quad + \frac{c_2\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6)\right)}{x^{\frac{3}{2}}}
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^{\frac{3}{2}}\left(1 + \frac{x}{4} + \frac{x^2}{20} + \frac{x^3}{120} + \frac{x^4}{840} + \frac{x^5}{6720} + O(x^6)\right) \\
&\quad + \frac{c_2\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6)\right)}{x^{\frac{3}{2}}}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x^{\frac{3}{2}}\left(1 + \frac{x}{4} + \frac{x^2}{20} + \frac{x^3}{120} + \frac{x^4}{840} + \frac{x^5}{6720} + O(x^6)\right) \\
&\quad + \frac{c_2\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6)\right)}{x^{\frac{3}{2}}}
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1 x^{\frac{3}{2}}\left(1 + \frac{x}{4} + \frac{x^2}{20} + \frac{x^3}{120} + \frac{x^4}{840} + \frac{x^5}{6720} + O(x^6)\right) \\
&\quad + \frac{c_2\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6)\right)}{x^{\frac{3}{2}}}
\end{aligned}
$$

Verified OK.

18.29.1 Maple step by step solution

Let's solve

$4x^2y'' + (-4x^2 + 4x)\,y' + (2x - 9)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(x-1)y'}{x} - \frac{(2x-9)y}{4x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(x-1)y'}{x} + \frac{(2x-9)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{x-1}{x}, P_3(x) = \frac{2x-9}{4x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = -\frac{9}{4}$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2y'' - 4x(x-1)\,y' + (2x-9)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- > k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k- > k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(3+2r)\,(-3+2r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r+3)\,(2k+2r-3) - 2a_{k-1}(2k+2r-3)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(3+2r)\,(-3+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{-\frac{3}{2}, \frac{3}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$4\left(\left(k+r+\frac{3}{2}\right) a_k - a_{k-1}\right) \left(k+r-\frac{3}{2}\right) = 0$$

- Shift index using $k- > k+1$

$$4\left(\left(k+\frac{5}{2}+r\right) a_{k+1} - a_k\right) \left(k-\frac{1}{2}+r\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{2a_k}{2k+5+2r}$$

- Recursion relation for $r = -\frac{3}{2}$

$$a_{k+1} = \frac{2a_k}{2k+2}$$

- Solution for $r = -\frac{3}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{3}{2}}, a_{k+1} = \frac{2a_k}{2k+2}\right]$$

- Recursion relation for $r = \frac{3}{2}$

$$a_{k+1} = \frac{2a_k}{2k+8}$$

- Solution for $r = \frac{3}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+1} = \frac{2a_k}{2k+8}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{3}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{2}}\right), a_{1+k} = \frac{2a_k}{2+2k}, b_{1+k} = \frac{2b_k}{2k+8}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 47

```
Order:=6;
dsolve(4*x^2*diff(y(x),x$2)+4*x*(1-x)*diff(y(x),x)+(2*x-9)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_1 x^3 \left(1 + \frac{1}{4}x + \frac{1}{20}x^2 + \frac{1}{120}x^3 + \frac{1}{840}x^4 + \frac{1}{6720}x^5 + \mathrm{O}\left(x^6\right)\right) + c_2\left(12 + 12x + 6x^2 + 2x^3 + \frac{1}{2}x^4 + \frac{1}{10}x^5 + \mathrm{O}\left(\right.\right.}{x^{\frac{3}{2}}}$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 90

```
AsymptoticDSolveValue[4*x^2*y''[x]+4*x*(1-x)*y'[x]+(2*x-9)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(\frac{x^{5/2}}{24} + \frac{x^{3/2}}{6} + \frac{1}{x^{3/2}} + \frac{\sqrt{x}}{2} + \frac{1}{\sqrt{x}} \right) + c_2 \left(\frac{x^{11/2}}{840} + \frac{x^{9/2}}{120} + \frac{x^{7/2}}{20} + \frac{x^{5/2}}{4} + x^{3/2} \right)$$

18.30 problem 24

Internal problem ID [2966]
Internal file name [`OUTPUT/2458_Sunday_June_05_2022_03_14_17_AM_30109653/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.5. page 771
Problem number: 24.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + 2x(x+2)\, y' + 2y(x+1) = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + \left(2x^2 + 4x\right) y' + (2 + 2x)\, y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{2x+4}{x}$$
$$q(x) = \frac{2+2x}{x^2}$$

Table 508: Table $p(x), q(x)$ singularites.

$p(x) = \frac{2x+4}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{2+2x}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + \left(2x^2 + 4x\right) y' + (2 + 2x) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}\right) + \left(2x^2 + 4x\right) \left(\sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}\right) + (2 + 2x) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)\,(n+r-1)\right)+\left(\sum_{n=0}^{\infty} 2x^{1+n+r} a_n(n+r)\right) \\
&+\left(\sum_{n=0}^{\infty} 4x^{n+r} a_n(n+r)\right)+\left(\sum_{n=0}^{\infty} 2a_n x^{n+r}\right)+\left(\sum_{n=0}^{\infty} 2x^{1+n+r} a_n\right)=0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} 2x^{1+n+r} a_n(n+r) &= \sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)\,x^{n+r} \\
\sum_{n=0}^{\infty} 2x^{1+n+r} a_n &= \sum_{n=1}^{\infty} 2a_{n-1} x^{n+r}
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)\,(n+r-1)\right)+\left(\sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)\,x^{n+r}\right) \\
&+\left(\sum_{n=0}^{\infty} 4x^{n+r} a_n(n+r)\right)+\left(\sum_{n=0}^{\infty} 2a_n x^{n+r}\right)+\left(\sum_{n=1}^{\infty} 2a_{n-1} x^{n+r}\right)=0
\end{aligned} \tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
x^{n+r} a_n(n+r)\,(n+r-1)+4x^{n+r} a_n(n+r)+2a_n x^{n+r}=0
$$

When $n=0$ the above becomes

$$
x^r a_0 r(-1+r)+4x^r a_0 r+2a_0 x^r=0
$$

Or

$$
\left(x^r r(-1+r)+4x^r r+2x^r\right) a_0=0
$$

Since $a_0 \neq 0$ then the above simplifies to

$$
\left(r^2+3r+2\right) x^r=0
$$

Since the above is true for all x then the indicial equation becomes

$$r^2 + 3r + 2 = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= -1 \\ r_2 &= -2 \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(r^2 + 3r + 2\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \frac{\sum_{n=0}^{\infty} a_n x^n}{x} \\ y_2(x) &= C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x^2} \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n-1} \\ y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-2}\right) \end{aligned}$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n + r)(n + r - 1) + 2a_{n-1}(n + r - 1) + 4a_n(n + r) + 2a_n + 2a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{2a_{n-1}(n+r)}{n^2+2nr+r^2+3n+3r+2} \tag{4}$$

Which for the root $r=-1$ becomes

$$a_n = -\frac{2a_{n-1}(n-1)}{n(1+n)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r=-1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n=1$, using the above recursive equation gives

$$a_1 = \frac{-2-2r}{r^2+5r+6}$$

Which for the root $r=-1$ becomes

$$a_1 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2-2r}{r^2+5r+6}$	0

For $n=2$, using the above recursive equation gives

$$a_2 = \frac{4+4r}{(r+4)(r+3)^2}$$

Which for the root $r=-1$ becomes

$$a_2 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2-2r}{r^2+5r+6}$	0
a_2	$\frac{4+4r}{(r+4)(r+3)^2}$	0

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-8 - 8r}{(r+4)^2 (r+3)(5+r)}$$

Which for the root $r = -1$ becomes

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2-2r}{r^2+5r+6}$	0
a_2	$\frac{4+4r}{(r+4)(r+3)^2}$	0
a_3	$\frac{-8-8r}{(r+4)^2(r+3)(5+r)}$	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{16 + 16r}{(r+4)(r+3)(5+r)^2 (6+r)}$$

Which for the root $r = -1$ becomes

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2-2r}{r^2+5r+6}$	0
a_2	$\frac{4+4r}{(r+4)(r+3)^2}$	0
a_3	$\frac{-8-8r}{(r+4)^2(r+3)(5+r)}$	0
a_4	$\frac{16+16r}{(r+4)(r+3)(5+r)^2(6+r)}$	0

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-32 - 32r}{(r+4)(r+3)(5+r)(6+r)^2(r+7)}$$

Which for the root $r = -1$ becomes

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2-2r}{r^2+5r+6}$	0
a_2	$\frac{4+4r}{(r+4)(r+3)^2}$	0
a_3	$\frac{-8-8r}{(r+4)^2(r+3)(5+r)}$	0
a_4	$\frac{16+16r}{(r+4)(r+3)(5+r)^2(6+r)}$	0
a_5	$\frac{-32-32r}{(r+4)(r+3)(5+r)(6+r)^2(r+7)}$	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= \frac{1}{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \ldots\right) \\ &= \frac{1 + O(x^6)}{x} \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_1 \\ &= \frac{-2 - 2r}{r^2 + 5r + 6} \end{aligned}$$

Therefore

$$\lim_{r\to r_2}\frac{-2-2r}{r^2+5r+6}=\lim_{r\to -2}\frac{-2-2r}{r^2+5r+6}$$
$$= \textit{undefined}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x)=Cy_1(x)\ln(x)+\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)$$

Therefore

$$\frac{d}{dx}y_2(x)=Cy_1'(x)\ln(x)+\frac{Cy_1(x)}{x}+\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)$$
$$=Cy_1'(x)\ln(x)+\frac{Cy_1(x)}{x}+\left(\sum_{n=0}^{\infty}x^{-1+n+r_2}b_n(n+r_2)\right)$$
$$\frac{d^2}{dx^2}y_2(x)=Cy_1''(x)\ln(x)+\frac{2Cy_1'(x)}{x}-\frac{Cy_1(x)}{x^2}+\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)$$
$$=Cy_1''(x)\ln(x)+\frac{2Cy_1'(x)}{x}-\frac{Cy_1(x)}{x^2}+\left(\sum_{n=0}^{\infty}x^{-2+n+r_2}b_n(n+r_2)(-1+n+r_2)\right)$$

Substituting these back into the given ode $x^2y''+(2x^2+4x)\,y'+(2+2x)\,y=0$ gives

$$x^2\left(Cy_1''(x)\ln(x)+\frac{2Cy_1'(x)}{x}-\frac{Cy_1(x)}{x^2}+\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)+(2x^2+4x)\left(Cy_1'(x)\ln(x)+\frac{Cy_1(x)}{x}+\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right)+(2+2x)\left(Cy_1(x)\ln(x)+\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)\right)=0$$

Which can be written as

$$\left(\left(x^2 y_1''(x) + \left(2x^2+4x\right) y_1'(x) + (2+2x)\, y_1(x)\right) \ln(x) + x^2 \left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right) + \frac{\left(2x^2+4x\right) y_1(x)}{x}\right) C + x^2 \left(\sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2} (n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2} (n+r_2)}{x^2}\right)\right) + \left(2x^2+4x\right) \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2} (n+r_2)}{x}\right) + (2+2x) \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0 \tag{7}$$

But since $y_1(x)$ is a solution to the ode, then

$$x^2 y_1''(x) + \left(2x^2+4x\right) y_1'(x) + (2+2x)\, y_1(x) = 0$$

Eq (7) simplifes to

$$\left(x^2 \left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right) + \frac{\left(2x^2+4x\right) y_1(x)}{x}\right) C + x^2 \left(\sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2} (n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2} (n+r_2)}{x^2}\right)\right) + \left(2x^2+4x\right) \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2} (n+r_2)}{x}\right) + (2+2x) \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0 \tag{8}$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\left(2\left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n (n+r_1)\right) x + (2x+3) \left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right)\right) C + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right) x^2 + 2\left(x^2+2x\right) \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) + 2(x+1) \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0 \tag{9}$$

Since $r_1 = -1$ and $r_2 = -2$ then the above becomes

$$
\begin{aligned}
&\left(2\left(\sum_{n=0}^{\infty} x^{n-2} a_n (n-1)\right) x + (2x+3)\left(\sum_{n=0}^{\infty} a_n x^{n-1}\right)\right) C \\
&+ \left(\sum_{n=0}^{\infty} x^{-4+n} b_n (n-2)(-3+n)\right) x^2 \\
&+ 2(x^2+2x)\left(\sum_{n=0}^{\infty} x^{-3+n} b_n (n-2)\right) + 2(x+1)\left(\sum_{n=0}^{\infty} b_n x^{n-2}\right) = 0
\end{aligned}
\tag{10}
$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} 2C\, x^{n-1} a_n (n-1)\right) + \left(\sum_{n=0}^{\infty} 2C a_n x^n\right) + \left(\sum_{n=0}^{\infty} 3C\, x^{n-1} a_n\right) \\
&+ \left(\sum_{n=0}^{\infty} x^{n-2} b_n (n^2 - 5n + 6)\right) + \left(\sum_{n=0}^{\infty} 2x^{n-1} b_n (n-2)\right) \\
&+ \left(\sum_{n=0}^{\infty} 4x^{n-2} b_n (n-2)\right) + \left(\sum_{n=0}^{\infty} 2x^{n-1} b_n\right) + \left(\sum_{n=0}^{\infty} 2b_n x^{n-2}\right) = 0
\end{aligned}
\tag{2A}
$$

The next step is to make all powers of x be $n-2$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-2} and adjusting the power and the corresponding index gives

$$
\sum_{n=0}^{\infty} 2C\, x^{n-1} a_n (n-1) = \sum_{n=1}^{\infty} 2C a_{n-1} (n-2)\, x^{n-2}
$$

$$
\sum_{n=0}^{\infty} 2C a_n x^n = \sum_{n=2}^{\infty} 2C a_{n-2} x^{n-2}
$$

$$
\sum_{n=0}^{\infty} 3C\, x^{n-1} a_n = \sum_{n=1}^{\infty} 3C a_{n-1} x^{n-2}
$$

$$
\sum_{n=0}^{\infty} 2x^{n-1} b_n (n-2) = \sum_{n=1}^{\infty} 2b_{n-1} (-3+n)\, x^{n-2}
$$

$$
\sum_{n=0}^{\infty} 2x^{n-1} b_n = \sum_{n=1}^{\infty} 2b_{n-1} x^{n-2}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-2$.

$$\begin{aligned}&\left(\sum_{n=1}^{\infty} 2C a_{n-1}(n-2)\, x^{n-2}\right)+\left(\sum_{n=2}^{\infty} 2C a_{n-2} x^{n-2}\right)+\left(\sum_{n=1}^{\infty} 3C a_{n-1} x^{n-2}\right)\\&+\left(\sum_{n=0}^{\infty} x^{n-2} b_n \left(n^2-5n+6\right)\right)+\left(\sum_{n=1}^{\infty} 2 b_{n-1}(-3+n)\, x^{n-2}\right)\\&+\left(\sum_{n=0}^{\infty} 4 x^{n-2} b_n (n-2)\right)+\left(\sum_{n=1}^{\infty} 2 b_{n-1} x^{n-2}\right)+\left(\sum_{n=0}^{\infty} 2 b_n x^{n-2}\right)=0\end{aligned} \tag{2B}$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0=1$. For $n=N$, where $N=1$ which is the difference between the two roots, we are free to choose $b_1=0$. Hence for $n=1$, Eq (2B) gives

$$C-2=0$$

Which is solved for C. Solving for C gives

$$C=2$$

For $n=2$, Eq (2B) gives

$$(2a_0+3a_1)\, C+2b_2=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$4+2b_2=0$$

Solving the above for b_2 gives

$$b_2=-2$$

For $n=3$, Eq (2B) gives

$$(2a_1+5a_2)\, C+2b_2+6b_3=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-4+6b_3=0$$

Solving the above for b_3 gives

$$b_3=\frac{2}{3}$$

For $n=4$, Eq (2B) gives

$$(2a_2+7a_3)\, C+4b_3+12b_4=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{8}{3} + 12b_4 = 0$$

Solving the above for b_4 gives

$$b_4 = -\frac{2}{9}$$

For $n = 5$, Eq (2B) gives

$$(2a_3 + 9a_4)\, C + 6b_4 + 20b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-\frac{4}{3} + 20b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = \frac{1}{15}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = 2$ and all b_n, then the second solution becomes

$$y_2(x) = 2\left(\frac{1 + O(x^6)}{x}\right) \ln(x) + \frac{1 - 2x^2 + \frac{2x^3}{3} - \frac{2x^4}{9} + \frac{x^5}{15} + O(x^6)}{x^2}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= \frac{c_1(1 + O(x^6))}{x} + c_2\left(2\left(\frac{1 + O(x^6)}{x}\right) \ln(x) + \frac{1 - 2x^2 + \frac{2x^3}{3} - \frac{2x^4}{9} + \frac{x^5}{15} + O(x^6)}{x^2}\right)
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= \frac{c_1(1 + O(x^6))}{x} + c_2\left(\frac{2(1 + O(x^6)) \ln(x)}{x} + \frac{1 - 2x^2 + \frac{2x^3}{3} - \frac{2x^4}{9} + \frac{x^5}{15} + O(x^6)}{x^2}\right)
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1(1+O(x^6))}{x} + c_2\left(\frac{2(1+O(x^6))\ln(x)}{x} + \frac{1-2x^2+\frac{2x^3}{3}-\frac{2x^4}{9}+\frac{x^5}{15}+O(x^6)}{x^2}\right) \tag{1}$$

Verification of solutions

$$y = \frac{c_1(1+O(x^6))}{x} + c_2\left(\frac{2(1+O(x^6))\ln(x)}{x} + \frac{1-2x^2+\frac{2x^3}{3}-\frac{2x^4}{9}+\frac{x^5}{15}+O(x^6)}{x^2}\right)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 51

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+2*x*(2+x)*diff(y(x),x)+2*(1+x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{\ln(x)\left(2x+\mathrm{O}\left(x^6\right)\right)c_2 + c_1x(1+\mathrm{O}\left(x^6\right)) + \left(1-2x-2x^2+\frac{2}{3}x^3-\frac{2}{9}x^4+\frac{1}{15}x^5+\mathrm{O}\left(x^6\right)\right)c_2}{x^2}$$

✓ Solution by Mathematica

Time used: 0.03 (sec). Leaf size: 48

```
AsymptoticDSolveValue[x^2*y''[x]+2*x*(2+x)*y'[x]+2*(1+x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(\frac{2\log(x)}{x} - \frac{2x^4 - 6x^3 + 18x^2 + 36x - 9}{9x^2} \right) + \frac{c_2}{x}$$

18.31 problem 25

Internal problem ID [2967]
Internal file name [`OUTPUT/2459_Sunday_June_05_2022_03_14_24_AM_60169280/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.5. page 771
Problem number: 25.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - x(1-x)\,y' + (1-x)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + \left(x^2 - x\right)y' + (1-x)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{x-1}{x}$$
$$q(x) = -\frac{x-1}{x^2}$$

Table 509: Table $p(x), q(x)$ singularites.

$p(x) = \frac{x-1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{x-1}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2 y'' + (x^2 - x) y' + (1 - x) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}\right) + (x^2 - x) \left(\sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}\right) + (1 - x) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)\right) \\ + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} a_{n-1}(n+r-1)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n\right) = \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1)\, x^{n+r}\right) \\ + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) + \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) - x^{n+r} a_n (n+r) + a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) - x^r a_0 r + a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) - x^r r + x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(-1+r)^2 x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(-1+r)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = 1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(-1+r)^2 x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = 1$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{1+n}$$
$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{1+n}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_{n-1}(n+r-1) - a_n(n+r) + a_n - a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}(n+r-2)}{n^2+2nr+r^2-2n-2r+1} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = -\frac{a_{n-1}(n-1)}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{1-r}{r^2}$$

Which for the root $r = 1$ becomes

$$a_1 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1-r}{r^2}$	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{-1+r}{r\,(r+1)^2}$$

Which for the root $r = 1$ becomes

$$a_2 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1-r}{r^2}$	0
a_2	$\frac{-1+r}{r(r+1)^2}$	0

For $n=3$, using the above recursive equation gives

$$a_3 = \frac{1-r}{r\left(r+1\right)\left(r+2\right)^2}$$

Which for the root $r=1$ becomes

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1-r}{r^2}$	0
a_2	$\frac{-1+r}{r(r+1)^2}$	0
a_3	$\frac{1-r}{r(r+1)(r+2)^2}$	0

For $n=4$, using the above recursive equation gives

$$a_4 = \frac{-1+r}{\left(r+2\right)\left(r+1\right)r\left(3+r\right)^2}$$

Which for the root $r=1$ becomes

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1-r}{r^2}$	0
a_2	$\frac{-1+r}{r(r+1)^2}$	0
a_3	$\frac{1-r}{r(r+1)(r+2)^2}$	0
a_4	$\frac{-1+r}{(r+2)(r+1)r(3+r)^2}$	0

For $n=5$, using the above recursive equation gives

$$a_5 = \frac{1-r}{r\left(r+1\right)\left(r+2\right)\left(3+r\right)\left(r+4\right)^2}$$

Which for the root $r=1$ becomes

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1-r}{r^2}$	0
a_2	$\frac{-1+r}{r(r+1)^2}$	0
a_3	$\frac{1-r}{r(r+1)(r+2)^2}$	0
a_4	$\frac{-1+r}{(r+2)(r+1)r(3+r)^2}$	0
a_5	$\frac{1-r}{r(r+1)(r+2)(3+r)(r+4)^2}$	0

Using the above table, then the first solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= x\left(1 + O\left(x^6\right)\right)
\end{aligned}
$$

Now the second solution is found. The second solution is given by

$$
y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)
$$

Where b_n is found using

$$
b_n = \frac{d}{dr} a_{n,r}
$$

And the above is then evaluated at $r = 1$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	$b_n(r = 1)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	$\frac{1-r}{r^2}$	0	$\frac{r-2}{r^3}$	-1
b_2	$\frac{-1+r}{r(r+1)^2}$	0	$\frac{-2r^2+3r+1}{r^2(r+1)^3}$	$\frac{1}{4}$
b_3	$\frac{1-r}{r(r+1)(r+2)^2}$	0	$\frac{3r^3-7r-2}{r^2(r+1)^2(r+2)^3}$	$-\frac{1}{18}$
b_4	$\frac{-1+r}{(r+2)(r+1)r(3+r)^2}$	0	$\frac{-4r^4-10r^3+8r^2+24r+6}{r^2(r+1)^2(r+2)^2(3+r)^3}$	$\frac{1}{96}$
b_5	$\frac{1-r}{r(r+1)(r+2)(3+r)(r+4)^2}$	0	$\frac{5r^5+30r^4+35r^3-60r^2-106r-24}{r^2(r+1)^2(r+2)^2(3+r)^2(r+4)^3}$	$-\frac{1}{600}$

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6\dots \\
&= x\left(1 + O(x^6)\right)\ln(x) + x\left(-x + \frac{x^2}{4} - \frac{x^3}{18} + \frac{x^4}{96} - \frac{x^5}{600} + O(x^6)\right)
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1y_1(x) + c_2y_2(x) \\
&= c_1x\left(1+O(x^6)\right)+c_2\left(x\left(1+O(x^6)\right)\ln(x)+x\left(-x+\frac{x^2}{4}-\frac{x^3}{18}+\frac{x^4}{96}-\frac{x^5}{600}+O(x^6)\right)\right)
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= c_1x\left(1+O(x^6)\right)+c_2\left(x\left(1+O(x^6)\right)\ln(x)+x\left(-x+\frac{x^2}{4}-\frac{x^3}{18}+\frac{x^4}{96}-\frac{x^5}{600}+O(x^6)\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}
y &= c_1x\left(1 + O(x^6)\right) \\
&\quad + c_2\left(x\left(1 + O(x^6)\right)\ln(x) + x\left(-x + \frac{x^2}{4} - \frac{x^3}{18} + \frac{x^4}{96} - \frac{x^5}{600} + O(x^6)\right)\right)
\end{aligned} \tag{1}$$

Verification of solutions

$$y=c_1x\left(1+O(x^6)\right)+c_2\left(x\left(1+O(x^6)\right)\ln(x)+x\left(-x+\frac{x^2}{4}-\frac{x^3}{18}+\frac{x^4}{96}-\frac{x^5}{600}+O(x^6)\right)\right)$$

Verified OK.

18.31.1 Maple step by step solution

Let's solve

$x^2y'' + (x^2 - x)\,y' + (1 - x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(x-1)y}{x^2} - \frac{(x-1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(x-1)y'}{x} - \frac{(x-1)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{x-1}{x}, P_3(x) = -\frac{x-1}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 1$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $x^2y'' + x(x-1)\,y' + (1-x)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k->k-m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  - Shift index using $k->k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(-1+r)^2\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r-1)^2 + a_{k-1}(k-2+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = 1$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r-1)^2 + a_{k-1}(k-2+r) = 0$

- Shift index using $k- >k+1$

  $a_{k+1}(k+r)^2 + a_k(k+r-1) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k(k+r-1)}{(k+r)^2}$

- Recursion relation for $r = 1$

  $a_{k+1} = -\frac{a_k k}{(k+1)^2}$

- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = -\frac{a_k k}{(k+1)^2}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 43

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)-x*(1-x)*diff(y(x),x)+(1-x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left((c_2 \ln(x) + c_1)\left(1 + \mathrm{O}\left(x^6\right)\right) + \left(-x + \frac{1}{4}x^2 - \frac{1}{18}x^3 + \frac{1}{96}x^4 - \frac{1}{600}x^5 + \mathrm{O}\left(x^6\right)\right) c_2\right) x$$

✓ Solution by Mathematica
Time used: 0.004 (sec). Leaf size: 50

```
AsymptoticDSolveValue[x^2*y''[x]-x*(1-x)*y'[x]+(1-x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(x\left(-\frac{x^5}{600} + \frac{x^4}{96} - \frac{x^3}{18} + \frac{x^2}{4} - x\right) + x\log(x)\right) + c_1 x$$

18.32 problem 26

Internal problem ID [2968]
Internal file name [`OUTPUT/2460_Sunday_June_05_2022_03_14_27_AM_97326943/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.5. page 771
Problem number: 26.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^{2}y'' + 4x(1+2x)\, y' + (-1+4x)\, y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$4x^{2}y'' + \left(8x^{2}+4x\right) y' + (-1+4x)\, y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$\begin{aligned} p(x) &= \frac{1+2x}{x} \\ q(x) &= \frac{-1+4x}{4x^{2}} \end{aligned}$$

Table 511: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1+2x}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{-1+4x}{4x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$4x^2 y'' + \left(8x^2 + 4x\right) y' + (-1 + 4x) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$4x^2 \left(\sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}\right) + \left(8x^2 + 4x\right) \left(\sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}\right) + (-1 + 4x) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 8x^{1+n+r} a_n (n+r)\right) \\ + \left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 4x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 8x^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} 8a_{n-1}(n+r-1)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} 4x^{1+n+r} a_n = \sum_{n=1}^{\infty} 4a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} 8a_{n-1}(n+r-1)\, x^{n+r}\right) \\ + \left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) + \left(\sum_{n=1}^{\infty} 4a_{n-1} x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$4x^{n+r} a_n (n+r)(n+r-1) + 4x^{n+r} a_n (n+r) - a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$4x^r a_0 r(-1+r) + 4x^r a_0 r - a_0 x^r = 0$$

Or

$$\left(4x^r r(-1+r) + 4x^r r - x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(4r^2-1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$4r^2 - 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{2}$$
$$r_2 = -\frac{1}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(4r^2 - 1)\, x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sqrt{x} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{\sqrt{x}}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-\frac{1}{2}}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$4a_n(n+r)(n+r-1) + 8a_{n-1}(n+r-1) + 4a_n(n+r) - a_n + 4a_{n-1} = 0 \qquad (3)$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{4a_{n-1}}{2n + 2r + 1} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = -\frac{2a_{n-1}}{1 + n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{4}{3 + 2r}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_1 = -1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{4}{3+2r}$	-1

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{16}{4r^2 + 16r + 15}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = \frac{2}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{4}{3+2r}$	-1
a_2	$\frac{16}{4r^2+16r+15}$	$\frac{2}{3}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{64}{8r^3 + 60r^2 + 142r + 105}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_3 = -\frac{1}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{4}{3+2r}$	-1
a_2	$\frac{16}{4r^2+16r+15}$	$\frac{2}{3}$
a_3	$-\frac{64}{8r^3+60r^2+142r+105}$	$-\frac{1}{3}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{256}{(4r^2 + 16r + 15)(7 + 2r)(9 + 2r)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = \frac{2}{15}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{4}{3+2r}$	-1
a_2	$\frac{16}{4r^2+16r+15}$	$\frac{2}{3}$
a_3	$-\frac{64}{8r^3+60r^2+142r+105}$	$-\frac{1}{3}$
a_4	$\frac{256}{(4r^2+16r+15)(7+2r)(9+2r)}$	$\frac{2}{15}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{1024}{(4r^2 + 16r + 15)(7 + 2r)(11 + 2r)(9 + 2r)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_5 = -\frac{2}{45}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{4}{3+2r}$	-1
a_2	$\frac{16}{4r^2+16r+15}$	$\frac{2}{3}$
a_3	$-\frac{64}{8r^3+60r^2+142r+105}$	$-\frac{1}{3}$
a_4	$\frac{256}{(4r^2+16r+15)(7+2r)(9+2r)}$	$\frac{2}{15}$
a_5	$-\frac{1024}{(4r^2+16r+15)(7+2r)(11+2r)(9+2r)}$	$-\frac{2}{45}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}
y_1(x) &= \sqrt{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= \sqrt{x}\left(1 - x + \frac{2x^2}{3} - \frac{x^3}{3} + \frac{2x^4}{15} - \frac{2x^5}{45} + O(x^6)\right)
\end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned}
a_N &= a_1 \\
&= -\frac{4}{3 + 2r}
\end{aligned}$$

Therefore

$$\begin{aligned}
\lim_{r \to r_2} -\frac{4}{3 + 2r} &= \lim_{r \to -\frac{1}{2}} -\frac{4}{3 + 2r} \\
&= -2
\end{aligned}$$

The limit is -2. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\ &= \sum_{n=0}^{\infty} b_n x^{n-\frac{1}{2}} \end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$4b_n(n+r)(n+r-1) + 8b_{n-1}(n+r-1) + 4b_n(n+r) - b_n + 4b_{n-1} = 0 \tag{4}$$

Which for for the root $r = -\frac{1}{2}$ becomes

$$4b_n\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right) + 8b_{n-1}\left(n-\frac{3}{2}\right) + 4b_n\left(n-\frac{1}{2}\right) - b_n + 4b_{n-1} = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = -\frac{4b_{n-1}}{2n+2r+1} \tag{5}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_n = -\frac{2b_{n-1}}{n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = -\frac{4}{3+2r}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_1 = -2$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{4}{3+2r}$	-2

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{16}{(3+2r)(5+2r)}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_2 = 2$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{4}{3+2r}$	-2
b_2	$\frac{16}{4r^2+16r+15}$	2

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{64}{(3+2r)(7+2r)(5+2r)}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_3 = -\frac{4}{3}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{4}{3+2r}$	-2
b_2	$\frac{16}{4r^2+16r+15}$	2
b_3	$-\frac{64}{8r^3+60r^2+142r+105}$	$-\frac{4}{3}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{256}{(3+2r)(5+2r)(7+2r)(9+2r)}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_4 = \frac{2}{3}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{4}{3+2r}$	-2
b_2	$\frac{16}{4r^2+16r+15}$	2
b_3	$-\frac{64}{8r^3+60r^2+142r+105}$	$-\frac{4}{3}$
b_4	$\frac{256}{16r^4+192r^3+824r^2+1488r+945}$	$\frac{2}{3}$

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{1024}{(3+2r)(7+2r)(5+2r)(11+2r)(9+2r)}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_5 = -\frac{4}{15}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{4}{3+2r}$	-2
b_2	$\frac{16}{4r^2+16r+15}$	2
b_3	$-\frac{64}{8r^3+60r^2+142r+105}$	$-\frac{4}{3}$
b_4	$\frac{256}{16r^4+192r^3+824r^2+1488r+945}$	$\frac{2}{3}$
b_5	$-\frac{1024}{(3+2r)(7+2r)(5+2r)(11+2r)(9+2r)}$	$-\frac{4}{15}$

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= \sqrt{x}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\
&= \frac{1 - 2x + 2x^2 - \frac{4x^3}{3} + \frac{2x^4}{3} - \frac{4x^5}{15} + O(x^6)}{\sqrt{x}}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 \sqrt{x}\left(1 - x + \frac{2x^2}{3} - \frac{x^3}{3} + \frac{2x^4}{15} - \frac{2x^5}{45} + O(x^6)\right) \\
&\quad + \frac{c_2\left(1 - 2x + 2x^2 - \frac{4x^3}{3} + \frac{2x^4}{3} - \frac{4x^5}{15} + O(x^6)\right)}{\sqrt{x}}
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 \sqrt{x}\left(1 - x + \frac{2x^2}{3} - \frac{x^3}{3} + \frac{2x^4}{15} - \frac{2x^5}{45} + O(x^6)\right) \\
&\quad + \frac{c_2\left(1 - 2x + 2x^2 - \frac{4x^3}{3} + \frac{2x^4}{3} - \frac{4x^5}{15} + O(x^6)\right)}{\sqrt{x}}
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 \sqrt{x}\left(1 - x + \frac{2x^2}{3} - \frac{x^3}{3} + \frac{2x^4}{15} - \frac{2x^5}{45} + O(x^6)\right) \\
&\quad + \frac{c_2\left(1 - 2x + 2x^2 - \frac{4x^3}{3} + \frac{2x^4}{3} - \frac{4x^5}{15} + O(x^6)\right)}{\sqrt{x}}
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1 \sqrt{x}\left(1 - x + \frac{2x^2}{3} - \frac{x^3}{3} + \frac{2x^4}{15} - \frac{2x^5}{45} + O(x^6)\right) \\
&\quad + \frac{c_2\left(1 - 2x + 2x^2 - \frac{4x^3}{3} + \frac{2x^4}{3} - \frac{4x^5}{15} + O(x^6)\right)}{\sqrt{x}}
\end{aligned}
$$

Verified OK.

18.32.1 Maple step by step solution

Let's solve

$4x^2y'' + (8x^2 + 4x)\,y' + (-1 + 4x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(-1+4x)y}{4x^2} - \frac{(1+2x)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(1+2x)y'}{x} + \frac{(-1+4x)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{1+2x}{x}, P_3(x) = \frac{-1+4x}{4x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -\frac{1}{4}$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2y'' + 4x(1 + 2x)\,y' + (-1 + 4x)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 (1+2r)\,(-1+2r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k (2k+2r+1)\,(2k+2r-1) + 4a_{k-1}(2k+2r-1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+2r)\,(-1+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$4\left(k+r-\frac{1}{2}\right)\left(\left(k+r+\frac{1}{2}\right) a_k + 2a_{k-1}\right) = 0$$

- Shift index using $k- >k+1$

$$4\left(k+r+\frac{1}{2}\right)\left(\left(k+\frac{3}{2}+r\right) a_{k+1} + 2a_k\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{4a_k}{2k+3+2r}$$

- Recursion relation for $r = -\frac{1}{2}$

$$a_{k+1} = -\frac{4a_k}{2k+2}$$

- Solution for $r = -\frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+1} = -\frac{4a_k}{2k+2}\right]$$

- Recursion relation for $r = \frac{1}{2}$

$$a_{k+1} = -\frac{4a_k}{2k+4}$$

- Solution for $r = \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = -\frac{4a_k}{2k+4}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{1+k} = -\frac{4a_k}{2+2k}, b_{1+k} = -\frac{4b_k}{2k+4}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 47

```
Order:=6;
dsolve(4*x^2*diff(y(x),x$2)+4*x*(1+2*x)*diff(y(x),x)+(4*x-1)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_1 x \left(1 - x + \frac{2}{3}x^2 - \frac{1}{3}x^3 + \frac{2}{15}x^4 - \frac{2}{45}x^5 + \mathrm{O}\left(x^6\right)\right) + c_2 \left(1 - 2x + 2x^2 - \frac{4}{3}x^3 + \frac{2}{3}x^4 - \frac{4}{15}x^5 + \mathrm{O}\left(x^6\right)\right)}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 88

```
AsymptoticDSolveValue[4*x^2*y''[x]+4*x*(1+2*x)*y'[x]+(4*x-1)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{2x^{7/2}}{3}-\frac{4x^{5/2}}{3}+2x^{3/2}-2\sqrt{x}+\frac{1}{\sqrt{x}}\right)+c_2\left(\frac{2x^{9/2}}{15}-\frac{x^{7/2}}{3}+\frac{2x^{5/2}}{3}-x^{3/2}+\sqrt{x}\right)$$

18.33 problem 27

Internal problem ID [2969]
Internal file name [`OUTPUT/2461_Sunday_June_05_2022_03_14_31_AM_35036725/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.5. page 771
Problem number: 27.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2y'' - (4x+3)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$4x^2y'' + (-4x-3)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$\begin{aligned}p(x) &= 0\\ q(x) &= -\frac{4x+3}{4x^2}\end{aligned}$$

Table 513: Table $p(x), q(x)$ singularites.

$p(x) = 0$	
singularity	type

$q(x) = -\frac{4x+3}{4x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$4x^2y'' + (-4x - 3)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\, (n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$4x^2 \left(\sum_{n=0}^{\infty} (n+r)\, (n+r-1)\, a_n x^{n+r-2}\right) + (-4x - 3) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \qquad (1)$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)\, (n+r-1)\right) + \sum_{n\,=0}^{\infty} \left(-4x^{1+n+r} a_n\right) + \sum_{n\,=0}^{\infty} \left(-3a_n x^{n+r}\right) = 0 \quad \text{(2A)}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-4x^{1+n+r} a_n\right) = \sum_{n=1}^{\infty} \left(-4a_{n-1} x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r) (n+r-1)\right) + \sum_{n=1}^{\infty} \left(-4a_{n-1} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-3a_n x^{n+r}\right) = 0 \quad \text{(2B)}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$4x^{n+r} a_n (n+r) (n+r-1) - 3a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$4x^r a_0 r(-1+r) - 3a_0 x^r = 0$$

Or

$$\left(4x^r r(-1+r) - 3x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(4r^2 - 4r - 3\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$4r^2 - 4r - 3 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{3}{2}$$
$$r_2 = -\frac{1}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(4r^2 - 4r - 3\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 2$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = Cy_1(x)\ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x^{\frac{3}{2}}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = Cy_1(x)\ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{\sqrt{x}}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{3}{2}}$$
$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-\frac{1}{2}}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$4a_n(n+r)(n+r-1) - 4a_{n-1} - 3a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{4a_{n-1}}{4n^2 + 8nr + 4r^2 - 4n - 4r - 3} \tag{4}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_n = \frac{a_{n-1}}{n(n+2)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{3}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{4}{4r^2 + 4r - 3}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_1 = \frac{1}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{4}{4r^2+4r-3}$	$\frac{1}{3}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{16}{(4r^2 + 4r - 3)(4r^2 + 12r + 5)}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_2 = \frac{1}{24}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{4}{4r^2+4r-3}$	$\frac{1}{3}$
a_2	$\frac{16}{(4r^2+4r-3)(4r^2+12r+5)}$	$\frac{1}{24}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{64}{(4r^2 + 4r - 3)(4r^2 + 12r + 5)(4r^2 + 20r + 21)}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_3 = \frac{1}{360}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{4}{4r^2+4r-3}$	$\frac{1}{3}$
a_2	$\frac{16}{(4r^2+4r-3)(4r^2+12r+5)}$	$\frac{1}{24}$
a_3	$\frac{64}{(4r^2+4r-3)(4r^2+12r+5)(4r^2+20r+21)}$	$\frac{1}{360}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{256}{(4r^2+4r-3)(4r^2+12r+5)(4r^2+20r+21)(4r^2+28r+45)}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_4 = \frac{1}{8640}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{4}{4r^2+4r-3}$	$\frac{1}{3}$
a_2	$\frac{16}{(4r^2+4r-3)(4r^2+12r+5)}$	$\frac{1}{24}$
a_3	$\frac{64}{(4r^2+4r-3)(4r^2+12r+5)(4r^2+20r+21)}$	$\frac{1}{360}$
a_4	$\frac{256}{(4r^2+4r-3)(4r^2+12r+5)(4r^2+20r+21)(4r^2+28r+45)}$	$\frac{1}{8640}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{1024}{(2r-1)(2r+1)(2r+5)^2(2r+7)^2(2r+3)^2(2r+9)(11+2r)}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_5 = \frac{1}{302400}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{4}{4r^2+4r-3}$	$\frac{1}{3}$
a_2	$\frac{16}{(4r^2+4r-3)(4r^2+12r+5)}$	$\frac{1}{24}$
a_3	$\frac{64}{(4r^2+4r-3)(4r^2+12r+5)(4r^2+20r+21)}$	$\frac{1}{360}$
a_4	$\frac{256}{(4r^2+4r-3)(4r^2+12r+5)(4r^2+20r+21)(4r^2+28r+45)}$	$\frac{1}{8640}$
a_5	$\frac{1024}{(2r-1)(2r+1)(2r+5)^2(2r+7)^2(2r+3)^2(2r+9)(11+2r)}$	$\frac{1}{302400}$

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x^{\frac{3}{2}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= x^{\frac{3}{2}}\left(1 + \frac{x}{3} + \frac{x^2}{24} + \frac{x^3}{360} + \frac{x^4}{8640} + \frac{x^5}{302400} + O(x^6)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Let

$$
r_1 - r_2 = N
$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 2$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_2(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$
\begin{aligned}
a_N &= a_2 \\
&= \frac{16}{(4r^2 + 4r - 3)(4r^2 + 12r + 5)}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\lim_{r\to r_2} \frac{16}{(4r^2 + 4r - 3)(4r^2 + 12r + 5)} &= \lim_{r\to -\frac{1}{2}} \frac{16}{(4r^2 + 4r - 3)(4r^2 + 12r + 5)} \\
&= \textit{undefined}
\end{aligned}
$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$
y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)
$$

Therefore

$$
\begin{aligned}
\frac{d}{dx}y_2(x) &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\
&= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right) \\
\frac{d^2}{dx^2}y_2(x) &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\
&= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right)
\end{aligned}
$$

Substituting these back into the given ode $4x^2y'' + (-4x-3)\,y = 0$ gives

$$
\begin{aligned}
&4x^2\left(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2}\right. \\
&\left.+ \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) \\
&+ (-4x-3)\left(Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)\right) = 0
\end{aligned}
$$

Which can be written as

$$
\begin{aligned}
&\left(\left(4x^2y_1''(x) + (-4x-3)\,y_1(x)\right)\ln(x) + 4x^2\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right)\right)C \\
&+ 4x^2\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) \\
&+ (-4x-3)\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned} \tag{7}
$$

But since $y_1(x)$ is a solution to the ode, then

$$
4x^2y_1''(x) + (-4x-3)\,y_1(x) = 0
$$

Eq (7) simplifes to

$$
\begin{aligned}
&4x^2\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)C+4x^2\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) \\
&+(-4x-3)\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)=0
\end{aligned} \tag{8}
$$

Substituting $y_1=\sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$
\begin{aligned}
&\left(8\left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n(n+r_1)\right)x-4\left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right)\right)C \\
&+4\left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right)x^2+(-4x-3)\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)=0
\end{aligned} \tag{9}
$$

Since $r_1=\frac{3}{2}$ and $r_2=-\frac{1}{2}$ then the above becomes

$$
\begin{aligned}
&\left(8\left(\sum_{n=0}^{\infty} x^{\frac{1}{2}+n} a_n\left(n+\frac{3}{2}\right)\right)x-4\left(\sum_{n=0}^{\infty} a_n x^{n+\frac{3}{2}}\right)\right)C \\
&+4\left(\sum_{n=0}^{\infty} x^{-\frac{5}{2}+n} b_n\left(n-\frac{1}{2}\right)\left(-\frac{3}{2}+n\right)\right)x^2+(-4x-3)\left(\sum_{n=0}^{\infty} b_n x^{n-\frac{1}{2}}\right)=0
\end{aligned} \tag{10}
$$

Expanding $-4\sqrt{x}$ as Taylor series around $x=0$ and keeping only the first 6 terms gives

$$
\begin{aligned}
-4\sqrt{x} &= -4\sqrt{x}+\dots \\
&= -4\sqrt{x}
\end{aligned}
$$

Expanding $-\frac{3}{\sqrt{x}}$ as Taylor series around $x=0$ and keeping only the first 6 terms gives

$$
\begin{aligned}
-\frac{3}{\sqrt{x}} &= -\frac{3}{\sqrt{x}}+\dots \\
&= -\frac{3}{\sqrt{x}}
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} (8n+12)\, C a_n x^{n+\frac{3}{2}}\right) + \sum_{n=0}^{\infty} \left(-4 C a_n x^{n+\frac{3}{2}}\right) \\
&+ \left(\sum_{n=0}^{\infty} x^{n-\frac{1}{2}} b_n \left(4n^2-8n+3\right)\right) + \sum_{n=0}^{\infty} \left(-4 x^{\frac{1}{2}+n} b_n\right) + \sum_{n=0}^{\infty} \left(-3 b_n x^{n-\frac{1}{2}}\right) = 0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n-\frac{1}{2}$ in each summation term. Going over each summation term above with power of x in it which is not already $x^{n-\frac{1}{2}}$ and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} (8n+12)\, C a_n x^{n+\frac{3}{2}} &= \sum_{n=2}^{\infty} C a_{n-2} (8n-4)\, x^{n-\frac{1}{2}} \\
\sum_{n=0}^{\infty} \left(-4 C a_n x^{n+\frac{3}{2}}\right) &= \sum_{n=2}^{\infty} \left(-4 C a_{n-2} x^{n-\frac{1}{2}}\right) \\
\sum_{n=0}^{\infty} \left(-4 x^{\frac{1}{2}+n} b_n\right) &= \sum_{n=1}^{\infty} \left(-4 b_{n-1} x^{n-\frac{1}{2}}\right)
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-\frac{1}{2}$.

$$
\begin{aligned}
&\left(\sum_{n=2}^{\infty} C a_{n-2} (8n-4)\, x^{n-\frac{1}{2}}\right) + \sum_{n=2}^{\infty} \left(-4 C a_{n-2} x^{n-\frac{1}{2}}\right) \\
&+ \left(\sum_{n=0}^{\infty} x^{n-\frac{1}{2}} b_n \left(4n^2-8n+3\right)\right) \\
&+ \sum_{n=1}^{\infty} \left(-4 b_{n-1} x^{n-\frac{1}{2}}\right) + \sum_{n=0}^{\infty} \left(-3 b_n x^{n-\frac{1}{2}}\right) = 0
\end{aligned} \tag{2B}
$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0 = 1$. For $n=1$, Eq (2B) gives

$$
-4b_1 - 4b_0 = 0
$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$
-4b_1 - 4 = 0
$$

Solving the above for b_1 gives

$$b_1 = -1$$

For $n = N$, where $N = 2$ which is the difference between the two roots, we are free to choose $b_2 = 0$. Hence for $n = 2$, Eq (2B) gives

$$8C + 4 = 0$$

Which is solved for C. Solving for C gives

$$C = -\frac{1}{2}$$

For $n = 3$, Eq (2B) gives

$$16Ca_1 - 4b_2 + 12b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$12b_3 - \frac{8}{3} = 0$$

Solving the above for b_3 gives

$$b_3 = \frac{2}{9}$$

For $n = 4$, Eq (2B) gives

$$24Ca_2 - 4b_3 + 32b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$32b_4 - \frac{25}{18} = 0$$

Solving the above for b_4 gives

$$b_4 = \frac{25}{576}$$

For $n = 5$, Eq (2B) gives

$$32Ca_3 - 4b_4 + 60b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$60b_5 - \frac{157}{720} = 0$$

Solving the above for b_5 gives

$$b_5 = \frac{157}{43200}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -\frac{1}{2}$ and all b_n, then the second solution becomes

$$
\begin{aligned}
y_2(x) = {} & -\frac{1}{2}\left(x^{\frac{3}{2}}\left(1+\frac{x}{3}+\frac{x^2}{24}+\frac{x^3}{360}+\frac{x^4}{8640}+\frac{x^5}{302400}+O(x^6)\right)\right)\ln(x) \\
& + \frac{1-x+\frac{2x^3}{9}+\frac{25x^4}{576}+\frac{157x^5}{43200}+O(x^6)}{\sqrt{x}}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^{\frac{3}{2}}\left(1+\frac{x}{3}+\frac{x^2}{24}+\frac{x^3}{360}+\frac{x^4}{8640}+\frac{x^5}{302400}+O(x^6)\right) \\
&\quad + c_2\left(-\frac{1}{2}\left(x^{\frac{3}{2}}\left(1+\frac{x}{3}+\frac{x^2}{24}+\frac{x^3}{360}+\frac{x^4}{8640}+\frac{x^5}{302400}+O(x^6)\right)\right)\ln(x)\right. \\
&\qquad \left.+ \frac{1-x+\frac{2x^3}{9}+\frac{25x^4}{576}+\frac{157x^5}{43200}+O(x^6)}{\sqrt{x}}\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^{\frac{3}{2}}\left(1+\frac{x}{3}+\frac{x^2}{24}+\frac{x^3}{360}+\frac{x^4}{8640}+\frac{x^5}{302400}+O(x^6)\right) \\
&\quad + c_2\left(-\frac{x^{\frac{3}{2}}\left(1+\frac{x}{3}+\frac{x^2}{24}+\frac{x^3}{360}+\frac{x^4}{8640}+\frac{x^5}{302400}+O(x^6)\right)\ln(x)}{2}\right. \\
&\qquad \left.+ \frac{1-x+\frac{2x^3}{9}+\frac{25x^4}{576}+\frac{157x^5}{43200}+O(x^6)}{\sqrt{x}}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x^{\frac{3}{2}}\left(1+\frac{x}{3}+\frac{x^2}{24}+\frac{x^3}{360}+\frac{x^4}{8640}+\frac{x^5}{302400}+O(x^6)\right) \\
&\quad + c_2\left(-\frac{x^{\frac{3}{2}}\left(1+\frac{x}{3}+\frac{x^2}{24}+\frac{x^3}{360}+\frac{x^4}{8640}+\frac{x^5}{302400}+O(x^6)\right)\ln(x)}{2}\right. \\
&\qquad \left.+ \frac{1-x+\frac{2x^3}{9}+\frac{25x^4}{576}+\frac{157x^5}{43200}+O(x^6)}{\sqrt{x}}\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$y = c_1 x^{\frac{3}{2}} \left(1+\frac{x}{3}+\frac{x^2}{24}+\frac{x^3}{360}+\frac{x^4}{8640}+\frac{x^5}{302400}+O(x^6)\right) + c_2 \left(-\frac{x^{\frac{3}{2}} \left(1+\frac{x}{3}+\frac{x^2}{24}+\frac{x^3}{360}+\frac{x^4}{8640}+\frac{x^5}{302400}+O(x^6)\right) \ln(x)}{2} + \frac{1-x+\frac{2x^3}{9}+\frac{25x^4}{576}+\frac{157x^5}{43200}+O(x^6)}{\sqrt{x}}\right)$$

Verified OK.

18.33.1 Maple step by step solution

Let's solve

$4x^2y''+(-4x-3)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{(4x+3)y}{4x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{(4x+3)y}{4x^2}=0$

□ Check to see if $x_0=0$ is a regular singular point

  - Define functions

    $\left[P_2(x)=0, P_3(x)=-\frac{4x+3}{4x^2}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=0$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2\cdot P_3(x))\right|_{x=0}=-\frac{3}{4}$

  - $x=0$is a regular singular point

    Check to see if $x_0=0$ is a regular singular point

    $x_0=0$

- Multiply by denominators

  $4x^2y'' + (-4x-3)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ◦ Shift index using $k- >k-m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  ◦ Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)\,(-3+2r)\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r+1)\,(2k+2r-3) - 4a_{k-1}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)\,(-3+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{3}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $4\left(k+r+\frac{1}{2}\right)\left(k+r-\frac{3}{2}\right) a_k - 4a_{k-1} = 0$

- Shift index using $k- >k+1$

  $4\left(k+\frac{3}{2}+r\right)\left(k-\frac{1}{2}+r\right) a_{k+1} - 4a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{4a_k}{(2k+3+2r)(2k-1+2r)}$

- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+1} = \frac{4a_k}{(2k+2)(2k-2)}$

- Series not valid for $r = -\frac{1}{2}$, division by 0 in the recursion relation at $k = 1$

$$a_{k+1} = \frac{4a_k}{(2k+2)(2k-2)}$$

- Recursion relation for $r = \frac{3}{2}$

$$a_{k+1} = \frac{4a_k}{(2k+6)(2k+2)}$$

- Solution for $r = \frac{3}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+1} = \frac{4a_k}{(2k+6)(2k+2)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 65

```
Order:=6;
dsolve(4*x^2*diff(y(x),x$2)-(3+4*x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_1 x^2 \left(1 + \frac{1}{3}x + \frac{1}{24}x^2 + \frac{1}{360}x^3 + \frac{1}{8640}x^4 + \frac{1}{302400}x^5 + \mathrm{O}\left(x^6\right)\right) + c_2 \left(\ln\left(x\right)\left(x^2 + \frac{1}{3}x^3 + \frac{1}{24}x^4 + \frac{1}{360}x^5 + \mathrm{O}\left(x^6 \right.\right.}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 101

```
AsymptoticDSolveValue[4*x^2*y''[x]-(3+4*x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{x^{11/2}}{8640}+\frac{x^{9/2}}{360}+\frac{x^{7/2}}{24}+\frac{x^{5/2}}{3}+x^{3/2}\right)+c_1\left(\frac{31x^4+176x^3+144x^2-576x+576}{576\sqrt{x}}-\frac{1}{48}x^{3/2}\left(x^2+8x+24\right)\log(x)\right)$$

18.34 problem 28

Internal problem ID [2970]
Internal file name [`OUTPUT/2462_Sunday_June_05_2022_03_14_35_AM_36856821/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.5. page 771
Problem number: 28.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[_Laguerre, [_2nd_order, _linear, `_with_symmetry_[0,F(x)]`]]
```

$$\boxed{xy'' - xy' + y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' - xy' + y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -1$$
$$q(x) = \frac{1}{x}$$

Table 515: Table $p(x), q(x)$ singularites.

$p(x) = -1$	
singularity	type

$q(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' - xy' + y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}\right) x - x\left(\sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r) (n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \quad \text{(2A)}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)\, x^{n+r-1}\right)$$
$$\sum_{n=0}^{\infty} a_n x^{n+r} = \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)\,(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)\, x^{n+r-1}\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)\,(n+r-1) = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) = 0$$

Or

$$x^{-1+r} a_0 r(-1+r) = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^{-1+r} r(-1+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(-1+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^{-1+r}r(-1+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) - a_{n-1}(n+r-1) + a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}(n+r-2)}{(n+r)(n+r-1)} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = \frac{a_{n-1}(n-1)}{(n+1)\,n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-1+r}{(1+r)\,r}$$

Which for the root $r = 1$ becomes

$$a_1 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1+r}{(1+r)r}$	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{-1+r}{(1+r)^2\,(2+r)}$$

Which for the root $r = 1$ becomes

$$a_2 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1+r}{(1+r)r}$	0
a_2	$\frac{-1+r}{(1+r)^2(2+r)}$	0

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-1+r}{(1+r)\,(2+r)^2\,(3+r)}$$

Which for the root $r = 1$ becomes

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1+r}{(1+r)r}$	0
a_2	$\frac{-1+r}{(1+r)^2(2+r)}$	0
a_3	$\frac{-1+r}{(1+r)(2+r)^2(3+r)}$	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{-1+r}{(1+r)(2+r)(3+r)^2(4+r)}$$

Which for the root $r = 1$ becomes

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1+r}{(1+r)r}$	0
a_2	$\frac{-1+r}{(1+r)^2(2+r)}$	0
a_3	$\frac{-1+r}{(1+r)(2+r)^2(3+r)}$	0
a_4	$\frac{-1+r}{(1+r)(2+r)(3+r)^2(4+r)}$	0

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-1+r}{(1+r)(2+r)(3+r)(4+r)^2(5+r)}$$

Which for the root $r = 1$ becomes

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1+r}{(1+r)r}$	0
a_2	$\frac{-1+r}{(1+r)^2(2+r)}$	0
a_3	$\frac{-1+r}{(1+r)(2+r)^2(3+r)}$	0
a_4	$\frac{-1+r}{(1+r)(2+r)(3+r)^2(4+r)}$	0
a_5	$\frac{-1+r}{(1+r)(2+r)(3+r)(4+r)^2(5+r)}$	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}y_1(x) &= x\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots\right)\\&= x\left(1 + O(x^6)\right)\end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned}a_N &= a_1\\&= \frac{-1+r}{(1+r)\,r}\end{aligned}$$

Therefore

$$\begin{aligned}\lim_{r\to r_2} \frac{-1+r}{(1+r)\,r} &= \lim_{r\to 0} \frac{-1+r}{(1+r)\,r}\\&= \textit{undefined}\end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$
\begin{aligned}
\frac{d}{dx}y_2(x) &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\
&= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right) \\
\frac{d^2}{dx^2}y_2(x) &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\
&= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right)
\end{aligned}
$$

Substituting these back into the given ode $xy'' - xy' + y = 0$ gives

$$
\begin{aligned}
&\left(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2}\right. \\
&\left.\quad + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x \\
&\quad - x\left(Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right) \\
&\quad + Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned}
$$

Which can be written as

$$
\begin{aligned}
&\left(\left(-y_1'(x)\,x + y_1''(x)\,x + y_1(x)\right)\ln(x) + \left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right)x - y_1(x)\right)C \\
&\quad + \left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x \\
&\quad - x\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned}
\tag{7}
$$

But since $y_1(x)$ is a solution to the ode, then

$$-y_1'(x)\,x + y_1''(x)\,x + y_1(x) = 0$$

Eq (7) simplifes to

$$\begin{aligned}&\left(\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right)x - y_1(x)\right)C\\&+\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x\\&-x\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)+\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0\end{aligned}\tag{8}$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\begin{aligned}&\frac{\left(2\left(\sum_{n=0}^{\infty}x^{-1+n+r_1}a_n(n+r_1)\right)x-(x+1)\left(\sum_{n=0}^{\infty}a_n x^{n+r_1}\right)\right)C}{x}\\&+\frac{-\left(\sum_{n=0}^{\infty}x^{-1+n+r_2}b_n(n+r_2)\right)x^2+\left(\sum_{n=0}^{\infty}x^{-2+n+r_2}b_n(n+r_2)(-1+n+r_2)\right)x^2+\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)x}{x}\\&=0\end{aligned}\tag{9}$$

Since $r_1 = 1$ and $r_2 = 0$ then the above becomes

$$\begin{aligned}&\frac{\left(2\left(\sum_{n=0}^{\infty}x^{n}a_n(n+1)\right)x-(x+1)\left(\sum_{n=0}^{\infty}a_n x^{n+1}\right)\right)C}{x}\\&+\frac{-\left(\sum_{n=0}^{\infty}x^{n-1}b_n n\right)x^2+\left(\sum_{n=0}^{\infty}x^{-2+n}b_n n(n-1)\right)x^2+\left(\sum_{n=0}^{\infty}b_n x^{n}\right)x}{x}=0\end{aligned}\tag{10}$$

Which simplifies to

$$\begin{aligned}&\left(\sum_{n=0}^{\infty}2C\,x^n a_n(n+1)\right)+\sum_{n=0}^{\infty}\left(-C\,x^{n+1}a_n\right)+\sum_{n=0}^{\infty}\left(-Ca_n x^n\right)\\&+\sum_{n=0}^{\infty}\left(-x^n b_n n\right)+\left(\sum_{n=0}^{\infty}n\,x^{n-1}b_n(n-1)\right)+\left(\sum_{n=0}^{\infty}b_n x^n\right)=0\end{aligned}\tag{2A}$$

The next step is to make all powers of x be $n-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-1} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} 2C x^{n} a_{n} (n+1) &= \sum_{n=1}^{\infty} 2C a_{n-1} n x^{n-1} \\
\sum_{n=0}^{\infty} \left(-C x^{n+1} a_{n}\right) &= \sum_{n=2}^{\infty} \left(-C a_{-2+n} x^{n-1}\right) \\
\sum_{n=0}^{\infty} \left(-C a_{n} x^{n}\right) &= \sum_{n=1}^{\infty} \left(-C a_{n-1} x^{n-1}\right) \\
\sum_{n=0}^{\infty} \left(-x^{n} b_{n} n\right) &= \sum_{n=1}^{\infty} \left(-(n-1) b_{n-1} x^{n-1}\right) \\
\sum_{n=0}^{\infty} b_{n} x^{n} &= \sum_{n=1}^{\infty} b_{n-1} x^{n-1}
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-1$.

$$
\begin{aligned}
&\left(\sum_{n=1}^{\infty} 2C a_{n-1} n x^{n-1}\right) + \sum_{n=2}^{\infty} \left(-C a_{-2+n} x^{n-1}\right) + \sum_{n=1}^{\infty} \left(-C a_{n-1} x^{n-1}\right) \\
&\quad + \sum_{n=1}^{\infty} \left(-(n-1) b_{n-1} x^{n-1}\right) + \left(\sum_{n=0}^{\infty} n x^{n-1} b_{n} (n-1)\right) + \left(\sum_{n=1}^{\infty} b_{n-1} x^{n-1}\right) = 0
\end{aligned} \tag{2B}
$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0 = 1$. For $n = N$, where $N = 1$ which is the difference between the two roots, we are free to choose $b_1 = 0$. Hence for $n = 1$, Eq (2B) gives

$$
C + 1 = 0
$$

Which is solved for C. Solving for C gives

$$
C = -1
$$

For $n = 2$, Eq (2B) gives

$$
(-a_0 + 3a_1) C + 2b_2 = 0
$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$1 + 2b_2 = 0$$

Solving the above for b_2 gives

$$b_2 = -\frac{1}{2}$$

For $n = 3$, Eq (2B) gives

$$(-a_1 + 5a_2)\, C - b_2 + 6b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{1}{2} + 6b_3 = 0$$

Solving the above for b_3 gives

$$b_3 = -\frac{1}{12}$$

For $n = 4$, Eq (2B) gives

$$(-a_2 + 7a_3)\, C - 2b_3 + 12b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{1}{6} + 12b_4 = 0$$

Solving the above for b_4 gives

$$b_4 = -\frac{1}{72}$$

For $n = 5$, Eq (2B) gives

$$(-a_3 + 9a_4)\, C - 3b_4 + 20b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{1}{24} + 20b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = -\frac{1}{480}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -1$ and all b_n, then the second solution becomes

$$y_2(x) = (-1)\left(x\left(1+O\left(x^6\right)\right)\right)\ln\left(x\right)+1-\frac{x^2}{2}-\frac{x^3}{12}-\frac{x^4}{72}-\frac{x^5}{480}+O\left(x^6\right)$$

Therefore the homogeneous solution is

$$\begin{aligned}y_h(x) &= c_1 y_1(x) + c_2 y_2(x)\\ &= c_1 x\left(1+O\left(x^6\right)\right)+c_2\left((-1)\left(x\left(1+O\left(x^6\right)\right)\right)\ln\left(x\right)+1-\frac{x^2}{2}-\frac{x^3}{12}-\frac{x^4}{72}-\frac{x^5}{480}+O\left(x^6\right)\right)\end{aligned}$$

Hence the final solution is

$$\begin{aligned}y &= y_h\\ &= c_1 x\left(1+O\left(x^6\right)\right)+c_2\left(-x\left(1+O\left(x^6\right)\right)\ln\left(x\right)+1-\frac{x^2}{2}-\frac{x^3}{12}-\frac{x^4}{72}-\frac{x^5}{480}+O\left(x^6\right)\right)\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x\left(1+O\left(x^6\right)\right)+c_2\left(-x\left(1+O\left(x^6\right)\right)\ln\left(x\right)+1-\frac{x^2}{2}-\frac{x^3}{12}-\frac{x^4}{72}-\frac{x^5}{480}+O\left(x^6\right)\right) \tag{1}$$

Verification of solutions

$$y = c_1 x\left(1+O\left(x^6\right)\right)+c_2\left(-x\left(1+O\left(x^6\right)\right)\ln\left(x\right)+1-\frac{x^2}{2}-\frac{x^3}{12}-\frac{x^4}{72}-\frac{x^5}{480}+O\left(x^6\right)\right)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 42

```
Order:=6;
dsolve(x*diff(y(x),x$2)-x*diff(y(x),x)+y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \ln(x)\left(-x + \mathrm{O}\left(x^6\right)\right)c_2 + c_1 x\left(1 + \mathrm{O}\left(x^6\right)\right) + \left(1 + x - \frac{1}{2}x^2 - \frac{1}{12}x^3 - \frac{1}{72}x^4 - \frac{1}{480}x^5 + \mathrm{O}\left(x^6\right)\right)c_2$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 41

```
AsymptoticDSolveValue[x*y''[x]-x*y'[x]+y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{1}{72}\left(-x^4 - 6x^3 - 36x^2 + 144x + 72\right) - x\log(x)\right) + c_2 x$$

18.35 problem 29

Internal problem ID [2971]
Internal file name [`OUTPUT/2463_Sunday_June_05_2022_03_14_40_AM_41738625/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.5. page 771
Problem number: 29.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + x(x+4)\,y' + (x+2)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + \left(x^2 + 4x\right) y' + (x+2)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{x+4}{x}$$
$$q(x) = \frac{x+2}{x^2}$$

Table 516: Table $p(x), q(x)$ singularites.

$p(x) = \frac{x+4}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{x+2}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + (x^2 + 4x)\, y' + (x + 2)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2} \right) + (x^2 + 4x) \left(\sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1} \right) + (x + 2) \left(\sum_{n=0}^{\infty} a_n x^{n+r} \right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n\right) + \left(\sum_{n=0}^{\infty} 2a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} a_{n-1}(n+r-1)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n = \sum_{n=1}^{\infty} a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 2a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) + 4x^{n+r} a_n (n+r) + 2a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) + 4x^r a_0 r + 2a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) + 4x^r r + 2x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2+3r+2\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 + 3r + 2 = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= -1 \\ r_2 &= -2 \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(r^2 + 3r + 2\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \frac{\sum_{n=0}^{\infty} a_n x^n}{x} \\ y_2(x) &= C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x^2} \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n-1} \\ y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-2}\right) \end{aligned}$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n + r)(n + r - 1) + a_{n-1}(n + r - 1) + 4a_n(n + r) + a_{n-1} + 2a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}(n+r)}{n^2 + 2nr + r^2 + 3n + 3r + 2} \tag{4}$$

Which for the root $r = -1$ becomes

$$a_n = -\frac{a_{n-1}(n-1)}{n\,(1+n)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-1-r}{r^2 + 5r + 6}$$

Which for the root $r = -1$ becomes

$$a_1 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-r}{r^2+5r+6}$	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1+r}{(r+4)\,(r+3)^2}$$

Which for the root $r = -1$ becomes

$$a_2 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-r}{r^2+5r+6}$	0
a_2	$\frac{1+r}{(r+4)(r+3)^2}$	0

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-1-r}{(r+4)^2 (r+3) (5+r)}$$

Which for the root $r = -1$ becomes

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-r}{r^2+5r+6}$	0
a_2	$\frac{1+r}{(r+4)(r+3)^2}$	0
a_3	$\frac{-1-r}{(r+4)^2(r+3)(5+r)}$	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1+r}{(r+4) (r+3) (5+r)^2 (r+6)}$$

Which for the root $r = -1$ becomes

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-r}{r^2+5r+6}$	0
a_2	$\frac{1+r}{(r+4)(r+3)^2}$	0
a_3	$\frac{-1-r}{(r+4)^2(r+3)(5+r)}$	0
a_4	$\frac{1+r}{(r+4)(r+3)(5+r)^2(r+6)}$	0

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-1 - r}{(r+4)(r+3)(5+r)(r+6)^2(r+7)}$$

Which for the root $r = -1$ becomes

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-r}{r^2+5r+6}$	0
a_2	$\frac{1+r}{(r+4)(r+3)^2}$	0
a_3	$\frac{-1-r}{(r+4)^2(r+3)(5+r)}$	0
a_4	$\frac{1+r}{(r+4)(r+3)(5+r)^2(r+6)}$	0
a_5	$\frac{-1-r}{(r+4)(r+3)(5+r)(r+6)^2(r+7)}$	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= \frac{1}{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= \frac{1 + O(x^6)}{x} \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_1 \\ &= \frac{-1 - r}{r^2 + 5r + 6} \end{aligned}$$

Therefore

$$\lim_{r \to r_2} \frac{-1-r}{r^2+5r+6} = \lim_{r \to -2} \frac{-1-r}{r^2+5r+6}$$
$$= \textit{undefined}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$\begin{aligned}
\frac{d}{dx} y_2(x) &= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\
&= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) \\
\frac{d^2}{dx^2} y_2(x) &= C y_1''(x) \ln(x) + \frac{2 C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\
&= C y_1''(x) \ln(x) + \frac{2 C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right)
\end{aligned}$$

Substituting these back into the given ode $x^2 y'' + (x^2+4x) y' + (x+2) y = 0$ gives

$$\begin{aligned}
& x^2 \Bigg(C y_1''(x) \ln(x) + \frac{2 C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\Bigg) \\
&\quad + (x^2+4x) \left(C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right) \\
&\quad + (x+2) \left(C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)\right) = 0
\end{aligned}$$

Which can be written as

$$\begin{aligned}&\left(\left(\left(x^{2} y_{1}''(x)+\left(x^{2}+4 x\right) y_{1}'(x)+(x+2) y_{1}(x)\right) \ln (x)+x^{2}\left(\frac{2 y_{1}'(x)}{x}-\frac{y_{1}(x)}{x^{2}}\right)\right.\right.\\&\left.+\frac{\left(x^{2}+4 x\right) y_{1}(x)}{x}\right) C+x^{2}\left(\sum_{n=0}^{\infty}\left(\frac{b_{n} x^{n+r_{2}}\left(n+r_{2}\right)^{2}}{x^{2}}-\frac{b_{n} x^{n+r_{2}}\left(n+r_{2}\right)}{x^{2}}\right)\right)\Bigg)\\&+\left(x^{2}+4 x\right)\left(\sum_{n=0}^{\infty} \frac{b_{n} x^{n+r_{2}}\left(n+r_{2}\right)}{x}\right)+(x+2)\left(\sum_{n=0}^{\infty} b_{n} x^{n+r_{2}}\right)=0\end{aligned} \tag{7}$$

But since $y_1(x)$ is a solution to the ode, then

$$x^{2} y_{1}''(x)+\left(x^{2}+4 x\right) y_{1}'(x)+(x+2) y_{1}(x)=0$$

Eq (7) simplifes to

$$\begin{aligned}&\left(x^{2}\left(\frac{2 y_{1}'(x)}{x}-\frac{y_{1}(x)}{x^{2}}\right)+\frac{\left(x^{2}+4 x\right) y_{1}(x)}{x}\right) C\\&+x^{2}\left(\sum_{n=0}^{\infty}\left(\frac{b_{n} x^{n+r_{2}}\left(n+r_{2}\right)^{2}}{x^{2}}-\frac{b_{n} x^{n+r_{2}}\left(n+r_{2}\right)}{x^{2}}\right)\right)\\&+\left(x^{2}+4 x\right)\left(\sum_{n=0}^{\infty} \frac{b_{n} x^{n+r_{2}}\left(n+r_{2}\right)}{x}\right)+(x+2)\left(\sum_{n=0}^{\infty} b_{n} x^{n+r_{2}}\right)=0\end{aligned} \tag{8}$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\begin{aligned}&\left(2\left(\sum_{n=0}^{\infty} x^{-1+n+r_{1}} a_{n}\left(n+r_{1}\right)\right) x+(x+3)\left(\sum_{n=0}^{\infty} a_{n} x^{n+r_{1}}\right)\right) C\\&+\left(\sum_{n=0}^{\infty} x^{-2+n+r_{2}} b_{n}\left(n+r_{2}\right)\left(-1+n+r_{2}\right)\right) x^{2}\\&+\left(x^{2}+4 x\right)\left(\sum_{n=0}^{\infty} x^{-1+n+r_{2}} b_{n}\left(n+r_{2}\right)\right)+(x+2)\left(\sum_{n=0}^{\infty} b_{n} x^{n+r_{2}}\right)=0\end{aligned} \tag{9}$$

Since $r_1 = -1$ and $r_2 = -2$ then the above becomes

$$\begin{aligned}
&\left(2\left(\sum_{n=0}^{\infty} x^{n-2} a_n (n-1)\right) x + (x+3)\left(\sum_{n=0}^{\infty} a_n x^{n-1}\right)\right) C \\
&+ \left(\sum_{n=0}^{\infty} x^{-4+n} b_n (n-2)(-3+n)\right) x^2 \\
&+ \left(x^2+4x\right)\left(\sum_{n=0}^{\infty} x^{-3+n} b_n (n-2)\right) + (x+2)\left(\sum_{n=0}^{\infty} b_n x^{n-2}\right) = 0
\end{aligned} \tag{10}$$

Which simplifies to

$$\begin{aligned}
&\left(\sum_{n=0}^{\infty} 2C\, x^{n-1} a_n (n-1)\right) + \left(\sum_{n=0}^{\infty} C a_n x^n\right) + \left(\sum_{n=0}^{\infty} 3C\, x^{n-1} a_n\right) \\
&+ \left(\sum_{n=0}^{\infty} x^{n-2} b_n \left(n^2-5n+6\right)\right) + \left(\sum_{n=0}^{\infty} x^{n-1} b_n (n-2)\right) \\
&+ \left(\sum_{n=0}^{\infty} 4x^{n-2} b_n (n-2)\right) + \left(\sum_{n=0}^{\infty} x^{n-1} b_n\right) + \left(\sum_{n=0}^{\infty} 2b_n x^{n-2}\right) = 0
\end{aligned} \tag{2A}$$

The next step is to make all powers of x be $n-2$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-2} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2C\, x^{n-1} a_n (n-1) = \sum_{n=1}^{\infty} 2C a_{n-1} (n-2)\, x^{n-2}$$

$$\sum_{n=0}^{\infty} C a_n x^n = \sum_{n=2}^{\infty} C a_{n-2} x^{n-2}$$

$$\sum_{n=0}^{\infty} 3C\, x^{n-1} a_n = \sum_{n=1}^{\infty} 3C a_{n-1} x^{n-2}$$

$$\sum_{n=0}^{\infty} x^{n-1} b_n (n-2) = \sum_{n=1}^{\infty} b_{n-1} (-3+n)\, x^{n-2}$$

$$\sum_{n=0}^{\infty} x^{n-1} b_n = \sum_{n=1}^{\infty} b_{n-1} x^{n-2}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-2$.

$$\left(\sum_{n=1}^{\infty} 2C a_{n-1}(n-2)\, x^{n-2}\right)+\left(\sum_{n=2}^{\infty} C a_{n-2} x^{n-2}\right)+\left(\sum_{n=1}^{\infty} 3C a_{n-1} x^{n-2}\right) + \left(\sum_{n=0}^{\infty} x^{n-2} b_n \left(n^2-5n+6\right)\right)+\left(\sum_{n=1}^{\infty} b_{n-1}(-3+n)\, x^{n-2}\right) + \left(\sum_{n=0}^{\infty} 4x^{n-2} b_n (n-2)\right)+\left(\sum_{n=1}^{\infty} b_{n-1} x^{n-2}\right)+\left(\sum_{n=0}^{\infty} 2b_n x^{n-2}\right)=0 \tag{2B}$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0=1$. For $n=N$, where $N=1$ which is the difference between the two roots, we are free to choose $b_1=0$. Hence for $n=1$, Eq (2B) gives

$$C-1=0$$

Which is solved for C. Solving for C gives

$$C=1$$

For $n=2$, Eq (2B) gives

$$(a_0+3a_1)\, C+2b_2=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$1+2b_2=0$$

Solving the above for b_2 gives

$$b_2=-\frac{1}{2}$$

For $n=3$, Eq (2B) gives

$$(a_1+5a_2)\, C+b_2+6b_3=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-\frac{1}{2}+6b_3=0$$

Solving the above for b_3 gives

$$b_3=\frac{1}{12}$$

For $n=4$, Eq (2B) gives

$$(a_2+7a_3)\, C+2b_3+12b_4=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{1}{6}+12b_4 = 0$$

Solving the above for b_4 gives

$$b_4 = -\frac{1}{72}$$

For $n = 5$, Eq (2B) gives

$$(a_3+9a_4)\, C+3b_4+20b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-\frac{1}{24}+20b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = \frac{1}{480}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = C y_1(x) \ln(x)+\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = 1$ and all b_n, then the second solution becomes

$$y_2(x) = 1\left(\frac{1+O(x^6)}{x}\right) \ln(x)+\frac{1-\frac{x^2}{2}+\frac{x^3}{12}-\frac{x^4}{72}+\frac{x^5}{480}+O(x^6)}{x^2}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x)+c_2 y_2(x)\\
&= \frac{c_1(1+O(x^6))}{x}+c_2\left(1\left(\frac{1+O(x^6)}{x}\right) \ln(x)+\frac{1-\frac{x^2}{2}+\frac{x^3}{12}-\frac{x^4}{72}+\frac{x^5}{480}+O(x^6)}{x^2}\right)
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h\\
&= \frac{c_1(1+O(x^6))}{x}+c_2\left(\frac{(1+O(x^6)) \ln(x)}{x}+\frac{1-\frac{x^2}{2}+\frac{x^3}{12}-\frac{x^4}{72}+\frac{x^5}{480}+O(x^6)}{x^2}\right)
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1(1+O(x^6))}{x} + c_2\left(\frac{(1+O(x^6))\ln(x)}{x} + \frac{1-\frac{x^2}{2}+\frac{x^3}{12}-\frac{x^4}{72}+\frac{x^5}{480}+O(x^6)}{x^2}\right) \quad (1)$$

Verification of solutions

$$y = \frac{c_1(1+O(x^6))}{x} + c_2\left(\frac{(1+O(x^6))\ln(x)}{x} + \frac{1-\frac{x^2}{2}+\frac{x^3}{12}-\frac{x^4}{72}+\frac{x^5}{480}+O(x^6)}{x^2}\right)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.031 (sec). Leaf size: 51

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+x*(4+x)*diff(y(x),x)+(2+x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{\ln(x)\left(x+\mathrm{O}\left(x^6\right)\right)c_2 + c_1x(1+\mathrm{O}\left(x^6\right)) + \left(1-x-\frac{1}{2}x^2+\frac{1}{12}x^3-\frac{1}{72}x^4+\frac{1}{480}x^5+\mathrm{O}\left(x^6\right)\right)c_2}{x^2}$$

✓ Solution by Mathematica

Time used: 0.031 (sec). Leaf size: 45

```
AsymptoticDSolveValue[x^2*y''[x]+x*(4+x)*y'[x]+(2+x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(\frac{\log(x)}{x} - \frac{x^4 - 6x^3 + 36x^2 + 144x - 72}{72x^2} \right) + \frac{c_2}{x}$$

19 Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.6. page 783

19.1 problem 2

Internal problem ID [2972]
Internal file name [OUTPUT/2464_Sunday_June_05_2022_03_14_46_AM_43825527/index.tex]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.6. page 783
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + xy' + \left(x^2 - \frac{9}{4}\right)y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + xy' + \left(x^2 - \frac{9}{4}\right)y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$\begin{aligned}p(x) &= \frac{1}{x}\\ q(x) &= \frac{4x^2 - 9}{4x^2}\end{aligned}$$

Table 517: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{4x^2-9}{4x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + xy' + \left(x^2 - \frac{9}{4}\right) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + x \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \left(x^2 - \frac{9}{4}\right) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)\,(n+r-1)\right)+\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)\right) \tag{2A}$$
$$+\left(\sum_{n=0}^{\infty} x^{n+r+2} a_n\right)+\sum_{n=0}^{\infty}\left(-\frac{9 a_n x^{n+r}}{4}\right)=0$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n=\sum_{n=2}^{\infty} a_{n-2} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)\,(n+r-1)\right)+\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)\right) \tag{2B}$$
$$+\left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r}\right)+\sum_{n=0}^{\infty}\left(-\frac{9 a_n x^{n+r}}{4}\right)=0$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n(n+r)\,(n+r-1)+x^{n+r} a_n(n+r)-\frac{9 a_n x^{n+r}}{4}=0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r)+x^r a_0 r-\frac{9 a_0 x^r}{4}=0$$

Or

$$\left(x^r r(-1+r)+x^r r-\frac{9 x^r}{4}\right) a_0=0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\frac{(4 r^2-9)\, x^r}{4}=0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - \frac{9}{4} = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{3}{2}$$
$$r_2 = -\frac{3}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\frac{(4r^2 - 9)\,x^r}{4} = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 3$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x)\ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x^{\frac{3}{2}}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x)\ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x^{\frac{3}{2}}}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{3}{2}}$$
$$y_2(x) = C y_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-\frac{3}{2}}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1)+a_n(n+r)+a_{n-2}-\frac{9a_n}{4}=0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n=-\frac{4a_{n-2}}{4n^2+8nr+4r^2-9} \tag{4}$$

Which for the root $r=\frac{3}{2}$ becomes

$$a_n=-\frac{a_{n-2}}{n(n+3)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r=\frac{3}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n=2$, using the above recursive equation gives

$$a_2=-\frac{4}{4r^2+16r+7}$$

Which for the root $r=\frac{3}{2}$ becomes

$$a_2=-\frac{1}{10}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{4}{4r^2+16r+7}$	$-\frac{1}{10}$

For $n=3$, using the above recursive equation gives

$$a_3=0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{4}{4r^2+16r+7}$	$-\frac{1}{10}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{16}{(4r^2+16r+7)(4r^2+32r+55)}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_4 = \frac{1}{280}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{4}{4r^2+16r+7}$	$-\frac{1}{10}$
a_3	0	0
a_4	$\frac{16}{(4r^2+16r+7)(4r^2+32r+55)}$	$\frac{1}{280}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{4}{4r^2+16r+7}$	$-\frac{1}{10}$
a_3	0	0
a_4	$\frac{16}{(4r^2+16r+7)(4r^2+32r+55)}$	$\frac{1}{280}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\frac{3}{2}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^{\frac{3}{2}}\left(1 - \frac{x^2}{10} + \frac{x^4}{280} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 3$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_3(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_3 \\ &= 0 \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r \to r_2} 0 &= \lim_{r \to -\frac{3}{2}} 0 \\ &= 0 \end{aligned}$$

The limit is 0. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\ &= \sum_{n=0}^{\infty} b_n x^{n-\frac{3}{2}} \end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq(3) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$b_n(n+r)(n+r-1) + b_n(n+r) + b_{n-2} - \frac{9b_n}{4} = 0 \tag{4}$$

Which for for the root $r = -\frac{3}{2}$ becomes

$$b_n \left(n - \frac{3}{2}\right) \left(n - \frac{5}{2}\right) + b_n \left(n - \frac{3}{2}\right) + b_{n-2} - \frac{9b_n}{4} = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = -\frac{4b_{n-2}}{4n^2 + 8nr + 4r^2 - 9} \tag{5}$$

Which for the root $r = -\frac{3}{2}$ becomes

$$b_n = -\frac{4b_{n-2}}{4n^2 - 12n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\frac{3}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = -\frac{4}{4r^2 + 16r + 7}$$

Which for the root $r = -\frac{3}{2}$ becomes

$$b_2 = \frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{4}{4r^2+16r+7}$	$\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{4}{4r^2+16r+7}$	$\frac{1}{2}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{16}{(4r^2+16r+7)(4r^2+32r+55)}$$

Which for the root $r = -\frac{3}{2}$ becomes

$$b_4 = -\frac{1}{8}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{4}{4r^2+16r+7}$	$\frac{1}{2}$
b_3	0	0
b_4	$\frac{16}{(4r^2+16r+7)(4r^2+32r+55)}$	$-\frac{1}{8}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{4}{4r^2+16r+7}$	$\frac{1}{2}$
b_3	0	0
b_4	$\frac{16}{(4r^2+16r+7)(4r^2+32r+55)}$	$-\frac{1}{8}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= x^{\frac{3}{2}}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\
&= \frac{1 + \frac{x^2}{2} - \frac{x^4}{8} + O(x^6)}{x^{\frac{3}{2}}}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^{\frac{3}{2}}\left(1 - \frac{x^2}{10} + \frac{x^4}{280} + O(x^6)\right) + \frac{c_2\left(1 + \frac{x^2}{2} - \frac{x^4}{8} + O(x^6)\right)}{x^{\frac{3}{2}}}
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^{\frac{3}{2}}\left(1 - \frac{x^2}{10} + \frac{x^4}{280} + O(x^6)\right) + \frac{c_2\left(1 + \frac{x^2}{2} - \frac{x^4}{8} + O(x^6)\right)}{x^{\frac{3}{2}}}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x^{\frac{3}{2}}\left(1 - \frac{x^2}{10} + \frac{x^4}{280} + O(x^6)\right) + \frac{c_2\left(1 + \frac{x^2}{2} - \frac{x^4}{8} + O(x^6)\right)}{x^{\frac{3}{2}}} \tag{1}
$$

Verification of solutions

$$
y = c_1 x^{\frac{3}{2}}\left(1 - \frac{x^2}{10} + \frac{x^4}{280} + O(x^6)\right) + \frac{c_2\left(1 + \frac{x^2}{2} - \frac{x^4}{8} + O(x^6)\right)}{x^{\frac{3}{2}}}
$$

Verified OK.

19.1.1 Maple step by step solution

Let's solve

$x^2 y'' + x y' + \left(x^2 - \frac{9}{4}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{(4x^2-9)y}{4x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{(4x^2-9)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- ○ Define functions

  $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{4x^2-9}{4x^2}\right]$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = 1$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = -\frac{9}{4}$

- ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2y'' + 4xy' + (4x^2 - 9)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- ○ Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- ○ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r}$

- ○ Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(3+2r)(-3+2r) x^r + a_1(5+2r)(-1+2r) x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+3)(2k+2r-3) + 4a$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(3+2r)(-3+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{3}{2}, \frac{3}{2}\right\}$

- Each term must be 0

  $a_1(5+2r)(-1+2r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2 + 8kr + 4r^2 - 9) + 4a_{k-2} = 0$

- Shift index using $k- > k+2$

  $a_{k+2}\left(4(k+2)^2 + 8(k+2) r + 4r^2 - 9\right) + 4a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4a_k}{4k^2+8kr+4r^2+16k+16r+7}$

- Recursion relation for $r = -\frac{3}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+4k-8}$

- Solution for $r = -\frac{3}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{3}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+4k-8}, a_1 = 0\right]$

- Recursion relation for $r = \frac{3}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+28k+40}$

- Solution for $r = \frac{3}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+28k+40}, a_1 = 0\right]$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-\frac{3}{2}}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{2}}\right), a_{2+k}=-\frac{4a_k}{4k^2+4k-8}, a_1=0, b_{2+k}=-\frac{4b_k}{4k^2+28k+40}, b_1=0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 35

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+(x^2-9/4)*y(x)=0,y(x),type='series',x=0);
```

$$y(x)=\frac{c_1 x^3\left(1-\frac{1}{10}x^2+\frac{1}{280}x^4+\mathrm{O}\left(x^6\right)\right)+c_2\left(12+6x^2-\frac{3}{2}x^4+\mathrm{O}\left(x^6\right)\right)}{x^{\frac{3}{2}}}$$

✓ Solution by Mathematica
Time used: 0.011 (sec). Leaf size: 58

```
AsymptoticDSolveValue[x^2*y''[x]+x*y'[x]+(x^2-9/4)*y[x]==0,y[x],{x,0,5}]
```

$$y(x)\to c_1\left(-\frac{x^{5/2}}{8}+\frac{1}{x^{3/2}}+\frac{\sqrt{x}}{2}\right)+c_2\left(\frac{x^{11/2}}{280}-\frac{x^{7/2}}{10}+x^{3/2}\right)$$

19.2 problem 3

Internal problem ID [2973]
Internal file name [`OUTPUT/2465_Sunday_June_05_2022_03_14_49_AM_639990/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Exercises for 11.6. page 783
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

`[_Lienard]`

$$\boxed{xy'' - y' + yx = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' - y' + yx = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{1}{x}$$
$$q(x) = 1$$

Table 519: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = 1$	
singularity	type

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' - y' + yx = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) x - \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) x = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n = \sum_{n=2}^{\infty} a_{n-2} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) - (n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) - r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) - r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}(-2+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(-2+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 2$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\, x^{-1+r}(-2+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 2$ is an integer, then we can construct two linearly independent solutions

$$
\begin{aligned}
y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right) \\
y_2(x) &= C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= x^2 \left(\sum_{n=0}^{\infty} a_n x^n\right) \\
y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+2} \\
y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}
$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$
a_1 = 0
$$

For $2 \leq n$ the recursive equation is

$$
a_n(n+r)(n+r-1) - a_n(n+r) + a_{n-2} = 0 \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$
a_n = -\frac{a_{n-2}}{n^2 + 2nr + r^2 - 2n - 2r} \tag{4}
$$

Which for the root $r = 2$ becomes

$$
a_n = -\frac{a_{n-2}}{n(n+2)} \tag{5}
$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 2$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{1}{r(r+2)}$$

Which for the root $r = 2$ becomes

$$a_2 = -\frac{1}{8}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r(r+2)}$	$-\frac{1}{8}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r(r+2)}$	$-\frac{1}{8}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{r(r+2)^2(r+4)}$$

Which for the root $r = 2$ becomes

$$a_4 = \frac{1}{192}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r(r+2)}$	$-\frac{1}{8}$
a_3	0	0
a_4	$\frac{1}{r(r+2)^2(r+4)}$	$\frac{1}{192}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r(r+2)}$	$-\frac{1}{8}$
a_3	0	0
a_4	$\frac{1}{r(r+2)^2(r+4)}$	$\frac{1}{192}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^2 \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^2 \left(1 - \frac{x^2}{8} + \frac{x^4}{192} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 2$. Now we need to determine if

C is zero or not. This is done by finding $\lim_{r\to r_2} a_2(r)$. If this limit exists, then $C=0$, else we need to keep the log term and $C\neq 0$. The above table shows that

$$
\begin{aligned}
a_N &= a_2 \\
&= -\frac{1}{r\,(r+2)}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\lim_{r\to r_2} -\frac{1}{r\,(r+2)} &= \lim_{r\to 0} -\frac{1}{r\,(r+2)} \\
&= \textit{undefined}
\end{aligned}
$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$
y_2(x) = C y_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)
$$

Therefore

$$
\begin{aligned}
\frac{d}{dx} y_2(x) &= C y_1'(x)\ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\
&= C y_1'(x)\ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right) \\
\frac{d^2}{dx^2} y_2(x) &= C y_1''(x)\ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\
&= C y_1''(x)\ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right)
\end{aligned}
$$

Substituting these back into the given ode $xy'' - y' + yx = 0$ gives

$$
\begin{aligned}
&\left(C y_1''(x)\ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2}\right. \\
&\left. + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) x - C y_1'(x)\ln(x) - \frac{C y_1(x)}{x} \\
&- \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) + \left(C y_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)\right) x = 0
\end{aligned}
$$

Which can be written as

$$\begin{aligned}&\left(\left(y_1(x)\,x+y_1''(x)\,x-y_1'(x)\right)\ln(x)+\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)x-\frac{y_1(x)}{x}\right)C\\&+\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x\\&+\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)x-\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)=0\end{aligned}\tag{7}$$

But since $y_1(x)$ is a solution to the ode, then

$$y_1(x)\,x+y_1''(x)\,x-y_1'(x)=0$$

Eq (7) simplifes to

$$\begin{aligned}&\left(\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)x-\frac{y_1(x)}{x}\right)C\\&+\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x\\&+\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)x-\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)=0\end{aligned}\tag{8}$$

Substituting $y_1=\sum_{n=0}^{\infty}a_n x^{n+r_1}$ into the above gives

$$\begin{aligned}&\frac{\left(2\left(\sum_{n=0}^{\infty}x^{-1+n+r_1}a_n(n+r_1)\right)x-2\left(\sum_{n=0}^{\infty}a_n x^{n+r_1}\right)\right)C}{x}\\&+\frac{\left(\left(\sum_{n=0}^{\infty}x^{-2+n+r_2}b_n(n+r_2)(-1+n+r_2)\right)+\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)\right)x^2-\left(\sum_{n=0}^{\infty}x^{-1+n+r_2}b_n(n+r_2)\right)x}{x}\\&=0\end{aligned}\tag{9}$$

Since $r_1=2$ and $r_2=0$ then the above becomes

$$\begin{aligned}&\frac{\left(2\left(\sum_{n=0}^{\infty}x^{1+n}a_n(n+2)\right)x-2\left(\sum_{n=0}^{\infty}a_n x^{n+2}\right)\right)C}{x}\\&+\frac{\left(\left(\sum_{n=0}^{\infty}x^{n-2}b_n n(-1+n)\right)+\left(\sum_{n=0}^{\infty}b_n x^{n}\right)\right)x^2-\left(\sum_{n=0}^{\infty}x^{-1+n}b_n n\right)x}{x}=0\end{aligned}\tag{10}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2C\, x^{1+n} a_n (n+2)\right) + \sum_{n=0}^{\infty} \left(-2C\, x^{1+n} a_n\right) + \left(\sum_{n=0}^{\infty} n\, x^{-1+n} b_n (-1+n)\right) + \left(\sum_{n=0}^{\infty} x^{1+n} b_n\right) + \sum_{n=0}^{\infty} \left(-x^{-1+n} b_n n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $-1+n$ in each summation term. Going over each summation term above with power of x in it which is not already x^{-1+n} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2C\, x^{1+n} a_n (n+2) = \sum_{n=2}^{\infty} 2C a_{n-2} n\, x^{-1+n}$$

$$\sum_{n=0}^{\infty} \left(-2C\, x^{1+n} a_n\right) = \sum_{n=2}^{\infty} \left(-2C a_{n-2} x^{-1+n}\right)$$

$$\sum_{n=0}^{\infty} x^{1+n} b_n = \sum_{n=2}^{\infty} b_{n-2} x^{-1+n}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $-1+n$.

$$\left(\sum_{n=2}^{\infty} 2C a_{n-2} n\, x^{-1+n}\right) + \sum_{n=2}^{\infty} \left(-2C a_{n-2} x^{-1+n}\right) + \left(\sum_{n=0}^{\infty} n\, x^{-1+n} b_n (-1+n)\right) + \left(\sum_{n=2}^{\infty} b_{n-2} x^{-1+n}\right) + \sum_{n=0}^{\infty} \left(-x^{-1+n} b_n n\right) = 0 \tag{2B}$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0 = 1$. For $n=1$, Eq (2B) gives

$$-b_1 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-b_1 = 0$$

Solving the above for b_1 gives

$$b_1 = 0$$

For $n = N$, where $N = 2$ which is the difference between the two roots, we are free to choose $b_2 = 0$. Hence for $n = 2$, Eq (2B) gives

$$2C + 1 = 0$$

Which is solved for C. Solving for C gives

$$C = -\frac{1}{2}$$

For $n = 3$, Eq (2B) gives

$$4Ca_1 + b_1 + 3b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$3b_3 = 0$$

Solving the above for b_3 gives

$$b_3 = 0$$

For $n = 4$, Eq (2B) gives

$$6Ca_2 + b_2 + 8b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$8b_4 + \frac{3}{8} = 0$$

Solving the above for b_4 gives

$$b_4 = -\frac{3}{64}$$

For $n = 5$, Eq (2B) gives

$$8Ca_3 + b_3 + 15b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$15b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = 0$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = Cy_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -\frac{1}{2}$ and all b_n, then the second solution becomes

$$y_2(x) = -\frac{1}{2}\left(x^2\left(1-\frac{x^2}{8}+\frac{x^4}{192}+O(x^6)\right)\right)\ln(x)+1-\frac{3x^4}{64}+O(x^6)$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1y_1(x)+c_2y_2(x) \\
&= c_1x^2\left(1-\frac{x^2}{8}+\frac{x^4}{192}+O(x^6)\right) \\
&\quad + c_2\left(-\frac{1}{2}\left(x^2\left(1-\frac{x^2}{8}+\frac{x^4}{192}+O(x^6)\right)\right)\ln(x)+1-\frac{3x^4}{64}+O(x^6)\right)
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= c_1x^2\left(1-\frac{x^2}{8}+\frac{x^4}{192}+O(x^6)\right) \\
&\quad + c_2\left(-\frac{x^2\left(1-\frac{x^2}{8}+\frac{x^4}{192}+O(x^6)\right)\ln(x)}{2}+1-\frac{3x^4}{64}+O(x^6)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}
y &= c_1x^2\left(1-\frac{x^2}{8}+\frac{x^4}{192}+O(x^6)\right) \\
&\quad + c_2\left(-\frac{x^2\left(1-\frac{x^2}{8}+\frac{x^4}{192}+O(x^6)\right)\ln(x)}{2}+1-\frac{3x^4}{64}+O(x^6)\right)
\end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned}
y &= c_1x^2\left(1-\frac{x^2}{8}+\frac{x^4}{192}+O(x^6)\right) \\
&\quad + c_2\left(-\frac{x^2\left(1-\frac{x^2}{8}+\frac{x^4}{192}+O(x^6)\right)\ln(x)}{2}+1-\frac{3x^4}{64}+O(x^6)\right)
\end{aligned}$$

Verified OK.

19.2.1 Maple step by step solution

Let's solve

$y''x - y' + yx = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{y'}{x} - y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{y'}{x} + y = 0$

- Simplify ODE

  $x^2 y'' - x y' + x^2 y = 0$

- Make a change of variables

  $y = x u(x)$

- Compute y'

  $y' = u(x) + x u'(x)$

- Compute y''

  $y'' = 2u'(x) + x u''(x)$

- Apply change of variables to the ODE

  $u''(x)\, x^2 + x^2 u(x) + x u'(x) - u(x) = 0$

- ODE is now of the Bessel form
- Solution to Bessel ODE

  $u(x) = c_1 \mathit{BesselJ}(1, x) + c_2 \mathit{BesselY}(1, x)$

- Make the change from y back to y

  $y = (c_1 \mathit{BesselJ}(1, x) + c_2 \mathit{BesselY}(1, x))\, x$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 42

```
Order:=6;
dsolve(x*diff(y(x),x$2)-diff(y(x),x)+x*y(x)=0,y(x),type='series',x=0);
```

$$
y(x) = c_1 x^2 \left(1 - \frac{1}{8}x^2 + \frac{1}{192}x^4 + \mathrm{O}\left(x^6\right)\right) + c_2\left(\ln(x)\left(x^2 - \frac{1}{8}x^4 + \mathrm{O}\left(x^6\right)\right) + \left(-2 + \frac{3}{32}x^4 + \mathrm{O}\left(x^6\right)\right)\right)
$$

✓ Solution by Mathematica

Time used: 0.009 (sec). Leaf size: 59

```
AsymptoticDSolveValue[x*y''[x]-y'[x]+x*y[x]==0,y[x],{x,0,5}]
```

$$
y(x) \to c_1\left(\frac{1}{16}\left(x^2-8\right)x^2\log(x) + \frac{1}{64}\left(-5x^4+16x^2+64\right)\right) + c_2\left(\frac{x^6}{192} - \frac{x^4}{8} + x^2\right)
$$

20 Chapter 11, Series Solutions to Linear Differential Equations. Additional problems. Section 11.7. page 788

20.1 problem 1

Internal problem ID [2974]
Internal file name [`OUTPUT/2466_Sunday_June_05_2022_03_14_53_AM_9917556/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Additional problems. Section 11.7. page 788
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_airy", "second_order_bessel_ode", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{y'' + yx = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using

Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f\Big|_{x_0,y_0,y_0'} + \frac{x^3}{3!} f'\Big|_{x_0,y_0,y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0,y_0,y_0'}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (726) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (727) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') &(4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f &(5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 &(6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -yx \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= -xy' - y \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= -2y' + x^2 y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= x(xy' + 4y) \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= -yx^3 + 6xy' + 4y
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= -y(0) \\
F_2 &= -2y'(0) \\
F_3 &= 0 \\
F_4 &= 4y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 - \frac{1}{6}x^3 + \frac{1}{180}x^6\right) y(0) + \left(x - \frac{1}{12}x^4\right) y'(0) + O(x^6)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard

power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = -\left(\sum_{n=0}^{\infty} a_n x^n\right) x \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=0}^{\infty} x^{1+n} a_n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n$$
$$\sum_{n=0}^{\infty} x^{1+n} a_n = \sum_{n=1}^{\infty} a_{n-1} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^n\right) = 0 \tag{3}$$

For $1 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(1+n) + a_{n-1} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_{n-1}}{(n+2)(1+n)} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 + a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{a_0}{6}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{a_1}{12}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = 0$$

For $n = 4$ the recurrence equation gives

$$30a_6 + a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0}{180}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{a_1}{504}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{1}{6} a_0 x^3 - \frac{1}{12} a_1 x^4 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{x^3}{6}\right) a_0 + \left(x - \frac{1}{12} x^4\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{x^3}{6}\right) c_1 + \left(x - \frac{1}{12} x^4\right) c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = \left(1 - \frac{1}{6} x^3 + \frac{1}{180} x^6\right) y(0) + \left(x - \frac{1}{12} x^4\right) y'(0) + O(x^6) \tag{1}$$

$$y = \left(1 - \frac{x^3}{6}\right) c_1 + \left(x - \frac{1}{12} x^4\right) c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1 - \frac{1}{6} x^3 + \frac{1}{180} x^6\right) y(0) + \left(x - \frac{1}{12} x^4\right) y'(0) + O(x^6)$$

Verified OK.

$$y = \left(1 - \frac{x^3}{6}\right) c_1 + \left(x - \frac{1}{12} x^4\right) c_2 + O(x^6)$$

Verified OK.

20.1.1 Maple step by step solution

Let's solve

$y'' = -yx$

- Highest derivative means the order of the ODE is 2

y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' + yx = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x \cdot y$ to series expansion

  $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

  ◦ Shift index using $k->k-1$

  $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$

  ◦ Convert y'' to series expansion

  $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k->k+2$

  $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

Rewrite ODE with series expansions

$2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_{k-1}\right) x^k\right) = 0$

- Each term must be 0

$2a_2 = 0$

- Each term in the series must be 0, giving the recursion relation

$(k^2 + 3k + 2)\, a_{k+2} + a_{k-1} = 0$

- Shift index using $k->k+1$

$$\left((k+1)^{2}+3k+5\right)a_{k+3}+a_{k}=0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y=\sum_{k=0}^{\infty} a_{k} x^{k}, a_{k+3}=-\frac{a_{k}}{k^{2}+5k+6}, 2a_{2}=0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
Order:=6;
dsolve(diff(y(x),x$2)+x*y(x)=0,y(x),type='series',x=0);
```

$$y(x)=\left(1-\frac{x^{3}}{6}\right)y(0)+\left(x-\frac{1}{12}x^{4}\right)D(y)(0)+O(x^{6})$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 28

```
AsymptoticDSolveValue[y''[x]+x*y[x]==0,y[x],{x,0,5}]
```

$$y(x)\to c_{2}\left(x-\frac{x^{4}}{12}\right)+c_{1}\left(1-\frac{x^{3}}{6}\right)$$

20.2 problem 2

Internal problem ID [2975]
Internal file name [`OUTPUT/2467_Sunday_June_05_2022_03_14_55_AM_24180021/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Additional problems. Section 11.7. page 788
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{y'' - x^2 y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using

Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x-x_0)\,y'(x_0) + \frac{(x-x_0)^2}{2}y''(x_0) + \frac{(x-x_0)^3}{3!}y'''(x_0) + \cdots \\
&= y_0 + xy_0' + \frac{x^2}{2}\,f|_{x_0,y_0,y_0'} + \frac{x^3}{3!}\,f'|_{x_0,y_0,y_0'} + \cdots \\
&= y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}\,\frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y_0'}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (729)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (730)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0 \qquad (5)\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \qquad (7)$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= x^2 y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= x(xy' + 2y) \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= yx^4 + 4xy' + 2y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= y'x^4 + 8yx^3 + 6y' \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= 12y'x^3 + x^2 y\left(x^4 + 30\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= 0 \\
F_2 &= 2y(0) \\
F_3 &= 6y'(0) \\
F_4 &= 0
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 + \frac{x^4}{12}\right) y(0) + \left(x + \frac{1}{20} x^5\right) y'(0) + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard

power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = x^2 \left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \sum_{n=0}^{\infty} \left(-x^{n+2} a_n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$
$$\sum_{n=0}^{\infty} \left(-x^{n+2} a_n\right) = \sum_{n=2}^{\infty} \left(-a_{n-2} x^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \sum_{n=2}^{\infty} \left(-a_{n-2} x^n\right) = 0 \tag{3}$$

For $2 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) - a_{n-2} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{a_{n-2}}{(n+2)(n+1)} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$12a_4 - a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{12}$$

For $n = 3$ the recurrence equation gives

$$20a_5 - a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_1}{20}$$

For $n = 4$ the recurrence equation gives

$$30a_6 - a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = 0$$

For $n = 5$ the recurrence equation gives

$$42a_7 - a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = 0$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \frac{1}{12} a_0 x^4 + \frac{1}{20} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + \frac{x^4}{12}\right) a_0 + \left(x + \frac{1}{20} x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 + \frac{x^4}{12}\right) c_1 + \left(x + \frac{1}{20} x^5\right) c_2 + O(x^6)$$

Summary
The solution(s) found are the following

$$y = \left(1 + \frac{x^4}{12}\right) y(0) + \left(x + \frac{1}{20} x^5\right) y'(0) + O(x^6) \tag{1}$$

$$y = \left(1 + \frac{x^4}{12}\right) c_1 + \left(x + \frac{1}{20} x^5\right) c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1 + \frac{x^4}{12}\right) y(0) + \left(x + \frac{1}{20} x^5\right) y'(0) + O(x^6)$$

Verified OK.

$$y = \left(1 + \frac{x^4}{12}\right) c_1 + \left(x + \frac{1}{20} x^5\right) c_2 + O(x^6)$$

Verified OK.

20.2.1 Maple step by step solution

Let's solve

$y'' = x^2 y$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$y'' - x^2 y = 0$$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^k$$

□ Rewrite ODE with series expansions

- ○ Convert $x^2 \cdot y$ to series expansion

$$x^2 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+2}$$

- ○ Shift index using $k- >k-2$

$$x^2 \cdot y = \sum_{k=2}^{\infty} a_{k-2} x^k$$

- ○ Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- ○ Shift index using $k- >k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$$

Rewrite ODE with series expansions

$$6a_3 x + 2a_2 + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - a_{k-2}\right) x^k\right) = 0$$

- The coefficients of each power of x must be 0

$$[2a_2 = 0, 6a_3 = 0]$$

- Solve for the dependent coefficient(s)

$$\{a_2 = 0, a_3 = 0\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k^2 + 3k + 2)\, a_{k+2} - a_{k-2} = 0$$

- Shift index using $k- >k+2$

$$\left((k+2)^2 + 3k + 8\right) a_{k+4} - a_k = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = \frac{a_k}{k^2+7k+12}, a_2 = 0, a_3 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
Order:=6;
dsolve(diff(y(x),x$2)-x^2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 + \frac{x^4}{12}\right) y(0) + \left(x + \frac{1}{20}x^5\right) D(y)\left(0\right) + O\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 28

```
AsymptoticDSolveValue[y''[x]-x^2*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{x^5}{20} + x\right) + c_1\left(\frac{x^4}{12} + 1\right)$$

20.3 problem 3

Internal problem ID [2976]
Internal file name [`OUTPUT/2468_Sunday_June_05_2022_03_14_57_AM_10648152/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Additional problems. Section 11.7. page 788
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{(-x^2+1)\,y''-6xy'-4y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using

Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0,y_0,y_0'} + \frac{x^3}{3!} f'|_{x_0,y_0,y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0,y_0,y_0'}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (732) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (733) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{2(3xy' + 2y)}{x^2 - 1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{38y'x^2 + 32yx + 10y'}{(x^2 - 1)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{-272y'x^3 - 248x^2y - 208xy' - 72y}{(x^2 - 1)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{(2200x^4 + 3280x^2 + 280)\, y' + (2080x^3 + 1760x)\, y}{(x^2 - 1)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{(-19920x^5 - 48480x^3 - 12240x)\, y' - 19200y\left(x^4 + \frac{33}{20}x^2 + \frac{3}{20}\right)}{(x^2 - 1)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 4y(0) \\
F_1 &= 10y'(0) \\
F_2 &= 72y(0) \\
F_3 &= 280y'(0) \\
F_4 &= 2880y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(4x^6 + 3x^4 + 2x^2 + 1\right) y(0) + \left(x + \frac{5}{3}x^3 + \frac{7}{3}x^5\right) y'(0) + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$\left(-x^2+1\right)y''-6xy'-4y=0$$

Let the solution be represented as power series of the form

$$y=\sum_{n=0}^{\infty} a_n x^n$$

Then

$$y'=\sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y''=\sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\left(-x^2+1\right)\left(\sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}\right)-6x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right)-4\left(\sum_{n=0}^{\infty} a_n x^n\right)=0 \tag{1}$$

Which simplifies to

$$\sum_{n=2}^{\infty}\left(-x^n a_n n(n-1)\right)+\left(\sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}\right)+\sum_{n=1}^{\infty}\left(-6n a_n x^n\right)+\sum_{n=0}^{\infty}\left(-4a_n x^n\right)=0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)\,a_{n+2}(n+1)\,x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\begin{aligned}\sum_{n=2}^{\infty}\left(-x^n a_n n(n-1)\right)+\left(\sum_{n=0}^{\infty}(n+2)\,a_{n+2}(n+1)\,x^n\right)\\+\sum_{n=1}^{\infty}\left(-6n a_n x^n\right)+\sum_{n=0}^{\infty}\left(-4a_n x^n\right)=0\end{aligned} \tag{3}$$

$n = 0$ gives

$$2a_2 - 4a_0 = 0$$

$$a_2 = 2a_0$$

$n = 1$ gives

$$6a_3 - 10a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = \frac{5a_1}{3}$$

For $2 \leq n$, the recurrence equation is

$$-na_n(n-1) + (n+2)\,a_{n+2}(n+1) - 6na_n - 4a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{(n+4)\,a_n}{n+2} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$-18a_2 + 12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = 3a_0$$

For $n = 3$ the recurrence equation gives

$$-28a_3 + 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{7a_1}{3}$$

For $n = 4$ the recurrence equation gives

$$-40a_4 + 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = 4a_0$$

For $n = 5$ the recurrence equation gives

$$-54a_5 + 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = 3a_1$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + 2a_0 x^2 + \frac{5}{3} a_1 x^3 + 3a_0 x^4 + \frac{7}{3} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(3x^4 + 2x^2 + 1\right) a_0 + \left(x + \frac{5}{3}x^3 + \frac{7}{3}x^5\right) a_1 + O\left(x^6\right) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(3x^4 + 2x^2 + 1\right) c_1 + \left(x + \frac{5}{3}x^3 + \frac{7}{3}x^5\right) c_2 + O\left(x^6\right)$$

Summary
The solution(s) found are the following

$$y = \left(4x^6 + 3x^4 + 2x^2 + 1\right) y(0) + \left(x + \frac{5}{3}x^3 + \frac{7}{3}x^5\right) y'(0) + O\left(x^6\right) \tag{1}$$

$$y = \left(3x^4 + 2x^2 + 1\right) c_1 + \left(x + \frac{5}{3}x^3 + \frac{7}{3}x^5\right) c_2 + O\left(x^6\right) \tag{2}$$

Verification of solutions

$$y = (4x^6 + 3x^4 + 2x^2 + 1)\, y(0) + \left(x + \frac{5}{3}x^3 + \frac{7}{3}x^5\right) y'(0) + O(x^6)$$

Verified OK.

$$y = (3x^4 + 2x^2 + 1)\, c_1 + \left(x + \frac{5}{3}x^3 + \frac{7}{3}x^5\right) c_2 + O(x^6)$$

Verified OK.

20.3.1 Maple step by step solution

Let's solve

$(-x^2+1)\, y'' - 6xy' - 4y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{6xy'}{x^2-1} - \frac{4y}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{6xy'}{x^2-1} + \frac{4y}{x^2-1} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{6x}{x^2-1}, P_3(x) = \frac{4}{x^2-1}\right]$

  ◦ $(x+1) \cdot P_2(x)$ is analytic at $x = -1$

  $\left((x+1) \cdot P_2(x)\right)\Big|_{x=-1} = 3$

  ◦ $(x+1)^2 \cdot P_3(x)$ is analytic at $x = -1$

  $\left((x+1)^2 \cdot P_3(x)\right)\Big|_{x=-1} = 0$

  ◦ $x = -1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

$y''(x^2-1)+6xy'+4y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

$(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right)+(6u-6)\left(\frac{d}{du}y(u)\right)+4y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  ◦ Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-2+m}$

  ◦ Shift index using $k->k+2-m$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $-2a_0r(2+r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-2a_{k+1}(k+r+1)(k+3+r)+a_k(k+r+4)(k+r+1)\right)u^{k+r}\right)=$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$-2r(2+r)=0$

- Values of r that satisfy the indicial equation

$r\in\{-2,0\}$

- Each term in the series must be 0, giving the recursion relation

$(k+r+1)\left((-2k-2r-6)\,a_{k+1}+a_k(k+r+4)\right)=0$

- Recursion relation that defines series solution to ODE

$a_{k+1}=\frac{a_k(k+r+4)}{2(k+3+r)}$

- Recursion relation for $r=-2$

$a_{k+1} = \frac{a_k(k+2)}{2(k+1)}$

- Solution for $r = -2$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-2}, a_{k+1} = \frac{a_k(k+2)}{2(k+1)}\right]$$

- Revert the change of variables $u = x + 1$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+1)^{k-2}, a_{k+1} = \frac{a_k(k+2)}{2(k+1)}\right]$$

- Recursion relation for $r = 0$

$a_{k+1} = \frac{a_k(k+4)}{2(k+3)}$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k}, a_{k+1} = \frac{a_k(k+4)}{2(k+3)}\right]$$

- Revert the change of variables $u = x + 1$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+1)^{k}, a_{k+1} = \frac{a_k(k+4)}{2(k+3)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (x+1)^{k-2}\right) + \left(\sum_{k=0}^{\infty} b_k (x+1)^{k}\right), a_{1+k} = \frac{a_k(2+k)}{2(1+k)}, b_{1+k} = \frac{b_k(k+4)}{2(k+3)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
Order:=6;
dsolve((1-x^2)*diff(y(x),x$2)-6*x*diff(y(x),x)-4*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = (3x^4 + 2x^2 + 1)\, y(0) + \left(x + \frac{5}{3}x^3 + \frac{7}{3}x^5\right) D(y)\,(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 38

```
AsymptoticDSolveValue[(1-x^2)*y''[x]-6*x*y'[x]-4*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{7x^5}{3} + \frac{5x^3}{3} + x\right) + c_1(3x^4 + 2x^2 + 1)$$

20.4 problem 4

Internal problem ID [2977]
Internal file name [`OUTPUT/2469_Sunday_June_05_2022_03_14_59_AM_29872371/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Additional problems. Section 11.7. page 788
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{xy'' + y' + 2y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + y' + 2y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = \frac{2}{x}$$

Table 524: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{2}{x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + y' + 2y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) x + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + 2\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} 2 a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2a_n x^{n+r} = \sum_{n=1}^{\infty} 2a_{n-1} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=1}^{\infty} 2a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) + (n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) + r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) + r\,x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^{-1+r} r^2 = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^{-1+r} r^2 = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_n(n+r) + 2a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{2a_{n-1}}{n^2 + 2nr + r^2} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = -\frac{2a_{n-1}}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{2}{(r+1)^2}$$

Which for the root $r = 0$ becomes

$$a_1 = -2$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{(r+1)^2}$	-2

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{4}{(r+1)^2 (r+2)^2}$$

Which for the root $r = 0$ becomes

$$a_2 = 1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{(r+1)^2}$	-2
a_2	$\frac{4}{(r+1)^2(r+2)^2}$	1

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{8}{(r+1)^2 (r+2)^2 (3+r)^2}$$

Which for the root $r = 0$ becomes

$$a_3 = -\frac{2}{9}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{(r+1)^2}$	-2
a_2	$\frac{4}{(r+1)^2(r+2)^2}$	1
a_3	$-\frac{8}{(r+1)^2(r+2)^2(3+r)^2}$	$-\frac{2}{9}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{16}{(r+1)^2 (r+2)^2 (3+r)^2 (r+4)^2}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{1}{36}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{(r+1)^2}$	-2
a_2	$\frac{4}{(r+1)^2(r+2)^2}$	1
a_3	$-\frac{8}{(r+1)^2(r+2)^2(3+r)^2}$	$-\frac{2}{9}$
a_4	$\frac{16}{(r+1)^2(r+2)^2(3+r)^2(r+4)^2}$	$\frac{1}{36}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{32}{(r+1)^2 (r+2)^2 (3+r)^2 (r+4)^2 (5+r)^2}$$

Which for the root $r = 0$ becomes

$$a_5 = -\frac{1}{450}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{(r+1)^2}$	-2
a_2	$\frac{4}{(r+1)^2(r+2)^2}$	1
a_3	$-\frac{8}{(r+1)^2(r+2)^2(3+r)^2}$	$-\frac{2}{9}$
a_4	$\frac{16}{(r+1)^2(r+2)^2(3+r)^2(r+4)^2}$	$\frac{1}{36}$
a_5	$-\frac{32}{(r+1)^2(r+2)^2(3+r)^2(r+4)^2(5+r)^2}$	$-\frac{1}{450}$

Using the above table, then the first solution $y_1(x)$ becomes

$$
\begin{aligned}
y_1(x) &= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots \\
&= x^2 - 2x + 1 - \frac{2x^3}{9} + \frac{x^4}{36} - \frac{x^5}{450} + O(x^6)
\end{aligned}
$$

Now the second solution is found. The second solution is given by

$$
y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)
$$

Where b_n is found using

$$
b_n = \frac{d}{dr} a_{n,r}
$$

And the above is then evaluated at $r = 0$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr}a_{n,r}$	$b_n(r=0)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	$-\frac{2}{(r+1)^2}$	-2	$\frac{4}{(r+1)^3}$	4
b_2	$\frac{4}{(r+1)^2(r+2)^2}$	1	$\frac{-16r-24}{(r+1)^3(r+2)^3}$	-3
b_3	$-\frac{8}{(r+1)^2(r+2)^2(3+r)^2}$	$-\frac{2}{9}$	$\frac{48r^2+192r+176}{(r+1)^3(r+2)^3(3+r)^3}$	$\frac{22}{27}$
b_4	$\frac{16}{(r+1)^2(r+2)^2(3+r)^2(r+4)^2}$	$\frac{1}{36}$	$-\frac{64(2r^3+15r^2+35r+25)}{(r+1)^3(r+2)^3(3+r)^3(r+4)^3}$	$-\frac{25}{216}$
b_5	$-\frac{32}{(r+1)^2(r+2)^2(3+r)^2(r+4)^2(5+r)^2}$	$-\frac{1}{450}$	$\frac{320r^4+3840r^3+16320r^2+28800r+17536}{(r+1)^3(r+2)^3(3+r)^3(r+4)^3(5+r)^3}$	$\frac{137}{13500}$

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x) \ln(x) + b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 \dots \\
&= \left(x^2 - 2x + 1 - \frac{2x^3}{9} + \frac{x^4}{36} - \frac{x^5}{450} + O(x^6)\right) \ln(x) - 3x^2 + 4x + \frac{22x^3}{27} - \frac{25x^4}{216} + \frac{137x^5}{13500} \\
&\quad + O(x^6)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1\left(x^2 - 2x + 1 - \frac{2x^3}{9} + \frac{x^4}{36} - \frac{x^5}{450} + O\left(x^6\right)\right) \\
&\quad + c_2\left(\left(x^2 - 2x + 1 - \frac{2x^3}{9} + \frac{x^4}{36} - \frac{x^5}{450} + O\left(x^6\right)\right)\ln\left(x\right) - 3x^2 + 4x + \frac{22x^3}{27} \right. \\
&\quad \left. - \frac{25x^4}{216} + \frac{137x^5}{13500} + O\left(x^6\right)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1\left(x^2 - 2x + 1 - \frac{2x^3}{9} + \frac{x^4}{36} - \frac{x^5}{450} + O\left(x^6\right)\right) \\
&\quad + c_2\left(\left(x^2 - 2x + 1 - \frac{2x^3}{9} + \frac{x^4}{36} - \frac{x^5}{450} + O\left(x^6\right)\right)\ln\left(x\right) - 3x^2 + 4x + \frac{22x^3}{27} - \frac{25x^4}{216} \right. \\
&\quad \left. + \frac{137x^5}{13500} + O\left(x^6\right)\right)
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1\left(x^2 - 2x + 1 - \frac{2x^3}{9} + \frac{x^4}{36} - \frac{x^5}{450} + O\left(x^6\right)\right) \\
&\quad + c_2\left(\left(x^2 - 2x + 1 - \frac{2x^3}{9} + \frac{x^4}{36} - \frac{x^5}{450} + O\left(x^6\right)\right)\ln\left(x\right) - 3x^2 + 4x + \frac{22x^3}{27} \right. \\
&\quad \left. - \frac{25x^4}{216} + \frac{137x^5}{13500} + O\left(x^6\right)\right)
\end{aligned} \tag{1}
$$

<u>Verification of solutions</u>

$$
\begin{aligned}
y &= c_1\left(x^2 - 2x + 1 - \frac{2x^3}{9} + \frac{x^4}{36} - \frac{x^5}{450} + O\left(x^6\right)\right) \\
&\quad + c_2\left(\left(x^2 - 2x + 1 - \frac{2x^3}{9} + \frac{x^4}{36} - \frac{x^5}{450} + O\left(x^6\right)\right)\ln\left(x\right) - 3x^2 + 4x + \frac{22x^3}{27} - \frac{25x^4}{216} \right. \\
&\quad \left. + \frac{137x^5}{13500} + O\left(x^6\right)\right)
\end{aligned}
$$

Verified OK.

20.4.1 Maple step by step solution

Let's solve

$y''x + y' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2y}{x} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{2y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{2}{x}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = 0$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $y''x + y' + 2y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert y' to series expansion

    $y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1}$

  - Shift index using $k- >k + 1$

$$y' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

$$x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k->k+1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r^2 x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)^2 + 2a_k\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = 0$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k+1}(k+1)^2 + 2a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{2a_k}{(k+1)^2}$$

- Recursion relation for $r = 0$

$$a_{k+1} = -\frac{2a_k}{(k+1)^2}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{2a_k}{(k+1)^2}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 59

```
Order:=6;
dsolve(x*diff(y(x),x$2)+diff(y(x),x)+2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = (c_2 \ln(x) + c_1)\left(1 - 2x + x^2 - \frac{2}{9}x^3 + \frac{1}{36}x^4 - \frac{1}{450}x^5 + \mathrm{O}\left(x^6\right)\right) + \left(4x - 3x^2 + \frac{22}{27}x^3 - \frac{25}{216}x^4 + \frac{137}{13500}x^5 + \mathrm{O}\left(x^6\right)\right)c_2$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 101

```
AsymptoticDSolveValue[x*y''[x]+y'[x]+2*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(-\frac{x^5}{450} + \frac{x^4}{36} - \frac{2x^3}{9} + x^2 - 2x + 1\right) + c_2\left(\frac{137x^5}{13500} - \frac{25x^4}{216} + \frac{22x^3}{27} - 3x^2 + \left(-\frac{x^5}{450} + \frac{x^4}{36} - \frac{2x^3}{9} + x^2 - 2x + 1\right)\log(x) + 4x\right)$$

20.5 problem 5

Internal problem ID [2978]
Internal file name [`OUTPUT/2470_Sunday_June_05_2022_03_15_02_AM_33677441/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Additional problems. Section 11.7. page 788
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

`[_Lienard]`

$$\boxed{xy'' + 2y' + yx = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + 2y' + yx = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{2}{x}$$
$$q(x) = 1$$

Table 526: Table $p(x), q(x)$ singularites.

$p(x) = \frac{2}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = 1$	
singularity	type

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + 2y' + yx = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) x + 2\left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) x = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n = \sum_{n=2}^{\infty} a_{n-2} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n(n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2(n+r)\, a_n x^{n+r-1}\right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n(n+r)(n+r-1) + 2(n+r)\, a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) + 2r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) + 2r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}(1+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(1+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = -1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\, x^{-1+r}(1+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^n$$
$$y_2(x) = C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^n$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-1} \right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n + r)(n + r - 1) + 2a_n(n + r) + a_{n-2} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-2}}{n^2 + 2nr + r^2 + n + r} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = -\frac{a_{n-2}}{n(1 + n)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{1}{r^2 + 5r + 6}$$

Which for the root $r = 0$ becomes

$$a_2 = -\frac{1}{6}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+5r+6}$	$-\frac{1}{6}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+5r+6}$	$-\frac{1}{6}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{r^4 + 14r^3 + 71r^2 + 154r + 120}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{1}{120}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+5r+6}$	$-\frac{1}{6}$
a_3	0	0
a_4	$\frac{1}{r^4+14r^3+71r^2+154r+120}$	$\frac{1}{120}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+5r+6}$	$-\frac{1}{6}$
a_3	0	0
a_4	$\frac{1}{r^4+14r^3+71r^2+154r+120}$	$\frac{1}{120}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots \\ &= 1 - \frac{x^2}{6} + \frac{x^4}{120} + O(x^6) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if

C is zero or not. This is done by finding $\lim_{r\to r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$
\begin{aligned}
a_N &= a_1 \\
&= 0
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\lim_{r\to r_2} 0 &= \lim_{r\to -1} 0 \\
&= 0
\end{aligned}
$$

The limit is 0. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$
\begin{aligned}
y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\
&= \sum_{n=0}^{\infty} b_n x^{n-1}
\end{aligned}
$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq(3) gives

$$
b_1 = 0
$$

For $2 \leq n$ the recursive equation is

$$
b_n(n+r)(n+r-1) + 2(n+r)b_n + b_{n-2} = 0 \tag{4}
$$

Which for for the root $r = -1$ becomes

$$
b_n(n-1)(n-2) + 2(n-1)b_n + b_{n-2} = 0 \tag{4A}
$$

Solving for b_n from the recursive equation (4) gives

$$
b_n = -\frac{b_{n-2}}{n^2 + 2nr + r^2 + n + r} \tag{5}
$$

Which for the root $r = -1$ becomes

$$
b_n = -\frac{b_{n-2}}{n^2 - n} \tag{6}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = -\frac{1}{r^2 + 5r + 6}$$

Which for the root $r = -1$ becomes

$$b_2 = -\frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{r^2+5r+6}$	$-\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{r^2+5r+6}$	$-\frac{1}{2}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{(r^2 + 5r + 6)(r^2 + 9r + 20)}$$

Which for the root $r = -1$ becomes

$$b_4 = \frac{1}{24}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{r^2+5r+6}$	$-\frac{1}{2}$
b_3	0	0
b_4	$\frac{1}{r^4+14r^3+71r^2+154r+120}$	$\frac{1}{24}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{r^2+5r+6}$	$-\frac{1}{2}$
b_3	0	0
b_4	$\frac{1}{r^4+14r^3+71r^2+154r+120}$	$\frac{1}{24}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= 1\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= \frac{1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6)}{x} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 \left(1 - \frac{x^2}{6} + \frac{x^4}{120} + O(x^6)\right) + \frac{c_2 \left(1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6)\right)}{x} \end{aligned}$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1\left(1-\frac{x^2}{6}+\frac{x^4}{120}+O(x^6)\right)+\frac{c_2\left(1-\frac{x^2}{2}+\frac{x^4}{24}+O(x^6)\right)}{x}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1\left(1-\frac{x^2}{6}+\frac{x^4}{120}+O(x^6)\right)+\frac{c_2\left(1-\frac{x^2}{2}+\frac{x^4}{24}+O(x^6)\right)}{x} \tag{1}
$$

Verification of solutions

$$
y = c_1\left(1-\frac{x^2}{6}+\frac{x^4}{120}+O(x^6)\right)+\frac{c_2\left(1-\frac{x^2}{2}+\frac{x^4}{24}+O(x^6)\right)}{x}
$$

Verified OK.

20.5.1 Maple step by step solution

Let's solve

$y''x+2y'+yx=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{2y'}{x}-y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{2y'}{x}+y=0$

□ Check to see if $x_0=0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{2}{x}, P_3(x)=1\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

    $(x\cdot P_2(x))\Big|_{x=0}=2$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y''x + 2y' + yx = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions

  - Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$

  - Shift index using $k- >k-1$

    $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$

  - Convert y' to series expansion

    $y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$

  - Shift index using $k- >k+1$

    $y' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\, x^{k+r}$

  - Convert $x \cdot y''$ to series expansion

    $x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$

  - Shift index using $k- >k+1$

    $x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $$a_0 r(1+r)\, x^{-1+r} + a_1(1+r)\,(2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k+2+r) + a_{k-1}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(1+r)=0$

- Values of r that satisfy the indicial equation

$r \in \{-1, 0\}$

- Each term must be 0

$a_1(1+r)(2+r)=0$

- Each term in the series must be 0, giving the recursion relation

$a_{k+1}(k+r+1)(k+2+r)+a_{k-1}=0$

- Shift index using $k->k+1$

$a_{k+2}(k+2+r)(k+3+r)+a_k=0$

- Recursion relation that defines series solution to ODE

$a_{k+2}=-\frac{a_k}{(k+2+r)(k+3+r)}$

- Recursion relation for $r=-1$

$a_{k+2}=-\frac{a_k}{(k+1)(k+2)}$

- Solution for $r=-1$

$\left[y=\sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2}=-\frac{a_k}{(k+1)(k+2)}, 0=0\right]$

- Recursion relation for $r=0$

$a_{k+2}=-\frac{a_k}{(k+2)(k+3)}$

- Solution for $r=0$

$\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+2}=-\frac{a_k}{(k+2)(k+3)}, 2a_1=0\right]$

- Combine solutions and rename parameters

$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-1}\right)+\left(\sum_{k=0}^{\infty} b_k x^k\right), a_{2+k}=-\frac{a_k}{(1+k)(2+k)}, 0=0, b_{2+k}=-\frac{b_k}{(2+k)(k+3)}, 2b_1=0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 32

```
Order:=6;
dsolve(x*diff(y(x),x$2)+2*diff(y(x),x)+x*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1\left(1 - \frac{1}{6}x^2 + \frac{1}{120}x^4 + O\left(x^6\right)\right) + \frac{c_2\left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + O\left(x^6\right)\right)}{x}$$

✓ Solution by Mathematica
Time used: 0.01 (sec). Leaf size: 42

```
AsymptoticDSolveValue[x*y''[x]+2*y'[x]+x*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{x^3}{24} - \frac{x}{2} + \frac{1}{x}\right) + c_2\left(\frac{x^4}{120} - \frac{x^2}{6} + 1\right)$$

20.6 problem 6

Internal problem ID [2979]
Internal file name [`OUTPUT/2471_Sunday_June_05_2022_03_15_05_AM_21211276/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Additional problems. Section 11.7. page 788
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2xy'' + 5(1-2x)\,y' - 5y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2xy'' + (-10x+5)\,y' - 5y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$\begin{aligned} p(x) &= -\frac{5(2x-1)}{2x} \\ q(x) &= -\frac{5}{2x} \end{aligned}$$

Table 528: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{5(2x-1)}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{5}{2x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$
2xy'' + (-10x + 5) y' - 5y = 0
$$

Let the solution be represented as Frobenius power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^{n+r}
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} \\
y'' &= \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\begin{aligned}
&2\left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) x \\
&+ (-10x+5)\left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) - 5\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0
\end{aligned} \tag{1}
$$

Which simplifies to

$$
\left(\sum_{n=0}^{\infty} 2x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-10x^{n+r} a_n (n+r)\right) \\
+ \left(\sum_{n=0}^{\infty} 5(n+r) a_n x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-5a_n x^{n+r}\right) = 0 \tag{2A}
$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$
\sum_{n=0}^{\infty} \left(-10x^{n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-10a_{n-1}(n+r-1) x^{n+r-1}\right)
$$

$$
\sum_{n=0}^{\infty} \left(-5a_n x^{n+r}\right) = \sum_{n=1}^{\infty} \left(-5a_{n-1} x^{n+r-1}\right)
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$
\left(\sum_{n=0}^{\infty} 2x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-10a_{n-1}(n+r-1) x^{n+r-1}\right) \\
+ \left(\sum_{n=0}^{\infty} 5(n+r) a_n x^{n+r-1}\right) + \sum_{n=1}^{\infty} \left(-5a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
2x^{n+r-1} a_n (n+r)(n+r-1) + 5(n+r) a_n x^{n+r-1} = 0
$$

When $n=0$ the above becomes

$$
2x^{-1+r} a_0 r(-1+r) + 5r a_0 x^{-1+r} = 0
$$

Or

$$
\left(2x^{-1+r} r(-1+r) + 5r\, x^{-1+r}\right) a_0 = 0
$$

Since $a_0 \neq 0$ then the above simplifies to

$$
r\, x^{-1+r}(3+2r) = 0
$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 + 3r = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = -\frac{3}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\,x^{-1+r}(3 + 2r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{3}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^n$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-\frac{3}{2}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$2a_n(n + r)(n + r - 1) - 10a_{n-1}(n + r - 1) + 5a_n(n + r) - 5a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{5a_{n-1}(2n + 2r - 1)}{2n^2 + 4nr + 2r^2 + 3n + 3r} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = \frac{(10n - 5)\,a_{n-1}}{n\,(2n + 3)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{10r + 5}{2r^2 + 7r + 5}$$

Which for the root $r = 0$ becomes

$$a_1 = 1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{10r+5}{2r^2+7r+5}$	1

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{100r^2 + 200r + 75}{(2r^2 + 7r + 5)(2r^2 + 11r + 14)}$$

Which for the root $r = 0$ becomes

$$a_2 = \frac{15}{14}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{10r+5}{2r^2+7r+5}$	1
a_2	$\frac{100r^2+200r+75}{(2r^2+7r+5)(2r^2+11r+14)}$	$\frac{15}{14}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{500r^2 + 1000r + 375}{4r^5 + 56r^4 + 299r^3 + 754r^2 + 885r + 378}$$

Which for the root $r = 0$ becomes

$$a_3 = \frac{125}{126}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{10r+5}{2r^2+7r+5}$	1
a_2	$\frac{100r^2+200r+75}{(2r^2+7r+5)(2r^2+11r+14)}$	$\frac{15}{14}$
a_3	$\frac{500r^2+1000r+375}{4r^5+56r^4+299r^3+754r^2+885r+378}$	$\frac{125}{126}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{2500r^2 + 5000r + 1875}{4r^6 + 80r^5 + 639r^4 + 2590r^3 + 5561r^2 + 5910r + 2376}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{625}{792}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{10r+5}{2r^2+7r+5}$	1
a_2	$\frac{100r^2+200r+75}{(2r^2+7r+5)(2r^2+11r+14)}$	$\frac{15}{14}$
a_3	$\frac{500r^2+1000r+375}{4r^5+56r^4+299r^3+754r^2+885r+378}$	$\frac{125}{126}$
a_4	$\frac{2500r^2+5000r+1875}{4r^6+80r^5+639r^4+2590r^3+5561r^2+5910r+2376}$	$\frac{625}{792}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{12500r^2 + 25000r + 9375}{4r^7 + 108r^6 + 1203r^5 + 7125r^4 + 24051r^3 + 45807r^2 + 44942r + 17160}$$

Which for the root $r = 0$ becomes

$$a_5 = \frac{625}{1144}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{10r+5}{2r^2+7r+5}$	1
a_2	$\frac{100r^2+200r+75}{(2r^2+7r+5)(2r^2+11r+14)}$	$\frac{15}{14}$
a_3	$\frac{500r^2+1000r+375}{4r^5+56r^4+299r^3+754r^2+885r+378}$	$\frac{125}{126}$
a_4	$\frac{2500r^2+5000r+1875}{4r^6+80r^5+639r^4+2590r^3+5561r^2+5910r+2376}$	$\frac{625}{792}$
a_5	$\frac{12500r^2+25000r+9375}{4r^7+108r^6+1203r^5+7125r^4+24051r^3+45807r^2+44942r+17160}$	$\frac{625}{1144}$

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots \\
&= 1 + x + \frac{15x^2}{14} + \frac{125x^3}{126} + \frac{625x^4}{792} + \frac{625x^5}{1144} + O(x^6)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$
2b_n(n+r)(n+r-1) - 10b_{n-1}(n+r-1) + 5(n+r)b_n - 5b_{n-1} = 0 \tag{3}
$$

Solving for b_n from recursive equation (4) gives

$$
b_n = \frac{5b_{n-1}(2n+2r-1)}{2n^2+4nr+2r^2+3n+3r} \tag{4}
$$

Which for the root $r = -\frac{3}{2}$ becomes

$$
b_n = \frac{10b_{n-1}(n-2)}{n(2n-3)} \tag{5}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\frac{3}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{10r + 5}{2r^2 + 7r + 5}$$

Which for the root $r = -\frac{3}{2}$ becomes

$$b_1 = 10$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{10r+5}{2r^2+7r+5}$	10

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{100r^2 + 200r + 75}{(2r^2 + 7r + 5)(2r^2 + 11r + 14)}$$

Which for the root $r = -\frac{3}{2}$ becomes

$$b_2 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{10r+5}{2r^2+7r+5}$	10
b_2	$\frac{100r^2+200r+75}{(2r^2+7r+5)(2r^2+11r+14)}$	0

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{500r^2 + 1000r + 375}{4r^5 + 56r^4 + 299r^3 + 754r^2 + 885r + 378}$$

Which for the root $r = -\frac{3}{2}$ becomes

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{10r+5}{2r^2+7r+5}$	10
b_2	$\frac{100r^2+200r+75}{(2r^2+7r+5)(2r^2+11r+14)}$	0
b_3	$\frac{500r^2+1000r+375}{4r^5+56r^4+299r^3+754r^2+885r+378}$	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{2500r^2 + 5000r + 1875}{4r^6 + 80r^5 + 639r^4 + 2590r^3 + 5561r^2 + 5910r + 2376}$$

Which for the root $r = -\frac{3}{2}$ becomes

$$b_4 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{10r+5}{2r^2+7r+5}$	10
b_2	$\frac{100r^2+200r+75}{(2r^2+7r+5)(2r^2+11r+14)}$	0
b_3	$\frac{500r^2+1000r+375}{4r^5+56r^4+299r^3+754r^2+885r+378}$	0
b_4	$\frac{2500r^2+5000r+1875}{4r^6+80r^5+639r^4+2590r^3+5561r^2+5910r+2376}$	0

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{12500r^2 + 25000r + 9375}{4r^7 + 108r^6 + 1203r^5 + 7125r^4 + 24051r^3 + 45807r^2 + 44942r + 17160}$$

Which for the root $r = -\frac{3}{2}$ becomes

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{10r+5}{2r^2+7r+5}$	10
b_2	$\frac{100r^2+200r+75}{(2r^2+7r+5)(2r^2+11r+14)}$	0
b_3	$\frac{500r^2+1000r+375}{4r^5+56r^4+299r^3+754r^2+885r+378}$	0
b_4	$\frac{2500r^2+5000r+1875}{4r^6+80r^5+639r^4+2590r^3+5561r^2+5910r+2376}$	0
b_5	$\frac{12500r^2+25000r+9375}{4r^7+108r^6+1203r^5+7125r^4+24051r^3+45807r^2+44942r+17160}$	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= 1\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= \frac{1 + 10x + O(x^6)}{x^{\frac{3}{2}}} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 \left(1 + x + \frac{15x^2}{14} + \frac{125x^3}{126} + \frac{625x^4}{792} + \frac{625x^5}{1144} + O(x^6)\right) + \frac{c_2(1 + 10x + O(x^6))}{x^{\frac{3}{2}}} \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 \left(1 + x + \frac{15x^2}{14} + \frac{125x^3}{126} + \frac{625x^4}{792} + \frac{625x^5}{1144} + O(x^6)\right) + \frac{c_2(1 + 10x + O(x^6))}{x^{\frac{3}{2}}} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \left(1 + x + \frac{15x^2}{14} + \frac{125x^3}{126} + \frac{625x^4}{792} + \frac{625x^5}{1144} + O(x^6)\right) + \frac{c_2(1 + 10x + O(x^6))}{x^{\frac{3}{2}}} \tag{1}$$

Verification of solutions

$$y = c_1 \left(1 + x + \frac{15x^2}{14} + \frac{125x^3}{126} + \frac{625x^4}{792} + \frac{625x^5}{1144} + O(x^6)\right) + \frac{c_2(1 + 10x + O(x^6))}{x^{\frac{3}{2}}}$$

Verified OK.

20.6.1 Maple step by step solution

Let's solve

$2y''x + (-10x + 5)\, y' - 5y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{5y}{2x} + \frac{5(2x-1)y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{5(2x-1)y'}{2x} - \frac{5y}{2x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -\frac{5(2x-1)}{2x}, P_3(x) = -\frac{5}{2x}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = \frac{5}{2}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2y''x + (-10x + 5)\, y' - 5y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^{m}\cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x\cdot y''$ to series expansion

  $x\cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$

- Shift index using $k->k+1$

  $x\cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(3+2r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(2k+5+2r) - 5a_k(2k+2r+1)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(3+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{0, -\frac{3}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $2\left(k+\frac{5}{2}+r\right)(k+1+r)\, a_{k+1} - 10a_k\left(k+r+\frac{1}{2}\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{5a_k(2k+2r+1)}{(2k+5+2r)(k+1+r)}$

- Recursion relation for $r=0$

  $a_{k+1} = \frac{5a_k(2k+1)}{(2k+5)(k+1)}$

- Solution for $r=0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k}, a_{k+1} = \frac{5a_k(2k+1)}{(2k+5)(k+1)}\right]$

- Recursion relation for $r=-\frac{3}{2}$; series terminates at $k=1$

  $a_{k+1} = \frac{5a_k(2k-2)}{(2k+2)\left(k-\frac{1}{2}\right)}$

- Apply recursion relation for $k=0$

  $a_1 = 10a_0$

- Terminating series solution of the ODE for $r=-\frac{3}{2}$. Use reduction of order to find the second

  $y = a_0\cdot(1+10x)$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^k\right)+b_0\cdot(1+10x)\,,a_{1+k}=\frac{5a_k(2k+1)}{(2k+5)(1+k)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 36

```
Order:=6;
dsolve(2*x*diff(y(x),x$2)+5*(1-2*x)*diff(y(x),x)-5*y(x)=0,y(x),type='series',x=0);
```

$$y(x)=\frac{c_1(1+10x+\mathrm{O}(x^6))}{x^{\frac{3}{2}}}+c_2\left(1+x+\frac{15}{14}x^2+\frac{125}{126}x^3+\frac{625}{792}x^4+\frac{625}{1144}x^5+\mathrm{O}(x^6)\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 51

```
AsymptoticDSolveValue[2*x*y''[x]+5*(1-2*x)*y'[x]-5*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to \frac{c_2(10x+1)}{x^{3/2}} + c_1\left(\frac{625x^5}{1144} + \frac{625x^4}{792} + \frac{125x^3}{126} + \frac{15x^2}{14} + x + 1\right)$$

20.7 problem 7

Internal problem ID [2980]
Internal file name [`OUTPUT/2472_Sunday_June_05_2022_03_15_09_AM_84619757/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Additional problems. Section 11.7. page 788
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

`[_Lienard]`

$$\boxed{xy'' + y' + yx = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + y' + yx = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = 1$$

Table 530: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = 1$	
singularity	type

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + y' + yx = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) x + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) x = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n = \sum_{n=2}^{\infty} a_{n-2} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) + (n+r)\, a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) + r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) + r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^{-1+r} r^2 = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^{-1+r} r^2 = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_n(n+r) + a_{n-2} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-2}}{n^2 + 2nr + r^2} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = -\frac{a_{n-2}}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{1}{(r+2)^2}$$

Which for the root $r = 0$ becomes

$$a_2 = -\frac{1}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{(r+2)^2}$	$-\frac{1}{4}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{(r+2)^2}$	$-\frac{1}{4}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(r+2)^2 (r+4)^2}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{1}{64}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{(r+2)^2}$	$-\frac{1}{4}$
a_3	0	0
a_4	$\frac{1}{(r+2)^2(r+4)^2}$	$\frac{1}{64}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{(r+2)^2}$	$-\frac{1}{4}$
a_3	0	0
a_4	$\frac{1}{(r+2)^2(r+4)^2}$	$\frac{1}{64}$
a_5	0	0

Using the above table, then the first solution $y_1(x)$ becomes

$$\begin{aligned} y_1(x) &= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots \\ &= 1 - \frac{x^2}{4} + \frac{x^4}{64} + O(x^6) \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = 0$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr}a_{n,r}$	$b_n(r=0)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	0	0	0	0
b_2	$-\frac{1}{(r+2)^2}$	$-\frac{1}{4}$	$\frac{2}{(r+2)^3}$	$\frac{1}{4}$
b_3	0	0	0	0
b_4	$\frac{1}{(r+2)^2(r+4)^2}$	$\frac{1}{64}$	$\frac{-4r-12}{(r+2)^3(r+4)^3}$	$-\frac{3}{128}$
b_5	0	0	0	0

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6\dots \\
&= \left(1 - \frac{x^2}{4} + \frac{x^4}{64} + O(x^6)\right)\ln(x) + \frac{x^2}{4} - \frac{3x^4}{128} + O(x^6)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1y_1(x) + c_2y_2(x) \\
&= c_1\left(1-\frac{x^2}{4}+\frac{x^4}{64}+O(x^6)\right)+c_2\left(\left(1-\frac{x^2}{4}+\frac{x^4}{64}+O(x^6)\right)\ln(x)+\frac{x^2}{4}-\frac{3x^4}{128}+O(x^6)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1\left(1-\frac{x^2}{4}+\frac{x^4}{64}+O(x^6)\right)+c_2\left(\left(1-\frac{x^2}{4}+\frac{x^4}{64}+O(x^6)\right)\ln(x)+\frac{x^2}{4}-\frac{3x^4}{128}+O(x^6)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = c_1&\left(1 - \frac{x^2}{4} + \frac{x^4}{64} + O(x^6)\right) \\
&+ c_2\left(\left(1 - \frac{x^2}{4} + \frac{x^4}{64} + O(x^6)\right)\ln(x) + \frac{x^2}{4} - \frac{3x^4}{128} + O(x^6)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$y = c_1\left(1-\frac{x^2}{4}+\frac{x^4}{64}+O\left(x^6\right)\right)+c_2\left(\left(1-\frac{x^2}{4}+\frac{x^4}{64}+O\left(x^6\right)\right)\ln\left(x\right)+\frac{x^2}{4}-\frac{3x^4}{128}+O\left(x^6\right)\right)$$

Verified OK.

20.7.1 Maple step by step solution

Let's solve

$y''x + y' + yx = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y'}{x} - y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + y = 0$

- □ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = 1\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y''x + y' + yx = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$
  - Shift index using $k->k-1$

    $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$
  - Convert y' to series expansion

    $y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$
  - Shift index using $k->k+1$

    $y' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\, x^{k+r}$
  - Convert $x \cdot y''$ to series expansion

    $x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$
  - Shift index using $k->k+1$

    $x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r^2 x^{-1+r} + a_1 (1+r)^2\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)^2 + a_{k-1}\right) x^{k+r}\right) = 0$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r^2 = 0$
- Values of r that satisfy the indicial equation

  $r = 0$
- Each term must be 0

  $a_1 (1+r)^2 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1)^2 + a_{k-1} = 0$
- Shift index using $k->k+1$

  $a_{k+2}(k+2)^2 + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{(k+2)^2}$

- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{a_k}{(k+2)^2}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k}{(k+2)^2}, a_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 41

```
Order:=6;
dsolve(x*diff(y(x),x$2)+diff(y(x),x)+x*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = (c_2 \ln(x) + c_1)\left(1 - \frac{1}{4}x^2 + \frac{1}{64}x^4 + \mathrm{O}\left(x^6\right)\right) + \left(\frac{1}{4}x^2 - \frac{3}{128}x^4 + \mathrm{O}\left(x^6\right)\right)c_2$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 60

```
AsymptoticDSolveValue[x*y''[x]+y'[x]+x*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{x^4}{64}-\frac{x^2}{4}+1\right)+c_2\left(-\frac{3x^4}{128}+\frac{x^2}{4}+\left(\frac{x^4}{64}-\frac{x^2}{4}+1\right)\log(x)\right)$$

20.8 problem 8

Internal problem ID [2981]
Internal file name [`OUTPUT/2473_Sunday_June_05_2022_03_15_13_AM_48290938/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Additional problems. Section 11.7. page 788
Problem number: 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{(4x^2+1)\,y''-8y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y_0'$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots\\
&= y_0+xy_0'+\frac{x^2}{2}f|_{x_0,y_0,y_0'}+\frac{x^3}{3!}f'|_{x_0,y_0,y_0'}+\cdots\\
&= y_0+xy_0'+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}\left.\frac{d^nf}{dx^n}\right|_{x_0,y_0,y_0'}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \qquad (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \qquad (739)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (740)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \qquad (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \qquad (3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4) \\
F_1 &= \frac{df}{dx} \\
&= \frac{dF_0}{dx} \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \qquad (5) \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right) \\
&= \frac{d}{dx}(F_1) \\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y'' \\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0 \\
&\vdots \\
F_n &= \frac{d}{dx}(F_{n-1}) \\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y'' \\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= \frac{8y}{4x^2+1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{32y'x^2 - 64yx + 8y'}{(4x^2+1)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= -\frac{128(4y'x^2 - 8yx + y')\, x}{(4x^2+1)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{10240\left(x^2 - \frac{1}{20}\right)\left(\left(x^2 + \frac{1}{4}\right) y' - 2yx\right)}{(4x^2+1)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= -\frac{245760\left(x^2 - \frac{3}{20}\right)\left(\left(x^2 + \frac{1}{4}\right) y' - 2yx\right) x}{(4x^2+1)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 8y(0) \\
F_1 &= 8y'(0) \\
F_2 &= 0 \\
F_3 &= -128y'(0) \\
F_4 &= 0
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(4x^2+1\right) y(0) + \left(x + \frac{4}{3}x^3 - \frac{16}{15}x^5\right) y'(0) + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(4x^2 + 1) y'' - 8y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the above back into the ode gives

$$(4x^2 + 1) \left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) - 8\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} 4x^n a_n n(n-1)\right) + \left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \sum_{n=0}^{\infty} (-8a_n x^n) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} 4x^n a_n n(n-1)\right) + \left(\sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n\right) + \sum_{n=0}^{\infty} (-8a_n x^n) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 - 8a_0 = 0$$

$$a_2 = 4a_0$$

$n = 1$ gives

$$6a_3 - 8a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = \frac{4a_1}{3}$$

For $2 \leq n$, the recurrence equation is

$$4na_n(n-1) + (n+2)\, a_{n+2}(n+1) - 8a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{4(n-2)\, a_n}{n+2} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = 0$$

For $n = 3$ the recurrence equation gives

$$16a_3 + 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{16a_1}{15}$$

For $n = 4$ the recurrence equation gives

$$40a_4 + 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = 0$$

For $n = 5$ the recurrence equation gives

$$72a_5 + 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{64a_1}{35}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + 4a_0 x^2 + \frac{4}{3} a_1 x^3 - \frac{16}{15} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = (4x^2 + 1) a_0 + \left(x + \frac{4}{3}x^3 - \frac{16}{15}x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = (4x^2 + 1) c_1 + \left(x + \frac{4}{3}x^3 - \frac{16}{15}x^5\right) c_2 + O(x^6)$$

<u>Summary</u>
The solution(s) found are the following

$$y = (4x^2 + 1) y(0) + \left(x + \frac{4}{3}x^3 - \frac{16}{15}x^5\right) y'(0) + O(x^6) \tag{1}$$

$$y = (4x^2 + 1) c_1 + \left(x + \frac{4}{3}x^3 - \frac{16}{15}x^5\right) c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$
y = (4x^2+1)\, y(0) + \left(x + \frac{4}{3}x^3 - \frac{16}{15}x^5\right) y'(0) + O(x^6)
$$

Verified OK.

$$
y = (4x^2+1)\, c_1 + \left(x + \frac{4}{3}x^3 - \frac{16}{15}x^5\right) c_2 + O(x^6)
$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 29

```
Order:=6;
dsolve((1+4*x^2)*diff(y(x),x$2)-8*y(x)=0,y(x),type='series',x=0);
```

$$
y(x) = (4x^2+1)\, y(0) + \left(x + \frac{4}{3}x^3 - \frac{16}{15}x^5\right) D(y)(0) + O(x^6)
$$

✓ Solution by Mathematica
Time used: 0.001 (sec). Leaf size: 33

```
AsymptoticDSolveValue[(1+4*x^2)*y''[x]-8*y[x]==0,y[x],{x,0,5}]
```

$$
y(x) \to c_1(4x^2+1) + c_2\left(-\frac{16x^5}{15} + \frac{4x^3}{3} + x\right)
$$

20.9 problem 9

Internal problem ID [2982]
Internal file name [`OUTPUT/2474_Sunday_June_05_2022_03_15_15_AM_1559149/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Additional problems. Section 11.7. page 788
Problem number: 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= \frac{4x^2 - 1}{4x^2} \end{aligned}$$

Table 532: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{4x^2-1}{4x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2 y'' + xy' + \left(x^2 - \frac{1}{4}\right) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}\right) + x \left(\sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}\right) + \left(x^2 - \frac{1}{4}\right) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) \tag{2A}$$
$$+ \left(\sum_{n=0}^{\infty} x^{n+r+2} a_n\right) + \sum_{n=0}^{\infty} \left(-\frac{a_n x^{n+r}}{4}\right) = 0$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n = \sum_{n=2}^{\infty} a_{n-2} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) \tag{2B}$$
$$+ \left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-\frac{a_n x^{n+r}}{4}\right) = 0$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) + x^{n+r} a_n (n+r) - \frac{a_n x^{n+r}}{4} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) + x^r a_0 r - \frac{a_0 x^r}{4} = 0$$

Or

$$\left(x^r r(-1+r) + x^r r - \frac{x^r}{4}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\frac{(4r^2-1)\, x^r}{4} = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - \frac{1}{4} = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{2}$$
$$r_2 = -\frac{1}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\frac{(4r^2 - 1)\, x^r}{4} = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right)$$

Or

$$y_1(x) = \sqrt{x} \left(\sum_{n=0}^{\infty} a_n x^n \right)$$
$$y_2(x) = C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{\sqrt{x}}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-\frac{1}{2}} \right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1)+a_n(n+r)+a_{n-2}-\frac{a_n}{4}=0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n=-\frac{4a_{n-2}}{4n^2+8nr+4r^2-1} \tag{4}$$

Which for the root $r=\frac{1}{2}$ becomes

$$a_n=-\frac{a_{n-2}}{n(n+1)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r=\frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n=2$, using the above recursive equation gives

$$a_2=-\frac{4}{4r^2+16r+15}$$

Which for the root $r=\frac{1}{2}$ becomes

$$a_2=-\frac{1}{6}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{4}{4r^2+16r+15}$	$-\frac{1}{6}$

For $n=3$, using the above recursive equation gives

$$a_3=0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{4}{4r^2+16r+15}$	$-\frac{1}{6}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{16}{(4r^2 + 16r + 15)(4r^2 + 32r + 63)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = \frac{1}{120}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{4}{4r^2+16r+15}$	$-\frac{1}{6}$
a_3	0	0
a_4	$\frac{16}{(4r^2+16r+15)(4r^2+32r+63)}$	$\frac{1}{120}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{4}{4r^2+16r+15}$	$-\frac{1}{6}$
a_3	0	0
a_4	$\frac{16}{(4r^2+16r+15)(4r^2+32r+63)}$	$\frac{1}{120}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}y_1(x) &= \sqrt{x}\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots\right) \\ &= \sqrt{x}\left(1 - \frac{x^2}{6} + \frac{x^4}{120} + O(x^6)\right)\end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned}a_N &= a_1 \\ &= 0\end{aligned}$$

Therefore

$$\begin{aligned}\lim_{r\to r_2} 0 &= \lim_{r\to -\frac{1}{2}} 0 \\ &= 0\end{aligned}$$

The limit is 0. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned}y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\ &= \sum_{n=0}^{\infty} b_n x^{n-\frac{1}{2}}\end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq(3) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$b_n(n + r)(n + r - 1) + b_n(n + r) + b_{n-2} - \frac{b_n}{4} = 0 \tag{4}$$

Which for for the root $r = -\frac{1}{2}$ becomes

$$b_n \left(n-\frac{1}{2}\right) \left(n-\frac{3}{2}\right) + b_n \left(n-\frac{1}{2}\right) + b_{n-2} - \frac{b_n}{4} = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = -\frac{4b_{n-2}}{4n^2 + 8nr + 4r^2 - 1} \tag{5}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_n = -\frac{4b_{n-2}}{4n^2 - 4n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = -\frac{4}{4r^2 + 16r + 15}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_2 = -\frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{4}{4r^2+16r+15}$	$-\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{4}{4r^2+16r+15}$	$-\frac{1}{2}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{16}{(4r^2 + 16r + 15)(4r^2 + 32r + 63)}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_4 = \frac{1}{24}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{4}{4r^2+16r+15}$	$-\frac{1}{2}$
b_3	0	0
b_4	$\frac{16}{(4r^2+16r+15)(4r^2+32r+63)}$	$\frac{1}{24}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{4}{4r^2+16r+15}$	$-\frac{1}{2}$
b_3	0	0
b_4	$\frac{16}{(4r^2+16r+15)(4r^2+32r+63)}$	$\frac{1}{24}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= \sqrt{x}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\
&= \frac{1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6)}{\sqrt{x}}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 \sqrt{x}\left(1 - \frac{x^2}{6} + \frac{x^4}{120} + O(x^6)\right) + \frac{c_2 \left(1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6)\right)}{\sqrt{x}}
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 \sqrt{x}\left(1 - \frac{x^2}{6} + \frac{x^4}{120} + O(x^6)\right) + \frac{c_2 \left(1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6)\right)}{\sqrt{x}}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \sqrt{x}\left(1 - \frac{x^2}{6} + \frac{x^4}{120} + O(x^6)\right) + \frac{c_2 \left(1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6)\right)}{\sqrt{x}} \tag{1}
$$

Verification of solutions

$$
y = c_1 \sqrt{x}\left(1 - \frac{x^2}{6} + \frac{x^4}{120} + O(x^6)\right) + \frac{c_2 \left(1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6)\right)}{\sqrt{x}}
$$

Verified OK.

20.9.1 Maple step by step solution

Let's solve

$x^2 y'' + x y' + \left(x^2 - \frac{1}{4}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(4x^2-1)y}{4x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{(4x^2-1)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{4x^2-1}{4x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = -\frac{1}{4}$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $4x^2y'' + 4xy' + (4x^2 - 1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(1+2r)\,(-1+2r)\,x^r + a_1(3+2r)\,(1+2r)\,x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+1)\,(2k+2r-1) + 4a_{k-}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(1+2r)\,(-1+2r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$$

- Each term must be 0

  $$a_1(3+2r)\,(1+2r) = 0$$

- Solve for the dependent coefficient(s)

  $$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$a_k(4k^2 + 8kr + 4r^2 - 1) + 4a_{k-2} = 0$$

- Shift index using $k- >k+2$

  $$a_{k+2}\left(4(k+2)^2 + 8(k+2)\,r + 4r^2 - 1\right) + 4a_k = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = -\frac{4a_k}{4k^2+8kr+4r^2+16k+16r+15}$$

- Recursion relation for $r = -\frac{1}{2}$

  $$a_{k+2} = -\frac{4a_k}{4k^2+12k+8}$$

- Solution for $r = -\frac{1}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0\right]$$

- Recursion relation for $r = \frac{1}{2}$

  $$a_{k+2} = -\frac{4a_k}{4k^2+20k+24}$$

- Solution for $r = \frac{1}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+20k+24}, a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty}a_k x^{k-\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+\frac{1}{2}}\right), a_{2+k}=-\frac{4a_k}{4k^2+12k+8}, a_1=0, b_{2+k}=-\frac{4b_k}{4k^2+20k+24}, b_1=0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 35

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+(x^2-1/4)*y(x)=0,y(x),type='series',x=0);
```

$$y(x)=\frac{c_1\left(1-\frac{1}{6}x^2+\frac{1}{120}x^4+\mathrm{O}\left(x^6\right)\right)x+c_2\left(1-\frac{1}{2}x^2+\frac{1}{24}x^4+\mathrm{O}\left(x^6\right)\right)}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 58

```
AsymptoticDSolveValue[x^2*y''[x]+x*y'[x]+(x^2-1/4)*y[x]==0,y[x],{x,0,5}]
```

$$y(x)\to c_1\left(\frac{x^{7/2}}{24}-\frac{x^{3/2}}{2}+\frac{1}{\sqrt{x}}\right)+c_2\left(\frac{x^{9/2}}{120}-\frac{x^{5/2}}{6}+\sqrt{x}\right)$$

20.10 problem 10

Internal problem ID [2983]
Internal file name [`OUTPUT/2475_Sunday_June_05_2022_03_15_18_AM_47132089/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Additional problems. Section 11.7. page 788
Problem number: 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{4xy'' + 3y' + 3y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$4xy'' + 3y' + 3y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$\begin{aligned} p(x) &= \frac{3}{4x} \\ q(x) &= \frac{3}{4x} \end{aligned}$$

Table 534: Table $p(x), q(x)$ singularites.

$p(x) = \frac{3}{4x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{3}{4x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$4xy'' + 3y' + 3y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$4\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x + 3\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + 3\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 4x^{n+r-1} a_n (n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 3(n+r)\, a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} 3a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 3a_n x^{n+r} = \sum_{n=1}^{\infty} 3a_{n-1} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} 4x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 3(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=1}^{\infty} 3a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$4x^{n+r-1} a_n (n+r)(n+r-1) + 3(n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$4x^{-1+r} a_0 r(-1+r) + 3r a_0 x^{-1+r} = 0$$

Or

$$\left(4x^{-1+r} r(-1+r) + 3r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}(-1+4r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$4r^2 - r = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{4}$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\, x^{-1+r}(-1+4r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{1}{4}$ is not an integer, then we can construct two linearly independent solutions

$$
\begin{aligned}
y_1(x) &= x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right) \\
y_2(x) &= x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{4}} \\
y_2(x) &= \sum_{n=0}^{\infty} b_n x^n
\end{aligned}
$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$
4a_n(n+r)(n+r-1) + 3a_n(n+r) + 3a_{n-1} = 0 \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$
a_n = -\frac{3a_{n-1}}{4n^2 + 8nr + 4r^2 - n - r} \tag{4}
$$

Which for the root $r = \frac{1}{4}$ becomes

$$
a_n = -\frac{3a_{n-1}}{4n^2 + n} \tag{5}
$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{4}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$
a_1 = -\frac{3}{4r^2 + 7r + 3}
$$

Which for the root $r = \frac{1}{4}$ becomes

$$a_1 = -\frac{3}{5}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3}{4r^2+7r+3}$	$-\frac{3}{5}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{9}{(4r^2 + 7r + 3)(4r^2 + 15r + 14)}$$

Which for the root $r = \frac{1}{4}$ becomes

$$a_2 = \frac{1}{10}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3}{4r^2+7r+3}$	$-\frac{3}{5}$
a_2	$\frac{9}{(4r^2+7r+3)(4r^2+15r+14)}$	$\frac{1}{10}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{27}{(4r^2 + 7r + 3)(4r^2 + 15r + 14)(4r^2 + 23r + 33)}$$

Which for the root $r = \frac{1}{4}$ becomes

$$a_3 = -\frac{1}{130}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3}{4r^2+7r+3}$	$-\frac{3}{5}$
a_2	$\frac{9}{(4r^2+7r+3)(4r^2+15r+14)}$	$\frac{1}{10}$
a_3	$-\frac{27}{(4r^2+7r+3)(4r^2+15r+14)(4r^2+23r+33)}$	$-\frac{1}{130}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{81}{(4r^2 + 7r + 3)(4r^2 + 15r + 14)(4r^2 + 23r + 33)(4r^2 + 31r + 60)}$$

Which for the root $r = \frac{1}{4}$ becomes

$$a_4 = \frac{3}{8840}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3}{4r^2+7r+3}$	$-\frac{3}{5}$
a_2	$\frac{9}{(4r^2+7r+3)(4r^2+15r+14)}$	$\frac{1}{10}$
a_3	$-\frac{27}{(4r^2+7r+3)(4r^2+15r+14)(4r^2+23r+33)}$	$-\frac{1}{130}$
a_4	$\frac{81}{(4r^2+7r+3)(4r^2+15r+14)(4r^2+23r+33)(4r^2+31r+60)}$	$\frac{3}{8840}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{243}{(4r^2 + 7r + 3)(4r^2 + 15r + 14)(4r^2 + 23r + 33)(4r^2 + 31r + 60)(4r^2 + 39r + 95)}$$

Which for the root $r = \frac{1}{4}$ becomes

$$a_5 = -\frac{3}{309400}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3}{4r^2+7r+3}$	$-\frac{3}{5}$
a_2	$\frac{9}{(4r^2+7r+3)(4r^2+15r+14)}$	$\frac{1}{10}$
a_3	$-\frac{27}{(4r^2+7r+3)(4r^2+15r+14)(4r^2+23r+33)}$	$-\frac{1}{130}$
a_4	$\frac{81}{(4r^2+7r+3)(4r^2+15r+14)(4r^2+23r+33)(4r^2+31r+60)}$	$\frac{3}{8840}$
a_5	$-\frac{243}{(4r^2+7r+3)(4r^2+15r+14)(4r^2+23r+33)(4r^2+31r+60)(4r^2+39r+95)}$	$-\frac{3}{309400}$

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x^{\frac{1}{4}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= x^{\frac{1}{4}}\left(1 - \frac{3x}{5} + \frac{x^2}{10} - \frac{x^3}{130} + \frac{3x^4}{8840} - \frac{3x^5}{309400} + O(x^6)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$
4b_n(n+r)(n+r-1) + 3(n+r)\, b_n + 3b_{n-1} = 0 \tag{3}
$$

Solving for b_n from recursive equation (4) gives

$$
b_n = -\frac{3b_{n-1}}{4n^2 + 8nr + 4r^2 - n - r} \tag{4}
$$

Which for the root $r = 0$ becomes

$$
b_n = -\frac{3b_{n-1}}{n(4n-1)} \tag{5}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$
b_1 = -\frac{3}{4r^2 + 7r + 3}
$$

Which for the root $r = 0$ becomes

$$
b_1 = -1
$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{3}{4r^2+7r+3}$	-1

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{9}{(4r^2+7r+3)(4r^2+15r+14)}$$

Which for the root $r = 0$ becomes

$$b_2 = \frac{3}{14}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{3}{4r^2+7r+3}$	-1
b_2	$\frac{9}{(4r^2+7r+3)(4r^2+15r+14)}$	$\frac{3}{14}$

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{27}{(4r^2+7r+3)(4r^2+15r+14)(4r^2+23r+33)}$$

Which for the root $r = 0$ becomes

$$b_3 = -\frac{3}{154}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{3}{4r^2+7r+3}$	-1
b_2	$\frac{9}{(4r^2+7r+3)(4r^2+15r+14)}$	$\frac{3}{14}$
b_3	$-\frac{27}{(4r^2+7r+3)(4r^2+15r+14)(4r^2+23r+33)}$	$-\frac{3}{154}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{81}{(4r^2+7r+3)(4r^2+15r+14)(4r^2+23r+33)(4r^2+31r+60)}$$

Which for the root $r = 0$ becomes

$$b_4 = \frac{3}{3080}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{3}{4r^2+7r+3}$	-1
b_2	$\frac{9}{(4r^2+7r+3)(4r^2+15r+14)}$	$\frac{3}{14}$
b_3	$-\frac{27}{(4r^2+7r+3)(4r^2+15r+14)(4r^2+23r+33)}$	$-\frac{3}{154}$
b_4	$\frac{81}{(4r^2+7r+3)(4r^2+15r+14)(4r^2+23r+33)(4r^2+31r+60)}$	$\frac{3}{3080}$

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{243}{(4r^2+7r+3)(4r^2+15r+14)(4r^2+23r+33)(4r^2+31r+60)(4r^2+39r+95)}$$

Which for the root $r = 0$ becomes

$$b_5 = -\frac{9}{292600}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{3}{4r^2+7r+3}$	-1
b_2	$\frac{9}{(4r^2+7r+3)(4r^2+15r+14)}$	$\frac{3}{14}$
b_3	$-\frac{27}{(4r^2+7r+3)(4r^2+15r+14)(4r^2+23r+33)}$	$-\frac{3}{154}$
b_4	$\frac{81}{(4r^2+7r+3)(4r^2+15r+14)(4r^2+23r+33)(4r^2+31r+60)}$	$\frac{3}{3080}$
b_5	$-\frac{243}{(4r^2+7r+3)(4r^2+15r+14)(4r^2+23r+33)(4r^2+31r+60)(4r^2+39r+95)}$	$-\frac{9}{292600}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\ &= 1 - x + \frac{3x^2}{14} - \frac{3x^3}{154} + \frac{3x^4}{3080} - \frac{9x^5}{292600} + O(x^6) \end{aligned}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^{\frac{1}{4}} \left(1 - \frac{3x}{5} + \frac{x^2}{10} - \frac{x^3}{130} + \frac{3x^4}{8840} - \frac{3x^5}{309400} + O(x^6)\right) \\
&\quad + c_2 \left(1 - x + \frac{3x^2}{14} - \frac{3x^3}{154} + \frac{3x^4}{3080} - \frac{9x^5}{292600} + O(x^6)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^{\frac{1}{4}} \left(1 - \frac{3x}{5} + \frac{x^2}{10} - \frac{x^3}{130} + \frac{3x^4}{8840} - \frac{3x^5}{309400} + O(x^6)\right) \\
&\quad + c_2 \left(1 - x + \frac{3x^2}{14} - \frac{3x^3}{154} + \frac{3x^4}{3080} - \frac{9x^5}{292600} + O(x^6)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
\begin{aligned}
y = {} & c_1 x^{\frac{1}{4}} \left(1 - \frac{3x}{5} + \frac{x^2}{10} - \frac{x^3}{130} + \frac{3x^4}{8840} - \frac{3x^5}{309400} + O(x^6)\right) \\
& + c_2 \left(1 - x + \frac{3x^2}{14} - \frac{3x^3}{154} + \frac{3x^4}{3080} - \frac{9x^5}{292600} + O(x^6)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = {} & c_1 x^{\frac{1}{4}} \left(1 - \frac{3x}{5} + \frac{x^2}{10} - \frac{x^3}{130} + \frac{3x^4}{8840} - \frac{3x^5}{309400} + O(x^6)\right) \\
& + c_2 \left(1 - x + \frac{3x^2}{14} - \frac{3x^3}{154} + \frac{3x^4}{3080} - \frac{9x^5}{292600} + O(x^6)\right)
\end{aligned}
$$

Verified OK.

20.10.1 Maple step by step solution

Let's solve

$4y''x + 3y' + 3y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{3y}{4x} - \frac{3y'}{4x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{3y'}{4x} + \frac{3y}{4x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{3}{4x}, P_3(x) = \frac{3}{4x}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = \frac{3}{4}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4y''x + 3y' + 3y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert y' to series expansion

  $y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$

  ◦ Shift index using $k- >k+1$

$$y' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

$$x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k- >k+1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-1+4r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(4k+3+4r) + 3a_k\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(-1+4r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, \frac{1}{4}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$4(k+1+r)\left(k+\frac{3}{4}+r\right) a_{k+1} + 3a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{3a_k}{(k+1+r)(4k+3+4r)}$$

- Recursion relation for $r=0$

$$a_{k+1} = -\frac{3a_k}{(k+1)(4k+3)}$$

- Solution for $r=0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{3a_k}{(k+1)(4k+3)}\right]$$

- Recursion relation for $r=\frac{1}{4}$

$$a_{k+1} = -\frac{3a_k}{\left(k+\frac{5}{4}\right)(4k+4)}$$

- Solution for $r=\frac{1}{4}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}, a_{k+1} = -\frac{3a_k}{\left(k+\frac{5}{4}\right)(4k+4)}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^k\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{4}}\right), a_{1+k}=-\frac{3a_k}{(1+k)(4k+3)}, b_{1+k}=-\frac{3b_k}{\left(k+\frac{5}{4}\right)(4k+4)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 44

```
Order:=6;
dsolve(4*x*diff(y(x),x$2)+3*diff(y(x),x)+3*y(x)=0,y(x),type='series',x=0);
```

$$y(x)=c_1 x^{\frac{1}{4}}\left(1-\frac{3}{5}x+\frac{1}{10}x^2-\frac{1}{130}x^3+\frac{3}{8840}x^4-\frac{3}{309400}x^5+\mathrm{O}\left(x^6\right)\right)+c_2\left(1-x+\frac{3}{14}x^2-\frac{3}{154}x^3+\frac{3}{3080}x^4-\frac{9}{292600}x^5+\mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 83

```
AsymptoticDSolveValue[4*x*y''[x]+3*y'[x]+3*y[x]==0,y[x],{x,0,5}]
```

$$y(x)\to c_1\sqrt[4]{x}\left(-\frac{3x^5}{309400}+\frac{3x^4}{8840}-\frac{x^3}{130}+\frac{x^2}{10}-\frac{3x}{5}+1\right)+c_2\left(-\frac{9x^5}{292600}+\frac{3x^4}{3080}-\frac{3x^3}{154}+\frac{3x^2}{14}-x+1\right)$$

20.11 problem 11

Internal problem ID [2984]
Internal file name [`OUTPUT/2476_Sunday_June_05_2022_03_15_21_AM_73552895/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Additional problems. Section 11.7. page 788
Problem number: 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y''+\frac{3xy'}{2}-\frac{y(x+1)}{2}=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y''+\frac{3xy'}{2}+\left(-\frac{x}{2}-\frac{1}{2}\right)y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{3}{2x}$$
$$q(x)=-\frac{x+1}{2x^2}$$

Table 536: Table $p(x), q(x)$ singularites.

$p(x) = \frac{3}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{x+1}{2x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2 y'' + \frac{3xy'}{2} + \left(-\frac{x}{2} - \frac{1}{2}\right) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) + \frac{3x \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right)}{2} + \left(-\frac{x}{2} - \frac{1}{2}\right) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty} \frac{3x^{n+r} a_n (n+r)}{2}\right) \\ +\sum_{n=0}^{\infty}\left(-\frac{x^{1+n+r} a_n}{2}\right)+\sum_{n=0}^{\infty}\left(-\frac{a_n x^{n+r}}{2}\right)=0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty}\left(-\frac{x^{1+n+r} a_n}{2}\right)=\sum_{n=1}^{\infty}\left(-\frac{a_{n-1} x^{n+r}}{2}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty} \frac{3x^{n+r} a_n (n+r)}{2}\right) \\ +\sum_{n=1}^{\infty}\left(-\frac{a_{n-1} x^{n+r}}{2}\right)+\sum_{n=0}^{\infty}\left(-\frac{a_n x^{n+r}}{2}\right)=0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1)+\frac{3x^{n+r} a_n (n+r)}{2}-\frac{a_n x^{n+r}}{2}=0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r)+\frac{3x^r a_0 r}{2}-\frac{a_0 x^r}{2}=0$$

Or

$$\left(x^r r(-1+r)+\frac{3x^r r}{2}-\frac{x^r}{2}\right) a_0=0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\frac{(2r^2+r-1)\, x^r}{2}=0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 + \frac{1}{2}r - \frac{1}{2} = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{2}$$
$$r_2 = -1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\frac{(2r^2 + r - 1)\, x^r}{2} = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{3}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-1}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + \frac{3a_n(n+r)}{2} - \frac{a_{n-1}}{2} - \frac{a_n}{2} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}}{2n^2 + 4nr + 2r^2 + n + r - 1} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = \frac{a_{n-1}}{n(2n+3)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{1}{2r^2 + 5r + 2}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_1 = \frac{1}{5}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2r^2+5r+2}$	$\frac{1}{5}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{4r^4 + 28r^3 + 67r^2 + 63r + 18}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = \frac{1}{70}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2r^2+5r+2}$	$\frac{1}{5}$
a_2	$\frac{1}{4r^4+28r^3+67r^2+63r+18}$	$\frac{1}{70}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{1}{8r^6 + 108r^5 + 578r^4 + 1557r^3 + 2195r^2 + 1494r + 360}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_3 = \frac{1}{1890}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2r^2+5r+2}$	$\frac{1}{5}$
a_2	$\frac{1}{4r^4+28r^3+67r^2+63r+18}$	$\frac{1}{70}$
a_3	$\frac{1}{8r^6+108r^5+578r^4+1557r^3+2195r^2+1494r+360}$	$\frac{1}{1890}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{16r^8 + 352r^7 + 3272r^6 + 16720r^5 + 51089r^4 + 94798r^3 + 102943r^2 + 58410r + 12600}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = \frac{1}{83160}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2r^2+5r+2}$	$\frac{1}{5}$
a_2	$\frac{1}{4r^4+28r^3+67r^2+63r+18}$	$\frac{1}{70}$
a_3	$\frac{1}{8r^6+108r^5+578r^4+1557r^3+2195r^2+1494r+360}$	$\frac{1}{1890}$
a_4	$\frac{1}{16r^8+352r^7+3272r^6+16720r^5+51089r^4+94798r^3+102943r^2+58410r+12600}$	$\frac{1}{83160}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{1}{32r^{10} + 1040r^9 + 14800r^8 + 121160r^7 + 629986r^6 + 2165345r^5 + 4955450r^4 + 7397715r^3 + 6810732}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_5 = \frac{1}{5405400}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2r^2+5r+2}$	$\frac{1}{5}$
a_2	$\frac{1}{4r^4+28r^3+67r^2+63r+18}$	$\frac{1}{70}$
a_3	$\frac{1}{8r^6+108r^5+578r^4+1557r^3+2195r^2+1494r+360}$	$\frac{1}{1890}$
a_4	$\frac{1}{16r^8+352r^7+3272r^6+16720r^5+51089r^4+94798r^3+102943r^2+58410r+12600}$	$\frac{1}{83160}$
a_5	$\frac{1}{32r^{10}+1040r^9+14800r^8+121160r^7+629986r^6+2165345r^5+4955450r^4+7397715r^3+6810732r^2+3418740r+680400}$	$\frac{1}{5405400}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= \sqrt{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= \sqrt{x}\left(1 + \frac{x}{5} + \frac{x^2}{70} + \frac{x^3}{1890} + \frac{x^4}{83160} + \frac{x^5}{5405400} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$b_n(n+r)(n+r-1) + \frac{3b_n(n+r)}{2} - \frac{b_{n-1}}{2} - \frac{b_n}{2} = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = \frac{b_{n-1}}{2n^2 + 4nr + 2r^2 + n + r - 1} \tag{4}$$

Which for the root $r = -1$ becomes

$$b_n = \frac{b_{n-1}}{n(2n-3)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{1}{2r^2 + 5r + 2}$$

Which for the root $r = -1$ becomes

$$b_1 = -1$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{2r^2+5r+2}$	-1

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1}{4r^4 + 28r^3 + 67r^2 + 63r + 18}$$

Which for the root $r = -1$ becomes

$$b_2 = -\frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{2r^2+5r+2}$	-1
b_2	$\frac{1}{4r^4+28r^3+67r^2+63r+18}$	$-\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{1}{8r^6 + 108r^5 + 578r^4 + 1557r^3 + 2195r^2 + 1494r + 360}$$

Which for the root $r = -1$ becomes

$$b_3 = -\frac{1}{18}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{2r^2+5r+2}$	-1
b_2	$\frac{1}{4r^4+28r^3+67r^2+63r+18}$	$-\frac{1}{2}$
b_3	$\frac{1}{8r^6+108r^5+578r^4+1557r^3+2195r^2+1494r+360}$	$-\frac{1}{18}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{16r^8 + 352r^7 + 3272r^6 + 16720r^5 + 51089r^4 + 94798r^3 + 102943r^2 + 58410r + 12600}$$

Which for the root $r = -1$ becomes

$$b_4 = -\frac{1}{360}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{2r^2+5r+2}$	-1
b_2	$\frac{1}{4r^4+28r^3+67r^2+63r+18}$	$-\frac{1}{2}$
b_3	$\frac{1}{8r^6+108r^5+578r^4+1557r^3+2195r^2+1494r+360}$	$-\frac{1}{18}$
b_4	$\frac{1}{16r^8+352r^7+3272r^6+16720r^5+51089r^4+94798r^3+102943r^2+58410r+12600}$	$-\frac{1}{360}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{1}{32r^{10} + 1040r^9 + 14800r^8 + 121160r^7 + 629986r^6 + 2165345r^5 + 4955450r^4 + 7397715r^3 + 6810732}$$

Which for the root $r = -1$ becomes

$$b_5 = -\frac{1}{12600}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{2r^2+5r+2}$	-1
b_2	$\frac{1}{4r^4+28r^3+67r^2+63r+18}$	$-\frac{1}{2}$
b_3	$\frac{1}{8r^6+108r^5+578r^4+1557r^3+2195r^2+1494r+360}$	$-\frac{1}{18}$
b_4	$\frac{1}{16r^8+352r^7+3272r^6+16720r^5+51089r^4+94798r^3+102943r^2+58410r+12600}$	$-\frac{1}{360}$
b_5	$\frac{1}{32r^{10}+1040r^9+14800r^8+121160r^7+629986r^6+2165345r^5+4955450r^4+7397715r^3+6810732r^2+3418740r+680400}$	$-\frac{1}{12600}$

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= \sqrt{x}\left(b_0+b_1x+b_2x^2+b_3x^3+b_4x^4+b_5x^5+b_6x^6\dots\right)\\
&= \frac{1-x-\frac{x^2}{2}-\frac{x^3}{18}-\frac{x^4}{360}-\frac{x^5}{12600}+O(x^6)}{x}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1y_1(x)+c_2y_2(x)\\
&= c_1\sqrt{x}\left(1+\frac{x}{5}+\frac{x^2}{70}+\frac{x^3}{1890}+\frac{x^4}{83160}+\frac{x^5}{5405400}+O(x^6)\right)\\
&+\frac{c_2\left(1-x-\frac{x^2}{2}-\frac{x^3}{18}-\frac{x^4}{360}-\frac{x^5}{12600}+O(x^6)\right)}{x}
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h\\
&= c_1\sqrt{x}\left(1+\frac{x}{5}+\frac{x^2}{70}+\frac{x^3}{1890}+\frac{x^4}{83160}+\frac{x^5}{5405400}+O(x^6)\right)\\
&+\frac{c_2\left(1-x-\frac{x^2}{2}-\frac{x^3}{18}-\frac{x^4}{360}-\frac{x^5}{12600}+O(x^6)\right)}{x}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1\sqrt{x}\left(1+\frac{x}{5}+\frac{x^2}{70}+\frac{x^3}{1890}+\frac{x^4}{83160}+\frac{x^5}{5405400}+O(x^6)\right)\\
&+\frac{c_2\left(1-x-\frac{x^2}{2}-\frac{x^3}{18}-\frac{x^4}{360}-\frac{x^5}{12600}+O(x^6)\right)}{x}
\end{aligned}
\tag{1}
$$

Verification of solutions

$$y = c_1\sqrt{x}\left(1+\frac{x}{5}+\frac{x^2}{70}+\frac{x^3}{1890}+\frac{x^4}{83160}+\frac{x^5}{5405400}+O(x^6)\right) + \frac{c_2\left(1-x-\frac{x^2}{2}-\frac{x^3}{18}-\frac{x^4}{360}-\frac{x^5}{12600}+O(x^6)\right)}{x}$$

Verified OK.

20.11.1 Maple step by step solution

Let's solve

$x^2y''+\frac{3xy'}{2}+\left(-\frac{x}{2}-\frac{1}{2}\right)y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{3y'}{2x}+\frac{(x+1)y}{2x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{3y'}{2x}-\frac{(x+1)y}{2x^2}=0$

□ Check to see if $x_0=0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{3}{2x}, P_3(x)=-\frac{x+1}{2x^2}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $(x\cdot P_2(x))\Big|_{x=0}=\frac{3}{2}$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $(x^2\cdot P_3(x))\Big|_{x=0}=-\frac{1}{2}$

  ◦ $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $2x^2y''+3xy'+(-x-1)y=0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x \cdot y'$ to series expansion

$$x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+r)\,(-1+2r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)\,(2k+2r-1) - a_{k-1}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+r)\,(-1+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{-1, \frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$2\left(k+r-\frac{1}{2}\right)(k+r+1)\, a_k - a_{k-1} = 0$$

- Shift index using $k- >k + 1$

$$2\left(k+\frac{1}{2}+r\right)(k+2+r)\, a_{k+1} - a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k}{(2k+1+2r)(k+2+r)}$$

- Recursion relation for $r = -1$

$$a_{k+1} = \frac{a_k}{(2k-1)(k+1)}$$

- Solution for $r = -1$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+1}=\frac{a_k}{(2k-1)(k+1)}\right]$$

- Recursion relation for $r=\frac{1}{2}$

$$a_{k+1}=\frac{a_k}{(2k+2)\left(k+\frac{5}{2}\right)}$$

- Solution for $r=\frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1}=\frac{a_k}{(2k+2)\left(k+\frac{5}{2}\right)}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-1}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{1+k}=\frac{a_k}{(2k-1)(1+k)}, b_{1+k}=\frac{b_k}{(2+2k)\left(k+\frac{5}{2}\right)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 47

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+3/2*x*diff(y(x),x)-1/2*(1+x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_2 x^{\frac{3}{2}}\left(1+\frac{1}{5}x+\frac{1}{70}x^2+\frac{1}{1890}x^3+\frac{1}{83160}x^4+\frac{1}{5405400}x^5+\mathrm{O}\left(x^6\right)\right)+c_1\left(1-x-\frac{1}{2}x^2-\frac{1}{18}x^3-\frac{1}{360}x^4-\frac{1}{12600}x\right.}{x}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 86

```
AsymptoticDSolveValue[x^2*y''[x]+3/2*x*y'[x]-1/2*(1+x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\sqrt{x}\left(\frac{x^5}{5405400}+\frac{x^4}{83160}+\frac{x^3}{1890}+\frac{x^2}{70}+\frac{x}{5}+1\right) + \frac{c_2\left(-\frac{x^5}{12600}-\frac{x^4}{360}-\frac{x^3}{18}-\frac{x^2}{2}-x+1\right)}{x}$$

20.12 problem 12

Internal problem ID [2985]
Internal file name [OUTPUT/2477_Sunday_June_05_2022_03_15_26_AM_26802/index.tex]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Additional problems. Section 11.7. page 788
Problem number: 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2 y'' - x(-x+2)\,y' + \left(x^2+2\right) y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2 y'' + \left(x^2 - 2x\right) y' + \left(x^2+2\right) y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$\begin{aligned} p(x) &= \frac{x-2}{x} \\ q(x) &= \frac{x^2+2}{x^2} \end{aligned}$$

Table 538: Table $p(x), q(x)$ singularites.

$p(x) = \frac{x-2}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{x^2+2}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + (x^2 - 2x)\, y' + (x^2 + 2)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned} & x^2 \left(\sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2}\right) \\ & + (x^2 - 2x) \left(\sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}\right) + (x^2 + 2) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \end{aligned} \tag{1}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)\right) \\
&+ \sum_{n=0}^{\infty} \left(-2x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{n+r+2} a_n\right) + \left(\sum_{n=0}^{\infty} 2a_n x^{n+r}\right) = 0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} a_{n-1} (n+r-1) x^{n+r}
$$

$$
\sum_{n=0}^{\infty} x^{n+r+2} a_n = \sum_{n=2}^{\infty} a_{n-2} x^{n+r}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} a_{n-1} (n+r-1) x^{n+r}\right) \\
&+ \sum_{n=0}^{\infty} \left(-2x^{n+r} a_n (n+r)\right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 2a_n x^{n+r}\right) = 0
\end{aligned} \tag{2B}
$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$
x^{n+r} a_n (n+r)(n+r-1) - 2x^{n+r} a_n (n+r) + 2a_n x^{n+r} = 0
$$

When $n = 0$ the above becomes

$$
x^r a_0 r(-1+r) - 2x^r a_0 r + 2a_0 x^r = 0
$$

Or

$$
\left(x^r r(-1+r) - 2x^r r + 2x^r\right) a_0 = 0
$$

Since $a_0 \neq 0$ then the above simplifies to

$$
\left(r^2 - 3r + 2\right) x^r = 0
$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - 3r + 2 = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned}r_1 &= 2\\ r_2 &= 1\end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(r^2 - 3r + 2\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$\begin{aligned}y_1(x) &= x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)\\ y_2(x) &= C y_1(x) \ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)\end{aligned}$$

Or

$$\begin{aligned}y_1(x) &= x^2\left(\sum_{n=0}^{\infty} a_n x^n\right)\\ y_2(x) &= C y_1(x) \ln(x) + x\left(\sum_{n=0}^{\infty} b_n x^n\right)\end{aligned}$$

Or

$$\begin{aligned}y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+2}\\ y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{1+n}\right)\end{aligned}$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = -\frac{1}{-1 + r}$$

For $2 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_{n-1}(n+r-1) - 2a_n(n+r) + a_{n-2} + 2a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{n a_{n-1} + r a_{n-1} + a_{n-2} - a_{n-1}}{n^2 + 2nr + r^2 - 3n - 3r + 2} \tag{4}$$

Which for the root $r = 2$ becomes

$$a_n = \frac{-n a_{n-1} - a_{n-2} - a_{n-1}}{n(1+n)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 2$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{-1+r}$	-1

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{2}{r^3 - r}$$

Which for the root $r = 2$ becomes

$$a_2 = \frac{1}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{-1+r}$	-1
a_2	$\frac{2}{r^3-r}$	$\frac{1}{3}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{r^2 - r - 4}{(r+2)(1+r)^2 r(-1+r)}$$

Which for the root $r = 2$ becomes

$$a_3 = -\frac{1}{36}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{-1+r}$	-1
a_2	$\frac{2}{r^3-r}$	$\frac{1}{3}$
a_3	$\frac{r^2-r-4}{(r+2)(1+r)^2 r(-1+r)}$	$-\frac{1}{36}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{-r^3 - 4r^2 + r + 8}{(3+r)(r+2)^2(-1+r)(1+r)^2 r}$$

Which for the root $r = 2$ becomes

$$a_4 = -\frac{7}{720}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{-1+r}$	-1
a_2	$\frac{2}{r^3-r}$	$\frac{1}{3}$
a_3	$\frac{r^2-r-4}{(r+2)(1+r)^2 r(-1+r)}$	$-\frac{1}{36}$
a_4	$\frac{-r^3-4r^2+r+8}{(3+r)(r+2)^2(-1+r)(1+r)^2 r}$	$-\frac{7}{720}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{4r^3 + 18r^2 + 14r - 8}{(3+r)^2(r+2)^2(-1+r)(1+r)^2 r(r+4)}$$

Which for the root $r = 2$ becomes

$$a_5 = \frac{31}{10800}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{-1+r}$	-1
a_2	$\frac{2}{r^3-r}$	$\frac{1}{3}$
a_3	$\frac{r^2-r-4}{(r+2)(1+r)^2 r(-1+r)}$	$-\frac{1}{36}$
a_4	$\frac{-r^3-4r^2+r+8}{(3+r)(r+2)^2(-1+r)(1+r)^2 r}$	$-\frac{7}{720}$
a_5	$\frac{4r^3+18r^2+14r-8}{(3+r)^2(r+2)^2(-1+r)(1+r)^2 r(r+4)}$	$\frac{31}{10800}$

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x^2\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= x^2\left(1 - x + \frac{x^2}{3} - \frac{x^3}{36} - \frac{7x^4}{720} + \frac{31x^5}{10800} + O(x^6)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Let

$$
r_1 - r_2 = N
$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$
\begin{aligned}
a_N &= a_1 \\
&= -\frac{1}{-1+r}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\lim_{r \to r_2} -\frac{1}{-1+r} &= \lim_{r \to 1} -\frac{1}{-1+r} \\
&= \textit{undefined}
\end{aligned}
$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$
y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)
$$

Therefore

$$
\begin{aligned}
\frac{d}{dx}y_2(x) &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\
&= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) \\
\frac{d^2}{dx^2}y_2(x) &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\
&= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right)
\end{aligned}
$$

Substituting these back into the given ode $x^2y'' + (x^2-2x)\,y' + (x^2+2)\,y = 0$ gives

$$
\begin{aligned}
&x^2\Bigg(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\Bigg) \\
&\quad + (x^2-2x)\left(Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right) \\
&\quad + (x^2+2)\left(Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)\right) = 0
\end{aligned}
$$

Which can be written as

$$
\begin{aligned}
&\Bigg(\left(x^2y_1''(x) + (x^2-2x)\,y_1'(x) + (x^2+2)\,y_1(x)\right)\ln(x) + x^2\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right) \\
&\quad + \frac{(x^2-2x)\,y_1(x)}{x}\Bigg)C + x^2\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) \\
&\quad + (x^2-2x)\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right) + (x^2+2)\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned} \tag{7}
$$

But since $y_1(x)$ is a solution to the ode, then

$$x^2 y_1''(x) + (x^2 - 2x)\, y_1'(x) + (x^2 + 2)\, y_1(x) = 0$$

Eq (7) simplifes to

$$\begin{aligned}
&\left(x^2 \left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right) + \frac{(x^2 - 2x)\, y_1(x)}{x}\right) C \\
&+ x^2 \left(\sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2} (n + r_2)^2}{x^2} - \frac{b_n x^{n+r_2} (n + r_2)}{x^2}\right)\right) \\
&+ (x^2 - 2x) \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2} (n + r_2)}{x}\right) + (x^2 + 2) \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned} \tag{8}$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\begin{aligned}
&\left(2 \left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n (n + r_1)\right) x + (x - 3) \left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right)\right) C \\
&+ \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n + r_2) (-1 + n + r_2)\right) x^2 \\
&+ (x^2 - 2x) \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n + r_2)\right) + (x^2 + 2) \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned} \tag{9}$$

Since $r_1 = 2$ and $r_2 = 1$ then the above becomes

$$\begin{aligned}
&\left(2 \left(\sum_{n=0}^{\infty} x^{1+n} a_n (n + 2)\right) x + (x - 3) \left(\sum_{n=0}^{\infty} a_n x^{n+2}\right)\right) C \\
&+ \left(\sum_{n=0}^{\infty} x^{n-1} b_n (1 + n)\, n\right) x^2 + (x^2 - 2x) \left(\sum_{n=0}^{\infty} x^n b_n (1 + n)\right) \\
&+ (x^2 + 2) \left(\sum_{n=0}^{\infty} b_n x^{1+n}\right) = 0
\end{aligned} \tag{10}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} 2C\, x^{n+2} a_n (n+2)\right)+\left(\sum_{n=0}^{\infty} C\, x^{n+3} a_n\right)+\sum_{n=0}^{\infty}\left(-3C\, x^{n+2} a_n\right) \\
&+\left(\sum_{n=0}^{\infty} n\, x^{1+n} b_n (1+n)\right)+\left(\sum_{n=0}^{\infty} x^{n+2} b_n (1+n)\right) \\
&+\sum_{n=0}^{\infty}\left(-2x^{1+n} b_n (1+n)\right)+\left(\sum_{n=0}^{\infty} x^{n+3} b_n\right)+\left(\sum_{n=0}^{\infty} 2b_n x^{1+n}\right)=0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $1+n$ in each summation term. Going over each summation term above with power of x in it which is not already x^{1+n} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} 2C\, x^{n+2} a_n (n+2) &= \sum_{n=1}^{\infty} 2C a_{n-1} (1+n)\, x^{1+n} \\
\sum_{n=0}^{\infty} C\, x^{n+3} a_n &= \sum_{n=2}^{\infty} C a_{n-2} x^{1+n} \\
\sum_{n=0}^{\infty} \left(-3C\, x^{n+2} a_n\right) &= \sum_{n=1}^{\infty} \left(-3C a_{n-1} x^{1+n}\right) \\
\sum_{n=0}^{\infty} x^{n+2} b_n (1+n) &= \sum_{n=1}^{\infty} b_{n-1} n\, x^{1+n} \\
\sum_{n=0}^{\infty} x^{n+3} b_n &= \sum_{n=2}^{\infty} b_{n-2} x^{1+n}
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $1+n$.

$$
\begin{aligned}
&\left(\sum_{n=1}^{\infty} 2C a_{n-1} (1+n)\, x^{1+n}\right)+\left(\sum_{n=2}^{\infty} C a_{n-2} x^{1+n}\right)+\sum_{n=1}^{\infty}\left(-3C a_{n-1} x^{1+n}\right) \\
&+\left(\sum_{n=0}^{\infty} n\, x^{1+n} b_n (1+n)\right)+\left(\sum_{n=1}^{\infty} b_{n-1} n\, x^{1+n}\right) \\
&+\sum_{n=0}^{\infty}\left(-2x^{1+n} b_n (1+n)\right)+\left(\sum_{n=2}^{\infty} b_{n-2} x^{1+n}\right)+\left(\sum_{n=0}^{\infty} 2b_n x^{1+n}\right)=0
\end{aligned} \tag{2B}
$$

For $n = 0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0 = 1$. For $n = N$, where $N = 1$ which is the difference between the two roots, we are free to choose $b_1 = 0$. Hence for $n = 1$, Eq (2B) gives

$$C + 1 = 0$$

Which is solved for C. Solving for C gives

$$C = -1$$

For $n = 2$, Eq (2B) gives

$$(a_0 + 3a_1) C + b_0 + 2b_1 + 2b_2 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$3 + 2b_2 = 0$$

Solving the above for b_2 gives

$$b_2 = -\frac{3}{2}$$

For $n = 3$, Eq (2B) gives

$$(a_1 + 5a_2) C + b_1 + 3b_2 + 6b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-\frac{31}{6} + 6b_3 = 0$$

Solving the above for b_3 gives

$$b_3 = \frac{31}{36}$$

For $n = 4$, Eq (2B) gives

$$(a_2 + 7a_3) C + b_2 + 4b_3 + 12b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{65}{36} + 12b_4 = 0$$

Solving the above for b_4 gives

$$b_4 = -\frac{65}{432}$$

For $n = 5$, Eq (2B) gives

$$(a_3 + 9a_4) C + b_3 + 5b_4 + 20b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{121}{540}+20b_5=0$$

Solving the above for b_5 gives

$$b_5=-\frac{121}{10800}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x)=Cy_1(x)\ln(x)+\left(\sum_{n=0}^{\infty}b_nx^{n+r_2}\right)$$

Using the above value found for $C=-1$ and all b_n, then the second solution becomes

$$\begin{aligned}y_2(x)=(-1)&\left(x^2\left(1-x+\frac{x^2}{3}-\frac{x^3}{36}-\frac{7x^4}{720}+\frac{31x^5}{10800}+O(x^6)\right)\right)\ln(x)\\&+x\left(1-\frac{3x^2}{2}+\frac{31x^3}{36}-\frac{65x^4}{432}-\frac{121x^5}{10800}+O(x^6)\right)\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}y_h(x)&=c_1y_1(x)+c_2y_2(x)\\&=c_1x^2\left(1-x+\frac{x^2}{3}-\frac{x^3}{36}-\frac{7x^4}{720}+\frac{31x^5}{10800}+O(x^6)\right)\\&\quad+c_2\left((-1)\left(x^2\left(1-x+\frac{x^2}{3}-\frac{x^3}{36}-\frac{7x^4}{720}+\frac{31x^5}{10800}+O(x^6)\right)\right)\ln(x)\right.\\&\qquad\left.+x\left(1-\frac{3x^2}{2}+\frac{31x^3}{36}-\frac{65x^4}{432}-\frac{121x^5}{10800}+O(x^6)\right)\right)\end{aligned}$$

Hence the final solution is

$$\begin{aligned}y&=y_h\\&=c_1x^2\left(1-x+\frac{x^2}{3}-\frac{x^3}{36}-\frac{7x^4}{720}+\frac{31x^5}{10800}+O(x^6)\right)\\&\quad+c_2\left(-x^2\left(1-x+\frac{x^2}{3}-\frac{x^3}{36}-\frac{7x^4}{720}+\frac{31x^5}{10800}+O(x^6)\right)\ln(x)\right.\\&\qquad\left.+x\left(1-\frac{3x^2}{2}+\frac{31x^3}{36}-\frac{65x^4}{432}-\frac{121x^5}{10800}+O(x^6)\right)\right)\end{aligned}$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & c_1 x^2 \left(1 - x + \frac{x^2}{3} - \frac{x^3}{36} - \frac{7x^4}{720} + \frac{31x^5}{10800} + O(x^6)\right) \\
& + c_2 \left(-x^2 \left(1 - x + \frac{x^2}{3} - \frac{x^3}{36} - \frac{7x^4}{720} + \frac{31x^5}{10800} + O(x^6)\right) \ln(x) \right. \\
& \left. + x \left(1 - \frac{3x^2}{2} + \frac{31x^3}{36} - \frac{65x^4}{432} - \frac{121x^5}{10800} + O(x^6)\right)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = {} & c_1 x^2 \left(1 - x + \frac{x^2}{3} - \frac{x^3}{36} - \frac{7x^4}{720} + \frac{31x^5}{10800} + O(x^6)\right) \\
& + c_2 \left(-x^2 \left(1 - x + \frac{x^2}{3} - \frac{x^3}{36} - \frac{7x^4}{720} + \frac{31x^5}{10800} + O(x^6)\right) \ln(x) \right. \\
& \left. + x \left(1 - \frac{3x^2}{2} + \frac{31x^3}{36} - \frac{65x^4}{432} - \frac{121x^5}{10800} + O(x^6)\right)\right)
\end{aligned}
$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Whittaker successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 65

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)-x*(2-x)*diff(y(x),x)+(2+x^2)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(c_1 x \left(1 - x + \frac{1}{3}x^2 - \frac{1}{36}x^3 - \frac{7}{720}x^4 + \frac{31}{10800}x^5 + \mathrm{O}\left(x^6\right)\right) + c_2\left(\ln\left(x\right)\left(-x + x^2 - \frac{1}{3}x^3 + \frac{1}{36}x^4 + \frac{7}{720}x^5 + \mathrm{O}\left(x^6\right)\right) + \left(1 - x - \frac{1}{2}x^2 + \frac{19}{36}x^3 - \frac{53}{432}x^4 - \frac{1}{675}x^5 + \mathrm{O}\left(x^6\right)\right)\right)\right)x$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 85

```
AsymptoticDSolveValue[x^2*y''[x]-x*(2-x)*y'[x]+(2+x^2)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{1}{36}x^2\left(x^3 - 12x^2 + 36x - 36\right)\log(x) - \frac{1}{432}x\left(65x^4 - 372x^3 + 648x^2 - 432\right)\right) + c_2\left(-\frac{7x^6}{720} - \frac{x^5}{36} + \frac{x^4}{3} - x^3 + x^2\right)$$

20.13 problem 13

Internal problem ID [2986]
Internal file name [`OUTPUT/2478_Sunday_June_05_2022_03_15_31_AM_15359822/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Additional problems. Section 11.7. page 788
Problem number: 13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - 3xy' + 4y(x+1) = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' - 3xy' + (4x+4)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$\begin{aligned}
p(x) &= -\frac{3}{x}\\
q(x) &= \frac{4x+4}{x^2}
\end{aligned}$$

Table 539: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{3}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{4x+4}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2 y'' - 3xy' + (4x + 4) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}\right) - 3x \left(\sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}\right) + (4x + 4) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-3 x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 4 x^{1+n+r} a_n\right) + \left(\sum_{n=0}^{\infty} 4 a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 4 x^{1+n+r} a_n = \sum_{n=1}^{\infty} 4 a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-3 x^{n+r} a_n (n+r)\right) + \left(\sum_{n=1}^{\infty} 4 a_{n-1} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 4 a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) - 3 x^{n+r} a_n (n+r) + 4 a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$x^r a_0 r(-1+r) - 3 x^r a_0 r + 4 a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) - 3 x^r r + 4 x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(r-2)^2 x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(r-2)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 2$$
$$r_2 = 2$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(r-2)^2 x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = 2$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+2}$$
$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+2}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) - 3a_n(n+r) + 4a_{n-1} + 4a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{4a_{n-1}}{n^2 + 2nr + r^2 - 4n - 4r + 4} \tag{4}$$

Which for the root $r = 2$ becomes

$$a_n = -\frac{4a_{n-1}}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 2$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{4}{(-1+r)^2}$$

Which for the root $r = 2$ becomes

$$a_1 = -4$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{4}{(-1+r)^2}$	-4

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{16}{(-1+r)^2 r^2}$$

Which for the root $r = 2$ becomes

$$a_2 = 4$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{4}{(-1+r)^2}$	-4
a_2	$\frac{16}{(-1+r)^2 r^2}$	4

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{64}{(-1+r)^2 r^2 (r+1)^2}$$

Which for the root $r = 2$ becomes

$$a_3 = -\frac{16}{9}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{4}{(-1+r)^2}$	-4
a_2	$\frac{16}{(-1+r)^2 r^2}$	4
a_3	$-\frac{64}{(-1+r)^2 r^2 (r+1)^2}$	$-\frac{16}{9}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{256}{(-1+r)^2 r^2 (r+1)^2 (r+2)^2}$$

Which for the root $r = 2$ becomes

$$a_4 = \frac{4}{9}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{4}{(-1+r)^2}$	-4
a_2	$\frac{16}{(-1+r)^2 r^2}$	4
a_3	$-\frac{64}{(-1+r)^2 r^2 (r+1)^2}$	$-\frac{16}{9}$
a_4	$\frac{256}{(-1+r)^2 r^2 (r+1)^2 (r+2)^2}$	$\frac{4}{9}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{1024}{(-1+r)^2 r^2 (r+1)^2 (r+2)^2 (3+r)^2}$$

Which for the root $r = 2$ becomes

$$a_5 = -\frac{16}{225}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{4}{(-1+r)^2}$	-4
a_2	$\frac{16}{(-1+r)^2 r^2}$	4
a_3	$-\frac{64}{(-1+r)^2 r^2 (r+1)^2}$	$-\frac{16}{9}$
a_4	$\frac{256}{(-1+r)^2 r^2 (r+1)^2 (r+2)^2}$	$\frac{4}{9}$
a_5	$-\frac{1024}{(-1+r)^2 r^2 (r+1)^2 (r+2)^2 (3+r)^2}$	$-\frac{16}{225}$

Using the above table, then the first solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^2 (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots) \\ &= x^2 \left(4x^2 - 4x + 1 - \frac{16x^3}{9} + \frac{4x^4}{9} - \frac{16x^5}{225} + O(x^6)\right) \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = 2$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr}a_{n,r}$	$b_n(r=2)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	$-\frac{4}{(-1+r)^2}$	-4	$\frac{8}{(-1+r)^3}$	8
b_2	$\frac{16}{(-1+r)^2 r^2}$	4	$\frac{-64r+32}{(-1+r)^3 r^3}$	-12
b_3	$-\frac{64}{(-1+r)^2 r^2 (r+1)^2}$	$-\frac{16}{9}$	$\frac{384r^2-128}{(-1+r)^3 r^3 (r+1)^3}$	$\frac{176}{27}$
b_4	$\frac{256}{(-1+r)^2 r^2 (r+1)^2 (r+2)^2}$	$\frac{4}{9}$	$\frac{-2048r^3-3072r^2+1024r+1024}{(-1+r)^3 r^3 (r+1)^3 (r+2)^3}$	$-\frac{50}{27}$
b_5	$-\frac{1024}{(-1+r)^2 r^2 (r+1)^2 (r+2)^2 (3+r)^2}$	$-\frac{16}{225}$	$\frac{10240r^4+40960r^3+30720r^2-20480r-12288}{(-1+r)^3 r^3 (r+1)^3 (r+2)^3 (3+r)^3}$	$\frac{1096}{3375}$

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\
&= x^2\left(4x^2 - 4x + 1 - \frac{16x^3}{9} + \frac{4x^4}{9} - \frac{16x^5}{225} + O(x^6)\right)\ln(x) \\
&\quad + x^2\left(-12x^2 + 8x + \frac{176x^3}{27} - \frac{50x^4}{27} + \frac{1096x^5}{3375} + O(x^6)\right)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^2\left(4x^2 - 4x + 1 - \frac{16x^3}{9} + \frac{4x^4}{9} - \frac{16x^5}{225} + O(x^6)\right) \\
&\quad + c_2\left(x^2\left(4x^2 - 4x + 1 - \frac{16x^3}{9} + \frac{4x^4}{9} - \frac{16x^5}{225} + O(x^6)\right)\ln(x)\right. \\
&\qquad \left. + x^2\left(-12x^2 + 8x + \frac{176x^3}{27} - \frac{50x^4}{27} + \frac{1096x^5}{3375} + O(x^6)\right)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^2\left(4x^2 - 4x + 1 - \frac{16x^3}{9} + \frac{4x^4}{9} - \frac{16x^5}{225} + O(x^6)\right) \\
&\quad + c_2\left(x^2\left(4x^2 - 4x + 1 - \frac{16x^3}{9} + \frac{4x^4}{9} - \frac{16x^5}{225} + O(x^6)\right)\ln(x)\right. \\
&\qquad \left. + x^2\left(-12x^2 + 8x + \frac{176x^3}{27} - \frac{50x^4}{27} + \frac{1096x^5}{3375} + O(x^6)\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & c_1 x^2 \left(4x^2 - 4x + 1 - \frac{16x^3}{9} + \frac{4x^4}{9} - \frac{16x^5}{225} + O(x^6)\right) \\
& + c_2 \left(x^2 \left(4x^2 - 4x + 1 - \frac{16x^3}{9} + \frac{4x^4}{9} - \frac{16x^5}{225} + O(x^6)\right) \ln(x) \right. \\
& \left. + x^2 \left(-12x^2 + 8x + \frac{176x^3}{27} - \frac{50x^4}{27} + \frac{1096x^5}{3375} + O(x^6)\right)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = {} & c_1 x^2 \left(4x^2 - 4x + 1 - \frac{16x^3}{9} + \frac{4x^4}{9} - \frac{16x^5}{225} + O(x^6)\right) \\
& + c_2 \left(x^2 \left(4x^2 - 4x + 1 - \frac{16x^3}{9} + \frac{4x^4}{9} - \frac{16x^5}{225} + O(x^6)\right) \ln(x) \right. \\
& \left. + x^2 \left(-12x^2 + 8x + \frac{176x^3}{27} - \frac{50x^4}{27} + \frac{1096x^5}{3375} + O(x^6)\right)\right)
\end{aligned}
$$

Verified OK.

20.13.1 Maple step by step solution

Let's solve

$x^2y'' - 3xy' + (4x + 4)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{3y'}{x} - \frac{4(x+1)y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{3y'}{x} + \frac{4(x+1)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -\frac{3}{x}, P_3(x) = \frac{4(x+1)}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = -3$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 4$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2 y'' - 3xy' + (4x + 4)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$

  - Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-2+r)^2 x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r-2)^2 + 4a_{k-1}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-2+r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = 2$

- Each term in the series must be 0, giving the recursion relation

$$a_k(k+r-2)^2 + 4a_{k-1} = 0$$

- Shift index using $k- >k+1$

$$a_{k+1}(k+r-1)^2 + 4a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{4a_k}{(k+r-1)^2}$$

- Recursion relation for $r = 2$

$$a_{k+1} = -\frac{4a_k}{(k+1)^2}$$

- Solution for $r = 2$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = -\frac{4a_k}{(k+1)^2}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 69

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)-3*x*diff(y(x),x)+4*(x+1)*y(x)=0,y(x),type='series',x=0);
```

$$
\begin{aligned}
y(x) = \Bigg(& (c_2 \ln(x) + c_1)\left(1 - 4x + 4x^2 - \frac{16}{9}x^3 + \frac{4}{9}x^4 - \frac{16}{225}x^5 + \mathrm{O}\left(x^6\right)\right) \\
& + \left(8x - 12x^2 + \frac{176}{27}x^3 - \frac{50}{27}x^4 + \frac{1096}{3375}x^5 + \mathrm{O}\left(x^6\right)\right) c_2 \Bigg) x^2
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 116

```
AsymptoticDSolveValue[x^2*y''[x]-3*x*y'[x]+4*(x+1)*y[x]==0,y[x],{x,0,5}]
```

$$
\begin{aligned}
y(x) \to\ & c_1\left(-\frac{16x^5}{225} + \frac{4x^4}{9} - \frac{16x^3}{9} + 4x^2 - 4x + 1\right)x^2 \\
& + c_2\left(\left(\frac{1096x^5}{3375} - \frac{50x^4}{27} + \frac{176x^3}{27} - 12x^2 + 8x\right)x^2 \right. \\
& \left. + \left(-\frac{16x^5}{225} + \frac{4x^4}{9} - \frac{16x^3}{9} + 4x^2 - 4x + 1\right)x^2 \log(x)\right)
\end{aligned}
$$

20.14 problem 20

Internal problem ID [2987]
Internal file name [`OUTPUT/2479_Sunday_June_05_2022_03_15_34_AM_55759182/index.tex`]

Book: Differential equations and linear algebra, Stephen W. Goode and Scott A Annin. Fourth edition, 2015
Section: Chapter 11, Series Solutions to Linear Differential Equations. Additional problems. Section 11.7. page 788
Problem number: 20.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + \left(1 - \frac{3}{4x^2}\right) y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$y'' + \left(1 - \frac{3}{4x^2}\right) y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = 0$$
$$q(x) = \frac{4x^2 - 3}{4x^2}$$

Table 541: Table $p(x), q(x)$ singularites.

$p(x) = 0$	
singularity	type

$q(x) = \frac{4x^2-3}{4x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$4x^2y'' + (4x^2 - 3)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$4x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + (4x^2 - 3) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r+2} a_n\right) + \sum_{n=0}^{\infty} \left(-3a_n x^{n+r}\right) = 0 \quad \text{(2A)}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 4x^{n+r+2}a_n = \sum_{n=2}^{\infty} 4a_{n-2}x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 4x^{n+r}a_n(n+r)(n+r-1)\right) + \left(\sum_{n=2}^{\infty} 4a_{n-2}x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-3a_n x^{n+r}\right) = 0 \quad \text{(2B)}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$4x^{n+r}a_n(n+r)(n+r-1) - 3a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$4x^r a_0 r(-1+r) - 3a_0 x^r = 0$$

Or

$$\left(4x^r r(-1+r) - 3x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(4r^2 - 4r - 3\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$4r^2 - 4r - 3 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{3}{2}$$
$$r_2 = -\frac{1}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(4r^2 - 4r - 3\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 2$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right)$$

Or

$$y_1(x) = x^{\frac{3}{2}} \left(\sum_{n=0}^{\infty} a_n x^n \right)$$
$$y_2(x) = C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{\sqrt{x}}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{3}{2}}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-\frac{1}{2}} \right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$4a_n(n+r)(n+r-1) + 4a_{n-2} - 3a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{4a_{n-2}}{4n^2 + 8nr + 4r^2 - 4n - 4r - 3} \tag{4}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_n = -\frac{a_{n-2}}{n(n+2)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{3}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{4}{4r^2 + 12r + 5}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_2 = -\frac{1}{8}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{4}{4r^2+12r+5}$	$-\frac{1}{8}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{4}{4r^2+12r+5}$	$-\frac{1}{8}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{16}{(4r^2 + 12r + 5)(4r^2 + 28r + 45)}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_4 = \frac{1}{192}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{4}{4r^2+12r+5}$	$-\frac{1}{8}$
a_3	0	0
a_4	$\frac{16}{(4r^2+12r+5)(4r^2+28r+45)}$	$\frac{1}{192}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{4}{4r^2+12r+5}$	$-\frac{1}{8}$
a_3	0	0
a_4	$\frac{16}{(4r^2+12r+5)(4r^2+28r+45)}$	$\frac{1}{192}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\frac{3}{2}} \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^{\frac{3}{2}} \left(1 - \frac{x^2}{8} + \frac{x^4}{192} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 2$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_2(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_2 \\ &= -\frac{4}{4r^2 + 12r + 5} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\to r_2} -\frac{4}{4r^2 + 12r + 5} &= \lim_{r\to -\frac{1}{2}} -\frac{4}{4r^2 + 12r + 5} \\ &= \textit{undefined} \end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$\begin{aligned} \frac{d}{dx} y_2(x) &= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\ &= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) \\ \frac{d^2}{dx^2} y_2(x) &= C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} \\ &\quad + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\ &= C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right) \end{aligned}$$

Substituting these back into the given ode $4x^2y'' + (4x^2 - 3)\, y = 0$ gives

$$4x^2 \left(Cy_1''(x) \ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2} (n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2} (n+r_2)}{x^2} \right) \right) + (4x^2 - 3) \left(Cy_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2} \right) \right) = 0$$

Which can be written as

$$\left(\left(4x^2 y_1''(x) + (4x^2 - 3)\, y_1(x) \right) \ln(x) + 4x^2 \left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2} \right) \right) C + 4x^2 \left(\sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2} (n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2} (n+r_2)}{x^2} \right) \right) + (4x^2 - 3) \left(\sum_{n=0}^{\infty} b_n x^{n+r_2} \right) = 0 \tag{7}$$

But since $y_1(x)$ is a solution to the ode, then

$$4x^2 y_1''(x) + (4x^2 - 3)\, y_1(x) = 0$$

Eq (7) simplifes to

$$4x^2 \left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2} \right) C + 4x^2 \left(\sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2} (n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2} (n+r_2)}{x^2} \right) \right) + (4x^2 - 3) \left(\sum_{n=0}^{\infty} b_n x^{n+r_2} \right) = 0 \tag{8}$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\left(8 \left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n (n+r_1) \right) x - 4 \left(\sum_{n=0}^{\infty} a_n x^{n+r_1} \right) \right) C + 4 \left(\sum_{n=0}^{\infty} b_n x^{n+r_2} \right) x^2 + 4 \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2) (-1+n+r_2) \right) x^2 - 3 \left(\sum_{n=0}^{\infty} b_n x^{n+r_2} \right) = 0 \tag{9}$$

Since $r_1 = \frac{3}{2}$ and $r_2 = -\frac{1}{2}$ then the above becomes

$$
\begin{aligned}
&\left(8\left(\sum_{n=0}^{\infty} x^{\frac{1}{2}+n} a_n \left(n+\frac{3}{2}\right)\right) x - 4\left(\sum_{n=0}^{\infty} a_n x^{n+\frac{3}{2}}\right)\right) C + 4\left(\sum_{n=0}^{\infty} b_n x^{n-\frac{1}{2}}\right) x^2 \\
&+ 4\left(\sum_{n=0}^{\infty} x^{-\frac{5}{2}+n} b_n \left(n-\frac{1}{2}\right)\left(-\frac{3}{2}+n\right)\right) x^2 - 3\left(\sum_{n=0}^{\infty} b_n x^{n-\frac{1}{2}}\right) = 0
\end{aligned} \tag{10}
$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} (8n+12)\, C a_n x^{n+\frac{3}{2}}\right) + \sum_{n=0}^{\infty} \left(-4 C a_n x^{n+\frac{3}{2}}\right) + \left(\sum_{n=0}^{\infty} 4 x^{n+\frac{3}{2}} b_n\right) \\
&+ \left(\sum_{n=0}^{\infty} x^{n-\frac{1}{2}} b_n \left(4n^2 - 8n + 3\right)\right) + \sum_{n=0}^{\infty} \left(-3 b_n x^{n-\frac{1}{2}}\right) = 0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n - \frac{1}{2}$ in each summation term. Going over each summation term above with power of x in it which is not already $x^{n-\frac{1}{2}}$ and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} (8n+12)\, C a_n x^{n+\frac{3}{2}} &= \sum_{n=2}^{\infty} C a_{n-2} (8n-4)\, x^{n-\frac{1}{2}} \\
\sum_{n=0}^{\infty} \left(-4 C a_n x^{n+\frac{3}{2}}\right) &= \sum_{n=2}^{\infty} \left(-4 C a_{n-2} x^{n-\frac{1}{2}}\right) \\
\sum_{n=0}^{\infty} 4 x^{n+\frac{3}{2}} b_n &= \sum_{n=2}^{\infty} 4 b_{n-2} x^{n-\frac{1}{2}}
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n - \frac{1}{2}$.

$$
\begin{aligned}
&\left(\sum_{n=2}^{\infty} C a_{n-2} (8n-4)\, x^{n-\frac{1}{2}}\right) + \sum_{n=2}^{\infty} \left(-4 C a_{n-2} x^{n-\frac{1}{2}}\right) + \left(\sum_{n=2}^{\infty} 4 b_{n-2} x^{n-\frac{1}{2}}\right) \\
&+ \left(\sum_{n=0}^{\infty} x^{n-\frac{1}{2}} b_n \left(4n^2 - 8n + 3\right)\right) + \sum_{n=0}^{\infty} \left(-3 b_n x^{n-\frac{1}{2}}\right) = 0
\end{aligned} \tag{2B}
$$

For $n = 0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0 = 1$. For $n = 1$, Eq (2B) gives

$$-4b_1 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-4b_1 = 0$$

Solving the above for b_1 gives

$$b_1 = 0$$

For $n = N$, where $N = 2$ which is the difference between the two roots, we are free to choose $b_2 = 0$. Hence for $n = 2$, Eq (2B) gives

$$8C + 4 = 0$$

Which is solved for C. Solving for C gives

$$C = -\frac{1}{2}$$

For $n = 3$, Eq (2B) gives

$$16Ca_1 + 4b_1 + 12b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$12b_3 = 0$$

Solving the above for b_3 gives

$$b_3 = 0$$

For $n = 4$, Eq (2B) gives

$$24Ca_2 + 4b_2 + 32b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$32b_4 + \frac{3}{2} = 0$$

Solving the above for b_4 gives

$$b_4 = -\frac{3}{64}$$

For $n = 5$, Eq (2B) gives

$$32Ca_3 + 4b_3 + 60b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$60b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = 0$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -\frac{1}{2}$ and all b_n, then the second solution becomes

$$y_2(x) = -\frac{1}{2}\left(x^{\frac{3}{2}}\left(1 - \frac{x^2}{8} + \frac{x^4}{192} + O(x^6)\right)\right)\ln(x) + \frac{1 - \frac{3x^4}{64} + O(x^6)}{\sqrt{x}}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^{\frac{3}{2}}\left(1 - \frac{x^2}{8} + \frac{x^4}{192} + O(x^6)\right) \\
&\quad + c_2\left(-\frac{1}{2}\left(x^{\frac{3}{2}}\left(1 - \frac{x^2}{8} + \frac{x^4}{192} + O(x^6)\right)\right)\ln(x) + \frac{1 - \frac{3x^4}{64} + O(x^6)}{\sqrt{x}}\right)
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= c_1 x^{\frac{3}{2}}\left(1 - \frac{x^2}{8} + \frac{x^4}{192} + O(x^6)\right) \\
&\quad + c_2\left(-\frac{x^{\frac{3}{2}}\left(1 - \frac{x^2}{8} + \frac{x^4}{192} + O(x^6)\right)\ln(x)}{2} + \frac{1 - \frac{3x^4}{64} + O(x^6)}{\sqrt{x}}\right)
\end{aligned}$$

<u>Summary</u>

The solution(s) found are the following

$$\begin{aligned}
y &= c_1 x^{\frac{3}{2}}\left(1 - \frac{x^2}{8} + \frac{x^4}{192} + O(x^6)\right) \\
&\quad + c_2\left(-\frac{x^{\frac{3}{2}}\left(1 - \frac{x^2}{8} + \frac{x^4}{192} + O(x^6)\right)\ln(x)}{2} + \frac{1 - \frac{3x^4}{64} + O(x^6)}{\sqrt{x}}\right)
\end{aligned} \tag{1}$$

Verification of solutions

$$y = c_1 x^{\frac{3}{2}} \left(1 - \frac{x^2}{8} + \frac{x^4}{192} + O(x^6)\right) + c_2 \left(-\frac{x^{\frac{3}{2}} \left(1 - \frac{x^2}{8} + \frac{x^4}{192} + O(x^6)\right) \ln(x)}{2} + \frac{1 - \frac{3x^4}{64} + O(x^6)}{\sqrt{x}}\right)$$

Verified OK.

20.14.1 Maple step by step solution

Let's solve

$4x^2y'' + (4x^2 - 3)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(4x^2-3)y}{4x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(4x^2-3)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = 0, P_3(x) = \frac{4x^2-3}{4x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 0$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = -\frac{3}{4}$

  ◦ $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $4x^2y'' + (4x^2 - 3)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- [ ] Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- > k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r) (k + r - 1) x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1 + 2r) (-3 + 2r) x^r + a_1(3 + 2r) (-1 + 2r) x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k + 2r + 1) (2k + 2r - 3) + 4a_{\ldots}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1 + 2r) (-3 + 2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{3}{2}\right\}$

- Each term must be 0

  $a_1(3 + 2r) (-1 + 2r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $4\left(k + r - \frac{3}{2}\right) \left(k + r + \frac{1}{2}\right) a_k + 4a_{k-2} = 0$

- Shift index using $k- > k + 2$

  $4\left(k + r + \frac{1}{2}\right) \left(k + \frac{5}{2} + r\right) a_{k+2} + 4a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4a_k}{(2k+2r+1)(2k+5+2r)}$

- Recursion relation for $r = -\frac{1}{2}$

$a_{k+2} = -\frac{2a_k}{k(2k+4)}$

- Solution for $r = -\frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{2a_k}{k(2k+4)}, a_1 = 0\right]$$

- Recursion relation for $r = \frac{3}{2}$

$a_{k+2} = -\frac{4a_k}{(2k+4)(2k+8)}$

- Solution for $r = \frac{3}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+2} = -\frac{4a_k}{(2k+4)(2k+8)}, a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{2}}\right), a_{2+k} = -\frac{2a_k}{k(2k+4)}, a_1 = 0, b_{2+k} = -\frac{4b_k}{(2k+4)(2k+8)}, b_1 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 49

```
Order:=6;
dsolve(diff(y(x),x$2)+(1-3/(4*x^2))*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_1 x^2 \left(1 - \frac{1}{8}x^2 + \frac{1}{192}x^4 + \mathrm{O}\left(x^6\right)\right) + c_2 \left(\ln\left(x\right)\left(x^2 - \frac{1}{8}x^4 + \mathrm{O}\left(x^6\right)\right) + \left(-2 + \frac{3}{32}x^4 + \mathrm{O}\left(x^6\right)\right)\right)}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 72

```
AsymptoticDSolveValue[y''[x]+(1-3/(4*x^2))*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2 \left(\frac{x^{11/2}}{192} - \frac{x^{7/2}}{8} + x^{3/2}\right) + c_1 \left(\frac{1}{16} x^{3/2} \left(x^2 - 8\right) \log(x) - \frac{5x^4 - 16x^2 - 64}{64\sqrt{x}}\right)$$