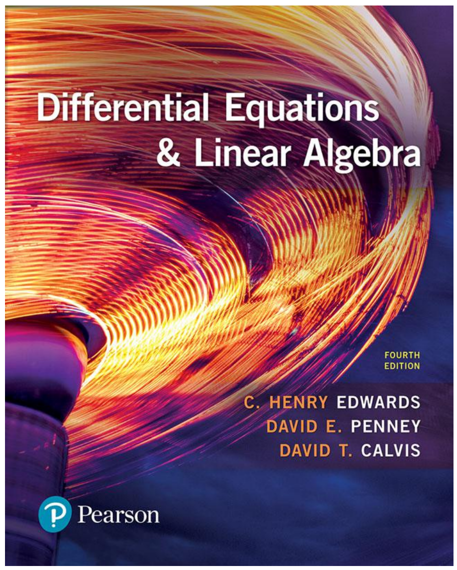
A Solution Manual For

Differential equations and linear algebra, 4th ed., Edwards and Penney



Nasser M. Abbasi

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Contents

1 Section 5.2, Higher-Order Linear Differential Equations. General solutions of Linear Equations. Page 288

1.1 problem problem 38

Internal problem ID [278]
Internal file name [`OUTPUT/278_Sunday_June_05_2022_01_38_11_AM_41602886/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 5.2, Higher-Order Linear Differential Equations. General solutions of Linear Equations. Page 288
Problem number: problem 38.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler], [_2nd_order, _linear, `_with_symmetry_[0,F(
   x)]`]]
```

$$\boxed{x^2y'' + y'x - 9y = 0}$$

Given that one solution of the ode is

$$y_1 = x^3$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1\left(\int \frac{e^{-(\int p dx)}}{y_1^2} dx\right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y'' + p(x)\, y' + q(x)\, y = f(x)$$

Looking at the ode to solve shows that

$$p(x) = \frac{1}{x}$$

Therefore

$$y_2(x) = x^3 \left(\int \frac{e^{-\left(\int \frac{1}{x} dx\right)}}{x^6} dx \right)$$

$$y_2(x) = x^3 \int \frac{\frac{1}{x}}{x^6}, dx$$

$$y_2(x) = x^3 \left(\int \frac{1}{x^7} dx \right)$$

$$y_2(x) = -\frac{1}{6x^3}$$

Hence the solution is

$$\begin{aligned} y &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^3 - \frac{c_2}{6x^3} \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 x^3 - \frac{c_2}{6x^3} \tag{1}$$

Verification of solutions

$$y = c_1 x^3 - \frac{c_2}{6x^3}$$

Verified OK.

1.1.1 Maple step by step solution

Let's solve

$x^2 y'' + y' x - 9y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y'}{x} + \frac{9y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} - \frac{9y}{x^2} = 0$

- Multiply by denominators of the ODE

  $x^2y'' + y'x - 9y = 0$

- Make a change of variables

  $t = \ln(x)$

☐ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right)t'(x)$

  - Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2}y(t)\right)t'(x)^2 + t''(x)\left(\frac{d}{dt}y(t)\right)$

  - Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) + \frac{d}{dt}y(t) - 9y(t) = 0$

- Simplify

  $\frac{d^2}{dt^2}y(t) - 9y(t) = 0$

- Characteristic polynomial of ODE

  $r^2 - 9 = 0$

- Factor the characteristic polynomial

  $(r-3)(r+3) = 0$

- Roots of the characteristic polynomial

  $r = (-3, 3)$

- 1st solution of the ODE

  $y_1(t) = e^{-3t}$

- 2nd solution of the ODE

  $y_2(t) = e^{3t}$

- General solution of the ODE

$y(t) = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y(t) = c_1 \mathrm{e}^{-3t} + c_2 \mathrm{e}^{3t}$

- Change variables back using $t = \ln(x)$

  $y = \frac{c_1}{x^3} + c_2 x^3$

- Simplify

  $y = \frac{c_1}{x^3} + c_2 x^3$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve([x^2*diff(y(x),x$2)+x*diff(y(x),x)-9*y(x)=0,x^3],singsol=all)
```

$$y(x) = \frac{c_2 x^6 + c_1}{x^3}$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 18

```
DSolve[x^2*y''[x]+x*y'[x]-9*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_2 x^6 + c_1}{x^3}$$

1.2 problem problem 39

Internal problem ID [279]
Internal file name [OUTPUT/279_Sunday_June_05_2022_01_38_11_AM_2821455/index.tex]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 5.2, Higher-Order Linear Differential Equations. General solutions of Linear Equations. Page 288
Problem number: problem 39.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{4y'' - 4y' + y = 0}$$

Given that one solution of the ode is

$$y_1 = \mathrm{e}^{\frac{x}{2}}$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1 \left(\int \frac{\mathrm{e}^{-(\int p dx)}}{y_1^2} dx \right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y'' + p(x)\, y' + q(x)\, y = f(x)$$

Looking at the ode to solve shows that

$$p(x) = -1$$

Therefore

$$y_2(x) = e^{\frac{x}{2}} \left(\int e^{-\left(\int (-1) dx\right)} e^{-x} dx \right)$$

$$y_2(x) = e^{\frac{x}{2}} \int \frac{e^{x}}{e^{x}}, dx$$

$$y_2(x) = e^{\frac{x}{2}} \left(\int 1 dx \right)$$

$$y_2(x) = e^{\frac{x}{2}} x$$

Hence the solution is

$$\begin{aligned} y &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 e^{\frac{x}{2}} + c_2 e^{\frac{x}{2}} x \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{\frac{x}{2}} + c_2 e^{\frac{x}{2}} x \tag{1}$$

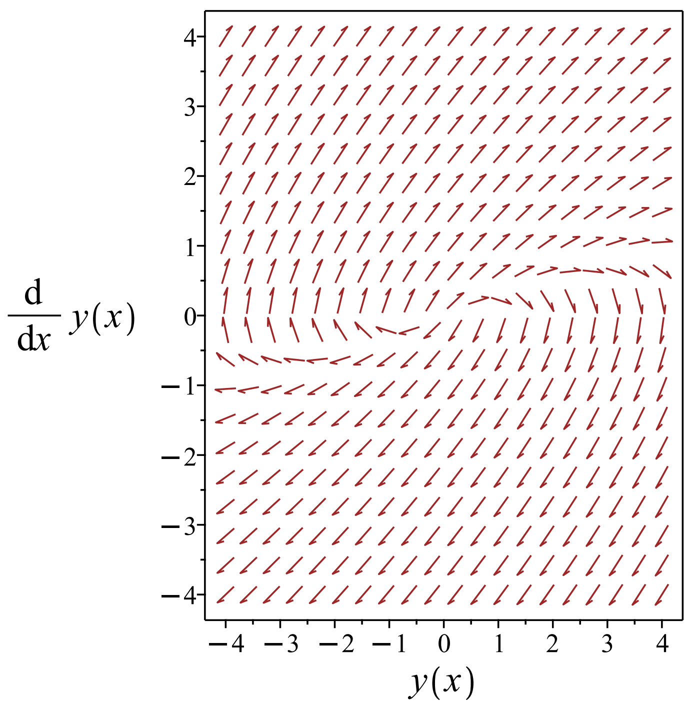


Figure 1: Slope field plot

Verification of solutions

$$y = c_1 e^{\frac{x}{2}} + c_2 e^{\frac{x}{2}} x$$

Verified OK.

1.2.1 Maple step by step solution

Let's solve

$4y'' - 4y' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = y' - \frac{y}{4}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - y' + \frac{y}{4} = 0$

- Characteristic polynomial of ODE

  $r^2 - r + \frac{1}{4} = 0$

- Factor the characteristic polynomial

  $\frac{(2r-1)^2}{4} = 0$

- Root of the characteristic polynomial

  $r = \frac{1}{2}$

- 1st solution of the ODE

  $y_1(x) = e^{\frac{x}{2}}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = e^{\frac{x}{2}} x$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{\frac{x}{2}} + c_2 e^{\frac{x}{2}} x$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve([4*diff(y(x),x$2)-4*diff(y(x),x)+y(x)=0,exp(x/2)],singsol=all)
```

$$y(x) = e^{\frac{x}{2}}(c_2x + c_1)$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 20

```
DSolve[4*y''[x]-4*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{x/2}(c_2x + c_1)$$

1.3 problem problem 40

Internal problem ID [280]
Internal file name [`OUTPUT/280_Sunday_June_05_2022_01_38_12_AM_3318965/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 5.2, Higher-Order Linear Differential Equations. General solutions of Linear Equations. Page 288
Problem number: problem 40.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "second_order_change_of_variable_on_y_method_1", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - x(2+x)\,y' + (2+x)\,y = 0}$$

Given that one solution of the ode is

$$y_1 = x$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1\left(\int \frac{\mathrm{e}^{-(\int p dx)}}{y_1^2} dx\right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y'' + p(x)\,y' + q(x)\,y = f(x)$$

Looking at the ode to solve shows that

$$p(x) = \frac{-x^2 - 2x}{x^2}$$

Therefore

$$y_2(x) = x\left(\int \frac{e^{-\left(\int \frac{-x^2-2x}{x^2}dx\right)}}{x^2}dx\right)$$

$$y_2(x) = x\int \frac{e^{x+2\ln(x)}}{x^2}, dx$$

$$y_2(x) = x\left(\int e^x dx\right)$$

$$y_2(x) = x\,e^x$$

Hence the solution is

$$\begin{aligned} y &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x + c_2 x\,e^x \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x + c_2 x\,e^x \tag{1}$$

Verification of solutions

$$y = c_1 x + c_2 x\,e^x$$

Verified OK.

1.3.1 Maple step by step solution

Let's solve

$x^2 y'' + (-x^2 - 2x)\,y' + (2 + x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(2+x)y}{x^2} + \frac{(2+x)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(2+x)y'}{x} + \frac{(2+x)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

$\left[P_2(x)=-\frac{2+x}{x}, P_3(x)=\frac{2+x}{x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x=0$

  $$\left.\left(x \cdot P_2(x)\right)\right|_{x=0}=-2$$

- $x^2 \cdot P_3(x)$ is analytic at $x=0$

  $$\left.\left(x^2 \cdot P_3(x)\right)\right|_{x=0}=2$$

- $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $$x^2y''-x(2+x)\,y'+(2+x)\,y=0$$

- Assume series solution for y

  $$y=\sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m=0..1$

  $$x^m \cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k->k-m$

  $$x^m \cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m=1..2$

  $$x^m \cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

  $$x^m \cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$$

  Rewrite ODE with series expansions

$$a_0(-1+r)(-2+r)x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r-1)(k+r-2) - a_{k-1}(k+r-2)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)(-2+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{1, 2\}$
- Each term in the series must be 0, giving the recursion relation

  $(k+r-2)(a_k(k+r-1) - a_{k-1}) = 0$
- Shift index using $k- >k+1$

  $(k+r-1)(a_{k+1}(k+r) - a_k) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k}{k+r}$
- Recursion relation for $r = 1$

  $a_{k+1} = \frac{a_k}{k+1}$
- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = \frac{a_k}{k+1}\right]$
- Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_k}{k+2}$
- Solution for $r = 2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = \frac{a_k}{k+2}\right]$
- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+1} = \frac{a_k}{k+1}, b_{k+1} = \frac{b_k}{k+2}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 12

```
dsolve([x^2*diff(y(x),x$2)-x*(x+2)*diff(y(x),x)+(x+2)*y(x)=0,x],singsol=all)
```

$$y(x) = x(c_1 + e^x c_2)$$

✓ Solution by Mathematica
Time used: 0.029 (sec). Leaf size: 16

```
DSolve[x^2*y''[x]-x*(x+2)*y'[x]+(x+2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x(c_2 e^x + c_1)$$

1.4 problem problem 41

Internal problem ID [281]
Internal file name [`OUTPUT/281_Sunday_June_05_2022_01_38_12_AM_1218278/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 5.2, Higher-Order Linear Differential Equations. General solutions of Linear Equations. Page 288
Problem number: problem 41.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x+1)\,y''-(2+x)\,y'+y=0}$$

Given that one solution of the ode is

$$y_1 = \mathrm{e}^{x}$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1\left(\int \frac{\mathrm{e}^{-\left(\int p dx\right)}}{y_1^2}dx\right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y''+p(x)\,y'+q(x)\,y=f(x)$$

Looking at the ode to solve shows that

$$p(x) = \frac{-x-2}{x+1}$$

Therefore

$$y_2(x) = \mathrm{e}^x \left(\int \mathrm{e}^{-\left(\int \frac{-x-2}{x+1} dx\right)} \mathrm{e}^{-2x} dx \right)$$

$$y_2(x) = \mathrm{e}^x \int \frac{\mathrm{e}^{x+\ln(x+1)}}{\mathrm{e}^{2x}}, dx$$

$$y_2(x) = \mathrm{e}^x \left(\int (x+1)\,\mathrm{e}^{-x} dx \right)$$

$$y_2(x) = -\mathrm{e}^x \mathrm{e}^{-x}(2+x)$$

Hence the solution is

$$\begin{aligned} y &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 \mathrm{e}^x - c_2 \mathrm{e}^x \mathrm{e}^{-x}(2+x) \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1 \mathrm{e}^x - c_2 \mathrm{e}^x \mathrm{e}^{-x}(2+x) \tag{1}$$

<u>Verification of solutions</u>

$$y = c_1 \mathrm{e}^x - c_2 \mathrm{e}^x \mathrm{e}^{-x}(2+x)$$

Verified OK.

1.4.1 Maple step by step solution

Let's solve

$(x+1)\,y'' + (-x-2)\,y' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y}{x+1} + \frac{(2+x)y'}{x+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(2+x)y'}{x+1} + \frac{y}{x+1} = 0$

□ Check to see if $x_0 = -1$ is a regular singular point

  ○ Define functions

$\left[P_2(x) = -\frac{2+x}{x+1}, P_3(x) = \frac{1}{x+1}\right]$

- ○ $(x+1) \cdot P_2(x)$ is analytic at $x = -1$

  $$\left((x+1) \cdot P_2(x)\right)\bigg|_{x=-1} = -1$$

- ○ $(x+1)^2 \cdot P_3(x)$ is analytic at $x = -1$

  $$\left((x+1)^2 \cdot P_3(x)\right)\bigg|_{x=-1} = 0$$

- ○ $x = -1$is a regular singular point

  Check to see if $x_0 = -1$ is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $$(x+1)\, y'' + (-x-2)\, y' + y = 0$$

- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

  $$u\left(\frac{d^2}{du^2}y(u)\right) + (-u-1)\left(\frac{d}{du}y(u)\right) + y(u) = 0$$

- Assume series solution for $y(u)$

  $$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

☐ Rewrite ODE with series expansions

- ○ Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1+m}$$

- ○ Shift index using $k- >k+1-m$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- ○ Convert $u \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

  $$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, u^{k+r-1}$$

- ○ Shift index using $k- >k+1$

  $$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, u^{k+r}$$

  Rewrite ODE with series expansions

$$a_0 r(-2+r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(k+r-1) - a_k(k+r-1)\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\,(a_{k+1}(k+1+r) - a_k) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k}{k+1+r}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k}{k+1}$

- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k}{k+1}\right]$

- Revert the change of variables $u = x+1$

  $\left[y = \sum_{k=0}^{\infty} a_k (x+1)^k, a_{k+1} = \frac{a_k}{k+1}\right]$

- Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_k}{k+3}$

- Solution for $r = 2$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$

- Revert the change of variables $u = x+1$

  $\left[y = \sum_{k=0}^{\infty} a_k (x+1)^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k (x+1)^k\right) + \left(\sum_{k=0}^{\infty} b_k (x+1)^{k+2}\right), a_{k+1} = \frac{a_k}{k+1}, b_{k+1} = \frac{b_k}{k+3}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve([(x+1)*diff(y(x),x$2)-(x+2)*diff(y(x),x)+y(x)=0,exp(x)],singsol=all)
```

$$y(x) = c_1(2 + x) + e^x c_2$$

✓ Solution by Mathematica

Time used: 0.143 (sec). Leaf size: 29

```
DSolve[(x+1)*y''[x]-(x+2)*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_1 e^{x+1} - 2c_2(x + 2)}{\sqrt{2e}}$$

1.5 problem problem 42

Internal problem ID [282]
Internal file name [OUTPUT/282_Sunday_June_05_2022_01_38_13_AM_3169935/index.tex]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 5.2, Higher-Order Linear Differential Equations. General solutions of Linear Equations. Page 288
Problem number: problem 42.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[_Gegenbauer]
```

$$\boxed{\left(-x^{2}+1\right) y''+2y'x-2y=0}$$

Given that one solution of the ode is

$$y_1 = x$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1 \left(\int \frac{\mathrm{e}^{-\left(\int p dx\right)}}{y_1^2} dx\right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y'' + p(x)\, y' + q(x)\, y = f(x)$$

Looking at the ode to solve shows that

$$p(x) = \frac{2x}{-x^2+1}$$

Therefore

$$y_2(x) = x\left(\int \frac{\mathrm{e}^{-\left(\int \frac{2x}{-x^2+1}dx\right)}}{x^2}dx\right)$$

$$y_2(x) = x\int \frac{\mathrm{e}^{\ln(x-1)+\ln(x+1)}}{x^2}, dx$$

$$y_2(x) = x\left(\int \frac{x^2-1}{x^2}dx\right)$$

$$y_2(x) = x\left(x+\frac{1}{x}\right)$$

Hence the solution is

$$\begin{aligned} y &= c_1y_1(x) + c_2y_2(x) \\ &= c_1x + c_2x\left(x+\frac{1}{x}\right) \end{aligned}$$

<u>Summary</u>

The solution(s) found are the following

$$y = c_1x + c_2x\left(x+\frac{1}{x}\right) \tag{1}$$

<u>Verification of solutions</u>

$$y = c_1x + c_2x\left(x+\frac{1}{x}\right)$$

Verified OK.

1.5.1 Maple step by step solution

Let's solve

$(-x^2+1)\,y'' + 2y'x - 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{2xy'}{x^2-1} - \frac{2y}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' - \frac{2xy'}{x^2-1} + \frac{2y}{x^2-1} = 0$

☐ Check to see if x_0 is a regular singular point

- Define functions

  $\left[P_2(x) = -\frac{2x}{x^2-1}, P_3(x) = \frac{2}{x^2-1}\right]$

- $(x+1) \cdot P_2(x)$ is analytic at $x = -1$

  $$\left((x+1) \cdot P_2(x)\right)\Big|_{x=-1} = -1$$

- $(x+1)^2 \cdot P_3(x)$ is analytic at $x = -1$

  $$\left((x+1)^2 \cdot P_3(x)\right)\Big|_{x=-1} = 0$$

- $x = -1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $(x^2-1)\, y'' - 2y'x + 2y = 0$

- Change variables using $x = u-1$ so that the regular singular point is at $u = 0$

  $(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right) + (-2u+2)\left(\frac{d}{du}y(u)\right) + 2y(u) = 0$

- Assume series solution for $y(u)$

  $$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$-2a_0 r(-2+r)\,u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2a_{k+1}(k+1+r)(k+r-1) + a_k(k+r-1)(k+r-2)\right)u^{k+r}\right)$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\left((-2k-2r-2)\,a_{k+1} + a_k(k+r-2)\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r-2)}{2(k+1+r)}$

- Recursion relation for $r = 0$; series terminates at $k = 2$

  $a_{k+1} = \frac{a_k(k-2)}{2(k+1)}$

- Apply recursion relation for $k = 0$

  $a_1 = -a_0$

- Apply recursion relation for $k = 1$

  $a_2 = -\frac{a_1}{4}$

- Express in terms of a_0

  $a_2 = \frac{a_0}{4}$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot \left(1 - u + \frac{1}{4}u^2\right)$

- Revert the change of variables $u = x + 1$

  $\left[y = \frac{a_0(x-1)^2}{4}\right]$

- Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_k k}{2(k+3)}$

- Solution for $r = 2$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = \frac{a_k k}{2(k+3)}\right]$

- Revert the change of variables $u = x + 1$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+1)^{k+2}, a_{k+1} = \frac{a_k k}{2(k+3)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \frac{a_0 (x-1)^2}{4} + \left(\sum_{k=0}^{\infty} b_k (x+1)^{k+2}\right), b_{k+1} = \frac{b_k k}{2(k+3)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve([(1-x^2)*diff(y(x),x$2)+2*x*diff(y(x),x)-2*y(x)=0,x],singsol=all)
```

$$y(x) = c_2 x^2 + c_1 x + c_2$$

✓ Solution by Mathematica

Time used: 0.08 (sec). Leaf size: 39

```
DSolve[(1-x^2)*y''[x]+2*x*y'[x]-2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\sqrt{x^2-1}(c_1(x-1)^2 + c_2 x)}{\sqrt{1-x^2}}$$

1.6 problem problem 43

Internal problem ID [283]
Internal file name [`OUTPUT/283_Sunday_June_05_2022_01_38_13_AM_28435787/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 5.2, Higher-Order Linear Differential Equations. General solutions of Linear Equations. Page 288
Problem number: problem 43.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[_Gegenbauer]
```

$$\boxed{\left(-x^{2}+1\right) y''-2y'x+2y=0}$$

Given that one solution of the ode is

$$y_1 = x$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1\left(\int \frac{\mathrm{e}^{-\left(\int p dx\right)}}{y_1^2} dx\right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y''+p(x)\, y'+q(x)\, y=f(x)$$

Looking at the ode to solve shows that

$$p(x) = -\frac{2x}{-x^{2}+1}$$

Therefore

$$y_2(x) = x\left(\int \frac{e^{-\left(\int -\frac{2x}{-x^2+1}dx\right)}}{x^2}dx\right)$$

$$y_2(x) = x\int \frac{e^{-\ln(x-1)-\ln(x+1)}}{x^2}, dx$$

$$y_2(x) = x\left(\int \frac{1}{x^2\left(x^2-1\right)}dx\right)$$

$$y_2(x) = x\left(-\frac{\ln\left(x+1\right)}{2}+\frac{1}{x}+\frac{\ln\left(x-1\right)}{2}\right)$$

Hence the solution is

$$\begin{aligned} y &= c_1y_1(x)+c_2y_2(x) \\ &= c_1x+c_2x\left(-\frac{\ln\left(x+1\right)}{2}+\frac{1}{x}+\frac{\ln\left(x-1\right)}{2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1x+c_2x\left(-\frac{\ln\left(x+1\right)}{2}+\frac{1}{x}+\frac{\ln\left(x-1\right)}{2}\right) \tag{1}$$

Verification of solutions

$$y = c_1x+c_2x\left(-\frac{\ln\left(x+1\right)}{2}+\frac{1}{x}+\frac{\ln\left(x-1\right)}{2}\right)$$

Verified OK.

1.6.1 Maple step by step solution

Let's solve

$\left(-x^2+1\right)y''-2y'x+2y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{2xy'}{x^2-1}+\frac{2y}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2xy'}{x^2-1} - \frac{2y}{x^2-1} = 0$

☐ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{2x}{x^2-1}, P_3(x) = -\frac{2}{x^2-1}\right]$

  - $(x+1) \cdot P_2(x)$ is analytic at $x = -1$

    $\left((x+1) \cdot P_2(x)\right)\bigg|_{x=-1} = 1$

  - $(x+1)^2 \cdot P_3(x)$ is analytic at $x = -1$

    $\left((x+1)^2 \cdot P_3(x)\right)\bigg|_{x=-1} = 0$

  - $x = -1$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = -1$

- Multiply by denominators

  $(x^2-1)\, y'' + 2y'x - 2y = 0$

- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

  $(u^2 - 2u)\left(\frac{d^2}{du^2}y(u)\right) + (2u-2)\left(\frac{d}{du}y(u)\right) - 2y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

  - Shift index using $k- >k+1-m$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

  - Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

    $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$

- Shift index using $k- >k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)u^{k+r}$

  Rewrite ODE with series expansions

  $-2a_0r^2u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2a_{k+1}(k+1+r)^2 + a_k(k+r+2)(k+r-1)\right)u^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r^2 = 0$

- Values of r that satisfy the indicial equation

  $r = 0$

- Each term in the series must be 0, giving the recursion relation

  $-2a_{k+1}(k+1)^2 + a_k(k+2)(k-1) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+2)(k-1)}{2(k+1)^2}$

- Recursion relation for $r = 0$; series terminates at $k = 1$

  $a_{k+1} = \frac{a_k(k+2)(k-1)}{2(k+1)^2}$

- Apply recursion relation for $k = 0$

  $a_1 = -a_0$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot (-u+1)$

- Revert the change of variables $u = x+1$

  $[y = -a_0x]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 25

```
dsolve([(1-x^2)*diff(y(x),x$2)-2*x*diff(y(x),x)+2*y(x)=0,x],singsol=all)
```

$$y(x) = \frac{c_2 \ln(x-1)\, x}{2} - \frac{c_2 \ln(x+1)\, x}{2} + c_1 x + c_2$$

✓ Solution by Mathematica
Time used: 0.021 (sec). Leaf size: 33

```
DSolve[(1-x^2)*y''[x]-2*x*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 x - \frac{1}{2} c_2 (x \log(1-x) - x \log(x+1) + 2)$$

1.7 problem problem 44

Internal problem ID [284]
Internal file name [`OUTPUT/284_Sunday_June_05_2022_01_38_14_AM_92845526/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 5.2, Higher-Order Linear Differential Equations. General solutions of Linear Equations. Page 288
Problem number: problem 44.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + y'x + \left(x^2 - \frac{1}{4}\right) y = 0}$$

Given that one solution of the ode is

$$y_1 = \frac{\cos(x)}{\sqrt{x}}$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1 \left(\int \frac{\mathrm{e}^{-(\int p dx)}}{y_1^2} dx \right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y'' + p(x)\, y' + q(x)\, y = f(x)$$

Looking at the ode to solve shows that

$$p(x) = \frac{1}{x}$$

Therefore

$$y_2(x) = \frac{\cos(x)\left(\int \frac{e^{-\left(\int \frac{1}{x}dx\right)}x}{\cos(x)^2}dx\right)}{\sqrt{x}}$$

$$y_2(x) = \frac{\cos(x)}{\sqrt{x}} \int \frac{\frac{1}{x}}{\frac{\cos(x)^2}{x}}, dx$$

$$y_2(x) = \frac{\cos(x)\left(\int \sec(x)^2 \, dx\right)}{\sqrt{x}}$$

$$y_2(x) = \frac{\cos(x)\tan(x)}{\sqrt{x}}$$

Hence the solution is

$$\begin{aligned} y &= c_1 y_1(x) + c_2 y_2(x) \\ &= \frac{\cos(x)\, c_1}{\sqrt{x}} + \frac{c_2 \cos(x)\tan(x)}{\sqrt{x}} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{\cos(x)\, c_1}{\sqrt{x}} + \frac{c_2 \cos(x)\tan(x)}{\sqrt{x}} \tag{1}$$

Verification of solutions

$$y = \frac{\cos(x)\, c_1}{\sqrt{x}} + \frac{c_2 \cos(x)\tan(x)}{\sqrt{x}}$$

Verified OK.

1.7.1 Maple step by step solution

Let's solve

$x^2y'' + y'x + \left(x^2 - \frac{1}{4}\right)y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(4x^2-1)y}{4x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$y'' + \frac{y'}{x} + \frac{(4x^2-1)y}{4x^2} = 0$$

☐ Check to see if $x_0 = 0$ is a regular singular point

- ○ Define functions

$$\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{4x^2-1}{4x^2}\right]$$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

$$\left.(x \cdot P_2(x))\right|_{x=0} = 1$$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left.(x^2 \cdot P_3(x))\right|_{x=0} = -\frac{1}{4}$$

- ○ $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

$$4x^2y'' + 4y'x + (4x^2-1)\,y = 0$$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- ○ Shift index using $k->k-m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- ○ Convert $x \cdot y'$ to series expansion

$$x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$$

- ○ Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+2r)(-1+2r)x^r + a_1(3+2r)(1+2r)x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k+2r+1)(2k+2r-1) + 4a_{k-}\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)(-1+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$
- Each term must be 0

  $a_1(3+2r)(1+2r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2 + 8kr + 4r^2 - 1) + 4a_{k-2} = 0$
- Shift index using $k- >k+2$

  $a_{k+2}\left(4(k+2)^2 + 8(k+2)r + 4r^2 - 1\right) + 4a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4a_k}{4k^2+8kr+4r^2+16k+16r+15}$
- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+12k+8}$
- Solution for $r = -\frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0\right]$
- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+20k+24}$
- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+20k+24}, a_1 = 0\right]$
- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0, b_{k+2} = -\frac{4b_k}{4k^2+20k+24}, b_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 17

```
dsolve([x^2*diff(y(x),x$2)+x*diff(y(x),x)+(x^2-1/4)*y(x)=0,x^(-1/2)*cos(x)],singsol=all)
```

$$y(x) = \frac{c_1 \sin(x) + c_2 \cos(x)}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.018 (sec). Leaf size: 33

```
DSolve[(1-x^2)*y''[x]-2*x*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 x - \frac{1}{2} c_2 (x \log(1-x) - x \log(x+1) + 2)$$

2 Section 5.3, Higher-Order Linear Differential Equations. Homogeneous Equations with Constant Coefficients. Page 300

2.1 problem problem 10

Internal problem ID [285]
Internal file name [`OUTPUT/285_Sunday_June_05_2022_01_38_15_AM_78402842/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 5.3, Higher-Order Linear Differential Equations. Homogeneous Equations with Constant Coefficients. Page 300
Problem number: problem 10.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{5y'''' + 3y''' = 0}$$

The characteristic equation is

$$5\lambda^4 + 3\lambda^3 = 0$$

The roots of the above equation are

$$\begin{aligned}\lambda_1 &= -\frac{3}{5}\\ \lambda_2 &= 0\\ \lambda_3 &= 0\\ \lambda_4 &= 0\end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = c_3x^2 + c_2x + c_1 + e^{-\frac{3x}{5}}c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned}y_1 &= 1\\ y_2 &= x\\ y_3 &= x^2\\ y_4 &= e^{-\frac{3x}{5}}\end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_3 x^2 + c_2 x + c_1 + \mathrm{e}^{-\frac{3x}{5}} c_4 \tag{1}$$

Verification of solutions

$$y = c_3 x^2 + c_2 x + c_1 + \mathrm{e}^{-\frac{3x}{5}} c_4$$

Verified OK.

2.1.1 Maple step by step solution

Let's solve

$5y'''' + 3y''' = 0$

- Highest derivative means the order of the ODE is 4

  y''''

- Isolate 4th derivative

  $y'''' = -\frac{3y'''}{5}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'''' + \frac{3y'''}{5} = 0$

- □ Convert linear ODE into a system of first order ODEs

  - Define new variable $y_1(x)$

    $y_1(x) = y$

  - Define new variable $y_2(x)$

    $y_2(x) = y'$

  - Define new variable $y_3(x)$

    $y_3(x) = y''$

  - Define new variable $y_4(x)$

    $y_4(x) = y'''$

  - Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = -\frac{3y_4(x)}{5}$

  Convert linear ODE into a system of first order ODEs

  $\left[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = -\frac{3y_4(x)}{5}\right]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\frac{3}{5} \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\frac{3}{5} \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-\frac{3}{5}, \begin{bmatrix} -\frac{125}{27} \\ \frac{25}{9} \\ -\frac{5}{3} \\ 1 \end{bmatrix}\right], \left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right], \left[0, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right], \left[0, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-\frac{3}{5}, \begin{bmatrix} -\frac{125}{27} \\ \frac{25}{9} \\ -\frac{5}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = e^{-\frac{3x}{5}} \cdot \begin{bmatrix} -\frac{125}{27} \\ \frac{25}{9} \\ -\frac{5}{3} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix} 0, & \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix} 0, & \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3 + c_4\vec{y}_4$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-\frac{3x}{5}} \cdot \begin{bmatrix} -\frac{125}{27} \\ \frac{25}{9} \\ -\frac{5}{3} \\ 1 \end{bmatrix} + \begin{bmatrix} c_2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = -\frac{125 c_1 e^{-\frac{3x}{5}}}{27} + c_2$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve(5*diff(y(x),x$4)+3*diff(y(x),x$3)=0,y(x), singsol=all)
```

$$y(x) = c_1 + c_2 x + c_3 x^2 + c_4 \mathrm{e}^{-\frac{3x}{5}}$$

✓ Solution by Mathematica

Time used: 0.037 (sec). Leaf size: 30

```
DSolve[5*y''''[x]+3*y'''[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{125}{27} c_1 e^{-3x/5} + x(c_4 x + c_3) + c_2$$

2.2 problem problem 11

Internal problem ID [286]
Internal file name [`OUTPUT/286_Sunday_June_05_2022_01_38_15_AM_85335422/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 5.3, Higher-Order Linear Differential Equations. Homogeneous Equations with Constant Coefficients. Page 300
Problem number: problem 11.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' - 8y''' + 16y'' = 0}$$

The characteristic equation is

$$\lambda^4 - 8\lambda^3 + 16\lambda^2 = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 0$$
$$\lambda_3 = 4$$
$$\lambda_4 = 4$$

Therefore the homogeneous solution is

$$y_h(x) = c_2 x + c_1 + e^{4x} c_3 + x\,e^{4x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = 1$$
$$y_2 = x$$
$$y_3 = e^{4x}$$
$$y_4 = x\,e^{4x}$$

Summary
The solution(s) found are the following

$$y = c_2 x + c_1 + e^{4x} c_3 + x\, e^{4x} c_4 \tag{1}$$

Verification of solutions

$$y = c_2 x + c_1 + e^{4x} c_3 + x\, e^{4x} c_4$$

Verified OK.

2.2.1 Maple step by step solution

Let's solve

$y'''' - 8y''' + 16y'' = 0$

- Highest derivative means the order of the ODE is 4

  y''''

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$
  - Define new variable $y_4(x)$

    $y_4(x) = y'''$
  - Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = 8y_4(x) - 16y_3(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = 8y_4(x) - 16y_3(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -16 & 8 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -16 & 8 \end{bmatrix}$$

- Rewrite the system as

$\vec{y}'(x) = A \cdot \vec{y}(x)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right], \left[0, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right], \left[4, \begin{bmatrix} \frac{1}{64} \\ \frac{1}{16} \\ \frac{1}{4} \\ 1 \end{bmatrix}\right], \left[4, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[4, \begin{bmatrix} \frac{1}{64} \\ \frac{1}{16} \\ \frac{1}{4} \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue 4

$$\overrightarrow{y}_3(x) = \mathrm{e}^{4x} \cdot \begin{bmatrix} \frac{1}{64} \\ \frac{1}{16} \\ \frac{1}{4} \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\overrightarrow{p}$ is to be solved for, $\lambda = 4$ is the eigenvalue, an

$$\overrightarrow{y}_4(x) = \mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right)$$

- Note that the x multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obt
- Substitute $\overrightarrow{y}_4(x)$ into the homogeneous system

$$\lambda \mathrm{e}^{\lambda x}\left(x\vec{v}+\vec{p}\right)+\mathrm{e}^{\lambda x}\vec{v}=\left(\mathrm{e}^{\lambda x}A\right)\cdot\left(x\vec{v}+\vec{p}\right)$$

- Use the fact that $\vec{v}$ is an eigenvector of A

$$\lambda \mathrm{e}^{\lambda x}\left(x\vec{v}+\vec{p}\right)+\mathrm{e}^{\lambda x}\vec{v}=\mathrm{e}^{\lambda x}\left(\lambda x\vec{v}+A\cdot\vec{p}\right)$$

- Simplify equation

$$\lambda\vec{p}+\vec{v}=A\cdot\vec{p}$$

- Make use of the identity matrix I

$$(\lambda\cdot I)\cdot\vec{p}+\vec{v}=A\cdot\vec{p}$$

- Condition $\vec{p}$ must meet for $\vec{y}_4(x)$ to be a solution to the homogeneous system

$$(A-\lambda\cdot I)\cdot\vec{p}=\vec{v}$$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue 4

$$\left(\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&-16&8\end{bmatrix}-4\cdot\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\right)\cdot\vec{p}=\begin{bmatrix}\frac{1}{64}\\\frac{1}{16}\\\frac{1}{4}\\1\end{bmatrix}$$

- Choice of $\vec{p}$

$$\vec{p}=\begin{bmatrix}-\frac{1}{256}\\0\\0\\0\end{bmatrix}$$

- Second solution from eigenvalue 4

$$\vec{y}_4(x)=\mathrm{e}^{4x}\cdot\left(x\cdot\begin{bmatrix}\frac{1}{64}\\\frac{1}{16}\\\frac{1}{4}\\1\end{bmatrix}+\begin{bmatrix}-\frac{1}{256}\\0\\0\\0\end{bmatrix}\right)$$

- General solution to the system of ODEs

$$\vec{y}=c_1\vec{y}_1+c_2\vec{y}_2+c_3\vec{y}_3(x)+c_4\vec{y}_4(x)$$

- Substitute solutions into the general solution

$$\vec{y} = e^{4x} c_3 \cdot \begin{bmatrix} \frac{1}{64} \\ \frac{1}{16} \\ \frac{1}{4} \\ 1 \end{bmatrix} + c_4 e^{4x} \cdot \left(x \cdot \begin{bmatrix} \frac{1}{64} \\ \frac{1}{16} \\ \frac{1}{4} \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{1}{256} \\ 0 \\ 0 \\ 0 \end{bmatrix} \right) + \begin{bmatrix} c_1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

  $y = \frac{((4x-1)c_4+4c_3)e^{4x}}{256} + c_1$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(diff(y(x),x$4)-8*diff(y(x),x$3)+16*diff(y(x),x$2)=0,y(x), singsol=all)
```

$$y(x) = (c_4 x + c_3) e^{4x} + c_2 x + c_1$$

✓ Solution by Mathematica

Time used: 0.081 (sec). Leaf size: 34

```
DSolve[y''''[x]-8*y'''[x]+16*y''[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{32} e^{4x}(c_2(2x-1) + 2c_1) + c_4 x + c_3$$

2.3 problem problem 12

Internal problem ID [287]
Internal file name [`OUTPUT/287_Sunday_June_05_2022_01_38_16_AM_97937637/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 5.3, Higher-Order Linear Differential Equations. Homogeneous Equations with Constant Coefficients. Page 300
Problem number: problem 12.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

`[[_high_order, _missing_x]]`

$$\boxed{y'''' - 3y''' + 3y'' - y' = 0}$$

The characteristic equation is

$$\lambda^4 - 3\lambda^3 + 3\lambda^2 - \lambda = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 1$$
$$\lambda_3 = 1$$
$$\lambda_4 = 1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 + c_2 \mathrm{e}^x + x\,\mathrm{e}^x c_3 + x^2 \mathrm{e}^x c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = 1$$
$$y_2 = \mathrm{e}^x$$
$$y_3 = x\,\mathrm{e}^x$$
$$y_4 = x^2 \mathrm{e}^x$$

Summary

The solution(s) found are the following

$$y = c_1 + c_2 e^x + x e^x c_3 + x^2 e^x c_4 \tag{1}$$

Verification of solutions

$$y = c_1 + c_2 e^x + x e^x c_3 + x^2 e^x c_4$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(diff(y(x),x$4)-3*diff(y(x),x$3)+3*diff(y(x),x$2)-diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = (c_4 x^2 + c_3 x + c_2) e^x + c_1$$

✓ Solution by Mathematica

Time used: 0.028 (sec). Leaf size: 32

```
DSolve[y''''[x]-3*y'''[x]+3*y''[x]-y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^x (c_3 (x^2 - 2x + 2) + c_2 (x - 1) + c_1) + c_4$$

2.4 problem problem 13

Internal problem ID [288]
Internal file name [`OUTPUT/288_Sunday_June_05_2022_01_38_16_AM_53478549/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 5.3, Higher-Order Linear Differential Equations. Homogeneous Equations with Constant Coefficients. Page 300
Problem number: problem 13.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{9y''' + 12y'' + 4y' = 0}$$

The characteristic equation is

$$9\lambda^3 + 12\lambda^2 + 4\lambda = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = -\frac{2}{3}$$
$$\lambda_3 = -\frac{2}{3}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 + \mathrm{e}^{-\frac{2x}{3}} c_2 + x\,\mathrm{e}^{-\frac{2x}{3}} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = 1$$
$$y_2 = \mathrm{e}^{-\frac{2x}{3}}$$
$$y_3 = x\,\mathrm{e}^{-\frac{2x}{3}}$$

Summary

The solution(s) found are the following

$$y = c_1 + e^{-\frac{2x}{3}} c_2 + x\, e^{-\frac{2x}{3}} c_3 \tag{1}$$

Verification of solutions

$$y = c_1 + e^{-\frac{2x}{3}} c_2 + x\, e^{-\frac{2x}{3}} c_3$$

Verified OK.

2.4.1 Maple step by step solution

Let's solve

$9y''' + 12y'' + 4y' = 0$

- Highest derivative means the order of the ODE is 3

  y'''

- Isolate 3rd derivative

  $y''' = -\frac{4y''}{3} - \frac{4y'}{9}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''' + \frac{4y''}{3} + \frac{4y'}{9} = 0$

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$

  ◦ Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = -\frac{4y_3(x)}{3} - \frac{4y_2(x)}{9}$

  Convert linear ODE into a system of first order ODEs

  $\left[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = -\frac{4y_3(x)}{3} - \frac{4y_2(x)}{9}\right]$

- Define vector

$$\vec{y}(x)=\left[\begin{array}{c} y_1(x) \\ y_2(x) \\ y_3(x) \end{array}\right]$$

- System to solve

$$\vec{y}'(x)=\left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\frac{4}{9} & -\frac{4}{3} \end{array}\right]\cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A=\left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\frac{4}{9} & -\frac{4}{3} \end{array}\right]$$

- Rewrite the system as

$$\vec{y}'(x)=A\cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-\frac{2}{3}, \left[\begin{array}{c} \frac{9}{4} \\ -\frac{3}{2} \\ 1 \end{array}\right]\right], \left[-\frac{2}{3}, \left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right]\right], \left[0, \left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right]\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[-\frac{2}{3}, \left[\begin{array}{c} \frac{9}{4} \\ -\frac{3}{2} \\ 1 \end{array}\right]\right]$$

- First solution from eigenvalue $-\frac{2}{3}$

$$\vec{y}_1(x)=\mathrm{e}^{-\frac{2x}{3}}\cdot \left[\begin{array}{c} \frac{9}{4} \\ -\frac{3}{2} \\ 1 \end{array}\right]$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda=-\frac{2}{3}$ is the eigenvalue, a

$\overrightarrow{y}_2(x) = \mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right)$

- Note that the x multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obt
- Substitute $\overrightarrow{y}_2(x)$ into the homogeneous system

  $\lambda\,\mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda x}\overrightarrow{v} = \left(\mathrm{e}^{\lambda x}A\right)\cdot\left(x\overrightarrow{v} + \overrightarrow{p}\right)$
- Use the fact that $\overrightarrow{v}$ is an eigenvector of A

  $\lambda\,\mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda x}\overrightarrow{v} = \mathrm{e}^{\lambda x}\left(\lambda x\overrightarrow{v} + A\cdot\overrightarrow{p}\right)$
- Simplify equation

  $\lambda\overrightarrow{p} + \overrightarrow{v} = A\cdot\overrightarrow{p}$
- Make use of the identity matrix I

  $(\lambda\cdot I)\cdot\overrightarrow{p} + \overrightarrow{v} = A\cdot\overrightarrow{p}$
- Condition $\overrightarrow{p}$ must meet for $\overrightarrow{y}_2(x)$ to be a solution to the homogeneous system

  $(A-\lambda\cdot I)\cdot\overrightarrow{p} = \overrightarrow{v}$
- Choose $\overrightarrow{p}$ to use in the second solution to the homogeneous system from eigenvalue $-\frac{2}{3}$

  $$\left(\left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\frac{4}{9} & -\frac{4}{3} \end{array}\right] - -\frac{2}{3}\cdot\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]\right)\cdot\overrightarrow{p} = \left[\begin{array}{c} \frac{9}{4} \\ -\frac{3}{2} \\ 1 \end{array}\right]$$
- Choice of $\overrightarrow{p}$

  $$\overrightarrow{p} = \left[\begin{array}{c} \frac{27}{8} \\ 0 \\ 0 \end{array}\right]$$
- Second solution from eigenvalue $-\frac{2}{3}$

  $$\overrightarrow{y}_2(x) = \mathrm{e}^{-\frac{2x}{3}}\cdot\left(x\cdot\left[\begin{array}{c} \frac{9}{4} \\ -\frac{3}{2} \\ 1 \end{array}\right] + \left[\begin{array}{c} \frac{27}{8} \\ 0 \\ 0 \end{array}\right]\right)$$
- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{y} = c_1\vec{y}_1(x) + c_2\vec{y}_2(x) + c_3\vec{y}_3$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-\frac{2x}{3}} \cdot \begin{bmatrix} \frac{9}{4} \\ -\frac{3}{2} \\ 1 \end{bmatrix} + e^{-\frac{2x}{3}} c_2 \cdot \left(x \cdot \begin{bmatrix} \frac{9}{4} \\ -\frac{3}{2} \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{27}{8} \\ 0 \\ 0 \end{bmatrix}\right) + \begin{bmatrix} c_3 \\ 0 \\ 0 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{9((2x+3)c_2 + 2c_1)e^{-\frac{2x}{3}}}{8} + c_3$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 16

```
dsolve(9*diff(y(x),x$3)+12*diff(y(x),x$2)+4*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = (c_3x + c_2)\,e^{-\frac{2x}{3}} + c_1$$

✓ Solution by Mathematica

Time used: 0.047 (sec). Leaf size: 32

```
DSolve[9*y'''[x]+12*y''[x]+4*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_3 - \frac{3}{4}e^{-2x/3}(c_2(2x+3)+2c_1)$$

2.5 problem problem 14

Internal problem ID [289]
Internal file name [`OUTPUT/289_Sunday_June_05_2022_01_38_17_AM_80529967/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 5.3, Higher-Order Linear Differential Equations. Homogeneous Equations with Constant Coefficients. Page 300
Problem number: problem 14.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' + 3y'' - 4y = 0}$$

The characteristic equation is

$$\lambda^4 + 3\lambda^2 - 4 = 0$$

The roots of the above equation are

$$\lambda_1 = 2i$$
$$\lambda_2 = -2i$$
$$\lambda_3 = 1$$
$$\lambda_4 = -1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + c_2 e^{x} + e^{2ix} c_3 + e^{-2ix} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-x}$$
$$y_2 = e^{x}$$
$$y_3 = e^{2ix}$$
$$y_4 = e^{-2ix}$$

Summary
The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + \mathrm{e}^{2ix} c_3 + \mathrm{e}^{-2ix} c_4 \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + \mathrm{e}^{2ix} c_3 + \mathrm{e}^{-2ix} c_4$$

Verified OK.

2.5.1 Maple step by step solution

Let's solve

$y'''' + 3y'' - 4y = 0$

- Highest derivative means the order of the ODE is 4

  y''''

□ Convert linear ODE into a system of first order ODEs

  - Define new variable $y_1(x)$

    $y_1(x) = y$

  - Define new variable $y_2(x)$

    $y_2(x) = y'$

  - Define new variable $y_3(x)$

    $y_3(x) = y''$

  - Define new variable $y_4(x)$

    $y_4(x) = y'''$

  - Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = -3y_3(x) + 4y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = -3y_3(x) + 4y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 4 & 0 & -3 & 0 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 4 & 0 & -3 & 0 \end{bmatrix}$$

- Rewrite the system as

$\vec{y}'(x) = A \cdot \vec{y}(x)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[-2\,\mathrm{I}, \begin{bmatrix} -\frac{\mathrm{I}}{8} \\ -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right], \left[2\,\mathrm{I}, \begin{bmatrix} \frac{\mathrm{I}}{8} \\ -\frac{1}{4} \\ -\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = e^{-x} \cdot \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-2\,\mathrm{I}, \begin{bmatrix} -\frac{\mathrm{I}}{8} \\ -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{-2\,\mathrm{I}x} \cdot \begin{bmatrix} -\frac{\mathrm{I}}{8} \\ -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$(\cos(2x) - \mathrm{I}\sin(2x)) \cdot \begin{bmatrix} -\frac{\mathrm{I}}{8} \\ -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} -\frac{\mathrm{I}}{8}(\cos(2x) - \mathrm{I}\sin(2x)) \\ -\frac{\cos(2x)}{4} + \frac{\mathrm{I}\sin(2x)}{4} \\ \frac{\mathrm{I}}{2}(\cos(2x) - \mathrm{I}\sin(2x)) \\ \cos(2x) - \mathrm{I}\sin(2x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(x) = \begin{bmatrix} -\frac{\sin(2x)}{8} \\ -\frac{\cos(2x)}{4} \\ \frac{\sin(2x)}{2} \\ \cos(2x) \end{bmatrix}, \vec{y}_4(x) = \begin{bmatrix} -\frac{\cos(2x)}{8} \\ \frac{\sin(2x)}{4} \\ \frac{\cos(2x)}{2} \\ -\sin(2x) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3(x) + c_4\vec{y}_4(x)$$

- Substitute solutions into the general solution

$$\vec{y} = c_1\mathrm{e}^{-x} \cdot \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} + c_2\mathrm{e}^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{c_3\sin(2x)}{8} - \frac{c_4\cos(2x)}{8} \\ -\frac{c_3\cos(2x)}{4} + \frac{c_4\sin(2x)}{4} \\ \frac{c_3\sin(2x)}{2} + \frac{c_4\cos(2x)}{2} \\ c_3\cos(2x) - c_4\sin(2x) \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = -c_1\mathrm{e}^{-x} + c_2\mathrm{e}^{x} - \frac{c_4\cos(2x)}{8} - \frac{c_3\sin(2x)}{8}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(diff(y(x),x$4)+3*diff(y(x),x$2)-4*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^x c_1 + c_2 e^{-x} + c_3 \sin(2x) + c_4 \cos(2x)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 34

```
DSolve[y''''[x]+3*y''[x]-4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_3 e^{-x} + c_4 e^x + c_1 \cos(2x) + c_2 \sin(2x)$$

2.6 problem problem 15

Internal problem ID [290]
Internal file name [`OUTPUT/290_Sunday_June_05_2022_01_38_17_AM_37975917/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 5.3, Higher-Order Linear Differential Equations. Homogeneous Equations with Constant Coefficients. Page 300
Problem number: problem 15.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' - 16y'' + 16y = 0}$$

The characteristic equation is

$$\lambda^4 - 16\lambda^2 + 16 = 0$$

The roots of the above equation are

$$\lambda_1 = \sqrt{2} - \sqrt{2}\,\sqrt{3}$$
$$\lambda_2 = -\sqrt{2} + \sqrt{2}\,\sqrt{3}$$
$$\lambda_3 = \sqrt{2} + \sqrt{2}\,\sqrt{3}$$
$$\lambda_4 = -\sqrt{2} - \sqrt{2}\,\sqrt{3}$$

Therefore the homogeneous solution is

$$y_h(x) = \mathrm{e}^{\left(\sqrt{2}-\sqrt{2}\,\sqrt{3}\right)x} c_1 + \mathrm{e}^{\left(-\sqrt{2}+\sqrt{2}\,\sqrt{3}\right)x} c_2 + \mathrm{e}^{\left(\sqrt{2}+\sqrt{2}\,\sqrt{3}\right)x} c_3 + \mathrm{e}^{\left(-\sqrt{2}-\sqrt{2}\,\sqrt{3}\right)x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{\left(\sqrt{2}-\sqrt{2}\,\sqrt{3}\right)x}$$
$$y_2 = e^{\left(-\sqrt{2}+\sqrt{2}\,\sqrt{3}\right)x}$$
$$y_3 = e^{\left(\sqrt{2}+\sqrt{2}\,\sqrt{3}\right)x}$$
$$y_4 = e^{\left(-\sqrt{2}-\sqrt{2}\,\sqrt{3}\right)x}$$

Summary

The solution(s) found are the following

$$y = e^{\left(\sqrt{2}-\sqrt{2}\,\sqrt{3}\right)x}c_1 + e^{\left(-\sqrt{2}+\sqrt{2}\,\sqrt{3}\right)x}c_2 + e^{\left(\sqrt{2}+\sqrt{2}\,\sqrt{3}\right)x}c_3 + e^{\left(-\sqrt{2}-\sqrt{2}\,\sqrt{3}\right)x}c_4 \tag{1}$$

Verification of solutions

$$y = e^{\left(\sqrt{2}-\sqrt{2}\,\sqrt{3}\right)x}c_1 + e^{\left(-\sqrt{2}+\sqrt{2}\,\sqrt{3}\right)x}c_2 + e^{\left(\sqrt{2}+\sqrt{2}\,\sqrt{3}\right)x}c_3 + e^{\left(-\sqrt{2}-\sqrt{2}\,\sqrt{3}\right)x}c_4$$

Verified OK.

2.6.1 Maple step by step solution

Let's solve

$y'''' - 16y'' + 16y = 0$

- Highest derivative means the order of the ODE is 4

  y''''

- ☐ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$
  - Define new variable $y_4(x)$

    $y_4(x) = y'''$
  - Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = 16y_3(x) - 16y_1(x)$

Convert linear ODE into a system of first order ODEs

$$[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = 16y_3(x) - 16y_1(x)]$$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -16 & 0 & 16 & 0 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -16 & 0 & 16 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-\sqrt{2} - \sqrt{2}\,\sqrt{3}, \begin{bmatrix} \frac{1}{\left(-\sqrt{2}-\sqrt{2}\,\sqrt{3}\right)^3} \\ \frac{1}{\left(-\sqrt{2}-\sqrt{2}\,\sqrt{3}\right)^2} \\ \frac{1}{-\sqrt{2}-\sqrt{2}\,\sqrt{3}} \\ 1 \end{bmatrix}\right], \left[-\sqrt{2} + \sqrt{2}\,\sqrt{3}, \begin{bmatrix} \frac{1}{\left(-\sqrt{2}+\sqrt{2}\,\sqrt{3}\right)^3} \\ \frac{1}{\left(-\sqrt{2}+\sqrt{2}\,\sqrt{3}\right)^2} \\ \frac{1}{-\sqrt{2}+\sqrt{2}\,\sqrt{3}} \\ 1 \end{bmatrix}\right], \left[\sqrt{2} - \sqrt{2}\,\sqrt{3}, \right.\right.$$

- Consider eigenpair

$$\left[-\sqrt{2}-\sqrt{2}\,\sqrt{3}, \left[\begin{array}{c} \frac{1}{\left(-\sqrt{2}-\sqrt{2}\,\sqrt{3}\right)^{3}} \\ \frac{1}{\left(-\sqrt{2}-\sqrt{2}\,\sqrt{3}\right)^{2}} \\ \frac{1}{-\sqrt{2}-\sqrt{2}\,\sqrt{3}} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_{1} = \mathrm{e}^{\left(-\sqrt{2}-\sqrt{2}\,\sqrt{3}\right)x} \cdot \left[\begin{array}{c} \frac{1}{\left(-\sqrt{2}-\sqrt{2}\,\sqrt{3}\right)^{3}} \\ \frac{1}{\left(-\sqrt{2}-\sqrt{2}\,\sqrt{3}\right)^{2}} \\ \frac{1}{-\sqrt{2}-\sqrt{2}\,\sqrt{3}} \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[-\sqrt{2}+\sqrt{2}\,\sqrt{3}, \left[\begin{array}{c} \frac{1}{\left(-\sqrt{2}+\sqrt{2}\,\sqrt{3}\right)^{3}} \\ \frac{1}{\left(-\sqrt{2}+\sqrt{2}\,\sqrt{3}\right)^{2}} \\ \frac{1}{-\sqrt{2}+\sqrt{2}\,\sqrt{3}} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_{2} = \mathrm{e}^{\left(-\sqrt{2}+\sqrt{2}\,\sqrt{3}\right)x} \cdot \left[\begin{array}{c} \frac{1}{\left(-\sqrt{2}+\sqrt{2}\,\sqrt{3}\right)^{3}} \\ \frac{1}{\left(-\sqrt{2}+\sqrt{2}\,\sqrt{3}\right)^{2}} \\ \frac{1}{-\sqrt{2}+\sqrt{2}\,\sqrt{3}} \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[\sqrt{2}-\sqrt{2}\,\sqrt{3}, \left[\begin{array}{c} \frac{1}{\left(\sqrt{2}-\sqrt{2}\,\sqrt{3}\right)^{3}} \\ \frac{1}{\left(\sqrt{2}-\sqrt{2}\,\sqrt{3}\right)^{2}} \\ \frac{1}{\sqrt{2}-\sqrt{2}\,\sqrt{3}} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_{3} = \mathrm{e}^{\left(\sqrt{2}-\sqrt{2}\,\sqrt{3}\right)x} \cdot \left[\begin{array}{c} \frac{1}{\left(\sqrt{2}-\sqrt{2}\,\sqrt{3}\right)^{3}} \\ \frac{1}{\left(\sqrt{2}-\sqrt{2}\,\sqrt{3}\right)^{2}} \\ \frac{1}{\sqrt{2}-\sqrt{2}\,\sqrt{3}} \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[\sqrt{2}+\sqrt{2}\,\sqrt{3}, \left[\begin{array}{c} \frac{1}{\left(\sqrt{2}+\sqrt{2}\,\sqrt{3}\right)^{3}} \\ \frac{1}{\left(\sqrt{2}+\sqrt{2}\,\sqrt{3}\right)^{2}} \\ \frac{1}{\sqrt{2}+\sqrt{2}\,\sqrt{3}} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_{4} = \mathrm{e}^{\left(\sqrt{2}+\sqrt{2}\,\sqrt{3}\right)x} \cdot \left[\begin{array}{c} \frac{1}{\left(\sqrt{2}+\sqrt{2}\,\sqrt{3}\right)^{3}} \\ \frac{1}{\left(\sqrt{2}+\sqrt{2}\,\sqrt{3}\right)^{2}} \\ \frac{1}{\sqrt{2}+\sqrt{2}\,\sqrt{3}} \\ 1 \end{array}\right]$$

- General solution to the system of ODEs

$$\vec{y} = c_{1}\vec{y}_{1} + c_{2}\vec{y}_{2} + c_{3}\vec{y}_{3} + c_{4}\vec{y}_{4}$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{\left(-\sqrt{2}-\sqrt{2}\sqrt{3}\right)x} \cdot \begin{bmatrix} \frac{1}{\left(-\sqrt{2}-\sqrt{2}\sqrt{3}\right)^3} \\ \frac{1}{\left(-\sqrt{2}-\sqrt{2}\sqrt{3}\right)^2} \\ \frac{1}{-\sqrt{2}-\sqrt{2}\sqrt{3}} \\ 1 \end{bmatrix} + e^{\left(-\sqrt{2}+\sqrt{2}\sqrt{3}\right)x} c_2 \cdot \begin{bmatrix} \frac{1}{\left(-\sqrt{2}+\sqrt{2}\sqrt{3}\right)^3} \\ \frac{1}{\left(-\sqrt{2}+\sqrt{2}\sqrt{3}\right)^2} \\ \frac{1}{-\sqrt{2}+\sqrt{2}\sqrt{3}} \\ 1 \end{bmatrix} + c_3 e^{\left(\sqrt{2}-\sqrt{2}\sqrt{3}\right)x}$$

- First component of the vector is the solution to the ODE

$$y = -\frac{3\left(c_1\left(\sqrt{3}-\frac{5}{3}\right)e^{-\sqrt{2}\left(1+\sqrt{3}\right)x} + c_3\left(\sqrt{3}+\frac{5}{3}\right)e^{-\sqrt{2}\left(\sqrt{3}-1\right)x} - \left(\sqrt{3}-\frac{5}{3}\right)c_4 e^{\sqrt{2}\left(1+\sqrt{3}\right)x} - \left(\sqrt{3}+\frac{5}{3}\right)c_2 e^{\sqrt{2}\left(\sqrt{3}-1\right)x}\right)\sqrt{2}}{2\left(1+\sqrt{3}\right)^3\left(\sqrt{3}-1\right)^3}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 59

```
dsolve(diff(y(x),x$4)-16*diff(y(x),x$2)+16*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{-\sqrt{2}\left(1+\sqrt{3}\right)x} + c_2 e^{\sqrt{2}\left(1+\sqrt{3}\right)x} + c_3 e^{-\sqrt{2}\left(\sqrt{3}-1\right)x} + c_4 e^{\sqrt{2}\left(\sqrt{3}-1\right)x}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 86

```
DSolve[y''''[x]-16*y''[x]+16*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{2\sqrt{2-\sqrt{3}}x} + c_2 e^{-2\sqrt{2-\sqrt{3}}x} + c_3 e^{2\sqrt{2+\sqrt{3}}x} + c_4 e^{-2\sqrt{2+\sqrt{3}}x}$$

2.7 problem problem 16

Internal problem ID [291]
Internal file name [`OUTPUT/291_Sunday_June_05_2022_01_38_18_AM_11065771/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 5.3, Higher-Order Linear Differential Equations. Homogeneous Equations with Constant Coefficients. Page 300
Problem number: problem 16.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

`[[_high_order, _missing_x]]`

$$\boxed{y'''' + 18y'' + 81y = 0}$$

The characteristic equation is

$$\lambda^4 + 18\lambda^2 + 81 = 0$$

The roots of the above equation are

$$\begin{aligned}\lambda_1 &= 3i\\ \lambda_2 &= -3i\\ \lambda_3 &= 3i\\ \lambda_4 &= -3i\end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = \mathrm{e}^{-3ix} c_1 + x\,\mathrm{e}^{-3ix} c_2 + \mathrm{e}^{3ix} c_3 + x\,\mathrm{e}^{3ix} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned}y_1 &= \mathrm{e}^{-3ix}\\ y_2 &= x\,\mathrm{e}^{-3ix}\\ y_3 &= \mathrm{e}^{3ix}\\ y_4 &= x\,\mathrm{e}^{3ix}\end{aligned}$$

Summary
The solution(s) found are the following

$$y = e^{-3ix} c_1 + x e^{-3ix} c_2 + e^{3ix} c_3 + x e^{3ix} c_4 \tag{1}$$

Verification of solutions

$$y = e^{-3ix} c_1 + x e^{-3ix} c_2 + e^{3ix} c_3 + x e^{3ix} c_4$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 25

```
dsolve(diff(y(x),x$4)+18*diff(y(x),x$2)+81*y(x)=0,y(x), singsol=all)
```

$$y(x) = (c_4 x + c_2) \cos(3x) + \sin(3x) (c_3 x + c_1)$$

✓ Solution by Mathematica
Time used: 0.003 (sec). Leaf size: 30

```
DSolve[y''''[x]+18*y''[x]+81*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to (c_2 x + c_1) \cos(3x) + (c_4 x + c_3) \sin(3x)$$

2.8 problem problem 17

Internal problem ID [292]
Internal file name [`OUTPUT/292_Sunday_June_05_2022_01_38_18_AM_83684402/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 5.3, Higher-Order Linear Differential Equations. Homogeneous Equations with Constant Coefficients. Page 300
Problem number: problem 17.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{6y'''' + 11y'' + 4y = 0}$$

The characteristic equation is

$$6\lambda^4 + 11\lambda^2 + 4 = 0$$

The roots of the above equation are

$$\lambda_1 = \frac{i\sqrt{2}}{2}$$
$$\lambda_2 = -\frac{i\sqrt{2}}{2}$$
$$\lambda_3 = \frac{2i\sqrt{3}}{3}$$
$$\lambda_4 = -\frac{2i\sqrt{3}}{3}$$

Therefore the homogeneous solution is

$$y_h(x) = e^{\frac{i\sqrt{2}\,x}{2}} c_1 + e^{-\frac{2i\sqrt{3}\,x}{3}} c_2 + e^{\frac{2i\sqrt{3}\,x}{3}} c_3 + e^{-\frac{i\sqrt{2}\,x}{2}} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= \mathrm{e}^{\frac{i\sqrt{2}\,x}{2}} \\
y_2 &= \mathrm{e}^{-\frac{2i\sqrt{3}\,x}{3}} \\
y_3 &= \mathrm{e}^{\frac{2i\sqrt{3}\,x}{3}} \\
y_4 &= \mathrm{e}^{-\frac{i\sqrt{2}\,x}{2}}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \mathrm{e}^{\frac{i\sqrt{2}\,x}{2}} c_1 + \mathrm{e}^{-\frac{2i\sqrt{3}\,x}{3}} c_2 + \mathrm{e}^{\frac{2i\sqrt{3}\,x}{3}} c_3 + \mathrm{e}^{-\frac{i\sqrt{2}\,x}{2}} c_4 \tag{1}
$$

Verification of solutions

$$
y = \mathrm{e}^{\frac{i\sqrt{2}\,x}{2}} c_1 + \mathrm{e}^{-\frac{2i\sqrt{3}\,x}{3}} c_2 + \mathrm{e}^{\frac{2i\sqrt{3}\,x}{3}} c_3 + \mathrm{e}^{-\frac{i\sqrt{2}\,x}{2}} c_4
$$

Verified OK.

2.8.1 Maple step by step solution

Let's solve

$6y'''' + 11y'' + 4y = 0$

- Highest derivative means the order of the ODE is 4

  y''''

- Isolate 4th derivative

  $y'''' = -\frac{11y''}{6} - \frac{2y}{3}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'''' + \frac{11y''}{6} + \frac{2y}{3} = 0$

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$

- Define new variable $y_4(x)$

  $y_4(x) = y'''$

- Isolate for $y_4'(x)$ using original ODE

  $y_4'(x) = -\frac{11y_3(x)}{6} - \frac{2y_1(x)}{3}$

  Convert linear ODE into a system of first order ODEs

  $\left[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = -\frac{11y_3(x)}{6} - \frac{2y_1(x)}{3}\right]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{2}{3} & 0 & -\frac{11}{6} & 0 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{2}{3} & 0 & -\frac{11}{6} & 0 \end{bmatrix}$$

- Rewrite the system as

  $\vec{y}'(x) = A \cdot \vec{y}(x)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-\frac{2I}{3}\sqrt{3}, \begin{bmatrix} -\frac{3I}{8}\sqrt{3} \\ -\frac{3}{4} \\ \frac{I}{2}\sqrt{3} \\ 1 \end{bmatrix}\right], \left[-\frac{I}{2}\sqrt{2}, \begin{bmatrix} -2I\sqrt{2} \\ -2 \\ I\sqrt{2} \\ 1 \end{bmatrix}\right], \left[\frac{I}{2}\sqrt{2}, \begin{bmatrix} 2I\sqrt{2} \\ -2 \\ -I\sqrt{2} \\ 1 \end{bmatrix}\right], \left[\frac{2I}{3}\sqrt{3}, \begin{bmatrix} \frac{3I}{8}\sqrt{3} \\ -\frac{3}{4} \\ -\frac{I}{2}\sqrt{3} \\ 1 \end{bmatrix}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\frac{2I}{3}\sqrt{3}, \begin{bmatrix} -\frac{3I}{8}\sqrt{3} \\ -\frac{3}{4} \\ \frac{I}{2}\sqrt{3} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{-\frac{2I}{3}\sqrt{3}x} \cdot \begin{bmatrix} -\frac{3I}{8}\sqrt{3} \\ -\frac{3}{4} \\ \frac{I}{2}\sqrt{3} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$\left(\cos\left(\frac{2\sqrt{3}x}{3}\right) - I\sin\left(\frac{2\sqrt{3}x}{3}\right)\right) \cdot \begin{bmatrix} -\frac{3I}{8}\sqrt{3} \\ -\frac{3}{4} \\ \frac{I}{2}\sqrt{3} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} -\frac{3I}{8}\left(\cos\left(\frac{2\sqrt{3}x}{3}\right) - I\sin\left(\frac{2\sqrt{3}x}{3}\right)\right)\sqrt{3} \\ -\frac{3\cos\left(\frac{2\sqrt{3}x}{3}\right)}{4} + \frac{3I\sin\left(\frac{2\sqrt{3}x}{3}\right)}{4} \\ \frac{I}{2}\left(\cos\left(\frac{2\sqrt{3}x}{3}\right) - I\sin\left(\frac{2\sqrt{3}x}{3}\right)\right)\sqrt{3} \\ \cos\left(\frac{2\sqrt{3}x}{3}\right) - I\sin\left(\frac{2\sqrt{3}x}{3}\right) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_1(x) = \left[\begin{array}{c} -\frac{3\sqrt{3}\,\sin\left(\frac{2\sqrt{3}\,x}{3}\right)}{8} \\ -\frac{3\cos\left(\frac{2\sqrt{3}\,x}{3}\right)}{4} \\ \frac{\sqrt{3}\,\sin\left(\frac{2\sqrt{3}\,x}{3}\right)}{2} \\ \cos\left(\frac{2\sqrt{3}\,x}{3}\right) \end{array}\right], \vec{y}_2(x) = \left[\begin{array}{c} -\frac{3\sqrt{3}\,\cos\left(\frac{2\sqrt{3}\,x}{3}\right)}{8} \\ \frac{3\sin\left(\frac{2\sqrt{3}\,x}{3}\right)}{4} \\ \frac{\sqrt{3}\,\cos\left(\frac{2\sqrt{3}\,x}{3}\right)}{2} \\ -\sin\left(\frac{2\sqrt{3}\,x}{3}\right) \end{array}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\frac{I}{2}\sqrt{2}, \left[\begin{array}{c} -2\,I\sqrt{2} \\ -2 \\ I\sqrt{2} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$e^{-\frac{I}{2}\sqrt{2}\,x} \cdot \left[\begin{array}{c} -2\,I\sqrt{2} \\ -2 \\ I\sqrt{2} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\left(\cos\left(\frac{\sqrt{2}\,x}{2}\right) - I\sin\left(\frac{\sqrt{2}\,x}{2}\right)\right) \cdot \left[\begin{array}{c} -2\,I\sqrt{2} \\ -2 \\ I\sqrt{2} \\ 1 \end{array}\right]$$

- Simplify expression

$$\left[\begin{array}{c} -2\,I\left(\cos\left(\frac{\sqrt{2}\,x}{2}\right) - I\sin\left(\frac{\sqrt{2}\,x}{2}\right)\right)\sqrt{2} \\ -2\cos\left(\frac{\sqrt{2}\,x}{2}\right) + 2\,I\sin\left(\frac{\sqrt{2}\,x}{2}\right) \\ I\left(\cos\left(\frac{\sqrt{2}\,x}{2}\right) - I\sin\left(\frac{\sqrt{2}\,x}{2}\right)\right)\sqrt{2} \\ \cos\left(\frac{\sqrt{2}\,x}{2}\right) - I\sin\left(\frac{\sqrt{2}\,x}{2}\right) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(x) = \begin{bmatrix} -2\sqrt{2}\,\sin\left(\frac{\sqrt{2}\,x}{2}\right) \\ -2\cos\left(\frac{\sqrt{2}\,x}{2}\right) \\ \sqrt{2}\,\sin\left(\frac{\sqrt{2}\,x}{2}\right) \\ \cos\left(\frac{\sqrt{2}\,x}{2}\right) \end{bmatrix}, \vec{y}_4(x) = \begin{bmatrix} -2\sqrt{2}\,\cos\left(\frac{\sqrt{2}\,x}{2}\right) \\ 2\sin\left(\frac{\sqrt{2}\,x}{2}\right) \\ \sqrt{2}\,\cos\left(\frac{\sqrt{2}\,x}{2}\right) \\ -\sin\left(\frac{\sqrt{2}\,x}{2}\right) \end{bmatrix}\right]$$

- General solution to the system of ODEs

  $$\vec{y} = c_1\vec{y}_1(x) + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + c_4\vec{y}_4(x)$$

- Substitute solutions into the general solution

  $$\vec{y} = \begin{bmatrix} -2c_4\sqrt{2}\,\cos\left(\frac{\sqrt{2}\,x}{2}\right) - 2c_3\sqrt{2}\,\sin\left(\frac{\sqrt{2}\,x}{2}\right) - \frac{3c_2\sqrt{3}\,\cos\left(\frac{2\sqrt{3}\,x}{3}\right)}{8} - \frac{3c_1\sqrt{3}\,\sin\left(\frac{2\sqrt{3}\,x}{3}\right)}{8} \\ 2c_4\sin\left(\frac{\sqrt{2}\,x}{2}\right) - 2c_3\cos\left(\frac{\sqrt{2}\,x}{2}\right) + \frac{3c_2\sin\left(\frac{2\sqrt{3}\,x}{3}\right)}{4} - \frac{3c_1\cos\left(\frac{2\sqrt{3}\,x}{3}\right)}{4} \\ c_4\sqrt{2}\,\cos\left(\frac{\sqrt{2}\,x}{2}\right) + c_3\sqrt{2}\,\sin\left(\frac{\sqrt{2}\,x}{2}\right) + \frac{c_2\sqrt{3}\,\cos\left(\frac{2\sqrt{3}\,x}{3}\right)}{2} + \frac{c_1\sqrt{3}\,\sin\left(\frac{2\sqrt{3}\,x}{3}\right)}{2} \\ -c_4\sin\left(\frac{\sqrt{2}\,x}{2}\right) + c_3\cos\left(\frac{\sqrt{2}\,x}{2}\right) - c_2\sin\left(\frac{2\sqrt{3}\,x}{3}\right) + c_1\cos\left(\frac{2\sqrt{3}\,x}{3}\right) \end{bmatrix}$$

- First component of the vector is the solution to the ODE

  $$y = -2c_4\sqrt{2}\,\cos\left(\frac{\sqrt{2}\,x}{2}\right) - 2c_3\sqrt{2}\,\sin\left(\frac{\sqrt{2}\,x}{2}\right) - \frac{3c_2\sqrt{3}\,\cos\left(\frac{2\sqrt{3}\,x}{3}\right)}{8} - \frac{3c_1\sqrt{3}\,\sin\left(\frac{2\sqrt{3}\,x}{3}\right)}{8}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 41

```
dsolve(6*diff(y(x),x$4)+11*diff(y(x),x$2)+4*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \sin\left(\frac{2\sqrt{3}\,x}{3}\right) + c_2 \cos\left(\frac{2\sqrt{3}\,x}{3}\right) + c_3 \sin\left(\frac{\sqrt{2}\,x}{2}\right) + c_4 \cos\left(\frac{\sqrt{2}\,x}{2}\right)$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 94

```
DSolve[y''''[x]+11*y''[x]+4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_3 \cos\left(\sqrt{\frac{1}{2}\left(11-\sqrt{105}\right)}x\right) + c_1 \cos\left(\sqrt{\frac{1}{2}\left(11+\sqrt{105}\right)}x\right) + c_4 \sin\left(\sqrt{\frac{1}{2}\left(11-\sqrt{105}\right)}x\right) + c_2 \sin\left(\sqrt{\frac{1}{2}\left(11+\sqrt{105}\right)}x\right)$$

2.9 problem problem 18

Internal problem ID [293]
Internal file name [`OUTPUT/293_Sunday_June_05_2022_01_38_19_AM_47421981/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 5.3, Higher-Order Linear Differential Equations. Homogeneous Equations with Constant Coefficients. Page 300
Problem number: problem 18.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' - 16y = 0}$$

The characteristic equation is

$$\lambda^4 - 16 = 0$$

The roots of the above equation are

$$\lambda_1 = 2$$
$$\lambda_2 = -2$$
$$\lambda_3 = 2i$$
$$\lambda_4 = -2i$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{2x} + \mathrm{e}^{2ix} c_3 + \mathrm{e}^{-2ix} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = \mathrm{e}^{-2x}$$
$$y_2 = \mathrm{e}^{2x}$$
$$y_3 = \mathrm{e}^{2ix}$$
$$y_4 = \mathrm{e}^{-2ix}$$

Summary
The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{2x} + \mathrm{e}^{2ix} c_3 + \mathrm{e}^{-2ix} c_4 \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{2x} + \mathrm{e}^{2ix} c_3 + \mathrm{e}^{-2ix} c_4$$

Verified OK.

2.9.1 Maple step by step solution

Let's solve

$y'''' - 16y = 0$

- Highest derivative means the order of the ODE is 4

  y''''

□ Convert linear ODE into a system of first order ODEs

  ∘ Define new variable $y_1(x)$

    $y_1(x) = y$

  ∘ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ∘ Define new variable $y_3(x)$

    $y_3(x) = y''$

  ∘ Define new variable $y_4(x)$

    $y_4(x) = y'''$

  ∘ Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = 16y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = 16y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 16 & 0 & 0 & 0 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 16 & 0 & 0 & 0 \end{bmatrix}$$

- Rewrite the system as

$\vec{y}'(x) = A \cdot \vec{y}(x)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right], \left[-2\,\mathrm{I}, \begin{bmatrix} -\frac{\mathrm{I}}{8} \\ -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right], \left[2\,\mathrm{I}, \begin{bmatrix} \frac{\mathrm{I}}{8} \\ -\frac{1}{4} \\ -\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-2, \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = e^{-2x} \cdot \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^{2x} \cdot \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-2\,\mathrm{I}, \begin{bmatrix} -\frac{\mathrm{I}}{8} \\ -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{-2\,\mathrm{I}x} \cdot \begin{bmatrix} -\frac{\mathrm{I}}{8} \\ -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\left(\cos \left(2 x\right)-\mathrm{I} \sin \left(2 x\right)\right) \cdot \left[\begin{array}{c}-\frac{\mathrm{I}}{8} \\ -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1\end{array}\right]$$

- Simplify expression

$$\left[\begin{array}{c}-\frac{\mathrm{I}}{8}\left(\cos \left(2 x\right)-\mathrm{I} \sin \left(2 x\right)\right) \\ -\frac{\cos(2 x)}{4}+\frac{\mathrm{I} \sin(2 x)}{4} \\ \frac{\mathrm{I}}{2}\left(\cos \left(2 x\right)-\mathrm{I} \sin \left(2 x\right)\right) \\ \cos \left(2 x\right)-\mathrm{I} \sin \left(2 x\right)\end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\overrightarrow{y}_{3}(x)=\left[\begin{array}{c}-\frac{\sin(2 x)}{8} \\ -\frac{\cos(2 x)}{4} \\ \frac{\sin(2 x)}{2} \\ \cos \left(2 x\right)\end{array}\right], \overrightarrow{y}_{4}(x)=\left[\begin{array}{c}-\frac{\cos(2 x)}{8} \\ \frac{\sin(2 x)}{4} \\ \frac{\cos(2 x)}{2} \\ -\sin \left(2 x\right)\end{array}\right]\right]$$

- General solution to the system of ODEs

$$\overrightarrow{y}=c_{1} \overrightarrow{y}_{1}+c_{2} \overrightarrow{y}_{2}+c_{3} \overrightarrow{y}_{3}(x)+c_{4} \overrightarrow{y}_{4}(x)$$

- Substitute solutions into the general solution

$$\overrightarrow{y}=c_{1} \mathrm{e}^{-2 x} \cdot \left[\begin{array}{c}-\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1\end{array}\right]+c_{2} \mathrm{e}^{2 x} \cdot \left[\begin{array}{c}\frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1\end{array}\right]+\left[\begin{array}{c}-\frac{c_{3} \sin(2 x)}{8}-\frac{c_{4} \cos(2 x)}{8} \\ -\frac{c_{3} \cos(2 x)}{4}+\frac{c_{4} \sin(2 x)}{4} \\ \frac{c_{3} \sin(2 x)}{2}+\frac{c_{4} \cos(2 x)}{2} \\ c_{3} \cos \left(2 x\right)-c_{4} \sin \left(2 x\right)\end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y=-\frac{c_{1} \mathrm{e}^{-2 x}}{8}+\frac{c_{2} \mathrm{e}^{2 x}}{8}-\frac{c_{4} \cos(2 x)}{8}-\frac{c_{3} \sin(2 x)}{8}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 29

```
dsolve(diff(y(x),x$4)=16*y(x),y(x), singsol=all)
```

$$y(x) = e^{2x}c_1 + c_2e^{-2x} + c_3\sin(2x) + c_4\cos(2x)$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 36

```
DSolve[y''''[x]==16*y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1e^{2x} + c_3e^{-2x} + c_2\cos(2x) + c_4\sin(2x)$$

2.10 problem problem 19

Internal problem ID [294]
Internal file name [OUTPUT/294_Sunday_June_05_2022_01_38_20_AM_8986700/index.tex]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 5.3, Higher-Order Linear Differential Equations. Homogeneous Equations with Constant Coefficients. Page 300
Problem number: problem 19.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' + y'' - y' - y = 0}$$

The characteristic equation is

$$\lambda^3 + \lambda^2 - \lambda - 1 = 0$$

The roots of the above equation are

$$\begin{aligned}\lambda_1 &= 1\\ \lambda_2 &= -1\\ \lambda_3 &= -1\end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-x} + x\,\mathrm{e}^{-x} c_2 + c_3 \mathrm{e}^{x}$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned}y_1 &= \mathrm{e}^{-x}\\ y_2 &= x\,\mathrm{e}^{-x}\\ y_3 &= \mathrm{e}^{x}\end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-x} + x\,e^{-x} c_2 + c_3 e^{x} \tag{1}$$

Verification of solutions

$$y = c_1 e^{-x} + x\,e^{-x} c_2 + c_3 e^{x}$$

Verified OK.

2.10.1 Maple step by step solution

Let's solve

$y''' + y'' - y' - y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Convert linear ODE into a system of first order ODEs
  - ○ Define new variable $y_1(x)$

    $y_1(x) = y$
  - ○ Define new variable $y_2(x)$

    $y_2(x) = y'$
  - ○ Define new variable $y_3(x)$

    $y_3(x) = y''$
  - ○ Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = -y_3(x) + y_2(x) + y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = -y_3(x) + y_2(x) + y_1(x)]$

- Define vector

  $\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[-1, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue -1

$$\vec{y}_1(x) = e^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = -1$ is the eigenvalue, a

$$\vec{y}_2(x) = e^{\lambda x}\left(x\vec{v} + \vec{p}\right)$$

- Note that the x multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obt
- Substitute $\vec{y}_2(x)$ into the homogeneous system

$$\lambda e^{\lambda x}\left(x\vec{v} + \vec{p}\right) + e^{\lambda x}\vec{v} = \left(e^{\lambda x} A\right) \cdot \left(x\vec{v} + \vec{p}\right)$$

- Use the fact that $\vec{v}$ is an eigenvector of A

$$\lambda\, \mathrm{e}^{\lambda x}\left(x\vec{v}+\vec{p}\right)+\mathrm{e}^{\lambda x}\vec{v}=\mathrm{e}^{\lambda x}\left(\lambda x\vec{v}+A\cdot\vec{p}\right)$$

- Simplify equation

$$\lambda\vec{p}+\vec{v}=A\cdot\vec{p}$$

- Make use of the identity matrix I

$$(\lambda\cdot I)\cdot\vec{p}+\vec{v}=A\cdot\vec{p}$$

- Condition $\vec{p}$ must meet for $\vec{y}_2(x)$ to be a solution to the homogeneous system

$$(A-\lambda\cdot I)\cdot\vec{p}=\vec{v}$$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue -1

$$\left(\begin{bmatrix}0&1&0\\0&0&1\\1&1&-1\end{bmatrix}-(-1)\cdot\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\right)\cdot\vec{p}=\begin{bmatrix}1\\-1\\1\end{bmatrix}$$

- Choice of $\vec{p}$

$$\vec{p}=\begin{bmatrix}1\\0\\0\end{bmatrix}$$

- Second solution from eigenvalue -1

$$\vec{y}_2(x)=\mathrm{e}^{-x}\cdot\left(x\cdot\begin{bmatrix}1\\-1\\1\end{bmatrix}+\begin{bmatrix}1\\0\\0\end{bmatrix}\right)$$

- Consider eigenpair

$$\left[1,\begin{bmatrix}1\\1\\1\end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3=\mathrm{e}^{x}\cdot\begin{bmatrix}1\\1\\1\end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{y} = c_1\vec{y}_1(x) + c_2\vec{y}_2(x) + c_3\vec{y}_3$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_2 e^{-x} \cdot \left(x \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) + c_3 e^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = ((x+1)\, c_2 + c_1)\, e^{-x} + c_3 e^{x}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(diff(y(x),x$3)+diff(y(x),x$2)-diff(y(x),x)-y(x)=0,y(x), singsol=all)
```

$$y(x) = (c_3 x + c_2)\, e^{-x} + e^{x} c_1$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 26

```
DSolve[y'''[x]+y''[x]-y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x} \left(c_2 x + c_3 e^{2x} + c_1\right)$$

2.11 problem problem 20

Internal problem ID [295]
Internal file name [`OUTPUT/295_Sunday_June_05_2022_01_38_20_AM_54787276/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 5.3, Higher-Order Linear Differential Equations. Homogeneous Equations with Constant Coefficients. Page 300
Problem number: problem 20.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' + 2y''' + 3y'' + 2y' + y = 0}$$

The characteristic equation is

$$\lambda^4 + 2\lambda^3 + 3\lambda^2 + 2\lambda + 1 = 0$$

The roots of the above equation are

$$\begin{aligned}\lambda_1 &= -\frac{1}{2} - \frac{i\sqrt{3}}{2} \\ \lambda_2 &= -\frac{1}{2} + \frac{i\sqrt{3}}{2} \\ \lambda_3 &= -\frac{1}{2} - \frac{i\sqrt{3}}{2} \\ \lambda_4 &= -\frac{1}{2} + \frac{i\sqrt{3}}{2}\end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = \mathrm{e}^{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x} c_1 + x\,\mathrm{e}^{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x} c_2 + \mathrm{e}^{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x} c_3 + x\,\mathrm{e}^{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= \mathrm{e}^{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x} \\
y_2 &= x\,\mathrm{e}^{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x} \\
y_3 &= \mathrm{e}^{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x} \\
y_4 &= x\,\mathrm{e}^{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \mathrm{e}^{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x} c_1 + x\,\mathrm{e}^{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x} c_2 + \mathrm{e}^{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x} c_3 + x\,\mathrm{e}^{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x} c_4 \tag{1}
$$

Verification of solutions

$$
y = \mathrm{e}^{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x} c_1 + x\,\mathrm{e}^{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x} c_2 + \mathrm{e}^{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x} c_3 + x\,\mathrm{e}^{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x} c_4
$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 36

```
dsolve(diff(y(x),x$4)+2*diff(y(x),x$3)+3*diff(y(x),x$2)+2*diff(y(x),x)+y(x)=0,y(x), singsol=
```

$$
y(x) = \mathrm{e}^{-\frac{x}{2}} \left((c_4 x + c_2) \cos\left(\frac{\sqrt{3}\,x}{2}\right) + \sin\left(\frac{\sqrt{3}\,x}{2}\right) (c_3 x + c_1) \right)
$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 52

```
DSolve[y''''[x]+2*y'''[x]+3*y''[x]+2*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x/2}\left((c_4x + c_3)\cos\left(\frac{\sqrt{3}x}{2}\right) + (c_2x + c_1)\sin\left(\frac{\sqrt{3}x}{2}\right)\right)$$

2.12 problem problem 24

Internal problem ID [296]
Internal file name [OUTPUT/296_Sunday_June_05_2022_01_38_21_AM_28338752/index.tex]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 5.3, Higher-Order Linear Differential Equations. Homogeneous Equations with Constant Coefficients. Page 300
Problem number: problem 24.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{2y''' - 3y'' - 2y' = 0}$$

With initial conditions

$$[y(0) = 1, y'(0) = -1, y''(0) = 3]$$

The characteristic equation is

$$2\lambda^3 - 3\lambda^2 - 2\lambda = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 2$$
$$\lambda_3 = -\frac{1}{2}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 + c_2 e^{2x} + e^{-\frac{x}{2}} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= 1 \\
y_2 &= \mathrm{e}^{2x} \\
y_3 &= \mathrm{e}^{-\frac{x}{2}}
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 + c_2 \mathrm{e}^{2x} + \mathrm{e}^{-\frac{x}{2}} c_3 \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$
1 = c_1 + c_2 + c_3 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = 2c_2 \mathrm{e}^{2x} - \frac{\mathrm{e}^{-\frac{x}{2}} c_3}{2}
$$

substituting $y' = -1$ and $x = 0$ in the above gives

$$
-1 = 2c_2 - \frac{c_3}{2} \tag{2A}
$$

Taking two derivatives of the solution gives

$$
y'' = 4c_2 \mathrm{e}^{2x} + \frac{\mathrm{e}^{-\frac{x}{2}} c_3}{4}
$$

substituting $y'' = 3$ and $x = 0$ in the above gives

$$
3 = 4c_2 + \frac{c_3}{4} \tag{3A}
$$

Equations {1A,2A,3A} are now solved for $\{c_1, c_2, c_3\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= -\frac{7}{2} \\
c_2 &= \frac{1}{2} \\
c_3 &= 4
\end{aligned}
$$

Substituting these values back in above solution results in

$$
y = -\frac{7}{2} + \frac{\mathrm{e}^{2x}}{2} + 4\,\mathrm{e}^{-\frac{x}{2}}
$$

Summary
The solution(s) found are the following

$$y = -\frac{7}{2} + \frac{e^{2x}}{2} + 4\,e^{-\frac{x}{2}} \tag{1}$$

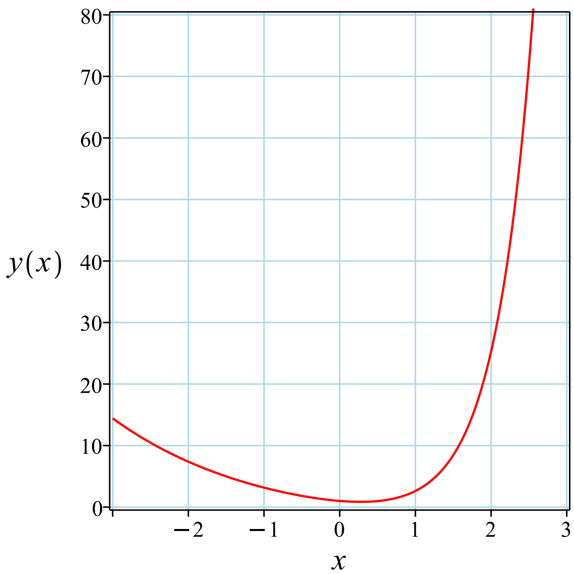


Figure 2: Solution plot

Verification of solutions

$$y = -\frac{7}{2} + \frac{e^{2x}}{2} + 4\,e^{-\frac{x}{2}}$$

Verified OK.

2.12.1 Maple step by step solution

Let's solve

$$\left[2y''' - 3y'' - 2y' = 0, y(0) = 1, y'\Big|_{\{x=0\}} = -1, y''\Big|_{\{x=0\}} = 3\right]$$

- Highest derivative means the order of the ODE is 3

  y'''

- Isolate 3rd derivative

  $y''' = \frac{3y''}{2} + y'$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y''' - \frac{3y''}{2} - y' = 0$

- ☐ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$
  - Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = \frac{3y_3(x)}{2} + y_2(x)$

  Convert linear ODE into a system of first order ODEs

  $\left[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = \frac{3y_3(x)}{2} + y_2(x)\right]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & \frac{3}{2} \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & \frac{3}{2} \end{bmatrix}$$

- Rewrite the system as

  $\vec{y}'(x) = A \cdot \vec{y}(x)$
- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\begin{bmatrix}\begin{bmatrix}-\frac{1}{2}, & \begin{bmatrix}4\\-2\\1\end{bmatrix}\end{bmatrix}, & \begin{bmatrix}0, & \begin{bmatrix}1\\0\\0\end{bmatrix}\end{bmatrix}, & \begin{bmatrix}2, & \begin{bmatrix}\frac{1}{4}\\\frac{1}{2}\\1\end{bmatrix}\end{bmatrix}\end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix}-\frac{1}{2}, & \begin{bmatrix}4\\-2\\1\end{bmatrix}\end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{-\frac{x}{2}} \cdot \begin{bmatrix}4\\-2\\1\end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix}0, & \begin{bmatrix}1\\0\\0\end{bmatrix}\end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \begin{bmatrix}1\\0\\0\end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix}2, & \begin{bmatrix}\frac{1}{4}\\\frac{1}{2}\\1\end{bmatrix}\end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = \mathrm{e}^{2x} \cdot \begin{bmatrix}\frac{1}{4}\\\frac{1}{2}\\1\end{bmatrix}$$

- General solution to the system of ODEs

  $\overrightarrow{y} = c_1\overrightarrow{y}_1 + c_2\overrightarrow{y}_2 + c_3\overrightarrow{y}_3$

- Substitute solutions into the general solution

  $\overrightarrow{y} = c_1 \mathrm{e}^{-\frac{x}{2}} \cdot \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix} + c_3 \mathrm{e}^{2x} \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} + \begin{bmatrix} c_2 \\ 0 \\ 0 \end{bmatrix}$

- First component of the vector is the solution to the ODE

  $y = 4c_1 \mathrm{e}^{-\frac{x}{2}} + \frac{c_3 \mathrm{e}^{2x}}{4} + c_2$

- Use the initial condition $y(0) = 1$

  $1 = 4c_1 + \frac{c_3}{4} + c_2$

- Calculate the 1st derivative of the solution

  $y' = -2c_1 \mathrm{e}^{-\frac{x}{2}} + \frac{c_3 \mathrm{e}^{2x}}{2}$

- Use the initial condition $y'\Big|_{\{x=0\}} = -1$

  $-1 = -2c_1 + \frac{c_3}{2}$

- Calculate the 2nd derivative of the solution

  $y'' = c_1 \mathrm{e}^{-\frac{x}{2}} + c_3 \mathrm{e}^{2x}$

- Use the initial condition $y''\Big|_{\{x=0\}} = 3$

  $3 = c_1 + c_3$

- Solve for the unknown coefficients

  $\left\{c_1 = 1, c_2 = -\frac{7}{2}, c_3 = 2\right\}$

- Solution to the IVP

  $y = -\frac{7}{2} + \frac{\mathrm{e}^{2x}}{2} + 4\,\mathrm{e}^{-\frac{x}{2}}$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 18

```
dsolve([2*diff(y(x),x$3)-3*diff(y(x),x$2)-2*diff(y(x),x)=0,y(0) = 1, D(y)(0) = -1, (D@@2)(y)
```

$$y(x) = -\frac{7}{2} + 4\,\mathrm{e}^{-\frac{x}{2}} + \frac{\mathrm{e}^{2x}}{2}$$

✓ Solution by Mathematica

Time used: 0.351 (sec). Leaf size: 70

```
DSolve[{2*y'''[x]-3*y''[x]-3*y'[x]==0,{y[0]==1,y'[0]==-1,y''[0]==3}},y[x],x,IncludeSingularS
```

$$y(x) \to \frac{1}{66} e^{-\frac{1}{4}\left(\sqrt{33}-3\right)x}\left(\left(99-13\sqrt{33}\right)e^{\frac{\sqrt{33}x}{2}} - 132e^{\frac{1}{4}\left(\sqrt{33}-3\right)x} + 99 + 13\sqrt{33}\right)$$

2.13 problem problem 25

Internal problem ID [297]
Internal file name [`OUTPUT/297_Sunday_June_05_2022_01_38_22_AM_69806262/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 5.3, Higher-Order Linear Differential Equations. Homogeneous Equations with Constant Coefficients. Page 300
Problem number: problem 25.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{3y''' + 2y'' = 0}$$

With initial conditions

$$[y(0) = -1, y'(0) = 0, y''(0) = 1]$$

The characteristic equation is

$$3\lambda^3 + 2\lambda^2 = 0$$

The roots of the above equation are

$$\lambda_1 = -\frac{2}{3}$$
$$\lambda_2 = 0$$
$$\lambda_3 = 0$$

Therefore the homogeneous solution is

$$y_h(x) = c_2 x + c_1 + \mathrm{e}^{-\frac{2x}{3}} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= 1 \\
y_2 &= x \\
y_3 &= \mathrm{e}^{-\frac{2x}{3}}
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_2 x + c_1 + \mathrm{e}^{-\frac{2x}{3}} c_3 \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -1$ and $x = 0$ in the above gives

$$
-1 = c_1 + c_3 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = c_2 - \frac{2\,\mathrm{e}^{-\frac{2x}{3}} c_3}{3}
$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$
0 = c_2 - \frac{2c_3}{3} \tag{2A}
$$

Taking two derivatives of the solution gives

$$
y'' = \frac{4\,\mathrm{e}^{-\frac{2x}{3}} c_3}{9}
$$

substituting $y'' = 1$ and $x = 0$ in the above gives

$$
1 = \frac{4c_3}{9} \tag{3A}
$$

Equations {1A,2A,3A} are now solved for $\{c_1, c_2, c_3\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= -\frac{13}{4} \\
c_2 &= \frac{3}{2} \\
c_3 &= \frac{9}{4}
\end{aligned}
$$

Substituting these values back in above solution results in

$$y = -\frac{13}{4} + \frac{3x}{2} + \frac{9\,e^{-\frac{2x}{3}}}{4}$$

Summary
The solution(s) found are the following

$$y = -\frac{13}{4} + \frac{3x}{2} + \frac{9\,e^{-\frac{2x}{3}}}{4} \tag{1}$$

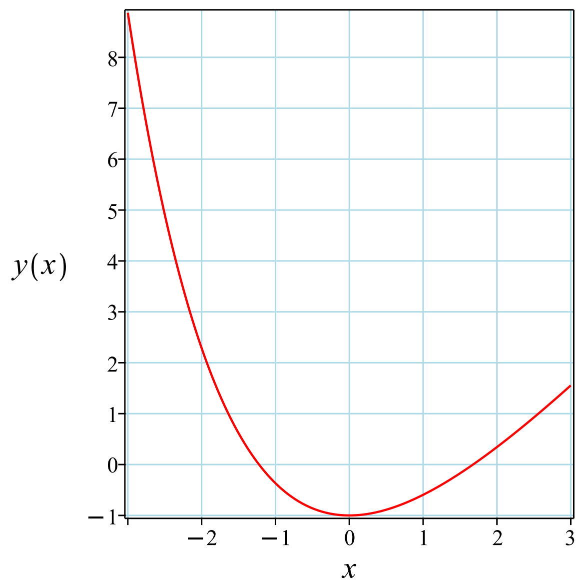


Figure 3: Solution plot

Verification of solutions

$$y = -\frac{13}{4} + \frac{3x}{2} + \frac{9\,e^{-\frac{2x}{3}}}{4}$$

Verified OK.

2.13.1 Maple step by step solution

Let's solve

$$\left[3y''' + 2y'' = 0, y(0) = -1, y'\Big|_{\{x=0\}} = 0, y''\Big|_{\{x=0\}} = 1\right]$$

- Highest derivative means the order of the ODE is 3

y'''

- Isolate 3rd derivative

  $y''' = -\frac{2y''}{3}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''' + \frac{2y''}{3} = 0$

☐ Convert linear ODE into a system of first order ODEs

  - Define new variable $y_1(x)$

    $y_1(x) = y$

  - Define new variable $y_2(x)$

    $y_2(x) = y'$

  - Define new variable $y_3(x)$

    $y_3(x) = y''$

  - Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = -\frac{2y_3(x)}{3}$

  Convert linear ODE into a system of first order ODEs

  $\left[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = -\frac{2y_3(x)}{3}\right]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\frac{2}{3} \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\frac{2}{3} \end{bmatrix}$$

- Rewrite the system as

  $\overrightarrow{y}'(x) = A \cdot \overrightarrow{y}(x)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[-\frac{2}{3}, \left[\begin{array}{c} \frac{9}{4} \\ -\frac{3}{2} \\ 1 \end{array}\right]\right], \left[0, \left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right]\right], \left[0, \left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right]\right]\right]$$

- Consider eigenpair

  $$\left[-\frac{2}{3}, \left[\begin{array}{c} \frac{9}{4} \\ -\frac{3}{2} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

  $$\overrightarrow{y}_1 = \mathrm{e}^{-\frac{2x}{3}} \cdot \left[\begin{array}{c} \frac{9}{4} \\ -\frac{3}{2} \\ 1 \end{array}\right]$$

- Consider eigenpair

  $$\left[0, \left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

  $$\overrightarrow{y}_2 = \left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right]$$

- Consider eigenpair

  $$\left[0, \left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- General solution to the system of ODEs

$\vec{y} = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-\frac{2x}{3}} \cdot \begin{bmatrix} \frac{9}{4} \\ -\frac{3}{2} \\ 1 \end{bmatrix} + \begin{bmatrix} c_2 \\ 0 \\ 0 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$y = \frac{9c_1 e^{-\frac{2x}{3}}}{4} + c_2$

- Use the initial condition $y(0) = -1$

$-1 = \frac{9c_1}{4} + c_2$

- Calculate the 1st derivative of the solution

$y' = -\frac{3c_1 e^{-\frac{2x}{3}}}{2}$

- Use the initial condition $y'\Big|_{\{x=0\}} = 0$

$0 = -\frac{3c_1}{2}$

- Calculate the 2nd derivative of the solution

$y'' = c_1 e^{-\frac{2x}{3}}$

- Use the initial condition $y''\Big|_{\{x=0\}} = 1$

$1 = c_1$

- Solve for the unknown coefficients
- The solution does not satisfy the initial conditions

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 15

```
dsolve([3*diff(y(x),x$3)+2*diff(y(x),x$2)=0,y(0) = -1, D(y)(0) = 0, (D@@2)(y)(0) = 1],y(x),
```

$$y(x) = -\frac{13}{4} + \frac{3x}{2} + \frac{9\,\mathrm{e}^{-\frac{2x}{3}}}{4}$$

✓ Solution by Mathematica

Time used: 0.038 (sec). Leaf size: 23

```
DSolve[{3*y'''[x]+2*y''[x]==0,{y[0]==1,y'[0]==-1,y''[0]==3}},y[x],x,IncludeSingularSolutions
```

$$y(x) \to \frac{1}{4}\left(14x + 27e^{-2x/3} - 23\right)$$

2.14 problem problem 26

Internal problem ID [298]
Internal file name [`OUTPUT/298_Sunday_June_05_2022_01_38_23_AM_15154783/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 5.3, Higher-Order Linear Differential Equations. Homogeneous Equations with Constant Coefficients. Page 300
Problem number: problem 26.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' + 10y'' + 25y' = 0}$$

With initial conditions

$$[y(0) = 3, y'(0) = 4, y''(0) = 5]$$

The characteristic equation is

$$\lambda^3 + 10\lambda^2 + 25\lambda = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = -5$$
$$\lambda_3 = -5$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 + c_2 e^{-5x} + x\, e^{-5x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= 1 \\
y_2 &= \mathrm{e}^{-5x} \\
y_3 &= x\,\mathrm{e}^{-5x}
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 + c_2\mathrm{e}^{-5x} + x\,\mathrm{e}^{-5x}c_3 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 0$ in the above gives

$$3 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -5c_2\mathrm{e}^{-5x} + \mathrm{e}^{-5x}c_3 - 5x\,\mathrm{e}^{-5x}c_3$$

substituting $y' = 4$ and $x = 0$ in the above gives

$$4 = -5c_2 + c_3 \tag{2A}$$

Taking two derivatives of the solution gives

$$y'' = 25c_2\mathrm{e}^{-5x} - 10\,\mathrm{e}^{-5x}c_3 + 25x\,\mathrm{e}^{-5x}c_3$$

substituting $y'' = 5$ and $x = 0$ in the above gives

$$5 = 25c_2 - 10c_3 \tag{3A}$$

Equations {1A,2A,3A} are now solved for $\{c_1, c_2, c_3\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= \frac{24}{5} \\
c_2 &= -\frac{9}{5} \\
c_3 &= -5
\end{aligned}
$$

Substituting these values back in above solution results in

$$y = \frac{24}{5} - \frac{9\,\mathrm{e}^{-5x}}{5} - 5x\,\mathrm{e}^{-5x}$$

Summary

The solution(s) found are the following

$$y = \frac{24}{5} - \frac{9\,e^{-5x}}{5} - 5x\,e^{-5x} \tag{1}$$

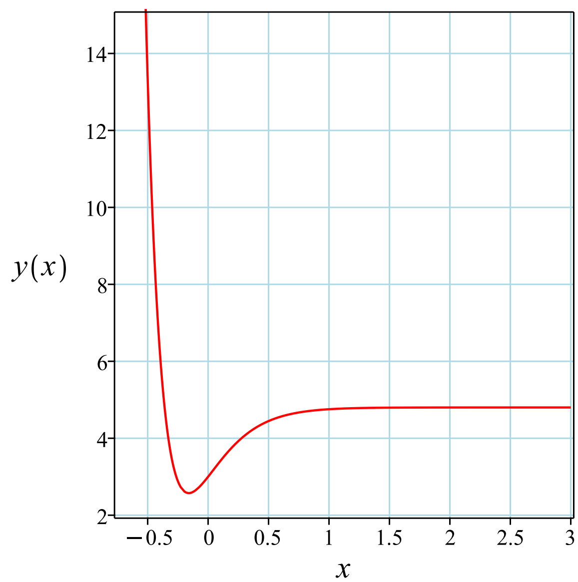


Figure 4: Solution plot

Verification of solutions

$$y = \frac{24}{5} - \frac{9\,e^{-5x}}{5} - 5x\,e^{-5x}$$

Verified OK.

2.14.1 Maple step by step solution

Let's solve

$$\left[y''' + 10y'' + 25y' = 0, y(0) = 3, y'\Big|_{\{x=0\}} = 4, y''\Big|_{\{x=0\}} = 5\right]$$

- Highest derivative means the order of the ODE is 3

  y'''

□ Convert linear ODE into a system of first order ODEs

  ○ Define new variable $y_1(x)$

    $y_1(x) = y$

- Define new variable $y_2(x)$

  $y_2(x) = y'$
- Define new variable $y_3(x)$

  $y_3(x) = y''$
- Isolate for $y_3'(x)$ using original ODE

  $y_3'(x) = -10y_3(x) - 25y_2(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = -10y_3(x) - 25y_2(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -25 & -10 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -25 & -10 \end{bmatrix}$$

- Rewrite the system as

  $\vec{y}'(x) = A \cdot \vec{y}(x)$
- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-5, \begin{bmatrix} \frac{1}{25} \\ -\frac{1}{5} \\ 1 \end{bmatrix}\right], \left[-5, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right], \left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[-5, \left[\begin{array}{c} \frac{1}{25} \\ -\frac{1}{5} \\ 1 \end{array}\right]\right]$$

- First solution from eigenvalue -5

$$\overrightarrow{y}_{1}(x) = \mathrm{e}^{-5x} \cdot \left[\begin{array}{c} \frac{1}{25} \\ -\frac{1}{5} \\ 1 \end{array}\right]$$

- Form of the 2nd homogeneous solution where $\overrightarrow{p}$ is to be solved for, $\lambda = -5$ is the eigenvalue, a

$$\overrightarrow{y}_{2}(x) = \mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right)$$

- Note that the x multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obt
- Substitute $\overrightarrow{y}_{2}(x)$ into the homogeneous system

$$\lambda\, \mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda x}\overrightarrow{v} = \left(\mathrm{e}^{\lambda x} A\right) \cdot \left(x\overrightarrow{v} + \overrightarrow{p}\right)$$

- Use the fact that $\overrightarrow{v}$ is an eigenvector of A

$$\lambda\, \mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda x}\overrightarrow{v} = \mathrm{e}^{\lambda x}\left(\lambda x\overrightarrow{v} + A \cdot \overrightarrow{p}\right)$$

- Simplify equation

$$\lambda\overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$$

- Make use of the identity matrix I

$$(\lambda \cdot I) \cdot \overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$$

- Condition $\overrightarrow{p}$ must meet for $\overrightarrow{y}_{2}(x)$ to be a solution to the homogeneous system

$$(A - \lambda \cdot I) \cdot \overrightarrow{p} = \overrightarrow{v}$$

- Choose $\overrightarrow{p}$ to use in the second solution to the homogeneous system from eigenvalue -5

$$\left(\left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -25 & -10 \end{array}\right] - (-5) \cdot \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]\right) \cdot \overrightarrow{p} = \left[\begin{array}{c} \frac{1}{25} \\ -\frac{1}{5} \\ 1 \end{array}\right]$$

- Choice of $\overrightarrow{p}$

$$\vec{p} = \begin{bmatrix} \frac{1}{125} \\ 0 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue -5

$$\vec{y}_2(x) = \mathrm{e}^{-5x} \cdot \left(x \cdot \begin{bmatrix} \frac{1}{25} \\ -\frac{1}{5} \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{125} \\ 0 \\ 0 \end{bmatrix} \right)$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{y} = c_1 \vec{y}_1(x) + c_2 \vec{y}_2(x) + c_3 \vec{y}_3$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 \mathrm{e}^{-5x} \cdot \begin{bmatrix} \frac{1}{25} \\ -\frac{1}{5} \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{-5x} \cdot \left(x \cdot \begin{bmatrix} \frac{1}{25} \\ -\frac{1}{5} \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{125} \\ 0 \\ 0 \end{bmatrix} \right) + \begin{bmatrix} c_3 \\ 0 \\ 0 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{((5x+1)c_2 + 5c_1)\mathrm{e}^{-5x}}{125} + c_3$$

- Use the initial condition $y(0) = 3$

$$3 = \frac{c_1}{25} + \frac{c_2}{125} + c_3$$

- Calculate the 1st derivative of the solution

$$y' = \frac{c_2 \mathrm{e}^{-5x}}{25} - \frac{((5x+1)c_2 + 5c_1)\mathrm{e}^{-5x}}{25}$$

- Use the initial condition $y'\Big|_{\{x=0\}} = 4$

  $4 = -\frac{c_1}{5}$
- Calculate the 2nd derivative of the solution

  $y'' = -\frac{2c_2 e^{-5x}}{5} + \frac{((5x+1)c_2+5c_1)e^{-5x}}{5}$
- Use the initial condition $y''\Big|_{\{x=0\}} = 5$

  $5 = -\frac{c_2}{5} + c_1$
- Solve for the unknown coefficients

  $\{c_1 = -20, c_2 = -125, c_3 = \frac{24}{5}\}$
- Solution to the IVP

  $y = \frac{24}{5} - \frac{9\,e^{-5x}}{5} - 5x\,e^{-5x}$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve([diff(y(x),x$3)+10*diff(y(x),x$2)+25*diff(y(x),x)=0,y(0) = 3, D(y)(0) = 4, (D@@2)(y)(
```

$$y(x) = \frac{24}{5} - \frac{9\,e^{-5x}}{5} - 5\,e^{-5x}x$$

✓ Solution by Mathematica

Time used: 0.045 (sec). Leaf size: 26

```
DSolve[{y'''[x]+10*y''[x]+25*y'[x]==0,{y[0]==3,y'[0]==4,y''[0]==5}},y[x],x,IncludeSingularSo
```

$$y(x) \to \frac{1}{5}e^{-5x}\left(-25x + 24e^{5x} - 9\right)$$

2.15 problem problem 27

Internal problem ID [299]
Internal file name [`OUTPUT/299_Sunday_June_05_2022_01_38_23_AM_18885951/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 5.3, Higher-Order Linear Differential Equations. Homogeneous Equations with Constant Coefficients. Page 300
Problem number: problem 27.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' + 3y'' - 4y = 0}$$

The characteristic equation is

$$\lambda^3 + 3\lambda^2 - 4 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = -2$$
$$\lambda_3 = -2$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-2x} + x\,\mathrm{e}^{-2x} c_2 + c_3 \mathrm{e}^{x}$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = \mathrm{e}^{-2x}$$
$$y_2 = \mathrm{e}^{-2x} x$$
$$y_3 = \mathrm{e}^{x}$$

Summary
The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-2x} + x\,\mathrm{e}^{-2x} c_2 + c_3 \mathrm{e}^{x} \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{-2x} + x\,\mathrm{e}^{-2x} c_2 + c_3 \mathrm{e}^{x}$$

Verified OK.

2.15.1 Maple step by step solution

Let's solve

$y''' + 3y'' - 4y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Convert linear ODE into a system of first order ODEs
  - ◦ Define new variable $y_1(x)$

    $y_1(x) = y$
  - ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$
  - ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$
  - ◦ Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = -3y_3(x) + 4y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = -3y_3(x) + 4y_1(x)]$

- Define vector

  $$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\overrightarrow{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & 0 & -3 \end{bmatrix} \cdot \overrightarrow{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & 0 & -3 \end{bmatrix}$$

- Rewrite the system as

$\overrightarrow{y}'(x) = A \cdot \overrightarrow{y}(x)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right], \left[-2, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[-2, \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue -2

$$\overrightarrow{y}_1(x) = e^{-2x} \cdot \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\overrightarrow{p}$ is to be solved for, $\lambda = -2$ is the eigenvalue, a

$\overrightarrow{y}_2(x) = e^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right)$

- Note that the x multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obt
- Substitute $\overrightarrow{y}_2(x)$ into the homogeneous system

$\lambda e^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right) + e^{\lambda x}\overrightarrow{v} = \left(e^{\lambda x}A\right) \cdot \left(x\overrightarrow{v} + \overrightarrow{p}\right)$

- Use the fact that $\vec{v}$ is an eigenvector of A

  $\lambda\, \mathrm{e}^{\lambda x}\left(x\vec{v}+\vec{p}\right)+\mathrm{e}^{\lambda x}\vec{v}=\mathrm{e}^{\lambda x}\left(\lambda x\vec{v}+A\cdot\vec{p}\right)$

- Simplify equation

  $\lambda\vec{p}+\vec{v}=A\cdot\vec{p}$

- Make use of the identity matrix I

  $(\lambda\cdot I)\cdot\vec{p}+\vec{v}=A\cdot\vec{p}$

- Condition $\vec{p}$ must meet for $\vec{y}_2(x)$ to be a solution to the homogeneous system

  $(A-\lambda\cdot I)\cdot\vec{p}=\vec{v}$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue -2

  $$\left(\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 4 & 0 & -3\end{bmatrix}-(-2)\cdot\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)\cdot\vec{p}=\begin{bmatrix}\frac{1}{4}\\ -\frac{1}{2}\\ 1\end{bmatrix}$$

- Choice of $\vec{p}$

  $$\vec{p}=\begin{bmatrix}\frac{1}{8}\\ 0\\ 0\end{bmatrix}$$

- Second solution from eigenvalue -2

  $$\vec{y}_2(x)=\mathrm{e}^{-2x}\cdot\left(x\cdot\begin{bmatrix}\frac{1}{4}\\ -\frac{1}{2}\\ 1\end{bmatrix}+\begin{bmatrix}\frac{1}{8}\\ 0\\ 0\end{bmatrix}\right)$$

- Consider eigenpair

  $$\left[1, \begin{bmatrix}1\\ 1\\ 1\end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

  $$\vec{y}_3=\mathrm{e}^{x}\cdot\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}$$

- General solution to the system of ODEs

  $\vec{y} = c_1 \vec{y}_1(x) + c_2 \vec{y}_2(x) + c_3 \vec{y}_3$

- Substitute solutions into the general solution

  $$\vec{y} = c_1 \mathrm{e}^{-2x} \cdot \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{-2x} \cdot \left(x \cdot \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{8} \\ 0 \\ 0 \end{bmatrix} \right) + c_3 \mathrm{e}^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

  $y = \frac{(8c_3 \mathrm{e}^{3x} + 2c_2 x + 2c_1 + c_2) \mathrm{e}^{-2x}}{8}$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve(diff(y(x),x$3)+3*diff(y(x),x$2)-4*y(x)=0,y(x), singsol=all)
```

$$y(x) = \left(c_1 \mathrm{e}^{3x} + c_3 x + c_2\right) \mathrm{e}^{-2x}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 26

```
DSolve[y'''[x]+3*y''[x]-4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-2x}\left(c_2 x + c_3 e^{3x} + c_1\right)$$

2.16 problem problem 28

Internal problem ID [300]
Internal file name [`OUTPUT/300_Sunday_June_05_2022_01_38_24_AM_29981424/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 5.3, Higher-Order Linear Differential Equations. Homogeneous Equations with Constant Coefficients. Page 300
Problem number: problem 28.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{2y''' - y'' - 5y' - 2y = 0}$$

The characteristic equation is

$$2\lambda^3 - \lambda^2 - 5\lambda - 2 = 0$$

The roots of the above equation are

$$\lambda_1 = 2$$
$$\lambda_2 = -\frac{1}{2}$$
$$\lambda_3 = -1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{2x} + \mathrm{e}^{-\frac{x}{2}} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = \mathrm{e}^{-x}$$
$$y_2 = \mathrm{e}^{2x}$$
$$y_3 = \mathrm{e}^{-\frac{x}{2}}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-x} + c_2 e^{2x} + e^{-\frac{x}{2}} c_3 \tag{1}$$

Verification of solutions

$$y = c_1 e^{-x} + c_2 e^{2x} + e^{-\frac{x}{2}} c_3$$

Verified OK.

2.16.1 Maple step by step solution

Let's solve

$2y''' - y'' - 5y' - 2y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

- Isolate 3rd derivative

  $y''' = \frac{y''}{2} + \frac{5y'}{2} + y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''' - \frac{y''}{2} - \frac{5y'}{2} - y = 0$

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$

  ◦ Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = \frac{y_3(x)}{2} + \frac{5y_2(x)}{2} + y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $\left[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = \frac{y_3(x)}{2} + \frac{5y_2(x)}{2} + y_1(x)\right]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & \frac{5}{2} & \frac{1}{2} \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & \frac{5}{2} & \frac{1}{2} \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[-\frac{1}{2}, \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[-\frac{1}{2}, \left[\begin{array}{c} 4 \\ -2 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_2 = \mathrm{e}^{-\frac{x}{2}} \cdot \left[\begin{array}{c} 4 \\ -2 \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[2, \left[\begin{array}{c} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_3 = \mathrm{e}^{2x} \cdot \left[\begin{array}{c} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{array}\right]$$

- General solution to the system of ODEs

$$\overrightarrow{y} = c_1 \overrightarrow{y}_1 + c_2 \overrightarrow{y}_2 + c_3 \overrightarrow{y}_3$$

- Substitute solutions into the general solution

$$\overrightarrow{y} = c_1 \mathrm{e}^{-x} \cdot \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right] + c_2 \mathrm{e}^{-\frac{x}{2}} \cdot \left[\begin{array}{c} 4 \\ -2 \\ 1 \end{array}\right] + c_3 \mathrm{e}^{2x} \cdot \left[\begin{array}{c} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y = \frac{\left(c_3 \mathrm{e}^{3x} + 16 c_2 \mathrm{e}^{\frac{x}{2}} + 4 c_1\right) \mathrm{e}^{-x}}{4}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 23

```
dsolve(2*diff(y(x),x$3)-diff(y(x),x$2)-5*diff(y(x),x)-2*y(x)=0,y(x), singsol=all)
```

$$y(x) = \left(c_2 e^{3x} + c_1 e^{\frac{x}{2}} + c_3\right) e^{-x}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 32

```
DSolve[2*y'''[x]-y''[x]-5*y'[x]-2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x}\left(c_1 e^{x/2} + c_3 e^{3x} + c_2\right)$$

2.17 problem problem 29

Internal problem ID [301]
Internal file name [`OUTPUT/301_Sunday_June_05_2022_01_38_24_AM_24282700/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 5.3, Higher-Order Linear Differential Equations. Homogeneous Equations with Constant Coefficients. Page 300
Problem number: problem 29.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' + 27y = 0}$$

The characteristic equation is

$$\lambda^3 + 27 = 0$$

The roots of the above equation are

$$\begin{aligned}
\lambda_1 &= -3 \\
\lambda_2 &= \frac{3}{2} - \frac{3i\sqrt{3}}{2} \\
\lambda_3 &= \frac{3}{2} + \frac{3i\sqrt{3}}{2}
\end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-3x} + \mathrm{e}^{\left(\frac{3}{2} - \frac{3i\sqrt{3}}{2}\right)x} c_2 + \mathrm{e}^{\left(\frac{3}{2} + \frac{3i\sqrt{3}}{2}\right)x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned}
y_1 &= \mathrm{e}^{-3x} \\
y_2 &= \mathrm{e}^{\left(\frac{3}{2} - \frac{3i\sqrt{3}}{2}\right)x} \\
y_3 &= \mathrm{e}^{\left(\frac{3}{2} + \frac{3i\sqrt{3}}{2}\right)x}
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-3x} + e^{\left(\frac{3}{2}-\frac{3i\sqrt{3}}{2}\right)x} c_2 + e^{\left(\frac{3}{2}+\frac{3i\sqrt{3}}{2}\right)x} c_3 \tag{1}$$

Verification of solutions

$$y = c_1 e^{-3x} + e^{\left(\frac{3}{2}-\frac{3i\sqrt{3}}{2}\right)x} c_2 + e^{\left(\frac{3}{2}+\frac{3i\sqrt{3}}{2}\right)x} c_3$$

Verified OK.

2.17.1 Maple step by step solution

Let's solve

$y''' + 27y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$

  ◦ Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = -27y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = -27y_1(x)]$

- Define vector

  $$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\overrightarrow{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -27 & 0 & 0 \end{bmatrix} \cdot \overrightarrow{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -27 & 0 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\overrightarrow{y}'(x) = A \cdot \overrightarrow{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-3, \begin{bmatrix} \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix}\right], \left[\frac{3}{2} - \frac{3\,\mathrm{I}\sqrt{3}}{2}, \begin{bmatrix} \frac{1}{\left(\frac{3}{2}-\frac{3\,\mathrm{I}\sqrt{3}}{2}\right)^2} \\ \frac{1}{\frac{3}{2}-\frac{3\,\mathrm{I}\sqrt{3}}{2}} \\ 1 \end{bmatrix}\right], \left[\frac{3}{2} + \frac{3\,\mathrm{I}\sqrt{3}}{2}, \begin{bmatrix} \frac{1}{\left(\frac{3}{2}+\frac{3\,\mathrm{I}\sqrt{3}}{2}\right)^2} \\ \frac{1}{\frac{3}{2}+\frac{3\,\mathrm{I}\sqrt{3}}{2}} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-3, \begin{bmatrix} \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = \mathrm{e}^{-3x} \cdot \begin{bmatrix} \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[\frac{3}{2} - \frac{3\,\mathrm{I}\sqrt{3}}{2}, \begin{bmatrix} \frac{1}{\left(\frac{3}{2}-\frac{3\,\mathrm{I}\sqrt{3}}{2}\right)^2} \\ \frac{1}{\frac{3}{2}-\frac{3\,\mathrm{I}\sqrt{3}}{2}} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{\left(\frac{3}{2}-\frac{3\,I\sqrt{3}}{2}\right)x}\cdot\left[\begin{array}{c}\frac{1}{\left(\frac{3}{2}-\frac{3\,I\sqrt{3}}{2}\right)^{2}}\\ \frac{1}{\frac{3}{2}-\frac{3\,I\sqrt{3}}{2}}\\ 1\end{array}\right]$$

- Use Euler identity to write solution in terms of sin and cos

$$e^{\frac{3x}{2}}\cdot\left(\cos\left(\frac{3\sqrt{3}\,x}{2}\right)-I\sin\left(\frac{3\sqrt{3}\,x}{2}\right)\right)\cdot\left[\begin{array}{c}\frac{1}{\left(\frac{3}{2}-\frac{3\,I\sqrt{3}}{2}\right)^{2}}\\ \frac{1}{\frac{3}{2}-\frac{3\,I\sqrt{3}}{2}}\\ 1\end{array}\right]$$

- Simplify expression

$$e^{\frac{3x}{2}}\cdot\left[\begin{array}{c}\frac{\cos\left(\frac{3\sqrt{3}\,x}{2}\right)-I\sin\left(\frac{3\sqrt{3}\,x}{2}\right)}{\left(\frac{3}{2}-\frac{3\,I\sqrt{3}}{2}\right)^{2}}\\ \frac{\cos\left(\frac{3\sqrt{3}\,x}{2}\right)-I\sin\left(\frac{3\sqrt{3}\,x}{2}\right)}{\frac{3}{2}-\frac{3\,I\sqrt{3}}{2}}\\ \cos\left(\frac{3\sqrt{3}\,x}{2}\right)-I\sin\left(\frac{3\sqrt{3}\,x}{2}\right)\end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_{2}(x)=e^{\frac{3x}{2}}\cdot\left[\begin{array}{c}-\frac{\cos\left(\frac{3\sqrt{3}\,x}{2}\right)}{18}+\frac{\sin\left(\frac{3\sqrt{3}\,x}{2}\right)\sqrt{3}}{18}\\ \frac{\cos\left(\frac{3\sqrt{3}\,x}{2}\right)}{6}+\frac{\sin\left(\frac{3\sqrt{3}\,x}{2}\right)\sqrt{3}}{6}\\ \cos\left(\frac{3\sqrt{3}\,x}{2}\right)\end{array}\right],\vec{y}_{3}(x)=e^{\frac{3x}{2}}\cdot\left[\begin{array}{c}\frac{\cos\left(\frac{3\sqrt{3}\,x}{2}\right)\sqrt{3}}{18}+\frac{\sin\left(\frac{3\sqrt{3}\,x}{2}\right)}{18}\\ \frac{\cos\left(\frac{3\sqrt{3}\,x}{2}\right)\sqrt{3}}{6}-\frac{\sin\left(\frac{3\sqrt{3}\,x}{2}\right)}{6}\\ -\sin\left(\frac{3\sqrt{3}\,x}{2}\right)\end{array}\right]\right]$$

- General solution to the system of ODEs

$$\vec{y}=c_{1}\vec{y}_{1}+c_{2}\vec{y}_{2}(x)+c_{3}\vec{y}_{3}(x)$$

- Substitute solutions into the general solution

$$\vec{y}=c_{1}e^{-3x}\cdot\left[\begin{array}{c}\frac{1}{9}\\ -\frac{1}{3}\\ 1\end{array}\right]+c_{2}e^{\frac{3x}{2}}\cdot\left[\begin{array}{c}-\frac{\cos\left(\frac{3\sqrt{3}\,x}{2}\right)}{18}+\frac{\sin\left(\frac{3\sqrt{3}\,x}{2}\right)\sqrt{3}}{18}\\ \frac{\cos\left(\frac{3\sqrt{3}\,x}{2}\right)}{6}+\frac{\sin\left(\frac{3\sqrt{3}\,x}{2}\right)\sqrt{3}}{6}\\ \cos\left(\frac{3\sqrt{3}\,x}{2}\right)\end{array}\right]+c_{3}e^{\frac{3x}{2}}\cdot\left[\begin{array}{c}\frac{\cos\left(\frac{3\sqrt{3}\,x}{2}\right)\sqrt{3}}{18}+\frac{\sin\left(\frac{3\sqrt{3}\,x}{2}\right)}{18}\\ \frac{\cos\left(\frac{3\sqrt{3}\,x}{2}\right)\sqrt{3}}{6}-\frac{\sin\left(\frac{3\sqrt{3}\,x}{2}\right)}{6}\\ -\sin\left(\frac{3\sqrt{3}\,x}{2}\right)\end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y = \frac{\left(-\frac{e^{\frac{9x}{2}}\left(-c_3\sqrt{3}+c_2\right)\cos\left(\frac{3\sqrt{3}\,x}{2}\right)}{2}+\frac{e^{\frac{9x}{2}}\left(\sqrt{3}\,c_2+c_3\right)\sin\left(\frac{3\sqrt{3}\,x}{2}\right)}{2}+c_1\right)e^{-3x}}{9}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 37

```
dsolve(diff(y(x),x$3)+27*y(x)=0,y(x), singsol=all)
```

$$y(x) = \left(c_2 e^{\frac{9x}{2}}\sin\left(\frac{3\sqrt{3}\,x}{2}\right)+c_3 e^{\frac{9x}{2}}\cos\left(\frac{3\sqrt{3}\,x}{2}\right)+c_1\right)e^{-3x}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 56

```
DSolve[y'''[x]+27*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-3x}\left(c_3 e^{9x/2}\cos\left(\frac{3\sqrt{3}x}{2}\right)+c_2 e^{9x/2}\sin\left(\frac{3\sqrt{3}x}{2}\right)+c_1\right)$$

2.18 problem problem 30

Internal problem ID [302]
Internal file name [`OUTPUT/302_Sunday_June_05_2022_01_38_25_AM_61857055/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 5.3, Higher-Order Linear Differential Equations. Homogeneous Equations with Constant Coefficients. Page 300
Problem number: problem 30.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' - y''' + y'' - 3y' - 6y = 0}$$

The characteristic equation is

$$\lambda^4 - \lambda^3 + \lambda^2 - 3\lambda - 6 = 0$$

The roots of the above equation are

$$\lambda_1 = 2$$
$$\lambda_2 = -1$$
$$\lambda_3 = i\sqrt{3}$$
$$\lambda_4 = -i\sqrt{3}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + c_2 e^{2x} + e^{i\sqrt{3}\, x} c_3 + e^{-i\sqrt{3}\, x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-x}$$
$$y_2 = e^{2x}$$
$$y_3 = e^{i\sqrt{3}\, x}$$
$$y_4 = e^{-i\sqrt{3}\, x}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-x} + c_2 e^{2x} + e^{i\sqrt{3}\, x} c_3 + e^{-i\sqrt{3}\, x} c_4 \tag{1}$$

Verification of solutions

$$y = c_1 e^{-x} + c_2 e^{2x} + e^{i\sqrt{3}\, x} c_3 + e^{-i\sqrt{3}\, x} c_4$$

Verified OK.

2.18.1 Maple step by step solution

Let's solve

$y'''' - y''' + y'' - 3y' - 6y = 0$

- Highest derivative means the order of the ODE is 4

  y''''

- □ Convert linear ODE into a system of first order ODEs
  - ◦ Define new variable $y_1(x)$

    $y_1(x) = y$
  - ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$
  - ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$
  - ◦ Define new variable $y_4(x)$

    $y_4(x) = y'''$
  - ◦ Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = y_4(x) - y_3(x) + 3y_2(x) + 6y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = y_4(x) - y_3(x) + 3y_2(x) + 6y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 6 & 3 & -1 & 1 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 6 & 3 & -1 & 1 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right], \left[-\mathrm{I}\sqrt{3}, \begin{bmatrix} -\frac{\mathrm{I}}{9}\sqrt{3} \\ -\frac{1}{3} \\ \frac{\mathrm{I}}{3}\sqrt{3} \\ 1 \end{bmatrix}\right], \left[\mathrm{I}\sqrt{3}, \begin{bmatrix} \frac{\mathrm{I}}{9}\sqrt{3} \\ -\frac{1}{3} \\ -\frac{\mathrm{I}}{3}\sqrt{3} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = e^{-x} \cdot \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^{2x} \cdot \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-I\sqrt{3}, \begin{bmatrix} -\frac{I}{9}\sqrt{3} \\ -\frac{1}{3} \\ \frac{I}{3}\sqrt{3} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{-I\sqrt{3}\,x} \cdot \begin{bmatrix} -\frac{I}{9}\sqrt{3} \\ -\frac{1}{3} \\ \frac{I}{3}\sqrt{3} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\left(\cos\left(\sqrt{3}\,x\right) - \mathrm{I}\sin\left(\sqrt{3}\,x\right)\right) \cdot \left[\begin{array}{c} -\frac{\mathrm{I}}{9}\sqrt{3} \\ -\frac{1}{3} \\ \frac{\mathrm{I}}{3}\sqrt{3} \\ 1 \end{array}\right]$$

- Simplify expression

$$\left[\begin{array}{c} -\frac{\mathrm{I}}{9}\left(\cos\left(\sqrt{3}\,x\right) - \mathrm{I}\sin\left(\sqrt{3}\,x\right)\right)\sqrt{3} \\ -\frac{\cos\left(\sqrt{3}\,x\right)}{3} + \frac{\mathrm{I}\sin\left(\sqrt{3}\,x\right)}{3} \\ \frac{\mathrm{I}}{3}\left(\cos\left(\sqrt{3}\,x\right) - \mathrm{I}\sin\left(\sqrt{3}\,x\right)\right)\sqrt{3} \\ \cos\left(\sqrt{3}\,x\right) - \mathrm{I}\sin\left(\sqrt{3}\,x\right) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(x) = \left[\begin{array}{c} -\frac{\sqrt{3}\,\sin\left(\sqrt{3}\,x\right)}{9} \\ -\frac{\cos\left(\sqrt{3}\,x\right)}{3} \\ \frac{\sqrt{3}\,\sin\left(\sqrt{3}\,x\right)}{3} \\ \cos\left(\sqrt{3}\,x\right) \end{array}\right], \vec{y}_4(x) = \left[\begin{array}{c} -\frac{\sqrt{3}\,\cos\left(\sqrt{3}\,x\right)}{9} \\ \frac{\sin\left(\sqrt{3}\,x\right)}{3} \\ \frac{\sqrt{3}\,\cos\left(\sqrt{3}\,x\right)}{3} \\ -\sin\left(\sqrt{3}\,x\right) \end{array}\right]\right]$$

- General solution to the system of ODEs

$$\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3(x) + c_4\vec{y}_4(x)$$

- Substitute solutions into the general solution

$$\vec{y} = c_1\mathrm{e}^{-x} \cdot \left[\begin{array}{c} -1 \\ 1 \\ -1 \\ 1 \end{array}\right] + c_2\mathrm{e}^{2x} \cdot \left[\begin{array}{c} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{array}\right] + \left[\begin{array}{c} -\frac{c_4\sqrt{3}\,\cos\left(\sqrt{3}\,x\right)}{9} - \frac{c_3\sqrt{3}\,\sin\left(\sqrt{3}\,x\right)}{9} \\ \frac{c_4\sin\left(\sqrt{3}\,x\right)}{3} - \frac{c_3\cos\left(\sqrt{3}\,x\right)}{3} \\ \frac{c_4\sqrt{3}\,\cos\left(\sqrt{3}\,x\right)}{3} + \frac{c_3\sqrt{3}\,\sin\left(\sqrt{3}\,x\right)}{3} \\ -c_4\sin\left(\sqrt{3}\,x\right) + c_3\cos\left(\sqrt{3}\,x\right) \end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y = -c_1\mathrm{e}^{-x} + \frac{c_2\mathrm{e}^{2x}}{8} - \frac{c_4\sqrt{3}\,\cos\left(\sqrt{3}\,x\right)}{9} - \frac{c_3\sqrt{3}\,\sin\left(\sqrt{3}\,x\right)}{9}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 33

```
dsolve(diff(y(x),x$4)-diff(y(x),x$3)+diff(y(x),x$2)-3*diff(y(x),x)-6*y(x)=0,y(x), singsol=al
```

$$y(x) = e^{2x}c_1 + c_2e^{-x} + c_3 \sin\left(\sqrt{3}\,x\right) + c_4 \cos\left(\sqrt{3}\,x\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 44

```
DSolve[y''''[x]-y'''[x]+y''[x]-3*y'[x]-6*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_3e^{-x} + c_4e^{2x} + c_1 \cos\left(\sqrt{3}x\right) + c_2 \sin\left(\sqrt{3}x\right)$$

2.19 problem problem 31

Internal problem ID [303]
Internal file name [`OUTPUT/303_Sunday_June_05_2022_01_38_25_AM_57161591/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 5.3, Higher-Order Linear Differential Equations. Homogeneous Equations with Constant Coefficients. Page 300
Problem number: problem 31.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' + 3y'' + 4y' - 8y = 0}$$

The characteristic equation is

$$\lambda^3 + 3\lambda^2 + 4\lambda - 8 = 0$$

The roots of the above equation are

$$\begin{aligned}\lambda_1 &= 1\\ \lambda_2 &= -2 - 2i\\ \lambda_3 &= -2 + 2i\end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^x + \mathrm{e}^{(-2+2i)x} c_2 + \mathrm{e}^{(-2-2i)x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned}y_1 &= \mathrm{e}^x\\ y_2 &= \mathrm{e}^{(-2+2i)x}\\ y_3 &= \mathrm{e}^{(-2-2i)x}\end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^x + e^{(-2+2i)x} c_2 + e^{(-2-2i)x} c_3 \tag{1}$$

Verification of solutions

$$y = c_1 e^x + e^{(-2+2i)x} c_2 + e^{(-2-2i)x} c_3$$

Verified OK.

2.19.1 Maple step by step solution

Let's solve

$y''' + 3y'' + 4y' - 8y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$

  ◦ Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = -3y_3(x) - 4y_2(x) + 8y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = -3y_3(x) - 4y_2(x) + 8y_1(x)]$

- Define vector

  $$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 8 & -4 & -3 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 8 & -4 & -3 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[-2-2\,\mathrm{I}, \begin{bmatrix} -\frac{\mathrm{I}}{8} \\ -\frac{1}{4}+\frac{\mathrm{I}}{4} \\ 1 \end{bmatrix}\right], \left[-2+2\,\mathrm{I}, \begin{bmatrix} \frac{\mathrm{I}}{8} \\ -\frac{1}{4}-\frac{\mathrm{I}}{4} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-2-2\,\mathrm{I}, \begin{bmatrix} -\frac{\mathrm{I}}{8} \\ -\frac{1}{4}+\frac{\mathrm{I}}{4} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{(-2-2\,I)x} \cdot \begin{bmatrix} -\frac{I}{8} \\ -\frac{1}{4}+\frac{I}{4} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$e^{-2x} \cdot (\cos(2x) - I\sin(2x)) \cdot \begin{bmatrix} -\frac{I}{8} \\ -\frac{1}{4}+\frac{I}{4} \\ 1 \end{bmatrix}$$

- Simplify expression

$$e^{-2x} \cdot \begin{bmatrix} -\frac{I}{8}(\cos(2x) - I\sin(2x)) \\ \left(-\frac{1}{4}+\frac{I}{4}\right)(\cos(2x) - I\sin(2x)) \\ \cos(2x) - I\sin(2x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_2(x) = e^{-2x} \cdot \begin{bmatrix} -\frac{\sin(2x)}{8} \\ -\frac{\cos(2x)}{4}+\frac{\sin(2x)}{4} \\ \cos(2x) \end{bmatrix}, \vec{y}_3(x) = e^{-2x} \cdot \begin{bmatrix} -\frac{\cos(2x)}{8} \\ \frac{\sin(2x)}{4}+\frac{\cos(2x)}{4} \\ -\sin(2x) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x)$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 e^{-2x} \cdot \begin{bmatrix} -\frac{\sin(2x)}{8} \\ -\frac{\cos(2x)}{4}+\frac{\sin(2x)}{4} \\ \cos(2x) \end{bmatrix} + c_3 e^{-2x} \cdot \begin{bmatrix} -\frac{\cos(2x)}{8} \\ \frac{\sin(2x)}{4}+\frac{\cos(2x)}{4} \\ -\sin(2x) \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{(8c_1 e^{3x} - c_2\sin(2x) - c_3\cos(2x))e^{-2x}}{8}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 28

```
dsolve(diff(y(x),x$3)+3*diff(y(x),x$2)+4*diff(y(x),x)-8*y(x)=0,y(x), singsol=all)
```

$$y(x) = \left(c_1 \mathrm{e}^{3x} + \sin(2x)\, c_2 + \cos(2x)\, c_3\right) \mathrm{e}^{-2x}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 34

```
DSolve[y'''[x]+3*y''[x]+4*y'[x]-8*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-2x}\left(c_3 e^{3x} + c_2 \cos(2x) + c_1 \sin(2x)\right)$$

2.20 problem problem 32

Internal problem ID [304]
Internal file name [`OUTPUT/304_Sunday_June_05_2022_01_38_26_AM_40263199/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 5.3, Higher-Order Linear Differential Equations. Homogeneous Equations with Constant Coefficients. Page 300
Problem number: problem 32.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

`[[_high_order, _missing_x]]`

$$\boxed{y'''' + y''' - 3y'' - 5y' - 2y = 0}$$

The characteristic equation is

$$\lambda^4 + \lambda^3 - 3\lambda^2 - 5\lambda - 2 = 0$$

The roots of the above equation are

$$\begin{aligned}\lambda_1 &= 2\\ \lambda_2 &= -1\\ \lambda_3 &= -1\\ \lambda_4 &= -1\end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-x} + x\,\mathrm{e}^{-x} c_2 + x^2 \mathrm{e}^{-x} c_3 + \mathrm{e}^{2x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned}y_1 &= \mathrm{e}^{-x}\\ y_2 &= x\,\mathrm{e}^{-x}\\ y_3 &= x^2 \mathrm{e}^{-x}\\ y_4 &= \mathrm{e}^{2x}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} + x\, e^{-x} c_2 + x^2 e^{-x} c_3 + e^{2x} c_4 \tag{1}$$

Verification of solutions

$$y = c_1 e^{-x} + x\, e^{-x} c_2 + x^2 e^{-x} c_3 + e^{2x} c_4$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve(diff(y(x),x$4)+diff(y(x),x$3)-3*diff(y(x),x$2)-5*diff(y(x),x)-2*y(x)=0,y(x), singsol=
```

$$y(x) = (c_4 x^2 + c_3 x + c_2)\, e^{-x} + e^{2x} c_1$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 32

```
DSolve[y''''[x]+y'''[x]-3*y''[x]-5*y'[x]-2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x} (c_3 x^2 + c_2 x + c_4 e^{3x} + c_1)$$

2.21 problem problem 38

Internal problem ID [305]
Internal file name [`OUTPUT/305_Sunday_June_05_2022_01_38_27_AM_94437083/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 5.3, Higher-Order Linear Differential Equations. Homogeneous Equations with Constant Coefficients. Page 300
Problem number: problem 38.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' - 5y'' + 100y' - 500y = 0}$$

With initial conditions

$$[y(0) = 0, y'(0) = 10, y''(0) = 250]$$

The characteristic equation is

$$\lambda^3 - 5\lambda^2 + 100\lambda - 500 = 0$$

The roots of the above equation are

$$\lambda_1 = 5$$
$$\lambda_2 = 10i$$
$$\lambda_3 = -10i$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{5x} + e^{10ix} c_2 + e^{-10ix} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= \mathrm{e}^{5x} \\
y_2 &= \mathrm{e}^{10ix} \\
y_3 &= \mathrm{e}^{-10ix}
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \mathrm{e}^{5x} + \mathrm{e}^{10ix} c_2 + \mathrm{e}^{-10ix} c_3 \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$
0 = c_1 + c_2 + c_3 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = 5c_1 \mathrm{e}^{5x} + 10i\mathrm{e}^{10ix} c_2 - 10i\mathrm{e}^{-10ix} c_3
$$

substituting $y' = 10$ and $x = 0$ in the above gives

$$
10 = 10c_2 i - 10c_3 i + 5c_1 \tag{2A}
$$

Taking two derivatives of the solution gives

$$
y'' = 25c_1 \mathrm{e}^{5x} - 100\,\mathrm{e}^{10ix} c_2 - 100\,\mathrm{e}^{-10ix} c_3
$$

substituting $y'' = 250$ and $x = 0$ in the above gives

$$
250 = 25c_1 - 100c_2 - 100c_3 \tag{3A}
$$

Equations {1A,2A,3A} are now solved for $\{c_1, c_2, c_3\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 2 \\
c_2 &= -1 \\
c_3 &= -1
\end{aligned}
$$

Substituting these values back in above solution results in

$$
y = 2\,\mathrm{e}^{5x} - 2\cos(10x)
$$

Summary

The solution(s) found are the following

$$y = 2\,e^{5x} - 2\cos(10x) \tag{1}$$

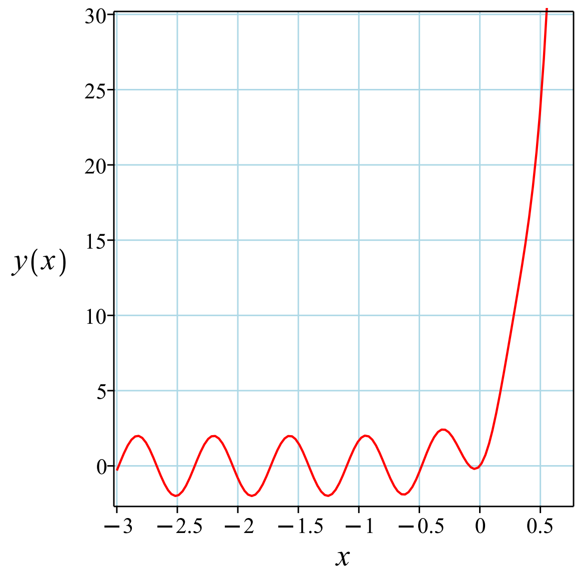


Figure 5: Solution plot

Verification of solutions

$$y = 2\,e^{5x} - 2\cos(10x)$$

Verified OK.

2.21.1 Maple step by step solution

Let's solve

$$\left[y''' - 5y'' + 100y' - 500y = 0, y(0) = 0, y'\Big|_{\{x=0\}} = 10, y''\Big|_{\{x=0\}} = 250\right]$$

- Highest derivative means the order of the ODE is 3

  y'''

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

$y_2(x) = y'$

- Define new variable $y_3(x)$

  $y_3(x) = y''$

- Isolate for $y_3'(x)$ using original ODE

  $y_3'(x) = 5y_3(x) - 100y_2(x) + 500y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = 5y_3(x) - 100y_2(x) + 500y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 500 & -100 & 5 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 500 & -100 & 5 \end{bmatrix}$$

- Rewrite the system as

  $\vec{y}'(x) = A \cdot \vec{y}(x)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[5, \begin{bmatrix} \frac{1}{25} \\ \frac{1}{5} \\ 1 \end{bmatrix}\right], \left[-10\,\mathrm{I}, \begin{bmatrix} -\frac{1}{100} \\ \frac{\mathrm{I}}{10} \\ 1 \end{bmatrix}\right], \left[10\,\mathrm{I}, \begin{bmatrix} -\frac{1}{100} \\ -\frac{\mathrm{I}}{10} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[5, \left[\begin{array}{c} \frac{1}{25} \\ \frac{1}{5} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = e^{5x} \cdot \left[\begin{array}{c} \frac{1}{25} \\ \frac{1}{5} \\ 1 \end{array}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-10\,\mathrm{I}, \left[\begin{array}{c} -\frac{1}{100} \\ \frac{\mathrm{I}}{10} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$e^{-10\,\mathrm{I}x} \cdot \left[\begin{array}{c} -\frac{1}{100} \\ \frac{\mathrm{I}}{10} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of sin and cos

$$\left(\cos\left(10x\right) - \mathrm{I}\sin\left(10x\right)\right) \cdot \left[\begin{array}{c} -\frac{1}{100} \\ \frac{\mathrm{I}}{10} \\ 1 \end{array}\right]$$

- Simplify expression

$$\left[\begin{array}{c} -\frac{\cos(10x)}{100} + \frac{\mathrm{I}\sin(10x)}{100} \\ \frac{\mathrm{I}}{10}\left(\cos\left(10x\right) - \mathrm{I}\sin\left(10x\right)\right) \\ \cos\left(10x\right) - \mathrm{I}\sin\left(10x\right) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_2(x) = \left[\begin{array}{c} -\frac{\cos(10x)}{100} \\ \frac{\sin(10x)}{10} \\ \cos\left(10x\right) \end{array}\right], \vec{y}_3(x) = \left[\begin{array}{c} \frac{\sin(10x)}{100} \\ \frac{\cos(10x)}{10} \\ -\sin\left(10x\right) \end{array}\right]\right]$$

- General solution to the system of ODEs

  $\overrightarrow{y} = c_1 \overrightarrow{y}_1 + c_2 \overrightarrow{y}_2(x) + c_3 \overrightarrow{y}_3(x)$

- Substitute solutions into the general solution

  $$\overrightarrow{y} = c_1 \mathrm{e}^{5x} \cdot \left[\begin{array}{c} \frac{1}{25} \\ \frac{1}{5} \\ 1 \end{array}\right] + \left[\begin{array}{c} -\frac{c_2 \cos(10x)}{100} + \frac{c_3 \sin(10x)}{100} \\ \frac{c_2 \sin(10x)}{10} + \frac{c_3 \cos(10x)}{10} \\ c_2 \cos(10x) - c_3 \sin(10x) \end{array}\right]$$

- First component of the vector is the solution to the ODE

  $y = \frac{c_1 \mathrm{e}^{5x}}{25} + \frac{c_3 \sin(10x)}{100} - \frac{c_2 \cos(10x)}{100}$

- Use the initial condition $y(0) = 0$

  $0 = \frac{c_1}{25} - \frac{c_2}{100}$

- Calculate the 1st derivative of the solution

  $y' = \frac{c_1 \mathrm{e}^{5x}}{5} + \frac{c_3 \cos(10x)}{10} + \frac{c_2 \sin(10x)}{10}$

- Use the initial condition $y'\Big|_{\{x=0\}} = 10$

  $10 = \frac{c_1}{5} + \frac{c_3}{10}$

- Calculate the 2nd derivative of the solution

  $y'' = c_1 \mathrm{e}^{5x} - c_3 \sin(10x) + c_2 \cos(10x)$

- Use the initial condition $y''\Big|_{\{x=0\}} = 250$

  $250 = c_1 + c_2$

- Solve for the unknown coefficients

  $\{c_1 = 50, c_2 = 200, c_3 = 0\}$

- Solution to the IVP

  $y = 2\,\mathrm{e}^{5x} - 2\cos(10x)$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 17

```
dsolve([diff(y(x),x$3)-5*diff(y(x),x$2)+100*diff(y(x),x)-500*y(x)=0,y(0) = 0, D(y)(0) = 10,
```

$$y(x) = 2\,\mathrm{e}^{5x} - 2\cos(10x)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 19

```
DSolve[{y'''[x]-5*y''[x]+100*y'[x]-500*y[x]==0,{y[0]==0,y'[0]==10,y''[0]==250}},y[x],x,Inclu
```

$$y(x) \to 2\left(e^{5x} - \cos(10x)\right)$$

2.22 problem problem 48

Internal problem ID [306]
Internal file name [`OUTPUT/306_Sunday_June_05_2022_01_38_28_AM_89837690/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 5.3, Higher-Order Linear Differential Equations. Homogeneous Equations with Constant Coefficients. Page 300
Problem number: problem 48.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' - y = 0}$$

With initial conditions

$$[y(0) = 1, y'(0) = 0, y''(0) = 0]$$

The characteristic equation is

$$\lambda^3 - 1 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$$
$$\lambda_3 = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^x + e^{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x} c_2 + e^{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= \mathrm{e}^{x} \\
y_2 &= \mathrm{e}^{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x} \\
y_3 &= \mathrm{e}^{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x}
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \mathrm{e}^{x} + \mathrm{e}^{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x} c_2 + \mathrm{e}^{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x} c_3 \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$
1 = c_1 + c_2 + c_3 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = c_1 \mathrm{e}^{x} + \left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right) \mathrm{e}^{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x} c_2 + \left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right) \mathrm{e}^{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x} c_3
$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$
0 = \frac{i(c_2 - c_3)\sqrt{3}}{2} + c_1 - \frac{c_2}{2} - \frac{c_3}{2} \tag{2A}
$$

Taking two derivatives of the solution gives

$$
y'' = c_1 \mathrm{e}^{x} + \left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^2 \mathrm{e}^{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x} c_2 + \left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)^2 \mathrm{e}^{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x} c_3
$$

substituting $y'' = 0$ and $x = 0$ in the above gives

$$
0 = \frac{i(-c_2 + c_3)\sqrt{3}}{2} + c_1 - \frac{c_2}{2} - \frac{c_3}{2} \tag{3A}
$$

Equations {1A,2A,3A} are now solved for $\{c_1, c_2, c_3\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= \frac{1}{3} \\
c_2 &= \frac{1}{3} \\
c_3 &= \frac{1}{3}
\end{aligned}
$$

Substituting these values back in above solution results in

$$y = \frac{e^x}{3} + \frac{e^{\frac{(i\sqrt{3}-1)x}{2}}}{3} + \frac{e^{-\frac{(1+i\sqrt{3})x}{2}}}{3}$$

Summary

The solution(s) found are the following

$$y = \frac{e^x}{3} + \frac{e^{\frac{(i\sqrt{3}-1)x}{2}}}{3} + \frac{e^{-\frac{(1+i\sqrt{3})x}{2}}}{3} \tag{1}$$

Verification of solutions

$$y = \frac{e^x}{3} + \frac{e^{\frac{(i\sqrt{3}-1)x}{2}}}{3} + \frac{e^{-\frac{(1+i\sqrt{3})x}{2}}}{3}$$

Verified OK.

2.22.1 Maple step by step solution

Let's solve

$$\left[y''' - y = 0, y(0) = 1, y'\Big|_{\{x=0\}} = 0, y''\Big|_{\{x=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Convert linear ODE into a system of first order ODEs
  - ○ Define new variable $y_1(x)$

    $y_1(x) = y$
  - ○ Define new variable $y_2(x)$

    $y_2(x) = y'$
  - ○ Define new variable $y_3(x)$

    $y_3(x) = y''$
  - ○ Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[-\frac{1}{2} - \frac{I\sqrt{3}}{2}, \begin{bmatrix} \frac{1}{\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^2} \\ \frac{1}{-\frac{1}{2}-\frac{I\sqrt{3}}{2}} \\ 1 \end{bmatrix}\right], \left[-\frac{1}{2} + \frac{I\sqrt{3}}{2}, \begin{bmatrix} \frac{1}{\left(-\frac{1}{2}+\frac{I\sqrt{3}}{2}\right)^2} \\ \frac{1}{-\frac{1}{2}+\frac{I\sqrt{3}}{2}} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = e^x \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}, \left[\begin{array}{c}\frac{1}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^{2}} \\ \frac{1}{-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}} \\ 1\end{array}\right]\right]$$

- Solution from eigenpair

$$\mathrm{e}^{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)x} \cdot \left[\begin{array}{c}\frac{1}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^{2}} \\ \frac{1}{-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}} \\ 1\end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\mathrm{e}^{-\frac{x}{2}} \cdot \left(\cos\left(\frac{\sqrt{3}\,x}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{3}\,x}{2}\right)\right) \cdot \left[\begin{array}{c}\frac{1}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^{2}} \\ \frac{1}{-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}} \\ 1\end{array}\right]$$

- Simplify expression

$$\mathrm{e}^{-\frac{x}{2}} \cdot \left[\begin{array}{c}\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^{2}} \\ \frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}} \\ \cos\left(\frac{\sqrt{3}\,x}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{3}\,x}{2}\right)\end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_{2}(x)=\mathrm{e}^{-\frac{x}{2}} \cdot \left[\begin{array}{c}-\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2}-\frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} \\ -\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2}+\frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} \\ \cos\left(\frac{\sqrt{3}\,x}{2}\right)\end{array}\right], \vec{y}_{3}(x)=\mathrm{e}^{-\frac{x}{2}} \cdot \left[\begin{array}{c}-\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2}+\frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} \\ \frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2}+\frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} \\ -\sin\left(\frac{\sqrt{3}\,x}{2}\right)\end{array}\right]\right]$$

- General solution to the system of ODEs

$$\vec{y}=c_{1}\vec{y}_{1}+c_{2}\vec{y}_{2}(x)+c_{3}\vec{y}_{3}(x)$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 \mathrm{e}^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{-\frac{x}{2}} \cdot \begin{bmatrix} -\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} - \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} \\ -\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} \\ \cos\left(\frac{\sqrt{3}\,x}{2}\right) \end{bmatrix} + \mathrm{e}^{-\frac{x}{2}} c_3 \cdot \begin{bmatrix} -\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} + \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} \\ \frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} + \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} \\ -\sin\left(\frac{\sqrt{3}\,x}{2}\right) \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = -\frac{\mathrm{e}^{-\frac{x}{2}}\left(c_3\sqrt{3}+c_2\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} - \frac{\mathrm{e}^{-\frac{x}{2}}\left(\sqrt{3}\,c_2-c_3\right)\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + c_1 \mathrm{e}^{x}$$

- Use the initial condition $y(0) = 1$

$$1 = -\frac{c_3\sqrt{3}}{2} - \frac{c_2}{2} + c_1$$

- Calculate the 1st derivative of the solution

$$y' = \frac{\mathrm{e}^{-\frac{x}{2}}\left(c_3\sqrt{3}+c_2\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{4} + \frac{\mathrm{e}^{-\frac{x}{2}}\left(c_3\sqrt{3}+c_2\right)\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{4} + \frac{\mathrm{e}^{-\frac{x}{2}}\left(\sqrt{3}\,c_2-c_3\right)\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{4} - \frac{\mathrm{e}^{-\frac{x}{2}}\left(\sqrt{3}\,c_2-c_3\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{4}$$

- Use the initial condition $y'\Big|_{\{x=0\}} = 0$

$$0 = \frac{c_3\sqrt{3}}{4} + \frac{c_2}{4} - \frac{\left(\sqrt{3}\,c_2-c_3\right)\sqrt{3}}{4} + c_1$$

- Calculate the 2nd derivative of the solution

$$y'' = \frac{\mathrm{e}^{-\frac{x}{2}}\left(c_3\sqrt{3}+c_2\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{4} - \frac{\mathrm{e}^{-\frac{x}{2}}\left(c_3\sqrt{3}+c_2\right)\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{4} + \frac{\mathrm{e}^{-\frac{x}{2}}\left(\sqrt{3}\,c_2-c_3\right)\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{4} + \frac{\mathrm{e}^{-\frac{x}{2}}\left(\sqrt{3}\,c_2-c_3\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{4}$$

- Use the initial condition $y''\Big|_{\{x=0\}} = 0$

$$0 = \frac{c_3\sqrt{3}}{4} + \frac{c_2}{4} + \frac{\left(\sqrt{3}\,c_2-c_3\right)\sqrt{3}}{4} + c_1$$

- Solve for the unknown coefficients

$$\left\{c_1 = \frac{1}{3}, c_2 = -\frac{1}{3}, c_3 = -\frac{\sqrt{3}}{3}\right\}$$

- Solution to the IVP

$$y = \frac{2\,\mathrm{e}^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{3} + \frac{\mathrm{e}^{x}}{3}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 22

```
dsolve([diff(y(x),x$3)=y(x),y(0) = 1, D(y)(0) = 0, (D@@2)(y)(0) = 0],y(x), singsol=all)
```

$$y(x) = \frac{e^x}{3} + \frac{2\,e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{3}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 33

```
DSolve[{y'''[x]==y[x],{y[0]==1,y'[0]==0,y''[0]==0}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{3}\left(e^x + 2e^{-x/2}\cos\left(\frac{\sqrt{3}x}{2}\right)\right)$$

2.23 problem problem 49

Internal problem ID [307]
Internal file name [`OUTPUT/307_Sunday_June_05_2022_01_38_29_AM_34480318/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 5.3, Higher-Order Linear Differential Equations. Homogeneous Equations with Constant Coefficients. Page 300
Problem number: problem 49.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' - y''' - y'' - y' - 2y = 0}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0, y''(0) = 0, y'''(0) = 15]$$

The characteristic equation is

$$\lambda^4 - \lambda^3 - \lambda^2 - \lambda - 2 = 0$$

The roots of the above equation are

$$\lambda_1 = 2$$
$$\lambda_2 = -1$$
$$\lambda_3 = i$$
$$\lambda_4 = -i$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{2x} + \mathrm{e}^{-ix} c_3 + \mathrm{e}^{ix} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= \mathrm{e}^{-x} \\
y_2 &= \mathrm{e}^{2x} \\
y_3 &= \mathrm{e}^{-ix} \\
y_4 &= \mathrm{e}^{ix}
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{2x} + \mathrm{e}^{-ix} c_3 + \mathrm{e}^{ix} c_4 \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$
0 = c_1 + c_2 + c_3 + c_4 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -c_1 \mathrm{e}^{-x} + 2c_2 \mathrm{e}^{2x} - i \mathrm{e}^{-ix} c_3 + i \mathrm{e}^{ix} c_4
$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$
0 = -c_3 i + c_4 i - c_1 + 2c_2 \tag{2A}
$$

Taking two derivatives of the solution gives

$$
y'' = c_1 \mathrm{e}^{-x} + 4c_2 \mathrm{e}^{2x} - \mathrm{e}^{-ix} c_3 - \mathrm{e}^{ix} c_4
$$

substituting $y'' = 0$ and $x = 0$ in the above gives

$$
0 = c_1 + 4c_2 - c_3 - c_4 \tag{3A}
$$

Taking three derivatives of the solution gives

$$
y''' = -c_1 \mathrm{e}^{-x} + 8c_2 \mathrm{e}^{2x} + i \mathrm{e}^{-ix} c_3 - i \mathrm{e}^{ix} c_4
$$

substituting $y''' = 15$ and $x = 0$ in the above gives

$$
15 = c_3 i - c_4 i - c_1 + 8c_2 \tag{4A}
$$

Equations {1A,2A,3A,4A} are now solved for $\{c_1, c_2, c_3, c_4\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= -\frac{5}{2} \\
c_2 &= 1 \\
c_3 &= \frac{3}{4} - \frac{9i}{4} \\
c_4 &= \frac{3}{4} + \frac{9i}{4}
\end{aligned}
$$

Substituting these values back in above solution results in

$$y = -\frac{5\,\mathrm{e}^{-x}}{2} + \mathrm{e}^{2x} + \frac{3\cos(x)}{2} - \frac{9\sin(x)}{2}$$

Summary

The solution(s) found are the following

$$y = -\frac{5\,\mathrm{e}^{-x}}{2} + \mathrm{e}^{2x} + \frac{3\cos(x)}{2} - \frac{9\sin(x)}{2} \tag{1}$$

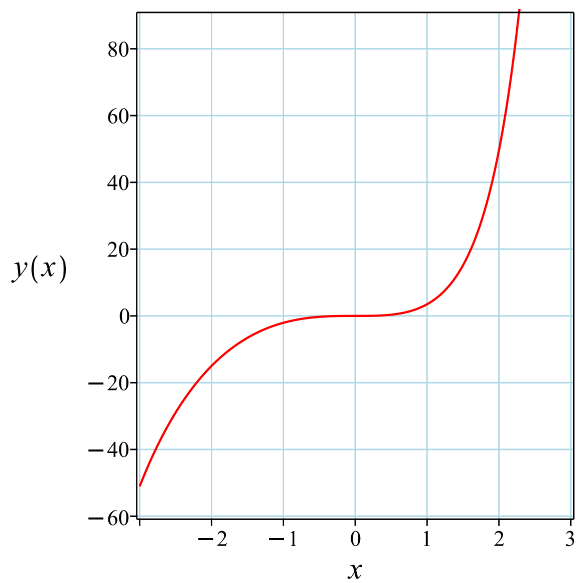


Figure 6: Solution plot

Verification of solutions

$$y = -\frac{5\,\mathrm{e}^{-x}}{2} + \mathrm{e}^{2x} + \frac{3\cos(x)}{2} - \frac{9\sin(x)}{2}$$

Verified OK.

2.23.1 Maple step by step solution

Let's solve

$$\left[y'''' - y''' - y'' - y' - 2y = 0, y(0) = 0, y'\Big|_{\{x=0\}} = 0, y''\Big|_{\{x=0\}} = 0, y'''\Big|_{\{x=0\}} = 15\right]$$

- Highest derivative means the order of the ODE is 4

  y''''

☐ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$

  ◦ Define new variable $y_4(x)$

    $y_4(x) = y'''$

  ◦ Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = y_4(x) + y_3(x) + y_2(x) + 2y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = y_4(x) + y_3(x) + y_2(x) + 2y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 1 & 1 & 1 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 1 & 1 & 1 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right], \left[-\mathrm{I}, \begin{bmatrix} -\mathrm{I} \\ -1 \\ \mathrm{I} \\ 1 \end{bmatrix}\right], \left[\mathrm{I}, \begin{bmatrix} \mathrm{I} \\ -1 \\ -\mathrm{I} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{-x} \cdot \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^{2x} \cdot \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\begin{bmatrix} -I, \begin{bmatrix} -I \\ -1 \\ I \\ 1 \end{bmatrix} \end{bmatrix}$$

- Solution from eigenpair

$$e^{-Ix} \cdot \begin{bmatrix} -I \\ -1 \\ I \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$(\cos(x) - I\sin(x)) \cdot \begin{bmatrix} -I \\ -1 \\ I \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} -I(\cos(x) - I\sin(x)) \\ -\cos(x) + I\sin(x) \\ I(\cos(x) - I\sin(x)) \\ \cos(x) - I\sin(x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(x)=\begin{bmatrix}-\sin(x)\\-\cos(x)\\\sin(x)\\\cos(x)\end{bmatrix},\vec{y}_4(x)=\begin{bmatrix}-\cos(x)\\\sin(x)\\\cos(x)\\-\sin(x)\end{bmatrix}\right]$$

- General solution to the system of ODEs

  $\vec{y}=c_1\vec{y}_1+c_2\vec{y}_2+c_3\vec{y}_3(x)+c_4\vec{y}_4(x)$
- Substitute solutions into the general solution

  $$\vec{y}=c_1\mathrm{e}^{-x}\cdot\begin{bmatrix}-1\\1\\-1\\1\end{bmatrix}+c_2\mathrm{e}^{2x}\cdot\begin{bmatrix}\frac{1}{8}\\\frac{1}{4}\\\frac{1}{2}\\1\end{bmatrix}+\begin{bmatrix}-c_3\sin(x)-c_4\cos(x)\\-c_3\cos(x)+c_4\sin(x)\\c_3\sin(x)+c_4\cos(x)\\c_3\cos(x)-c_4\sin(x)\end{bmatrix}$$
- First component of the vector is the solution to the ODE

  $y=-c_1\mathrm{e}^{-x}+\frac{c_2\mathrm{e}^{2x}}{8}-c_4\cos(x)-c_3\sin(x)$
- Use the initial condition $y(0)=0$

  $0=-c_1+\frac{c_2}{8}-c_4$
- Calculate the 1st derivative of the solution

  $y'=c_1\mathrm{e}^{-x}+\frac{c_2\mathrm{e}^{2x}}{4}+c_4\sin(x)-c_3\cos(x)$
- Use the initial condition $y'\Big|_{\{x=0\}}=0$

  $0=c_1+\frac{c_2}{4}-c_3$
- Calculate the 2nd derivative of the solution

  $y''=-c_1\mathrm{e}^{-x}+\frac{c_2\mathrm{e}^{2x}}{2}+c_4\cos(x)+c_3\sin(x)$
- Use the initial condition $y''\Big|_{\{x=0\}}=0$

  $0=-c_1+\frac{c_2}{2}+c_4$
- Calculate the 3rd derivative of the solution

  $y'''=c_1\mathrm{e}^{-x}+c_2\mathrm{e}^{2x}-c_4\sin(x)+c_3\cos(x)$
- Use the initial condition $y'''\Big|_{\{x=0\}}=15$

  $15=c_1+c_2+c_3$

- Solve for the unknown coefficients

  $\left\{c_1 = \frac{5}{2}, c_2 = 8, c_3 = \frac{9}{2}, c_4 = -\frac{3}{2}\right\}$
- Solution to the IVP

  $y = -\frac{5\,\mathrm{e}^{-x}}{2} + \mathrm{e}^{2x} + \frac{3\cos(x)}{2} - \frac{9\sin(x)}{2}$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 23

```
dsolve([diff(y(x),x$4)=diff(y(x),x$3)+diff(y(x),x$2)+diff(y(x),x)+2*y(x),y(0) = 0, D(y)(0) =
```

$$y(x) = \mathrm{e}^{2x} - \frac{5\,\mathrm{e}^{-x}}{2} - \frac{9\sin(x)}{2} + \frac{3\cos(x)}{2}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 33

```
DSolve[{y'''[x]==y[x],{y[0]==1,y'[0]==0,y''[0]==0}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{3}\left(e^x + 2e^{-x/2}\cos\left(\frac{\sqrt{3}x}{2}\right)\right)$$

2.24 problem problem 54

Internal problem ID [308]
Internal file name [OUTPUT/308_Sunday_June_05_2022_01_38_30_AM_77912269/index.tex]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 5.3, Higher-Order Linear Differential Equations. Homogeneous Equations with Constant Coefficients. Page 300
Problem number: problem 54.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_ODE_non_constant_coefficients_of_type_Euler"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_y]]
```

$$\boxed{x^3y''' + 6x^2y'' + 4y'x = 0}$$

This is Euler ODE of higher order. Let $y = x^\lambda$. Hence

$$\begin{aligned} y' &= \lambda\, x^{\lambda-1} \\ y'' &= \lambda(\lambda-1)\, x^{\lambda-2} \\ y''' &= \lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} \end{aligned}$$

Substituting these back into

$$x^3y''' + 6x^2y'' + 4y'x = 0$$

gives

$$4x\lambda\, x^{\lambda-1} + 6x^2\lambda(\lambda-1)\, x^{\lambda-2} + x^3\lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} = 0$$

Which simplifies to

$$4\lambda\, x^\lambda + 6\lambda(\lambda-1)\, x^\lambda + \lambda(\lambda-1)\,(\lambda-2)\, x^\lambda = 0$$

And since $x^\lambda \neq 0$ then dividing through by x^λ, the above becomes

$$4\lambda + 6\lambda(\lambda - 1) + \lambda(\lambda - 1)(\lambda - 2) = 0$$

Simplifying gives the characteristic equation as

$$\lambda^2(\lambda + 3) = 0$$

Solving the above gives the following roots

$$\lambda_1 = -3$$
$$\lambda_2 = 0$$
$$\lambda_3 = 0$$

This table summarises the result

root	multiplicity	type of root
0	2	real root
-3	1	real root

The solution is generated by going over the above table. For each real root λ of multiplicity one generates a $c_1 x^\lambda$ basis solution. Each real root of multiplicty two, generates $c_1 x^\lambda$ and $c_2 x^\lambda \ln(x)$ basis solutions. Each real root of multiplicty three, generates $c_1 x^\lambda$ and $c_2 x^\lambda \ln(x)$ and $c_3 x^\lambda \ln(x)^2$ basis solutions, and so on. Each complex root $\alpha \pm i\beta$ of multiplicity one generates $x^\alpha(c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity two generates $\ln(x)\, x^\alpha(c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity three generates $\ln(x)^2\, x^\alpha(c_1 \cos(\beta \ln(x)) + c_2$ basis solutions. And so on. Using the above show that the solution is

$$y = c_1 + c_2 \ln(x) + \frac{c_3}{x^3}$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = 1$$
$$y_2 = \ln(x)$$
$$y_3 = \frac{1}{x^3}$$

Summary
The solution(s) found are the following

$$y = c_1 + c_2 \ln(x) + \frac{c_3}{x^3} \tag{1}$$

Verification of solutions

$$y = c_1 + c_2 \ln (x) + \frac{c_3}{x^3}$$

Verified OK.

2.24.1 Maple step by step solution

Let's solve

$x^3 y''' + 6x^2 y'' + 4y'x = 0$

- Highest derivative means the order of the ODE is 3

  y'''

- Isolate 3rd derivative

  $y''' = -\frac{2(3y''x+2y')}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''' + \frac{6y''}{x} + \frac{4y'}{x^2} = 0$

- Multiply by denominators of the ODE

  $y'''x^2 + 6y''x + 4y' = 0$

- Make a change of variables

  $t = \ln (x)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right) t'(x)$

  - Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2}y(t)\right) t'(x)^2 + t''(x) \left(\frac{d}{dt}y(t)\right)$

  - Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  - Calculate the 3rd derivative of y with respect to x , using the chain rule

    $y''' = \left(\frac{d^3}{dt^3}y(t)\right) t'(x)^3 + 3t'(x)\, t''(x) \left(\frac{d^2}{dt^2}y(t)\right) + t'''(x) \left(\frac{d}{dt}y(t)\right)$

- Compute derivative

  $y''' = \frac{\frac{d^3}{dt^3}y(t)}{x^3} - \frac{3\left(\frac{d^2}{dt^2}y(t)\right)}{x^3} + \frac{2\left(\frac{d}{dt}y(t)\right)}{x^3}$

  Substitute the change of variables back into the ODE

  $\left(\frac{\frac{d^3}{dt^3}y(t)}{x^3} - \frac{3\left(\frac{d^2}{dt^2}y(t)\right)}{x^3} + \frac{2\left(\frac{d}{dt}y(t)\right)}{x^3}\right)x^2 + 6\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right)x + \frac{4\left(\frac{d}{dt}y(t)\right)}{x} = 0$

- Simplify

  $\frac{\frac{d^3}{dt^3}y(t)+3\frac{d^2}{dt^2}y(t)}{x} = 0$

- Isolate 3rd derivative

  $\frac{d^3}{dt^3}y(t) = -3\frac{d^2}{dt^2}y(t)$

- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^3}{dt^3}y(t) + 3\frac{d^2}{dt^2}y(t) = 0$

- ☐ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(t)$

    $y_1(t) = y(t)$
  - Define new variable $y_2(t)$

    $y_2(t) = \frac{d}{dt}y(t)$
  - Define new variable $y_3(t)$

    $y_3(t) = \frac{d^2}{dt^2}y(t)$
  - Isolate for $\frac{d}{dt}y_3(t)$ using original ODE

    $\frac{d}{dt}y_3(t) = -3y_3(t)$

    Convert linear ODE into a system of first order ODEs

    $\left[y_2(t) = \frac{d}{dt}y_1(t), y_3(t) = \frac{d}{dt}y_2(t), \frac{d}{dt}y_3(t) = -3y_3(t)\right]$

- Define vector

  $\vec{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$

- System to solve

$$\frac{d}{dt}\vec{y}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -3 \end{bmatrix} \cdot \vec{y}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -3 \end{bmatrix}$$

- Rewrite the system as

$$\frac{d}{dt}\vec{y}(t) = A \cdot \vec{y}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-3, \begin{bmatrix} \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix}\right], \left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right], \left[0, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-3, \begin{bmatrix} \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{-3t} \cdot \begin{bmatrix} \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{y} = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 \mathrm{e}^{-3t} \cdot \begin{bmatrix} \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix} + \begin{bmatrix} c_2 \\ 0 \\ 0 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y(t) = \frac{c_1 \mathrm{e}^{-3t}}{9} + c_2$$

- Change variables back using $t = \ln(x)$

$$y = \frac{c_1}{9x^3} + c_2$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(x^3*diff(y(x),x$3)+6*x^2*diff(y(x),x$2)+4*x*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = c_1 + c_2 \ln(x) + \frac{c_3}{x^3}$$

✓ Solution by Mathematica

Time used: 0.017 (sec). Leaf size: 22

```
DSolve[x^3*y'''[x]+6*x^2*y''[x]+4*x*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{c_1}{3x^3} + c_2 \log(x) + c_3$$

2.25 problem problem 55

Internal problem ID [309]
Internal file name [OUTPUT/309_Sunday_June_05_2022_01_38_30_AM_4305051/index.tex]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 5.3, Higher-Order Linear Differential Equations. Homogeneous Equations with Constant Coefficients. Page 300
Problem number: problem 55.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_ODE_non_constant_coefficients_of_type_Euler"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_y]]
```

$$\boxed{x^3y''' - x^2y'' + y'x = 0}$$

This is Euler ODE of higher order. Let $y = x^\lambda$. Hence

$$\begin{aligned} y' &= \lambda\, x^{\lambda-1} \\ y'' &= \lambda(\lambda-1)\, x^{\lambda-2} \\ y''' &= \lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} \end{aligned}$$

Substituting these back into

$$x^3y''' - x^2y'' + y'x = 0$$

gives

$$x\lambda\, x^{\lambda-1} - x^2\lambda(\lambda-1)\, x^{\lambda-2} + x^3\lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} = 0$$

Which simplifies to

$$\lambda\, x^\lambda - \lambda(\lambda-1)\, x^\lambda + \lambda(\lambda-1)\,(\lambda-2)\, x^\lambda = 0$$

And since $x^\lambda \neq 0$ then dividing through by x^λ, the above becomes

$$\lambda - \lambda(\lambda - 1) + \lambda(\lambda - 1)(\lambda - 2) = 0$$

Simplifying gives the characteristic equation as

$$\lambda(\lambda - 2)^2 = 0$$

Solving the above gives the following roots

$$\lambda_1 = 0$$
$$\lambda_2 = 2$$
$$\lambda_3 = 2$$

This table summarises the result

root	multiplicity	type of root
0	1	real root
2	2	real root

The solution is generated by going over the above table. For each real root λ of multiplicity one generates a c_1x^λ basis solution. Each real root of multiplicty two, generates c_1x^λ and $c_2x^\lambda \ln(x)$ basis solutions. Each real root of multiplicty three, generates c_1x^λ and $c_2x^\lambda \ln(x)$ and $c_3x^\lambda \ln(x)^2$ basis solutions, and so on. Each complex root $\alpha \pm i\beta$ of multiplicity one generates $x^\alpha(c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity two generates $\ln(x)\, x^\alpha(c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity three generates $\ln(x)^2\, x^\alpha(c_1 \cos(\beta \ln(x)) + c_2$ basis solutions. And so on. Using the above show that the solution is

$$y = c_2x^2 + c_1 + c_3 \ln(x)\, x^2$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = 1$$
$$y_2 = x^2$$
$$y_3 = \ln(x)\, x^2$$

Summary

The solution(s) found are the following

$$y = c_2x^2 + c_1 + c_3 \ln(x)\, x^2 \tag{1}$$

Verification of solutions

$$y = c_2x^2 + c_1 + c_3 \ln(x)\, x^2$$

Verified OK.

2.25.1 Maple step by step solution

Let's solve

$x^3y''' - x^2y'' + y'x = 0$

- Highest derivative means the order of the ODE is 3

  y'''

- Isolate 3rd derivative

  $y''' = \frac{y''x - y'}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''' - \frac{y''}{x} + \frac{y'}{x^2} = 0$

- Multiply by denominators of the ODE

  $y'''x^2 - y''x + y' = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right) t'(x)$

  - Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2}y(t)\right) t'(x)^2 + t''(x)\left(\frac{d}{dt}y(t)\right)$

  - Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  - Calculate the 3rd derivative of y with respect to x , using the chain rule

    $y''' = \left(\frac{d^3}{dt^3}y(t)\right) t'(x)^3 + 3t'(x)\, t''(x) \left(\frac{d^2}{dt^2}y(t)\right) + t'''(x)\left(\frac{d}{dt}y(t)\right)$

  - Compute derivative

    $y''' = \frac{\frac{d^3}{dt^3}y(t)}{x^3} - \frac{3\left(\frac{d^2}{dt^2}y(t)\right)}{x^3} + \frac{2\left(\frac{d}{dt}y(t)\right)}{x^3}$

  Substitute the change of variables back into the ODE

$$\left(\frac{\frac{d^3}{dt^3}y(t)}{x^3}-\frac{3\left(\frac{d^2}{dt^2}y(t)\right)}{x^3}+\frac{2\left(\frac{d}{dt}y(t)\right)}{x^3}\right)x^2-\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2}-\frac{\frac{d}{dt}y(t)}{x^2}\right)x+\frac{\frac{d}{dt}y(t)}{x}=0$$

- Simplify

$$\frac{\frac{d^3}{dt^3}y(t)-4\frac{d^2}{dt^2}y(t)+4\frac{d}{dt}y(t)}{x}=0$$

- Isolate 3rd derivative

$$\frac{d^3}{dt^3}y(t)=4\frac{d^2}{dt^2}y(t)-4\frac{d}{dt}y(t)$$

- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$$\frac{d^3}{dt^3}y(t)-4\frac{d^2}{dt^2}y(t)+4\frac{d}{dt}y(t)=0$$

☐ Convert linear ODE into a system of first order ODEs

  ○ Define new variable $y_1(t)$

  $$y_1(t)=y(t)$$

  ○ Define new variable $y_2(t)$

  $$y_2(t)=\frac{d}{dt}y(t)$$

  ○ Define new variable $y_3(t)$

  $$y_3(t)=\frac{d^2}{dt^2}y(t)$$

  ○ Isolate for $\frac{d}{dt}y_3(t)$ using original ODE

  $$\frac{d}{dt}y_3(t)=4y_3(t)-4y_2(t)$$

  Convert linear ODE into a system of first order ODEs

  $$\left[y_2(t)=\frac{d}{dt}y_1(t), y_3(t)=\frac{d}{dt}y_2(t), \frac{d}{dt}y_3(t)=4y_3(t)-4y_2(t)\right]$$

- Define vector

$$\vec{y}(t)=\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

- System to solve

$$\frac{d}{dt}\vec{y}(t)=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -4 & 4 \end{bmatrix}\cdot\vec{y}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -4 & 4 \end{bmatrix}$$

- Rewrite the system as

  $\frac{d}{dt}\vec{y}(t) = A \cdot \vec{y}(t)$
- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\begin{bmatrix} \begin{bmatrix} 0, & \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \end{bmatrix}, & \begin{bmatrix} 2, & \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} \end{bmatrix}, & \begin{bmatrix} 2, & \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{bmatrix} \end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix} 0, & \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\begin{bmatrix} 2, & \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} \end{bmatrix}$$

- First solution from eigenvalue 2

$$\vec{y}_2(t) = \mathrm{e}^{2t} \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = 2$ is the eigenvalue, an

$$\vec{y}_3(t) = e^{\lambda t}\left(t\vec{v} + \vec{p}\right)$$

- Note that the t multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obta
- Substitute $\vec{y}_3(t)$ into the homogeneous system

$$\lambda e^{\lambda t}\left(t\vec{v} + \vec{p}\right) + e^{\lambda t}\vec{v} = \left(e^{\lambda t}A\right) \cdot \left(t\vec{v} + \vec{p}\right)$$

- Use the fact that $\vec{v}$ is an eigenvector of A

$$\lambda e^{\lambda t}\left(t\vec{v} + \vec{p}\right) + e^{\lambda t}\vec{v} = e^{\lambda t}\left(\lambda t\vec{v} + A \cdot \vec{p}\right)$$

- Simplify equation

$$\lambda\vec{p} + \vec{v} = A \cdot \vec{p}$$

- Make use of the identity matrix I

$$(\lambda \cdot I) \cdot \vec{p} + \vec{v} = A \cdot \vec{p}$$

- Condition $\vec{p}$ must meet for $\vec{y}_3(t)$ to be a solution to the homogeneous system

$$(A - \lambda \cdot I) \cdot \vec{p} = \vec{v}$$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue 2

$$\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -4 & 4 \end{bmatrix} - 2 \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \cdot \vec{p} = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- Choice of $\vec{p}$

$$\vec{p} = \begin{bmatrix} -\frac{1}{8} \\ 0 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue 2

$$\vec{y}_3(t) = e^{2t} \cdot \left(t \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{1}{8} \\ 0 \\ 0 \end{bmatrix}\right)$$

- General solution to the system of ODEs

$$\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2(t) + c_3\vec{y}_3(t)$$

- Substitute solutions into the general solution

$$\vec{y} = c_2 e^{2t} \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} + c_3 e^{2t} \cdot \left(t \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{1}{8} \\ 0 \\ 0 \end{bmatrix} \right) + \begin{bmatrix} c_1 \\ 0 \\ 0 \end{bmatrix}$$

- First component of the vector is the solution to the ODE
  $y(t) = \frac{((2t-1)c_3+2c_2)e^{2t}}{8} + c_1$
- Change variables back using $t = \ln(x)$
  $y = \frac{((2\ln(x)-1)c_3+2c_2)x^2}{8} + c_1$
- Simplify
  $y = \frac{c_3 \ln(x) x^2}{4} + \frac{c_2 x^2}{4} - \frac{c_3 x^2}{8} + c_1$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve(x^3*diff(y(x),x$3)-x^2*diff(y(x),x$2)+x*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = c_1 + c_2 x^2 + c_3 x^2 \ln(x)$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 35

```
DSolve[x^3*y'''[x]-x^2*y''[x]+x*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4}(2c_1 - c_2)x^2 + \frac{1}{2}c_2 x^2 \log(x) + c_3$$

2.26 problem problem 56

Internal problem ID [310]
Internal file name [`OUTPUT/310_Sunday_June_05_2022_01_38_31_AM_34334670/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 5.3, Higher-Order Linear Differential Equations. Homogeneous Equations with Constant Coefficients. Page 300
Problem number: problem 56.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_ODE_non_constant_coefficients_of_type_Euler"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_y]]
```

$$\boxed{x^3y''' + 3x^2y'' + y'x = 0}$$

This is Euler ODE of higher order. Let $y = x^\lambda$. Hence

$$\begin{aligned} y' &= \lambda\, x^{\lambda-1} \\ y'' &= \lambda(\lambda-1)\, x^{\lambda-2} \\ y''' &= \lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} \end{aligned}$$

Substituting these back into

$$x^3y''' + 3x^2y'' + y'x = 0$$

gives

$$x\lambda\, x^{\lambda-1} + 3x^2\lambda(\lambda-1)\, x^{\lambda-2} + x^3\lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} = 0$$

Which simplifies to

$$\lambda\, x^\lambda + 3\lambda(\lambda-1)\, x^\lambda + \lambda(\lambda-1)\,(\lambda-2)\, x^\lambda = 0$$

And since $x^\lambda \neq 0$ then dividing through by x^λ, the above becomes

$$\lambda + 3\lambda(\lambda - 1) + \lambda(\lambda - 1)(\lambda - 2) = 0$$

Simplifying gives the characteristic equation as

$$\lambda^3 = 0$$

Solving the above gives the following roots

$$\lambda_1 = 0$$
$$\lambda_2 = 0$$
$$\lambda_3 = 0$$

This table summarises the result

root	multiplicity	type of root
0	3	real root

The solution is generated by going over the above table. For each real root λ of multiplicity one generates a $c_1 x^\lambda$ basis solution. Each real root of multiplicty two, generates $c_1 x^\lambda$ and $c_2 x^\lambda \ln(x)$ basis solutions. Each real root of multiplicty three, generates $c_1 x^\lambda$ and $c_2 x^\lambda \ln(x)$ and $c_3 x^\lambda \ln(x)^2$ basis solutions, and so on. Each complex root $\alpha \pm i\beta$ of multiplicity one generates $x^\alpha (c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity two generates $\ln(x) x^\alpha (c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity three generates $\ln(x)^2 x^\alpha (c_1 \cos(\beta \ln(x)) + c_2$ basis solutions. And so on. Using the above show that the solution is

$$y = c_1 + c_2 \ln(x) + c_3 \ln(x)^2$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = 1$$
$$y_2 = \ln(x)$$
$$y_3 = \ln(x)^2$$

Summary

The solution(s) found are the following

$$y = c_1 + c_2 \ln(x) + c_3 \ln(x)^2 \tag{1}$$

Verification of solutions

$$y = c_1 + c_2 \ln(x) + c_3 \ln(x)^2$$

Verified OK.

2.26.1 Maple step by step solution

Let's solve

$x^3y''' + 3x^2y'' + y'x = 0$

- Highest derivative means the order of the ODE is 3

  y'''

- Isolate 3rd derivative

  $y''' = -\frac{3y''x+y'}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''' + \frac{3y''}{x} + \frac{y'}{x^2} = 0$

- Multiply by denominators of the ODE

  $y'''x^2 + 3y''x + y' = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  ◦ Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right)t'(x)$

  ◦ Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$

  ◦ Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2}y(t)\right)t'(x)^2 + t''(x)\left(\frac{d}{dt}y(t)\right)$

  ◦ Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  ◦ Calculate the 3rd derivative of y with respect to x , using the chain rule

    $y''' = \left(\frac{d^3}{dt^3}y(t)\right)t'(x)^3 + 3t'(x)\,t''(x)\left(\frac{d^2}{dt^2}y(t)\right) + t'''(x)\left(\frac{d}{dt}y(t)\right)$

  ◦ Compute derivative

    $y''' = \frac{\frac{d^3}{dt^3}y(t)}{x^3} - \frac{3\left(\frac{d^2}{dt^2}y(t)\right)}{x^3} + \frac{2\left(\frac{d}{dt}y(t)\right)}{x^3}$

  Substitute the change of variables back into the ODE

$$\left(\frac{\frac{d^3}{dt^3}y(t)}{x^3}-\frac{3\left(\frac{d^2}{dt^2}y(t)\right)}{x^3}+\frac{2\left(\frac{d}{dt}y(t)\right)}{x^3}\right)x^2+3\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2}-\frac{\frac{d}{dt}y(t)}{x^2}\right)x+\frac{\frac{d}{dt}y(t)}{x}=0$$

- Simplify

$$\frac{\frac{d^3}{dt^3}y(t)}{x}=0$$

- Isolate 3rd derivative

$$\frac{d^3}{dt^3}y(t)=0$$

☐ Convert linear ODE into a system of first order ODEs

  - Define new variable $y_1(t)$

$$y_1(t)=y(t)$$

  - Define new variable $y_2(t)$

$$y_2(t)=\frac{d}{dt}y(t)$$

  - Define new variable $y_3(t)$

$$y_3(t)=\frac{d^2}{dt^2}y(t)$$

  - Isolate for $\frac{d}{dt}y_3(t)$ using original ODE

$$\frac{d}{dt}y_3(t)=0$$

Convert linear ODE into a system of first order ODEs

$$\left[y_2(t)=\frac{d}{dt}y_1(t), y_3(t)=\frac{d}{dt}y_2(t), \frac{d}{dt}y_3(t)=0\right]$$

- Define vector

$$\vec{y}(t)=\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

- System to solve

$$\frac{d}{dt}\vec{y}(t)=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\cdot\vec{y}(t)$$

- Define the coefficient matrix

$$A=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

- Rewrite the system as

  $\frac{d}{dt}\overrightarrow{y}(t) = A \cdot \overrightarrow{y}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[0, \left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right]\right], \left[0, \left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right]\right], \left[0, \left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right]\right]\right]$$

- Consider eigenpair

  $$\left[0, \left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

  $$\overrightarrow{y}_1 = \left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right]$$

- Consider eigenpair

  $$\left[0, \left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

  $$\overrightarrow{y}_2 = \left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right]$$

- Consider eigenpair

  $$\left[0, \left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- General solution to the system of ODEs

  $\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3$

- Substitute solutions into the general solution

$$\vec{y} = \begin{bmatrix} c_1 \\ 0 \\ 0 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

  $y(t) = c_1$

- Change variables back using $t = \ln(x)$

  $y = c_1$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 16

```
dsolve(x^3*diff(y(x),x$3)+3*x^2*diff(y(x),x$2)+x*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = c_3 \ln(x)^2 + c_2 \ln(x) + c_1$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 23

```
DSolve[x^3*y'''[x]+3*x^2*y''[x]+x*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}c_2 \log^2(x) + c_1 \log(x) + c_3$$

2.27 problem problem 57

Internal problem ID [311]
Internal file name [`OUTPUT/311_Sunday_June_05_2022_01_38_32_AM_14832962/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 5.3, Higher-Order Linear Differential Equations. Homogeneous Equations with Constant Coefficients. Page 300
Problem number: problem 57.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_ODE_non_constant_coefficients_of_type_Euler"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_y]]
```

$$\boxed{x^3y''' - 3x^2y'' + y'x = 0}$$

This is Euler ODE of higher order. Let $y = x^\lambda$. Hence

$$\begin{aligned}
y' &= \lambda\, x^{\lambda-1} \\
y'' &= \lambda(\lambda-1)\, x^{\lambda-2} \\
y''' &= \lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3}
\end{aligned}$$

Substituting these back into

$$x^3y''' - 3x^2y'' + y'x = 0$$

gives

$$x\lambda\, x^{\lambda-1} - 3x^2\lambda(\lambda-1)\, x^{\lambda-2} + x^3\lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} = 0$$

Which simplifies to

$$\lambda\, x^\lambda - 3\lambda(\lambda-1)\, x^\lambda + \lambda(\lambda-1)\,(\lambda-2)\, x^\lambda = 0$$

And since $x^\lambda \neq 0$ then dividing through by x^λ, the above becomes

$$\lambda - 3\lambda(\lambda - 1) + \lambda(\lambda - 1)(\lambda - 2) = 0$$

Simplifying gives the characteristic equation as

$$\lambda^3 - 6\lambda^2 + 6\lambda = 0$$

Solving the above gives the following roots

$$\lambda_1 = 0$$
$$\lambda_2 = 3 + \sqrt{3}$$
$$\lambda_3 = 3 - \sqrt{3}$$

This table summarises the result

root	multiplicity	type of root
0	1	real root
$3 - \sqrt{3}$	1	real root
$3 + \sqrt{3}$	1	real root

The solution is generated by going over the above table. For each real root λ of multiplicity one generates a $c_1 x^\lambda$ basis solution. Each real root of multiplicty two, generates $c_1 x^\lambda$ and $c_2 x^\lambda \ln(x)$ basis solutions. Each real root of multiplicty three, generates $c_1 x^\lambda$ and $c_2 x^\lambda \ln(x)$ and $c_3 x^\lambda \ln(x)^2$ basis solutions, and so on. Each complex root $\alpha \pm i\beta$ of multiplicity one generates $x^\alpha (c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity two generates $\ln(x)\, x^\alpha (c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity three generates $\ln(x)^2 x^\alpha (c_1 \cos(\beta \ln(x)) + c_2$ basis solutions. And so on. Using the above show that the solution is

$$y = c_1 + c_2 x^{3-\sqrt{3}} + c_3 x^{3+\sqrt{3}}$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = 1$$
$$y_2 = x^{3-\sqrt{3}}$$
$$y_3 = x^{3+\sqrt{3}}$$

Summary

The solution(s) found are the following

$$y = c_1 + c_2 x^{3-\sqrt{3}} + c_3 x^{3+\sqrt{3}} \tag{1}$$

Verification of solutions

$$y = c_1 + c_2 x^{3-\sqrt{3}} + c_3 x^{3+\sqrt{3}}$$

Verified OK.

2.27.1 Maple step by step solution

Let's solve

$x^3 y''' - 3x^2 y'' + y'x = 0$

- Highest derivative means the order of the ODE is 3

  y'''

- Isolate 3rd derivative

  $y''' = \frac{3y''x - y'}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''' - \frac{3y''}{x} + \frac{y'}{x^2} = 0$

- Multiply by denominators of the ODE

  $y'''x^2 - 3y''x + y' = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  ◦ Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right) t'(x)$

  ◦ Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$

  ◦ Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2}y(t)\right) t'(x)^2 + t''(x) \left(\frac{d}{dt}y(t)\right)$

  ◦ Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  ◦ Calculate the 3rd derivative of y with respect to x , using the chain rule

    $y''' = \left(\frac{d^3}{dt^3}y(t)\right) t'(x)^3 + 3t'(x)\, t''(x) \left(\frac{d^2}{dt^2}y(t)\right) + t'''(x) \left(\frac{d}{dt}y(t)\right)$

- Compute derivative

  $y''' = \frac{\frac{d^3}{dt^3}y(t)}{x^3} - \frac{3\left(\frac{d^2}{dt^2}y(t)\right)}{x^3} + \frac{2\left(\frac{d}{dt}y(t)\right)}{x^3}$

  Substitute the change of variables back into the ODE

  $\left(\frac{\frac{d^3}{dt^3}y(t)}{x^3} - \frac{3\left(\frac{d^2}{dt^2}y(t)\right)}{x^3} + \frac{2\left(\frac{d}{dt}y(t)\right)}{x^3}\right)x^2 - 3\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right)x + \frac{\frac{d}{dt}y(t)}{x} = 0$

• Simplify

  $\frac{\frac{d^3}{dt^3}y(t) - 6\frac{d^2}{dt^2}y(t) + 6\frac{d}{dt}y(t)}{x} = 0$

• Isolate 3rd derivative

  $\frac{d^3}{dt^3}y(t) = 6\frac{d^2}{dt^2}y(t) - 6\frac{d}{dt}y(t)$

• Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^3}{dt^3}y(t) - 6\frac{d^2}{dt^2}y(t) + 6\frac{d}{dt}y(t) = 0$

☐ Convert linear ODE into a system of first order ODEs

- Define new variable $y_1(t)$

  $y_1(t) = y(t)$

- Define new variable $y_2(t)$

  $y_2(t) = \frac{d}{dt}y(t)$

- Define new variable $y_3(t)$

  $y_3(t) = \frac{d^2}{dt^2}y(t)$

- Isolate for $\frac{d}{dt}y_3(t)$ using original ODE

  $\frac{d}{dt}y_3(t) = 6y_3(t) - 6y_2(t)$

  Convert linear ODE into a system of first order ODEs

  $\left[y_2(t) = \frac{d}{dt}y_1(t), y_3(t) = \frac{d}{dt}y_2(t), \frac{d}{dt}y_3(t) = 6y_3(t) - 6y_2(t)\right]$

• Define vector

  $\vec{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$

• System to solve

$$\frac{d}{dt}\vec{y}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -6 & 6 \end{bmatrix} \cdot \vec{y}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -6 & 6 \end{bmatrix}$$

- Rewrite the system as

$$\frac{d}{dt}\vec{y}(t) = A \cdot \vec{y}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right], \left[3-\sqrt{3}, \begin{bmatrix} \frac{1}{\left(3-\sqrt{3}\right)^2} \\ \frac{1}{3-\sqrt{3}} \\ 1 \end{bmatrix}\right], \left[3+\sqrt{3}, \begin{bmatrix} \frac{1}{\left(3+\sqrt{3}\right)^2} \\ \frac{1}{3+\sqrt{3}} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\left[3-\sqrt{3}, \begin{bmatrix} \frac{1}{\left(3-\sqrt{3}\right)^2} \\ \frac{1}{3-\sqrt{3}} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_2 = \mathrm{e}^{\left(3-\sqrt{3}\right)t} \cdot \begin{bmatrix} \frac{1}{\left(3-\sqrt{3}\right)^2} \\ \frac{1}{3-\sqrt{3}} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[3+\sqrt{3}, \begin{bmatrix} \frac{1}{\left(3+\sqrt{3}\right)^2} \\ \frac{1}{3+\sqrt{3}} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_3 = \mathrm{e}^{\left(3+\sqrt{3}\right)t} \cdot \begin{bmatrix} \frac{1}{\left(3+\sqrt{3}\right)^2} \\ \frac{1}{3+\sqrt{3}} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\overrightarrow{y} = c_1\overrightarrow{y}_1 + c_2\overrightarrow{y}_2 + c_3\overrightarrow{y}_3$$

- Substitute solutions into the general solution

$$\overrightarrow{y} = c_2\mathrm{e}^{\left(3-\sqrt{3}\right)t} \cdot \begin{bmatrix} \frac{1}{\left(3-\sqrt{3}\right)^2} \\ \frac{1}{3-\sqrt{3}} \\ 1 \end{bmatrix} + c_3\mathrm{e}^{\left(3+\sqrt{3}\right)t} \cdot \begin{bmatrix} \frac{1}{\left(3+\sqrt{3}\right)^2} \\ \frac{1}{3+\sqrt{3}} \\ 1 \end{bmatrix} + \begin{bmatrix} c_1 \\ 0 \\ 0 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y(t) = \frac{c_2\left(2+\sqrt{3}\right)\mathrm{e}^{-\left(-3+\sqrt{3}\right)t}}{6} - \frac{c_3\left(\sqrt{3}-2\right)\mathrm{e}^{\left(3+\sqrt{3}\right)t}}{6} + c_1$$

- Change variables back using $t = \ln(x)$

$$y = \frac{c_2\left(2+\sqrt{3}\right)\mathrm{e}^{-\left(-3+\sqrt{3}\right)\ln(x)}}{6} - \frac{c_3\left(\sqrt{3}-2\right)\mathrm{e}^{\left(3+\sqrt{3}\right)\ln(x)}}{6} + c_1$$

- Simplify

$$y = \frac{x^{3-\sqrt{3}}\sqrt{3}\,c_2}{6} + \frac{c_2 x^{3-\sqrt{3}}}{3} - \frac{x^{3+\sqrt{3}}\sqrt{3}\,c_3}{6} + \frac{c_3 x^{3+\sqrt{3}}}{3} + c_1$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 26

```
dsolve(x^3*diff(y(x),x$3)-3*x^2*diff(y(x),x$2)+x*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = c_1 + c_2 x^{3+\sqrt{3}} + c_3 x^{3-\sqrt{3}}$$

✓ Solution by Mathematica

Time used: 0.136 (sec). Leaf size: 54

```
DSolve[x^3*y'''[x]-3*x^2*y''[x]+x*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_2 x^{3+\sqrt{3}}}{3+\sqrt{3}} + \frac{c_1 x^{3-\sqrt{3}}}{3-\sqrt{3}} + c_3$$

2.28 problem problem 58

Internal problem ID [312]
Internal file name [`OUTPUT/312_Sunday_June_05_2022_01_38_33_AM_14822218/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 5.3, Higher-Order Linear Differential Equations. Homogeneous Equations with Constant Coefficients. Page 300
Problem number: problem 58.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_ODE_non_constant_coefficients_of_type_Euler"**

Maple gives the following as the ode type

```
[[_3rd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{x^3y''' + 6x^2y'' + 7y'x + y = 0}$$

This is Euler ODE of higher order. Let $y = x^\lambda$. Hence

$$\begin{aligned}
y' &= \lambda\, x^{\lambda-1} \\
y'' &= \lambda(\lambda-1)\, x^{\lambda-2} \\
y''' &= \lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3}
\end{aligned}$$

Substituting these back into

$$x^3y''' + 6x^2y'' + 7y'x + y = 0$$

gives

$$7x\lambda\, x^{\lambda-1} + 6x^2\lambda(\lambda-1)\, x^{\lambda-2} + x^3\lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} + x^\lambda = 0$$

Which simplifies to

$$7\lambda\, x^\lambda + 6\lambda(\lambda-1)\, x^\lambda + \lambda(\lambda-1)\,(\lambda-2)\, x^\lambda + x^\lambda = 0$$

And since $x^\lambda \neq 0$ then dividing through by x^λ, the above becomes

$$7\lambda + 6\lambda(\lambda - 1) + \lambda(\lambda - 1)(\lambda - 2) + 1 = 0$$

Simplifying gives the characteristic equation as

$$(\lambda + 1)^3 = 0$$

Solving the above gives the following roots

$$\lambda_1 = -1$$
$$\lambda_2 = -1$$
$$\lambda_3 = -1$$

This table summarises the result

root	multiplicity	type of root
-1	3	real root

The solution is generated by going over the above table. For each real root λ of multiplicity one generates a $c_1 x^\lambda$ basis solution. Each real root of multiplicty two, generates $c_1 x^\lambda$ and $c_2 x^\lambda \ln(x)$ basis solutions. Each real root of multiplicty three, generates $c_1 x^\lambda$ and $c_2 x^\lambda \ln(x)$ and $c_3 x^\lambda \ln(x)^2$ basis solutions, and so on. Each complex root $\alpha \pm i\beta$ of multiplicity one generates $x^\alpha(c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity two generates $\ln(x)\, x^\alpha(c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity three generates $\ln(x)^2\, x^\alpha(c_1 \cos(\beta \ln(x)) + c_2$ basis solutions. And so on. Using the above show that the solution is

$$y = \frac{c_1}{x} + \frac{c_2 \ln(x)}{x} + \frac{c_3 \ln(x)^2}{x}$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = \frac{1}{x}$$
$$y_2 = \frac{\ln(x)}{x}$$
$$y_3 = \frac{\ln(x)^2}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{x} + \frac{c_2 \ln(x)}{x} + \frac{c_3 \ln(x)^2}{x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x} + \frac{c_2 \ln(x)}{x} + \frac{c_3 \ln(x)^2}{x}$$

Verified OK.

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve(x^3*diff(y(x),x$3)+6*x^2*diff(y(x),x$2)+7*x*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_3 \ln(x)^2 + c_2 \ln(x) + c_1}{x}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 24

```
DSolve[x^3*y'''[x]+6*x^2*y''[x]+7*x*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_3 \log^2(x) + c_2 \log(x) + c_1}{x}$$

3 Section 7.2, Matrices and Linear systems. Page 384

3.1 problem problem 13

Internal problem ID [313]
Internal file name [`OUTPUT/313_Sunday_June_05_2022_01_38_33_AM_86130880/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.2, Matrices and Linear systems. Page 384
Problem number: problem 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= 6x_1(t) \\
x_2'(t) &= -3x_1(t) - x_2(t)
\end{aligned}
$$

3.1.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ -3 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{6t} & 0 \\ -\frac{3\,e^{6t}}{7} + \frac{3\,e^{-t}}{7} & e^{-t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{6t} & 0 \\ -\frac{3\,e^{6t}}{7}+\frac{3\,e^{-t}}{7} & e^{-t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{6t}c_1 \\ \left(-\frac{3\,e^{6t}}{7}+\frac{3\,e^{-t}}{7}\right)c_1 + e^{-t}c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{6t}c_1 \\ \frac{(3c_1+7c_2)e^{-t}}{7} - \frac{3\,e^{6t}c_1}{7} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

3.1.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ -3 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 6 & 0 \\ -3 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 6-\lambda & 0 \\ -3 & -1-\lambda \end{bmatrix}\right) = 0$$

Since the matrix A is triangular matrix, then the determinant is the product of the elements along the diagonal. Therefore the above becomes

$$(6-\lambda)(-1-\lambda)=0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1$$
$$\lambda_2 = 6$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
6	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 6 & 0 \\ -3 & -1 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 7 & 0 \\ -3 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 7 & 0 & 0 \\ -3 & 0 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{3R_1}{7} \implies \left[\begin{array}{cc|c} 7 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 7 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 6$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 6 & 0 \\ -3 & -1 \end{bmatrix} - (6) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ -3 & -7 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ -3 & -7 & 0 \end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{cc|c} -3 & -7 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -3 & -7 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{7t}{3}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{7t}{3} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{7t}{3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{7t}{3} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{7}{3} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{7t}{3} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{7}{3} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{7t}{3} \\ t \end{bmatrix} = \begin{bmatrix} -7 \\ 3 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
-1	1	1	No	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$
6	1	1	No	$\begin{bmatrix} -\frac{7}{3} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{-t} \\ &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-t}\end{aligned}$$

Since eigenvalue 6 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{6t} \\ &= \begin{bmatrix} -\frac{7}{3} \\ 1 \end{bmatrix} e^{6t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ e^{-t} \end{bmatrix} + c_2 \begin{bmatrix} -\frac{7\,e^{6t}}{3} \\ e^{6t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -\frac{7 c_2 e^{6t}}{3} \\ c_1 e^{-t} + c_2 e^{6t} \end{bmatrix}$$

The following is the phase plot of the system.

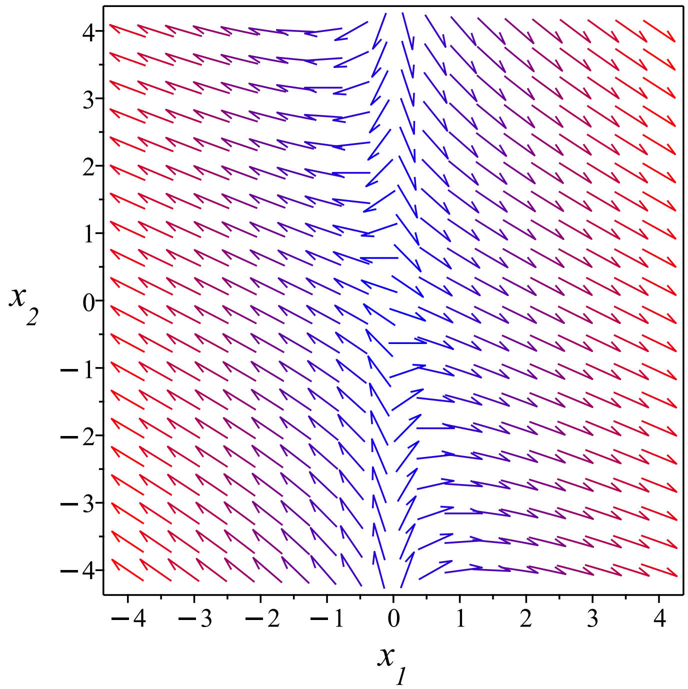


Figure 7: Phase plot

3.1.3 Maple step by step solution

Let's solve

$[x_1'(t) = 6x_1(t), x_2'(t) = -3x_1(t) - x_2(t)]$

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x}^{\rightarrow}{}'(t) = \begin{bmatrix} 6 & 0 \\ -3 & -1 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{x}^{\rightarrow}{}'(t) = \begin{bmatrix} 6 & 0 \\ -3 & -1 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 6 & 0 \\ -3 & -1 \end{bmatrix}$$

- Rewrite the system as

  $\underline{x}^{\rightarrow\prime}(t) = A \cdot \underline{x}^{\rightarrow}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[-1, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right], \left[6, \begin{bmatrix} -\frac{7}{3} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

  $$\left[-1, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

  $$\underline{x}^{\rightarrow}_{1} = \mathrm{e}^{-t} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- Consider eigenpair

  $$\left[6, \begin{bmatrix} -\frac{7}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

  $$\underline{x}^{\rightarrow}_{2} = \mathrm{e}^{6t} \cdot \begin{bmatrix} -\frac{7}{3} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

  $\underline{x}^{\rightarrow} = c_1 \underline{x}^{\rightarrow}_{1} + c_2 \underline{x}^{\rightarrow}_{2}$

- Substitute solutions into the general solution

  $$\underline{x}^{\rightarrow} = c_1 \mathrm{e}^{-t} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{6t} \cdot \begin{bmatrix} -\frac{7}{3} \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -\frac{7c_2 e^{6t}}{3} \\ c_1 e^{-t} + c_2 e^{6t} \end{bmatrix}$$

- Solution to the system of ODEs

  $$\left\{x_1(t) = -\frac{7c_2 e^{6t}}{3}, x_2(t) = c_1 e^{-t} + c_2 e^{6t}\right\}$$

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 28

```
dsolve([diff(x__1(t),t)=4*x__1(t)+2*x__1(t),diff(x__2(t),t)=-3*x__1(t)-x__2(t)],singsol=all)
```

$$x_1(t) = c_2 e^{6t}$$
$$x_2(t) = -\frac{3c_2 e^{6t}}{7} + e^{-t} c_1$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 56

```
DSolve[{x1'[t]==4*x1[t]+2*x2[t],x2'[t]==-3*x1[t]-x2[t]},{x1[t],x2[t]},t,IncludeSingularSolut
```

$$\text{x1}(t) \to e^t\left(c_1\left(3e^t - 2\right) + 2c_2\left(e^t - 1\right)\right)$$
$$\text{x2}(t) \to e^t\left(c_2\left(3 - 2e^t\right) - 3c_1\left(e^t - 1\right)\right)$$

3.2 problem problem 14

Internal problem ID [314]
Internal file name [`OUTPUT/314_Sunday_June_05_2022_01_38_34_AM_23480794/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.2, Matrices and Linear systems. Page 384
Problem number: problem 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= -3x_1(t) + 2x_2(t) \\
x_2'(t) &= -3x_1(t) + 4x_2(t)
\end{aligned}
$$

3.2.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} -\frac{(\mathrm{e}^{5t}-6)\mathrm{e}^{-2t}}{5} & \frac{2(\mathrm{e}^{5t}-1)\mathrm{e}^{-2t}}{5} \\ -\frac{3(\mathrm{e}^{5t}-1)\mathrm{e}^{-2t}}{5} & \frac{(6\,\mathrm{e}^{5t}-1)\mathrm{e}^{-2t}}{5} \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} -\frac{(e^{5t}-6)e^{-2t}}{5} & \frac{2(e^{5t}-1)e^{-2t}}{5} \\ -\frac{3(e^{5t}-1)e^{-2t}}{5} & \frac{(6\,e^{5t}-1)e^{-2t}}{5} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} -\frac{(e^{5t}-6)e^{-2t}c_1}{5} + \frac{2(e^{5t}-1)e^{-2t}c_2}{5} \\ -\frac{3(e^{5t}-1)e^{-2t}c_1}{5} + \frac{(6\,e^{5t}-1)e^{-2t}c_2}{5} \end{bmatrix} \\
&= \begin{bmatrix} -\frac{((c_1-2c_2)e^{5t}-6c_1+2c_2)e^{-2t}}{5} \\ -\frac{3((c_1-2c_2)e^{5t}-c_1+\frac{c_2}{3})e^{-2t}}{5} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

3.2.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -3 & 2 \\ -3 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -3-\lambda & 2 \\ -3 & 4-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - \lambda - 6 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -2$$
$$\lambda_2 = 3$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-2	1	real eigenvalue
3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -3 & 2 \\ -3 & 4 \end{bmatrix} - (-2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -1 & 2 \\ -3 & 6 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -1 & 2 & 0 \\ -3 & 6 & 0 \end{array}\right]$$

$$R_2 = R_2 - 3R_1 \Longrightarrow \left[\begin{array}{cc|c} -1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2t\}$

Hence the solution is

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -3 & 2 \\ -3 & 4 \end{bmatrix} - (3) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -6 & 2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -6 & 2 & 0 \\ -3 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{2} \implies \left[\begin{array}{cc|c} -6 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -6 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{3}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{3} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-2	1	1	No	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$
3	1	1	No	$\begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -2 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{-2t} \\ &= \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-2t}\end{aligned}$$

Since eigenvalue 3 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{3t} \\ &= \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix} e^{3t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} 2\,\mathrm{e}^{-2t} \\ \mathrm{e}^{-2t} \end{bmatrix} + c_2 \begin{bmatrix} \frac{\mathrm{e}^{3t}}{3} \\ \mathrm{e}^{3t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{(c_2 \mathrm{e}^{5t} + 6c_1)\mathrm{e}^{-2t}}{3} \\ (c_2 \mathrm{e}^{5t} + c_1)\,\mathrm{e}^{-2t} \end{bmatrix}$$

The following is the phase plot of the system.

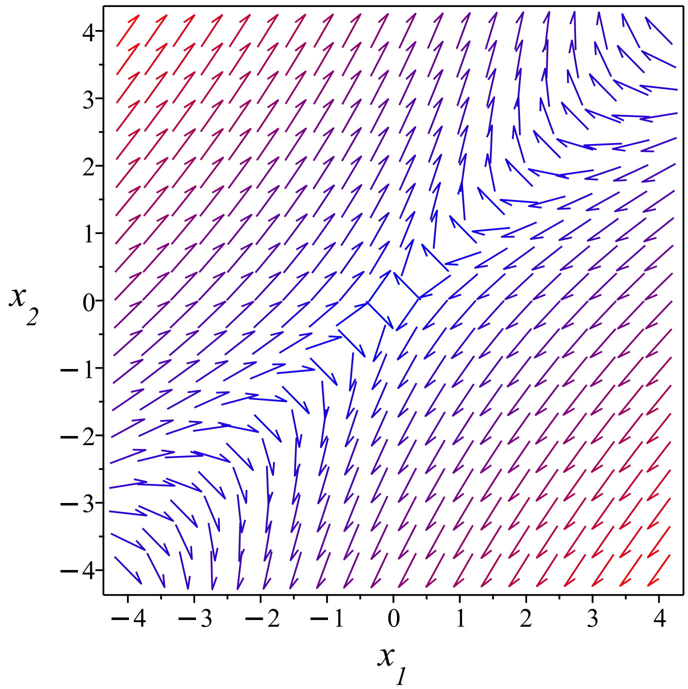


Figure 8: Phase plot

3.2.3 Maple step by step solution

Let's solve

$[x_1'(t) = -3x_1(t) + 2x_2(t), x_2'(t) = -3x_1(t) + 4x_2(t)]$

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x}^{\rightarrow}{}'(t) = \begin{bmatrix} -3 & 2 \\ -3 & 4 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{x}^{\rightarrow}{}'(t) = \begin{bmatrix} -3 & 2 \\ -3 & 4 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -3 & 2 \\ -3 & 4 \end{bmatrix}$$

- Rewrite the system as

$$\underline{\vec{x}}'(t) = A \cdot \underline{\vec{x}}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right], \left[3, \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-2, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_1 = \mathrm{e}^{-2t} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[3, \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_2 = \mathrm{e}^{3t} \cdot \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\underline{\vec{x}} = c_1 \underline{\vec{x}}_1 + c_2 \underline{\vec{x}}_2$$

- Substitute solutions into the general solution

$$\underline{\vec{x}} = c_1 \mathrm{e}^{-2t} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{3t} \cdot \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{(c_2 e^{5t}+6c_1)e^{-2t}}{3} \\ (c_2 e^{5t} + c_1) e^{-2t} \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x_1(t) = \frac{(c_2 e^{5t}+6c_1)e^{-2t}}{3}, x_2(t) = (c_2 e^{5t} + c_1) e^{-2t}\right\}$$

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 36

```
dsolve([diff(x__1(t),t)=-3*x__1(t)+2*x__2(t),diff(x__2(t),t)=-3*x__1(t)+4*x__2(t)],singsol=a
```

$$x_1(t) = c_1 e^{3t} + c_2 e^{-2t}$$
$$x_2(t) = 3c_1 e^{3t} + \frac{c_2 e^{-2t}}{2}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 73

```
DSolve[{x1'[t]==-3*x1[t]+2*x2[t],x2'[t]==-3*x1[t]+4*x2[t]},{x1[t],x2[t]},t,IncludeSingularSo
```

$$\text{x1}(t) \to \frac{1}{5} e^{-2t} \left(2c_2 \left(e^{5t} - 1\right) - c_1 \left(e^{5t} - 6\right)\right)$$
$$\text{x2}(t) \to \frac{1}{5} e^{-2t} \left(c_2 \left(6e^{5t} - 1\right) - 3c_1 \left(e^{5t} - 1\right)\right)$$

4 Section 7.3, The eigenvalue method for linear systems. Page 395

4.1 problem problem 1

Internal problem ID [315]
Internal file name [`OUTPUT/315_Sunday_June_05_2022_01_38_35_AM_13749895/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.3, The eigenvalue method for linear systems. Page 395
Problem number: problem 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x_1'(t) = x_1(t) + 2x_2(t)$$
$$x_2'(t) = 2x_1(t) + x_2(t)$$

4.1.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{e^{-t}}{2} + \frac{e^{3t}}{2} & \frac{e^{3t}}{2} - \frac{e^{-t}}{2} \\ \frac{e^{3t}}{2} - \frac{e^{-t}}{2} & \frac{e^{-t}}{2} + \frac{e^{3t}}{2} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{e^{-t}}{2} + \frac{e^{3t}}{2} & \frac{e^{3t}}{2} - \frac{e^{-t}}{2} \\ \frac{e^{3t}}{2} - \frac{e^{-t}}{2} & \frac{e^{-t}}{2} + \frac{e^{3t}}{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \left(\frac{e^{-t}}{2} + \frac{e^{3t}}{2}\right) c_1 + \left(\frac{e^{3t}}{2} - \frac{e^{-t}}{2}\right) c_2 \\ \left(\frac{e^{3t}}{2} - \frac{e^{-t}}{2}\right) c_1 + \left(\frac{e^{-t}}{2} + \frac{e^{3t}}{2}\right) c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(c_1 - c_2)e^{-t}}{2} + \frac{e^{3t}(c_1 + c_2)}{2} \\ \frac{(-c_1 + c_2)e^{-t}}{2} + \frac{e^{3t}(c_1 + c_2)}{2} \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

4.1.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1 - \lambda & 2 \\ 2 & 1 - \lambda \end{bmatrix} \right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 2\lambda - 3 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 3$$
$$\lambda_2 = -1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2 & 2 & 0 \\ 2 & 2 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 2 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} - (3) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -2 & 2 & 0 \\ 2 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cc|c} -2 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
3	1	1	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
-1	1	1	No	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 3 is real and distinct then the

corresponding eigenvector solution is

$$\vec{x}_1(t) = \vec{v}_1 e^{3t}$$
$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t}$$

Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_2(t) = \vec{v}_2 e^{-t}$$
$$= \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} \mathrm{e}^{3t} \\ \mathrm{e}^{3t} \end{bmatrix} + c_2 \begin{bmatrix} -\mathrm{e}^{-t} \\ \mathrm{e}^{-t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} c_1 \mathrm{e}^{3t} - c_2 \mathrm{e}^{-t} \\ c_1 \mathrm{e}^{3t} + c_2 \mathrm{e}^{-t} \end{bmatrix}$$

The following is the phase plot of the system.

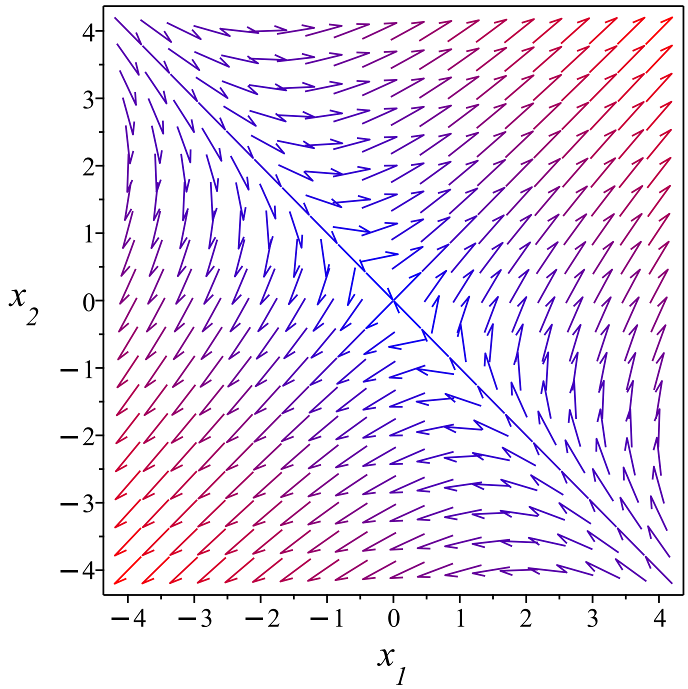


Figure 9: Phase plot

4.1.3 Maple step by step solution

Let's solve

$[x_1'(t) = x_1(t) + 2x_2(t), x_2'(t) = 2x_1(t) + x_2(t)]$

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x}^{\rightarrow}{}'(t) = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{x}^{\rightarrow}{}'(t) = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

- Rewrite the system as

$$\underline{x}^{\rightarrow\prime}(t) = A \cdot \underline{x}^{\rightarrow}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right], \left[3, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{x}^{\rightarrow}_{1} = \mathrm{e}^{-t} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[3, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{x}^{\rightarrow}_{2} = \mathrm{e}^{3t} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\underline{x}^{\rightarrow} = c_1 \underline{x}^{\rightarrow}_{1} + c_2 \underline{x}^{\rightarrow}_{2}$$

- Substitute solutions into the general solution

$$\underline{x}^{\rightarrow} = c_1 \mathrm{e}^{-t} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{3t} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{3t} \\ c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{3t} \end{bmatrix}$$

- Solution to the system of ODEs

  $\{x_1(t) = -c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{3t}, x_2(t) = c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{3t}\}$

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 35

```
dsolve([diff(x__1(t),t)=x__1(t)+2*x__2(t),diff(x__2(t),t)=2*x__1(t)+x__2(t)],singsol=all)
```

$$x_1(t) = c_1 \mathrm{e}^{3t} + c_2 \mathrm{e}^{-t}$$
$$x_2(t) = c_1 \mathrm{e}^{3t} - c_2 \mathrm{e}^{-t}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 68

```
DSolve[{x1'[t]==x1[t]+2*x2[t],x2'[t]==2*x1[t]+x2[t]},{x1[t],x2[t]},t,IncludeSingularSolution
```

$$\mathrm{x1}(t) \to \frac{1}{2} e^{-t} \left(c_1 \left(e^{4t} + 1\right) + c_2 \left(e^{4t} - 1\right)\right)$$
$$\mathrm{x2}(t) \to \frac{1}{2} e^{-t} \left(c_1 \left(e^{4t} - 1\right) + c_2 \left(e^{4t} + 1\right)\right)$$

4.2 problem problem 2

Internal problem ID [316]
Internal file name [`OUTPUT/316_Sunday_June_05_2022_01_38_36_AM_89229362/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.3, The eigenvalue method for linear systems. Page 395
Problem number: problem 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x_1'(t) = 2x_1(t) + 3x_2(t)$$
$$x_2'(t) = 2x_1(t) + x_2(t)$$

4.2.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{2\,e^{-t}}{5} + \frac{3\,e^{4t}}{5} & \frac{3\,e^{4t}}{5} - \frac{3\,e^{-t}}{5} \\ \frac{2\,e^{4t}}{5} - \frac{2\,e^{-t}}{5} & \frac{3\,e^{-t}}{5} + \frac{2\,e^{4t}}{5} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{2\,\mathrm{e}^{-t}}{5}+\frac{3\,\mathrm{e}^{4t}}{5} & \frac{3\,\mathrm{e}^{4t}}{5}-\frac{3\,\mathrm{e}^{-t}}{5} \\ \frac{2\,\mathrm{e}^{4t}}{5}-\frac{2\,\mathrm{e}^{-t}}{5} & \frac{3\,\mathrm{e}^{-t}}{5}+\frac{2\,\mathrm{e}^{4t}}{5} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \left(\frac{2\,\mathrm{e}^{-t}}{5}+\frac{3\,\mathrm{e}^{4t}}{5}\right)c_1+\left(\frac{3\,\mathrm{e}^{4t}}{5}-\frac{3\,\mathrm{e}^{-t}}{5}\right)c_2 \\ \left(\frac{2\,\mathrm{e}^{4t}}{5}-\frac{2\,\mathrm{e}^{-t}}{5}\right)c_1+\left(\frac{3\,\mathrm{e}^{-t}}{5}+\frac{2\,\mathrm{e}^{4t}}{5}\right)c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(2c_1-3c_2)\mathrm{e}^{-t}}{5}+\frac{3\,\mathrm{e}^{4t}(c_1+c_2)}{5} \\ \frac{(-2c_1+3c_2)\mathrm{e}^{-t}}{5}+\frac{2\,\mathrm{e}^{4t}(c_1+c_2)}{5} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

4.2.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A-\lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 2-\lambda & 3 \\ 2 & 1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 3\lambda - 4 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 4$$
$$\lambda_2 = -1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
4	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 3 & 3 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 3 & 3 & 0 \\ 2 & 2 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{2R_1}{3} \Longrightarrow \left[\begin{array}{cc|c} 3 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3 & 3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 4$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} - (4) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 3 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -2 & 3 & 0 \\ 2 & -3 & 0 \end{array} \right]$$

$$R_2 = R_2 + R_1 \implies \left[\begin{array}{cc|c} -2 & 3 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{3t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{3t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{3t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{3t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{3t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{3t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
4	1	1	No	$\begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$
-1	1	1	No	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 4 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{4t} \\
&= \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix} e^{4t}
\end{aligned}
$$

Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{-t} \\
&= \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} \frac{3\,\mathrm{e}^{4t}}{2} \\ \mathrm{e}^{4t} \end{bmatrix} + c_2 \begin{bmatrix} -\mathrm{e}^{-t} \\ \mathrm{e}^{-t} \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{3c_1 \mathrm{e}^{4t}}{2} - c_2 \mathrm{e}^{-t} \\ c_1 \mathrm{e}^{4t} + c_2 \mathrm{e}^{-t} \end{bmatrix}
$$

The following is the phase plot of the system.

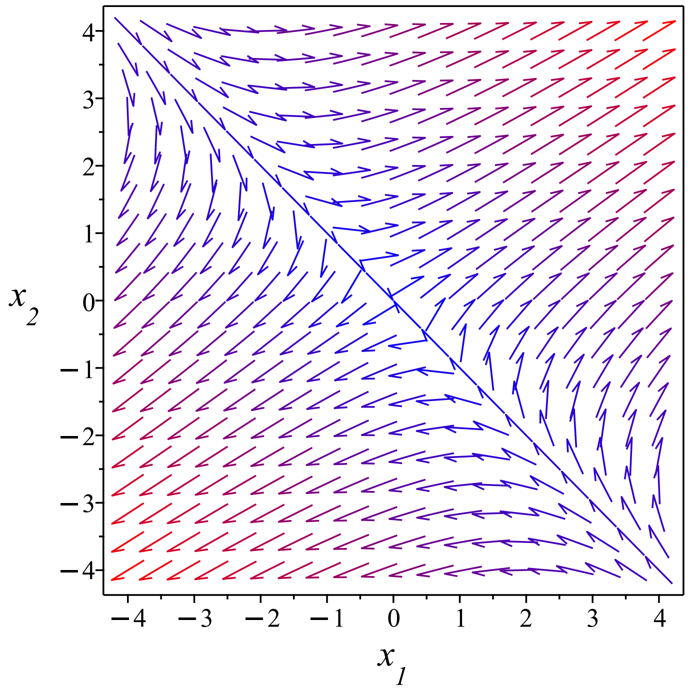


Figure 10: Phase plot

4.2.3 Maple step by step solution

Let's solve

$[x_1'(t) = 2x_1(t) + 3x_2(t), x_2'(t) = 2x_1(t) + x_2(t)]$

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x}^{\rightarrow}{}'(t) = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{x}^{\rightarrow}{}'(t) = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}$$

- Rewrite the system as

  $\underline{\vec{x}}'(t) = A \cdot \underline{\vec{x}}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[-1, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right], \left[4, \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

  $$\left[-1, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

  $$\underline{\vec{x}}_1 = \mathrm{e}^{-t} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

  $$\left[4, \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

  $$\underline{\vec{x}}_2 = \mathrm{e}^{4t} \cdot \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

  $\underline{\vec{x}} = c_1 \underline{\vec{x}}_1 + c_2 \underline{\vec{x}}_2$

- Substitute solutions into the general solution

  $$\underline{\vec{x}} = c_1 \mathrm{e}^{-t} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{4t} \cdot \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -c_1 \mathrm{e}^{-t} + \frac{3c_2 \mathrm{e}^{4t}}{2} \\ c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{4t} \end{bmatrix}$$

- Solution to the system of ODEs

  $\left\{x_1(t) = -c_1 \mathrm{e}^{-t} + \frac{3c_2 \mathrm{e}^{4t}}{2}, x_2(t) = c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{4t}\right\}$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 36

```
dsolve([diff(x__1(t),t)=2*x__1(t)+3*x__2(t),diff(x__2(t),t)=2*x__1(t)+x__2(t)],singsol=all)
```

$$x_1(t) = c_1 \mathrm{e}^{4t} + c_2 \mathrm{e}^{-t}$$
$$x_2(t) = \frac{2c_1 \mathrm{e}^{4t}}{3} - c_2 \mathrm{e}^{-t}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 74

```
DSolve[{x1'[t]==2*x1[t]+3*x2[t],x2'[t]==2*x1[t]+x2[t]},{x1[t],x2[t]},t,IncludeSingularSoluti
```

$$\mathrm{x1}(t) \to \frac{1}{5} e^{-t} \left(c_1 \left(3e^{5t} + 2\right) + 3c_2 \left(e^{5t} - 1\right)\right)$$
$$\mathrm{x2}(t) \to \frac{1}{5} e^{-t} \left(2c_1 \left(e^{5t} - 1\right) + c_2 \left(2e^{5t} + 3\right)\right)$$

4.3 problem problem 3

Internal problem ID [317]
Internal file name [`OUTPUT/317_Sunday_June_05_2022_01_38_37_AM_23964630/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.3, The eigenvalue method for linear systems. Page 395
Problem number: problem 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= 3x_1(t) + 4x_2(t) \\
x_2'(t) &= 3x_1(t) + 2x_2(t)
\end{aligned}
$$

With initial conditions

$$[x_1(0) = 1, x_2(0) = 1]$$

4.3.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{3\,e^{-t}}{7} + \frac{4\,e^{6t}}{7} & \frac{4\,e^{6t}}{7} - \frac{4\,e^{-t}}{7} \\ \frac{3\,e^{6t}}{7} - \frac{3\,e^{-t}}{7} & \frac{4\,e^{-t}}{7} + \frac{3\,e^{6t}}{7} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} \frac{3\,e^{-t}}{7}+\frac{4\,e^{6t}}{7} & \frac{4\,e^{6t}}{7}-\frac{4\,e^{-t}}{7} \\ \frac{3\,e^{6t}}{7}-\frac{3\,e^{-t}}{7} & \frac{4\,e^{-t}}{7}+\frac{3\,e^{6t}}{7} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} -\frac{e^{-t}}{7}+\frac{8\,e^{6t}}{7} \\ \frac{6\,e^{6t}}{7}+\frac{e^{-t}}{7} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

4.3.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det\left(A - \lambda I\right) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} 3 & 4 \\ 3 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} 3-\lambda & 4 \\ 3 & 2-\lambda \end{bmatrix}\right) = 0
$$

Which gives the characteristic equation

$$
\lambda^2 - 5\lambda - 6 = 0
$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1$$
$$\lambda_2 = 6$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
6	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & 4 \\ 3 & 2 \end{bmatrix} - (-1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 4 & 4 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 4 & 4 & 0 \\ 3 & 3 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{3R_1}{4} \Longrightarrow \left[\begin{array}{cc|c} 4 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 4 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 6$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & 4 \\ 3 & 2 \end{bmatrix} - (6) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 4 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -3 & 4 & 0 \\ 3 & -4 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cc|c} -3 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -3 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{4t}{3}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{4t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{4t}{3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{4t}{3} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{4}{3} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{4t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{4}{3} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{4t}{3} \\ t \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-1	1	1	No	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$
6	1	1	No	$\begin{bmatrix} \frac{4}{3} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -1 is real and distinct then the

corresponding eigenvector solution is

$$\vec{x}_1(t) = \vec{v}_1 e^{-t}$$
$$= \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t}$$

Since eigenvalue 6 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_2(t) = \vec{v}_2 e^{6t}$$
$$= \begin{bmatrix} \frac{4}{3} \\ 1 \end{bmatrix} e^{6t}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} -\mathrm{e}^{-t} \\ \mathrm{e}^{-t} \end{bmatrix} + c_2 \begin{bmatrix} \frac{4\,\mathrm{e}^{6t}}{3} \\ \mathrm{e}^{6t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -c_1 \mathrm{e}^{-t} + \frac{4c_2 \mathrm{e}^{6t}}{3} \\ c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{6t} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x_1(0) = 1 \\ x_2(0) = 1 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -c_1 + \frac{4c_2}{3} \\ c_1 + c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = \frac{1}{7} \\ c_2 = \frac{6}{7} \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -\frac{e^{-t}}{7} + \frac{8\,e^{6t}}{7} \\ \frac{6\,e^{6t}}{7} + \frac{e^{-t}}{7} \end{bmatrix}$$

The following is the phase plot of the system.

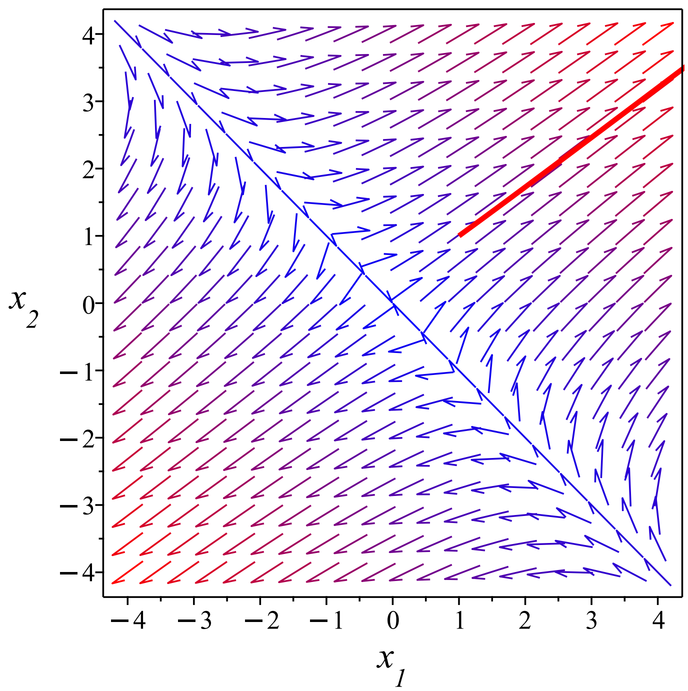
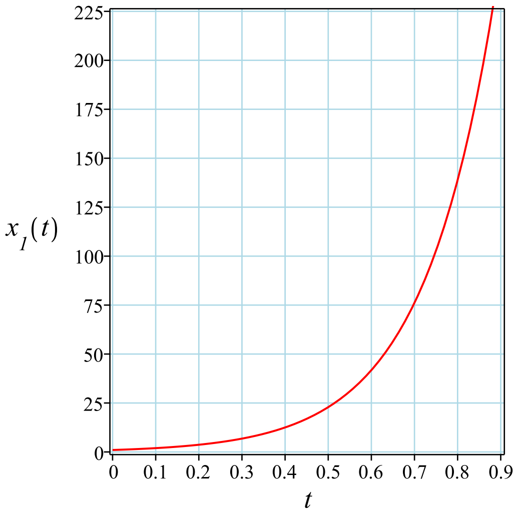
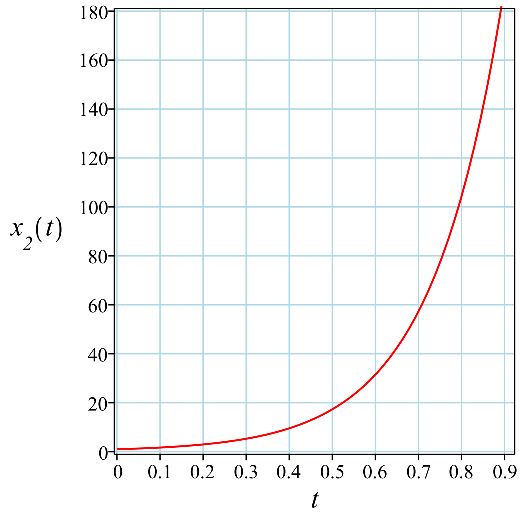


Figure 11: Phase plot

The following are plots of each solution.





✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 34

```
dsolve([diff(x__1(t),t) = 3*x__1(t)+4*x__2(t), diff(x__2(t),t) = 3*x__1(t)+2*x__2(t), x__1(0
```

$$x_1(t) = -\frac{e^{-t}}{7} + \frac{8\,e^{6t}}{7}$$
$$x_2(t) = \frac{e^{-t}}{7} + \frac{6\,e^{6t}}{7}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 44

```
DSolve[{x1'[t]==3*x1[t]+4*x2[t],x2'[t]==3*x1[t]+2*x2[t]},{x1[0]==1,x2[0]==1},{x1[t],x2[t]},t
```

$$\mathrm{x1}(t) \to \frac{1}{7}e^{-t}\left(8e^{7t} - 1\right)$$
$$\mathrm{x2}(t) \to \frac{1}{7}e^{-t}\left(6e^{7t} + 1\right)$$

4.4 problem problem 4

Internal problem ID [318]
Internal file name [`OUTPUT/318_Sunday_June_05_2022_01_38_39_AM_31992579/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.3, The eigenvalue method for linear systems. Page 395
Problem number: problem 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x_1'(t) = 4x_1(t) + x_2(t)$$
$$x_2'(t) = 6x_1(t) - x_2(t)$$

4.4.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 6 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{(6\,e^{7t}+1)e^{-2t}}{7} & \frac{(e^{7t}-1)e^{-2t}}{7} \\ \frac{6(e^{7t}-1)e^{-2t}}{7} & \frac{(e^{7t}+6)e^{-2t}}{7} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{(6\,e^{7t}+1)\,e^{-2t}}{7} & \frac{(e^{7t}-1)\,e^{-2t}}{7} \\ \frac{6(e^{7t}-1)\,e^{-2t}}{7} & \frac{(e^{7t}+6)\,e^{-2t}}{7} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(6\,e^{7t}+1)\,e^{-2t}c_1}{7} + \frac{(e^{7t}-1)\,e^{-2t}c_2}{7} \\ \frac{6(e^{7t}-1)\,e^{-2t}c_1}{7} + \frac{(e^{7t}+6)\,e^{-2t}c_2}{7} \end{bmatrix} \\
&= \begin{bmatrix} \frac{e^{-2t}((6c_1+c_2)e^{7t}+c_1-c_2)}{7} \\ \frac{6\,e^{-2t}((c_1+\frac{c_2}{6})e^{7t}-c_1+c_2)}{7} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

4.4.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 6 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det(A - \lambda I) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} 4 & 1 \\ 6 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} 4-\lambda & 1 \\ 6 & -1-\lambda \end{bmatrix}\right) = 0
$$

Which gives the characteristic equation

$$\lambda^2 - 3\lambda - 10 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -2$$
$$\lambda_2 = 5$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-2	1	real eigenvalue
5	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 4 & 1 \\ 6 & -1 \end{bmatrix} - (-2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 6 & 1 \\ 6 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 6 & 1 & 0 \\ 6 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 6 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 6 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{t}{6}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{t}{6} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{t}{6} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{t}{6} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{6} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{t}{6} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{6} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{t}{6} \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 5$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 4 & 1 \\ 6 & -1 \end{bmatrix} - (5)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \\ 6 & -6 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 6 & -6 & 0 \end{array}\right]$$

$$R_2 = R_2 + 6R_1 \implies \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-2	1	1	No	$\begin{bmatrix} -\frac{1}{6} \\ 1 \end{bmatrix}$
5	1	1	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -2 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{-2t} \\
&= \begin{bmatrix} -\frac{1}{6} \\ 1 \end{bmatrix} e^{-2t}
\end{aligned}
$$

Since eigenvalue 5 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{5t} \\
&= \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{5t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} -\frac{e^{-2t}}{6} \\ e^{-2t} \end{bmatrix} + c_2 \begin{bmatrix} e^{5t} \\ e^{5t} \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -\frac{(-6c_2 e^{7t} + c_1) e^{-2t}}{6} \\ (c_2 e^{7t} + c_1) e^{-2t} \end{bmatrix}
$$

The following is the phase plot of the system.

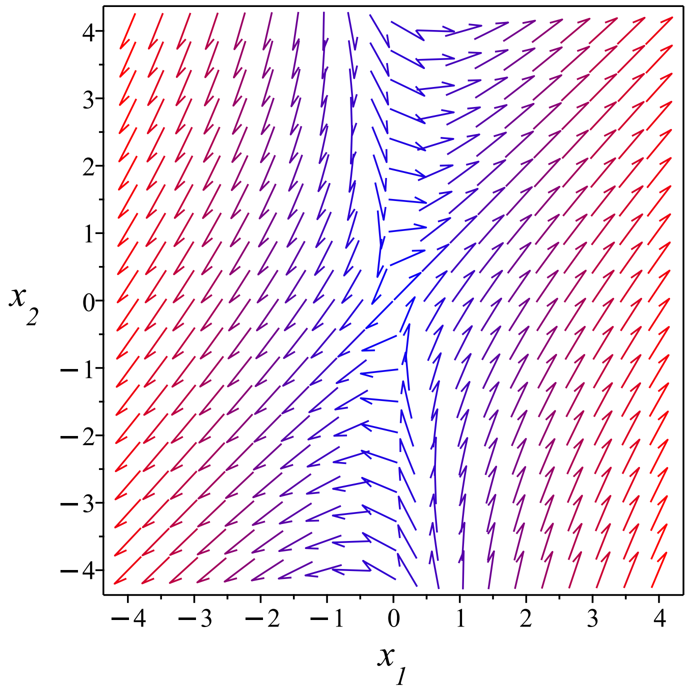


Figure 12: Phase plot

4.4.3 Maple step by step solution

Let's solve

$[x_1'(t) = 4x_1(t) + x_2(t), x_2'(t) = 6x_1(t) - x_2(t)]$

- Define vector

$$\underline{\vec{x}}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{\vec{x}}'(t) = \begin{bmatrix} 4 & 1 \\ 6 & -1 \end{bmatrix} \cdot \underline{\vec{x}}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{\vec{x}}'(t) = \begin{bmatrix} 4 & 1 \\ 6 & -1 \end{bmatrix} \cdot \underline{\vec{x}}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 4 & 1 \\ 6 & -1 \end{bmatrix}$$

- Rewrite the system as

$$\underline{x}^{\rightarrow\prime}(t) = A \cdot \underline{x}^{\rightarrow}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \begin{bmatrix} -\frac{1}{6} \\ 1 \end{bmatrix}\right], \left[5, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-2, \begin{bmatrix} -\frac{1}{6} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{x}^{\rightarrow}{}_1 = e^{-2t} \cdot \begin{bmatrix} -\frac{1}{6} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[5, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{x}^{\rightarrow}{}_2 = e^{5t} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\underline{x}^{\rightarrow} = c_1 \underline{x}^{\rightarrow}{}_1 + c_2 \underline{x}^{\rightarrow}{}_2$$

- Substitute solutions into the general solution

$$\underline{x}^{\rightarrow} = c_1 e^{-2t} \cdot \begin{bmatrix} -\frac{1}{6} \\ 1 \end{bmatrix} + c_2 e^{5t} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -\frac{(-6c_2\mathrm{e}^{7t}+c_1)\mathrm{e}^{-2t}}{6} \\ (c_2\mathrm{e}^{7t}+c_1)\,\mathrm{e}^{-2t} \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x_1(t) = -\frac{(-6c_2\mathrm{e}^{7t}+c_1)\mathrm{e}^{-2t}}{6}, x_2(t) = (c_2\mathrm{e}^{7t}+c_1)\,\mathrm{e}^{-2t}\right\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 35

```
dsolve([diff(x__1(t),t)=4*x__1(t)+x__2(t),diff(x__2(t),t)=6*x__1(t)-x__2(t)],singsol=all)
```

$$x_1(t) = c_1\mathrm{e}^{-2t} + c_2\mathrm{e}^{5t}$$
$$x_2(t) = -6c_1\mathrm{e}^{-2t} + c_2\mathrm{e}^{5t}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 71

```
DSolve[{x1'[t]==4*x1[t]+x2[t],x2'[t]==6*x1[t]-x2[t]},{x1[t],x2[t]},t,IncludeSingularSolution
```

$$\mathrm{x1}(t) \to \frac{1}{7}e^{-2t}\left(c_1\left(6e^{7t}+1\right)+c_2\left(e^{7t}-1\right)\right)$$
$$\mathrm{x2}(t) \to \frac{1}{7}e^{-2t}\left(6c_1\left(e^{7t}-1\right)+c_2\left(e^{7t}+6\right)\right)$$

4.5 problem problem 5

Internal problem ID [319]
Internal file name [`OUTPUT/319_Sunday_June_05_2022_01_38_40_AM_26380635/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.3, The eigenvalue method for linear systems. Page 395
Problem number: problem 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x_1'(t) = 6x_1(t) - 7x_2(t)$$
$$x_2'(t) = x_1(t) - 2x_2(t)$$

4.5.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 6 & -7 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} -\frac{e^{-t}}{6} + \frac{7\,e^{5t}}{6} & -\frac{7\,e^{5t}}{6} + \frac{7\,e^{-t}}{6} \\ \frac{e^{5t}}{6} - \frac{e^{-t}}{6} & \frac{7\,e^{-t}}{6} - \frac{e^{5t}}{6} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} -\frac{e^{-t}}{6}+\frac{7\,e^{5t}}{6} & -\frac{7\,e^{5t}}{6}+\frac{7\,e^{-t}}{6} \\ \frac{e^{5t}}{6}-\frac{e^{-t}}{6} & \frac{7\,e^{-t}}{6}-\frac{e^{5t}}{6} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \left(-\frac{e^{-t}}{6}+\frac{7\,e^{5t}}{6}\right)c_1+\left(-\frac{7\,e^{5t}}{6}+\frac{7\,e^{-t}}{6}\right)c_2 \\ \left(\frac{e^{5t}}{6}-\frac{e^{-t}}{6}\right)c_1+\left(\frac{7\,e^{-t}}{6}-\frac{e^{5t}}{6}\right)c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(-c_1+7c_2)e^{-t}}{6}+\frac{7\,e^{5t}(c_1-c_2)}{6} \\ \frac{(-c_1+7c_2)e^{-t}}{6}+\frac{e^{5t}(c_1-c_2)}{6} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

4.5.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 6 & -7 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det(A-\lambda I) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} 6 & -7 \\ 1 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} 6-\lambda & -7 \\ 1 & -2-\lambda \end{bmatrix}\right) = 0
$$

Which gives the characteristic equation

$$\lambda^2 - 4\lambda - 5 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1$$
$$\lambda_2 = 5$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
5	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 6 & -7 \\ 1 & -2 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 7 & -7 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 7 & -7 & 0 \\ 1 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{7} \Longrightarrow \left[\begin{array}{cc|c} 7 & -7 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 7 & -7 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 5$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 6 & -7 \\ 1 & -2 \end{bmatrix} - (5) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -7 \\ 1 & -7 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & -7 & 0 \\ 1 & -7 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 1 & -7 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & -7 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 7t\}$

Hence the solution is

$$\begin{bmatrix} 7t \\ t \end{bmatrix} = \begin{bmatrix} 7t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 7t \\ t \end{bmatrix} = t \begin{bmatrix} 7 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 7t \\ t \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-1	1	1	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
5	1	1	No	$\begin{bmatrix} 7 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -1 is real and distinct then the

corresponding eigenvector solution is

$$\vec{x}_1(t) = \vec{v}_1 e^{-t}$$
$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}$$

Since eigenvalue 5 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_2(t) = \vec{v}_2 e^{5t}$$
$$= \begin{bmatrix} 7 \\ 1 \end{bmatrix} e^{5t}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} \mathrm{e}^{-t} \\ \mathrm{e}^{-t} \end{bmatrix} + c_2 \begin{bmatrix} 7\,\mathrm{e}^{5t} \\ \mathrm{e}^{5t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} c_1 \mathrm{e}^{-t} + 7 c_2 \mathrm{e}^{5t} \\ c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{5t} \end{bmatrix}$$

The following is the phase plot of the system.

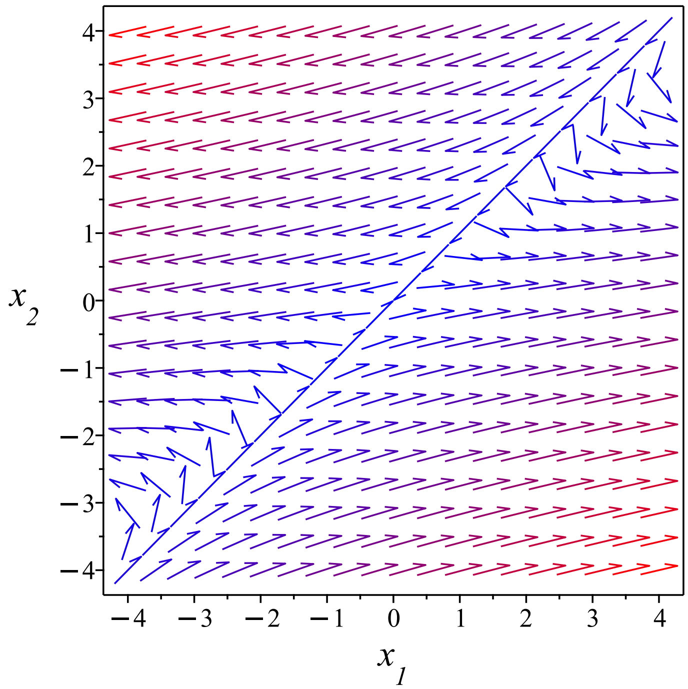


Figure 13: Phase plot

4.5.3 Maple step by step solution

Let's solve

$[x_1'(t) = 6x_1(t) - 7x_2(t), x_2'(t) = x_1(t) - 2x_2(t)]$

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x}^{\rightarrow\prime}(t) = \begin{bmatrix} 6 & -7 \\ 1 & -2 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{x}^{\rightarrow\prime}(t) = \begin{bmatrix} 6 & -7 \\ 1 & -2 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 6 & -7 \\ 1 & -2 \end{bmatrix}$$

- Rewrite the system as

$$\underline{\vec{x}}'(t) = A \cdot \underline{\vec{x}}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right], \left[5, \begin{bmatrix} 7 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_1 = \mathrm{e}^{-t} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[5, \begin{bmatrix} 7 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_2 = \mathrm{e}^{5t} \cdot \begin{bmatrix} 7 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\underline{\vec{x}} = c_1 \underline{\vec{x}}_1 + c_2 \underline{\vec{x}}_2$$

- Substitute solutions into the general solution

$$\underline{\vec{x}} = c_1 \mathrm{e}^{-t} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{5t} \cdot \begin{bmatrix} 7 \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{-t} + 7c_2 e^{5t} \\ c_1 e^{-t} + c_2 e^{5t} \end{bmatrix}$$

- Solution to the system of ODEs

  $\{x_1(t) = c_1 e^{-t} + 7c_2 e^{5t}, x_2(t) = c_1 e^{-t} + c_2 e^{5t}\}$

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 35

```
dsolve([diff(x__1(t),t)=6*x__1(t)-7*x__2(t),diff(x__2(t),t)=x__1(t)-2*x__2(t)],singsol=all)
```

$$x_1(t) = e^{-t} c_1 + c_2 e^{5t}$$
$$x_2(t) = e^{-t} c_1 + \frac{c_2 e^{5t}}{7}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 72

```
DSolve[{x1'[t]==6*x1[t]-7*x2[t],x2'[t]==x1[t]-2*x2[t]},{x1[t],x2[t]},t,IncludeSingularSoluti
```

$$\text{x1}(t) \to \frac{1}{6} e^{-t} \left(c_1 \left(7e^{6t} - 1\right) - 7c_2 \left(e^{6t} - 1\right)\right)$$
$$\text{x2}(t) \to \frac{1}{6} e^{-t} \left(c_1 \left(e^{6t} - 1\right) - c_2 \left(e^{6t} - 7\right)\right)$$

4.6 problem problem 6

Internal problem ID [320]
Internal file name [`OUTPUT/320_Sunday_June_05_2022_01_38_41_AM_84692260/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.3, The eigenvalue method for linear systems. Page 395
Problem number: problem 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= 9x_1(t) + 5x_2(t) \\
x_2'(t) &= -6x_1(t) - 2x_2(t)
\end{aligned}
$$

With initial conditions

$$[x_1(0) = 1, x_2(0) = 0]$$

4.6.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 9 & 5 \\ -6 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} -5\,e^{3t} + 6\,e^{4t} & 5\,e^{4t} - 5\,e^{3t} \\ -6\,e^{4t} + 6\,e^{3t} & 6\,e^{3t} - 5\,e^{4t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} -5\,\mathrm{e}^{3t} + 6\,\mathrm{e}^{4t} & 5\,\mathrm{e}^{4t} - 5\,\mathrm{e}^{3t} \\ -6\,\mathrm{e}^{4t} + 6\,\mathrm{e}^{3t} & 6\,\mathrm{e}^{3t} - 5\,\mathrm{e}^{4t} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} -5\,\mathrm{e}^{3t} + 6\,\mathrm{e}^{4t} \\ -6\,\mathrm{e}^{4t} + 6\,\mathrm{e}^{3t} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

4.6.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 9 & 5 \\ -6 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 9 & 5 \\ -6 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 9-\lambda & 5 \\ -6 & -2-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 7\lambda + 12 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 3$$
$$\lambda_2 = 4$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
3	1	real eigenvalue
4	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 9 & 5 \\ -6 & -2 \end{bmatrix} - (3) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 6 & 5 \\ -6 & -5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 6 & 5 & 0 \\ -6 & -5 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cc|c} 6 & 5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 6 & 5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{5t}{6}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{5t}{6} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{5t}{6} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{5t}{6} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{5}{6} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{5t}{6} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{5}{6} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{5t}{6} \\ t \end{bmatrix} = \begin{bmatrix} -5 \\ 6 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 4$

We need to solve $A\vec{v} = \lambda \vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 9 & 5 \\ -6 & -2 \end{bmatrix} - (4) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 5 \\ -6 & -6 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 5 & 5 & 0 \\ -6 & -6 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{6R_1}{5} \Longrightarrow \left[\begin{array}{cc|c} 5 & 5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 5 & 5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
3	1	1	No	$\begin{bmatrix} -\frac{5}{6} \\ 1 \end{bmatrix}$
4	1	1	No	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 3 is real and distinct then the

corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{3t} \\
&= \begin{bmatrix} -\frac{5}{6} \\ 1 \end{bmatrix} e^{3t}
\end{aligned}
$$

Since eigenvalue 4 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{4t} \\
&= \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{4t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} -\frac{5\,\mathrm{e}^{3t}}{6} \\ \mathrm{e}^{3t} \end{bmatrix} + c_2 \begin{bmatrix} -\mathrm{e}^{4t} \\ \mathrm{e}^{4t} \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -\frac{5c_1\mathrm{e}^{3t}}{6} - c_2\mathrm{e}^{4t} \\ c_1\mathrm{e}^{3t} + c_2\mathrm{e}^{4t} \end{bmatrix}
$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$
\begin{bmatrix} x_1(0) = 1 \\ x_2(0) = 0 \end{bmatrix} \tag{1}
$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$
\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{5c_1}{6} - c_2 \\ c_1 + c_2 \end{bmatrix}
$$

Solving for the constants of integrations gives

$$
\begin{bmatrix} c_1 = 6 \\ c_2 = -6 \end{bmatrix}
$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -5\,\mathrm{e}^{3t} + 6\,\mathrm{e}^{4t} \\ -6\,\mathrm{e}^{4t} + 6\,\mathrm{e}^{3t} \end{bmatrix}$$

The following is the phase plot of the system.

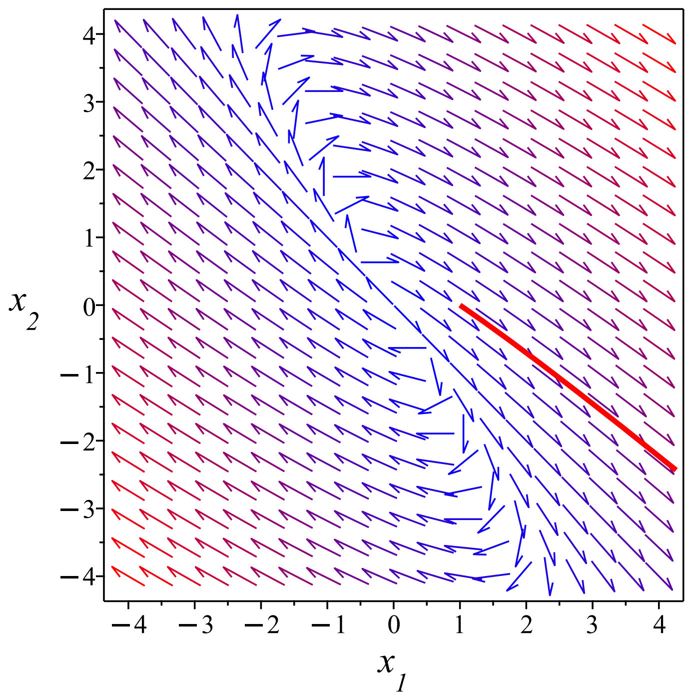
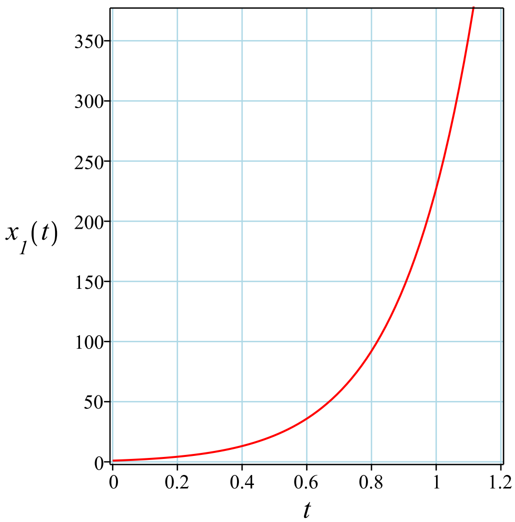
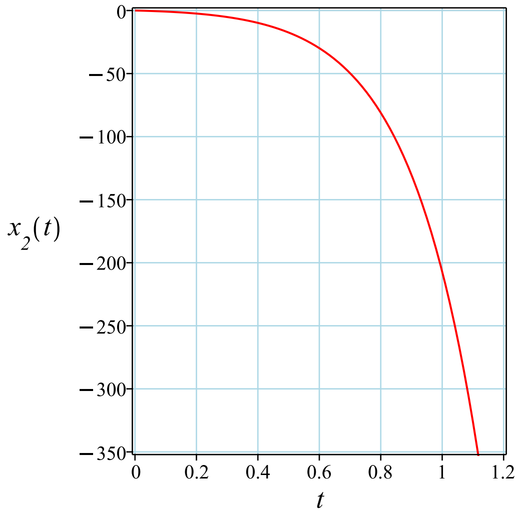


Figure 14: Phase plot

The following are plots of each solution.





✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 34

```
dsolve([diff(x__1(t),t) = 9*x__1(t)+5*x__2(t), diff(x__2(t),t) = -6*x__1(t)-2*x__2(t), x__1(
```

$$x_1(t) = 6\,\mathrm{e}^{4t} - 5\,\mathrm{e}^{3t}$$
$$x_2(t) = -6\,\mathrm{e}^{4t} + 6\,\mathrm{e}^{3t}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 33

```
DSolve[{x1'[t]==9*x1[t]+5*x2[t],x2'[t]==-6*x1[t]-2*x2[t]},{x1[0]==1,x2[0]==0},{x1[t],x2[t]},
```

$$\text{x1}(t) \to e^{3t}\left(6e^t - 5\right)$$
$$\text{x2}(t) \to -6e^{3t}\left(e^t - 1\right)$$

4.7 problem problem 7

Internal problem ID [321]
Internal file name [`OUTPUT/321_Sunday_June_05_2022_01_38_42_AM_39793054/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.3, The eigenvalue method for linear systems. Page 395
Problem number: problem 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x_1'(t) = -3x_1(t) + 4x_2(t)$$
$$x_2'(t) = 6x_1(t) - 5x_2(t)$$

4.7.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ 6 & -5 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{(3\,e^{10t}+2)\,e^{-9t}}{5} & \frac{2(e^{10t}-1)\,e^{-9t}}{5} \\ \frac{3(e^{10t}-1)\,e^{-9t}}{5} & \frac{(2\,e^{10t}+3)\,e^{-9t}}{5} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{(3\,e^{10t}+2)\,e^{-9t}}{5} & \frac{2(e^{10t}-1)\,e^{-9t}}{5} \\ \frac{3(e^{10t}-1)\,e^{-9t}}{5} & \frac{(2\,e^{10t}+3)\,e^{-9t}}{5} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(3\,e^{10t}+2)\,e^{-9t}c_1}{5} + \frac{2(e^{10t}-1)\,e^{-9t}c_2}{5} \\ \frac{3(e^{10t}-1)\,e^{-9t}c_1}{5} + \frac{(2\,e^{10t}+3)\,e^{-9t}c_2}{5} \end{bmatrix} \\
&= \begin{bmatrix} \frac{e^{-9t}((3c_1+2c_2)e^{10t}+2c_1-2c_2)}{5} \\ \frac{3\left(\left(c_1+\frac{2c_2}{3}\right)e^{10t}-c_1+c_2\right)e^{-9t}}{5} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

4.7.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ 6 & -5 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det(A - \lambda I) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} -3 & 4 \\ 6 & -5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} -3-\lambda & 4 \\ 6 & -5-\lambda \end{bmatrix}\right) = 0
$$

Which gives the characteristic equation

$$\lambda^2 + 8\lambda - 9 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 1$$
$$\lambda_2 = -9$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue
-9	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -9$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -3 & 4 \\ 6 & -5 \end{bmatrix} - (-9)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 6 & 4 \\ 6 & 4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 6 & 4 & 0 \\ 6 & 4 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 6 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 6 & 4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -\frac{2t}{3}\}$

Hence the solution is

$$\begin{bmatrix} -\frac{2t}{3} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2t}{3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{2t}{3} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{2t}{3} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{2t}{3} \\ t \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -3 & 4 \\ 6 & -5 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 4 \\ 6 & -6 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -4 & 4 & 0 \\ 6 & -6 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{3R_1}{2} \Longrightarrow \left[\begin{array}{cc|c} -4 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -4 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
1	1	1	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
-9	1	1	No	$\begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{t} \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t}\end{aligned}$$

Since eigenvalue -9 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{-9t} \\ &= \begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix} e^{-9t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} \mathrm{e}^{t} \\ \mathrm{e}^{t} \end{bmatrix} + c_2 \begin{bmatrix} -\frac{2\,\mathrm{e}^{-9t}}{3} \\ \mathrm{e}^{-9t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{(3c_1 \mathrm{e}^{10t} - 2c_2)\mathrm{e}^{-9t}}{3} \\ (c_1 \mathrm{e}^{10t} + c_2)\,\mathrm{e}^{-9t} \end{bmatrix}$$

The following is the phase plot of the system.

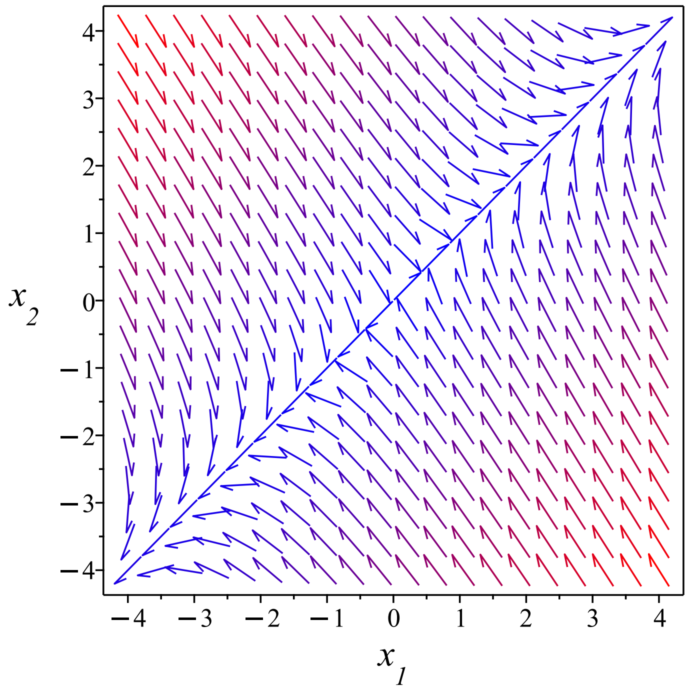


Figure 15: Phase plot

4.7.3 Maple step by step solution

Let's solve

$[x_1'(t) = -3x_1(t) + 4x_2(t), x_2'(t) = 6x_1(t) - 5x_2(t)]$

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x}^{\rightarrow}{}'(t) = \begin{bmatrix} -3 & 4 \\ 6 & -5 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{x}^{\rightarrow}{}'(t) = \begin{bmatrix} -3 & 4 \\ 6 & -5 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -3 & 4 \\ 6 & -5 \end{bmatrix}$$

- Rewrite the system as

$$\underline{x^{\rightarrow}}'(t) = A \cdot \underline{x^{\rightarrow}}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-9, \begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-9, \begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{x^{\rightarrow}}_1 = \mathrm{e}^{-9t} \cdot \begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{x^{\rightarrow}}_2 = \mathrm{e}^{t} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\underline{x^{\rightarrow}} = c_1 \underline{x^{\rightarrow}}_1 + c_2 \underline{x^{\rightarrow}}_2$$

- Substitute solutions into the general solution

$$\underline{x^{\rightarrow}} = c_1 \mathrm{e}^{-9t} \cdot \begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{t} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{-9t} c_2 \mathrm{e}^{10t} - \frac{2c_1 \mathrm{e}^{-9t}}{3} \\ (c_2 \mathrm{e}^{10t} + c_1) \mathrm{e}^{-9t} \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x_1(t) = \mathrm{e}^{-9t} c_2 \mathrm{e}^{10t} - \frac{2c_1 \mathrm{e}^{-9t}}{3}, x_2(t) = (c_2 \mathrm{e}^{10t} + c_1) \mathrm{e}^{-9t}\right\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 31

```
dsolve([diff(x__1(t),t)=-3*x__1(t)+4*x__2(t),diff(x__2(t),t)=6*x__1(t)-5*x__2(t)],singsol=al
```

$$x_1(t) = c_1 \mathrm{e}^{-9t} + c_2 \mathrm{e}^{t}$$
$$x_2(t) = -\frac{3c_1 \mathrm{e}^{-9t}}{2} + c_2 \mathrm{e}^{t}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 74

```
DSolve[{x1'[t]==-3*x1[t]+4*x2[t],x2'[t]==6*x1[t]-5*x2[t]},{x1[t],x2[t]},t,IncludeSingularSol
```

$$\mathrm{x1}(t) \to \frac{1}{5} e^{-9t} \left(c_1 \left(3e^{10t} + 2\right) + 2c_2 \left(e^{10t} - 1\right)\right)$$
$$\mathrm{x2}(t) \to \frac{1}{5} e^{-9t} \left(3c_1 \left(e^{10t} - 1\right) + c_2 \left(2e^{10t} + 3\right)\right)$$

4.8 problem problem 8

Internal problem ID [322]
Internal file name [`OUTPUT/322_Sunday_June_05_2022_01_38_43_AM_66449525/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.3, The eigenvalue method for linear systems. Page 395
Problem number: problem 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= x_1(t) - 5x_2(t)\\
x_2'(t) &= x_1(t) - x_2(t)
\end{aligned}
$$

4.8.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & -5 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \cos(2t) + \frac{\sin(2t)}{2} & -\frac{5\sin(2t)}{2} \\ \frac{\sin(2t)}{2} & \cos(2t) - \frac{\sin(2t)}{2} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \cos(2t) + \frac{\sin(2t)}{2} & -\frac{5\sin(2t)}{2} \\ \frac{\sin(2t)}{2} & \cos(2t) - \frac{\sin(2t)}{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \left(\cos(2t) + \frac{\sin(2t)}{2}\right) c_1 - \frac{5\sin(2t)c_2}{2} \\ \frac{\sin(2t)c_1}{2} + \left(\cos(2t) - \frac{\sin(2t)}{2}\right) c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(c_1-5c_2)\sin(2t)}{2} + c_1\cos(2t) \\ \frac{\sin(2t)(c_1-c_2)}{2} + c_2\cos(2t) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

4.8.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & -5 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & -5 \\ 1 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & -5 \\ 1 & -1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 4 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2i$$
$$\lambda_2 = -2i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$2i$	1	complex eigenvalue
$-2i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -2i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -5 \\ 1 & -1 \end{bmatrix} - (-2i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1+2i & -5 \\ 1 & -1+2i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1+2i & -5 & 0 \\ 1 & -1+2i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{1}{5} + \frac{2i}{5}\right) R_1 \Longrightarrow \left[\begin{array}{cc|c} 1+2i & -5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1+2i & -5 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = (1-2i)\,t\}$

Hence the solution is

$$\begin{bmatrix} (1-2\,\mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} (1-2i)\,t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} (1-2\,\mathrm{I})\,t \\ t \end{bmatrix} = t \begin{bmatrix} 1-2i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} (1-2\,\mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} 1-2i \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 2i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -5 \\ 1 & -1 \end{bmatrix} - (2i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1-2i & -5 \\ 1 & -1-2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1-2i & -5 & 0 \\ 1 & -1-2i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{1}{5} - \frac{2i}{5}\right) R_1 \Longrightarrow \left[\begin{array}{cc|c} 1-2i & -5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1-2i & -5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = (1 + 2i)\, t\}$

Hence the solution is

$$\begin{bmatrix} (1+2\ \mathrm{I})\, t \\ t \end{bmatrix} = \begin{bmatrix} (1+2i)\, t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} (1+2\ \mathrm{I})\, t \\ t \end{bmatrix} = t \begin{bmatrix} 1+2i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} (1+2\ \mathrm{I})\, t \\ t \end{bmatrix} = \begin{bmatrix} 1+2i \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$2i$	1	1	No	$\begin{bmatrix} 1+2i \\ 1 \end{bmatrix}$
$-2i$	1	1	No	$\begin{bmatrix} 1-2i \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} (1+2i)\,e^{2it} \\ e^{2it} \end{bmatrix} + c_2 \begin{bmatrix} (1-2i)\,e^{-2it} \\ e^{-2it} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} (1+2i)\,c_1 e^{2it} + (1-2i)\,c_2 e^{-2it} \\ c_1 e^{2it} + c_2 e^{-2it} \end{bmatrix}$$

The following is the phase plot of the system.

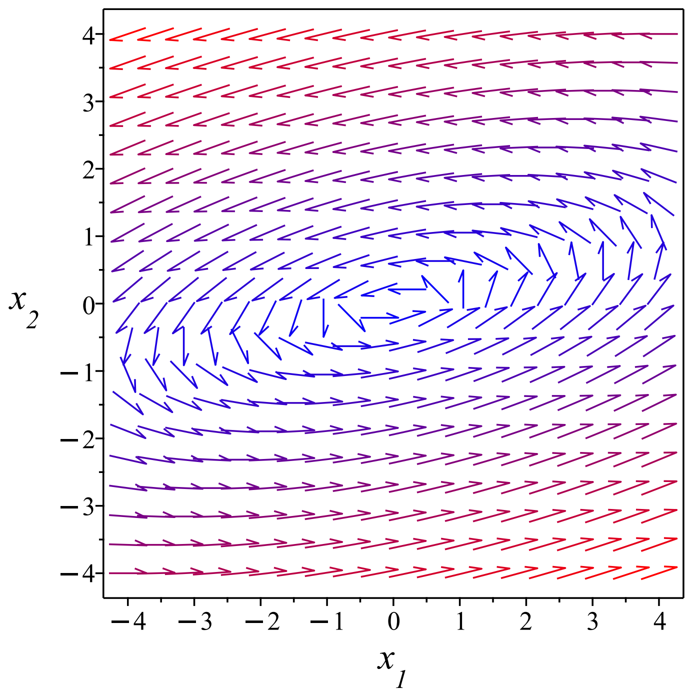


Figure 16: Phase plot

4.8.3 Maple step by step solution

Let's solve

$[x_1'(t) = x_1(t) - 5x_2(t)\,, x_2'(t) = x_1(t) - x_2(t)]$

- Define vector

$$\underline{x^{\rightarrow}}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x^{\rightarrow}}'(t) = \begin{bmatrix} 1 & -5 \\ 1 & -1 \end{bmatrix} \cdot \underline{x^{\rightarrow}}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{x^{\rightarrow}}'(t) = \begin{bmatrix} 1 & -5 \\ 1 & -1 \end{bmatrix} \cdot \underline{x^{\rightarrow}}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & -5 \\ 1 & -1 \end{bmatrix}$$

- Rewrite the system as

$$\underline{x^{\rightarrow}}'(t) = A \cdot \underline{x^{\rightarrow}}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2\,\mathrm{I}, \begin{bmatrix} 1-2\,\mathrm{I} \\ 1 \end{bmatrix}\right], \left[2\,\mathrm{I}, \begin{bmatrix} 1+2\,\mathrm{I} \\ 1 \end{bmatrix}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-2\,\mathrm{I}, \begin{bmatrix} 1-2\,\mathrm{I} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{-2\,\mathrm{I}t} \cdot \begin{bmatrix} 1-2\,\mathrm{I} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$(\cos(2t) - \mathrm{I}\sin(2t)) \cdot \begin{bmatrix} 1-2\,\mathrm{I} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\left[\begin{array}{c}(1-2\,\mathrm{I})\left(\cos\left(2t\right)-\mathrm{I}\sin\left(2t\right)\right)\\ \cos\left(2t\right)-\mathrm{I}\sin\left(2t\right)\end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\underline{x}^{\rightarrow}{}_{1}(t)=\left[\begin{array}{c}\cos\left(2t\right)-2\sin\left(2t\right)\\ \cos\left(2t\right)\end{array}\right],\underline{x}^{\rightarrow}{}_{2}(t)=\left[\begin{array}{c}-2\cos\left(2t\right)-\sin\left(2t\right)\\ -\sin\left(2t\right)\end{array}\right]\right]$$

- General solution to the system of ODEs

$$\underline{x}^{\rightarrow}=c_{1}\underline{x}^{\rightarrow}{}_{1}(t)+c_{2}\underline{x}^{\rightarrow}{}_{2}(t)$$

- Substitute solutions into the general solution

$$\underline{x}^{\rightarrow}=\left[\begin{array}{c}c_{2}\left(-2\cos\left(2t\right)-\sin\left(2t\right)\right)+c_{1}\left(\cos\left(2t\right)-2\sin\left(2t\right)\right)\\ -c_{2}\sin\left(2t\right)+c_{1}\cos\left(2t\right)\end{array}\right]$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c}x_{1}(t)\\ x_{2}(t)\end{array}\right]=\left[\begin{array}{c}\left(c_{1}-2c_{2}\right)\cos\left(2t\right)-2\left(c_{1}+\frac{c_{2}}{2}\right)\sin\left(2t\right)\\ -c_{2}\sin\left(2t\right)+c_{1}\cos\left(2t\right)\end{array}\right]$$

- Solution to the system of ODEs

$$\left\{x_{1}(t)=\left(c_{1}-2c_{2}\right)\cos\left(2t\right)-2\left(c_{1}+\frac{c_{2}}{2}\right)\sin\left(2t\right),x_{2}(t)=-c_{2}\sin\left(2t\right)+c_{1}\cos\left(2t\right)\right\}$$

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 50

```
dsolve([diff(x__1(t),t)=x__1(t)-5*x__2(t),diff(x__2(t),t)=x__1(t)-x__2(t)],singsol=all)
```

$$x_{1}(t)=c_{1}\sin\left(2t\right)+c_{2}\cos\left(2t\right)$$
$$x_{2}(t)=-\frac{2c_{1}\cos\left(2t\right)}{5}+\frac{2c_{2}\sin\left(2t\right)}{5}+\frac{c_{1}\sin\left(2t\right)}{5}+\frac{c_{2}\cos\left(2t\right)}{5}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 48

```
DSolve[{x1'[t]==x1[t]-5*x2[t],x2'[t]==x1[t]-x2[t]},{x1[t],x2[t]},t,IncludeSingularSolutions
```

$$\text{x1}(t)\to c_{1}\cos(2t)+(c_{1}-5c_{2})\sin(t)\cos(t)$$
$$\text{x2}(t)\to c_{2}\cos(2t)+(c_{1}-c_{2})\sin(t)\cos(t)$$

4.9 problem problem 9

Internal problem ID [323]
Internal file name [OUTPUT/323_Sunday_June_05_2022_01_38_44_AM_41357565/index.tex]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.3, The eigenvalue method for linear systems. Page 395
Problem number: problem 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x_1'(t) = 2x_1(t) - 5x_2(t)$$
$$x_2'(t) = 4x_1(t) - 2x_2(t)$$

With initial conditions

$$[x_1(0) = 2, x_2(0) = 3]$$

4.9.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \cos(4t) + \frac{\sin(4t)}{2} & -\frac{5\sin(4t)}{4} \\ \sin(4t) & \cos(4t) - \frac{\sin(4t)}{2} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} \cos(4t) + \frac{\sin(4t)}{2} & -\frac{5\sin(4t)}{4} \\ \sin(4t) & \cos(4t) - \frac{\sin(4t)}{2} \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} \\
&= \begin{bmatrix} 2\cos(4t) - \frac{11\sin(4t)}{4} \\ \frac{\sin(4t)}{2} + 3\cos(4t) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

4.9.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 2 & -5 \\ 4 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 2-\lambda & -5 \\ 4 & -2-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 16 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 4i$$
$$\lambda_2 = -4i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-4i$	1	complex eigenvalue
$4i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -4i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & -5 \\ 4 & -2 \end{bmatrix} - (-4i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2+4i & -5 \\ 4 & -2+4i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2+4i & -5 & 0 \\ 4 & -2+4i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{2}{5} + \frac{4i}{5}\right) R_1 \Longrightarrow \left[\begin{array}{cc|c} 2+4i & -5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2+4i & -5 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(\frac{1}{2} - i\right) t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(\frac{1}{2} - \mathrm{I}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2} - i\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(\frac{1}{2} - \mathrm{I}\right) t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} - i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(\frac{1}{2} - \mathrm{I}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - i \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(\frac{1}{2} - \mathrm{I}\right) t \\ t \end{bmatrix} = \begin{bmatrix} 1 - 2i \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 4i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & -5 \\ 4 & -2 \end{bmatrix} - (4i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 - 4i & -5 \\ 4 & -2 - 4i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2 - 4i & -5 & 0 \\ 4 & -2 - 4i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{2}{5} - \frac{4i}{5} \right) R_1 \Longrightarrow \left[\begin{array}{cc|c} 2 - 4i & -5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 - 4i & -5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = \left(\frac{1}{2} + i\right) t\}$

Hence the solution is

$$\begin{bmatrix} \left(\frac{1}{2} + \mathrm{I}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2} + i\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(\frac{1}{2} + \mathrm{I}\right) t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} + i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(\frac{1}{2} + \mathrm{I}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} + i \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(\frac{1}{2} + \mathrm{I}\right) t \\ t \end{bmatrix} = \begin{bmatrix} 1 + 2i \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$4i$	1	1	No	$\begin{bmatrix} \frac{1}{2} + i \\ 1 \end{bmatrix}$
$-4i$	1	1	No	$\begin{bmatrix} \frac{1}{2} - i \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} \left(\frac{1}{2}+i\right) \mathrm{e}^{4it} \\ \mathrm{e}^{4it} \end{bmatrix} + c_2 \begin{bmatrix} \left(\frac{1}{2}-i\right) \mathrm{e}^{-4it} \\ \mathrm{e}^{-4it} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2}+i\right) c_1\mathrm{e}^{4it} + \left(\frac{1}{2}-i\right) c_2\mathrm{e}^{-4it} \\ c_1\mathrm{e}^{4it} + c_2\mathrm{e}^{-4it} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x_1(0) = 2 \\ x_2(0) = 3 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2}+i\right) c_1 + \left(\frac{1}{2}-i\right) c_2 \\ c_1 + c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = \frac{3}{2} - \frac{i}{4} \\ c_2 = \frac{3}{2} + \frac{i}{4} \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \left(1+\frac{11i}{8}\right) \mathrm{e}^{4it} + \left(1-\frac{11i}{8}\right) \mathrm{e}^{-4it} \\ \left(\frac{3}{2}-\frac{i}{4}\right) \mathrm{e}^{4it} + \left(\frac{3}{2}+\frac{i}{4}\right) \mathrm{e}^{-4it} \end{bmatrix}$$

The following is the phase plot of the system.

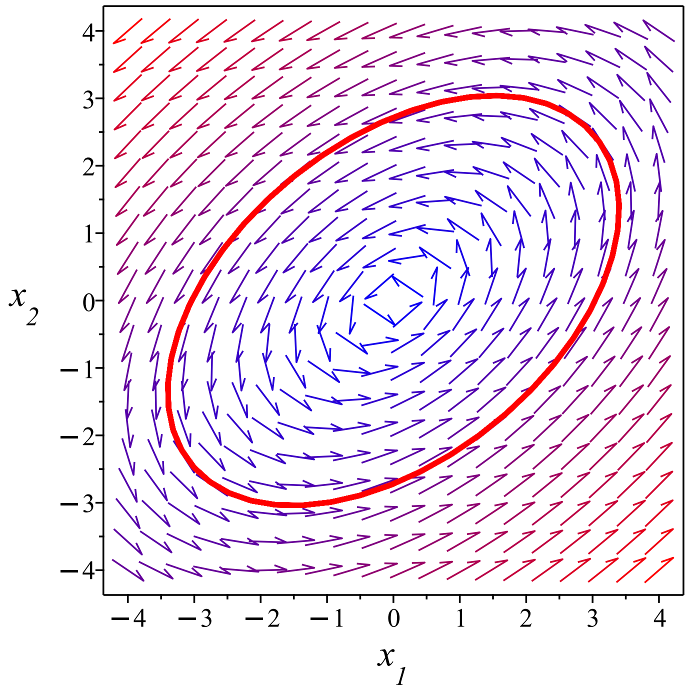


Figure 17: Phase plot

The following are plots of each solution.

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 34

```
dsolve([diff(x__1(t),t) = 2*x__1(t)-5*x__2(t), diff(x__2(t),t) = 4*x__1(t)-2*x__2(t), x__1(0
```

$$x_1(t) = -\frac{11\sin(4t)}{4} + 2\cos(4t)$$
$$x_2(t) = 3\cos(4t) + \frac{\sin(4t)}{2}$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 34

```
DSolve[{x1'[t]==x1[t]-5*x2[t],x2'[t]==x1[t]-x2[t]},{x1[0]==2,x2[0]==3},{x1[t],x2[t]},t,Inclu
```

$$\text{x1}(t) \to 2\cos(2t) - 13\sin(t)\cos(t)$$
$$\text{x2}(t) \to 3\cos(2t) - \sin(t)\cos(t)$$

4.10 problem problem 10

Internal problem ID [324]
Internal file name [`OUTPUT/324_Sunday_June_05_2022_01_38_45_AM_72543614/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.3, The eigenvalue method for linear systems. Page 395
Problem number: problem 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= -3x_1(t) - 2x_2(t) \\
x_2'(t) &= 9x_1(t) + 3x_2(t)
\end{aligned}
$$

4.10.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -3 & -2 \\ 9 & 3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \cos(3t) - \sin(3t) & -\frac{2\sin(3t)}{3} \\ 3\sin(3t) & \sin(3t) + \cos(3t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \cos(3t) - \sin(3t) & -\frac{2\sin(3t)}{3} \\ 3\sin(3t) & \sin(3t) + \cos(3t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} (\cos(3t) - \sin(3t))\, c_1 - \frac{2\sin(3t)c_2}{3} \\ 3\sin(3t)\, c_1 + (\sin(3t) + \cos(3t))\, c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(-3c_1-2c_2)\sin(3t)}{3} + c_1\cos(3t) \\ (3c_1 + c_2)\sin(3t) + c_2\cos(3t) \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

4.10.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -3 & -2 \\ 9 & 3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -3 & -2 \\ 9 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -3-\lambda & -2 \\ 9 & 3-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 9 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 3i$$
$$\lambda_2 = -3i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-3i$	1	complex eigenvalue
$3i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -3i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -3 & -2 \\ 9 & 3 \end{bmatrix} - (-3i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -3+3i & -2 \\ 9 & 3+3i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -3+3i & -2 & 0 \\ 9 & 3+3i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(\frac{3}{2} + \frac{3i}{2}\right)R_1 \Longrightarrow \left[\begin{array}{cc|c} -3+3i & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -3+3i & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(-\frac{1}{3} - \frac{i}{3}\right) t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(-\frac{1}{3} - \frac{I}{3}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \left(-\frac{1}{3} - \frac{i}{3}\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(-\frac{1}{3} - \frac{I}{3}\right) t \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{3} - \frac{i}{3} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(-\frac{1}{3} - \frac{I}{3}\right) t \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} - \frac{i}{3} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(-\frac{1}{3} - \frac{I}{3}\right) t \\ t \end{bmatrix} = \begin{bmatrix} -1 - i \\ 3 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 3i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -3 & -2 \\ 9 & 3 \end{bmatrix} - (3i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 - 3i & -2 \\ 9 & 3 - 3i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -3 - 3i & -2 & 0 \\ 9 & 3 - 3i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(\frac{3}{2} - \frac{3i}{2}\right) R_1 \Longrightarrow \left[\begin{array}{cc|c} -3 - 3i & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -3-3i & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(-\frac{1}{3} + \frac{i}{3}\right) t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(-\frac{1}{3} + \frac{I}{3}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \left(-\frac{1}{3} + \frac{i}{3}\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(-\frac{1}{3} + \frac{I}{3}\right) t \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{3} + \frac{i}{3} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(-\frac{1}{3} + \frac{I}{3}\right) t \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} + \frac{i}{3} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(-\frac{1}{3} + \frac{I}{3}\right) t \\ t \end{bmatrix} = \begin{bmatrix} -1+i \\ 3 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$3i$	1	1	No	$\begin{bmatrix} -\frac{1}{3}+\frac{i}{3} \\ 1 \end{bmatrix}$
$-3i$	1	1	No	$\begin{bmatrix} -\frac{1}{3}-\frac{i}{3} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} \left(-\frac{1}{3}+\frac{i}{3}\right) e^{3it} \\ e^{3it} \end{bmatrix} + c_2 \begin{bmatrix} \left(-\frac{1}{3}-\frac{i}{3}\right) e^{-3it} \\ e^{-3it} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \left(-\frac{1}{3}+\frac{i}{3}\right) c_1 e^{3it} + \left(-\frac{1}{3}-\frac{i}{3}\right) c_2 e^{-3it} \\ c_1 e^{3it} + c_2 e^{-3it} \end{bmatrix}$$

The following is the phase plot of the system.

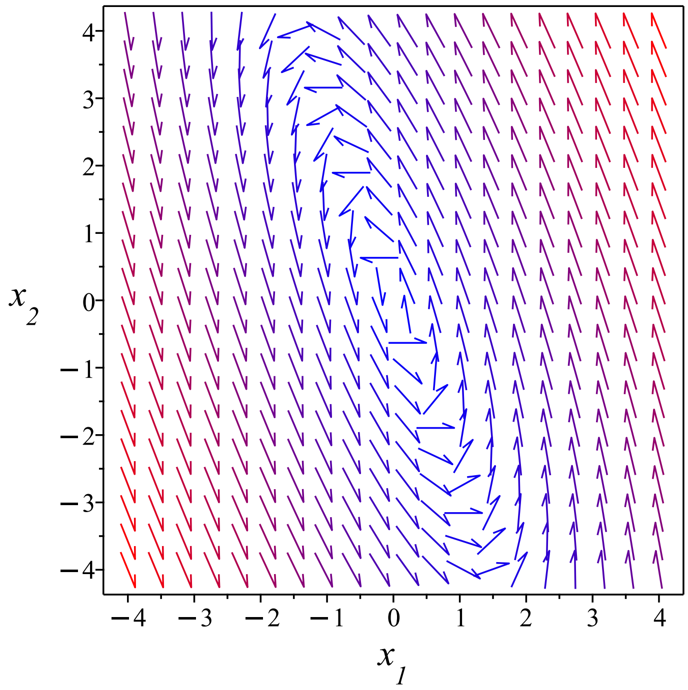


Figure 18: Phase plot

4.10.3 Maple step by step solution

Let's solve

$[x_1'(t) = -3x_1(t) - 2x_2(t), x_2'(t) = 9x_1(t) + 3x_2(t)]$

- Define vector

$$\underline{\vec{x}}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{\vec{x}}'(t) = \begin{bmatrix} -3 & -2 \\ 9 & 3 \end{bmatrix} \cdot \underline{\vec{x}}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{\vec{x}}'(t) = \begin{bmatrix} -3 & -2 \\ 9 & 3 \end{bmatrix} \cdot \underline{\vec{x}}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -3 & -2 \\ 9 & 3 \end{bmatrix}$$

- Rewrite the system as

$$\underline{x^{\rightarrow}}'(t) = A \cdot \underline{x^{\rightarrow}}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-3\,\mathrm{I}, \begin{bmatrix} -\frac{1}{3} - \frac{\mathrm{I}}{3} \\ 1 \end{bmatrix}\right], \left[3\,\mathrm{I}, \begin{bmatrix} -\frac{1}{3} + \frac{\mathrm{I}}{3} \\ 1 \end{bmatrix}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-3\,\mathrm{I}, \begin{bmatrix} -\frac{1}{3} - \frac{\mathrm{I}}{3} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{-3\,\mathrm{I}t} \cdot \begin{bmatrix} -\frac{1}{3} - \frac{\mathrm{I}}{3} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$(\cos(3t) - \mathrm{I}\sin(3t)) \cdot \begin{bmatrix} -\frac{1}{3} - \frac{\mathrm{I}}{3} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} \left(-\frac{1}{3} - \frac{\mathrm{I}}{3}\right)(\cos(3t) - \mathrm{I}\sin(3t)) \\ \cos(3t) - \mathrm{I}\sin(3t) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\underline{x^{\rightarrow}}_1(t) = \begin{bmatrix} -\frac{\cos(3t)}{3} - \frac{\sin(3t)}{3} \\ \cos(3t) \end{bmatrix}, \underline{x^{\rightarrow}}_2(t) = \begin{bmatrix} \frac{\sin(3t)}{3} - \frac{\cos(3t)}{3} \\ -\sin(3t) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\underline{x^{\rightarrow}} = c_1\underline{x^{\rightarrow}}_1(t) + c_2\underline{x^{\rightarrow}}_2(t)$$

- Substitute solutions into the general solution

$$\underline{x}^{\rightarrow} = \begin{bmatrix} c_2\left(\frac{\sin(3t)}{3}-\frac{\cos(3t)}{3}\right)+c_1\left(-\frac{\cos(3t)}{3}-\frac{\sin(3t)}{3}\right) \\ -c_2\sin(3t)+c_1\cos(3t) \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{(-c_1-c_2)\cos(3t)}{3}-\frac{\sin(3t)(c_1-c_2)}{3} \\ -c_2\sin(3t)+c_1\cos(3t) \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x_1(t)=\frac{(-c_1-c_2)\cos(3t)}{3}-\frac{\sin(3t)(c_1-c_2)}{3}, x_2(t)=-c_2\sin(3t)+c_1\cos(3t)\right\}$$

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 50

```
dsolve([diff(x__1(t),t)=-3*x__1(t)-2*x__2(t),diff(x__2(t),t)=9*x__1(t)+3*x__2(t)],singsol=al
```

$$x_1(t) = c_1\sin(3t)+c_2\cos(3t)$$
$$x_2(t) = -\frac{3c_1\cos(3t)}{2}+\frac{3c_2\sin(3t)}{2}-\frac{3c_1\sin(3t)}{2}-\frac{3c_2\cos(3t)}{2}$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 53

```
DSolve[{x1'[t]==-3*x1[t]-2*x2[t],x2'[t]==9*x1[t]+3*x2[t]},{x1[t],x2[t]},t,IncludeSingularSol
```

$$\text{x1}(t) \to c_1\cos(3t)-\frac{1}{3}(3c_1+2c_2)\sin(3t)$$
$$\text{x2}(t) \to c_2\cos(3t)+(3c_1+c_2)\sin(3t)$$

4.11 problem problem 11

Internal problem ID [325]
Internal file name [`OUTPUT/325_Sunday_June_05_2022_01_38_46_AM_67850612/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.3, The eigenvalue method for linear systems. Page 395
Problem number: problem 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= x_1(t) - 2x_2(t) \\
x_2'(t) &= 2x_1(t) + x_2(t)
\end{aligned}
$$

With initial conditions

$$
[x_1(0) = 0, x_2(0) = 4]
$$

4.11.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} \mathrm{e}^t \cos(2t) & -\mathrm{e}^t \sin(2t) \\ \mathrm{e}^t \sin(2t) & \mathrm{e}^t \cos(2t) \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} \mathrm{e}^t \cos(2t) & -\mathrm{e}^t \sin(2t) \\ \mathrm{e}^t \sin(2t) & \mathrm{e}^t \cos(2t) \end{bmatrix} \begin{bmatrix} 0 \\ 4 \end{bmatrix} \\
&= \begin{bmatrix} -4\,\mathrm{e}^t \sin(2t) \\ 4\,\mathrm{e}^t \cos(2t) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

4.11.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & -2 \\ 2 & 1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 2\lambda + 5 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 1 + 2i$$
$$\lambda_2 = 1 - 2i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$1 + 2i$	1	complex eigenvalue
$1 - 2i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 1 - 2i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} - (1 - 2i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2i & -2 \\ 2 & 2i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2i & -2 & 0 \\ 2 & 2i & 0 \end{array}\right]$$

$$R_2 = iR_1 + R_2 \Longrightarrow \left[\begin{array}{cc|c} 2i & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2i & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -it\}$

Hence the solution is

$$\begin{bmatrix} \text{-I}\,t \\ t \end{bmatrix} = \begin{bmatrix} -it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \text{-I}\,t \\ t \end{bmatrix} = t \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \text{-I}\,t \\ t \end{bmatrix} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 1 + 2i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} - (1+2i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2i & -2 \\ 2 & -2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -2i & -2 & 0 \\ 2 & -2i & 0 \end{array}\right]$$

$$R_2 = -iR_1 + R_2 \Longrightarrow \left[\begin{array}{cc|c} -2i & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2i & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = it\}$

Hence the solution is

$$\begin{bmatrix} \mathrm{I}t \\ t \end{bmatrix} = \begin{bmatrix} it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \mathrm{I}t \\ t \end{bmatrix} = t \begin{bmatrix} i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \mathrm{I}t \\ t \end{bmatrix} = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$1 + 2i$	1	1	No	$\begin{bmatrix} i \\ 1 \end{bmatrix}$
$1 - 2i$	1	1	No	$\begin{bmatrix} -i \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} i\mathrm{e}^{(1+2i)t} \\ \mathrm{e}^{(1+2i)t} \end{bmatrix} + c_2 \begin{bmatrix} -i\mathrm{e}^{(1-2i)t} \\ \mathrm{e}^{(1-2i)t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -i\left(c_2 e^{(1-2i)t} - c_1 e^{(1+2i)t}\right) \\ c_1 e^{(1+2i)t} + c_2 e^{(1-2i)t} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x_1(0) = 0 \\ x_2(0) = 4 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 0 \\ 4 \end{bmatrix} = \begin{bmatrix} i(c_1 - c_2) \\ c_1 + c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = 2 \\ c_2 = 2 \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -i\left(2\,e^{(1-2i)t} - 2\,e^{(1+2i)t}\right) \\ 2\,e^{(1+2i)t} + 2\,e^{(1-2i)t} \end{bmatrix}$$

The following is the phase plot of the system.

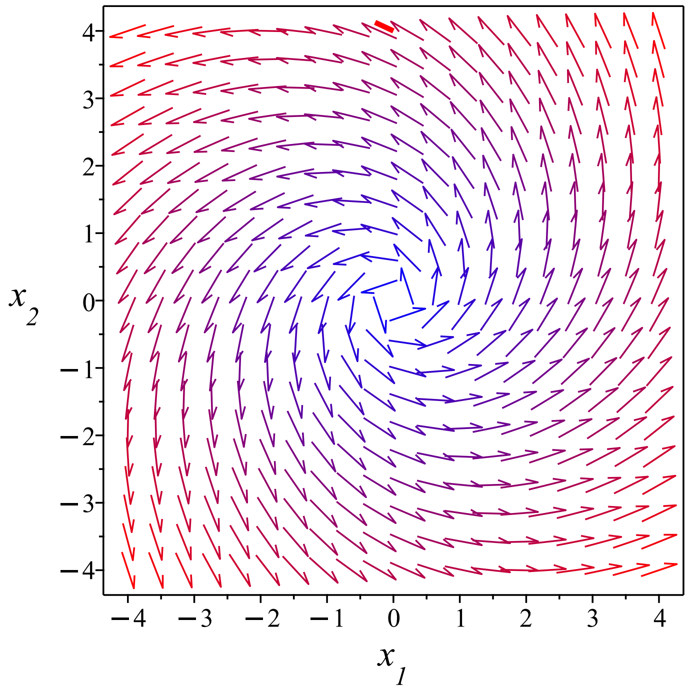


Figure 19: Phase plot

The following are plots of each solution.

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 24

```
dsolve([diff(x__1(t),t) = x__1(t)-2*x__2(t), diff(x__2(t),t) = 2*x__1(t)+x__2(t), x__1(0) =
```

$$x_1(t) = -4\,e^t \sin(2t)$$
$$x_2(t) = 4\,e^t \cos(2t)$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 26

```
DSolve[{x1'[t]==x1[t]-2*x2[t],x2'[t]==2*x1[t]+x2[t]},{x1[0]==0,x2[0]==4},{x1[t],x2[t]},t,Inc
```

$$\text{x1}(t) \to -4e^t \sin(2t)$$
$$\text{x2}(t) \to 4e^t \cos(2t)$$

4.12 problem problem 12

Internal problem ID [326]
Internal file name [`OUTPUT/326_Sunday_June_05_2022_01_38_48_AM_91837534/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.3, The eigenvalue method for linear systems. Page 395
Problem number: problem 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x_1'(t) = x_1(t) - 5x_2(t)$$
$$x_2'(t) = x_1(t) + 3x_2(t)$$

4.12.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & -5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$\begin{aligned} e^{At} &= \begin{bmatrix} e^{2t}\cos(2t) - \frac{e^{2t}\sin(2t)}{2} & -\frac{5\,e^{2t}\sin(2t)}{2} \\ \frac{e^{2t}\sin(2t)}{2} & e^{2t}\cos(2t) + \frac{e^{2t}\sin(2t)}{2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{e^{2t}(2\cos(2t)-\sin(2t))}{2} & -\frac{5\,e^{2t}\sin(2t)}{2} \\ \frac{e^{2t}\sin(2t)}{2} & \frac{e^{2t}(2\cos(2t)+\sin(2t))}{2} \end{bmatrix} \end{aligned}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{e^{2t}(2\cos(2t)-\sin(2t))}{2} & -\frac{5\,e^{2t}\sin(2t)}{2} \\ \frac{e^{2t}\sin(2t)}{2} & \frac{e^{2t}(2\cos(2t)+\sin(2t))}{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{e^{2t}(2\cos(2t)-\sin(2t))c_1}{2} - \frac{5\,e^{2t}\sin(2t)c_2}{2} \\ \frac{e^{2t}\sin(2t)c_1}{2} + \frac{e^{2t}(2\cos(2t)+\sin(2t))c_2}{2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{e^{2t}(-c_1-5c_2)\sin(2t)}{2} + e^{2t}\cos(2t)\,c_1 \\ \frac{((c_1+c_2)\sin(2t)+2c_2\cos(2t))e^{2t}}{2} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

4.12.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & -5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & -5 \\ 1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & -5 \\ 1 & 3-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 4\lambda + 8 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2 + 2i$$
$$\lambda_2 = 2 - 2i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$2 - 2i$	1	complex eigenvalue
$2 + 2i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 2 - 2i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -5 \\ 1 & 3 \end{bmatrix} - (2 - 2i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -1 + 2i & -5 \\ 1 & 1 + 2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -1 + 2i & -5 & 0 \\ 1 & 1 + 2i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(\frac{1}{5} + \frac{2i}{5}\right) R_1 \Longrightarrow \left[\begin{array}{cc|c} -1 + 2i & -5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 + 2i & -5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = (-1 - 2i)\,t\}$

Hence the solution is

$$\begin{bmatrix} (-1 - 2\,\mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} (-1 - 2i)\,t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} (-1 - 2\,\mathrm{I})\,t \\ t \end{bmatrix} = t \begin{bmatrix} -1 - 2i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} (-1 - 2\,\mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} -1 - 2i \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 2 + 2i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -5 \\ 1 & 3 \end{bmatrix} - (2 + 2i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 - 2i & -5 \\ 1 & 1 - 2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -1 - 2i & -5 & 0 \\ 1 & 1 - 2i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(\frac{1}{5} - \frac{2i}{5}\right) R_1 \Longrightarrow \left[\begin{array}{cc|c} -1 - 2i & -5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 - 2i & -5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = (-1+2i)\,t\}$

Hence the solution is

$$\begin{bmatrix} (-1+2\ \mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} (-1+2i)\,t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} (-1+2\ \mathrm{I})\,t \\ t \end{bmatrix} = t \begin{bmatrix} -1+2i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} (-1+2\ \mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} -1+2i \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$2+2i$	1	1	No	$\begin{bmatrix} -1+2i \\ 1 \end{bmatrix}$
$2-2i$	1	1	No	$\begin{bmatrix} -1-2i \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} (-1+2i)\,e^{(2+2i)t} \\ e^{(2+2i)t} \end{bmatrix} + c_2 \begin{bmatrix} (-1-2i)\,e^{(2-2i)t} \\ e^{(2-2i)t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} (-1+2i)\,c_1 e^{(2+2i)t} + (-1-2i)\,c_2 e^{(2-2i)t} \\ c_1 e^{(2+2i)t} + c_2 e^{(2-2i)t} \end{bmatrix}$$

The following is the phase plot of the system.

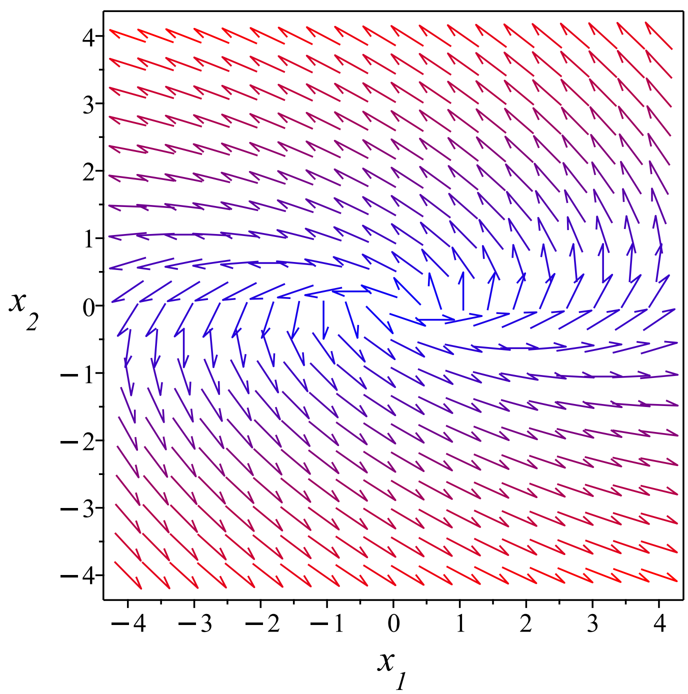


Figure 20: Phase plot

4.12.3 Maple step by step solution

Let's solve

$[x_1'(t) = x_1(t) - 5x_2(t)\,, x_2'(t) = x_1(t) + 3x_2(t)]$

- Define vector

$$\underline{x^{\rightarrow}}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x^{\rightarrow}}'(t) = \begin{bmatrix} 1 & -5 \\ 1 & 3 \end{bmatrix} \cdot \underline{x^{\rightarrow}}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{x^{\rightarrow}}'(t) = \begin{bmatrix} 1 & -5 \\ 1 & 3 \end{bmatrix} \cdot \underline{x^{\rightarrow}}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & -5 \\ 1 & 3 \end{bmatrix}$$

- Rewrite the system as

$$\underline{x^{\rightarrow}}'(t) = A \cdot \underline{x^{\rightarrow}}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[2-2\,\mathrm{I}, \begin{bmatrix} -1-2\,\mathrm{I} \\ 1 \end{bmatrix}\right], \left[2+2\,\mathrm{I}, \begin{bmatrix} -1+2\,\mathrm{I} \\ 1 \end{bmatrix}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[2-2\,\mathrm{I}, \begin{bmatrix} -1-2\,\mathrm{I} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{(2-2\,\mathrm{I})t} \cdot \begin{bmatrix} -1-2\,\mathrm{I} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\mathrm{e}^{2t} \cdot (\cos(2t) - \mathrm{I}\sin(2t)) \cdot \begin{bmatrix} -1-2\,\mathrm{I} \\ 1 \end{bmatrix}$$

- Simplify expression

$$e^{2t}\cdot\begin{bmatrix}(-1-2\mathrm{I})(\cos(2t)-\mathrm{I}\sin(2t))\\ \cos(2t)-\mathrm{I}\sin(2t)\end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\underline{\vec{x}}_1(t)=e^{2t}\cdot\begin{bmatrix}-\cos(2t)-2\sin(2t)\\ \cos(2t)\end{bmatrix},\underline{\vec{x}}_2(t)=e^{2t}\cdot\begin{bmatrix}-2\cos(2t)+\sin(2t)\\ -\sin(2t)\end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\underline{\vec{x}}=c_1\underline{\vec{x}}_1(t)+c_2\underline{\vec{x}}_2(t)$$

- Substitute solutions into the general solution

$$\underline{\vec{x}}=c_1e^{2t}\cdot\begin{bmatrix}-\cos(2t)-2\sin(2t)\\ \cos(2t)\end{bmatrix}+c_2e^{2t}\cdot\begin{bmatrix}-2\cos(2t)+\sin(2t)\\ -\sin(2t)\end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}=\begin{bmatrix}-\left((c_1+2c_2)\cos(2t)+2\left(c_1-\frac{c_2}{2}\right)\sin(2t)\right)e^{2t}\\ e^{2t}(-c_2\sin(2t)+c_1\cos(2t))\end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x_1(t)=-\left((c_1+2c_2)\cos(2t)+2\left(c_1-\frac{c_2}{2}\right)\sin(2t)\right)e^{2t},x_2(t)=e^{2t}(-c_2\sin(2t)+c_1\cos(2t))\right\}$$

✓ Solution by Maple

Time used: 0.094 (sec). Leaf size: 59

```
dsolve([diff(x__1(t),t)=x__1(t)-5*x__2(t),diff(x__2(t),t)=x__1(t)+3*x__2(t)],singsol=all)
```

$$x_1(t)=e^{2t}(c_1\sin(2t)+c_2\cos(2t))$$
$$x_2(t)=-\frac{e^{2t}(2c_1\cos(2t)+c_2\cos(2t)+c_1\sin(2t)-2c_2\sin(2t))}{5}$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 67

```
DSolve[{x1'[t]==x1[t]-5*x2[t],x2'[t]==x1[t]+3*x2[t]},{x1[t],x2[t]},t,IncludeSingularSolution
```

$$
\begin{aligned}
\mathrm{x1}(t) &\to \frac{1}{2}e^{2t}(2c_1\cos(2t) - (c_1 + 5c_2)\sin(2t)) \\
\mathrm{x2}(t) &\to \frac{1}{2}e^{2t}(2c_2\cos(2t) + (c_1 + c_2)\sin(2t))
\end{aligned}
$$

4.13 problem problem 13

Internal problem ID [327]
Internal file name [`OUTPUT/327_Sunday_June_05_2022_01_38_49_AM_39023998/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.3, The eigenvalue method for linear systems. Page 395
Problem number: problem 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x_1'(t) = 5x_1(t) - 9x_2(t)$$
$$x_2'(t) = 2x_1(t) - x_2(t)$$

4.13.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 5 & -9 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$\begin{aligned} e^{At} &= \begin{bmatrix} \mathrm{e}^{2t}\cos(3t) + \mathrm{e}^{2t}\sin(3t) & -3\,\mathrm{e}^{2t}\sin(3t) \\ \frac{2\,\mathrm{e}^{2t}\sin(3t)}{3} & \mathrm{e}^{2t}\cos(3t) - \mathrm{e}^{2t}\sin(3t) \end{bmatrix} \\ &= \begin{bmatrix} \mathrm{e}^{2t}(\sin(3t) + \cos(3t)) & -3\,\mathrm{e}^{2t}\sin(3t) \\ \frac{2\,\mathrm{e}^{2t}\sin(3t)}{3} & \mathrm{e}^{2t}(\cos(3t) - \sin(3t)) \end{bmatrix} \end{aligned}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{2t}(\sin(3t)+\cos(3t)) & -3\,e^{2t}\sin(3t) \\ \frac{2\,e^{2t}\sin(3t)}{3} & e^{2t}(\cos(3t)-\sin(3t)) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{2t}(\sin(3t)+\cos(3t))\,c_1 - 3\,e^{2t}\sin(3t)\,c_2 \\ \frac{2\,e^{2t}\sin(3t)c_1}{3} + e^{2t}(\cos(3t)-\sin(3t))\,c_2 \end{bmatrix} \\
&= \begin{bmatrix} ((c_1-3c_2)\sin(3t)+c_1\cos(3t))\,e^{2t} \\ \frac{e^{2t}(2c_1-3c_2)\sin(3t)}{3} + e^{2t}\cos(3t)\,c_2 \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

4.13.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 5 & -9 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 5 & -9 \\ 2 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 5-\lambda & -9 \\ 2 & -1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 4\lambda + 13 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2 + 3i$$
$$\lambda_2 = 2 - 3i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$2 + 3i$	1	complex eigenvalue
$2 - 3i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 2 - 3i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 5 & -9 \\ 2 & -1 \end{bmatrix} - (2 - 3i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 3 + 3i & -9 \\ 2 & -3 + 3i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 3 + 3i & -9 & 0 \\ 2 & -3 + 3i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{1}{3} + \frac{i}{3}\right) R_1 \Longrightarrow \left[\begin{array}{cc|c} 3 + 3i & -9 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3 + 3i & -9 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(\frac{3}{2} - \frac{3i}{2}\right)t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(\frac{3}{2} - \frac{3\,\mathrm{I}}{2}\right)t \\ t \end{bmatrix} = \begin{bmatrix} \left(\frac{3}{2} - \frac{3i}{2}\right)t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(\frac{3}{2} - \frac{3\,\mathrm{I}}{2}\right)t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{3}{2} - \frac{3i}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(\frac{3}{2} - \frac{3\,\mathrm{I}}{2}\right)t \\ t \end{bmatrix} = \begin{bmatrix} \frac{3}{2} - \frac{3i}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(\frac{3}{2} - \frac{3\,\mathrm{I}}{2}\right)t \\ t \end{bmatrix} = \begin{bmatrix} 3 - 3i \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 2 + 3i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 5 & -9 \\ 2 & -1 \end{bmatrix} - (2+3i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 - 3i & -9 \\ 2 & -3 - 3i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 3 - 3i & -9 & 0 \\ 2 & -3 - 3i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{1}{3} - \frac{i}{3}\right)R_1 \Longrightarrow \left[\begin{array}{cc|c} 3 - 3i & -9 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3-3i & -9 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(\frac{3}{2} + \frac{3i}{2}\right) t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(\frac{3}{2} + \frac{3\,\mathrm{I}}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \left(\frac{3}{2} + \frac{3i}{2}\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(\frac{3}{2} + \frac{3\,\mathrm{I}}{2}\right) t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{3}{2} + \frac{3i}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(\frac{3}{2} + \frac{3\,\mathrm{I}}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \frac{3}{2} + \frac{3i}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(\frac{3}{2} + \frac{3\,\mathrm{I}}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} 3+3i \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity: algebraic m	multiplicity: geometric k	defective?	eigenvectors
$2+3i$	1	1	No	$\left[\begin{array}{c}\frac{3}{2}+\frac{3i}{2} \\ 1\end{array}\right]$
$2-3i$	1	1	No	$\left[\begin{array}{c}\frac{3}{2}-\frac{3i}{2} \\ 1\end{array}\right]$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\left[\begin{array}{c}x_1(t) \\ x_2(t)\end{array}\right] = c_1\left[\begin{array}{c}\left(\frac{3}{2}+\frac{3i}{2}\right)\mathrm{e}^{(2+3i)t} \\ \mathrm{e}^{(2+3i)t}\end{array}\right] + c_2\left[\begin{array}{c}\left(\frac{3}{2}-\frac{3i}{2}\right)\mathrm{e}^{(2-3i)t} \\ \mathrm{e}^{(2-3i)t}\end{array}\right]$$

Which becomes

$$\left[\begin{array}{c}x_1(t) \\ x_2(t)\end{array}\right] = \left[\begin{array}{c}\left(\frac{3}{2}+\frac{3i}{2}\right)c_1\mathrm{e}^{(2+3i)t} + \left(\frac{3}{2}-\frac{3i}{2}\right)c_2\mathrm{e}^{(2-3i)t} \\ c_1\mathrm{e}^{(2+3i)t} + c_2\mathrm{e}^{(2-3i)t}\end{array}\right]$$

The following is the phase plot of the system.

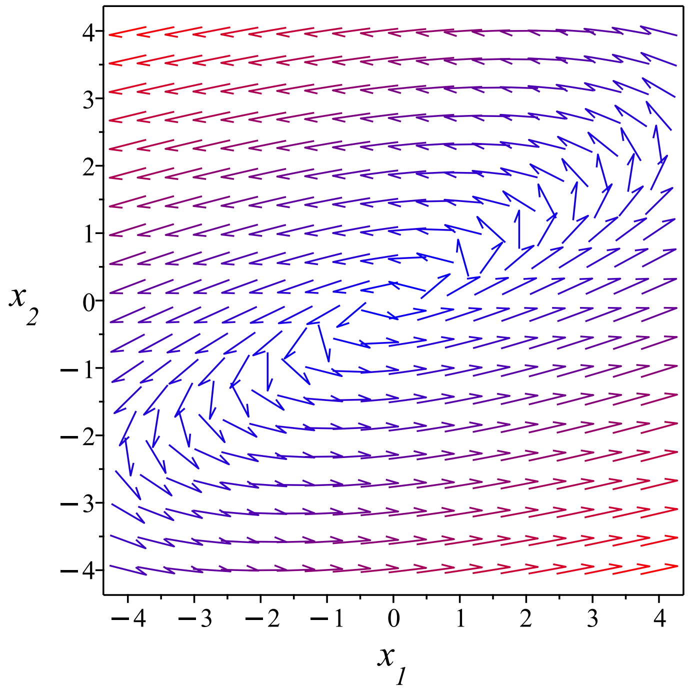


Figure 21: Phase plot

4.13.3 Maple step by step solution

Let's solve

$[x_1'(t) = 5x_1(t) - 9x_2(t), x_2'(t) = 2x_1(t) - x_2(t)]$

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x}^{\rightarrow\prime}(t) = \begin{bmatrix} 5 & -9 \\ 2 & -1 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{x}^{\rightarrow\prime}(t) = \begin{bmatrix} 5 & -9 \\ 2 & -1 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 5 & -9 \\ 2 & -1 \end{bmatrix}$$

- Rewrite the system as

  $$\underline{\vec{x}}'(t) = A \cdot \underline{\vec{x}}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[2-3\,\mathrm{I}, \begin{bmatrix} \frac{3}{2}-\frac{3\,\mathrm{I}}{2} \\ 1 \end{bmatrix}\right], \left[2+3\,\mathrm{I}, \begin{bmatrix} \frac{3}{2}+\frac{3\,\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

  $$\left[2-3\,\mathrm{I}, \begin{bmatrix} \frac{3}{2}-\frac{3\,\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

  $$\mathrm{e}^{(2-3\,\mathrm{I})t} \cdot \begin{bmatrix} \frac{3}{2}-\frac{3\,\mathrm{I}}{2} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

  $$\mathrm{e}^{2t} \cdot (\cos(3t) - \mathrm{I}\sin(3t)) \cdot \begin{bmatrix} \frac{3}{2}-\frac{3\,\mathrm{I}}{2} \\ 1 \end{bmatrix}$$

- Simplify expression

  $$\mathrm{e}^{2t} \cdot \begin{bmatrix} \left(\frac{3}{2}-\frac{3\,\mathrm{I}}{2}\right)(\cos(3t) - \mathrm{I}\sin(3t)) \\ \cos(3t) - \mathrm{I}\sin(3t) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

  $$\left[\underline{\vec{x}}_1(t) = \mathrm{e}^{2t} \cdot \begin{bmatrix} \frac{3\cos(3t)}{2} - \frac{3\sin(3t)}{2} \\ \cos(3t) \end{bmatrix}, \underline{\vec{x}}_2(t) = \mathrm{e}^{2t} \cdot \begin{bmatrix} -\frac{3\sin(3t)}{2} - \frac{3\cos(3t)}{2} \\ -\sin(3t) \end{bmatrix}\right]$$

- General solution to the system of ODEs

  $$\underline{\vec{x}} = c_1\underline{\vec{x}}_1(t) + c_2\underline{\vec{x}}_2(t)$$

- Substitute solutions into the general solution

$$\vec{x} = c_1 e^{2t} \cdot \begin{bmatrix} \frac{3\cos(3t)}{2} - \frac{3\sin(3t)}{2} \\ \cos(3t) \end{bmatrix} + c_2 e^{2t} \cdot \begin{bmatrix} -\frac{3\sin(3t)}{2} - \frac{3\cos(3t)}{2} \\ -\sin(3t) \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{3((c_1-c_2)\cos(3t)-(c_1+c_2)\sin(3t))e^{2t}}{2} \\ e^{2t}(-c_2\sin(3t)+c_1\cos(3t)) \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x_1(t) = \frac{3((c_1-c_2)\cos(3t)-(c_1+c_2)\sin(3t))e^{2t}}{2}, x_2(t) = e^{2t}(-c_2\sin(3t)+c_1\cos(3t))\right\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 58

```
dsolve([diff(x__1(t),t)=5*x__1(t)-9*x__2(t),diff(x__2(t),t)=2*x__1(t)-x__2(t)],singsol=all)
```

$$x_1(t) = e^{2t}(c_1\sin(3t) + c_2\cos(3t))$$
$$x_2(t) = \frac{e^{2t}(c_1\sin(3t) + c_2\sin(3t) - c_1\cos(3t) + c_2\cos(3t))}{3}$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 66

```
DSolve[{x1'[t]==5*x1[t]-9*x2[t],x2'[t]==2*x1[t]-x2[t]},{x1[t],x2[t]},t,IncludeSingularSoluti
```

$$\text{x1}(t) \to e^{2t}(c_1\cos(3t) + (c_1 - 3c_2)\sin(3t))$$
$$\text{x2}(t) \to \frac{1}{3}e^{2t}(3c_2\cos(3t) + (2c_1 - 3c_2)\sin(3t))$$

4.14 problem problem 14

Internal problem ID [328]
Internal file name [`OUTPUT/328_Sunday_June_05_2022_01_38_50_AM_99833809/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.3, The eigenvalue method for linear systems. Page 395
Problem number: problem 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x_1'(t) = 3x_1(t) - 4x_2(t)$$
$$x_2'(t) = 4x_1(t) + 3x_2(t)$$

4.14.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{3t}\cos(4t) & -e^{3t}\sin(4t) \\ e^{3t}\sin(4t) & e^{3t}\cos(4t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{3t}\cos(4t) & -e^{3t}\sin(4t) \\ e^{3t}\sin(4t) & e^{3t}\cos(4t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{3t}\cos(4t)\, c_1 - e^{3t}\sin(4t)\, c_2 \\ e^{3t}\sin(4t)\, c_1 + e^{3t}\cos(4t)\, c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{3t}(\cos(4t)\, c_1 - \sin(4t)\, c_2) \\ e^{3t}(\sin(4t)\, c_1 + \cos(4t)\, c_2) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

4.14.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det(A - \lambda I) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} 3-\lambda & -4 \\ 4 & 3-\lambda \end{bmatrix}\right) = 0
$$

Which gives the characteristic equation

$$\lambda^2 - 6\lambda + 25 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 3 + 4i$$
$$\lambda_2 = 3 - 4i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$3 + 4i$	1	complex eigenvalue
$3 - 4i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 3 - 4i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix} - (3 - 4i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 4i & -4 \\ 4 & 4i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 4i & -4 & 0 \\ 4 & 4i & 0 \end{array}\right]$$

$$R_2 = iR_1 + R_2 \Longrightarrow \left[\begin{array}{cc|c} 4i & -4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 4i & -4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -it\}$

Hence the solution is

$$\begin{bmatrix} \text{-I}\,t \\ t \end{bmatrix} = \begin{bmatrix} -it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \text{-I}\,t \\ t \end{bmatrix} = t \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \text{-I}\,t \\ t \end{bmatrix} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 3 + 4i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix} - (3+4i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4i & -4 \\ 4 & -4i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -4i & -4 & 0 \\ 4 & -4i & 0 \end{array}\right]$$

$$R_2 = -iR_1 + R_2 \Longrightarrow \left[\begin{array}{cc|c} -4i & -4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -4i & -4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = it\}$

Hence the solution is

$$\begin{bmatrix} \mathrm{I}t \\ t \end{bmatrix} = \begin{bmatrix} it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \mathrm{I}t \\ t \end{bmatrix} = t \begin{bmatrix} i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \mathrm{I}t \\ t \end{bmatrix} = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$3 + 4i$	1	1	No	$\begin{bmatrix} i \\ 1 \end{bmatrix}$
$3 - 4i$	1	1	No	$\begin{bmatrix} -i \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} i\mathrm{e}^{(3+4i)t} \\ \mathrm{e}^{(3+4i)t} \end{bmatrix} + c_2 \begin{bmatrix} -i\mathrm{e}^{(3-4i)t} \\ \mathrm{e}^{(3-4i)t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -i\left(c_2\mathrm{e}^{(3-4i)t} - c_1\mathrm{e}^{(3+4i)t}\right) \\ c_1\mathrm{e}^{(3+4i)t} + c_2\mathrm{e}^{(3-4i)t} \end{bmatrix}$$

The following is the phase plot of the system.

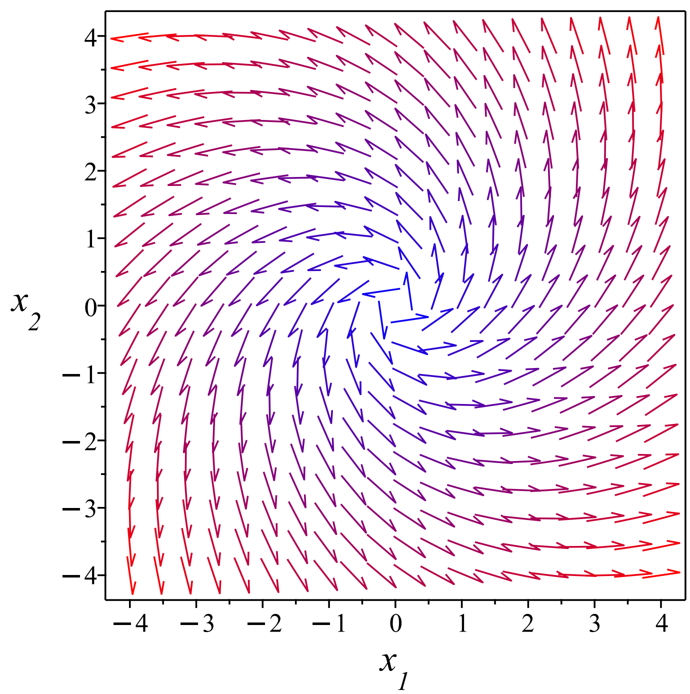


Figure 22: Phase plot

4.14.3 Maple step by step solution

Let's solve

$[x_1'(t) = 3x_1(t) - 4x_2(t), x_2'(t) = 4x_1(t) + 3x_2(t)]$

- Define vector

$$\underline{x^{\rightarrow}}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x^{\rightarrow}}'(t) = \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix} \cdot \underline{x^{\rightarrow}}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{x^{\rightarrow}}'(t) = \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix} \cdot \underline{x^{\rightarrow}}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$$

- Rewrite the system as

$$\underline{x^{\rightarrow}}'(t) = A \cdot \underline{x^{\rightarrow}}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[3 - 4\,\mathrm{I}, \begin{bmatrix} -\mathrm{I} \\ 1 \end{bmatrix}\right], \left[3 + 4\,\mathrm{I}, \begin{bmatrix} \mathrm{I} \\ 1 \end{bmatrix}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[3 - 4\,\mathrm{I}, \begin{bmatrix} -\mathrm{I} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{(3-4\,\mathrm{I})t} \cdot \begin{bmatrix} -\mathrm{I} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$\mathrm{e}^{3t} \cdot (\cos(4t) - \mathrm{I}\sin(4t)) \cdot \begin{bmatrix} -\mathrm{I} \\ 1 \end{bmatrix}$$

- Simplify expression

$$e^{3t} \cdot \begin{bmatrix} -\mathrm{I}(\cos(4t) - \mathrm{I}\sin(4t)) \\ \cos(4t) - \mathrm{I}\sin(4t) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\underline{\overrightarrow{x}}_{1}(t) = e^{3t} \cdot \begin{bmatrix} -\sin(4t) \\ \cos(4t) \end{bmatrix}, \underline{\overrightarrow{x}}_{2}(t) = e^{3t} \cdot \begin{bmatrix} -\cos(4t) \\ -\sin(4t) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\underline{\overrightarrow{x}} = c_1 \underline{\overrightarrow{x}}_{1}(t) + c_2 \underline{\overrightarrow{x}}_{2}(t)$$

- Substitute solutions into the general solution

$$\underline{\overrightarrow{x}} = c_1 e^{3t} \cdot \begin{bmatrix} -\sin(4t) \\ \cos(4t) \end{bmatrix} + c_2 e^{3t} \cdot \begin{bmatrix} -\cos(4t) \\ -\sin(4t) \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -e^{3t}(c_1 \sin(4t) + c_2 \cos(4t)) \\ e^{3t}(c_1 \cos(4t) - c_2 \sin(4t)) \end{bmatrix}$$

- Solution to the system of ODEs

$$\{x_1(t) = -e^{3t}(c_1 \sin(4t) + c_2 \cos(4t)), x_2(t) = e^{3t}(c_1 \cos(4t) - c_2 \sin(4t))\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 46

```
dsolve([diff(x__1(t),t)=3*x__1(t)-4*x__2(t),diff(x__2(t),t)=4*x__1(t)+3*x__2(t)],singsol=all
```

$$x_1(t) = e^{3t}(c_1 \sin(4t) + c_2 \cos(4t))$$
$$x_2(t) = -e^{3t}(c_1 \cos(4t) - c_2 \sin(4t))$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 51

```
DSolve[{x1'[t]==3*x1[t]-4*x2[t],x2'[t]==4*x1[t]+3*x2[t]},{x1[t],x2[t]},t,IncludeSingularSolu
```

$$\text{x1}(t) \to e^{3t}(c_1 \cos(4t) - c_2 \sin(4t))$$
$$\text{x2}(t) \to e^{3t}(c_2 \cos(4t) + c_1 \sin(4t))$$

4.15 problem problem 15

Internal problem ID [329]
Internal file name [`OUTPUT/329_Sunday_June_05_2022_01_38_51_AM_23405210/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.3, The eigenvalue method for linear systems. Page 395
Problem number: problem 15.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= 7x_1(t) - 5x_2(t) \\
x_2'(t) &= 4x_1(t) + 3x_2(t)
\end{aligned}
$$

4.15.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 7 & -5 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$
\begin{aligned}
e^{At} &= \begin{bmatrix} e^{5t}\cos(4t) + \frac{e^{5t}\sin(4t)}{2} & -\frac{5\,e^{5t}\sin(4t)}{4} \\ e^{5t}\sin(4t) & e^{5t}\cos(4t) - \frac{e^{5t}\sin(4t)}{2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{e^{5t}(2\cos(4t)+\sin(4t))}{2} & -\frac{5\,e^{5t}\sin(4t)}{4} \\ e^{5t}\sin(4t) & \frac{e^{5t}(2\cos(4t)-\sin(4t))}{2} \end{bmatrix}
\end{aligned}
$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{e^{5t}(2\cos(4t)+\sin(4t))}{2} & -\frac{5\,e^{5t}\sin(4t)}{4} \\ e^{5t}\sin(4t) & \frac{e^{5t}(2\cos(4t)-\sin(4t))}{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{e^{5t}(2\cos(4t)+\sin(4t))c_1}{2} - \frac{5\,e^{5t}\sin(4t)c_2}{4} \\ e^{5t}\sin(4t)\,c_1 + \frac{e^{5t}(2\cos(4t)-\sin(4t))c_2}{2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{e^{5t}(2c_1-5c_2)\sin(4t)}{4} + e^{5t}\cos(4t)\,c_1 \\ \left(\left(c_1 - \frac{c_2}{2}\right)\sin(4t) + c_2\cos(4t)\right)e^{5t} \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

4.15.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 7 & -5 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 7 & -5 \\ 4 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 7-\lambda & -5 \\ 4 & 3-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 10\lambda + 41 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 5 + 4i$$
$$\lambda_2 = 5 - 4i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$5 - 4i$	1	complex eigenvalue
$5 + 4i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 5 - 4i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 7 & -5 \\ 4 & 3 \end{bmatrix} - (5 - 4i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2 + 4i & -5 \\ 4 & -2 + 4i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2 + 4i & -5 & 0 \\ 4 & -2 + 4i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{2}{5} + \frac{4i}{5}\right) R_1 \Longrightarrow \left[\begin{array}{cc|c} 2 + 4i & -5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 + 4i & -5 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = \left(\frac{1}{2} - i\right) t\}$

Hence the solution is

$$\begin{bmatrix} \left(\frac{1}{2} - \mathrm{I}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2} - i\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(\frac{1}{2} - \mathrm{I}\right) t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} - i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(\frac{1}{2} - \mathrm{I}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - i \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(\frac{1}{2} - \mathrm{I}\right) t \\ t \end{bmatrix} = \begin{bmatrix} 1 - 2i \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 5 + 4i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 7 & -5 \\ 4 & 3 \end{bmatrix} - (5 + 4i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 - 4i & -5 \\ 4 & -2 - 4i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2 - 4i & -5 & 0 \\ 4 & -2 - 4i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{2}{5} - \frac{4i}{5}\right) R_1 \Longrightarrow \left[\begin{array}{cc|c} 2 - 4i & -5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2-4i & -5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(\frac{1}{2} + i\right) t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(\frac{1}{2} + \mathrm{I}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2} + i\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(\frac{1}{2} + \mathrm{I}\right) t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} + i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(\frac{1}{2} + \mathrm{I}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} + i \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(\frac{1}{2} + \mathrm{I}\right) t \\ t \end{bmatrix} = \begin{bmatrix} 1 + 2i \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$5+4i$	1	1	No	$\begin{bmatrix} \frac{1}{2}+i \\ 1 \end{bmatrix}$
$5-4i$	1	1	No	$\begin{bmatrix} \frac{1}{2}-i \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} \left(\frac{1}{2}+i\right) e^{(5+4i)t} \\ e^{(5+4i)t} \end{bmatrix} + c_2 \begin{bmatrix} \left(\frac{1}{2}-i\right) e^{(5-4i)t} \\ e^{(5-4i)t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2}+i\right) c_1 e^{(5+4i)t} + \left(\frac{1}{2}-i\right) c_2 e^{(5-4i)t} \\ c_1 e^{(5+4i)t} + c_2 e^{(5-4i)t} \end{bmatrix}$$

The following is the phase plot of the system.

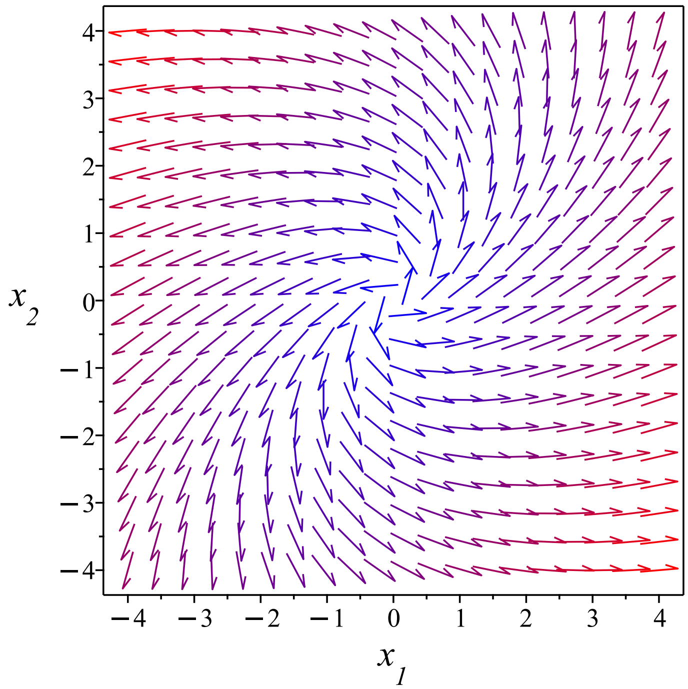


Figure 23: Phase plot

4.15.3 Maple step by step solution

Let's solve

$[x_1'(t) = 7x_1(t) - 5x_2(t), x_2'(t) = 4x_1(t) + 3x_2(t)]$

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x}^{\rightarrow}{}'(t) = \begin{bmatrix} 7 & -5 \\ 4 & 3 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{x}^{\rightarrow}{}'(t) = \begin{bmatrix} 7 & -5 \\ 4 & 3 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 7 & -5 \\ 4 & 3 \end{bmatrix}$$

- Rewrite the system as

$$\underline{x}^{\rightarrow\prime}(t) = A \cdot \underline{x}^{\rightarrow}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[5-4\,\mathrm{I}, \begin{bmatrix} \frac{1}{2}-\mathrm{I} \\ 1 \end{bmatrix}\right], \left[5+4\,\mathrm{I}, \begin{bmatrix} \frac{1}{2}+\mathrm{I} \\ 1 \end{bmatrix}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[5-4\,\mathrm{I}, \begin{bmatrix} \frac{1}{2}-\mathrm{I} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{(5-4\,\mathrm{I})t} \cdot \begin{bmatrix} \frac{1}{2}-\mathrm{I} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$\mathrm{e}^{5t} \cdot (\cos(4t) - \mathrm{I}\sin(4t)) \cdot \begin{bmatrix} \frac{1}{2}-\mathrm{I} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\mathrm{e}^{5t} \cdot \begin{bmatrix} \left(\frac{1}{2}-\mathrm{I}\right)(\cos(4t) - \mathrm{I}\sin(4t)) \\ \cos(4t) - \mathrm{I}\sin(4t) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\underline{x}^{\rightarrow}{}_1(t) = \mathrm{e}^{5t} \cdot \begin{bmatrix} \frac{\cos(4t)}{2} - \sin(4t) \\ \cos(4t) \end{bmatrix}, \underline{x}^{\rightarrow}{}_2(t) = \mathrm{e}^{5t} \cdot \begin{bmatrix} -\frac{\sin(4t)}{2} - \cos(4t) \\ -\sin(4t) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\underline{x}^{\rightarrow} = c_1 \underline{x}^{\rightarrow}{}_1(t) + c_2 \underline{x}^{\rightarrow}{}_2(t)$$

- Substitute solutions into the general solution

$$\vec{x} = c_1 e^{5t} \cdot \begin{bmatrix} \frac{\cos(4t)}{2} - \sin(4t) \\ \cos(4t) \end{bmatrix} + c_2 e^{5t} \cdot \begin{bmatrix} -\frac{\sin(4t)}{2} - \cos(4t) \\ -\sin(4t) \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{\left((c_1-2c_2)\cos(4t)-2\left(c_1+\frac{c_2}{2}\right)\sin(4t)\right)e^{5t}}{2} \\ e^{5t}(c_1 \cos(4t) - c_2 \sin(4t)) \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x_1(t) = \frac{\left((c_1-2c_2)\cos(4t)-2\left(c_1+\frac{c_2}{2}\right)\sin(4t)\right)e^{5t}}{2}, x_2(t) = e^{5t}(c_1 \cos(4t) - c_2 \sin(4t))\right\}$$

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 61

```
dsolve([diff(x__1(t),t)=7*x__1(t)-5*x__2(t),diff(x__2(t),t)=4*x__1(t)+3*x__2(t)],singsol=all
```

$$x_1(t) = e^{5t}(c_1 \sin(4t) + c_2 \cos(4t))$$
$$x_2(t) = -\frac{2\,e^{5t}(2c_1 \cos(4t) - c_2 \cos(4t) - c_1 \sin(4t) - 2c_2 \sin(4t))}{5}$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 72

```
DSolve[{x1'[t]==7*x1[t]-5*x2[t],x2'[t]==4*x1[t]+3*x2[t]},{x1[t],x2[t]},t,IncludeSingularSolu
```

$$\text{x1}(t) \to \frac{1}{4}e^{5t}(4c_1 \cos(4t) + (2c_1 - 5c_2)\sin(4t))$$
$$\text{x2}(t) \to \frac{1}{2}e^{5t}(2c_2 \cos(4t) + (2c_1 - c_2)\sin(4t))$$

4.16 problem problem 16

Internal problem ID [330]
Internal file name [`OUTPUT/330_Sunday_June_05_2022_01_38_53_AM_18915798/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.3, The eigenvalue method for linear systems. Page 395
Problem number: problem 16.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= -50x_1(t) + 20x_2(t) \\
x_2'(t) &= 100x_1(t) - 60x_2(t)
\end{aligned}
$$

4.16.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -50 & 20 \\ 100 & -60 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{4\,\mathrm{e}^{-100t}}{9} + \frac{5\,\mathrm{e}^{-10t}}{9} & \frac{2\,\mathrm{e}^{-10t}}{9} - \frac{2\,\mathrm{e}^{-100t}}{9} \\ \frac{10\,\mathrm{e}^{-10t}}{9} - \frac{10\,\mathrm{e}^{-100t}}{9} & \frac{5\,\mathrm{e}^{-100t}}{9} + \frac{4\,\mathrm{e}^{-10t}}{9} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{4\,\mathrm{e}^{-100t}}{9}+\frac{5\,\mathrm{e}^{-10t}}{9} & \frac{2\,\mathrm{e}^{-10t}}{9}-\frac{2\,\mathrm{e}^{-100t}}{9} \\ \frac{10\,\mathrm{e}^{-10t}}{9}-\frac{10\,\mathrm{e}^{-100t}}{9} & \frac{5\,\mathrm{e}^{-100t}}{9}+\frac{4\,\mathrm{e}^{-10t}}{9} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \left(\frac{4\,\mathrm{e}^{-100t}}{9}+\frac{5\,\mathrm{e}^{-10t}}{9}\right)c_1+\left(\frac{2\,\mathrm{e}^{-10t}}{9}-\frac{2\,\mathrm{e}^{-100t}}{9}\right)c_2 \\ \left(\frac{10\,\mathrm{e}^{-10t}}{9}-\frac{10\,\mathrm{e}^{-100t}}{9}\right)c_1+\left(\frac{5\,\mathrm{e}^{-100t}}{9}+\frac{4\,\mathrm{e}^{-10t}}{9}\right)c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(4c_1-2c_2)\mathrm{e}^{-100t}}{9}+\frac{5\left(c_1+\frac{2c_2}{5}\right)\mathrm{e}^{-10t}}{9} \\ \frac{5(-2c_1+c_2)\mathrm{e}^{-100t}}{9}+\frac{10\left(c_1+\frac{2c_2}{5}\right)\mathrm{e}^{-10t}}{9} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

4.16.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -50 & 20 \\ 100 & -60 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -50 & 20 \\ 100 & -60 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -50-\lambda & 20 \\ 100 & -60-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 110\lambda + 1000 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -10$$
$$\lambda_2 = -100$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-10	1	real eigenvalue
-100	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -100$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -50 & 20 \\ 100 & -60 \end{bmatrix} - (-100) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 50 & 20 \\ 100 & 40 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 50 & 20 & 0 \\ 100 & 40 & 0 \end{array}\right]$$

$$R_2 = R_2 - 2R_1 \Longrightarrow \left[\begin{array}{cc|c} 50 & 20 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 50 & 20 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{2t}{5}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{2t}{5} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2t}{5} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{2t}{5} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{2}{5} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{2t}{5} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2}{5} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{2t}{5} \\ t \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -10$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -50 & 20 \\ 100 & -60 \end{bmatrix} - (-10) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -40 & 20 \\ 100 & -50 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -40 & 20 & 0 \\ 100 & -50 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{5R_1}{2} \implies \left[\begin{array}{cc|c} -40 & 20 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -40 & 20 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-10	1	1	No	$\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$
-100	1	1	No	$\begin{bmatrix} -\frac{2}{5} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -10 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{-10t} \\ &= \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} e^{-10t} \end{aligned}$$

Since eigenvalue -100 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_2(t) &= \vec{v}_2 e^{-100t} \\ &= \begin{bmatrix} -\frac{2}{5} \\ 1 \end{bmatrix} e^{-100t} \end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} \frac{e^{-10t}}{2} \\ e^{-10t} \end{bmatrix} + c_2 \begin{bmatrix} -\frac{2\,e^{-100t}}{5} \\ e^{-100t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{c_1 e^{-10t}}{2} - \frac{2 c_2 e^{-100t}}{5} \\ c_1 e^{-10t} + c_2 e^{-100t} \end{bmatrix}$$

The following is the phase plot of the system.

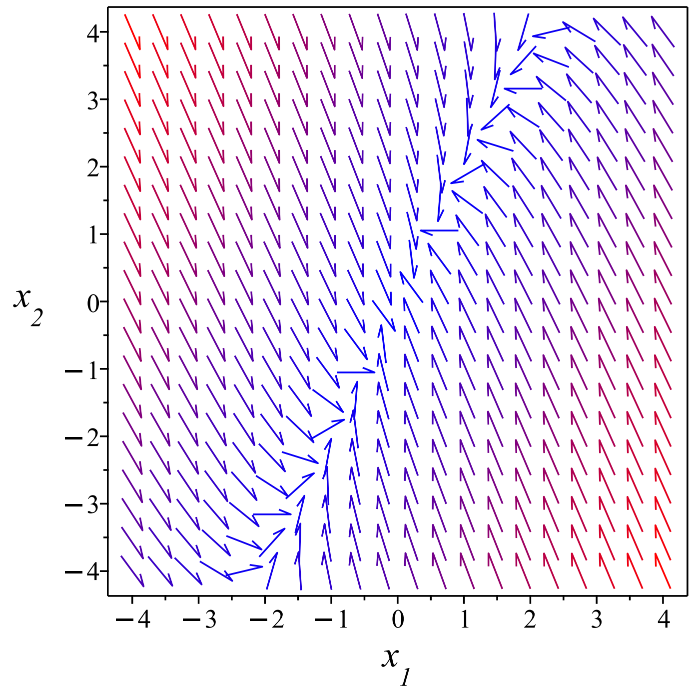


Figure 24: Phase plot

4.16.3 Maple step by step solution

Let's solve

$[x_1'(t) = -50x_1(t) + 20x_2(t), x_2'(t) = 100x_1(t) - 60x_2(t)]$

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x}^{\rightarrow}{}'(t) = \begin{bmatrix} -50 & 20 \\ 100 & -60 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{x}^{\rightarrow}{}'(t) = \begin{bmatrix} -50 & 20 \\ 100 & -60 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -50 & 20 \\ 100 & -60 \end{bmatrix}$$

- Rewrite the system as

$$\underline{\vec{x}}'(t) = A \cdot \underline{\vec{x}}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-100, \begin{bmatrix} -\frac{2}{5} \\ 1 \end{bmatrix}\right], \left[-10, \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-100, \begin{bmatrix} -\frac{2}{5} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_1 = \mathrm{e}^{-100t} \cdot \begin{bmatrix} -\frac{2}{5} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[-10, \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_2 = \mathrm{e}^{-10t} \cdot \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\underline{\vec{x}} = c_1 \underline{\vec{x}}_1 + c_2 \underline{\vec{x}}_2$$

- Substitute solutions into the general solution

$$\underline{\vec{x}} = c_1 \mathrm{e}^{-100t} \cdot \begin{bmatrix} -\frac{2}{5} \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{-10t} \cdot \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -\frac{2c_1 e^{-100t}}{5} + \frac{c_2 e^{-10t}}{2} \\ c_1 e^{-100t} + c_2 e^{-10t} \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x_1(t) = -\frac{2c_1 e^{-100t}}{5} + \frac{c_2 e^{-10t}}{2}, x_2(t) = c_1 e^{-100t} + c_2 e^{-10t}\right\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 36

```
dsolve([diff(x__1(t),t)=-50*x__1(t)+20*x__2(t),diff(x__2(t),t)=100*x__1(t)-60*x__2(t)],sings
```

$$x_1(t) = c_1 e^{-100t} + c_2 e^{-10t}$$
$$x_2(t) = -\frac{5c_1 e^{-100t}}{2} + 2c_2 e^{-10t}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 74

```
DSolve[{x1'[t]==-50*x1[t]+20*x2[t],x2'[t]==100*x1[t]-60*x2[t]},{x1[t],x2[t]},t,IncludeSingul
```

$$\text{x1}(t) \to \frac{1}{9}e^{-100t}\left(c_1\left(5e^{90t} + 4\right) + 2c_2\left(e^{90t} - 1\right)\right)$$
$$\text{x2}(t) \to \frac{1}{9}e^{-100t}\left(10c_1\left(e^{90t} - 1\right) + c_2\left(4e^{90t} + 5\right)\right)$$

4.17 problem problem 17

Internal problem ID [331]
Internal file name [`OUTPUT/331_Sunday_June_05_2022_01_38_54_AM_11743124/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.3, The eigenvalue method for linear systems. Page 395
Problem number: problem 17.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= 4x_1(t) + x_2(t) + 4x_3(t)\\
x_2'(t) &= x_1(t) + 7x_2(t) + x_3(t)\\
x_3'(t) &= 4x_1(t) + x_2(t) + 4x_3(t)
\end{aligned}
$$

4.17.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 4 & 1 & 4 \\ 1 & 7 & 1 \\ 4 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{1}{2}+\frac{e^{6t}}{6}+\frac{e^{9t}}{3} & \frac{e^{9t}}{3}-\frac{e^{6t}}{3} & \frac{e^{9t}}{3}+\frac{e^{6t}}{6}-\frac{1}{2} \\ \frac{e^{9t}}{3}-\frac{e^{6t}}{3} & \frac{e^{9t}}{3}+\frac{2\,e^{6t}}{3} & \frac{e^{9t}}{3}-\frac{e^{6t}}{3} \\ \frac{e^{9t}}{3}+\frac{e^{6t}}{6}-\frac{1}{2} & \frac{e^{9t}}{3}-\frac{e^{6t}}{3} & \frac{1}{2}+\frac{e^{6t}}{6}+\frac{e^{9t}}{3} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{1}{2}+\frac{e^{6t}}{6}+\frac{e^{9t}}{3} & \frac{e^{9t}}{3}-\frac{e^{6t}}{3} & \frac{e^{9t}}{3}+\frac{e^{6t}}{6}-\frac{1}{2} \\ \frac{e^{9t}}{3}-\frac{e^{6t}}{3} & \frac{e^{9t}}{3}+\frac{2\,e^{6t}}{3} & \frac{e^{9t}}{3}-\frac{e^{6t}}{3} \\ \frac{e^{9t}}{3}+\frac{e^{6t}}{6}-\frac{1}{2} & \frac{e^{9t}}{3}-\frac{e^{6t}}{3} & \frac{1}{2}+\frac{e^{6t}}{6}+\frac{e^{9t}}{3} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} \left(\frac{1}{2}+\frac{e^{6t}}{6}+\frac{e^{9t}}{3}\right)c_1+\left(\frac{e^{9t}}{3}-\frac{e^{6t}}{3}\right)c_2+\left(\frac{e^{9t}}{3}+\frac{e^{6t}}{6}-\frac{1}{2}\right)c_3 \\ \left(\frac{e^{9t}}{3}-\frac{e^{6t}}{3}\right)c_1+\left(\frac{e^{9t}}{3}+\frac{2\,e^{6t}}{3}\right)c_2+\left(\frac{e^{9t}}{3}-\frac{e^{6t}}{3}\right)c_3 \\ \left(\frac{e^{9t}}{3}+\frac{e^{6t}}{6}-\frac{1}{2}\right)c_1+\left(\frac{e^{9t}}{3}-\frac{e^{6t}}{3}\right)c_2+\left(\frac{1}{2}+\frac{e^{6t}}{6}+\frac{e^{9t}}{3}\right)c_3 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(c_1-2c_2+c_3)e^{6t}}{6}+\frac{(2c_1+2c_2+2c_3)e^{9t}}{6}+\frac{c_1}{2}-\frac{c_3}{2} \\ \frac{(-c_1+2c_2-c_3)e^{6t}}{3}+\frac{(c_1+c_2+c_3)e^{9t}}{3} \\ \frac{(c_1-2c_2+c_3)e^{6t}}{6}+\frac{(2c_1+2c_2+2c_3)e^{9t}}{6}-\frac{c_1}{2}+\frac{c_3}{2} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

4.17.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 4 & 1 & 4 \\ 1 & 7 & 1 \\ 4 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 4 & 1 & 4 \\ 1 & 7 & 1 \\ 4 & 1 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 4-\lambda & 1 & 4 \\ 1 & 7-\lambda & 1 \\ 4 & 1 & 4-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 15\lambda^2 + 54\lambda = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 0$$
$$\lambda_2 = 6$$
$$\lambda_3 = 9$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
0	1	real eigenvalue
6	1	real eigenvalue
9	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 0$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 4 & 1 & 4 \\ 1 & 7 & 1 \\ 4 & 1 & 4 \end{bmatrix} - (0)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 1 & 4 \\ 1 & 7 & 1 \\ 4 & 1 & 4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 4 & 1 & 4 & 0 \\ 1 & 7 & 1 & 0 \\ 4 & 1 & 4 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{4} \Longrightarrow \left[\begin{array}{ccc|c} 4 & 1 & 4 & 0 \\ 0 & \frac{27}{4} & 0 & 0 \\ 4 & 1 & 4 & 0 \end{array}\right]$$

$$R_3 = R_3 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} 4 & 1 & 4 & 0 \\ 0 & \frac{27}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 4 & 1 & 4 \\ 0 & \frac{27}{4} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t, v_2 = 0\}$

Hence the solution is

$$\begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} -t \\ 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 6$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 4 & 1 & 4 \\ 1 & 7 & 1 \\ 4 & 1 & 4 \end{bmatrix} - (6)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 & 4 \\ 1 & 1 & 1 \\ 4 & 1 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -2 & 1 & 4 & 0 \\ 1 & 1 & 1 & 0 \\ 4 & 1 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} -2 & 1 & 4 & 0 \\ 0 & \frac{3}{2} & 3 & 0 \\ 4 & 1 & -2 & 0 \end{array}\right]$$

$$R_3 = R_3 + 2R_1 \Longrightarrow \left[\begin{array}{ccc|c} -2 & 1 & 4 & 0 \\ 0 & \frac{3}{2} & 3 & 0 \\ 0 & 3 & 6 & 0 \end{array}\right]$$

$$R_3 = R_3 - 2R_2 \Longrightarrow \left[\begin{array}{ccc|c} -2 & 1 & 4 & 0 \\ 0 & \frac{3}{2} & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & 1 & 4 \\ 0 & \frac{3}{2} & 3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = -2t\}$

Hence the solution is

$$\begin{bmatrix} t \\ -2t \\ t \end{bmatrix} = \begin{bmatrix} t \\ -2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ -2t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ -2t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = 9$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 4 & 1 & 4 \\ 1 & 7 & 1 \\ 4 & 1 & 4 \end{bmatrix} - (9) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -5 & 1 & 4 \\ 1 & -2 & 1 \\ 4 & 1 & -5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -5 & 1 & 4 & 0 \\ 1 & -2 & 1 & 0 \\ 4 & 1 & -5 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{5} \Longrightarrow \left[\begin{array}{ccc|c} -5 & 1 & 4 & 0 \\ 0 & -\frac{9}{5} & \frac{9}{5} & 0 \\ 4 & 1 & -5 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{4R_1}{5} \Longrightarrow \left[\begin{array}{ccc|c} -5 & 1 & 4 & 0 \\ 0 & -\frac{9}{5} & \frac{9}{5} & 0 \\ 0 & \frac{9}{5} & -\frac{9}{5} & 0 \end{array}\right]$$

$$R_3 = R_3 + R_2 \Longrightarrow \left[\begin{array}{ccc|c} -5 & 1 & 4 & 0 \\ 0 & -\frac{9}{5} & \frac{9}{5} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -5 & 1 & 4 \\ 0 & -\frac{9}{5} & \frac{9}{5} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
0	1	1	No	$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$
6	1	1	No	$\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$
9	1	1	No	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 0 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{0} \\ &= \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{0}\end{aligned}$$

Since eigenvalue 6 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{6t} \\
&= \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} e^{6t}
\end{aligned}
$$

Since eigenvalue 9 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_3(t) &= \vec{v}_3 e^{9t} \\
&= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{9t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)
$$

Which is written as

$$
\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{6t} \\ -2\,\mathrm{e}^{6t} \\ \mathrm{e}^{6t} \end{bmatrix} + c_3 \begin{bmatrix} \mathrm{e}^{9t} \\ \mathrm{e}^{9t} \\ \mathrm{e}^{9t} \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} -c_1 + c_2 \mathrm{e}^{6t} + c_3 \mathrm{e}^{9t} \\ -2 c_2 \mathrm{e}^{6t} + c_3 \mathrm{e}^{9t} \\ c_1 + c_2 \mathrm{e}^{6t} + c_3 \mathrm{e}^{9t} \end{bmatrix}
$$

4.17.3 Maple step by step solution

Let's solve

$[x_1'(t) = 4x_1(t) + x_2(t) + 4x_3(t)\,, x_2'(t) = x_1(t) + 7x_2(t) + x_3(t)\,, x_3'(t) = 4x_1(t) + x_2(t) + 4x_3(t)]$

- Define vector

$$\underline{\vec{x}}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{\vec{x}}'(t) = \begin{bmatrix} 4 & 1 & 4 \\ 1 & 7 & 1 \\ 4 & 1 & 4 \end{bmatrix} \cdot \underline{\vec{x}}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{\vec{x}}'(t) = \begin{bmatrix} 4 & 1 & 4 \\ 1 & 7 & 1 \\ 4 & 1 & 4 \end{bmatrix} \cdot \underline{\vec{x}}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 4 & 1 & 4 \\ 1 & 7 & 1 \\ 4 & 1 & 4 \end{bmatrix}$$

- Rewrite the system as

$\underline{\vec{x}}'(t) = A \cdot \underline{\vec{x}}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right], \left[6, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\right], \left[9, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{x}_1^{\rightarrow} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[6, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{x}_2^{\rightarrow} = e^{6t} \cdot \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[9, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{x}_3^{\rightarrow} = e^{9t} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\underline{x}^{\rightarrow} = c_1 \underline{x}_1^{\rightarrow} + c_2 \underline{x}_2^{\rightarrow} + c_3 \underline{x}_3^{\rightarrow}$$

- Substitute solutions into the general solution

$$\underline{x}^{\rightarrow} = c_2 e^{6t} \cdot \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + c_3 e^{9t} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -c_1 \\ 0 \\ c_1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} -c_1 + c_2 e^{6t} + c_3 e^{9t} \\ -2c_2 e^{6t} + c_3 e^{9t} \\ c_1 + c_2 e^{6t} + c_3 e^{9t} \end{bmatrix}$$

- Solution to the system of ODEs

  $\{x_1(t) = -c_1 + c_2 e^{6t} + c_3 e^{9t}, x_2(t) = -2c_2 e^{6t} + c_3 e^{9t}, x_3(t) = c_1 + c_2 e^{6t} + c_3 e^{9t}\}$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 55

```
dsolve([diff(x__1(t),t)=4*x__1(t)+x__2(t)+4*x__3(t),diff(x__2(t),t)=x__1(t)+7*x__2(t)+x__3(t
```

$$x_1(t) = c_1 + c_2 e^{6t} + c_3 e^{9t}$$
$$x_2(t) = -2c_2 e^{6t} + c_3 e^{9t}$$
$$x_3(t) = c_2 e^{6t} + c_3 e^{9t} - c_1$$

✓ Solution by Mathematica

Time used: 0.008 (sec). Leaf size: 158

```
DSolve[{x1'[t]==4*x1[t]+x2[t]+4*x3[t],x2'[t]==x1[t]+7*x2[t]+x3[t],x3'[t]==4*x1[t]+x2[t]+4*x3
```

$$\text{x1}(t) \to \frac{1}{6}\left(c_1\left(e^{6t} + 2e^{9t} + 3\right) + \left(e^{3t} - 1\right)\left(3c_3 e^{3t} + 2(c_2 + c_3)e^{6t} + 3c_3\right)\right)$$
$$\text{x2}(t) \to \frac{1}{3}e^{6t}\left(c_1\left(e^{3t} - 1\right) + c_2\left(e^{3t} + 2\right) + c_3\left(e^{3t} - 1\right)\right)$$
$$\text{x3}(t) \to \frac{1}{6}\left(c_1\left(e^{6t} + 2e^{9t} - 3\right) + (c_3 - 2c_2)e^{6t} + 2(c_2 + c_3)e^{9t} + 3c_3\right)$$

4.18 problem problem 18

Internal problem ID [332]
Internal file name [`OUTPUT/332_Sunday_June_05_2022_01_38_55_AM_43808668/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.3, The eigenvalue method for linear systems. Page 395
Problem number: problem 18.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= x_1(t) + 2x_2(t) + 2x_3(t) \\
x_2'(t) &= 2x_1(t) + 7x_2(t) + x_3(t) \\
x_3'(t) &= 2x_1(t) + x_2(t) + 7x_3(t)
\end{aligned}
$$

4.18.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 7 & 1 \\ 2 & 1 & 7 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{8}{9} + \frac{e^{9t}}{9} & \frac{2\,e^{9t}}{9} - \frac{2}{9} & \frac{2\,e^{9t}}{9} - \frac{2}{9} \\ \frac{2\,e^{9t}}{9} - \frac{2}{9} & \frac{1}{18} + \frac{4\,e^{9t}}{9} + \frac{e^{6t}}{2} & \frac{4\,e^{9t}}{9} - \frac{e^{6t}}{2} + \frac{1}{18} \\ \frac{2\,e^{9t}}{9} - \frac{2}{9} & \frac{4\,e^{9t}}{9} - \frac{e^{6t}}{2} + \frac{1}{18} & \frac{1}{18} + \frac{4\,e^{9t}}{9} + \frac{e^{6t}}{2} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{8}{9}+\frac{e^{9t}}{9} & \frac{2\,e^{9t}}{9}-\frac{2}{9} & \frac{2\,e^{9t}}{9}-\frac{2}{9} \\ \frac{2\,e^{9t}}{9}-\frac{2}{9} & \frac{1}{18}+\frac{4\,e^{9t}}{9}+\frac{e^{6t}}{2} & \frac{4\,e^{9t}}{9}-\frac{e^{6t}}{2}+\frac{1}{18} \\ \frac{2\,e^{9t}}{9}-\frac{2}{9} & \frac{4\,e^{9t}}{9}-\frac{e^{6t}}{2}+\frac{1}{18} & \frac{1}{18}+\frac{4\,e^{9t}}{9}+\frac{e^{6t}}{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} \left(\frac{8}{9}+\frac{e^{9t}}{9}\right)c_1+\left(\frac{2\,e^{9t}}{9}-\frac{2}{9}\right)c_2+\left(\frac{2\,e^{9t}}{9}-\frac{2}{9}\right)c_3 \\ \left(\frac{2\,e^{9t}}{9}-\frac{2}{9}\right)c_1+\left(\frac{1}{18}+\frac{4\,e^{9t}}{9}+\frac{e^{6t}}{2}\right)c_2+\left(\frac{4\,e^{9t}}{9}-\frac{e^{6t}}{2}+\frac{1}{18}\right)c_3 \\ \left(\frac{2\,e^{9t}}{9}-\frac{2}{9}\right)c_1+\left(\frac{4\,e^{9t}}{9}-\frac{e^{6t}}{2}+\frac{1}{18}\right)c_2+\left(\frac{1}{18}+\frac{4\,e^{9t}}{9}+\frac{e^{6t}}{2}\right)c_3 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(c_1+2c_2+2c_3)e^{9t}}{9}+\frac{8c_1}{9}-\frac{2c_2}{9}-\frac{2c_3}{9} \\ \frac{2(c_1+2c_2+2c_3)e^{9t}}{9}+\frac{(c_2-c_3)e^{6t}}{2}-\frac{2c_1}{9}+\frac{c_2}{18}+\frac{c_3}{18} \\ \frac{2(c_1+2c_2+2c_3)e^{9t}}{9}+\frac{(-c_2+c_3)e^{6t}}{2}-\frac{2c_1}{9}+\frac{c_2}{18}+\frac{c_3}{18} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

4.18.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 7 & 1 \\ 2 & 1 & 7 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A-\lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & 2 & 2 \\ 2 & 7 & 1 \\ 2 & 1 & 7 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & 2 & 2 \\ 2 & 7-\lambda & 1 \\ 2 & 1 & 7-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 15\lambda^2 + 54\lambda = 0$$

The roots of the above are the eigenvalues.

$$\begin{aligned} \lambda_1 &= 0 \\ \lambda_2 &= 6 \\ \lambda_3 &= 9 \end{aligned}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
0	1	real eigenvalue
6	1	real eigenvalue
9	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 0$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 2 & 2 \\ 2 & 7 & 1 \\ 2 & 1 & 7 \end{bmatrix} - (0)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 7 & 1 \\ 2 & 1 & 7 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 2 & 7 & 1 & 0 \\ 2 & 1 & 7 & 0 \end{array}\right]$$

$$R_2 = R_2 - 2R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 0 & 3 & -3 & 0 \\ 2 & 1 & 7 & 0 \end{array}\right]$$

$$R_3 = R_3 - 2R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 0 & 3 & -3 & 0 \\ 0 & -3 & 3 & 0 \end{array}\right]$$

$$R_3 = R_3 + R_2 \Longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 0 & 3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -4t, v_2 = t\}$

Hence the solution is

$$\begin{bmatrix} -4t \\ t \\ t \end{bmatrix} = \begin{bmatrix} -4t \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -4t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -4t \\ t \\ t \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 6$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 2 & 2 \\ 2 & 7 & 1 \\ 2 & 1 & 7 \end{bmatrix} - (6)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -5 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -5 & 2 & 2 & 0 \\ 2 & 1 & 1 & 0 \\ 2 & 1 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{2R_1}{5} \Longrightarrow \left[\begin{array}{ccc|c} -5 & 2 & 2 & 0 \\ 0 & \frac{9}{5} & \frac{9}{5} & 0 \\ 2 & 1 & 1 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{2R_1}{5} \Longrightarrow \left[\begin{array}{ccc|c} -5 & 2 & 2 & 0 \\ 0 & \frac{9}{5} & \frac{9}{5} & 0 \\ 0 & \frac{9}{5} & \frac{9}{5} & 0 \end{array}\right]$$

$$R_3 = R_3 - R_2 \Longrightarrow \left[\begin{array}{ccc|c} -5 & 2 & 2 & 0 \\ 0 & \frac{9}{5} & \frac{9}{5} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -5 & 2 & 2 \\ 0 & \frac{9}{5} & \frac{9}{5} \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_2 = -t\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ -t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ -t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = 9$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 2 & 2 \\ 2 & 7 & 1 \\ 2 & 1 & 7 \end{bmatrix} - (9) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -8 & 2 & 2 \\ 2 & -2 & 1 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -8 & 2 & 2 & 0 \\ 2 & -2 & 1 & 0 \\ 2 & 1 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{4} \Longrightarrow \left[\begin{array}{ccc|c} -8 & 2 & 2 & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2} & 0 \\ 2 & 1 & -2 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{R_1}{4} \Longrightarrow \left[\begin{array}{ccc|c} -8 & 2 & 2 & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2} & 0 \\ 0 & \frac{3}{2} & -\frac{3}{2} & 0 \end{array}\right]$$

$$R_3 = R_3 + R_2 \Longrightarrow \left[\begin{array}{ccc|c} -8 & 2 & 2 & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -8 & 2 & 2 \\ 0 & -\frac{3}{2} & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{2}, v_2 = t\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{2} \\ t \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{2} \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{2} \\ t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} \\ 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{t}{2} \\ t \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{t}{2} \\ t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
0	1	1	No	$\begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix}$
6	1	1	No	$\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$
9	1	1	No	$\begin{bmatrix} \frac{1}{2} \\ 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 0 is real and distinct then the

corresponding eigenvector solution is

$$\vec{x}_1(t) = \vec{v}_1 e^{0}$$
$$= \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix} e^{0}$$

Since eigenvalue 6 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_2(t) = \vec{v}_2 e^{6t}$$
$$= \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} e^{6t}$$

Since eigenvalue 9 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_3(t) = \vec{v}_3 e^{9t}$$
$$= \begin{bmatrix} \frac{1}{2} \\ 1 \\ 1 \end{bmatrix} e^{9t}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ -\mathrm{e}^{6t} \\ \mathrm{e}^{6t} \end{bmatrix} + c_3 \begin{bmatrix} \frac{\mathrm{e}^{9t}}{2} \\ \mathrm{e}^{9t} \\ \mathrm{e}^{9t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} -4c_1 + \frac{c_3 \mathrm{e}^{9t}}{2} \\ c_1 - c_2 \mathrm{e}^{6t} + c_3 \mathrm{e}^{9t} \\ c_1 + c_2 \mathrm{e}^{6t} + c_3 \mathrm{e}^{9t} \end{bmatrix}$$

4.18.3 Maple step by step solution

Let's solve

$[x_1'(t) = x_1(t) + 2x_2(t) + 2x_3(t), x_2'(t) = 2x_1(t) + 7x_2(t) + x_3(t), x_3'(t) = 2x_1(t) + x_2(t) + 7x_3(t)]$

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x}^{\rightarrow\prime}(t) = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 7 & 1 \\ 2 & 1 & 7 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{x}^{\rightarrow\prime}(t) = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 7 & 1 \\ 2 & 1 & 7 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 7 & 1 \\ 2 & 1 & 7 \end{bmatrix}$$

- Rewrite the system as

$\underline{x}^{\rightarrow\prime}(t) = A \cdot \underline{x}^{\rightarrow}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix}\right], \left[6, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}\right], \left[9, \begin{bmatrix} \frac{1}{2} \\ 1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_1 = \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[6, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_2 = \mathrm{e}^{6t} \cdot \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[9, \begin{bmatrix} \frac{1}{2} \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_3 = \mathrm{e}^{9t} \cdot \begin{bmatrix} \frac{1}{2} \\ 1 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\underline{\vec{x}} = c_1\underline{\vec{x}}_1 + c_2\underline{\vec{x}}_2 + c_3\underline{\vec{x}}_3$$

- Substitute solutions into the general solution

$$\underline{x^{\rightarrow}} = c_2 e^{6t} \cdot \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + c_3 e^{9t} \cdot \begin{bmatrix} \frac{1}{2} \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -4c_1 \\ c_1 \\ c_1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} -4c_1 + \frac{c_3 e^{9t}}{2} \\ c_1 - c_2 e^{6t} + c_3 e^{9t} \\ c_1 + c_2 e^{6t} + c_3 e^{9t} \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x_1(t) = -4c_1 + \frac{c_3 e^{9t}}{2}, x_2(t) = c_1 - c_2 e^{6t} + c_3 e^{9t}, x_3(t) = c_1 + c_2 e^{6t} + c_3 e^{9t}\right\}$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 54

```
dsolve([diff(x__1(t),t)=x__1(t)+2*x__2(t)+2*x__3(t),diff(x__2(t),t)=2*x__1(t)+7*x__2(t)+x__3
```

$$x_1(t) = c_2 + c_3 e^{9t}$$
$$x_2(t) = 2c_3 e^{9t} + e^{6t} c_1 - \frac{c_2}{4}$$
$$x_3(t) = 2c_3 e^{9t} - e^{6t} c_1 - \frac{c_2}{4}$$

✓ Solution by Mathematica

Time used: 0.008 (sec). Leaf size: 148

```
DSolve[{x1'[t]==x1[t]+2*x2[t]+2*x3[t],x2'[t]==2*x1[t]+7*x2[t]+x3[t],x3'[t]==2*x1[t]+x2[t]+7*
```

$$\text{x1}(t) \to \frac{1}{9}\left(c_1\left(e^{9t} + 8\right) + 2(c_2 + c_3)\left(e^{9t} - 1\right)\right)$$
$$\text{x2}(t) \to \frac{1}{18}\left(4c_1\left(e^{9t} - 1\right) + c_2\left(9e^{6t} + 8e^{9t} + 1\right) + c_3\left(-9e^{6t} + 8e^{9t} + 1\right)\right)$$
$$\text{x3}(t) \to \frac{1}{18}\left(4c_1\left(e^{9t} - 1\right) + c_2\left(-9e^{6t} + 8e^{9t} + 1\right) + c_3\left(9e^{6t} + 8e^{9t} + 1\right)\right)$$

4.19 problem problem 19

Internal problem ID [333]
Internal file name [`OUTPUT/333_Sunday_June_05_2022_01_38_57_AM_83880904/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.3, The eigenvalue method for linear systems. Page 395
Problem number: problem 19.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned}
x_1'(t) &= 4x_1(t) + x_2(t) + x_3(t)\\
x_2'(t) &= x_1(t) + 4x_2(t) + x_3(t)\\
x_3'(t) &= x_1(t) + x_2(t) + 4x_3(t)
\end{aligned}$$

4.19.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t)\\ x_2'(t)\\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 4 & 1 & 1\\ 1 & 4 & 1\\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_1(t)\\ x_2(t)\\ x_3(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{2\,\mathrm{e}^{3t}}{3} + \frac{\mathrm{e}^{6t}}{3} & \frac{\mathrm{e}^{6t}}{3} - \frac{\mathrm{e}^{3t}}{3} & \frac{\mathrm{e}^{6t}}{3} - \frac{\mathrm{e}^{3t}}{3}\\ \frac{\mathrm{e}^{6t}}{3} - \frac{\mathrm{e}^{3t}}{3} & \frac{2\,\mathrm{e}^{3t}}{3} + \frac{\mathrm{e}^{6t}}{3} & \frac{\mathrm{e}^{6t}}{3} - \frac{\mathrm{e}^{3t}}{3}\\ \frac{\mathrm{e}^{6t}}{3} - \frac{\mathrm{e}^{3t}}{3} & \frac{\mathrm{e}^{6t}}{3} - \frac{\mathrm{e}^{3t}}{3} & \frac{2\,\mathrm{e}^{3t}}{3} + \frac{\mathrm{e}^{6t}}{3} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{2\,e^{3t}}{3}+\frac{e^{6t}}{3} & \frac{e^{6t}}{3}-\frac{e^{3t}}{3} & \frac{e^{6t}}{3}-\frac{e^{3t}}{3} \\ \frac{e^{6t}}{3}-\frac{e^{3t}}{3} & \frac{2\,e^{3t}}{3}+\frac{e^{6t}}{3} & \frac{e^{6t}}{3}-\frac{e^{3t}}{3} \\ \frac{e^{6t}}{3}-\frac{e^{3t}}{3} & \frac{e^{6t}}{3}-\frac{e^{3t}}{3} & \frac{2\,e^{3t}}{3}+\frac{e^{6t}}{3} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} \left(\frac{2\,e^{3t}}{3}+\frac{e^{6t}}{3}\right)c_1+\left(\frac{e^{6t}}{3}-\frac{e^{3t}}{3}\right)c_2+\left(\frac{e^{6t}}{3}-\frac{e^{3t}}{3}\right)c_3 \\ \left(\frac{e^{6t}}{3}-\frac{e^{3t}}{3}\right)c_1+\left(\frac{2\,e^{3t}}{3}+\frac{e^{6t}}{3}\right)c_2+\left(\frac{e^{6t}}{3}-\frac{e^{3t}}{3}\right)c_3 \\ \left(\frac{e^{6t}}{3}-\frac{e^{3t}}{3}\right)c_1+\left(\frac{e^{6t}}{3}-\frac{e^{3t}}{3}\right)c_2+\left(\frac{2\,e^{3t}}{3}+\frac{e^{6t}}{3}\right)c_3 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(2c_1-c_2-c_3)e^{3t}}{3}+\frac{(c_1+c_2+c_3)e^{6t}}{3} \\ \frac{(-c_1+2c_2-c_3)e^{3t}}{3}+\frac{(c_1+c_2+c_3)e^{6t}}{3} \\ \frac{(-c_1-c_2+2c_3)e^{3t}}{3}+\frac{(c_1+c_2+c_3)e^{6t}}{3} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

4.19.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A-\lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 4-\lambda & 1 & 1 \\ 1 & 4-\lambda & 1 \\ 1 & 1 & 4-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 12\lambda^2 + 45\lambda - 54 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 3$$
$$\lambda_2 = 6$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
3	1	real eigenvalue
6	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix} - (3)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right]$$

$$R_3 = R_3 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2, v_3\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Let $v_3 = s$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t - s\}$

Hence the solution is

$$\begin{bmatrix} -t - s \\ t \\ s \end{bmatrix} = \begin{bmatrix} -t - s \\ t \\ s \end{bmatrix}$$

Since there are two free Variable, we have found two eigenvectors associated with this eigenvalue. The above can be written as

$$\begin{aligned} \begin{bmatrix} -t - s \\ t \\ s \end{bmatrix} &= \begin{bmatrix} -t \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} -s \\ 0 \\ s \end{bmatrix} \\ &= t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

By letting $t = 1$ and $s = 1$ then the above becomes

$$\begin{bmatrix} -t - s \\ t \\ s \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Hence the two eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right)$$

Considering the eigenvalue $\lambda_2 = 6$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix} - (6) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2} & 0 \\ 1 & 1 & -2 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2} & 0 \\ 0 & \frac{3}{2} & -\frac{3}{2} & 0 \end{array}\right]$$

$$R_3 = R_3 + R_2 \Longrightarrow \left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & 1 & 1 \\ 0 & -\frac{3}{2} & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
3	2	2	No	$\begin{bmatrix} -1 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$
6	1	1	No	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 3 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

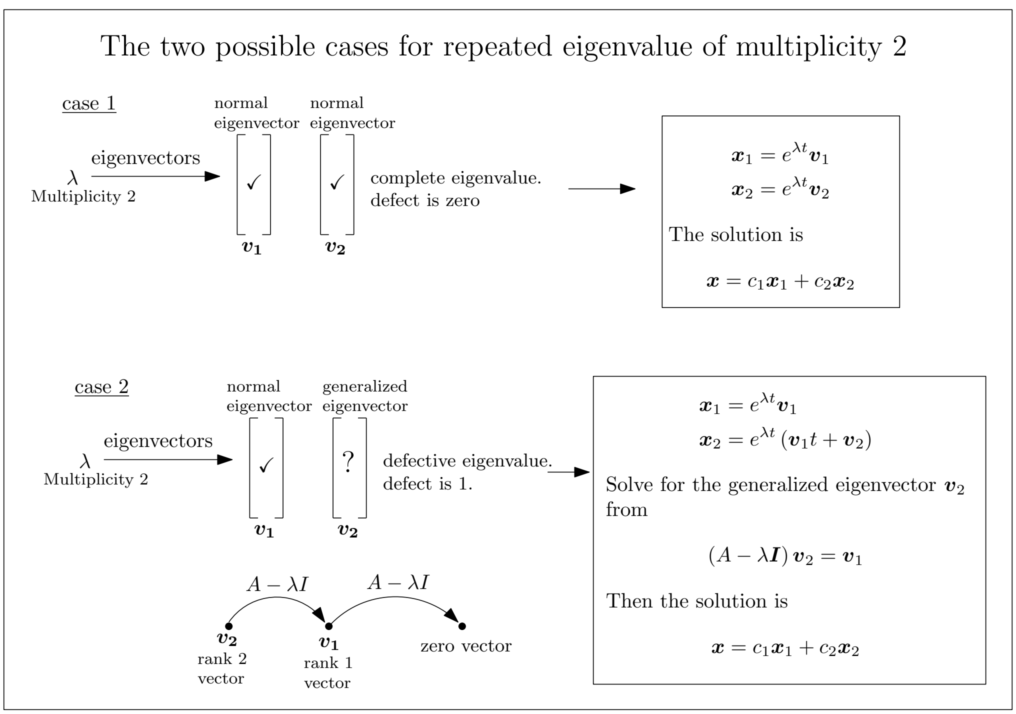


Figure 25: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2 which is the same as its geometric

multiplicity 2, then it is complete eigenvalue and this falls into case 1 shown above. Hence the corresponding eigenvector basis are

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{3t} \\ &= \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{3t}\end{aligned}$$

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{3t} \\ &= \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{3t}\end{aligned}$$

Since eigenvalue 6 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_3(t) &= \vec{v}_3 e^{6t} \\ &= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{6t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 \begin{bmatrix} -\mathrm{e}^{3t} \\ 0 \\ \mathrm{e}^{3t} \end{bmatrix} + c_2 \begin{bmatrix} -\mathrm{e}^{3t} \\ \mathrm{e}^{3t} \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} \mathrm{e}^{6t} \\ \mathrm{e}^{6t} \\ \mathrm{e}^{6t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} (-c_1 - c_2)\,\mathrm{e}^{3t} + c_3 \mathrm{e}^{6t} \\ c_2 \mathrm{e}^{3t} + c_3 \mathrm{e}^{6t} \\ c_1 \mathrm{e}^{3t} + c_3 \mathrm{e}^{6t} \end{bmatrix}$$

4.19.3 Maple step by step solution

Let's solve

$[x_1'(t) = 4x_1(t) + x_2(t) + x_3(t), x_2'(t) = x_1(t) + 4x_2(t) + x_3(t), x_3'(t) = x_1(t) + x_2(t) + 4x_3(t)]$

- Define vector

$$\underline{\vec{x}}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{\vec{x}}'(t) = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix} \cdot \underline{\vec{x}}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{\vec{x}}'(t) = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix} \cdot \underline{\vec{x}}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix}$$

- Rewrite the system as

$\underline{\vec{x}}'(t) = A \cdot \underline{\vec{x}}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[3, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right], \left[3, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right], \left[6, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[3, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue 3

$$\underline{\vec{x}}_1(t) = e^{3t} \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = 3$ is the eigenvalue, and

$$\underline{\vec{x}}_2(t) = e^{\lambda t}\left(t\vec{v} + \vec{p}\right)$$

- Note that the t multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\underline{\vec{x}}_2(t)$ into the homogeneous system

$$\lambda e^{\lambda t}\left(t\vec{v} + \vec{p}\right) + e^{\lambda t}\vec{v} = \left(e^{\lambda t}A\right) \cdot \left(t\vec{v} + \vec{p}\right)$$

- Use the fact that $\vec{v}$ is an eigenvector of A

$$\lambda e^{\lambda t}\left(t\vec{v} + \vec{p}\right) + e^{\lambda t}\vec{v} = e^{\lambda t}\left(\lambda t\vec{v} + A \cdot \vec{p}\right)$$

- Simplify equation

$$\lambda\vec{p} + \vec{v} = A \cdot \vec{p}$$

- Make use of the identity matrix I

$$(\lambda \cdot I) \cdot \vec{p} + \vec{v} = A \cdot \vec{p}$$

- Condition $\vec{p}$ must meet for $\underline{\vec{x}}_2(t)$ to be a solution to the homogeneous system

$$(A - \lambda \cdot I) \cdot \vec{p} = \vec{v}$$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue 3

$$\left(\begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix} - 3 \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \cdot \vec{p} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

- Choice of $\vec{p}$

$$\overrightarrow{p} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue 3

$$\overrightarrow{x\_\_\_}_{2}(t) = e^{3t} \cdot \left(t \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \right)$$

- Consider eigenpair

$$\begin{bmatrix} 6, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x\_\_\_}_{3} = e^{6t} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\overrightarrow{x\_\_\_} = c_1 \overrightarrow{x\_\_\_}_{1}(t) + c_2 \overrightarrow{x\_\_\_}_{2}(t) + c_3 \overrightarrow{x\_\_\_}_{3}$$

- Substitute solutions into the general solution

$$\overrightarrow{x\_\_\_} = c_1 e^{3t} \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_2 e^{3t} \cdot \left(t \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \right) + c_3 e^{6t} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} ((-t-1)\, c_2 - c_1)\, e^{3t} + c_3 e^{6t} \\ c_3 e^{6t} \\ (c_2 t + c_1)\, e^{3t} + c_3 e^{6t} \end{bmatrix}$$

- Solution to the system of ODEs

$$\{x_1(t) = ((-t-1)\, c_2 - c_1)\, e^{3t} + c_3 e^{6t}, x_2(t) = c_3 e^{6t}, x_3(t) = (c_2 t + c_1)\, e^{3t} + c_3 e^{6t}\}$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 64

```
dsolve([diff(x__1(t),t)=4*x__1(t)+1*x__2(t)+1*x__3(t),diff(x__2(t),t)=1*x__1(t)+4*x__2(t)+1*
```

$$x_1(t) = c_2 e^{3t} + c_3 e^{6t}$$
$$x_2(t) = c_2 e^{3t} + c_3 e^{6t} + c_1 e^{3t}$$
$$x_3(t) = -2c_2 e^{3t} + c_3 e^{6t} - c_1 e^{3t}$$

✓ Solution by Mathematica

Time used: 0.008 (sec). Leaf size: 124

```
DSolve[{x1'[t]==4*x1[t]+1*x2[t]+1*x3[t],x2'[t]==1*x1[t]+4*x2[t]+1*x3[t],x3'[t]==1*x1[t]+1*x2
```

$$\text{x1}(t) \to \frac{1}{3}e^{3t}\left(c_1\left(e^{3t}+2\right)+\left(c_2+c_3\right)\left(e^{3t}-1\right)\right)$$
$$\text{x2}(t) \to \frac{1}{3}e^{3t}\left(c_1\left(e^{3t}-1\right)+c_2\left(e^{3t}+2\right)+c_3\left(e^{3t}-1\right)\right)$$
$$\text{x3}(t) \to \frac{1}{3}e^{3t}\left(c_1\left(e^{3t}-1\right)+c_2\left(e^{3t}-1\right)+c_3\left(e^{3t}+2\right)\right)$$

4.20 problem problem 20

Internal problem ID [334]
Internal file name [`OUTPUT/334_Sunday_June_05_2022_01_38_58_AM_59671171/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.3, The eigenvalue method for linear systems. Page 395
Problem number: problem 20.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= 5x_1(t) + x_2(t) + 3x_3(t) \\
x_2'(t) &= x_1(t) + 7x_2(t) + x_3(t) \\
x_3'(t) &= 3x_1(t) + x_2(t) + 5x_3(t)
\end{aligned}
$$

4.20.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 5 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} \frac{e^{2t}}{2} + \frac{e^{6t}}{6} + \frac{e^{9t}}{3} & \frac{e^{9t}}{3} - \frac{e^{6t}}{3} & \frac{e^{9t}}{3} + \frac{e^{6t}}{6} - \frac{e^{2t}}{2} \\ \frac{e^{9t}}{3} - \frac{e^{6t}}{3} & \frac{e^{9t}}{3} + \frac{2\,e^{6t}}{3} & \frac{e^{9t}}{3} - \frac{e^{6t}}{3} \\ \frac{e^{9t}}{3} + \frac{e^{6t}}{6} - \frac{e^{2t}}{2} & \frac{e^{9t}}{3} - \frac{e^{6t}}{3} & \frac{e^{2t}}{2} + \frac{e^{6t}}{6} + \frac{e^{9t}}{3} \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{e^{2t}}{2}+\frac{e^{6t}}{6}+\frac{e^{9t}}{3} & \frac{e^{9t}}{3}-\frac{e^{6t}}{3} & \frac{e^{9t}}{3}+\frac{e^{6t}}{6}-\frac{e^{2t}}{2} \\ \frac{e^{9t}}{3}-\frac{e^{6t}}{3} & \frac{e^{9t}}{3}+\frac{2\,e^{6t}}{3} & \frac{e^{9t}}{3}-\frac{e^{6t}}{3} \\ \frac{e^{9t}}{3}+\frac{e^{6t}}{6}-\frac{e^{2t}}{2} & \frac{e^{9t}}{3}-\frac{e^{6t}}{3} & \frac{e^{2t}}{2}+\frac{e^{6t}}{6}+\frac{e^{9t}}{3} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} \left(\frac{e^{2t}}{2}+\frac{e^{6t}}{6}+\frac{e^{9t}}{3}\right)c_1+\left(\frac{e^{9t}}{3}-\frac{e^{6t}}{3}\right)c_2+\left(\frac{e^{9t}}{3}+\frac{e^{6t}}{6}-\frac{e^{2t}}{2}\right)c_3 \\ \left(\frac{e^{9t}}{3}-\frac{e^{6t}}{3}\right)c_1+\left(\frac{e^{9t}}{3}+\frac{2\,e^{6t}}{3}\right)c_2+\left(\frac{e^{9t}}{3}-\frac{e^{6t}}{3}\right)c_3 \\ \left(\frac{e^{9t}}{3}+\frac{e^{6t}}{6}-\frac{e^{2t}}{2}\right)c_1+\left(\frac{e^{9t}}{3}-\frac{e^{6t}}{3}\right)c_2+\left(\frac{e^{2t}}{2}+\frac{e^{6t}}{6}+\frac{e^{9t}}{3}\right)c_3 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(c_1-2c_2+c_3)e^{6t}}{6}+\frac{(2c_1+2c_2+2c_3)e^{9t}}{6}+\frac{e^{2t}(c_1-c_3)}{2} \\ \frac{(-c_1+2c_2-c_3)e^{6t}}{3}+\frac{(c_1+c_2+c_3)e^{9t}}{3} \\ \frac{(c_1-2c_2+c_3)e^{6t}}{6}+\frac{(2c_1+2c_2+2c_3)e^{9t}}{6}-\frac{e^{2t}(c_1-c_3)}{2} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

4.20.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 5 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A-\lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 5 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 5-\lambda & 1 & 3 \\ 1 & 7-\lambda & 1 \\ 3 & 1 & 5-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 17\lambda^2 + 84\lambda - 108 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 6$$
$$\lambda_2 = 2$$
$$\lambda_3 = 9$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
2	1	real eigenvalue
6	1	real eigenvalue
9	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 5 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 5 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 3 & 1 & 3 & 0 \\ 1 & 5 & 1 & 0 \\ 3 & 1 & 3 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{3} \Longrightarrow \left[\begin{array}{ccc|c} 3 & 1 & 3 & 0 \\ 0 & \frac{14}{3} & 0 & 0 \\ 3 & 1 & 3 & 0 \end{array}\right]$$

$$R_3 = R_3 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} 3 & 1 & 3 & 0 \\ 0 & \frac{14}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3 & 1 & 3 \\ 0 & \frac{14}{3} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t, v_2 = 0\}$

Hence the solution is

$$\begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} -t \\ 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 6$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 5 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 5 \end{bmatrix} - (6)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -1 & 1 & 3 \\ 1 & 1 & 1 \\ 3 & 1 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -1 & 1 & 3 & 0 \\ 1 & 1 & 1 & 0 \\ 3 & 1 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \implies \left[\begin{array}{ccc|c} -1 & 1 & 3 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 1 & -1 & 0 \end{array}\right]$$

$$R_3 = R_3 + 3R_1 \implies \left[\begin{array}{ccc|c} -1 & 1 & 3 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 4 & 8 & 0 \end{array}\right]$$

$$R_3 = R_3 - 2R_2 \implies \left[\begin{array}{ccc|c} -1 & 1 & 3 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 1 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = -2t\}$

Hence the solution is

$$\begin{bmatrix} t \\ -2t \\ t \end{bmatrix} = \begin{bmatrix} t \\ -2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ -2t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ -2t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = 9$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 5 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 5 \end{bmatrix} - (9) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 1 & 3 \\ 1 & -2 & 1 \\ 3 & 1 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -4 & 1 & 3 & 0 \\ 1 & -2 & 1 & 0 \\ 3 & 1 & -4 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{4} \Longrightarrow \left[\begin{array}{ccc|c} -4 & 1 & 3 & 0 \\ 0 & -\frac{7}{4} & \frac{7}{4} & 0 \\ 3 & 1 & -4 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{3R_1}{4} \Longrightarrow \left[\begin{array}{ccc|c} -4 & 1 & 3 & 0 \\ 0 & -\frac{7}{4} & \frac{7}{4} & 0 \\ 0 & \frac{7}{4} & -\frac{7}{4} & 0 \end{array}\right]$$

$$R_3 = R_3 + R_2 \Longrightarrow \left[\begin{array}{ccc|c} -4 & 1 & 3 & 0 \\ 0 & -\frac{7}{4} & \frac{7}{4} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -4 & 1 & 3 \\ 0 & -\frac{7}{4} & \frac{7}{4} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
6	1	1	No	$\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$
2	1	1	No	$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$
9	1	1	No	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 6 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{6t} \\ &= \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} e^{6t} \end{aligned}$$

Since eigenvalue 2 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{2t} \\ &= \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{2t}\end{aligned}$$

Since eigenvalue 9 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_3(t) &= \vec{v}_3 e^{9t} \\ &= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{9t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 \begin{bmatrix} e^{6t} \\ -2\, e^{6t} \\ e^{6t} \end{bmatrix} + c_2 \begin{bmatrix} -e^{2t} \\ 0 \\ e^{2t} \end{bmatrix} + c_3 \begin{bmatrix} e^{9t} \\ e^{9t} \\ e^{9t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{6t} - c_2 e^{2t} + c_3 e^{9t} \\ -2c_1 e^{6t} + c_3 e^{9t} \\ c_1 e^{6t} + c_2 e^{2t} + c_3 e^{9t} \end{bmatrix}$$

4.20.3 Maple step by step solution

Let's solve

$[x_1'(t) = 5x_1(t) + x_2(t) + 3x_3(t), x_2'(t) = x_1(t) + 7x_2(t) + x_3(t), x_3'(t) = 3x_1(t) + x_2(t) + 5x_3(t)]$

- Define vector

$$\underline{\vec{x}}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{\vec{x}}'(t) = \begin{bmatrix} 5 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 5 \end{bmatrix} \cdot \underline{\vec{x}}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{\vec{x}}'(t) = \begin{bmatrix} 5 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 5 \end{bmatrix} \cdot \underline{\vec{x}}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 5 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 5 \end{bmatrix}$$

- Rewrite the system as

$$\underline{\vec{x}}'(t) = A \cdot \underline{\vec{x}}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[2, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right], \left[6, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\right], \left[9, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_1 = e^{2t} \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[6, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_2 = e^{6t} \cdot \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[9, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_3 = e^{9t} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\underline{\vec{x}} = c_1 \underline{\vec{x}}_1 + c_2 \underline{\vec{x}}_2 + c_3 \underline{\vec{x}}_3$$

- Substitute solutions into the general solution

$$\underline{\vec{x}} = c_1 e^{2t} \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_2 e^{6t} \cdot \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + c_3 e^{9t} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} -c_1 e^{2t} + c_2 e^{6t} + c_3 e^{9t} \\ -2c_2 e^{6t} + c_3 e^{9t} \\ c_1 e^{2t} + c_2 e^{6t} + c_3 e^{9t} \end{bmatrix}$$

- Solution to the system of ODEs

$$\{x_1(t) = -c_1 e^{2t} + c_2 e^{6t} + c_3 e^{9t}, x_2(t) = -2c_2 e^{6t} + c_3 e^{9t}, x_3(t) = c_1 e^{2t} + c_2 e^{6t} + c_3 e^{9t}\}$$

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 64

```
dsolve([diff(x__1(t),t)=5*x__1(t)+1*x__2(t)+3*x__3(t),diff(x__2(t),t)=1*x__1(t)+7*x__2(t)+1*
```

$$x_1(t) = e^{6t} c_1 + c_2 e^{9t} + c_3 e^{2t}$$
$$x_2(t) = -2\, e^{6t} c_1 + c_2 e^{9t}$$
$$x_3(t) = e^{6t} c_1 + c_2 e^{9t} - c_3 e^{2t}$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 163

```
DSolve[{x1'[t]==5*x1[t]+1*x2[t]+3*x3[t],x2'[t]==1*x1[t]+7*x2[t]+1*x3[t],x3'[t]==3*x1[t]+1*x2
```

$$\text{x1}(t) \to \frac{1}{6} e^{2t} \left(c_1 \left(e^{4t} + 2e^{7t} + 3\right) + (c_3 - 2c_2) e^{4t} + 2(c_2 + c_3) e^{7t} - 3c_3\right)$$
$$\text{x2}(t) \to \frac{1}{3} e^{6t} \left(c_1 \left(e^{3t} - 1\right) + c_2 \left(e^{3t} + 2\right) + c_3 \left(e^{3t} - 1\right)\right)$$
$$\text{x3}(t) \to \frac{1}{6} e^{2t} \left(c_1 \left(e^{4t} + 2e^{7t} - 3\right) + (c_3 - 2c_2) e^{4t} + 2(c_2 + c_3) e^{7t} + 3c_3\right)$$

4.21 problem problem 21

Internal problem ID [335]
Internal file name [`OUTPUT/335_Sunday_June_05_2022_01_39_00_AM_76937184/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.3, The eigenvalue method for linear systems. Page 395
Problem number: problem 21.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= 5x_1(t) - 6x_3(t) \\
x_2'(t) &= 2x_1(t) - x_2(t) - 2x_3(t) \\
x_3'(t) &= 4x_1(t) - 2x_2(t) - 4x_3(t)
\end{aligned}
$$

4.21.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 5 & 0 & -6 \\ 2 & -1 & -2 \\ 4 & -2 & -4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} -2\,\mathrm{e}^{-t} + 3\,\mathrm{e}^{t} & -12 + 6\,\mathrm{e}^{t} + 6\,\mathrm{e}^{-t} & -6\,\mathrm{e}^{t} + 6 \\ \mathrm{e}^{t} - \mathrm{e}^{-t} & 3\,\mathrm{e}^{-t} + 2\,\mathrm{e}^{t} - 4 & -2\,\mathrm{e}^{t} + 2 \\ -2\,\mathrm{e}^{-t} + 2\,\mathrm{e}^{t} & -10 + 6\,\mathrm{e}^{-t} + 4\,\mathrm{e}^{t} & -4\,\mathrm{e}^{t} + 5 \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} -2\,\mathrm{e}^{-t}+3\,\mathrm{e}^{t} & -12+6\,\mathrm{e}^{t}+6\,\mathrm{e}^{-t} & -6\,\mathrm{e}^{t}+6 \\ \mathrm{e}^{t}-\mathrm{e}^{-t} & 3\,\mathrm{e}^{-t}+2\,\mathrm{e}^{t}-4 & -2\,\mathrm{e}^{t}+2 \\ -2\,\mathrm{e}^{-t}+2\,\mathrm{e}^{t} & -10+6\,\mathrm{e}^{-t}+4\,\mathrm{e}^{t} & -4\,\mathrm{e}^{t}+5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} (-2\,\mathrm{e}^{-t}+3\,\mathrm{e}^{t})\,c_1+(-12+6\,\mathrm{e}^{t}+6\,\mathrm{e}^{-t})\,c_2+(-6\,\mathrm{e}^{t}+6)\,c_3 \\ (\mathrm{e}^{t}-\mathrm{e}^{-t})\,c_1+(3\,\mathrm{e}^{-t}+2\,\mathrm{e}^{t}-4)\,c_2+(-2\,\mathrm{e}^{t}+2)\,c_3 \\ (-2\,\mathrm{e}^{-t}+2\,\mathrm{e}^{t})\,c_1+(-10+6\,\mathrm{e}^{-t}+4\,\mathrm{e}^{t})\,c_2+(-4\,\mathrm{e}^{t}+5)\,c_3 \end{bmatrix} \\
&= \begin{bmatrix} (-2c_1+6c_2)\,\mathrm{e}^{-t}+(3c_1+6c_2-6c_3)\,\mathrm{e}^{t}-12c_2+6c_3 \\ (-c_1+3c_2)\,\mathrm{e}^{-t}+(c_1+2c_2-2c_3)\,\mathrm{e}^{t}-4c_2+2c_3 \\ (-2c_1+6c_2)\,\mathrm{e}^{-t}+(2c_1+4c_2-4c_3)\,\mathrm{e}^{t}-10c_2+5c_3 \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

4.21.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 5 & 0 & -6 \\ 2 & -1 & -2 \\ 4 & -2 & -4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A-\lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 5 & 0 & -6 \\ 2 & -1 & -2 \\ 4 & -2 & -4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 5-\lambda & 0 & -6 \\ 2 & -1-\lambda & -2 \\ 4 & -2 & -4-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - \lambda = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1$$
$$\lambda_2 = 1$$
$$\lambda_3 = 0$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
0	1	real eigenvalue
1	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 5 & 0 & -6 \\ 2 & -1 & -2 \\ 4 & -2 & -4 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 0 & -6 \\ 2 & 0 & -2 \\ 4 & -2 & -3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 6 & 0 & -6 & 0 \\ 2 & 0 & -2 & 0 \\ 4 & -2 & -3 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{3} \Longrightarrow \left[\begin{array}{ccc|c} 6 & 0 & -6 & 0 \\ 0 & 0 & 0 & 0 \\ 4 & -2 & -3 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{2R_1}{3} \Longrightarrow \left[\begin{array}{ccc|c} 6 & 0 & -6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 \end{array}\right]$$

Since the current pivot $A(2,2)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{ccc|c} 6 & 0 & -6 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 6 & 0 & -6 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = t, v_2 = \frac{t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} t \\ \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} t \\ \frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ \frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} t \\ \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 0$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 5 & 0 & -6 \\ 2 & -1 & -2 \\ 4 & -2 & -4 \end{bmatrix} - (0) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 0 & -6 \\ 2 & -1 & -2 \\ 4 & -2 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 5 & 0 & -6 & 0 \\ 2 & -1 & -2 & 0 \\ 4 & -2 & -4 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{2R_1}{5} \Longrightarrow \left[\begin{array}{ccc|c} 5 & 0 & -6 & 0 \\ 0 & -1 & \frac{2}{5} & 0 \\ 4 & -2 & -4 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{4R_1}{5} \Longrightarrow \left[\begin{array}{ccc|c} 5 & 0 & -6 & 0 \\ 0 & -1 & \frac{2}{5} & 0 \\ 0 & -2 & \frac{4}{5} & 0 \end{array}\right]$$

$$R_3 = R_3 - 2R_2 \Longrightarrow \left[\begin{array}{ccc|c} 5 & 0 & -6 & 0 \\ 0 & -1 & \frac{2}{5} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 5 & 0 & -6 \\ 0 & -1 & \frac{2}{5} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{6t}{5}, v_2 = \frac{2t}{5}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{6t}{5} \\ \frac{2t}{5} \\ t \end{bmatrix} = \begin{bmatrix} \frac{6t}{5} \\ \frac{2t}{5} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{6t}{5} \\ \frac{2t}{5} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{6}{5} \\ \frac{2}{5} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{6t}{5} \\ \frac{2t}{5} \\ t \end{bmatrix} = \begin{bmatrix} \frac{6}{5} \\ \frac{2}{5} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{6t}{5} \\ \frac{2t}{5} \\ t \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 5 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 5 & 0 & -6 \\ 2 & -1 & -2 \\ 4 & -2 & -4 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 0 & -6 \\ 2 & -2 & -2 \\ 4 & -2 & -5 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 4 & 0 & -6 & 0 \\ 2 & -2 & -2 & 0 \\ 4 & -2 & -5 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} 4 & 0 & -6 & 0 \\ 0 & -2 & 1 & 0 \\ 4 & -2 & -5 & 0 \end{array}\right]$$

$$R_3 = R_3 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} 4 & 0 & -6 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & -2 & 1 & 0 \end{array}\right]$$

$$R_3 = R_3 - R_2 \Longrightarrow \left[\begin{array}{ccc|c} 4 & 0 & -6 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 4 & 0 & -6 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{3t}{2}, v_2 = \frac{t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{3t}{2} \\ \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{3t}{2} \\ \frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{3t}{2} \\ \frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{3}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{3t}{2} \\ \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{3t}{2} \\ \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
-1	1	1	No	$\begin{bmatrix} 1 \\ \frac{1}{2} \\ 1 \end{bmatrix}$
1	1	1	No	$\begin{bmatrix} \frac{3}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$
0	1	1	No	$\begin{bmatrix} \frac{6}{5} \\ \frac{2}{5} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{-t} \\
&= \begin{bmatrix} 1 \\ \frac{1}{2} \\ 1 \end{bmatrix} e^{-t}
\end{aligned}
$$

Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{t} \\
&= \begin{bmatrix} \frac{3}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix} e^{t}
\end{aligned}
$$

Since eigenvalue 0 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_3(t) &= \vec{v}_3 e^0 \\ &= \begin{bmatrix}\frac{6}{5} \\ \frac{2}{5} \\ 1\end{bmatrix} e^0\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix}x_1(t) \\ x_2(t) \\ x_3(t)\end{bmatrix} = c_1\begin{bmatrix}\mathrm{e}^{-t} \\ \frac{\mathrm{e}^{-t}}{2} \\ \mathrm{e}^{-t}\end{bmatrix} + c_2\begin{bmatrix}\frac{3\,\mathrm{e}^{t}}{2} \\ \frac{\mathrm{e}^{t}}{2} \\ \mathrm{e}^{t}\end{bmatrix} + c_3\begin{bmatrix}\frac{6}{5} \\ \frac{2}{5} \\ 1\end{bmatrix}$$

Which becomes

$$\begin{bmatrix}x_1(t) \\ x_2(t) \\ x_3(t)\end{bmatrix} = \begin{bmatrix}c_1\mathrm{e}^{-t} + \frac{3c_2\mathrm{e}^{t}}{2} + \frac{6c_3}{5} \\ \frac{c_1\mathrm{e}^{-t}}{2} + \frac{c_2\mathrm{e}^{t}}{2} + \frac{2c_3}{5} \\ c_1\mathrm{e}^{-t} + c_2\mathrm{e}^{t} + c_3\end{bmatrix}$$

4.21.3 Maple step by step solution

Let's solve

$[x_1'(t) = 5x_1(t) - 6x_3(t), x_2'(t) = 2x_1(t) - x_2(t) - 2x_3(t), x_3'(t) = 4x_1(t) - 2x_2(t) - 4x_3(t)]$

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix}x_1(t) \\ x_2(t) \\ x_3(t)\end{bmatrix}$$

- Convert system into a vector equation

$$\underline{\overrightarrow{x}}'(t) = \begin{bmatrix} 5 & 0 & -6 \\ 2 & -1 & -2 \\ 4 & -2 & -4 \end{bmatrix} \cdot \underline{\overrightarrow{x}}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{\overrightarrow{x}}'(t) = \begin{bmatrix} 5 & 0 & -6 \\ 2 & -1 & -2 \\ 4 & -2 & -4 \end{bmatrix} \cdot \underline{\overrightarrow{x}}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 5 & 0 & -6 \\ 2 & -1 & -2 \\ 4 & -2 & -4 \end{bmatrix}$$

- Rewrite the system as

$\underline{\overrightarrow{x}}'(t) = A \cdot \underline{\overrightarrow{x}}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} 1 \\ \frac{1}{2} \\ 1 \end{bmatrix}\right], \left[0, \begin{bmatrix} \frac{6}{5} \\ \frac{2}{5} \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} \frac{3}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} 1 \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\overrightarrow{x}}_1 = \mathrm{e}^{-t} \cdot \begin{bmatrix} 1 \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[0, \left[\begin{array}{c}\frac{6}{5}\\ \frac{2}{5}\\ 1\end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_{2} = \left[\begin{array}{c}\frac{6}{5}\\ \frac{2}{5}\\ 1\end{array}\right]$$

- Consider eigenpair

$$\left[1, \left[\begin{array}{c}\frac{3}{2}\\ \frac{1}{2}\\ 1\end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_{3} = \mathrm{e}^{t} \cdot \left[\begin{array}{c}\frac{3}{2}\\ \frac{1}{2}\\ 1\end{array}\right]$$

- General solution to the system of ODEs

$$\underline{\vec{x}} = c_1 \underline{\vec{x}}_{1} + c_2 \underline{\vec{x}}_{2} + c_3 \underline{\vec{x}}_{3}$$

- Substitute solutions into the general solution

$$\underline{\vec{x}} = c_1 \mathrm{e}^{-t} \cdot \left[\begin{array}{c}1\\ \frac{1}{2}\\ 1\end{array}\right] + c_3 \mathrm{e}^{t} \cdot \left[\begin{array}{c}\frac{3}{2}\\ \frac{1}{2}\\ 1\end{array}\right] + \left[\begin{array}{c}\frac{6c_2}{5}\\ \frac{2c_2}{5}\\ c_2\end{array}\right]$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c}x_1(t)\\ x_2(t)\\ x_3(t)\end{array}\right] = \left[\begin{array}{c}c_1 \mathrm{e}^{-t} + \frac{3c_3 \mathrm{e}^{t}}{2} + \frac{6c_2}{5}\\ \frac{c_1 \mathrm{e}^{-t}}{2} + \frac{c_3 \mathrm{e}^{t}}{2} + \frac{2c_2}{5}\\ c_1 \mathrm{e}^{-t} + c_3 \mathrm{e}^{t} + c_2\end{array}\right]$$

- Solution to the system of ODEs

$$\left\{x_1(t) = c_1 \mathrm{e}^{-t} + \frac{3c_3 \mathrm{e}^{t}}{2} + \frac{6c_2}{5}, x_2(t) = \frac{c_1 \mathrm{e}^{-t}}{2} + \frac{c_3 \mathrm{e}^{t}}{2} + \frac{2c_2}{5}, x_3(t) = c_1 \mathrm{e}^{-t} + c_3 \mathrm{e}^{t} + c_2\right\}$$

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 54

```
dsolve([diff(x__1(t),t)=5*x__1(t)+0*x__2(t)-6*x__3(t),diff(x__2(t),t)=2*x__1(t)-1*x__2(t)-2*
```

$$x_1(t) = c_1 + c_2\mathrm{e}^{-t} + c_3\mathrm{e}^{t}$$
$$x_2(t) = \frac{c_2\mathrm{e}^{-t}}{2} + \frac{c_3\mathrm{e}^{t}}{3} + \frac{c_1}{3}$$
$$x_3(t) = c_2\mathrm{e}^{-t} + \frac{2c_3\mathrm{e}^{t}}{3} + \frac{5c_1}{6}$$

✓ Solution by Mathematica

Time used: 0.008 (sec). Leaf size: 139

```
DSolve[{x1'[t]==5*x1[t]+0*x2[t]-6*x3[t],x2'[t]==2*x1[t]-1*x2[t]-2*x3[t],x3'[t]==4*x1[t]-2*x2
```

$$\mathrm{x1}(t) \to e^{-t}\left(c_1\left(3e^{2t} - 2\right) + 6\left(e^t - 1\right)\left(c_2\left(e^t - 1\right) - c_3e^t\right)\right)$$
$$\mathrm{x2}(t) \to e^{-t}\left(c_1\left(e^{2t} - 1\right) + c_2\left(-4e^t + 2e^{2t} + 3\right) - 2c_3e^t\left(e^t - 1\right)\right)$$
$$\mathrm{x3}(t) \to -2(c_1 - 3c_2)e^{-t} + 2(c_1 + 2c_2 - 2c_3)e^t + 5(c_3 - 2c_2)$$

4.22 problem problem 22

Internal problem ID [336]
Internal file name [`OUTPUT/336_Sunday_June_05_2022_01_39_01_AM_64273904/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.3, The eigenvalue method for linear systems. Page 395
Problem number: problem 22.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= 3x_1(t) + 2x_2(t) + 2x_3(t) \\
x_2'(t) &= -5x_1(t) - 4x_2(t) - 2x_3(t) \\
x_3'(t) &= 5x_1(t) + 5x_2(t) + 3x_3(t)
\end{aligned}
$$

4.22.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 3 & 2 & 2 \\ -5 & -4 & -2 \\ 5 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{3t} & e^{3t} - e^{t} & e^{3t} - e^{t} \\ -(e^{5t} - 1)\,e^{-2t} & (-e^{5t} + e^{3t} + 1)\,e^{-2t} & -e^{3t} + e^{t} \\ (e^{5t} - 1)\,e^{-2t} & (e^{5t} - 1)\,e^{-2t} & e^{3t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{3t} & e^{3t}-e^{t} & e^{3t}-e^{t} \\ -(e^{5t}-1)\,e^{-2t} & (-e^{5t}+e^{3t}+1)\,e^{-2t} & -e^{3t}+e^{t} \\ (e^{5t}-1)\,e^{-2t} & (e^{5t}-1)\,e^{-2t} & e^{3t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} e^{3t}c_1 + (e^{3t}-e^{t})\,c_2 + (e^{3t}-e^{t})\,c_3 \\ -(e^{5t}-1)\,e^{-2t}c_1 + (-e^{5t}+e^{3t}+1)\,e^{-2t}c_2 + (-e^{3t}+e^{t})\,c_3 \\ (e^{5t}-1)\,e^{-2t}c_1 + (e^{5t}-1)\,e^{-2t}c_2 + e^{3t}c_3 \end{bmatrix} \\
&= \begin{bmatrix} (c_1+c_2+c_3)\,e^{3t} - e^{t}(c_2+c_3) \\ -((c_1+c_2+c_3)\,e^{5t} + (-c_2-c_3)\,e^{3t} - c_1 - c_2)\,e^{-2t} \\ ((c_1+c_2+c_3)\,e^{5t} - c_1 - c_2)\,e^{-2t} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

4.22.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 3 & 2 & 2 \\ -5 & -4 & -2 \\ 5 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 3 & 2 & 2 \\ -5 & -4 & -2 \\ 5 & 5 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 3-\lambda & 2 & 2 \\ -5 & -4-\lambda & -2 \\ 5 & 5 & 3-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 1$$
$$\lambda_2 = 3$$
$$\lambda_3 = -2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-2	1	real eigenvalue
1	1	real eigenvalue
3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & 2 & 2 \\ -5 & -4 & -2 \\ 5 & 5 & 3 \end{bmatrix} - (-2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 2 & 2 \\ -5 & -2 & -2 \\ 5 & 5 & 5 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 5 & 2 & 2 & 0 \\ -5 & -2 & -2 & 0 \\ 5 & 5 & 5 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \implies \left[\begin{array}{ccc|c} 5 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 5 & 5 & 5 & 0 \end{array}\right]$$

$$R_3 = R_3 - R_1 \implies \left[\begin{array}{ccc|c} 5 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 3 & 0 \end{array}\right]$$

Since the current pivot $A(2,2)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{ccc|c} 5 & 2 & 2 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 5 & 2 & 2 \\ 0 & 3 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_2 = -t\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ -t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ -t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & 2 & 2 \\ -5 & -4 & -2 \\ 5 & 5 & 3 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 & 2 \\ -5 & -5 & -2 \\ 5 & 5 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ -5 & -5 & -2 & 0 \\ 5 & 5 & 2 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{5R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ 0 & 0 & 3 & 0 \\ 5 & 5 & 2 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{5R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & -3 & 0 \end{array}\right]$$

$$R_3 = R_3 + R_2 \Longrightarrow \left[\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & 2 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1, v_3\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} -t \\ t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = 3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & 2 & 2 \\ -5 & -4 & -2 \\ 5 & 5 & 3 \end{bmatrix} - (3) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 2 & 2 \\ -5 & -7 & -2 \\ 5 & 5 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & 2 & 2 & 0 \\ -5 & -7 & -2 & 0 \\ 5 & 5 & 0 & 0 \end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{ccc|c} -5 & -7 & -2 & 0 \\ 0 & 2 & 2 & 0 \\ 5 & 5 & 0 & 0 \end{array}\right]$$

$$R_3 = R_3 + R_1 \Longrightarrow \left[\begin{array}{ccc|c} -5 & -7 & -2 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & -2 & -2 & 0 \end{array}\right]$$

$$R_3 = R_3 + R_2 \Longrightarrow \left[\begin{array}{ccc|c} -5 & -7 & -2 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -5 & -7 & -2 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = -t\}$

Hence the solution is

$$\begin{bmatrix} t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} t \\ -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
1	1	1	No	$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$
3	1	1	No	$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$
-2	1	1	No	$\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{t}\\ &= \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{t}\end{aligned}$$

Since eigenvalue 3 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{3t}\\ &= \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^{3t}\end{aligned}$$

Since eigenvalue -2 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_3(t) &= \vec{v}_3 e^{-2t}\\ &= \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} e^{-2t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 \begin{bmatrix} -\mathrm{e}^{t} \\ \mathrm{e}^{t} \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{3t} \\ -\mathrm{e}^{3t} \\ \mathrm{e}^{3t} \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ -\mathrm{e}^{-2t} \\ \mathrm{e}^{-2t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} -c_1 \mathrm{e}^{t} + c_2 \mathrm{e}^{3t} \\ \left(-c_2 \mathrm{e}^{5t} + c_1 \mathrm{e}^{3t} - c_3\right) \mathrm{e}^{-2t} \\ \left(c_2 \mathrm{e}^{5t} + c_3\right) \mathrm{e}^{-2t} \end{bmatrix}$$

4.22.3 Maple step by step solution

Let's solve

$[x_1'(t) = 3x_1(t) + 2x_2(t) + 2x_3(t), x_2'(t) = -5x_1(t) - 4x_2(t) - 2x_3(t), x_3'(t) = 5x_1(t) + 5x_2(t) + 3x$

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x}^{\rightarrow}{}'(t) = \begin{bmatrix} 3 & 2 & 2 \\ -5 & -4 & -2 \\ 5 & 5 & 3 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{x}^{\rightarrow}{}'(t) = \begin{bmatrix} 3 & 2 & 2 \\ -5 & -4 & -2 \\ 5 & 5 & 3 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 3 & 2 & 2 \\ -5 & -4 & -2 \\ 5 & 5 & 3 \end{bmatrix}$$

- Rewrite the system as

$\underline{x}^{\rightarrow}{}'(t) = A \cdot \underline{x}^{\rightarrow}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right], \left[3, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-2, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_{1} = e^{-2t} \cdot \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_{2} = e^{t} \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\left[3, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_{3} = e^{3t} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\underline{\vec{x}} = c_1 \underline{\vec{x}}_{1} + c_2 \underline{\vec{x}}_{2} + c_3 \underline{\vec{x}}_{3}$$

- Substitute solutions into the general solution

$$\underline{\overrightarrow{x}} = c_1 e^{-2t} \cdot \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + c_2 e^{t} \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{3t} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} -c_2 e^{t} + c_3 e^{3t} \\ -(c_3 e^{5t} - c_2 e^{3t} + c_1) e^{-2t} \\ (c_3 e^{5t} + c_1) e^{-2t} \end{bmatrix}$$

- Solution to the system of ODEs

$$\{x_1(t) = -c_2 e^{t} + c_3 e^{3t}, x_2(t) = -(c_3 e^{5t} - c_2 e^{3t} + c_1) e^{-2t}, x_3(t) = (c_3 e^{5t} + c_1) e^{-2t}\}$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 55

```
dsolve([diff(x__1(t),t)=3*x__1(t)+2*x__2(t)+2*x__3(t),diff(x__2(t),t)=-5*x__1(t)-4*x__2(t)-2
```

$$x_1(t) = c_2 e^{3t} + c_3 e^{t}$$
$$x_2(t) = -c_2 e^{3t} - c_3 e^{t} + c_1 e^{-2t}$$
$$x_3(t) = c_2 e^{3t} - c_1 e^{-2t}$$

✓ Solution by Mathematica

Time used: 0.008 (sec). Leaf size: 123

```
DSolve[{x1'[t]==3*x1[t]+2*x2[t]+2*x3[t],x2'[t]==-5*x1[t]-4*x2[t]-2*x3[t],x3'[t]==5*x1[t]+5*x
```

$$\text{x1}(t) \to e^{t}\left((c_1 + c_2 + c_3)e^{2t} - c_2 - c_3\right)$$
$$\text{x2}(t) \to e^{-2t}\left(-\left(c_1\left(e^{5t} - 1\right)\right) + c_2\left(e^{3t} - e^{5t} + 1\right) - c_3 e^{3t}\left(e^{2t} - 1\right)\right)$$
$$\text{x3}(t) \to e^{-2t}\left(c_1\left(e^{5t} - 1\right) + c_2\left(e^{5t} - 1\right) + c_3 e^{5t}\right)$$

4.23 problem problem 23

Internal problem ID [337]
Internal file name [`OUTPUT/337_Sunday_June_05_2022_01_39_03_AM_51136371/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.3, The eigenvalue method for linear systems. Page 395
Problem number: problem 23.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= 3x_1(t) + x_2(t) + x_3(t) \\
x_2'(t) &= -5x_1(t) - 3x_2(t) - x_3(t) \\
x_3'(t) &= 5x_1(t) + 5x_2(t) + 3x_3(t)
\end{aligned}
$$

4.23.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 3 & 1 & 1 \\ -5 & -3 & -1 \\ 5 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{3t} & e^{3t} - e^{2t} & e^{3t} - e^{2t} \\ -(e^{5t} - 1)\,e^{-2t} & (-e^{5t} + e^{4t} + 1)\,e^{-2t} & -e^{3t} + e^{2t} \\ (e^{5t} - 1)\,e^{-2t} & (e^{5t} - 1)\,e^{-2t} & e^{3t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{3t} & e^{3t}-e^{2t} & e^{3t}-e^{2t} \\ -(e^{5t}-1)\,e^{-2t} & (-e^{5t}+e^{4t}+1)\,e^{-2t} & -e^{3t}+e^{2t} \\ (e^{5t}-1)\,e^{-2t} & (e^{5t}-1)\,e^{-2t} & e^{3t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} e^{3t}c_1 + (e^{3t}-e^{2t})\,c_2 + (e^{3t}-e^{2t})\,c_3 \\ -(e^{5t}-1)\,e^{-2t}c_1 + (-e^{5t}+e^{4t}+1)\,e^{-2t}c_2 + (-e^{3t}+e^{2t})\,c_3 \\ (e^{5t}-1)\,e^{-2t}c_1 + (e^{5t}-1)\,e^{-2t}c_2 + e^{3t}c_3 \end{bmatrix} \\
&= \begin{bmatrix} (c_1+c_2+c_3)\,e^{3t} - e^{2t}(c_2+c_3) \\ -((c_1+c_2+c_3)\,e^{5t} + (-c_2-c_3)\,e^{4t} - c_1 - c_2)\,e^{-2t} \\ ((c_1+c_2+c_3)\,e^{5t} - c_1 - c_2)\,e^{-2t} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

4.23.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 3 & 1 & 1 \\ -5 & -3 & -1 \\ 5 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 3 & 1 & 1 \\ -5 & -3 & -1 \\ 5 & 5 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 3-\lambda & 1 & 1 \\ -5 & -3-\lambda & -1 \\ 5 & 5 & 3-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 3\lambda^2 - 4\lambda + 12 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -2$$
$$\lambda_2 = 2$$
$$\lambda_3 = 3$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-2	1	real eigenvalue
2	1	real eigenvalue
3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & 1 & 1 \\ -5 & -3 & -1 \\ 5 & 5 & 3 \end{bmatrix} - (-2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 1 & 1 \\ -5 & -1 & -1 \\ 5 & 5 & 5 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 5 & 1 & 1 & 0 \\ -5 & -1 & -1 & 0 \\ 5 & 5 & 5 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \implies \left[\begin{array}{ccc|c} 5 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 5 & 5 & 5 & 0 \end{array}\right]$$

$$R_3 = R_3 - R_1 \implies \left[\begin{array}{ccc|c} 5 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 4 & 4 & 0 \end{array}\right]$$

Since the current pivot $A(2,2)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{ccc|c} 5 & 1 & 1 & 0 \\ 0 & 4 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 5 & 1 & 1 \\ 0 & 4 & 4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_2 = -t\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ -t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Let $t=1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ -t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & 1 & 1 \\ -5 & -3 & -1 \\ 5 & 5 & 3 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ -5 & -5 & -1 \\ 5 & 5 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ -5 & -5 & -1 & 0 \\ 5 & 5 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 + 5R_1 \implies \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 4 & 0 \\ 5 & 5 & 1 & 0 \end{array}\right]$$

$$R_3 = R_3 - 5R_1 \implies \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & -4 & 0 \end{array}\right]$$

$$R_3 = R_3 + R_2 \implies \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1, v_3\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} -t \\ t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = 3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & 1 & 1 \\ -5 & -3 & -1 \\ 5 & 5 & 3 \end{bmatrix} - (3) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 1 \\ -5 & -6 & -1 \\ 5 & 5 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ -5 & -6 & -1 & 0 \\ 5 & 5 & 0 & 0 \end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{ccc|c} -5 & -6 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 5 & 5 & 0 & 0 \end{array}\right]$$

$$R_3 = R_3 + R_1 \Longrightarrow \left[\begin{array}{ccc|c} -5 & -6 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right]$$

$$R_3 = R_3 + R_2 \Longrightarrow \left[\begin{array}{ccc|c} -5 & -6 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -5 & -6 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = -t\}$

Hence the solution is

$$\begin{bmatrix} t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} t \\ -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-2	1	1	No	$\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$
2	1	1	No	$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$
3	1	1	No	$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -2 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{-2t} \\ &= \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} e^{-2t}\end{aligned}$$

Since eigenvalue 2 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{2t} \\ &= \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{2t}\end{aligned}$$

Since eigenvalue 3 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_3(t) &= \vec{v}_3 e^{3t} \\ &= \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^{3t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ -\mathrm{e}^{-2t} \\ \mathrm{e}^{-2t} \end{bmatrix} + c_2 \begin{bmatrix} -\mathrm{e}^{2t} \\ \mathrm{e}^{2t} \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} \mathrm{e}^{3t} \\ -\mathrm{e}^{3t} \\ \mathrm{e}^{3t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} -c_2 \mathrm{e}^{2t} + c_3 \mathrm{e}^{3t} \\ -(c_3 \mathrm{e}^{5t} - c_2 \mathrm{e}^{4t} + c_1)\, \mathrm{e}^{-2t} \\ (c_3 \mathrm{e}^{5t} + c_1)\, \mathrm{e}^{-2t} \end{bmatrix}$$

4.23.3 Maple step by step solution

Let's solve

$[x_1'(t) = 3x_1(t) + x_2(t) + x_3(t), x_2'(t) = -5x_1(t) - 3x_2(t) - x_3(t), x_3'(t) = 5x_1(t) + 5x_2(t) + 3x_3(t$

- Define vector

$$\underline{\vec{x}}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{\vec{x}}'(t) = \begin{bmatrix} 3 & 1 & 1 \\ -5 & -3 & -1 \\ 5 & 5 & 3 \end{bmatrix} \cdot \underline{\vec{x}}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{\vec{x}}'(t) = \begin{bmatrix} 3 & 1 & 1 \\ -5 & -3 & -1 \\ 5 & 5 & 3 \end{bmatrix} \cdot \underline{\vec{x}}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 3 & 1 & 1 \\ -5 & -3 & -1 \\ 5 & 5 & 3 \end{bmatrix}$$

- Rewrite the system as

$\underline{\vec{x}}'(t) = A \cdot \underline{\vec{x}}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right], \left[3, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-2, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_{1} = \mathrm{e}^{-2t} \cdot \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_{2} = \mathrm{e}^{2t} \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\left[3, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_{3} = \mathrm{e}^{3t} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$\underline{\vec{x}} = c_1 \underline{\vec{x}}_{1} + c_2 \underline{\vec{x}}_{2} + c_3 \underline{\vec{x}}_{3}$

- Substitute solutions into the general solution

$$\underline{\vec{x}} = c_1 e^{-2t} \cdot \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + c_2 e^{2t} \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{3t} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} -c_2 e^{2t} + c_3 e^{3t} \\ -(c_3 e^{5t} - c_2 e^{4t} + c_1) e^{-2t} \\ (c_3 e^{5t} + c_1) e^{-2t} \end{bmatrix}$$

- Solution to the system of ODEs

$$\{x_1(t) = -c_2 e^{2t} + c_3 e^{3t}, x_2(t) = -(c_3 e^{5t} - c_2 e^{4t} + c_1) e^{-2t}, x_3(t) = (c_3 e^{5t} + c_1) e^{-2t}\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 59

```
dsolve([diff(x__1(t),t)=3*x__1(t)+1*x__2(t)+1*x__3(t),diff(x__2(t),t)=-5*x__1(t)-3*x__2(t)-1
```

$$\begin{aligned} x_1(t) &= c_2 e^{3t} + c_3 e^{2t} \\ x_2(t) &= -c_2 e^{3t} - c_3 e^{2t} + c_1 e^{-2t} \\ x_3(t) &= c_2 e^{3t} - c_1 e^{-2t} \end{aligned}$$

✓ Solution by Mathematica

Time used: 0.008 (sec). Leaf size: 121

```
DSolve[{x1'[t]==3*x1[t]+1*x2[t]+1*x3[t],x2'[t]==-5*x1[t]-3*x2[t]-1*x3[t],x3'[t]==5*x1[t]+5*x
```

$$\begin{aligned} \text{x1}(t) &\to e^{2t}((c_1 + c_2 + c_3)e^t - c_2 - c_3) \\ \text{x2}(t) &\to e^{-2t}(-(c_1(e^{5t} - 1)) + c_2(e^{4t} - e^{5t} + 1) - c_3 e^{4t}(e^t - 1)) \\ \text{x3}(t) &\to e^{-2t}(c_1(e^{5t} - 1) + c_2(e^{5t} - 1) + c_3 e^{5t}) \end{aligned}$$

4.24 problem problem 24

Internal problem ID [338]
Internal file name [`OUTPUT/338_Sunday_June_05_2022_01_39_04_AM_78020164/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.3, The eigenvalue method for linear systems. Page 395
Problem number: problem 24.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= 2x_1(t) + x_2(t) - x_3(t)\\
x_2'(t) &= -4x_1(t) - 3x_2(t) - x_3(t)\\
x_3'(t) &= 4x_1(t) + 4x_2(t) + 2x_3(t)
\end{aligned}
$$

4.24.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 2 & 1 & -1 \\ -4 & -3 & -1 \\ 4 & 4 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \cos(2t) + \sin(2t) & -\mathrm{e}^{t} + \cos(2t) + \sin(2t) & -\mathrm{e}^{t} + \cos(2t) \\ -2\sin(2t) & \mathrm{e}^{t} - 2\sin(2t) & \mathrm{e}^{t} - \cos(2t) - \sin(2t) \\ 2\sin(2t) & 2\sin(2t) & \cos(2t) + \sin(2t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \cos(2t)+\sin(2t) & -\mathrm{e}^t+\cos(2t)+\sin(2t) & -\mathrm{e}^t+\cos(2t) \\ -2\sin(2t) & \mathrm{e}^t-2\sin(2t) & \mathrm{e}^t-\cos(2t)-\sin(2t) \\ 2\sin(2t) & 2\sin(2t) & \cos(2t)+\sin(2t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} (\cos(2t)+\sin(2t))\,c_1+(-\mathrm{e}^t+\cos(2t)+\sin(2t))\,c_2+(-\mathrm{e}^t+\cos(2t))\,c_3 \\ -2\sin(2t)\,c_1+(\mathrm{e}^t-2\sin(2t))\,c_2+(\mathrm{e}^t-\cos(2t)-\sin(2t))\,c_3 \\ 2\sin(2t)\,c_1+2\sin(2t)\,c_2+(\cos(2t)+\sin(2t))\,c_3 \end{bmatrix} \\
&= \begin{bmatrix} (c_1+c_2+c_3)\cos(2t)+(c_1+c_2)\sin(2t)-\mathrm{e}^t(c_2+c_3) \\ (-2c_1-2c_2-c_3)\sin(2t)-c_3\cos(2t)+\mathrm{e}^t(c_2+c_3) \\ (2c_1+2c_2+c_3)\sin(2t)+c_3\cos(2t) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

4.24.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 2 & 1 & -1 \\ -4 & -3 & -1 \\ 4 & 4 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A-\lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 2 & 1 & -1 \\ -4 & -3 & -1 \\ 4 & 4 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 2-\lambda & 1 & -1 \\ -4 & -3-\lambda & -1 \\ 4 & 4 & 2-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - \lambda^2 + 4\lambda - 4 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2i$$
$$\lambda_2 = -2i$$
$$\lambda_3 = 1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue
$2i$	1	complex eigenvalue
$-2i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 1 & -1 \\ -4 & -3 & -1 \\ 4 & 4 & 2 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -1 \\ -4 & -4 & -1 \\ 4 & 4 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ -4 & -4 & -1 & 0 \\ 4 & 4 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 + 4R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 0 & -5 & 0 \\ 4 & 4 & 1 & 0 \end{array}\right]$$

$$R_3 = R_3 - 4R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 5 & 0 \end{array}\right]$$

$$R_3 = R_3 + R_2 \Longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & -5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1, v_3\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} -t \\ t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -2i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 1 & -1 \\ -4 & -3 & -1 \\ 4 & 4 & 2 \end{bmatrix} - (-2i)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2+2i & 1 & -1 \\ -4 & -3+2i & -1 \\ 4 & 4 & 2+2i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 2+2i & 1 & -1 & 0 \\ -4 & -3+2i & -1 & 0 \\ 4 & 4 & 2+2i & 0 \end{array}\right]$$

$$R_2 = R_2 + (1-i)\,R_1 \implies \left[\begin{array}{ccc|c} 2+2i & 1 & -1 & 0 \\ 0 & -2+i & -2+i & 0 \\ 4 & 4 & 2+2i & 0 \end{array}\right]$$

$$R_3 = R_3 + (-1+i)\,R_1 \implies \left[\begin{array}{ccc|c} 2+2i & 1 & -1 & 0 \\ 0 & -2+i & -2+i & 0 \\ 0 & 3+i & 3+i & 0 \end{array}\right]$$

$$R_3 = R_3 + (1+i)\,R_2 \implies \left[\begin{array}{ccc|c} 2+2i & 1 & -1 & 0 \\ 0 & -2+i & -2+i & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2+2i & 1 & -1 \\ 0 & -2+i & -2+i \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(\frac{1}{2} - \frac{i}{2}\right)t, v_2 = -t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(\frac{1}{2} - \frac{I}{2}\right)t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2} - \frac{i}{2}\right)t \\ -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(\frac{1}{2} - \frac{I}{2}\right)t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} - \frac{i}{2} \\ -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(\frac{1}{2} - \frac{I}{2}\right)t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - \frac{i}{2} \\ -1 \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(\frac{1}{2} - \frac{I}{2}\right)t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} 1 - i \\ -2 \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = 2i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 1 & -1 \\ -4 & -3 & -1 \\ 4 & 4 & 2 \end{bmatrix} - (2i) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 - 2i & 1 & -1 \\ -4 & -3 - 2i & -1 \\ 4 & 4 & 2 - 2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 2-2i & 1 & -1 & 0 \\ -4 & -3-2i & -1 & 0 \\ 4 & 4 & 2-2i & 0 \end{array}\right]$$

$$R_2 = R_2 + (1+i)\,R_1 \Longrightarrow \left[\begin{array}{ccc|c} 2-2i & 1 & -1 & 0 \\ 0 & -2-i & -2-i & 0 \\ 4 & 4 & 2-2i & 0 \end{array}\right]$$

$$R_3 = R_3 + (-1-i)\,R_1 \Longrightarrow \left[\begin{array}{ccc|c} 2-2i & 1 & -1 & 0 \\ 0 & -2-i & -2-i & 0 \\ 0 & 3-i & 3-i & 0 \end{array}\right]$$

$$R_3 = R_3 + (1-i)\,R_2 \Longrightarrow \left[\begin{array}{ccc|c} 2-2i & 1 & -1 & 0 \\ 0 & -2-i & -2-i & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2-2i & 1 & -1 \\ 0 & -2-i & -2-i \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(\frac{1}{2} + \frac{i}{2}\right)t, v_2 = -t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(\frac{1}{2} + \frac{I}{2}\right)t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2} + \frac{i}{2}\right)t \\ -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(\frac{1}{2}+\frac{I}{2}\right)t \\ -t \\ t \end{bmatrix} = t\begin{bmatrix} \frac{1}{2}+\frac{i}{2} \\ -1 \\ 1 \end{bmatrix}$$

Let $t=1$ the eigenvector becomes

$$\begin{bmatrix} \left(\frac{1}{2}+\frac{I}{2}\right)t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2}+\frac{i}{2} \\ -1 \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(\frac{1}{2}+\frac{I}{2}\right)t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} 1+i \\ -2 \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m>k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m-k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$2i$	1	1	No	$\begin{bmatrix} \frac{1}{2}+\frac{i}{2} \\ -1 \\ 1 \end{bmatrix}$
$-2i$	1	1	No	$\begin{bmatrix} \frac{1}{2}-\frac{i}{2} \\ -1 \\ 1 \end{bmatrix}$
1	1	1	No	$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^t \\ &= \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^t\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 \begin{bmatrix} \left(\frac{1}{2}+\frac{i}{2}\right) e^{2it} \\ -e^{2it} \\ e^{2it} \end{bmatrix} + c_2 \begin{bmatrix} \left(\frac{1}{2}-\frac{i}{2}\right) e^{-2it} \\ -e^{-2it} \\ e^{-2it} \end{bmatrix} + c_3 \begin{bmatrix} -e^t \\ e^t \\ 0 \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2}+\frac{i}{2}\right) c_1 e^{2it} + \left(\frac{1}{2}-\frac{i}{2}\right) c_2 e^{-2it} - c_3 e^{t} \\ -c_1 e^{2it} - c_2 e^{-2it} + c_3 e^{t} \\ c_1 e^{2it} + c_2 e^{-2it} \end{bmatrix}$$

4.24.3 Maple step by step solution

Let's solve

$[x_1'(t) = 2x_1(t) + x_2(t) - x_3(t), x_2'(t) = -4x_1(t) - 3x_2(t) - x_3(t), x_3'(t) = 4x_1(t) + 4x_2(t) + 2x_3(t$

- Define vector

$$\underline{\vec{x}}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{\vec{x}}'(t) = \begin{bmatrix} 2 & 1 & -1 \\ -4 & -3 & -1 \\ 4 & 4 & 2 \end{bmatrix} \cdot \underline{\vec{x}}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{\vec{x}}'(t) = \begin{bmatrix} 2 & 1 & -1 \\ -4 & -3 & -1 \\ 4 & 4 & 2 \end{bmatrix} \cdot \underline{\vec{x}}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 2 & 1 & -1 \\ -4 & -3 & -1 \\ 4 & 4 & 2 \end{bmatrix}$$

- Rewrite the system as

$\underline{\vec{x}}'(t) = A \cdot \underline{\vec{x}}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right], \left[-2\,\mathrm{I}, \begin{bmatrix} \frac{1}{2} - \frac{\mathrm{I}}{2} \\ -1 \\ 1 \end{bmatrix}\right], \left[2\,\mathrm{I}, \begin{bmatrix} \frac{1}{2} + \frac{\mathrm{I}}{2} \\ -1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{x^{\rightarrow}}_1 = \mathrm{e}^{t} \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-2\,\mathrm{I}, \begin{bmatrix} \frac{1}{2} - \frac{\mathrm{I}}{2} \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{-2\,\mathrm{I}t} \cdot \begin{bmatrix} \frac{1}{2} - \frac{\mathrm{I}}{2} \\ -1 \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$(\cos(2t) - \mathrm{I}\sin(2t)) \cdot \begin{bmatrix} \frac{1}{2} - \frac{\mathrm{I}}{2} \\ -1 \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} \left(\frac{1}{2} - \frac{\mathrm{I}}{2}\right)(\cos(2t) - \mathrm{I}\sin(2t)) \\ -\cos(2t) + \mathrm{I}\sin(2t) \\ \cos(2t) - \mathrm{I}\sin(2t) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\underline{\vec{x}}_2(t) = \begin{bmatrix} \frac{\cos(2t)}{2} - \frac{\sin(2t)}{2} \\ -\cos(2t) \\ \cos(2t) \end{bmatrix}, \underline{\vec{x}}_3(t) = \begin{bmatrix} -\frac{\sin(2t)}{2} - \frac{\cos(2t)}{2} \\ \sin(2t) \\ -\sin(2t) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\underline{\vec{x}} = c_1\underline{\vec{x}}_1 + c_2\underline{\vec{x}}_2(t) + c_3\underline{\vec{x}}_3(t)$$

- Substitute solutions into the general solution

$$\underline{\vec{x}} = c_1\mathrm{e}^t \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} c_2\left(\frac{\cos(2t)}{2} - \frac{\sin(2t)}{2}\right) + c_3\left(-\frac{\sin(2t)}{2} - \frac{\cos(2t)}{2}\right) \\ -c_2\cos(2t) + c_3\sin(2t) \\ -c_3\sin(2t) + c_2\cos(2t) \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} \frac{(c_2-c_3)\cos(2t)}{2} + \frac{(-c_2-c_3)\sin(2t)}{2} - c_1\mathrm{e}^t \\ c_1\mathrm{e}^t + c_3\sin(2t) - c_2\cos(2t) \\ -c_3\sin(2t) + c_2\cos(2t) \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x_1(t) = \frac{(c_2-c_3)\cos(2t)}{2} + \frac{(-c_2-c_3)\sin(2t)}{2} - c_1\mathrm{e}^t, x_2(t) = c_1\mathrm{e}^t + c_3\sin(2t) - c_2\cos(2t), x_3(t) = -c_3\sin\right.$$

✓ Solution by Maple

Time used: 0.11 (sec). Leaf size: 87

```
dsolve([diff(x__1(t),t)=2*x__1(t)+1*x__2(t)-1*x__3(t),diff(x__2(t),t)=-4*x__1(t)-3*x__2(t)-1
```

$$x_1(t) = c_1\mathrm{e}^t + c_2\sin(2t) + c_3\cos(2t)$$
$$x_2(t) = -c_1\mathrm{e}^t - c_2\sin(2t) - c_3\cos(2t) + c_2\cos(2t) - c_3\sin(2t)$$
$$x_3(t) = -c_2\cos(2t) + c_3\sin(2t) + c_2\sin(2t) + c_3\cos(2t)$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 103

```
DSolve[{x1'[t]==2*x1[t]+1*x2[t]-1*x3[t],x2'[t]==-4*x1[t]-3*x2[t]-1*x3[t],x3'[t]==4*x1[t]+4*x
```

$$\begin{aligned}
\text{x1}(t) &\to (c_2 + c_3)\left(-e^t\right) + (c_1 + c_2 + c_3)\cos(2t) + (c_1 + c_2)\sin(2t) \\
\text{x2}(t) &\to (c_2 + c_3)e^t - c_3\cos(2t) - (2c_1 + 2c_2 + c_3)\sin(2t) \\
\text{x3}(t) &\to c_3\cos(2t) + (2c_1 + 2c_2 + c_3)\sin(2t)
\end{aligned}$$

4.25 problem problem 25

Internal problem ID [339]
Internal file name [`OUTPUT/339_Sunday_June_05_2022_01_39_05_AM_97894964/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.3, The eigenvalue method for linear systems. Page 395
Problem number: problem 25.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= 5x_1(t) + 5x_2(t) + 2x_3(t) \\
x_2'(t) &= -6x_1(t) - 6x_2(t) - 5x_3(t) \\
x_3'(t) &= 6x_1(t) + 6x_2(t) + 5x_3(t)
\end{aligned}
$$

4.25.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 5 & 5 & 2 \\ -6 & -6 & -5 \\ 6 & 6 & 5 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{2t}\cos(3t) + e^{2t}\sin(3t) & e^{2t}\cos(3t) + e^{2t}\sin(3t) - 1 & e^{2t}\cos(3t) - 1 \\ -2\,e^{2t}\sin(3t) & 1 - 2\,e^{2t}\sin(3t) & -e^{2t}\cos(3t) - e^{2t}\sin(3t) + 1 \\ 2\,e^{2t}\sin(3t) & 2\,e^{2t}\sin(3t) & e^{2t}\cos(3t) + e^{2t}\sin(3t) \end{bmatrix}$$
$$= \begin{bmatrix} e^{2t}(\sin(3t) + \cos(3t)) & -1 + e^{2t}(\sin(3t) + \cos(3t)) & e^{2t}\cos(3t) - 1 \\ -2\,e^{2t}\sin(3t) & 1 - 2\,e^{2t}\sin(3t) & 1 + (-\sin(3t) - \cos(3t))\,e^{2t} \\ 2\,e^{2t}\sin(3t) & 2\,e^{2t}\sin(3t) & e^{2t}(\sin(3t) + \cos(3t)) \end{bmatrix}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = e^{At}\vec{c}$$
$$= \begin{bmatrix} e^{2t}(\sin(3t) + \cos(3t)) & -1 + e^{2t}(\sin(3t) + \cos(3t)) & e^{2t}\cos(3t) - 1 \\ -2\,e^{2t}\sin(3t) & 1 - 2\,e^{2t}\sin(3t) & 1 + (-\sin(3t) - \cos(3t))\,e^{2t} \\ 2\,e^{2t}\sin(3t) & 2\,e^{2t}\sin(3t) & e^{2t}(\sin(3t) + \cos(3t)) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$
$$= \begin{bmatrix} e^{2t}(\sin(3t) + \cos(3t))\,c_1 + (-1 + e^{2t}(\sin(3t) + \cos(3t)))\,c_2 + (e^{2t}\cos(3t) - 1)\,c_3 \\ -2\,e^{2t}\sin(3t)\,c_1 + (1 - 2\,e^{2t}\sin(3t))\,c_2 + (1 + (-\sin(3t) - \cos(3t))\,e^{2t})\,c_3 \\ 2\,e^{2t}\sin(3t)\,c_1 + 2\,e^{2t}\sin(3t)\,c_2 + e^{2t}(\sin(3t) + \cos(3t))\,c_3 \end{bmatrix}$$
$$= \begin{bmatrix} ((c_1 + c_2 + c_3)\cos(3t) + (c_1 + c_2)\sin(3t))\,e^{2t} - c_2 - c_3 \\ ((-2c_1 - 2c_2 - c_3)\sin(3t) - \cos(3t)\,c_3)\,e^{2t} + c_2 + c_3 \\ 2\left(\left(c_1 + c_2 + \frac{c_3}{2}\right)\sin(3t) + \frac{\cos(3t)c_3}{2}\right)e^{2t} \end{bmatrix}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

4.25.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 5 & 5 & 2 \\ -6 & -6 & -5 \\ 6 & 6 & 5 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 5 & 5 & 2 \\ -6 & -6 & -5 \\ 6 & 6 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 5-\lambda & 5 & 2 \\ -6 & -6-\lambda & -5 \\ 6 & 6 & 5-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 4\lambda^2 + 13\lambda = 0$$

The roots of the above are the eigenvalues.

$$\begin{aligned} \lambda_1 &= 2 + 3i \\ \lambda_2 &= 2 - 3i \\ \lambda_3 &= 0 \end{aligned}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
0	1	real eigenvalue
$2 + 3i$	1	complex eigenvalue
$2 - 3i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 0$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 5 & 5 & 2 \\ -6 & -6 & -5 \\ 6 & 6 & 5 \end{bmatrix} - (0)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 5 & 2 \\ -6 & -6 & -5 \\ 6 & 6 & 5 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 5 & 5 & 2 & 0 \\ -6 & -6 & -5 & 0 \\ 6 & 6 & 5 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{6R_1}{5} \implies \left[\begin{array}{ccc|c} 5 & 5 & 2 & 0 \\ 0 & 0 & -\frac{13}{5} & 0 \\ 6 & 6 & 5 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{6R_1}{5} \implies \left[\begin{array}{ccc|c} 5 & 5 & 2 & 0 \\ 0 & 0 & -\frac{13}{5} & 0 \\ 0 & 0 & \frac{13}{5} & 0 \end{array}\right]$$

$$R_3 = R_3 + R_2 \implies \left[\begin{array}{ccc|c} 5 & 5 & 2 & 0 \\ 0 & 0 & -\frac{13}{5} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 5 & 5 & 2 \\ 0 & 0 & -\frac{13}{5} \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1, v_3\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} -t \\ t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 2 - 3i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 5 & 5 & 2 \\ -6 & -6 & -5 \\ 6 & 6 & 5 \end{bmatrix} - (2 - 3i) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3+3i & 5 & 2 \\ -6 & -8+3i & -5 \\ 6 & 6 & 3+3i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 3+3i & 5 & 2 & 0 \\ -6 & -8+3i & -5 & 0 \\ 6 & 6 & 3+3i & 0 \end{array}\right]$$

$$R_2 = R_2 + (1-i)\,R_1 \Longrightarrow \left[\begin{array}{ccc|c} 3+3i & 5 & 2 & 0 \\ 0 & -3-2i & -3-2i & 0 \\ 6 & 6 & 3+3i & 0 \end{array}\right]$$

$$R_3 = R_3 + (-1+i)\,R_1 \Longrightarrow \left[\begin{array}{ccc|c} 3+3i & 5 & 2 & 0 \\ 0 & -3-2i & -3-2i & 0 \\ 0 & 1+5i & 1+5i & 0 \end{array}\right]$$

$$R_3 = R_3 + (1+i)\,R_2 \Longrightarrow \left[\begin{array}{ccc|c} 3+3i & 5 & 2 & 0 \\ 0 & -3-2i & -3-2i & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3+3i & 5 & 2 \\ 0 & -3-2i & -3-2i \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(\frac{1}{2} - \frac{i}{2}\right) t, v_2 = -t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(\frac{1}{2} - \frac{I}{2}\right) t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2} - \frac{i}{2}\right) t \\ -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(\frac{1}{2} - \frac{I}{2}\right) t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} - \frac{i}{2} \\ -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(\frac{1}{2} - \frac{I}{2}\right) t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - \frac{i}{2} \\ -1 \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(\frac{1}{2} - \frac{I}{2}\right) t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} 1-i \\ -2 \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = 2 + 3i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 5 & 5 & 2 \\ -6 & -6 & -5 \\ 6 & 6 & 5 \end{bmatrix} - (2+3i) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3-3i & 5 & 2 \\ -6 & -8-3i & -5 \\ 6 & 6 & 3-3i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 3-3i & 5 & 2 & 0 \\ -6 & -8-3i & -5 & 0 \\ 6 & 6 & 3-3i & 0 \end{array}\right]$$

$$R_2 = R_2 + (1+i)\,R_1 \Longrightarrow \left[\begin{array}{ccc|c} 3-3i & 5 & 2 & 0 \\ 0 & -3+2i & -3+2i & 0 \\ 6 & 6 & 3-3i & 0 \end{array}\right]$$

$$R_3 = R_3 + (-1-i)\,R_1 \Longrightarrow \left[\begin{array}{ccc|c} 3-3i & 5 & 2 & 0 \\ 0 & -3+2i & -3+2i & 0 \\ 0 & 1-5i & 1-5i & 0 \end{array}\right]$$

$$R_3 = R_3 + (1-i)\,R_2 \Longrightarrow \left[\begin{array}{ccc|c} 3-3i & 5 & 2 & 0 \\ 0 & -3+2i & -3+2i & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3-3i & 5 & 2 \\ 0 & -3+2i & -3+2i \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(\frac{1}{2} + \frac{i}{2}\right) t, v_2 = -t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(\frac{1}{2} + \frac{I}{2}\right) t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2} + \frac{i}{2}\right) t \\ -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(\frac{1}{2} + \frac{I}{2}\right) t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} + \frac{i}{2} \\ -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(\frac{1}{2} + \frac{I}{2}\right) t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} + \frac{i}{2} \\ -1 \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(\frac{1}{2} + \frac{I}{2}\right) t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} 1+i \\ -2 \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
$2+3i$	1	1	No	$\begin{bmatrix} \frac{1}{2}+\frac{i}{2} \\ -1 \\ 1 \end{bmatrix}$
$2-3i$	1	1	No	$\begin{bmatrix} \frac{1}{2}-\frac{i}{2} \\ -1 \\ 1 \end{bmatrix}$
0	1	1	No	$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 0 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{0} \\ &= \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{0}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 \begin{bmatrix} \left(\frac{1}{2}+\frac{i}{2}\right) e^{(2+3i)t} \\ -e^{(2+3i)t} \\ e^{(2+3i)t} \end{bmatrix} + c_2 \begin{bmatrix} \left(\frac{1}{2}-\frac{i}{2}\right) e^{(2-3i)t} \\ -e^{(2-3i)t} \\ e^{(2-3i)t} \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2}+\frac{i}{2}\right) c_1 e^{(2+3i)t} + \left(\frac{1}{2}-\frac{i}{2}\right) c_2 e^{(2-3i)t} - c_3 \\ -c_1 e^{(2+3i)t} - c_2 e^{(2-3i)t} + c_3 \\ c_1 e^{(2+3i)t} + c_2 e^{(2-3i)t} \end{bmatrix}$$

4.25.3 Maple step by step solution

Let's solve

$[x_1'(t) = 5x_1(t) + 5x_2(t) + 2x_3(t), x_2'(t) = -6x_1(t) - 6x_2(t) - 5x_3(t), x_3'(t) = 6x_1(t) + 6x_2(t) + 5x$

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x}^{\rightarrow\prime}(t) = \begin{bmatrix} 5 & 5 & 2 \\ -6 & -6 & -5 \\ 6 & 6 & 5 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{x}^{\rightarrow\prime}(t) = \begin{bmatrix} 5 & 5 & 2 \\ -6 & -6 & -5 \\ 6 & 6 & 5 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 5 & 5 & 2 \\ -6 & -6 & -5 \\ 6 & 6 & 5 \end{bmatrix}$$

- Rewrite the system as

$\underline{x}^{\rightarrow\prime}(t) = A \cdot \underline{x}^{\rightarrow}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right], \left[2-3\,\mathrm{I}, \begin{bmatrix} \frac{1}{2}-\frac{\mathrm{I}}{2} \\ -1 \\ 1 \end{bmatrix}\right], \left[2+3\,\mathrm{I}, \begin{bmatrix} \frac{1}{2}+\frac{\mathrm{I}}{2} \\ -1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_{1} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[2-3\,\mathrm{I}, \begin{bmatrix} \frac{1}{2}-\frac{\mathrm{I}}{2} \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{(2-3\,\mathrm{I})t} \cdot \begin{bmatrix} \frac{1}{2}-\frac{\mathrm{I}}{2} \\ -1 \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\mathrm{e}^{2t} \cdot (\cos(3t) - \mathrm{I}\sin(3t)) \cdot \begin{bmatrix} \frac{1}{2}-\frac{\mathrm{I}}{2} \\ -1 \\ 1 \end{bmatrix}$$

- Simplify expression

$$\mathrm{e}^{2t} \cdot \begin{bmatrix} \left(\frac{1}{2}-\frac{\mathrm{I}}{2}\right)(\cos(3t) - \mathrm{I}\sin(3t)) \\ -\cos(3t) + \mathrm{I}\sin(3t) \\ \cos(3t) - \mathrm{I}\sin(3t) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\underline{x}^{\rightarrow}{}_{2}(t) = e^{2t} \cdot \begin{bmatrix} \frac{\cos(3t)}{2} - \frac{\sin(3t)}{2} \\ -\cos(3t) \\ \cos(3t) \end{bmatrix}, \underline{x}^{\rightarrow}{}_{3}(t) = e^{2t} \cdot \begin{bmatrix} -\frac{\sin(3t)}{2} - \frac{\cos(3t)}{2} \\ \sin(3t) \\ -\sin(3t) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\underline{x}^{\rightarrow} = c_1 \underline{x}^{\rightarrow}{}_{1} + c_2 \underline{x}^{\rightarrow}{}_{2}(t) + c_3 \underline{x}^{\rightarrow}{}_{3}(t)$$

- Substitute solutions into the general solution

$$\underline{x}^{\rightarrow} = c_2 e^{2t} \cdot \begin{bmatrix} \frac{\cos(3t)}{2} - \frac{\sin(3t)}{2} \\ -\cos(3t) \\ \cos(3t) \end{bmatrix} + c_3 e^{2t} \cdot \begin{bmatrix} -\frac{\sin(3t)}{2} - \frac{\cos(3t)}{2} \\ \sin(3t) \\ -\sin(3t) \end{bmatrix} + \begin{bmatrix} -c_1 \\ c_1 \\ 0 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} \frac{((c_2-c_3)\cos(3t)-\sin(3t)(c_2+c_3))e^{2t}}{2} - c_1 \\ -c_2 e^{2t}\cos(3t) + c_3 e^{2t}\sin(3t) + c_1 \\ e^{2t}(c_2\cos(3t) - \sin(3t)\, c_3) \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x_1(t) = \frac{((c_2-c_3)\cos(3t)-\sin(3t)(c_2+c_3))e^{2t}}{2} - c_1, x_2(t) = -c_2 e^{2t}\cos(3t) + c_3 e^{2t}\sin(3t) + c_1, x_3(t) = e^{2t}(c\right.$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 111

```
dsolve([diff(x__1(t),t)=5*x__1(t)+5*x__2(t)+2*x__3(t),diff(x__2(t),t)=-6*x__1(t)-6*x__2(t)-5
```

$$x_1(t) = c_1 + c_2 e^{2t}\sin(3t) + c_3 e^{2t}\cos(3t)$$
$$x_2(t) = -c_2 e^{2t}\sin(3t) + c_2 e^{2t}\cos(3t) - c_3 e^{2t}\cos(3t) - c_3 e^{2t}\sin(3t) - c_1$$
$$x_3(t) = e^{2t}(c_2\sin(3t) + \sin(3t)\, c_3 - c_2\cos(3t) + \cos(3t)\, c_3)$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 122

```
DSolve[{x1'[t]==5*x1[t]+5*x2[t]+2*x3[t],x2'[t]==-6*x1[t]-6*x2[t]-5*x3[t],x3'[t]==6*x1[t]+6*x
```

$$\begin{aligned}
\text{x1}(t) &\to (c_1 + c_2 + c_3)e^{2t}\cos(3t) + (c_1 + c_2)e^{2t}\sin(3t) - c_2 - c_3 \\
\text{x2}(t) &\to -c_3 e^{2t}\cos(3t) - (2c_1 + 2c_2 + c_3)e^{2t}\sin(3t) + c_2 + c_3 \\
\text{x3}(t) &\to e^{2t}(c_3\cos(3t) + (2c_1 + 2c_2 + c_3)\sin(3t))
\end{aligned}$$

4.26 problem problem 26

Internal problem ID [340]
Internal file name [`OUTPUT/340_Sunday_June_05_2022_01_39_06_AM_11246814/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.3, The eigenvalue method for linear systems. Page 395
Problem number: problem 26.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= 3x_1(t) + x_3(t) \\
x_2'(t) &= 9x_1(t) - x_2(t) + 2x_3(t) \\
x_3'(t) &= -9x_1(t) + 4x_2(t) - x_3(t)
\end{aligned}
$$

With initial conditions

$$[x_1(0) = 0, x_2(0) = 0, x_3(0) = 17]$$

4.26.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 3 & 0 & 1 \\ 9 & -1 & 2 \\ -9 & 4 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{8\,e^{3t}}{17}+\frac{9\,e^{-t}\cos(t)}{17}+\frac{36\,e^{-t}\sin(t)}{17} & -\frac{4\,e^{-t}\cos(t)}{17}-\frac{16\,e^{-t}\sin(t)}{17}+\frac{4\,e^{3t}}{17} & -\frac{4\,e^{-t}\cos(t)}{17}+\frac{e^{-t}\sin(t)}{17}+\frac{4\,e^{3t}}{17} \\ -\frac{18\,e^{-t}\cos(t)}{17}+\frac{81\,e^{-t}\sin(t)}{17}+\frac{18\,e^{3t}}{17} & \frac{9\,e^{3t}}{17}+\frac{8\,e^{-t}\cos(t)}{17}-\frac{36\,e^{-t}\sin(t)}{17} & -\frac{9\,e^{-t}\cos(t)}{17}-\frac{2\,e^{-t}\sin(t)}{17}+\frac{9\,e^{3t}}{17} \\ -9\,e^{-t}\sin(t) & 4\,e^{-t}\sin(t) & e^{-t}\cos(t) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{9(\cos(t)+4\sin(t))e^{-t}}{17}+\frac{8\,e^{3t}}{17} & \frac{4(-\cos(t)-4\sin(t))e^{-t}}{17}+\frac{4\,e^{3t}}{17} & \frac{(-4\cos(t)+\sin(t))e^{-t}}{17}+\frac{4\,e^{3t}}{17} \\ \frac{9(-2\cos(t)+9\sin(t))e^{-t}}{17}+\frac{18\,e^{3t}}{17} & \frac{4(2\cos(t)-9\sin(t))e^{-t}}{17}+\frac{9\,e^{3t}}{17} & \frac{(-9\cos(t)-2\sin(t))e^{-t}}{17}+\frac{9\,e^{3t}}{17} \\ -9\,e^{-t}\sin(t) & 4\,e^{-t}\sin(t) & e^{-t}\cos(t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = e^{At}\vec{x}_0$$

$$= \begin{bmatrix} \frac{9(\cos(t)+4\sin(t))e^{-t}}{17}+\frac{8\,e^{3t}}{17} & \frac{4(-\cos(t)-4\sin(t))e^{-t}}{17}+\frac{4\,e^{3t}}{17} & \frac{(-4\cos(t)+\sin(t))e^{-t}}{17}+\frac{4\,e^{3t}}{17} \\ \frac{9(-2\cos(t)+9\sin(t))e^{-t}}{17}+\frac{18\,e^{3t}}{17} & \frac{4(2\cos(t)-9\sin(t))e^{-t}}{17}+\frac{9\,e^{3t}}{17} & \frac{(-9\cos(t)-2\sin(t))e^{-t}}{17}+\frac{9\,e^{3t}}{17} \\ -9\,e^{-t}\sin(t) & 4\,e^{-t}\sin(t) & e^{-t}\cos(t) \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 17 \end{bmatrix}$$

$$= \begin{bmatrix} (-4\cos(t)+\sin(t))\,e^{-t}+4\,e^{3t} \\ (-9\cos(t)-2\sin(t))\,e^{-t}+9\,e^{3t} \\ 17\,e^{-t}\cos(t) \end{bmatrix}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

4.26.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 3 & 0 & 1 \\ 9 & -1 & 2 \\ -9 & 4 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A-\lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 3 & 0 & 1 \\ 9 & -1 & 2 \\ -9 & 4 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 3-\lambda & 0 & 1 \\ 9 & -1-\lambda & 2 \\ -9 & 4 & -1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - \lambda^2 - 4\lambda - 6 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1 + i$$
$$\lambda_2 = -1 - i$$
$$\lambda_3 = 3$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-1 - i$	1	complex eigenvalue
3	1	real eigenvalue
$-1 + i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & 0 & 1 \\ 9 & -1 & 2 \\ -9 & 4 & -1 \end{bmatrix} - (3)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 9 & -4 & 2 \\ -9 & 4 & -4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 9 & -4 & 2 & 0 \\ -9 & 4 & -4 & 0 \end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{ccc|c} 9 & -4 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ -9 & 4 & -4 & 0 \end{array}\right]$$

$$R_3 = R_3 + R_1 \Longrightarrow \left[\begin{array}{ccc|c} 9 & -4 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right]$$

$$R_3 = R_3 + 2R_2 \Longrightarrow \left[\begin{array}{ccc|c} 9 & -4 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 9 & -4 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1, v_3\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{4t}{9}, v_3 = 0\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{4t}{9} \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{4t}{9} \\ t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{4t}{9} \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} \frac{4}{9} \\ 1 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{4t}{9} \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{4}{9} \\ 1 \\ 0 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{4t}{9} \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 0 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -1 - i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & 0 & 1 \\ 9 & -1 & 2 \\ -9 & 4 & -1 \end{bmatrix} - (-1 - i) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4+i & 0 & 1 \\ 9 & i & 2 \\ -9 & 4 & i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 4+i & 0 & 1 & 0 \\ 9 & i & 2 & 0 \\ -9 & 4 & i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{36}{17} + \frac{9i}{17}\right) R_1 \Longrightarrow \left[\begin{array}{ccc|c} 4+i & 0 & 1 & 0 \\ 0 & i & -\frac{2}{17} + \frac{9i}{17} & 0 \\ -9 & 4 & i & 0 \end{array}\right]$$

$$R_3 = R_3 + \left(\frac{36}{17} - \frac{9i}{17}\right) R_1 \Longrightarrow \left[\begin{array}{ccc|c} 4+i & 0 & 1 & 0 \\ 0 & i & -\frac{2}{17} + \frac{9i}{17} & 0 \\ 0 & 4 & \frac{36}{17} + \frac{8i}{17} & 0 \end{array}\right]$$

$$R_3 = 4iR_2 + R_3 \Longrightarrow \left[\begin{array}{ccc|c} 4+i & 0 & 1 & 0 \\ 0 & i & -\frac{2}{17} + \frac{9i}{17} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 4+i & 0 & 1 \\ 0 & i & -\frac{2}{17} + \frac{9i}{17} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(-\frac{4}{17} + \frac{i}{17}\right) t, v_2 = \left(-\frac{9}{17} - \frac{2i}{17}\right) t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(-\frac{4}{17} + \frac{\mathrm{I}}{17}\right) t \\ \left(-\frac{9}{17} - \frac{2\,\mathrm{I}}{17}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \left(-\frac{4}{17} + \frac{i}{17}\right) t \\ \left(-\frac{9}{17} - \frac{2i}{17}\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(-\frac{4}{17} + \frac{\mathrm{I}}{17}\right) t \\ \left(-\frac{9}{17} - \frac{2\,\mathrm{I}}{17}\right) t \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{4}{17} + \frac{i}{17} \\ -\frac{9}{17} - \frac{2i}{17} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(-\frac{4}{17} + \frac{\mathrm{I}}{17}\right) t \\ \left(-\frac{9}{17} - \frac{2\,\mathrm{I}}{17}\right) t \\ t \end{bmatrix} = \begin{bmatrix} -\frac{4}{17} + \frac{i}{17} \\ -\frac{9}{17} - \frac{2i}{17} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(-\frac{4}{17}+\frac{I}{17}\right)t \\ \left(-\frac{9}{17}-\frac{2I}{17}\right)t \\ t \end{bmatrix} = \begin{bmatrix} -4+i \\ -9-2i \\ 17 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = -1+i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A-\lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & 0 & 1 \\ 9 & -1 & 2 \\ -9 & 4 & -1 \end{bmatrix} - (-1+i) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4-i & 0 & 1 \\ 9 & -i & 2 \\ -9 & 4 & -i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 4-i & 0 & 1 & 0 \\ 9 & -i & 2 & 0 \\ -9 & 4 & -i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{36}{17} - \frac{9i}{17}\right) R_1 \Longrightarrow \left[\begin{array}{ccc|c} 4-i & 0 & 1 & 0 \\ 0 & -i & -\frac{2}{17}-\frac{9i}{17} & 0 \\ -9 & 4 & -i & 0 \end{array}\right]$$

$$R_3 = R_3 + \left(\frac{36}{17} + \frac{9i}{17}\right) R_1 \Longrightarrow \left[\begin{array}{ccc|c} 4-i & 0 & 1 & 0 \\ 0 & -i & -\frac{2}{17}-\frac{9i}{17} & 0 \\ 0 & 4 & \frac{36}{17}-\frac{8i}{17} & 0 \end{array}\right]$$

$$R_3 = -4iR_2 + R_3 \Longrightarrow \left[\begin{array}{ccc|c} 4-i & 0 & 1 & 0 \\ 0 & -i & -\frac{2}{17}-\frac{9i}{17} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 4-i & 0 & 1 \\ 0 & -i & -\frac{2}{17}-\frac{9i}{17} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(-\frac{4}{17}-\frac{i}{17}\right)t, v_2 = \left(-\frac{9}{17}+\frac{2i}{17}\right)t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(-\frac{4}{17}-\frac{\mathrm{I}}{17}\right)t \\ \left(-\frac{9}{17}+\frac{2\,\mathrm{I}}{17}\right)t \\ t \end{bmatrix} = \begin{bmatrix} \left(-\frac{4}{17}-\frac{i}{17}\right)t \\ \left(-\frac{9}{17}+\frac{2i}{17}\right)t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(-\frac{4}{17}-\frac{\mathrm{I}}{17}\right)t \\ \left(-\frac{9}{17}+\frac{2\,\mathrm{I}}{17}\right)t \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{4}{17}-\frac{i}{17} \\ -\frac{9}{17}+\frac{2i}{17} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(-\frac{4}{17}-\frac{\mathrm{I}}{17}\right)t \\ \left(-\frac{9}{17}+\frac{2\,\mathrm{I}}{17}\right)t \\ t \end{bmatrix} = \begin{bmatrix} -\frac{4}{17}-\frac{i}{17} \\ -\frac{9}{17}+\frac{2i}{17} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(-\frac{4}{17}-\frac{\mathrm{I}}{17}\right)t \\ \left(-\frac{9}{17}+\frac{2\,\mathrm{I}}{17}\right)t \\ t \end{bmatrix} = \begin{bmatrix} -4-i \\ -9+2i \\ 17 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m-k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity: algebraic m	multiplicity: geometric k	defective?	eigenvectors
$-1+i$	1	1	No	$\begin{bmatrix} -\frac{4}{17}-\frac{i}{17} \\ -\frac{9}{17}+\frac{2i}{17} \\ 1 \end{bmatrix}$
$-1-i$	1	1	No	$\begin{bmatrix} -\frac{4}{17}+\frac{i}{17} \\ -\frac{9}{17}-\frac{2i}{17} \\ 1 \end{bmatrix}$
3	1	1	No	$\begin{bmatrix} \frac{4}{9} \\ 1 \\ 0 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 3 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{3t} \\ &= \begin{bmatrix} \frac{4}{9} \\ 1 \\ 0 \end{bmatrix} e^{3t} \end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 \begin{bmatrix} \left(-\frac{4}{17}-\frac{i}{17}\right) \mathrm{e}^{(-1+i)t} \\ \left(-\frac{9}{17}+\frac{2i}{17}\right) \mathrm{e}^{(-1+i)t} \\ \mathrm{e}^{(-1+i)t} \end{bmatrix} + c_2 \begin{bmatrix} \left(-\frac{4}{17}+\frac{i}{17}\right) \mathrm{e}^{(-1-i)t} \\ \left(-\frac{9}{17}-\frac{2i}{17}\right) \mathrm{e}^{(-1-i)t} \\ \mathrm{e}^{(-1-i)t} \end{bmatrix} + c_3 \begin{bmatrix} \frac{4\,\mathrm{e}^{3t}}{9} \\ \mathrm{e}^{3t} \\ 0 \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} \left(-\frac{4}{17}-\frac{i}{17}\right)c_1\mathrm{e}^{(-1+i)t}+\left(-\frac{4}{17}+\frac{i}{17}\right)c_2\mathrm{e}^{(-1-i)t}+\frac{4c_3\mathrm{e}^{3t}}{9} \\ \left(-\frac{9}{17}+\frac{2i}{17}\right)c_1\mathrm{e}^{(-1+i)t}+\left(-\frac{9}{17}-\frac{2i}{17}\right)c_2\mathrm{e}^{(-1-i)t}+c_3\mathrm{e}^{3t} \\ c_1\mathrm{e}^{(-1+i)t}+c_2\mathrm{e}^{(-1-i)t} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x_1(0)=0 \\ x_2(0)=0 \\ x_3(0)=17 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 0 \\ 0 \\ 17 \end{bmatrix} = \begin{bmatrix} \left(-\frac{4}{17}-\frac{i}{17}\right)c_1+\left(-\frac{4}{17}+\frac{i}{17}\right)c_2+\frac{4c_3}{9} \\ \left(-\frac{9}{17}+\frac{2i}{17}\right)c_1+\left(-\frac{9}{17}-\frac{2i}{17}\right)c_2+c_3 \\ c_1+c_2 \end{bmatrix}$$

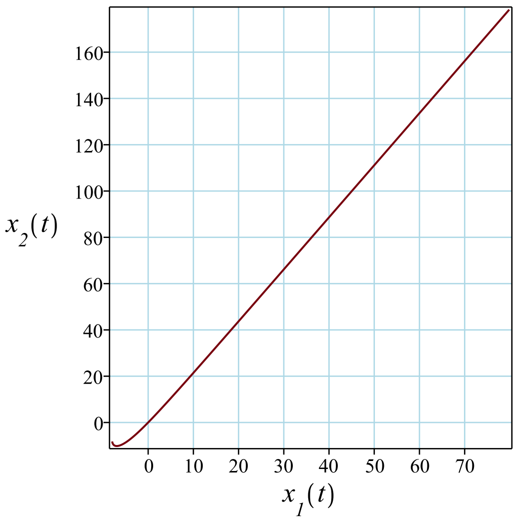
Solving for the constants of integrations gives

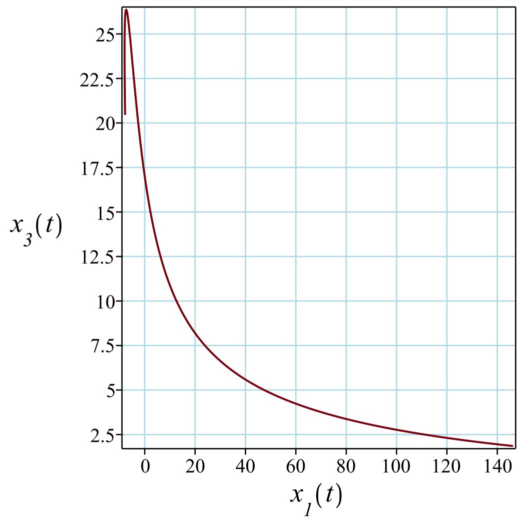
$$\begin{bmatrix} c_1=\frac{17}{2} \\ c_2=\frac{17}{2} \\ c_3=9 \end{bmatrix}$$

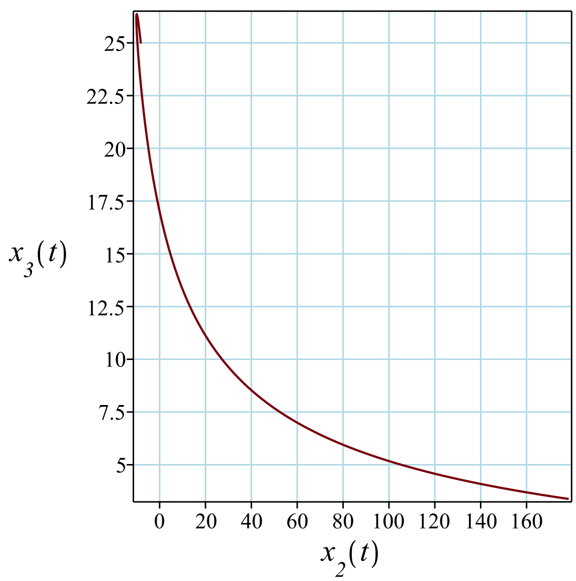
Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} \left(-2-\frac{i}{2}\right)\mathrm{e}^{(-1+i)t}+\left(-2+\frac{i}{2}\right)\mathrm{e}^{(-1-i)t}+4\,\mathrm{e}^{3t} \\ \left(-\frac{9}{2}+i\right)\mathrm{e}^{(-1+i)t}+\left(-\frac{9}{2}-i\right)\mathrm{e}^{(-1-i)t}+9\,\mathrm{e}^{3t} \\ \frac{17\,\mathrm{e}^{(-1+i)t}}{2}+\frac{17\,\mathrm{e}^{(-1-i)t}}{2} \end{bmatrix}$$

The following are plots of each solution against another.







The following are plots of each solution.

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 64

```
dsolve([diff(x__1(t),t) = 3*x__1(t)+x__3(t), diff(x__2(t),t) = 9*x__1(t)-x__2(t)+2*x__3(t),
```

$$x_1(t) = 4\,\mathrm{e}^{3t} + \mathrm{e}^{-t}\sin(t) - 4\,\mathrm{e}^{-t}\cos(t)$$
$$x_2(t) = 9\,\mathrm{e}^{3t} - 9\,\mathrm{e}^{-t}\cos(t) - 2\,\mathrm{e}^{-t}\sin(t)$$
$$x_3(t) = 17\,\mathrm{e}^{-t}\cos(t)$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 62

```
DSolve[{x1'[t]==3*x1[t]+0*x2[t]+1*x3[t],x2'[t]==9*x1[t]-1*x2[t]+2*x3[t],x3'[t]==-9*x1[t]+4*x
```

$$
\begin{aligned}
\text{x1}(t) &\to e^{-t}\left(4e^{4t}+\sin(t)-4\cos(t)\right)\\
\text{x2}(t) &\to e^{-t}\left(9e^{4t}-2\sin(t)-9\cos(t)\right)\\
\text{x3}(t) &\to 17e^{-t}\cos(t)
\end{aligned}
$$

4.27 problem problem 38

Internal problem ID [341]
Internal file name [`OUTPUT/341_Sunday_June_05_2022_01_39_08_AM_70749370/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.3, The eigenvalue method for linear systems. Page 395
Problem number: problem 38.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= x_1(t) \\
x_2'(t) &= 2x_1(t) + 2x_2(t) \\
x_3'(t) &= 3x_2(t) + 3x_3(t) \\
x_4'(t) &= 4x_3(t) + 4x_4(t)
\end{aligned}
$$

4.27.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \\ x_4'(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 4 & 4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{t} & 0 & 0 & 0 \\ 2\,e^{2t} - 2\,e^{t} & e^{2t} & 0 & 0 \\ 3\,e^{3t} - 6\,e^{2t} + 3\,e^{t} & -3\,e^{2t} + 3\,e^{3t} & e^{3t} & 0 \\ 4\,e^{4t} - 12\,e^{3t} + 12\,e^{2t} - 4\,e^{t} & -12\,e^{3t} + 6\,e^{2t} + 6\,e^{4t} & -4\,e^{3t} + 4\,e^{4t} & e^{4t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{t} & 0 & 0 & 0 \\ 2\,e^{2t} - 2\,e^{t} & e^{2t} & 0 & 0 \\ 3\,e^{3t} - 6\,e^{2t} + 3\,e^{t} & -3\,e^{2t} + 3\,e^{3t} & e^{3t} & 0 \\ 4\,e^{4t} - 12\,e^{3t} + 12\,e^{2t} - 4\,e^{t} & -12\,e^{3t} + 6\,e^{2t} + 6\,e^{4t} & -4\,e^{3t} + 4\,e^{4t} & e^{4t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} \\
&= \begin{bmatrix} e^{t}c_1 \\ (2\,e^{2t} - 2\,e^{t})\,c_1 + e^{2t}c_2 \\ (3\,e^{3t} - 6\,e^{2t} + 3\,e^{t})\,c_1 + (-3\,e^{2t} + 3\,e^{3t})\,c_2 + e^{3t}c_3 \\ (4\,e^{4t} - 12\,e^{3t} + 12\,e^{2t} - 4\,e^{t})\,c_1 + (-12\,e^{3t} + 6\,e^{2t} + 6\,e^{4t})\,c_2 + (-4\,e^{3t} + 4\,e^{4t})\,c_3 + e^{4t}c_4 \end{bmatrix} \\
&= \begin{bmatrix} e^{t}c_1 \\ e^{2t}(2c_1 + c_2) - 2\,e^{t}c_1 \\ (3c_1 + 3c_2 + c_3)\,e^{3t} + (-6c_1 - 3c_2)\,e^{2t} + 3\,e^{t}c_1 \\ (4c_1 + 6c_2 + 4c_3 + c_4)\,e^{4t} + 4(-3c_1 - 3c_2 - c_3)\,e^{3t} + 6\,e^{2t}(2c_1 + c_2) - 4\,e^{t}c_1 \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

4.27.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \\ x_4'(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 4 & 4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A-\lambda I)=0$$

Expanding gives

$$\det\left(\begin{bmatrix}1&0&0&0\\2&2&0&0\\0&3&3&0\\0&0&4&4\end{bmatrix}-\lambda\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\right)=0$$

Therefore

$$\det\left(\begin{bmatrix}1-\lambda&0&0&0\\2&2-\lambda&0&0\\0&3&3-\lambda&0\\0&0&4&4-\lambda\end{bmatrix}\right)=0$$

Since the matrix A is triangular matrix, then the determinant is the product of the elements along the diagonal. Therefore the above becomes

$$(1-\lambda)(2-\lambda)(3-\lambda)(4-\lambda)=0$$

The roots of the above are the eigenvalues.

$$\lambda_1=3$$
$$\lambda_2=2$$
$$\lambda_3=4$$
$$\lambda_4=1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue
2	1	real eigenvalue
3	1	real eigenvalue
4	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 4 & 4 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 4 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 0 & 3 & 2 & 0 & 0 \\ 0 & 0 & 4 & 3 & 0 \end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{cccc|c} 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 2 & 0 & 0 \\ 0 & 0 & 4 & 3 & 0 \end{array}\right]$$

Since the current pivot $A(2,2)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{cccc|c} 2 & 1 & 0 & 0 & 0 \\ 0 & 3 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 3 & 0 \end{array}\right]$$

Since the current pivot $A(3,3)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 3 and row 4 gives

$$\left[\begin{array}{cccc|c} 2 & 1 & 0 & 0 & 0 \\ 0 & 3 & 2 & 0 & 0 \\ 0 & 0 & 4 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_4\}$ and the leading variables are $\{v_1, v_2, v_3\}$. Let $v_4 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{t}{4}, v_2 = \frac{t}{2}, v_3 = -\frac{3t}{4}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{t}{4} \\ \frac{t}{2} \\ -\frac{3t}{4} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{t}{4} \\ \frac{t}{2} \\ -\frac{3t}{4} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{t}{4} \\ \frac{t}{2} \\ -\frac{3t}{4} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{4} \\ \frac{1}{2} \\ -\frac{3}{4} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{t}{4} \\ \frac{t}{2} \\ -\frac{3t}{4} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} \\ \frac{1}{2} \\ -\frac{3}{4} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{t}{4} \\ \frac{t}{2} \\ -\frac{3t}{4} \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -3 \\ 4 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 4 & 4 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} -1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 \end{array}\right]$$

$$R_2 = R_2 + 2R_1 \Longrightarrow \left[\begin{array}{cccc|c} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 \end{array}\right]$$

Since the current pivot $A(2,2)$ is zero, then the current pivot row is replaced with a

row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{cccc|c} -1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 \end{array}\right]$$

Since the current pivot $A(3,3)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 3 and row 4 gives

$$\left[\begin{array}{cccc|c} -1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_4\}$ and the leading variables are $\{v_1, v_2, v_3\}$. Let $v_4 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = 0, v_2 = \frac{t}{6}, v_3 = -\frac{t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ \frac{t}{6} \\ -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{t}{6} \\ -\frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ \frac{t}{6} \\ -\frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ \frac{1}{6} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ \frac{t}{6} \\ -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{6} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} 0 \\ \frac{t}{6} \\ -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -3 \\ 6 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = 3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 4 & 4 \end{bmatrix} - (3) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} -2 & 0 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 4 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cccc|c} -2 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 4 & 1 & 0 \end{array}\right]$$

$$R_3 = R_3 + 3R_2 \Longrightarrow \left[\begin{array}{cccc|c} -2 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 1 & 0 \end{array}\right]$$

Since the current pivot $A(3,3)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 3 and row 4 gives

$$\left[\begin{array}{cccc|c} -2 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_4\}$ and the leading variables are $\{v_1, v_2, v_3\}$. Let $v_4 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = 0, v_2 = 0, v_3 = -\frac{t}{4}\right\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ 0 \\ -\frac{t}{4} \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\frac{t}{4} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ 0 \\ -\frac{t}{4} \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{4} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ 0 \\ -\frac{t}{4} \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{4} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} 0 \\ 0 \\ -\frac{t}{4} \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 4 \end{bmatrix}$$

Considering the eigenvalue $\lambda_4 = 4$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 4 & 4 \end{bmatrix} - (4) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 0 & 0 & 0 \\ 2 & -2 & 0 & 0 \\ 0 & 3 & -1 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} -3 & 0 & 0 & 0 & 0 \\ 2 & -2 & 0 & 0 & 0 \\ 0 & 3 & -1 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{2R_1}{3} \implies \left[\begin{array}{cccc|c} -3 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 3 & -1 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{3R_2}{2} \implies \left[\begin{array}{cccc|c} -3 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \end{array}\right]$$

$$R_4 = R_4 + 4R_3 \implies \left[\begin{array}{cccc|c} -3 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -3 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_4\}$ and the leading variables are $\{v_1, v_2, v_3\}$. Let $v_4 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_2 = 0, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity algebraic m	multiplicity geometric k	defective?	eigenvectors
3	1	1	No	$\begin{bmatrix} 0 \\ 0 \\ -\frac{1}{4} \\ 1 \end{bmatrix}$
2	1	1	No	$\begin{bmatrix} 0 \\ \frac{1}{6} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$
4	1	1	No	$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$
1	1	1	No	$\begin{bmatrix} -\frac{1}{4} \\ \frac{1}{2} \\ -\frac{3}{4} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 3 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{3t} \\
&= \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{4} \\ 1 \end{bmatrix} e^{3t}
\end{aligned}$$

Since eigenvalue 2 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{2t} \\
&= \begin{bmatrix} 0 \\ \frac{1}{6} \\ -\frac{1}{2} \\ 1 \end{bmatrix} e^{2t}
\end{aligned}
$$

Since eigenvalue 4 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_3(t) &= \vec{v}_3 e^{4t} \\
&= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} e^{4t}
\end{aligned}
$$

Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_4(t) &= \vec{v}_4 e^{t} \\
&= \begin{bmatrix} -\frac{1}{4} \\ \frac{1}{2} \\ -\frac{3}{4} \\ 1 \end{bmatrix} e^{t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t) + c_4 \vec{x}_4(t)
$$

Which is written as

$$
\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ 0 \\ -\frac{e^{3t}}{4} \\ e^{3t} \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ \frac{e^{2t}}{6} \\ -\frac{e^{2t}}{2} \\ e^{2t} \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 0 \\ e^{4t} \end{bmatrix} + c_4 \begin{bmatrix} -\frac{e^{t}}{4} \\ \frac{e^{t}}{2} \\ -\frac{3\,e^{t}}{4} \\ e^{t} \end{bmatrix}
$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = \begin{bmatrix} -\frac{c_4 e^t}{4} \\ \frac{c_2 e^{2t}}{6} + \frac{c_4 e^t}{2} \\ -\frac{c_1 e^{3t}}{4} - \frac{c_2 e^{2t}}{2} - \frac{3c_4 e^t}{4} \\ c_1 e^{3t} + c_2 e^{2t} + c_3 e^{4t} + c_4 e^t \end{bmatrix}$$

4.27.3 Maple step by step solution

Let's solve

$[x_1'(t) = x_1(t), x_2'(t) = 2x_1(t) + 2x_2(t), x_3'(t) = 3x_2(t) + 3x_3(t), x_4'(t) = 4x_3(t) + 4x_4(t)]$

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x}^{\rightarrow\prime}(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 4 & 4 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{x}^{\rightarrow\prime}(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 4 & 4 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 4 & 4 \end{bmatrix}$$

- Rewrite the system as

  $\underline{\vec{x}}'(t) = A \cdot \underline{\vec{x}}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[1, \begin{bmatrix} -\frac{1}{4} \\ \frac{1}{2} \\ -\frac{3}{4} \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} 0 \\ \frac{1}{6} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right], \left[3, \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{4} \\ 1 \end{bmatrix}\right], \left[4, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

  $$\left[1, \begin{bmatrix} -\frac{1}{4} \\ \frac{1}{2} \\ -\frac{3}{4} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

  $$\underline{\vec{x}}_1 = \mathrm{e}^t \cdot \begin{bmatrix} -\frac{1}{4} \\ \frac{1}{2} \\ -\frac{3}{4} \\ 1 \end{bmatrix}$$

- Consider eigenpair

  $$\left[2, \begin{bmatrix} 0 \\ \frac{1}{6} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_{2} = e^{2t} \cdot \begin{bmatrix} 0 \\ \frac{1}{6} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix} 3, \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{4} \\ 1 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_{3} = e^{3t} \cdot \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{4} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix} 4, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_{4} = e^{4t} \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\underline{\vec{x}} = c_1 \underline{\vec{x}}_{1} + c_2 \underline{\vec{x}}_{2} + c_3 \underline{\vec{x}}_{3} + c_4 \underline{\vec{x}}_{4}$$

- Substitute solutions into the general solution

$$\underline{\vec{x}} = c_1 e^{t} \cdot \begin{bmatrix} -\frac{1}{4} \\ \frac{1}{2} \\ -\frac{3}{4} \\ 1 \end{bmatrix} + c_2 e^{2t} \cdot \begin{bmatrix} 0 \\ \frac{1}{6} \\ -\frac{1}{2} \\ 1 \end{bmatrix} + c_3 e^{3t} \cdot \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{4} \\ 1 \end{bmatrix} + c_4 e^{4t} \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = \begin{bmatrix} -\frac{c_1 e^{t}}{4} \\ \frac{c_1 e^{t}}{2} + \frac{c_2 e^{2t}}{6} \\ -\frac{3 c_1 e^{t}}{4} - \frac{c_2 e^{2t}}{2} - \frac{c_3 e^{3t}}{4} \\ c_1 e^{t} + c_2 e^{2t} + c_3 e^{3t} + c_4 e^{4t} \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x_1(t) = -\frac{c_1 e^{t}}{4}, x_2(t) = \frac{c_1 e^{t}}{2} + \frac{c_2 e^{2t}}{6}, x_3(t) = -\frac{3 c_1 e^{t}}{4} - \frac{c_2 e^{2t}}{2} - \frac{c_3 e^{3t}}{4}, x_4(t) = c_1 e^{t} + c_2 e^{2t} + c_3 e^{3t} + c_4 e$$

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 75

```
dsolve([diff(x__1(t),t)=x__1(t)+0*x__2(t)+0*x__3(t)+0*x__4(t),diff(x__2(t),t)=2*x__1(t)+2*x_
```

$$\begin{aligned} x_1(t) &= c_4 e^{t} \\ x_2(t) &= -2 c_4 e^{t} + c_3 e^{2t} \\ x_3(t) &= c_2 e^{3t} - 3 c_3 e^{2t} + 3 c_4 e^{t} \\ x_4(t) &= c_1 e^{4t} - 4 c_2 e^{3t} + 6 c_3 e^{2t} - 4 c_4 e^{t} \end{aligned}$$

✓ Solution by Mathematica

Time used: 0.008 (sec). Leaf size: 128

```
DSolve[{x1'[t]==1*x1[t]+0*x2[t]+0*x3[t]+0*x4[t],x2'[t]==2*x1[t]+2*x2[t]+0*x3[t]+0*x4[t],x3'[
```

$$\begin{aligned} \text{x1}(t) &\to c_1 e^{t} \\ \text{x2}(t) &\to e^{t}\left(2 c_1 \left(e^{t} - 1\right) + c_2 e^{t}\right) \\ \text{x3}(t) &\to e^{t}\left(3 c_1 \left(e^{t} - 1\right)^2 + e^{t}\left(3 c_2 \left(e^{t} - 1\right) + c_3 e^{t}\right)\right) \\ \text{x4}(t) &\to e^{t}\left(4 c_1 \left(e^{t} - 1\right)^3 + e^{t}\left(6 c_2 \left(e^{t} - 1\right)^2 + e^{t}\left(4 c_3 \left(e^{t} - 1\right) + c_4 e^{t}\right)\right)\right) \end{aligned}$$

4.28 problem problem 39

Internal problem ID [342]
Internal file name [`OUTPUT/342_Sunday_June_05_2022_01_39_09_AM_80092057/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.3, The eigenvalue method for linear systems. Page 395
Problem number: problem 39.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= -2x_1(t) + 9x_4(t) \\
x_2'(t) &= 4x_1(t) + 2x_2(t) - 10x_4(t) \\
x_3'(t) &= -x_3(t) + 8x_4(t) \\
x_4'(t) &= x_4(t)
\end{aligned}
$$

4.28.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \\ x_4'(t) \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 & 9 \\ 4 & 2 & 0 & -10 \\ 0 & 0 & -1 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{-2t} & 0 & 0 & 3(e^{3t}-1)\,e^{-2t} \\ e^{2t}-e^{-2t} & e^{2t} & 0 & (-e^{4t}-2\,e^{3t}+3)\,e^{-2t} \\ 0 & 0 & e^{-t} & 4\,e^{t}-4\,e^{-t} \\ 0 & 0 & 0 & e^{t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{-2t} & 0 & 0 & 3(e^{3t}-1)\,e^{-2t} \\ e^{2t}-e^{-2t} & e^{2t} & 0 & (-e^{4t}-2\,e^{3t}+3)\,e^{-2t} \\ 0 & 0 & e^{-t} & 4\,e^{t}-4\,e^{-t} \\ 0 & 0 & 0 & e^{t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} \\
&= \begin{bmatrix} e^{-2t}c_1 + 3(e^{3t}-1)\,e^{-2t}c_4 \\ (e^{2t}-e^{-2t})\,c_1 + e^{2t}c_2 + (-e^{4t}-2\,e^{3t}+3)\,e^{-2t}c_4 \\ e^{-t}c_3 + (4\,e^{t}-4\,e^{-t})\,c_4 \\ e^{t}c_4 \end{bmatrix} \\
&= \begin{bmatrix} (3\,e^{3t}c_4 + c_1 - 3c_4)\,e^{-2t} \\ ((c_1+c_2-c_4)\,e^{4t} - 2\,e^{3t}c_4 - c_1 + 3c_4)\,e^{-2t} \\ (c_3-4c_4)\,e^{-t} + 4\,e^{t}c_4 \\ e^{t}c_4 \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

4.28.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \\ x_4'(t) \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 & 9 \\ 4 & 2 & 0 & -10 \\ 0 & 0 & -1 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -2 & 0 & 0 & 9 \\ 4 & 2 & 0 & -10 \\ 0 & 0 & -1 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -2-\lambda & 0 & 0 & 9 \\ 4 & 2-\lambda & 0 & -10 \\ 0 & 0 & -1-\lambda & 8 \\ 0 & 0 & 0 & 1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^4 - 5\lambda^2 + 4 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2$$
$$\lambda_2 = -1$$
$$\lambda_3 = 1$$
$$\lambda_4 = -2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
-2	1	real eigenvalue
1	1	real eigenvalue
2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -2 & 0 & 0 & 9 \\ 4 & 2 & 0 & -10 \\ 0 & 0 & -1 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix} - (-2)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 9 \\ 4 & 4 & 0 & -10 \\ 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} 0 & 0 & 0 & 9 & 0 \\ 4 & 4 & 0 & -10 & 0 \\ 0 & 0 & 1 & 8 & 0 \\ 0 & 0 & 0 & 3 & 0 \end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{cccc|c} 4 & 4 & 0 & -10 & 0 \\ 0 & 0 & 0 & 9 & 0 \\ 0 & 0 & 1 & 8 & 0 \\ 0 & 0 & 0 & 3 & 0 \end{array}\right]$$

Since the current pivot $A(2,3)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{cccc|c} 4 & 4 & 0 & -10 & 0 \\ 0 & 0 & 1 & 8 & 0 \\ 0 & 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 3 & 0 \end{array}\right]$$

$$R_4 = R_4 - \frac{R_3}{3} \Longrightarrow \left[\begin{array}{cccc|c} 4 & 4 & 0 & -10 & 0 \\ 0 & 0 & 1 & 8 & 0 \\ 0 & 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 4 & 4 & 0 & -10 \\ 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1, v_3, v_4\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t, v_3 = 0, v_4 = 0\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -t \\ t \\ 0 \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \\ 0 \\ 0 \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -2 & 0 & 0 & 9 \\ 4 & 2 & 0 & -10 \\ 0 & 0 & -1 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix} - (-1) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 0 & 9 \\ 4 & 3 & 0 & -10 \\ 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} -1 & 0 & 0 & 9 & 0 \\ 4 & 3 & 0 & -10 & 0 \\ 0 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{array}\right]$$

$$R_2 = R_2 + 4R_1 \Longrightarrow \left[\begin{array}{cccc|c} -1 & 0 & 0 & 9 & 0 \\ 0 & 3 & 0 & 26 & 0 \\ 0 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{array}\right]$$

$$R_4 = R_4 - \frac{R_3}{4} \Longrightarrow \left[\begin{array}{cccc|c} -1 & 0 & 0 & 9 & 0 \\ 0 & 3 & 0 & 26 & 0 \\ 0 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 0 & 0 & 9 \\ 0 & 3 & 0 & 26 \\ 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2, v_4\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_2 = 0, v_4 = 0\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ 0 \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ 0 \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ 0 \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = 1$

We need to solve $A\vec{v} = \lambda \vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -2 & 0 & 0 & 9 \\ 4 & 2 & 0 & -10 \\ 0 & 0 & -1 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 0 & 0 & 9 \\ 4 & 1 & 0 & -10 \\ 0 & 0 & -2 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} -3 & 0 & 0 & 9 & 0 \\ 4 & 1 & 0 & -10 & 0 \\ 0 & 0 & -2 & 8 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{4R_1}{3} \Longrightarrow \left[\begin{array}{cccc|c} -3 & 0 & 0 & 9 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & -2 & 8 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -3 & 0 & 0 & 9 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -2 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_4\}$ and the leading variables are $\{v_1, v_2, v_3\}$. Let $v_4 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 3t, v_2 = -2t, v_3 = 4t\}$

Hence the solution is

$$\begin{bmatrix} 3t \\ -2t \\ 4t \\ t \end{bmatrix} = \begin{bmatrix} 3t \\ -2t \\ 4t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 3t \\ -2t \\ 4t \\ t \end{bmatrix} = t \begin{bmatrix} 3 \\ -2 \\ 4 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 3t \\ -2t \\ 4t \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 4 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_4 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -2 & 0 & 0 & 9 \\ 4 & 2 & 0 & -10 \\ 0 & 0 & -1 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 0 & 0 & 9 \\ 4 & 0 & 0 & -10 \\ 0 & 0 & -3 & 8 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} -4 & 0 & 0 & 9 & 0 \\ 4 & 0 & 0 & -10 & 0 \\ 0 & 0 & -3 & 8 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cccc|c} -4 & 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -3 & 8 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{array}\right]$$

Since the current pivot $A(2,3)$ is zero, then the current pivot row is replaced with a

row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{cccc|c} -4 & 0 & 0 & 9 & 0 \\ 0 & 0 & -3 & 8 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{array}\right]$$

$$R_4 = R_4 - R_3 \Longrightarrow \left[\begin{array}{cccc|c} -4 & 0 & 0 & 9 & 0 \\ 0 & 0 & -3 & 8 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -4 & 0 & 0 & 9 \\ 0 & 0 & -3 & 8 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1, v_3, v_4\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_3 = 0, v_4 = 0\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ t \\ 0 \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ t \\ 0 \\ 0 \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
2	1	1	No	$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$
-1	1	1	No	$\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$
1	1	1	No	$\begin{bmatrix} 3 \\ -2 \\ 4 \\ 1 \end{bmatrix}$
-2	1	1	No	$\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 2 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{2t}\\ &= \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} e^{2t}\end{aligned}$$

Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{-t}\\ &= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} e^{-t}\end{aligned}$$

Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_3(t) &= \vec{v}_3 e^{t}\\ &= \begin{bmatrix} 3 \\ -2 \\ 4 \\ 1 \end{bmatrix} e^{t}\end{aligned}$$

Since eigenvalue -2 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_4(t) &= \vec{v}_4 e^{-2t}\\ &= \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} e^{-2t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t) + c_4\vec{x}_4(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ e^{2t} \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 0 \\ e^{-t} \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 3\,e^{t} \\ -2\,e^{t} \\ 4\,e^{t} \\ e^{t} \end{bmatrix} + c_4 \begin{bmatrix} -e^{-2t} \\ e^{-2t} \\ 0 \\ 0 \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = \begin{bmatrix} (3c_3 e^{3t} - c_4)\,e^{-2t} \\ (c_1 e^{4t} - 2c_3 e^{3t} + c_4)\,e^{-2t} \\ c_2 e^{-t} + 4c_3 e^{t} \\ c_3 e^{t} \end{bmatrix}$$

4.28.3 Maple step by step solution

Let's solve

$[x_1'(t) = -2x_1(t) + 9x_4(t)\,, x_2'(t) = 4x_1(t) + 2x_2(t) - 10x_4(t)\,, x_3'(t) = -x_3(t) + 8x_4(t)\,, x_4'(t) = x_4$

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x}^{\rightarrow\prime}(t) = \begin{bmatrix} -2 & 0 & 0 & 9 \\ 4 & 2 & 0 & -10 \\ 0 & 0 & -1 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{\vec{x}}'(t) = \begin{bmatrix} -2 & 0 & 0 & 9 \\ 4 & 2 & 0 & -10 \\ 0 & 0 & -1 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \underline{\vec{x}}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -2 & 0 & 0 & 9 \\ 4 & 2 & 0 & -10 \\ 0 & 0 & -1 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Rewrite the system as

$\underline{\vec{x}}'(t) = A \cdot \underline{\vec{x}}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right], \left[-1, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right], \left[1, \begin{bmatrix} 3 \\ -2 \\ 4 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-2, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_1 = \mathrm{e}^{-2t} \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_{2} = \mathrm{e}^{-t} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 3 \\ -2 \\ 4 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_{3} = \mathrm{e}^{t} \cdot \begin{bmatrix} 3 \\ -2 \\ 4 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{x}^{\rightarrow}_{4} = e^{2t} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

- General solution to the system of ODEs

$$\underline{x}^{\rightarrow} = c_1 \underline{x}^{\rightarrow}_{1} + c_2 \underline{x}^{\rightarrow}_{2} + c_3 \underline{x}^{\rightarrow}_{3} + c_4 \underline{x}^{\rightarrow}_{4}$$

- Substitute solutions into the general solution

$$\underline{x}^{\rightarrow} = c_1 e^{-2t} \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 e^{-t} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{t} \cdot \begin{bmatrix} 3 \\ -2 \\ 4 \\ 1 \end{bmatrix} + c_4 e^{2t} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = \begin{bmatrix} -(-3c_3 e^{3t} + c_1) e^{-2t} \\ (c_4 e^{4t} - 2c_3 e^{3t} + c_1) e^{-2t} \\ c_2 e^{-t} + 4c_3 e^{t} \\ c_3 e^{t} \end{bmatrix}$$

- Solution to the system of ODEs

$$\{x_1(t) = -(-3c_3 e^{3t} + c_1) e^{-2t}, x_2(t) = (c_4 e^{4t} - 2c_3 e^{3t} + c_1) e^{-2t}, x_3(t) = c_2 e^{-t} + 4c_3 e^{t}, x_4(t) = c_3 e^{t}$$

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 61

```
dsolve([diff(x__1(t),t)=-2*x__1(t)+0*x__2(t)+0*x__3(t)+9*x__4(t),diff(x__2(t),t)=4*x__1(t)+2
```

$$x_1(t) = 3c_4 e^{t} + c_2 e^{-2t}$$
$$x_2(t) = c_1 e^{2t} - 2c_4 e^{t} - c_2 e^{-2t}$$
$$x_3(t) = 4c_4 e^{t} + c_3 e^{-t}$$
$$x_4(t) = c_4 e^{t}$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 103

```
DSolve[{x1'[t]==-2*x1[t]+0*x2[t]+0*x3[t]+9*x4[t],x2'[t]==4*x1[t]+2*x2[t]+0*x3[t]-10*x4[t],x3
```

$$\begin{aligned}
\mathrm{x1}(t) &\to e^{-2t}\left(3c_4\left(e^{3t}-1\right)+c_1\right)\\
\mathrm{x2}(t) &\to e^{-2t}\left(c_1\left(e^{4t}-1\right)+(c_2-c_4)e^{4t}-2c_4e^{3t}+3c_4\right)\\
\mathrm{x3}(t) &\to e^{-t}\left(4c_4\left(e^{2t}-1\right)+c_3\right)\\
\mathrm{x4}(t) &\to c_4e^t
\end{aligned}$$

4.29 problem problem 40

Internal problem ID [343]
Internal file name [`OUTPUT/343_Sunday_June_05_2022_01_39_11_AM_717931/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.3, The eigenvalue method for linear systems. Page 395
Problem number: problem 40.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= 2x_1(t) \\
x_2'(t) &= -21x_1(t) - 5x_2(t) - 27x_3(t) - 9x_4(t) \\
x_3'(t) &= 5x_3(t) \\
x_4'(t) &= -21x_3(t) - 2x_4(t)
\end{aligned}
$$

4.29.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \\ x_4'(t) \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ -21 & -5 & -27 & -9 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & -21 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{2t} & 0 & 0 & 0 \\ -3(e^{7t}-1)\,e^{-5t} & e^{-5t} & 9\,e^{-5t}-9\,e^{-2t} & -3\,e^{-2t}+3\,e^{-5t} \\ 0 & 0 & e^{5t} & 0 \\ 0 & 0 & -3(e^{7t}-1)\,e^{-2t} & e^{-2t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{2t} & 0 & 0 & 0 \\ -3(e^{7t}-1)\,e^{-5t} & e^{-5t} & 9\,e^{-5t}-9\,e^{-2t} & -3\,e^{-2t}+3\,e^{-5t} \\ 0 & 0 & e^{5t} & 0 \\ 0 & 0 & -3(e^{7t}-1)\,e^{-2t} & e^{-2t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} \\
&= \begin{bmatrix} e^{2t}c_1 \\ -3(e^{7t}-1)\,e^{-5t}c_1 + e^{-5t}c_2 + (9\,e^{-5t}-9\,e^{-2t})\,c_3 + (-3\,e^{-2t}+3\,e^{-5t})\,c_4 \\ e^{5t}c_3 \\ -3(e^{7t}-1)\,e^{-2t}c_3 + e^{-2t}c_4 \end{bmatrix} \\
&= \begin{bmatrix} e^{2t}c_1 \\ -3\left((3c_3+c_4)\,e^{3t} + e^{7t}c_1 - c_1 - \frac{c_2}{3} - 3c_3 - c_4\right) e^{-5t} \\ e^{5t}c_3 \\ -3\left(e^{7t}c_3 - c_3 - \frac{c_4}{3}\right) e^{-2t} \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

4.29.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \\ x_4'(t) \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ -21 & -5 & -27 & -9 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & -21 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 2 & 0 & 0 & 0 \\ -21 & -5 & -27 & -9 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & -21 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 2-\lambda & 0 & 0 & 0 \\ -21 & -5-\lambda & -27 & -9 \\ 0 & 0 & 5-\lambda & 0 \\ 0 & 0 & -21 & -2-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^4 - 29\lambda^2 + 100 = 0$$

The roots of the above are the eigenvalues.

$$\begin{aligned} \lambda_1 &= 2 \\ \lambda_2 &= -5 \\ \lambda_3 &= 5 \\ \lambda_4 &= -2 \end{aligned}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-2	1	real eigenvalue
2	1	real eigenvalue
-5	1	real eigenvalue
5	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -5$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 0 & 0 & 0 \\ -21 & -5 & -27 & -9 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & -21 & -2 \end{bmatrix} - (-5)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 7 & 0 & 0 & 0 \\ -21 & 0 & -27 & -9 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & -21 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} 7 & 0 & 0 & 0 & 0 \\ -21 & 0 & -27 & -9 & 0 \\ 0 & 0 & 10 & 0 & 0 \\ 0 & 0 & -21 & 3 & 0 \end{array}\right]$$

$$R_2 = R_2 + 3R_1 \Longrightarrow \left[\begin{array}{cccc|c} 7 & 0 & 0 & 0 & 0 \\ 0 & 0 & -27 & -9 & 0 \\ 0 & 0 & 10 & 0 & 0 \\ 0 & 0 & -21 & 3 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{10R_2}{27} \Longrightarrow \left[\begin{array}{cccc|c} 7 & 0 & 0 & 0 & 0 \\ 0 & 0 & -27 & -9 & 0 \\ 0 & 0 & 0 & -\frac{10}{3} & 0 \\ 0 & 0 & -21 & 3 & 0 \end{array}\right]$$

$$R_4 = R_4 - \frac{7R_2}{9} \Longrightarrow \left[\begin{array}{cccc|c} 7 & 0 & 0 & 0 & 0 \\ 0 & 0 & -27 & -9 & 0 \\ 0 & 0 & 0 & -\frac{10}{3} & 0 \\ 0 & 0 & 0 & 10 & 0 \end{array}\right]$$

$$R_4 = R_4 + 3R_3 \Longrightarrow \left[\begin{array}{cccc|c} 7 & 0 & 0 & 0 & 0 \\ 0 & 0 & -27 & -9 & 0 \\ 0 & 0 & 0 & -\frac{10}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 7 & 0 & 0 & 0 \\ 0 & 0 & -27 & -9 \\ 0 & 0 & 0 & -\frac{10}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1, v_3, v_4\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_3 = 0, v_4 = 0\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ t \\ 0 \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ t \\ 0 \\ 0 \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 0 & 0 & 0 \\ -21 & -5 & -27 & -9 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & -21 & -2 \end{bmatrix} - (-2) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 0 & 0 & 0 \\ -21 & -3 & -27 & -9 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & -21 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} 4 & 0 & 0 & 0 & 0 \\ -21 & -3 & -27 & -9 & 0 \\ 0 & 0 & 7 & 0 & 0 \\ 0 & 0 & -21 & 0 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{21R_1}{4} \Longrightarrow \left[\begin{array}{cccc|c} 4 & 0 & 0 & 0 & 0 \\ 0 & -3 & -27 & -9 & 0 \\ 0 & 0 & 7 & 0 & 0 \\ 0 & 0 & -21 & 0 & 0 \end{array}\right]$$

$$R_4 = R_4 + 3R_3 \Longrightarrow \left[\begin{array}{cccc|c} 4 & 0 & 0 & 0 & 0 \\ 0 & -3 & -27 & -9 & 0 \\ 0 & 0 & 7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & -3 & -27 & -9 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_4\}$ and the leading variables are $\{v_1, v_2, v_3\}$. Let $v_4 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_2 = -3t, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ -3t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -3t \\ 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ -3t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ -3t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 0 & 0 & 0 \\ -21 & -5 & -27 & -9 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & -21 & -2 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ -21 & -7 & -27 & -9 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & -21 & -4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ -21 & -7 & -27 & -9 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & -21 & -4 & 0 \end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{cccc|c} -21 & -7 & -27 & -9 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & -21 & -4 & 0 \end{array}\right]$$

Since the current pivot $A(2,3)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{cccc|c} -21 & -7 & -27 & -9 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -21 & -4 & 0 \end{array}\right]$$

$$R_4 = R_4 + 7R_2 \implies \left[\begin{array}{cccc|c} -21 & -7 & -27 & -9 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 \end{array}\right]$$

Since the current pivot $A(3,4)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 3 and row 4 gives

$$\left[\begin{array}{cccc|c} -21 & -7 & -27 & -9 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -21 & -7 & -27 & -9 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1, v_3, v_4\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{t}{3}, v_3 = 0, v_4 = 0\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{t}{3} \\ t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{t}{3} \\ t \\ 0 \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{t}{3} \\ t \\ 0 \\ 0 \end{bmatrix} = t \begin{bmatrix} -\frac{1}{3} \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{t}{3} \\ t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{t}{3} \\ t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 0 \\ 0 \end{bmatrix}$$

Considering the eigenvalue $\lambda_4 = 5$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 0 & 0 & 0 \\ -21 & -5 & -27 & -9 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & -21 & -2 \end{bmatrix} - (5) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 0 & 0 & 0 \\ -21 & -10 & -27 & -9 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -21 & -7 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} -3 & 0 & 0 & 0 & 0 \\ -21 & -10 & -27 & -9 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -21 & -7 & 0 \end{array}\right]$$

$$R_2 = R_2 - 7R_1 \implies \left[\begin{array}{cccc|c} -3 & 0 & 0 & 0 & 0 \\ 0 & -10 & -27 & -9 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -21 & -7 & 0 \end{array}\right]$$

Since the current pivot $A(3,3)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 3 and row 4 gives

$$\left[\begin{array}{cccc|c} -3 & 0 & 0 & 0 & 0 \\ 0 & -10 & -27 & -9 & 0 \\ 0 & 0 & -21 & -7 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -3 & 0 & 0 & 0 \\ 0 & -10 & -27 & -9 \\ 0 & 0 & -21 & -7 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_4\}$ and the leading variables are $\{v_1, v_2, v_3\}$. Let $v_4 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = 0, v_2 = 0, v_3 = -\frac{t}{3}\right\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ 0 \\ -\frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\frac{t}{3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ 0 \\ -\frac{t}{3} \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{3} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ 0 \\ -\frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{3} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} 0 \\ 0 \\ -\frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 3 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity: algebraic m	multiplicity: geometric k	defective?	eigenvectors
2	1	1	No	$\begin{bmatrix} -\frac{1}{3} \\ 1 \\ 0 \\ 0 \end{bmatrix}$
-5	1	1	No	$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$
5	1	1	No	$\begin{bmatrix} 0 \\ 0 \\ -\frac{1}{3} \\ 1 \end{bmatrix}$
-2	1	1	No	$\begin{bmatrix} 0 \\ -3 \\ 0 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 2 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{2t} \\ &= \begin{bmatrix} -\frac{1}{3} \\ 1 \\ 0 \\ 0 \end{bmatrix} e^{2t}\end{aligned}$$

Since eigenvalue -5 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_2(t) = \vec{v}_2 e^{-5t}$$
$$= \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} e^{-5t}$$

Since eigenvalue 5 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_3(t) = \vec{v}_3 e^{5t}$$
$$= \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{3} \\ 1 \end{bmatrix} e^{5t}$$

Since eigenvalue -2 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_4(t) = \vec{v}_4 e^{-2t}$$
$$= \begin{bmatrix} 0 \\ -3 \\ 0 \\ 1 \end{bmatrix} e^{-2t}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t) + c_4 \vec{x}_4(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = c_1 \begin{bmatrix} -\frac{e^{2t}}{3} \\ e^{2t} \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ e^{-5t} \\ 0 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ -\frac{e^{5t}}{3} \\ e^{5t} \end{bmatrix} + c_4 \begin{bmatrix} 0 \\ -3\,e^{-2t} \\ 0 \\ e^{-2t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = \begin{bmatrix} -\frac{c_1 \mathrm{e}^{2t}}{3} \\ \left(c_1 \mathrm{e}^{7t} - 3c_4 \mathrm{e}^{3t} + c_2\right) \mathrm{e}^{-5t} \\ -\frac{c_3 \mathrm{e}^{5t}}{3} \\ \left(c_3 \mathrm{e}^{7t} + c_4\right) \mathrm{e}^{-2t} \end{bmatrix}$$

4.29.3 Maple step by step solution

Let's solve

$[x_1'(t) = 2x_1(t), x_2'(t) = -21x_1(t) - 5x_2(t) - 27x_3(t) - 9x_4(t), x_3'(t) = 5x_3(t), x_4'(t) = -21x_3(t) -$

- Define vector

$$\underline{x^{\rightarrow}}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x^{\rightarrow}}'(t) = \begin{bmatrix} 2 & 0 & 0 & 0 \\ -21 & -5 & -27 & -9 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & -21 & -2 \end{bmatrix} \cdot \underline{x^{\rightarrow}}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{x^{\rightarrow}}'(t) = \begin{bmatrix} 2 & 0 & 0 & 0 \\ -21 & -5 & -27 & -9 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & -21 & -2 \end{bmatrix} \cdot \underline{x^{\rightarrow}}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ -21 & -5 & -27 & -9 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & -21 & -2 \end{bmatrix}$$

- Rewrite the system as

  $\underline{\vec{x}}'(t) = A \cdot \underline{\vec{x}}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-5, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right], \left[-2, \begin{bmatrix} 0 \\ -3 \\ 0 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} -\frac{1}{3} \\ 1 \\ 0 \\ 0 \end{bmatrix}\right], \left[5, \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{3} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-5, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_1 = \mathrm{e}^{-5t} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\left[-2, \begin{bmatrix} 0 \\ -3 \\ 0 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_2 = e^{-2t} \cdot \begin{bmatrix} 0 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix} 2, \begin{bmatrix} -\frac{1}{3} \\ 1 \\ 0 \\ 0 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_3 = e^{2t} \cdot \begin{bmatrix} -\frac{1}{3} \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix} 5, \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{3} \\ 1 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_4 = e^{5t} \cdot \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{3} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\underline{\vec{x}} = c_1\underline{\vec{x}}_1 + c_2\underline{\vec{x}}_2 + c_3\underline{\vec{x}}_3 + c_4\underline{\vec{x}}_4$$

- Substitute solutions into the general solution

$$\vec{x} = c_1 e^{-5t} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + e^{-2t} c_2 \cdot \begin{bmatrix} 0 \\ -3 \\ 0 \\ 1 \end{bmatrix} + c_3 e^{2t} \cdot \begin{bmatrix} -\frac{1}{3} \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_4 e^{5t} \cdot \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{3} \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = \begin{bmatrix} -\frac{c_3 e^{2t}}{3} \\ (c_3 e^{7t} - 3c_2 e^{3t} + c_1) e^{-5t} \\ -\frac{c_4 e^{5t}}{3} \\ (c_4 e^{7t} + c_2) e^{-2t} \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x_1(t) = -\frac{c_3 e^{2t}}{3}, x_2(t) = (c_3 e^{7t} - 3c_2 e^{3t} + c_1) e^{-5t}, x_3(t) = -\frac{c_4 e^{5t}}{3}, x_4(t) = (c_4 e^{7t} + c_2) e^{-2t}\right\}$$

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 61

```
dsolve([diff(x__1(t),t)=2*x__1(t)+0*x__2(t)+0*x__3(t)+0*x__4(t),diff(x__2(t),t)=-21*x__1(t)-
```

$$x_1(t) = c_4 e^{2t}$$
$$x_2(t) = -3c_4 e^{2t} - 3c_2 e^{-2t} + c_1 e^{-5t}$$
$$x_3(t) = c_3 e^{5t}$$
$$x_4(t) = -3c_3 e^{5t} + c_2 e^{-2t}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 86

```
DSolve[{x1'[t]==2*x1[t]+0*x2[t]+0*x3[t]+0*x4[t],x2'[t]==-21*x1[t]-5*x2[t]-27*x3[t]-9*x4[t],x
```

$$\text{x1}(t) \to c_1 e^{2t}$$
$$\text{x2}(t) \to e^{-5t}\left(-3c_1\left(e^{7t} - 1\right) - 3(3c_3 + c_4)\left(e^{3t} - 1\right) + c_2\right)$$
$$\text{x3}(t) \to c_3 e^{5t}$$
$$\text{x4}(t) \to e^{-2t}\left(c_4 - 3c_3\left(e^{7t} - 1\right)\right)$$

4.30 problem problem 41

Internal problem ID [344]
Internal file name [`OUTPUT/344_Sunday_June_05_2022_01_39_13_AM_79418298/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.3, The eigenvalue method for linear systems. Page 395
Problem number: problem 41.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= 4x_1(t) + x_2(t) + x_3(t) + 7x_4(t) \\
x_2'(t) &= x_1(t) + 4x_2(t) + 10x_3(t) + x_4(t) \\
x_3'(t) &= x_1(t) + 10x_2(t) + 4x_3(t) + x_4(t) \\
x_4'(t) &= 7x_1(t) + x_2(t) + x_3(t) + 4x_4(t)
\end{aligned}
$$

With initial conditions

$$[x_1(0) = 3, x_2(0) = 1, x_3(0) = 1, x_4(0) = 3]$$

4.30.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \\ x_4'(t) \end{bmatrix} = \begin{bmatrix} 4 & 1 & 1 & 7 \\ 1 & 4 & 10 & 1 \\ 1 & 10 & 4 & 1 \\ 7 & 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{(e^{18t}+4\,e^{13t}+5)e^{-3t}}{10} & \frac{e^{15t}}{5}-\frac{e^{10t}}{5} & \frac{e^{15t}}{5}-\frac{e^{10t}}{5} & \frac{(e^{18t}+4\,e^{13t}-5)e^{-3t}}{10} \\ \frac{e^{15t}}{5}-\frac{e^{10t}}{5} & \frac{(4\,e^{21t}+e^{16t}+5)e^{-6t}}{10} & \frac{(4\,e^{21t}+e^{16t}-5)e^{-6t}}{10} & \frac{e^{15t}}{5}-\frac{e^{10t}}{5} \\ \frac{e^{15t}}{5}-\frac{e^{10t}}{5} & \frac{(4\,e^{21t}+e^{16t}-5)e^{-6t}}{10} & \frac{(4\,e^{21t}+e^{16t}+5)e^{-6t}}{10} & \frac{e^{15t}}{5}-\frac{e^{10t}}{5} \\ \frac{(e^{18t}+4\,e^{13t}-5)e^{-3t}}{10} & \frac{e^{15t}}{5}-\frac{e^{10t}}{5} & \frac{e^{15t}}{5}-\frac{e^{10t}}{5} & \frac{(e^{18t}+4\,e^{13t}+5)e^{-3t}}{10} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{x}_0 \\ &= \begin{bmatrix} \frac{(e^{18t}+4\,e^{13t}+5)e^{-3t}}{10} & \frac{e^{15t}}{5}-\frac{e^{10t}}{5} & \frac{e^{15t}}{5}-\frac{e^{10t}}{5} & \frac{(e^{18t}+4\,e^{13t}-5)e^{-3t}}{10} \\ \frac{e^{15t}}{5}-\frac{e^{10t}}{5} & \frac{(4\,e^{21t}+e^{16t}+5)e^{-6t}}{10} & \frac{(4\,e^{21t}+e^{16t}-5)e^{-6t}}{10} & \frac{e^{15t}}{5}-\frac{e^{10t}}{5} \\ \frac{e^{15t}}{5}-\frac{e^{10t}}{5} & \frac{(4\,e^{21t}+e^{16t}-5)e^{-6t}}{10} & \frac{(4\,e^{21t}+e^{16t}+5)e^{-6t}}{10} & \frac{e^{15t}}{5}-\frac{e^{10t}}{5} \\ \frac{(e^{18t}+4\,e^{13t}-5)e^{-3t}}{10} & \frac{e^{15t}}{5}-\frac{e^{10t}}{5} & \frac{e^{15t}}{5}-\frac{e^{10t}}{5} & \frac{(e^{18t}+4\,e^{13t}+5)e^{-3t}}{10} \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 1 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} \frac{3(e^{18t}+4\,e^{13t}+5)e^{-3t}}{10} + \frac{2\,e^{15t}}{5} - \frac{2\,e^{10t}}{5} + \frac{3(e^{18t}+4\,e^{13t}-5)e^{-3t}}{10} \\ \frac{6\,e^{15t}}{5} - \frac{6\,e^{10t}}{5} + \frac{(4\,e^{21t}+e^{16t}+5)e^{-6t}}{10} + \frac{(4\,e^{21t}+e^{16t}-5)e^{-6t}}{10} \\ \frac{6\,e^{15t}}{5} - \frac{6\,e^{10t}}{5} + \frac{(4\,e^{21t}+e^{16t}+5)e^{-6t}}{10} + \frac{(4\,e^{21t}+e^{16t}-5)e^{-6t}}{10} \\ \frac{3(e^{18t}+4\,e^{13t}+5)e^{-3t}}{10} + \frac{2\,e^{15t}}{5} - \frac{2\,e^{10t}}{5} + \frac{3(e^{18t}+4\,e^{13t}-5)e^{-3t}}{10} \end{bmatrix} \\ &= \begin{bmatrix} e^{15t} + 2\,e^{10t} \\ 2\,e^{15t} - e^{10t} \\ 2\,e^{15t} - e^{10t} \\ e^{15t} + 2\,e^{10t} \end{bmatrix} \end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

4.30.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \\ x_4'(t) \end{bmatrix} = \begin{bmatrix} 4 & 1 & 1 & 7 \\ 1 & 4 & 10 & 1 \\ 1 & 10 & 4 & 1 \\ 7 & 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 4 & 1 & 1 & 7 \\ 1 & 4 & 10 & 1 \\ 1 & 10 & 4 & 1 \\ 7 & 1 & 1 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 4-\lambda & 1 & 1 & 7 \\ 1 & 4-\lambda & 10 & 1 \\ 1 & 10 & 4-\lambda & 1 \\ 7 & 1 & 1 & 4-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^4 - 16\lambda^3 - 57\lambda^2 + 900\lambda + 2700 = 0$$

The roots of the above are the eigenvalues.

$$\begin{aligned} \lambda_1 &= -3 \\ \lambda_2 &= 10 \\ \lambda_3 &= -6 \\ \lambda_4 &= 15 \end{aligned}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-3	1	real eigenvalue
-6	1	real eigenvalue
10	1	real eigenvalue
15	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -6$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 4 & 1 & 1 & 7 \\ 1 & 4 & 10 & 1 \\ 1 & 10 & 4 & 1 \\ 7 & 1 & 1 & 4 \end{bmatrix} - (-6)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 10 & 1 & 1 & 7 \\ 1 & 10 & 10 & 1 \\ 1 & 10 & 10 & 1 \\ 7 & 1 & 1 & 10 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} 10 & 1 & 1 & 7 & 0 \\ 1 & 10 & 10 & 1 & 0 \\ 1 & 10 & 10 & 1 & 0 \\ 7 & 1 & 1 & 10 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{10} \Longrightarrow \left[\begin{array}{cccc|c} 10 & 1 & 1 & 7 & 0 \\ 0 & \frac{99}{10} & \frac{99}{10} & \frac{3}{10} & 0 \\ 1 & 10 & 10 & 1 & 0 \\ 7 & 1 & 1 & 10 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{R_1}{10} \Longrightarrow \left[\begin{array}{cccc|c} 10 & 1 & 1 & 7 & 0 \\ 0 & \frac{99}{10} & \frac{99}{10} & \frac{3}{10} & 0 \\ 0 & \frac{99}{10} & \frac{99}{10} & \frac{3}{10} & 0 \\ 7 & 1 & 1 & 10 & 0 \end{array}\right]$$

$$R_4 = R_4 - \frac{7R_1}{10} \Longrightarrow \left[\begin{array}{cccc|c} 10 & 1 & 1 & 7 & 0 \\ 0 & \frac{99}{10} & \frac{99}{10} & \frac{3}{10} & 0 \\ 0 & \frac{99}{10} & \frac{99}{10} & \frac{3}{10} & 0 \\ 0 & \frac{3}{10} & \frac{3}{10} & \frac{51}{10} & 0 \end{array}\right]$$

$$R_3 = R_3 - R_2 \Longrightarrow \left[\begin{array}{cccc|c} 10 & 1 & 1 & 7 & 0 \\ 0 & \frac{99}{10} & \frac{99}{10} & \frac{3}{10} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{3}{10} & \frac{3}{10} & \frac{51}{10} & 0 \end{array}\right]$$

$$R_4 = R_4 - \frac{R_2}{33} \Longrightarrow \left[\begin{array}{cccc|c} 10 & 1 & 1 & 7 & 0 \\ 0 & \frac{99}{10} & \frac{99}{10} & \frac{3}{10} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{56}{11} & 0 \end{array}\right]$$

Since the current pivot $A(3,4)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 3 and row 4 gives

$$\left[\begin{array}{cccc|c} 10 & 1 & 1 & 7 & 0 \\ 0 & \frac{99}{10} & \frac{99}{10} & \frac{3}{10} & 0 \\ 0 & 0 & 0 & \frac{56}{11} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 10 & 1 & 1 & 7 \\ 0 & \frac{99}{10} & \frac{99}{10} & \frac{3}{10} \\ 0 & 0 & 0 & \frac{56}{11} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2, v_4\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_2 = -t, v_4 = 0\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ -t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -t \\ t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ -t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ -t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 4 & 1 & 1 & 7 \\ 1 & 4 & 10 & 1 \\ 1 & 10 & 4 & 1 \\ 7 & 1 & 1 & 4 \end{bmatrix} - (-3) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 7 & 1 & 1 & 7 \\ 1 & 7 & 10 & 1 \\ 1 & 10 & 7 & 1 \\ 7 & 1 & 1 & 7 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} 7 & 1 & 1 & 7 & 0 \\ 1 & 7 & 10 & 1 & 0 \\ 1 & 10 & 7 & 1 & 0 \\ 7 & 1 & 1 & 7 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{7} \Longrightarrow \left[\begin{array}{cccc|c} 7 & 1 & 1 & 7 & 0 \\ 0 & \frac{48}{7} & \frac{69}{7} & 0 & 0 \\ 1 & 10 & 7 & 1 & 0 \\ 7 & 1 & 1 & 7 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{R_1}{7} \Longrightarrow \left[\begin{array}{cccc|c} 7 & 1 & 1 & 7 & 0 \\ 0 & \frac{48}{7} & \frac{69}{7} & 0 & 0 \\ 0 & \frac{69}{7} & \frac{48}{7} & 0 & 0 \\ 7 & 1 & 1 & 7 & 0 \end{array}\right]$$

$$R_4 = R_4 - R_1 \Longrightarrow \left[\begin{array}{cccc|c} 7 & 1 & 1 & 7 & 0 \\ 0 & \frac{48}{7} & \frac{69}{7} & 0 & 0 \\ 0 & \frac{69}{7} & \frac{48}{7} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{23R_2}{16} \Longrightarrow \left[\begin{array}{cccc|c} 7 & 1 & 1 & 7 & 0 \\ 0 & \frac{48}{7} & \frac{69}{7} & 0 & 0 \\ 0 & 0 & -\frac{117}{16} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 7 & 1 & 1 & 7 \\ 0 & \frac{48}{7} & \frac{69}{7} & 0 \\ 0 & 0 & -\frac{117}{16} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_4\}$ and the leading variables are $\{v_1, v_2, v_3\}$. Let $v_4 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t, v_2 = 0, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} -t \\ 0 \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} -t \\ 0 \\ 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ 0 \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ 0 \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = 10$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 4 & 1 & 1 & 7 \\ 1 & 4 & 10 & 1 \\ 1 & 10 & 4 & 1 \\ 7 & 1 & 1 & 4 \end{bmatrix} - (10) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -6 & 1 & 1 & 7 \\ 1 & -6 & 10 & 1 \\ 1 & 10 & -6 & 1 \\ 7 & 1 & 1 & -6 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} -6 & 1 & 1 & 7 & 0 \\ 1 & -6 & 10 & 1 & 0 \\ 1 & 10 & -6 & 1 & 0 \\ 7 & 1 & 1 & -6 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{6} \implies \left[\begin{array}{cccc|c} -6 & 1 & 1 & 7 & 0 \\ 0 & -\frac{35}{6} & \frac{61}{6} & \frac{13}{6} & 0 \\ 1 & 10 & -6 & 1 & 0 \\ 7 & 1 & 1 & -6 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{R_1}{6} \implies \left[\begin{array}{cccc|c} -6 & 1 & 1 & 7 & 0 \\ 0 & -\frac{35}{6} & \frac{61}{6} & \frac{13}{6} & 0 \\ 0 & \frac{61}{6} & -\frac{35}{6} & \frac{13}{6} & 0 \\ 7 & 1 & 1 & -6 & 0 \end{array}\right]$$

$$R_4 = R_4 + \frac{7R_1}{6} \implies \left[\begin{array}{cccc|c} -6 & 1 & 1 & 7 & 0 \\ 0 & -\frac{35}{6} & \frac{61}{6} & \frac{13}{6} & 0 \\ 0 & \frac{61}{6} & -\frac{35}{6} & \frac{13}{6} & 0 \\ 0 & \frac{13}{6} & \frac{13}{6} & \frac{13}{6} & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{61R_2}{35} \implies \left[\begin{array}{cccc|c} -6 & 1 & 1 & 7 & 0 \\ 0 & -\frac{35}{6} & \frac{61}{6} & \frac{13}{6} & 0 \\ 0 & 0 & \frac{416}{35} & \frac{208}{35} & 0 \\ 0 & \frac{13}{6} & \frac{13}{6} & \frac{13}{6} & 0 \end{array}\right]$$

$$R_4 = R_4 + \frac{13R_2}{35} \implies \left[\begin{array}{cccc|c} -6 & 1 & 1 & 7 & 0 \\ 0 & -\frac{35}{6} & \frac{61}{6} & \frac{13}{6} & 0 \\ 0 & 0 & \frac{416}{35} & \frac{208}{35} & 0 \\ 0 & 0 & \frac{208}{35} & \frac{104}{35} & 0 \end{array}\right]$$

$$R_4 = R_4 - \frac{R_3}{2} \Longrightarrow \left[\begin{array}{cccc|c} -6 & 1 & 1 & 7 & 0 \\ 0 & -\frac{35}{6} & \frac{61}{6} & \frac{13}{6} & 0 \\ 0 & 0 & \frac{416}{35} & \frac{208}{35} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -6 & 1 & 1 & 7 \\ 0 & -\frac{35}{6} & \frac{61}{6} & \frac{13}{6} \\ 0 & 0 & \frac{416}{35} & \frac{208}{35} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_4\}$ and the leading variables are $\{v_1, v_2, v_3\}$. Let $v_4 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = t, v_2 = -\frac{t}{2}, v_3 = -\frac{t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} t \\ -\frac{t}{2} \\ -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} t \\ -\frac{t}{2} \\ -\frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ -\frac{t}{2} \\ -\frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ -\frac{t}{2} \\ -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} t \\ -\frac{t}{2} \\ -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -1 \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_4 = 15$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 4 & 1 & 1 & 7 \\ 1 & 4 & 10 & 1 \\ 1 & 10 & 4 & 1 \\ 7 & 1 & 1 & 4 \end{bmatrix} - (15) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -11 & 1 & 1 & 7 \\ 1 & -11 & 10 & 1 \\ 1 & 10 & -11 & 1 \\ 7 & 1 & 1 & -11 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} -11 & 1 & 1 & 7 & 0 \\ 1 & -11 & 10 & 1 & 0 \\ 1 & 10 & -11 & 1 & 0 \\ 7 & 1 & 1 & -11 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{11} \Longrightarrow \left[\begin{array}{cccc|c} -11 & 1 & 1 & 7 & 0 \\ 0 & -\frac{120}{11} & \frac{111}{11} & \frac{18}{11} & 0 \\ 1 & 10 & -11 & 1 & 0 \\ 7 & 1 & 1 & -11 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{R_1}{11} \Longrightarrow \left[\begin{array}{cccc|c} -11 & 1 & 1 & 7 & 0 \\ 0 & -\frac{120}{11} & \frac{111}{11} & \frac{18}{11} & 0 \\ 0 & \frac{111}{11} & -\frac{120}{11} & \frac{18}{11} & 0 \\ 7 & 1 & 1 & -11 & 0 \end{array}\right]$$

$$R_4 = R_4 + \frac{7R_1}{11} \Longrightarrow \left[\begin{array}{cccc|c} -11 & 1 & 1 & 7 & 0 \\ 0 & -\frac{120}{11} & \frac{111}{11} & \frac{18}{11} & 0 \\ 0 & \frac{111}{11} & -\frac{120}{11} & \frac{18}{11} & 0 \\ 0 & \frac{18}{11} & \frac{18}{11} & -\frac{72}{11} & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{37R_2}{40} \Longrightarrow \left[\begin{array}{cccc|c} -11 & 1 & 1 & 7 & 0 \\ 0 & -\frac{120}{11} & \frac{111}{11} & \frac{18}{11} & 0 \\ 0 & 0 & -\frac{63}{40} & \frac{63}{20} & 0 \\ 0 & \frac{18}{11} & \frac{18}{11} & -\frac{72}{11} & 0 \end{array}\right]$$

$$R_4 = R_4 + \frac{3R_2}{20} \Longrightarrow \left[\begin{array}{cccc|c} -11 & 1 & 1 & 7 & 0 \\ 0 & -\frac{120}{11} & \frac{111}{11} & \frac{18}{11} & 0 \\ 0 & 0 & -\frac{63}{40} & \frac{63}{20} & 0 \\ 0 & 0 & \frac{63}{20} & -\frac{63}{10} & 0 \end{array}\right]$$

$$R_4 = R_4 + 2R_3 \Longrightarrow \left[\begin{array}{cccc|c} -11 & 1 & 1 & 7 & 0 \\ 0 & -\frac{120}{11} & \frac{111}{11} & \frac{18}{11} & 0 \\ 0 & 0 & -\frac{63}{40} & \frac{63}{20} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -11 & 1 & 1 & 7 \\ 0 & -\frac{120}{11} & \frac{111}{11} & \frac{18}{11} \\ 0 & 0 & -\frac{63}{40} & \frac{63}{20} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_4\}$ and the leading variables are $\{v_1, v_2, v_3\}$. Let $v_4 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = 2t, v_3 = 2t\}$

Hence the solution is

$$\begin{bmatrix} t \\ 2t \\ 2t \\ t \end{bmatrix} = \begin{bmatrix} t \\ 2t \\ 2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ 2t \\ 2t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ 2t \\ 2t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity algebraic m	multiplicity geometric k	defective?	eigenvectors
-3	1	1	No	$\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$
10	1	1	No	$\begin{bmatrix} 1 \\ -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$
-6	1	1	No	$\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$
15	1	1	No	$\begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -3 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{-3t} \\ &= \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} e^{-3t} \end{aligned}$$

Since eigenvalue 10 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{10t} \\ &= \begin{bmatrix} 1 \\ -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix} e^{10t}\end{aligned}$$

Since eigenvalue -6 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_3(t) &= \vec{v}_3 e^{-6t} \\ &= \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} e^{-6t}\end{aligned}$$

Since eigenvalue 15 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_4(t) &= \vec{v}_4 e^{15t} \\ &= \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix} e^{15t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t) + c_4 \vec{x}_4(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = c_1 \begin{bmatrix} -\mathrm{e}^{-3t} \\ 0 \\ 0 \\ \mathrm{e}^{-3t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{10t} \\ -\frac{\mathrm{e}^{10t}}{2} \\ -\frac{\mathrm{e}^{10t}}{2} \\ \mathrm{e}^{10t} \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ -\mathrm{e}^{-6t} \\ \mathrm{e}^{-6t} \\ 0 \end{bmatrix} + c_4 \begin{bmatrix} \mathrm{e}^{15t} \\ 2\,\mathrm{e}^{15t} \\ 2\,\mathrm{e}^{15t} \\ \mathrm{e}^{15t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = \begin{bmatrix} -\left(-c_4 \mathrm{e}^{18t} - c_2 \mathrm{e}^{13t} + c_1\right) \mathrm{e}^{-3t} \\ -\frac{\left(-4c_4 \mathrm{e}^{21t} + c_2 \mathrm{e}^{16t} + 2c_3\right) \mathrm{e}^{-6t}}{2} \\ -\frac{\left(-4c_4 \mathrm{e}^{21t} + c_2 \mathrm{e}^{16t} - 2c_3\right) \mathrm{e}^{-6t}}{2} \\ \left(c_4 \mathrm{e}^{18t} + c_2 \mathrm{e}^{13t} + c_1\right) \mathrm{e}^{-3t} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x_1(0) = 3 \\ x_2(0) = 1 \\ x_3(0) = 1 \\ x_4(0) = 3 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 3 \\ 1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} c_4 + c_2 - c_1 \\ 2c_4 - \frac{c_2}{2} - c_3 \\ 2c_4 - \frac{c_2}{2} + c_3 \\ c_4 + c_2 + c_1 \end{bmatrix}$$

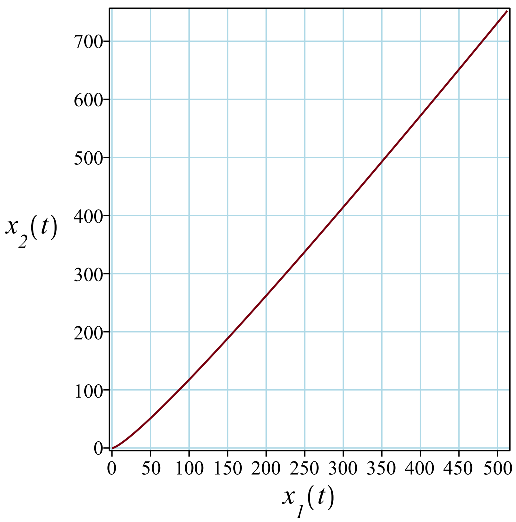
Solving for the constants of integrations gives

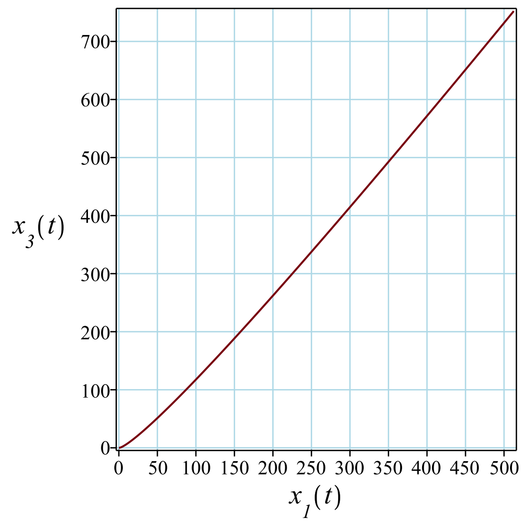
$$\begin{bmatrix} c_1 = 0 \\ c_2 = 2 \\ c_3 = 0 \\ c_4 = 1 \end{bmatrix}$$

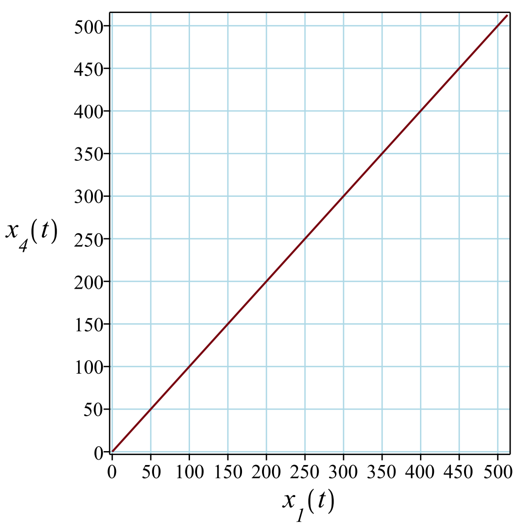
Substituting these constants back in original solution in Eq. (1) gives

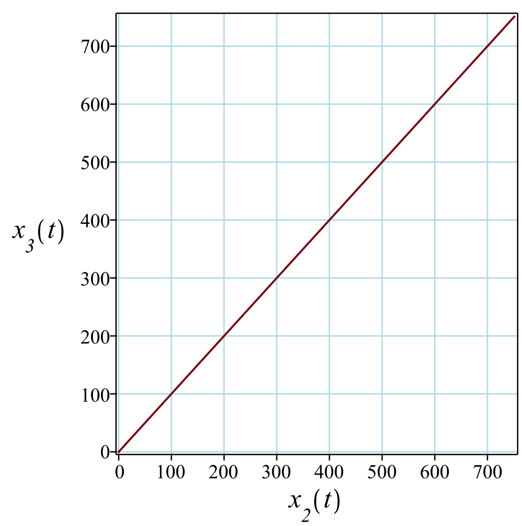
$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = \begin{bmatrix} -\left(-\mathrm{e}^{18t} - 2\,\mathrm{e}^{13t}\right) \mathrm{e}^{-3t} \\ -\frac{\left(-4\,\mathrm{e}^{21t} + 2\,\mathrm{e}^{16t}\right) \mathrm{e}^{-6t}}{2} \\ -\frac{\left(-4\,\mathrm{e}^{21t} + 2\,\mathrm{e}^{16t}\right) \mathrm{e}^{-6t}}{2} \\ \left(\mathrm{e}^{18t} + 2\,\mathrm{e}^{13t}\right) \mathrm{e}^{-3t} \end{bmatrix}$$

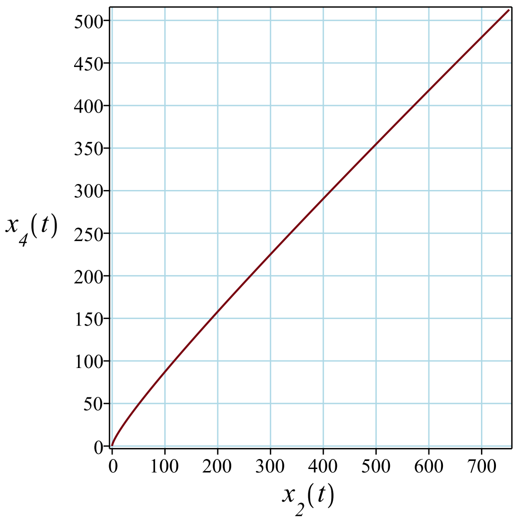
The following are plots of each solution against another.

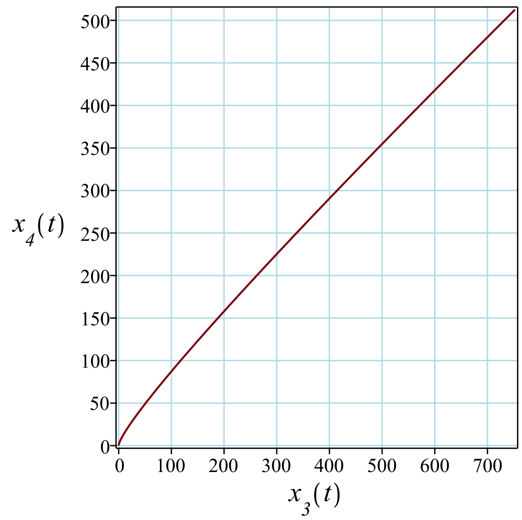




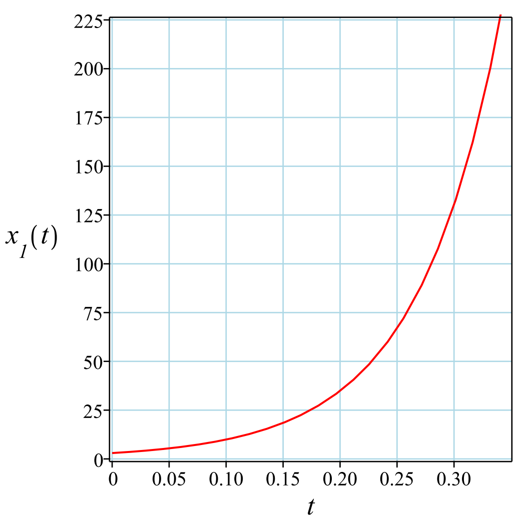


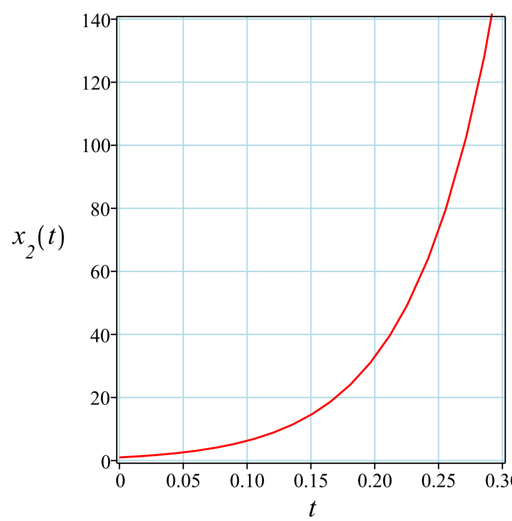


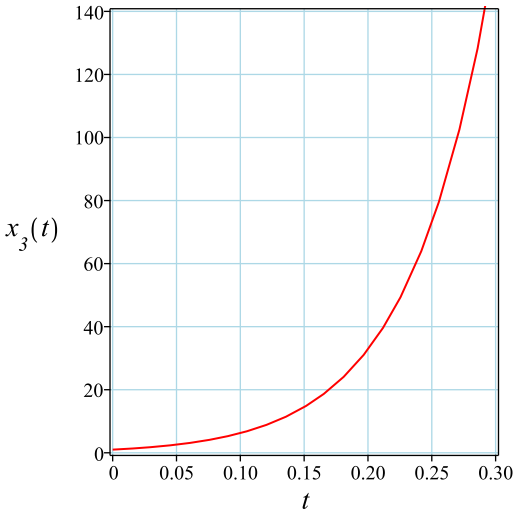


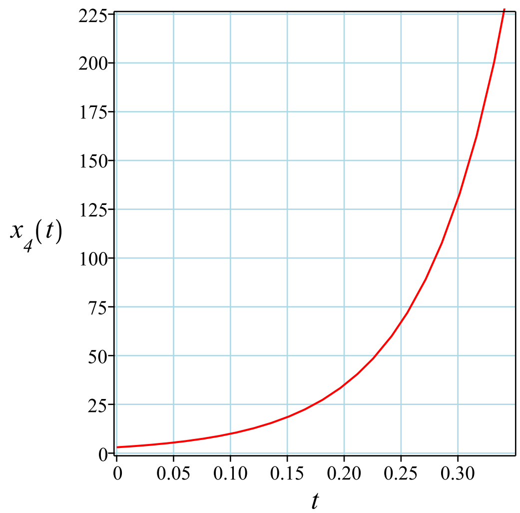


The following are plots of each solution.









✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 62

```
dsolve([diff(x__1(t),t) = 4*x__1(t)+x__2(t)+x__3(t)+7*x__4(t), diff(x__2(t),t) = x__1(t)+4*x
```

$$x_1(t) = e^{15t} + 2\,e^{10t}$$
$$x_2(t) = 2\,e^{15t} - e^{10t}$$
$$x_3(t) = 2\,e^{15t} - e^{10t}$$
$$x_4(t) = e^{15t} + 2\,e^{10t}$$

✓ Solution by Mathematica

Time used: 0.017 (sec). Leaf size: 70

```
DSolve[{x1'[t]==4*x1[t]+1*x2[t]+1*x3[t]+7*x4[t],x2'[t]==1*x1[t]+4*x2[t]+10*x3[t]+1*x4[t],x3'
```

$$\text{x1}(t) \to e^{10t}\left(e^{5t} + 2\right)$$
$$\text{x2}(t) \to e^{10t}\left(2e^{5t} - 1\right)$$
$$\text{x3}(t) \to e^{10t}\left(2e^{5t} - 1\right)$$
$$\text{x4}(t) \to e^{10t}\left(e^{5t} + 2\right)$$

4.31 problem problem 42

Internal problem ID [345]
Internal file name [`OUTPUT/345_Sunday_June_05_2022_01_39_14_AM_55224577/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.3, The eigenvalue method for linear systems. Page 395
Problem number: problem 42.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned}
x_1'(t) &= -40x_1(t) - 12x_2(t) + 54x_3(t) \\
x_2'(t) &= 35x_1(t) + 13x_2(t) - 46x_3(t) \\
x_3'(t) &= -25x_1(t) - 7x_2(t) + 34x_3(t)
\end{aligned}$$

4.31.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} -40 & -12 & 54 \\ 35 & 13 & -46 \\ -25 & -7 & 34 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} 12 - 5\,e^{2t} - 6\,e^{5t} & -2\,e^{5t} - e^{2t} + 3 & 8\,e^{5t} + 7\,e^{2t} - 15 \\ -5\,e^{2t} - 4 + 9\,e^{5t} & -1 + 3\,e^{5t} - e^{2t} & -12\,e^{5t} + 7\,e^{2t} + 5 \\ -5\,e^{2t} + 8 - 3\,e^{5t} & -e^{5t} - e^{2t} + 2 & -10 + 4\,e^{5t} + 7\,e^{2t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} 12-5\,\mathrm{e}^{2t}-6\,\mathrm{e}^{5t} & -2\,\mathrm{e}^{5t}-\mathrm{e}^{2t}+3 & 8\,\mathrm{e}^{5t}+7\,\mathrm{e}^{2t}-15 \\ -5\,\mathrm{e}^{2t}-4+9\,\mathrm{e}^{5t} & -1+3\,\mathrm{e}^{5t}-\mathrm{e}^{2t} & -12\,\mathrm{e}^{5t}+7\,\mathrm{e}^{2t}+5 \\ -5\,\mathrm{e}^{2t}+8-3\,\mathrm{e}^{5t} & -\mathrm{e}^{5t}-\mathrm{e}^{2t}+2 & -10+4\,\mathrm{e}^{5t}+7\,\mathrm{e}^{2t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} (12-5\,\mathrm{e}^{2t}-6\,\mathrm{e}^{5t})\,c_1+(-2\,\mathrm{e}^{5t}-\mathrm{e}^{2t}+3)\,c_2+(8\,\mathrm{e}^{5t}+7\,\mathrm{e}^{2t}-15)\,c_3 \\ (-5\,\mathrm{e}^{2t}-4+9\,\mathrm{e}^{5t})\,c_1+(-1+3\,\mathrm{e}^{5t}-\mathrm{e}^{2t})\,c_2+(-12\,\mathrm{e}^{5t}+7\,\mathrm{e}^{2t}+5)\,c_3 \\ (-5\,\mathrm{e}^{2t}+8-3\,\mathrm{e}^{5t})\,c_1+(-\mathrm{e}^{5t}-\mathrm{e}^{2t}+2)\,c_2+(-10+4\,\mathrm{e}^{5t}+7\,\mathrm{e}^{2t})\,c_3 \end{bmatrix} \\
&= \begin{bmatrix} (-5c_1-c_2+7c_3)\,\mathrm{e}^{2t}+(-6c_1-2c_2+8c_3)\,\mathrm{e}^{5t}+12c_1+3c_2-15c_3 \\ (-5c_1-c_2+7c_3)\,\mathrm{e}^{2t}+3(3c_1+c_2-4c_3)\,\mathrm{e}^{5t}-4c_1-c_2+5c_3 \\ (-5c_1-c_2+7c_3)\,\mathrm{e}^{2t}+(-3c_1-c_2+4c_3)\,\mathrm{e}^{5t}+8c_1+2c_2-10c_3 \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

4.31.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} -40 & -12 & 54 \\ 35 & 13 & -46 \\ -25 & -7 & 34 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A-\lambda I\right)=0$$

Expanding gives

$$\det\left(\begin{bmatrix} -40 & -12 & 54 \\ 35 & 13 & -46 \\ -25 & -7 & 34 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -40-\lambda & -12 & 54 \\ 35 & 13-\lambda & -46 \\ -25 & -7 & 34-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 7\lambda^2 + 10\lambda = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 0$$
$$\lambda_2 = 2$$
$$\lambda_3 = 5$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
0	1	real eigenvalue
2	1	real eigenvalue
5	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 0$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -40 & -12 & 54 \\ 35 & 13 & -46 \\ -25 & -7 & 34 \end{bmatrix} - (0)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -40 & -12 & 54 \\ 35 & 13 & -46 \\ -25 & -7 & 34 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -40 & -12 & 54 & 0 \\ 35 & 13 & -46 & 0 \\ -25 & -7 & 34 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{7R_1}{8} \Longrightarrow \left[\begin{array}{ccc|c} -40 & -12 & 54 & 0 \\ 0 & \frac{5}{2} & \frac{5}{4} & 0 \\ -25 & -7 & 34 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{5R_1}{8} \Longrightarrow \left[\begin{array}{ccc|c} -40 & -12 & 54 & 0 \\ 0 & \frac{5}{2} & \frac{5}{4} & 0 \\ 0 & \frac{1}{2} & \frac{1}{4} & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{R_2}{5} \Longrightarrow \left[\begin{array}{ccc|c} -40 & -12 & 54 & 0 \\ 0 & \frac{5}{2} & \frac{5}{4} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -40 & -12 & 54 \\ 0 & \frac{5}{2} & \frac{5}{4} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{3t}{2}, v_2 = -\frac{t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{3t}{2} \\ -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{3t}{2} \\ -\frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{3t}{2} \\ -\frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{3}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{3t}{2} \\ -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{3t}{2} \\ -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -40 & -12 & 54 \\ 35 & 13 & -46 \\ -25 & -7 & 34 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -42 & -12 & 54 \\ 35 & 11 & -46 \\ -25 & -7 & 32 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -42 & -12 & 54 & 0 \\ 35 & 11 & -46 & 0 \\ -25 & -7 & 32 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{5R_1}{6} \implies \left[\begin{array}{ccc|c} -42 & -12 & 54 & 0 \\ 0 & 1 & -1 & 0 \\ -25 & -7 & 32 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{25R_1}{42} \implies \left[\begin{array}{ccc|c} -42 & -12 & 54 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & \frac{1}{7} & -\frac{1}{7} & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{R_2}{7} \implies \left[\begin{array}{ccc|c} -42 & -12 & 54 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -42 & -12 & 54 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = 5$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -40 & -12 & 54 \\ 35 & 13 & -46 \\ -25 & -7 & 34 \end{bmatrix} - (5)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -45 & -12 & 54 \\ 35 & 8 & -46 \\ -25 & -7 & 29 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -45 & -12 & 54 & 0 \\ 35 & 8 & -46 & 0 \\ -25 & -7 & 29 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{7R_1}{9} \Longrightarrow \left[\begin{array}{ccc|c} -45 & -12 & 54 & 0 \\ 0 & -\frac{4}{3} & -4 & 0 \\ -25 & -7 & 29 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{5R_1}{9} \Longrightarrow \left[\begin{array}{ccc|c} -45 & -12 & 54 & 0 \\ 0 & -\frac{4}{3} & -4 & 0 \\ 0 & -\frac{1}{3} & -1 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{R_2}{4} \Longrightarrow \left[\begin{array}{ccc|c} -45 & -12 & 54 & 0 \\ 0 & -\frac{4}{3} & -4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -45 & -12 & 54 \\ 0 & -\frac{4}{3} & -4 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2t, v_2 = -3t\}$

Hence the solution is

$$\begin{bmatrix} 2t \\ -3t \\ t \end{bmatrix} = \begin{bmatrix} 2t \\ -3t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 2t \\ -3t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 2t \\ -3t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity algebraic m	multiplicity geometric k	defective?	eigenvectors
0	1	1	No	$\begin{bmatrix} \frac{3}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$
2	1	1	No	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$
5	1	1	No	$\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 0 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^0 \\
&= \begin{bmatrix} \frac{3}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix} e^0
\end{aligned}
$$

Since eigenvalue 2 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{2t} \\
&= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{2t}
\end{aligned}
$$

Since eigenvalue 5 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_3(t) &= \vec{v}_3 e^{5t}\\ &= \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} e^{5t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 \begin{bmatrix} \frac{3}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{bmatrix} + c_3 \begin{bmatrix} 2\,e^{5t} \\ -3\,e^{5t} \\ e^{5t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} \frac{3c_1}{2} + c_2 e^{2t} + 2c_3 e^{5t} \\ -\frac{c_1}{2} + c_2 e^{2t} - 3c_3 e^{5t} \\ c_1 + c_2 e^{2t} + c_3 e^{5t} \end{bmatrix}$$

4.31.3 Maple step by step solution

Let's solve

$[x_1'(t) = -40x_1(t) - 12x_2(t) + 54x_3(t)\,, x_2'(t) = 35x_1(t) + 13x_2(t) - 46x_3(t)\,, x_3'(t) = -25x_1(t) - 7$

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{\vec{x}}'(t) = \begin{bmatrix} -40 & -12 & 54 \\ 35 & 13 & -46 \\ -25 & -7 & 34 \end{bmatrix} \cdot \underline{\vec{x}}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{\vec{x}}'(t) = \begin{bmatrix} -40 & -12 & 54 \\ 35 & 13 & -46 \\ -25 & -7 & 34 \end{bmatrix} \cdot \underline{\vec{x}}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -40 & -12 & 54 \\ 35 & 13 & -46 \\ -25 & -7 & 34 \end{bmatrix}$$

- Rewrite the system as

$\underline{\vec{x}}'(t) = A \cdot \underline{\vec{x}}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \begin{bmatrix} \frac{3}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[5, \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} \frac{3}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_1 = \begin{bmatrix} \frac{3}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_{2} = \mathrm{e}^{2t} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[5, \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_{3} = \mathrm{e}^{5t} \cdot \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$\underline{\vec{x}} = c_1 \underline{\vec{x}}_{1} + c_2 \underline{\vec{x}}_{2} + c_3 \underline{\vec{x}}_{3}$

- Substitute solutions into the general solution

$$\underline{\vec{x}} = c_2 \mathrm{e}^{2t} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_3 \mathrm{e}^{5t} \cdot \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{3c_1}{2} \\ -\frac{c_1}{2} \\ c_1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} \frac{3c_1}{2} + c_2 \mathrm{e}^{2t} + 2c_3 \mathrm{e}^{5t} \\ -\frac{c_1}{2} + c_2 \mathrm{e}^{2t} - 3c_3 \mathrm{e}^{5t} \\ c_1 + c_2 \mathrm{e}^{2t} + c_3 \mathrm{e}^{5t} \end{bmatrix}$$

- Solution to the system of ODEs

$\left\{x_1(t) = \frac{3c_1}{2} + c_2 \mathrm{e}^{2t} + 2c_3 \mathrm{e}^{5t}, x_2(t) = -\frac{c_1}{2} + c_2 \mathrm{e}^{2t} - 3c_3 \mathrm{e}^{5t}, x_3(t) = c_1 + c_2 \mathrm{e}^{2t} + c_3 \mathrm{e}^{5t}\right\}$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 59

```
dsolve([diff(x__1(t),t)=-40*x__1(t)-12*x__2(t)+54*x__3(t),diff(x__2(t),t)=35*x__1(t)+13*x__2
```

$$
\begin{aligned}
x_1(t) &= c_1 + c_2 e^{2t} + c_3 e^{5t} \\
x_2(t) &= c_2 e^{2t} - \frac{3c_3 e^{5t}}{2} - \frac{c_1}{3} \\
x_3(t) &= c_2 e^{2t} + \frac{c_3 e^{5t}}{2} + \frac{2c_1}{3}
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 181

```
DSolve[{x1'[t]==-40*x1[t]-12*x2[t]+54*x3[t],x2'[t]==35*x1[t]+13*x2[t]-46*x3[t],x3'[t]==-25*x
```

$$
\begin{aligned}
\text{x1}(t) &\to c_1\left(-5e^{2t} - 6e^{5t} + 12\right) - c_2\left(e^{2t} + 2e^{5t} - 3\right) + c_3\left(7e^{2t} + 8e^{5t} - 15\right) \\
\text{x2}(t) &\to c_1\left(-5e^{2t} + 9e^{5t} - 4\right) + c_2\left(-e^{2t} + 3e^{5t} - 1\right) + c_3\left(7e^{2t} - 12e^{5t} + 5\right) \\
\text{x3}(t) &\to c_1\left(-5e^{2t} - 3e^{5t} + 8\right) - c_2\left(e^{2t} + e^{5t} - 2\right) + c_3\left(7e^{2t} + 4e^{5t} - 10\right)
\end{aligned}
$$

4.32 problem problem 43

Internal problem ID [346]
Internal file name [`OUTPUT/346_Sunday_June_05_2022_01_39_15_AM_48202215/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.3, The eigenvalue method for linear systems. Page 395
Problem number: problem 43.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= -20x_1(t) + 11x_2(t) + 13x_3(t) \\
x_2'(t) &= 12x_1(t) - x_2(t) - 7x_3(t) \\
x_3'(t) &= -48x_1(t) + 21x_2(t) + 31x_3(t)
\end{aligned}
$$

4.32.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} -20 & 11 & 13 \\ 12 & -1 & -7 \\ -48 & 21 & 31 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} 3\,e^{-2t} - \frac{e^{4t}}{2} - \frac{3\,e^{8t}}{2} & \frac{e^{8t}}{2} + e^{4t} - \frac{3\,e^{-2t}}{2} & e^{8t} + \frac{e^{4t}}{2} - \frac{3\,e^{-2t}}{2} \\ -\frac{e^{4t}}{2} - e^{-2t} + \frac{3\,e^{8t}}{2} & \frac{e^{-2t}}{2} - \frac{e^{8t}}{2} + e^{4t} & -e^{8t} + \frac{e^{4t}}{2} + \frac{e^{-2t}}{2} \\ -\frac{e^{4t}}{2} + 5\,e^{-2t} - \frac{9\,e^{8t}}{2} & \frac{3\,e^{8t}}{2} + e^{4t} - \frac{5\,e^{-2t}}{2} & -\frac{5\,e^{-2t}}{2} + 3\,e^{8t} + \frac{e^{4t}}{2} \end{bmatrix}
$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = e^{At}\vec{c}$$

$$= \begin{bmatrix} 3\,e^{-2t} - \frac{e^{4t}}{2} - \frac{3\,e^{8t}}{2} & \frac{e^{8t}}{2} + e^{4t} - \frac{3\,e^{-2t}}{2} & e^{8t} + \frac{e^{4t}}{2} - \frac{3\,e^{-2t}}{2} \\ -\frac{e^{4t}}{2} - e^{-2t} + \frac{3\,e^{8t}}{2} & \frac{e^{-2t}}{2} - \frac{e^{8t}}{2} + e^{4t} & -e^{8t} + \frac{e^{4t}}{2} + \frac{e^{-2t}}{2} \\ -\frac{e^{4t}}{2} + 5\,e^{-2t} - \frac{9\,e^{8t}}{2} & \frac{3\,e^{8t}}{2} + e^{4t} - \frac{5\,e^{-2t}}{2} & -\frac{5\,e^{-2t}}{2} + 3\,e^{8t} + \frac{e^{4t}}{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$= \begin{bmatrix} \left(3\,e^{-2t} - \frac{e^{4t}}{2} - \frac{3\,e^{8t}}{2}\right) c_1 + \left(\frac{e^{8t}}{2} + e^{4t} - \frac{3\,e^{-2t}}{2}\right) c_2 + \left(e^{8t} + \frac{e^{4t}}{2} - \frac{3\,e^{-2t}}{2}\right) c_3 \\ \left(-\frac{e^{4t}}{2} - e^{-2t} + \frac{3\,e^{8t}}{2}\right) c_1 + \left(\frac{e^{-2t}}{2} - \frac{e^{8t}}{2} + e^{4t}\right) c_2 + \left(-e^{8t} + \frac{e^{4t}}{2} + \frac{e^{-2t}}{2}\right) c_3 \\ \left(-\frac{e^{4t}}{2} + 5\,e^{-2t} - \frac{9\,e^{8t}}{2}\right) c_1 + \left(\frac{3\,e^{8t}}{2} + e^{4t} - \frac{5\,e^{-2t}}{2}\right) c_2 + \left(-\frac{5\,e^{-2t}}{2} + 3\,e^{8t} + \frac{e^{4t}}{2}\right) c_3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{(6c_1-3c_2-3c_3)e^{-2t}}{2} + \frac{(-c_1+2c_2+c_3)e^{4t}}{2} - \frac{3\,e^{8t}\left(-\frac{2c_3}{3}-\frac{c_2}{3}+c_1\right)}{2} \\ \frac{(-2c_1+c_2+c_3)e^{-2t}}{2} + \frac{(-c_1+2c_2+c_3)e^{4t}}{2} + \frac{3\,e^{8t}\left(-\frac{2c_3}{3}-\frac{c_2}{3}+c_1\right)}{2} \\ \frac{5(2c_1-c_2-c_3)e^{-2t}}{2} + \frac{(-c_1+2c_2+c_3)e^{4t}}{2} - \frac{9\,e^{8t}\left(-\frac{2c_3}{3}-\frac{c_2}{3}+c_1\right)}{2} \end{bmatrix}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

4.32.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} -20 & 11 & 13 \\ 12 & -1 & -7 \\ -48 & 21 & 31 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -20 & 11 & 13 \\ 12 & -1 & -7 \\ -48 & 21 & 31 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -20-\lambda & 11 & 13 \\ 12 & -1-\lambda & -7 \\ -48 & 21 & 31-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 10\lambda^2 + 8\lambda + 64 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -2$$
$$\lambda_2 = 8$$
$$\lambda_3 = 4$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-2	1	real eigenvalue
4	1	real eigenvalue
8	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -20 & 11 & 13 \\ 12 & -1 & -7 \\ -48 & 21 & 31 \end{bmatrix} - (-2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -18 & 11 & 13 \\ 12 & 1 & -7 \\ -48 & 21 & 33 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -18 & 11 & 13 & 0 \\ 12 & 1 & -7 & 0 \\ -48 & 21 & 33 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{2R_1}{3} \Longrightarrow \left[\begin{array}{ccc|c} -18 & 11 & 13 & 0 \\ 0 & \frac{25}{3} & \frac{5}{3} & 0 \\ -48 & 21 & 33 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{8R_1}{3} \Longrightarrow \left[\begin{array}{ccc|c} -18 & 11 & 13 & 0 \\ 0 & \frac{25}{3} & \frac{5}{3} & 0 \\ 0 & -\frac{25}{3} & -\frac{5}{3} & 0 \end{array}\right]$$

$$R_3 = R_3 + R_2 \Longrightarrow \left[\begin{array}{ccc|c} -18 & 11 & 13 & 0 \\ 0 & \frac{25}{3} & \frac{5}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -18 & 11 & 13 \\ 0 & \frac{25}{3} & \frac{5}{3} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{3t}{5}, v_2 = -\frac{t}{5}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{3t}{5} \\ -\frac{t}{5} \\ t \end{bmatrix} = \begin{bmatrix} \frac{3t}{5} \\ -\frac{t}{5} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{3t}{5} \\ -\frac{t}{5} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{3}{5} \\ -\frac{1}{5} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{3t}{5} \\ -\frac{t}{5} \\ t \end{bmatrix} = \begin{bmatrix} \frac{3}{5} \\ -\frac{1}{5} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{3t}{5} \\ -\frac{t}{5} \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 4$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -20 & 11 & 13 \\ 12 & -1 & -7 \\ -48 & 21 & 31 \end{bmatrix} - (4) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -24 & 11 & 13 \\ 12 & -5 & -7 \\ -48 & 21 & 27 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -24 & 11 & 13 & 0 \\ 12 & -5 & -7 & 0 \\ -48 & 21 & 27 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{2} \implies \left[\begin{array}{ccc|c} -24 & 11 & 13 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ -48 & 21 & 27 & 0 \end{array}\right]$$

$$R_3 = R_3 - 2R_1 \implies \left[\begin{array}{ccc|c} -24 & 11 & 13 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -1 & 1 & 0 \end{array}\right]$$

$$R_3 = R_3 + 2R_2 \Longrightarrow \left[\begin{array}{ccc|c} -24 & 11 & 13 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -24 & 11 & 13 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = 8$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -20 & 11 & 13 \\ 12 & -1 & -7 \\ -48 & 21 & 31 \end{bmatrix} - (8)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -28 & 11 & 13 \\ 12 & -9 & -7 \\ -48 & 21 & 23 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -28 & 11 & 13 & 0 \\ 12 & -9 & -7 & 0 \\ -48 & 21 & 23 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{3R_1}{7} \implies \left[\begin{array}{ccc|c} -28 & 11 & 13 & 0 \\ 0 & -\frac{30}{7} & -\frac{10}{7} & 0 \\ -48 & 21 & 23 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{12R_1}{7} \implies \left[\begin{array}{ccc|c} -28 & 11 & 13 & 0 \\ 0 & -\frac{30}{7} & -\frac{10}{7} & 0 \\ 0 & \frac{15}{7} & \frac{5}{7} & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{R_2}{2} \implies \left[\begin{array}{ccc|c} -28 & 11 & 13 & 0 \\ 0 & -\frac{30}{7} & -\frac{10}{7} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -28 & 11 & 13 \\ 0 & -\frac{30}{7} & -\frac{10}{7} \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{3}, v_2 = -\frac{t}{3}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{3} \\ -\frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{3} \\ -\frac{t}{3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{3} \\ -\frac{t}{3} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{t}{3} \\ -\frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{t}{3} \\ -\frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
-2	1	1	No	$\begin{bmatrix} \frac{3}{5} \\ -\frac{1}{5} \\ 1 \end{bmatrix}$
8	1	1	No	$\begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ 1 \end{bmatrix}$
4	1	1	No	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -2 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{-2t} \\ &= \begin{bmatrix} \frac{3}{5} \\ -\frac{1}{5} \\ 1 \end{bmatrix} e^{-2t} \end{aligned}$$

Since eigenvalue 8 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_2(t) &= \vec{v}_2 e^{8t} \\ &= \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ 1 \end{bmatrix} e^{8t} \end{aligned}$$

Since eigenvalue 4 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_3(t) = \vec{v}_3 e^{4t}$$
$$= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{4t}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 \begin{bmatrix} \frac{3\,\mathrm{e}^{-2t}}{5} \\ -\frac{\mathrm{e}^{-2t}}{5} \\ \mathrm{e}^{-2t} \end{bmatrix} + c_2 \begin{bmatrix} \frac{\mathrm{e}^{8t}}{3} \\ -\frac{\mathrm{e}^{8t}}{3} \\ \mathrm{e}^{8t} \end{bmatrix} + c_3 \begin{bmatrix} \mathrm{e}^{4t} \\ \mathrm{e}^{4t} \\ \mathrm{e}^{4t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} \frac{3c_1\mathrm{e}^{-2t}}{5} + \frac{c_2\mathrm{e}^{8t}}{3} + c_3\mathrm{e}^{4t} \\ -\frac{c_1\mathrm{e}^{-2t}}{5} - \frac{c_2\mathrm{e}^{8t}}{3} + c_3\mathrm{e}^{4t} \\ c_1\mathrm{e}^{-2t} + c_2\mathrm{e}^{8t} + c_3\mathrm{e}^{4t} \end{bmatrix}$$

4.32.3 Maple step by step solution

Let's solve

$[x_1'(t) = -20x_1(t) + 11x_2(t) + 13x_3(t)\,, x_2'(t) = 12x_1(t) - x_2(t) - 7x_3(t)\,, x_3'(t) = -48x_1(t) + 21x_2$

- Define vector

$$x\underline{\overset{\rightarrow}{\quad}}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x^{\rightarrow}}'(t) = \begin{bmatrix} -20 & 11 & 13 \\ 12 & -1 & -7 \\ -48 & 21 & 31 \end{bmatrix} \cdot \underline{x^{\rightarrow}}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{x^{\rightarrow}}'(t) = \begin{bmatrix} -20 & 11 & 13 \\ 12 & -1 & -7 \\ -48 & 21 & 31 \end{bmatrix} \cdot \underline{x^{\rightarrow}}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -20 & 11 & 13 \\ 12 & -1 & -7 \\ -48 & 21 & 31 \end{bmatrix}$$

- Rewrite the system as

$\underline{x^{\rightarrow}}'(t) = A \cdot \underline{x^{\rightarrow}}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \begin{bmatrix} \frac{3}{5} \\ -\frac{1}{5} \\ 1 \end{bmatrix}\right], \left[4, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[8, \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-2, \begin{bmatrix} \frac{3}{5} \\ -\frac{1}{5} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{x^{\rightarrow}}_1 = \mathrm{e}^{-2t} \cdot \begin{bmatrix} \frac{3}{5} \\ -\frac{1}{5} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[4, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\overrightarrow{x}}_{2} = e^{4t} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[8, \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\overrightarrow{x}}_{3} = e^{8t} \cdot \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\underline{\overrightarrow{x}} = c_1 \underline{\overrightarrow{x}}_{1} + c_2 \underline{\overrightarrow{x}}_{2} + c_3 \underline{\overrightarrow{x}}_{3}$$

- Substitute solutions into the general solution

$$\underline{\overrightarrow{x}} = c_1 e^{-2t} \cdot \begin{bmatrix} \frac{3}{5} \\ -\frac{1}{5} \\ 1 \end{bmatrix} + c_2 e^{4t} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_3 e^{8t} \cdot \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} \frac{3c_1 e^{-2t}}{5} + c_2 e^{4t} + \frac{c_3 e^{8t}}{3} \\ -\frac{c_1 e^{-2t}}{5} + c_2 e^{4t} - \frac{c_3 e^{8t}}{3} \\ c_1 e^{-2t} + c_2 e^{4t} + c_3 e^{8t} \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x_1(t) = \frac{3c_1 e^{-2t}}{5} + c_2 e^{4t} + \frac{c_3 e^{8t}}{3}, x_2(t) = -\frac{c_1 e^{-2t}}{5} + c_2 e^{4t} - \frac{c_3 e^{8t}}{3}, x_3(t) = c_1 e^{-2t} + c_2 e^{4t} + c_3 e^{8t}\right\}$$

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 72

```
dsolve([diff(x__1(t),t)=-20*x__1(t)+11*x__2(t)+13*x__3(t),diff(x__2(t),t)=12*x__1(t)-1*x__2(
```

$$x_1(t) = c_1 e^{4t} + c_2 e^{-2t} + c_3 e^{8t}$$
$$x_2(t) = c_1 e^{4t} - \frac{c_2 e^{-2t}}{3} - c_3 e^{8t}$$
$$x_3(t) = c_1 e^{4t} + \frac{5 c_2 e^{-2t}}{3} + 3 c_3 e^{8t}$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 554

```
DSolve[{x1'[t]==20*x1[t]+11*x2[t]+13*x3[t],x2'[t]==12*x1[t]-1*x2[t]-7*x3[t],x3'[t]==-48*x1[t
```

$$x1(t) \to c_2\text{RootSum}\left[\#1^3-50\#1^2+1208\#1-4576\&,\frac{11\#1e^{\#1t}-68e^{\#1t}}{3\#1^2-100\#1+1208}\&\right] + c_3\text{RootSum}\left[\#1^3-50\#1^2+1208\#1-4576\&,\frac{13\#1e^{\#1t}-64e^{\#1t}}{3\#1^2-100\#1+1208}\&\right] + c_1\text{RootSum}\left[\#1^3-50\#1^2+1208\#1-4576\&,\frac{\#1^2e^{\#1t}-30\#1e^{\#1t}+116e^{\#1t}}{3\#1^2-100\#1+1208}\&\right]$$

$$x2(t) \to 12c_1\text{RootSum}\left[\#1^3-50\#1^2+1208\#1-4576\&,\frac{\#1e^{\#1t}-3e^{\#1t}}{3\#1^2-100\#1+1208}\&\right] - c_3\text{RootSum}\left[\#1^3-50\#1^2+1208\#1-4576\&,\frac{7\#1e^{\#1t}-296e^{\#1t}}{3\#1^2-100\#1+1208}\&\right] + c_2\text{RootSum}\left[\#1^3-50\#1^2+1208\#1-4576\&,\frac{\#1^2e^{\#1t}-51\#1e^{\#1t}+1244e^{\#1t}}{3\#1^2-100\#1+1208}\&\right]$$

$$x3(t) \to -12c_1\text{RootSum}\left[\#1^3-50\#1^2+1208\#1-4576\&,\frac{4\#1e^{\#1t}-17e^{\#1t}}{3\#1^2-100\#1+1208}\&\right] + 3c_2\text{RootSum}\left[\#1^3-50\#1^2+1208\#1-4576\&,\frac{7\#1e^{\#1t}-316e^{\#1t}}{3\#1^2-100\#1+1208}\&\right] + c_3\text{RootSum}\left[\#1^3-50\#1^2+1208\#1-4576\&,\frac{\#1^2e^{\#1t}-19\#1e^{\#1t}-152e^{\#1t}}{3\#1^2-100\#1+1208}\&\right]$$

4.33 problem problem 44

Internal problem ID [347]
Internal file name [`OUTPUT/347_Sunday_June_05_2022_01_39_17_AM_8252262/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.3, The eigenvalue method for linear systems. Page 395
Problem number: problem 44.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned}
x_1'(t) &= 147x_1(t) + 23x_2(t) - 202x_3(t)\\
x_2'(t) &= -90x_1(t) - 9x_2(t) + 129x_3(t)\\
x_3'(t) &= 90x_1(t) + 15x_2(t) - 123x_3(t)
\end{aligned}$$

4.33.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 147 & 23 & -202 \\ -90 & -9 & 129 \\ 90 & 15 & -123 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} -9\,\mathrm{e}^{-3t} + 10\,\mathrm{e}^{12t} & \frac{5\,\mathrm{e}^{12t}}{6} + \frac{7\,\mathrm{e}^{6t}}{6} - 2\,\mathrm{e}^{-3t} & -\frac{85\,\mathrm{e}^{12t}}{6} + \frac{7\,\mathrm{e}^{6t}}{6} + 13\,\mathrm{e}^{-3t} \\ 6\,\mathrm{e}^{-3t} - 6\,\mathrm{e}^{12t} & \frac{4\,\mathrm{e}^{-3t}}{3} - \frac{\mathrm{e}^{12t}}{2} + \frac{\mathrm{e}^{6t}}{6} & \frac{17\,\mathrm{e}^{12t}}{2} + \frac{\mathrm{e}^{6t}}{6} - \frac{26\,\mathrm{e}^{-3t}}{3} \\ -6\,\mathrm{e}^{-3t} + 6\,\mathrm{e}^{12t} & \frac{\mathrm{e}^{12t}}{2} + \frac{5\,\mathrm{e}^{6t}}{6} - \frac{4\,\mathrm{e}^{-3t}}{3} & \frac{26\,\mathrm{e}^{-3t}}{3} - \frac{17\,\mathrm{e}^{12t}}{2} + \frac{5\,\mathrm{e}^{6t}}{6} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = e^{At}\vec{c}$$

$$= \begin{bmatrix} -9\,e^{-3t}+10\,e^{12t} & \frac{5\,e^{12t}}{6}+\frac{7\,e^{6t}}{6}-2\,e^{-3t} & -\frac{85\,e^{12t}}{6}+\frac{7\,e^{6t}}{6}+13\,e^{-3t} \\ 6\,e^{-3t}-6\,e^{12t} & \frac{4\,e^{-3t}}{3}-\frac{e^{12t}}{2}+\frac{e^{6t}}{6} & \frac{17\,e^{12t}}{2}+\frac{e^{6t}}{6}-\frac{26\,e^{-3t}}{3} \\ -6\,e^{-3t}+6\,e^{12t} & \frac{e^{12t}}{2}+\frac{5\,e^{6t}}{6}-\frac{4\,e^{-3t}}{3} & \frac{26\,e^{-3t}}{3}-\frac{17\,e^{12t}}{2}+\frac{5\,e^{6t}}{6} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$= \begin{bmatrix} \left(-9\,e^{-3t}+10\,e^{12t}\right)c_1+\left(\frac{5\,e^{12t}}{6}+\frac{7\,e^{6t}}{6}-2\,e^{-3t}\right)c_2+\left(-\frac{85\,e^{12t}}{6}+\frac{7\,e^{6t}}{6}+13\,e^{-3t}\right)c_3 \\ \left(6\,e^{-3t}-6\,e^{12t}\right)c_1+\left(\frac{4\,e^{-3t}}{3}-\frac{e^{12t}}{2}+\frac{e^{6t}}{6}\right)c_2+\left(\frac{17\,e^{12t}}{2}+\frac{e^{6t}}{6}-\frac{26\,e^{-3t}}{3}\right)c_3 \\ \left(-6\,e^{-3t}+6\,e^{12t}\right)c_1+\left(\frac{e^{12t}}{2}+\frac{5\,e^{6t}}{6}-\frac{4\,e^{-3t}}{3}\right)c_2+\left(\frac{26\,e^{-3t}}{3}-\frac{17\,e^{12t}}{2}+\frac{5\,e^{6t}}{6}\right)c_3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5(12c_1+c_2-17c_3)e^{12t}}{6}+(-9c_1-2c_2+13c_3)\,e^{-3t}+\frac{7\,e^{6t}(c_2+c_3)}{6} \\ \frac{(-12c_1-c_2+17c_3)e^{12t}}{2}+\frac{2(9c_1+2c_2-13c_3)e^{-3t}}{3}+\frac{e^{6t}(c_2+c_3)}{6} \\ \frac{(12c_1+c_2-17c_3)e^{12t}}{2}+\frac{2(-9c_1-2c_2+13c_3)e^{-3t}}{3}+\frac{5\,e^{6t}(c_2+c_3)}{6} \end{bmatrix}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

4.33.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 147 & 23 & -202 \\ -90 & -9 & 129 \\ 90 & 15 & -123 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A-\lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 147 & 23 & -202 \\ -90 & -9 & 129 \\ 90 & 15 & -123 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 147-\lambda & 23 & -202 \\ -90 & -9-\lambda & 129 \\ 90 & 15 & -123-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 15\lambda^2 + 18\lambda + 216 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 12$$
$$\lambda_2 = 6$$
$$\lambda_3 = -3$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-3	1	real eigenvalue
6	1	real eigenvalue
12	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 147 & 23 & -202 \\ -90 & -9 & 129 \\ 90 & 15 & -123 \end{bmatrix} - (-3)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 150 & 23 & -202 \\ -90 & -6 & 129 \\ 90 & 15 & -120 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 150 & 23 & -202 & 0 \\ -90 & -6 & 129 & 0 \\ 90 & 15 & -120 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{3R_1}{5} \implies \left[\begin{array}{ccc|c} 150 & 23 & -202 & 0 \\ 0 & \frac{39}{5} & \frac{39}{5} & 0 \\ 90 & 15 & -120 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{3R_1}{5} \implies \left[\begin{array}{ccc|c} 150 & 23 & -202 & 0 \\ 0 & \frac{39}{5} & \frac{39}{5} & 0 \\ 0 & \frac{6}{5} & \frac{6}{5} & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{2R_2}{13} \implies \left[\begin{array}{ccc|c} 150 & 23 & -202 & 0 \\ 0 & \frac{39}{5} & \frac{39}{5} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 150 & 23 & -202 \\ 0 & \frac{39}{5} & \frac{39}{5} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{3t}{2}, v_2 = -t\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{3t}{2} \\ -t \\ t \end{bmatrix} = \begin{bmatrix} \frac{3t}{2} \\ -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{3t}{2} \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{3}{2} \\ -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{3t}{2} \\ -t \\ t \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ -1 \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{3t}{2} \\ -t \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 6$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 147 & 23 & -202 \\ -90 & -9 & 129 \\ 90 & 15 & -123 \end{bmatrix} - (6) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 141 & 23 & -202 \\ -90 & -15 & 129 \\ 90 & 15 & -129 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 141 & 23 & -202 & 0 \\ -90 & -15 & 129 & 0 \\ 90 & 15 & -129 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{30R_1}{47} \Longrightarrow \left[\begin{array}{ccc|c} 141 & 23 & -202 & 0 \\ 0 & -\frac{15}{47} & \frac{3}{47} & 0 \\ 90 & 15 & -129 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{30R_1}{47} \Longrightarrow \left[\begin{array}{ccc|c} 141 & 23 & -202 & 0 \\ 0 & -\frac{15}{47} & \frac{3}{47} & 0 \\ 0 & \frac{15}{47} & -\frac{3}{47} & 0 \end{array}\right]$$

$$R_3 = R_3 + R_2 \Longrightarrow \left[\begin{array}{ccc|c} 141 & 23 & -202 & 0 \\ 0 & -\frac{15}{47} & \frac{3}{47} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 141 & 23 & -202 \\ 0 & -\frac{15}{47} & \frac{3}{47} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{7t}{5}, v_2 = \frac{t}{5}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{7t}{5} \\ \frac{t}{5} \\ t \end{bmatrix} = \begin{bmatrix} \frac{7t}{5} \\ \frac{t}{5} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{7t}{5} \\ \frac{t}{5} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{7}{5} \\ \frac{1}{5} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{7t}{5} \\ \frac{t}{5} \\ t \end{bmatrix} = \begin{bmatrix} \frac{7}{5} \\ \frac{1}{5} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{7t}{5} \\ \frac{t}{5} \\ t \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \\ 5 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = 12$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 147 & 23 & -202 \\ -90 & -9 & 129 \\ 90 & 15 & -123 \end{bmatrix} - (12)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 135 & 23 & -202 \\ -90 & -21 & 129 \\ 90 & 15 & -135 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 135 & 23 & -202 & 0 \\ -90 & -21 & 129 & 0 \\ 90 & 15 & -135 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{2R_1}{3} \Longrightarrow \left[\begin{array}{ccc|c} 135 & 23 & -202 & 0 \\ 0 & -\frac{17}{3} & -\frac{17}{3} & 0 \\ 90 & 15 & -135 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{2R_1}{3} \Longrightarrow \left[\begin{array}{ccc|c} 135 & 23 & -202 & 0 \\ 0 & -\frac{17}{3} & -\frac{17}{3} & 0 \\ 0 & -\frac{1}{3} & -\frac{1}{3} & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{R_2}{17} \Longrightarrow \left[\begin{array}{ccc|c} 135 & 23 & -202 & 0 \\ 0 & -\frac{17}{3} & -\frac{17}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 135 & 23 & -202 \\ 0 & -\frac{17}{3} & -\frac{17}{3} \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{5t}{3}, v_2 = -t\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{5t}{3} \\ -t \\ t \end{bmatrix} = \begin{bmatrix} \frac{5t}{3} \\ -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{5t}{3} \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{5}{3} \\ -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{5t}{3} \\ -t \\ t \end{bmatrix} = \begin{bmatrix} \frac{5}{3} \\ -1 \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{5t}{3} \\ -t \\ t \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ 3 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
12	1	1	No	$\begin{bmatrix} \frac{5}{3} \\ -1 \\ 1 \end{bmatrix}$
6	1	1	No	$\begin{bmatrix} \frac{7}{5} \\ \frac{1}{5} \\ 1 \end{bmatrix}$
-3	1	1	No	$\begin{bmatrix} \frac{3}{2} \\ -1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 12 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{12t} \\ &= \begin{bmatrix} \frac{5}{3} \\ -1 \\ 1 \end{bmatrix} e^{12t} \end{aligned}$$

Since eigenvalue 6 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_2(t) &= \vec{v}_2 e^{6t} \\ &= \begin{bmatrix} \frac{7}{5} \\ \frac{1}{5} \\ 1 \end{bmatrix} e^{6t} \end{aligned}$$

Since eigenvalue -3 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_3(t) &= \vec{v}_3 e^{-3t} \\
&= \begin{bmatrix} \frac{3}{2} \\ -1 \\ 1 \end{bmatrix} e^{-3t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)
$$

Which is written as

$$
\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 \begin{bmatrix} \frac{5\,\mathrm{e}^{12t}}{3} \\ -\mathrm{e}^{12t} \\ \mathrm{e}^{12t} \end{bmatrix} + c_2 \begin{bmatrix} \frac{7\,\mathrm{e}^{6t}}{5} \\ \frac{\mathrm{e}^{6t}}{5} \\ \mathrm{e}^{6t} \end{bmatrix} + c_3 \begin{bmatrix} \frac{3\,\mathrm{e}^{-3t}}{2} \\ -\mathrm{e}^{-3t} \\ \mathrm{e}^{-3t} \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} \frac{5c_1\mathrm{e}^{12t}}{3} + \frac{7c_2\mathrm{e}^{6t}}{5} + \frac{3c_3\mathrm{e}^{-3t}}{2} \\ -c_1\mathrm{e}^{12t} + \frac{c_2\mathrm{e}^{6t}}{5} - c_3\mathrm{e}^{-3t} \\ c_1\mathrm{e}^{12t} + c_2\mathrm{e}^{6t} + c_3\mathrm{e}^{-3t} \end{bmatrix}
$$

4.33.3 Maple step by step solution

Let's solve

$[x_1'(t) = 147x_1(t) + 23x_2(t) - 202x_3(t)\,, x_2'(t) = -90x_1(t) - 9x_2(t) + 129x_3(t)\,, x_3'(t) = 90x_1(t) +$

- Define vector

$$
\underline{x^{\rightarrow}}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}
$$

- Convert system into a vector equation

$$\underline{\vec{x}}'(t) = \begin{bmatrix} 147 & 23 & -202 \\ -90 & -9 & 129 \\ 90 & 15 & -123 \end{bmatrix} \cdot \underline{\vec{x}}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{\vec{x}}'(t) = \begin{bmatrix} 147 & 23 & -202 \\ -90 & -9 & 129 \\ 90 & 15 & -123 \end{bmatrix} \cdot \underline{\vec{x}}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 147 & 23 & -202 \\ -90 & -9 & 129 \\ 90 & 15 & -123 \end{bmatrix}$$

- Rewrite the system as

$\underline{\vec{x}}'(t) = A \cdot \underline{\vec{x}}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-3, \begin{bmatrix} \frac{3}{2} \\ -1 \\ 1 \end{bmatrix}\right], \left[6, \begin{bmatrix} \frac{7}{5} \\ \frac{1}{5} \\ 1 \end{bmatrix}\right], \left[12, \begin{bmatrix} \frac{5}{3} \\ -1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-3, \begin{bmatrix} \frac{3}{2} \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_1 = \mathrm{e}^{-3t} \cdot \begin{bmatrix} \frac{3}{2} \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[6, \begin{bmatrix} \frac{7}{5} \\ \frac{1}{5} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_2 = e^{6t} \cdot \begin{bmatrix} \frac{7}{5} \\ \frac{1}{5} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[12, \begin{bmatrix} \frac{5}{3} \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_3 = e^{12t} \cdot \begin{bmatrix} \frac{5}{3} \\ -1 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{x} = c_1 \vec{x}_1 + c_2 \vec{x}_2 + c_3 \vec{x}_3$$

- Substitute solutions into the general solution

$$\vec{x} = c_1 e^{-3t} \cdot \begin{bmatrix} \frac{3}{2} \\ -1 \\ 1 \end{bmatrix} + c_2 e^{6t} \cdot \begin{bmatrix} \frac{7}{5} \\ \frac{1}{5} \\ 1 \end{bmatrix} + c_3 e^{12t} \cdot \begin{bmatrix} \frac{5}{3} \\ -1 \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} \frac{3c_1 e^{-3t}}{2} + \frac{7c_2 e^{6t}}{5} + \frac{5c_3 e^{12t}}{3} \\ -c_1 e^{-3t} + \frac{c_2 e^{6t}}{5} - c_3 e^{12t} \\ c_1 e^{-3t} + c_2 e^{6t} + c_3 e^{12t} \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x_1(t) = \frac{3c_1 e^{-3t}}{2} + \frac{7c_2 e^{6t}}{5} + \frac{5c_3 e^{12t}}{3}, x_2(t) = -c_1 e^{-3t} + \frac{c_2 e^{6t}}{5} - c_3 e^{12t}, x_3(t) = c_1 e^{-3t} + c_2 e^{6t} + c_3 e^{12t}\right\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 74

```
dsolve([diff(x__1(t),t)=147*x__1(t)+23*x__2(t)-202*x__3(t),diff(x__2(t),t)=-90*x__1(t)-9*x__
```

$$x_1(t) = e^{6t}c_1 + c_2e^{-3t} + c_3e^{12t}$$
$$x_2(t) = \frac{e^{6t}c_1}{7} - \frac{2c_2e^{-3t}}{3} - \frac{3c_3e^{12t}}{5}$$
$$x_3(t) = \frac{5\,e^{6t}c_1}{7} + \frac{2c_2e^{-3t}}{3} + \frac{3c_3e^{12t}}{5}$$

✓ Solution by Mathematica

Time used: 0.009 (sec). Leaf size: 188

```
DSolve[{x1'[t]==147*x1[t]+23*x2[t]-202*x3[t],x2'[t]==-90*x1[t]-9*x2[t]+129*x3[t],x3'[t]==90*
```

$$\text{x1}(t) \to \frac{1}{6}e^{-3t}\left(6c_1\left(10e^{15t}-9\right)+c_2\left(7e^{9t}+5e^{15t}-12\right)-c_3\left(-7e^{9t}+85e^{15t}-78\right)\right)$$
$$\text{x2}(t) \to \frac{1}{6}e^{-3t}\left(-36c_1\left(e^{15t}-1\right)+c_2\left(e^{9t}-3e^{15t}+8\right)+c_3\left(e^{9t}+51e^{15t}-52\right)\right)$$
$$\text{x3}(t) \to \frac{1}{6}e^{-3t}\left(36c_1\left(e^{15t}-1\right)+c_2\left(5e^{9t}+3e^{15t}-8\right)-c_3\left(-5e^{9t}+51e^{15t}-52\right)\right)$$

4.34 problem problem 45

Internal problem ID [348]
Internal file name [`OUTPUT/348_Sunday_June_05_2022_01_39_18_AM_42100891/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.3, The eigenvalue method for linear systems. Page 395
Problem number: problem 45.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= 9x_1(t) - 7x_2(t) - 5x_3(t) \\
x_2'(t) &= -12x_1(t) + 7x_2(t) + 11x_3(t) + 9x_4(t) \\
x_3'(t) &= 24x_1(t) - 17x_2(t) - 19x_3(t) - 9x_4(t) \\
x_4'(t) &= -18x_1(t) + 13x_2(t) + 17x_3(t) + 9x_4(t)
\end{aligned}
$$

4.34.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \\ x_4'(t) \end{bmatrix} = \begin{bmatrix} 9 & -7 & -5 & 0 \\ -12 & 7 & 11 & 9 \\ 24 & -17 & -19 & -9 \\ -18 & 13 & 17 & 9 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} -e^{-3t}+2-2e^{3t}+2e^{6t} & -\frac{4e^{6t}}{3}+\frac{4e^{3t}}{3}-1+e^{-3t} & -\frac{5e^{6t}}{3}+\frac{8e^{3t}}{3}-2+e^{-3t} & -e^{6t}+2e^{3t}-1 \\ 4-e^{-3t}-e^{3t}-2e^{6t} & e^{-3t}+\frac{4e^{6t}}{3}+\frac{2e^{3t}}{3}-2 & \frac{5e^{6t}}{3}+\frac{4e^{3t}}{3}-4+e^{-3t} & e^{3t}-2+e^{6t} \\ -2-e^{-3t}-e^{3t}+4e^{6t} & -\frac{8e^{6t}}{3}+\frac{2e^{3t}}{3}+1+e^{-3t} & e^{-3t}-\frac{10e^{6t}}{3}+\frac{4e^{3t}}{3}+2 & -2e^{6t}+e^{3t}+1 \\ 2+e^{-3t}-e^{3t}-2e^{6t} & \frac{4e^{6t}}{3}+\frac{2e^{3t}}{3}-1-e^{-3t} & \frac{5e^{6t}}{3}+\frac{4e^{3t}}{3}-2-e^{-3t} & e^{6t}+e^{3t}-1 \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\vec{x}_h(t) = e^{At}\vec{c}
$$

$$
= \begin{bmatrix} -e^{-3t}+2-2e^{3t}+2e^{6t} & -\frac{4e^{6t}}{3}+\frac{4e^{3t}}{3}-1+e^{-3t} & -\frac{5e^{6t}}{3}+\frac{8e^{3t}}{3}-2+e^{-3t} & -e^{6t}+2e^{3t}-1 \\ 4-e^{-3t}-e^{3t}-2e^{6t} & e^{-3t}+\frac{4e^{6t}}{3}+\frac{2e^{3t}}{3}-2 & \frac{5e^{6t}}{3}+\frac{4e^{3t}}{3}-4+e^{-3t} & e^{3t}-2+e^{6t} \\ -2-e^{-3t}-e^{3t}+4e^{6t} & -\frac{8e^{6t}}{3}+\frac{2e^{3t}}{3}+1+e^{-3t} & e^{-3t}-\frac{10e^{6t}}{3}+\frac{4e^{3t}}{3}+2 & -2e^{6t}+e^{3t}+1 \\ 2+e^{-3t}-e^{3t}-2e^{6t} & \frac{4e^{6t}}{3}+\frac{2e^{3t}}{3}-1-e^{-3t} & \frac{5e^{6t}}{3}+\frac{4e^{3t}}{3}-2-e^{-3t} & e^{6t}+e^{3t}-1 \end{bmatrix}
$$

$$
= \begin{bmatrix} \left(-e^{-3t}+2-2e^{3t}+2e^{6t}\right)c_1+\left(-\frac{4e^{6t}}{3}+\frac{4e^{3t}}{3}-1+e^{-3t}\right)c_2+\left(-\frac{5e^{6t}}{3}+\frac{8e^{3t}}{3}-2+e^{-3t}\right)c_3+(\\ \left(4-e^{-3t}-e^{3t}-2e^{6t}\right)c_1+\left(e^{-3t}+\frac{4e^{6t}}{3}+\frac{2e^{3t}}{3}-2\right)c_2+\left(\frac{5e^{6t}}{3}+\frac{4e^{3t}}{3}-4+e^{-3t}\right)c_3+(e^3 \\ \left(-2-e^{-3t}-e^{3t}+4e^{6t}\right)c_1+\left(-\frac{8e^{6t}}{3}+\frac{2e^{3t}}{3}+1+e^{-3t}\right)c_2+\left(e^{-3t}-\frac{10e^{6t}}{3}+\frac{4e^{3t}}{3}+2\right)c_3+(- \\ \left(2+e^{-3t}-e^{3t}-2e^{6t}\right)c_1+\left(\frac{4e^{6t}}{3}+\frac{2e^{3t}}{3}-1-e^{-3t}\right)c_2+\left(\frac{5e^{6t}}{3}+\frac{4e^{3t}}{3}-2-e^{-3t}\right)c_3+(e^6 \end{bmatrix}
$$

$$
= \begin{bmatrix} \frac{2(-3c_1+2c_2+4c_3+3c_4)e^{3t}}{3}+\frac{(6c_1-4c_2-5c_3-3c_4)e^{6t}}{3}+(-c_1+c_2+c_3)e^{-3t}+2c_1-c_2-2c_3-c_4 \\ \frac{(-3c_1+2c_2+4c_3+3c_4)e^{3t}}{3}+\frac{(-6c_1+4c_2+5c_3+3c_4)e^{6t}}{3}+(-c_1+c_2+c_3)e^{-3t}+4c_1-2c_2-4c_3-2c_4 \\ \frac{(-3c_1+2c_2+4c_3+3c_4)e^{3t}}{3}+\frac{2(6c_1-4c_2-5c_3-3c_4)e^{6t}}{3}+(-c_1+c_2+c_3)e^{-3t}-2c_1+c_2+2c_3+c_4 \\ \frac{(-3c_1+2c_2+4c_3+3c_4)e^{3t}}{3}+\frac{(-6c_1+4c_2+5c_3+3c_4)e^{6t}}{3}+(c_1-c_2-c_3)e^{-3t}+2c_1-c_2-2c_3-c_4 \end{bmatrix}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

4.34.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \\ x_4'(t) \end{bmatrix} = \begin{bmatrix} 9 & -7 & -5 & 0 \\ -12 & 7 & 11 & 9 \\ 24 & -17 & -19 & -9 \\ -18 & 13 & 17 & 9 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 9 & -7 & -5 & 0 \\ -12 & 7 & 11 & 9 \\ 24 & -17 & -19 & -9 \\ -18 & 13 & 17 & 9 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 9-\lambda & -7 & -5 & 0 \\ -12 & 7-\lambda & 11 & 9 \\ 24 & -17 & -19-\lambda & -9 \\ -18 & 13 & 17 & 9-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^4 - 6\lambda^3 - 9\lambda^2 + 54\lambda = 0$$

The roots of the above are the eigenvalues.

$$\begin{aligned} \lambda_1 &= 3 \\ \lambda_2 &= 0 \\ \lambda_3 &= 6 \\ \lambda_4 &= -3 \end{aligned}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
0	1	real eigenvalue
-3	1	real eigenvalue
3	1	real eigenvalue
6	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 9 & -7 & -5 & 0 \\ -12 & 7 & 11 & 9 \\ 24 & -17 & -19 & -9 \\ -18 & 13 & 17 & 9 \end{bmatrix} - (-3)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 12 & -7 & -5 & 0 \\ -12 & 10 & 11 & 9 \\ 24 & -17 & -16 & -9 \\ -18 & 13 & 17 & 12 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} 12 & -7 & -5 & 0 & 0 \\ -12 & 10 & 11 & 9 & 0 \\ 24 & -17 & -16 & -9 & 0 \\ -18 & 13 & 17 & 12 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cccc|c} 12 & -7 & -5 & 0 & 0 \\ 0 & 3 & 6 & 9 & 0 \\ 24 & -17 & -16 & -9 & 0 \\ -18 & 13 & 17 & 12 & 0 \end{array}\right]$$

$$R_3 = R_3 - 2R_1 \Longrightarrow \left[\begin{array}{cccc|c} 12 & -7 & -5 & 0 & 0 \\ 0 & 3 & 6 & 9 & 0 \\ 0 & -3 & -6 & -9 & 0 \\ -18 & 13 & 17 & 12 & 0 \end{array}\right]$$

$$R_4 = R_4 + \frac{3R_1}{2} \Longrightarrow \left[\begin{array}{cccc|c} 12 & -7 & -5 & 0 & 0 \\ 0 & 3 & 6 & 9 & 0 \\ 0 & -3 & -6 & -9 & 0 \\ 0 & \frac{5}{2} & \frac{19}{2} & 12 & 0 \end{array}\right]$$

$$R_3 = R_3 + R_2 \Longrightarrow \left[\begin{array}{cccc|c} 12 & -7 & -5 & 0 & 0 \\ 0 & 3 & 6 & 9 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{5}{2} & \frac{19}{2} & 12 & 0 \end{array}\right]$$

$$R_4 = R_4 - \frac{5R_2}{6} \Longrightarrow \left[\begin{array}{cccc|c} 12 & -7 & -5 & 0 & 0 \\ 0 & 3 & 6 & 9 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{9}{2} & \frac{9}{2} & 0 \end{array}\right]$$

Since the current pivot $A(3,3)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 3 and row 4 gives

$$\left[\begin{array}{cccc|c} 12 & -7 & -5 & 0 & 0 \\ 0 & 3 & 6 & 9 & 0 \\ 0 & 0 & \frac{9}{2} & \frac{9}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 12 & -7 & -5 & 0 \\ 0 & 3 & 6 & 9 \\ 0 & 0 & \frac{9}{2} & \frac{9}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_4\}$ and the leading variables are $\{v_1, v_2, v_3\}$. Let $v_4 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t, v_2 = -t, v_3 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ -t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ -t \\ -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ -t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ -t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 0$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 9 & -7 & -5 & 0 \\ -12 & 7 & 11 & 9 \\ 24 & -17 & -19 & -9 \\ -18 & 13 & 17 & 9 \end{bmatrix} - (0) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 9 & -7 & -5 & 0 \\ -12 & 7 & 11 & 9 \\ 24 & -17 & -19 & -9 \\ -18 & 13 & 17 & 9 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} 9 & -7 & -5 & 0 & 0 \\ -12 & 7 & 11 & 9 & 0 \\ 24 & -17 & -19 & -9 & 0 \\ -18 & 13 & 17 & 9 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{4R_1}{3} \Longrightarrow \left[\begin{array}{cccc|c} 9 & -7 & -5 & 0 & 0 \\ 0 & -\frac{7}{3} & \frac{13}{3} & 9 & 0 \\ 24 & -17 & -19 & -9 & 0 \\ -18 & 13 & 17 & 9 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{8R_1}{3} \Longrightarrow \left[\begin{array}{cccc|c} 9 & -7 & -5 & 0 & 0 \\ 0 & -\frac{7}{3} & \frac{13}{3} & 9 & 0 \\ 0 & \frac{5}{3} & -\frac{17}{3} & -9 & 0 \\ -18 & 13 & 17 & 9 & 0 \end{array}\right]$$

$$R_4 = R_4 + 2R_1 \Longrightarrow \left[\begin{array}{cccc|c} 9 & -7 & -5 & 0 & 0 \\ 0 & -\frac{7}{3} & \frac{13}{3} & 9 & 0 \\ 0 & \frac{5}{3} & -\frac{17}{3} & -9 & 0 \\ 0 & -1 & 7 & 9 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{5R_2}{7} \Longrightarrow \left[\begin{array}{cccc|c} 9 & -7 & -5 & 0 & 0 \\ 0 & -\frac{7}{3} & \frac{13}{3} & 9 & 0 \\ 0 & 0 & -\frac{18}{7} & -\frac{18}{7} & 0 \\ 0 & -1 & 7 & 9 & 0 \end{array}\right]$$

$$R_4 = R_4 - \frac{3R_2}{7} \Longrightarrow \left[\begin{array}{cccc|c} 9 & -7 & -5 & 0 & 0 \\ 0 & -\frac{7}{3} & \frac{13}{3} & 9 & 0 \\ 0 & 0 & -\frac{18}{7} & -\frac{18}{7} & 0 \\ 0 & 0 & \frac{36}{7} & \frac{36}{7} & 0 \end{array}\right]$$

$$R_4 = R_4 + 2R_3 \Longrightarrow \left[\begin{array}{cccc|c} 9 & -7 & -5 & 0 & 0 \\ 0 & -\frac{7}{3} & \frac{13}{3} & 9 & 0 \\ 0 & 0 & -\frac{18}{7} & -\frac{18}{7} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 9 & -7 & -5 & 0 \\ 0 & -\frac{7}{3} & \frac{13}{3} & 9 \\ 0 & 0 & -\frac{18}{7} & -\frac{18}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_4\}$ and the leading variables are $\{v_1, v_2, v_3\}$. Let $v_4 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = 2t, v_3 = -t\}$

Hence the solution is

$$\begin{bmatrix} t \\ 2t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} t \\ 2t \\ -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ 2t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ 2t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = 3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 9 & -7 & -5 & 0 \\ -12 & 7 & 11 & 9 \\ 24 & -17 & -19 & -9 \\ -18 & 13 & 17 & 9 \end{bmatrix} - (3)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 6 & -7 & -5 & 0 \\ -12 & 4 & 11 & 9 \\ 24 & -17 & -22 & -9 \\ -18 & 13 & 17 & 6 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} 6 & -7 & -5 & 0 & 0 \\ -12 & 4 & 11 & 9 & 0 \\ 24 & -17 & -22 & -9 & 0 \\ -18 & 13 & 17 & 6 & 0 \end{array}\right]$$

$$R_2 = R_2 + 2R_1 \Longrightarrow \left[\begin{array}{cccc|c} 6 & -7 & -5 & 0 & 0 \\ 0 & -10 & 1 & 9 & 0 \\ 24 & -17 & -22 & -9 & 0 \\ -18 & 13 & 17 & 6 & 0 \end{array}\right]$$

$$R_3 = R_3 - 4R_1 \Longrightarrow \left[\begin{array}{cccc|c} 6 & -7 & -5 & 0 & 0 \\ 0 & -10 & 1 & 9 & 0 \\ 0 & 11 & -2 & -9 & 0 \\ -18 & 13 & 17 & 6 & 0 \end{array}\right]$$

$$R_4 = R_4 + 3R_1 \Longrightarrow \left[\begin{array}{cccc|c} 6 & -7 & -5 & 0 & 0 \\ 0 & -10 & 1 & 9 & 0 \\ 0 & 11 & -2 & -9 & 0 \\ 0 & -8 & 2 & 6 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{11R_2}{10} \Longrightarrow \left[\begin{array}{cccc|c} 6 & -7 & -5 & 0 & 0 \\ 0 & -10 & 1 & 9 & 0 \\ 0 & 0 & -\frac{9}{10} & \frac{9}{10} & 0 \\ 0 & -8 & 2 & 6 & 0 \end{array}\right]$$

$$R_4 = R_4 - \frac{4R_2}{5} \Longrightarrow \left[\begin{array}{cccc|c} 6 & -7 & -5 & 0 & 0 \\ 0 & -10 & 1 & 9 & 0 \\ 0 & 0 & -\frac{9}{10} & \frac{9}{10} & 0 \\ 0 & 0 & \frac{6}{5} & -\frac{6}{5} & 0 \end{array}\right]$$

$$R_4 = R_4 + \frac{4R_3}{3} \Longrightarrow \left[\begin{array}{cccc|c} 6 & -7 & -5 & 0 & 0 \\ 0 & -10 & 1 & 9 & 0 \\ 0 & 0 & -\frac{9}{10} & \frac{9}{10} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 6 & -7 & -5 & 0 \\ 0 & -10 & 1 & 9 \\ 0 & 0 & -\frac{9}{10} & \frac{9}{10} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_4\}$ and the leading variables are $\{v_1, v_2, v_3\}$. Let $v_4 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2t, v_2 = t, v_3 = t\}$

Hence the solution is

$$\begin{bmatrix} 2t \\ t \\ t \\ t \end{bmatrix} = \begin{bmatrix} 2t \\ t \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this

eigenvalue. The above can be written as

$$\begin{bmatrix} 2t \\ t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 2t \\ t \\ t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_4 = 6$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 9 & -7 & -5 & 0 \\ -12 & 7 & 11 & 9 \\ 24 & -17 & -19 & -9 \\ -18 & 13 & 17 & 9 \end{bmatrix} - (6) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -7 & -5 & 0 \\ -12 & 1 & 11 & 9 \\ 24 & -17 & -25 & -9 \\ -18 & 13 & 17 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} 3 & -7 & -5 & 0 & 0 \\ -12 & 1 & 11 & 9 & 0 \\ 24 & -17 & -25 & -9 & 0 \\ -18 & 13 & 17 & 3 & 0 \end{array}\right]$$

$$R_2 = R_2 + 4R_1 \implies \left[\begin{array}{cccc|c} 3 & -7 & -5 & 0 & 0 \\ 0 & -27 & -9 & 9 & 0 \\ 24 & -17 & -25 & -9 & 0 \\ -18 & 13 & 17 & 3 & 0 \end{array}\right]$$

$$R_3 = R_3 - 8R_1 \implies \left[\begin{array}{cccc|c} 3 & -7 & -5 & 0 & 0 \\ 0 & -27 & -9 & 9 & 0 \\ 0 & 39 & 15 & -9 & 0 \\ -18 & 13 & 17 & 3 & 0 \end{array}\right]$$

$$R_4 = R_4 + 6R_1 \implies \left[\begin{array}{cccc|c} 3 & -7 & -5 & 0 & 0 \\ 0 & -27 & -9 & 9 & 0 \\ 0 & 39 & 15 & -9 & 0 \\ 0 & -29 & -13 & 3 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{13R_2}{9} \implies \left[\begin{array}{cccc|c} 3 & -7 & -5 & 0 & 0 \\ 0 & -27 & -9 & 9 & 0 \\ 0 & 0 & 2 & 4 & 0 \\ 0 & -29 & -13 & 3 & 0 \end{array}\right]$$

$$R_4 = R_4 - \frac{29R_2}{27} \implies \left[\begin{array}{cccc|c} 3 & -7 & -5 & 0 & 0 \\ 0 & -27 & -9 & 9 & 0 \\ 0 & 0 & 2 & 4 & 0 \\ 0 & 0 & -\frac{10}{3} & -\frac{20}{3} & 0 \end{array}\right]$$

$$R_4 = R_4 + \frac{5R_3}{3} \implies \left[\begin{array}{cccc|c} 3 & -7 & -5 & 0 & 0 \\ 0 & -27 & -9 & 9 & 0 \\ 0 & 0 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3 & -7 & -5 & 0 \\ 0 & -27 & -9 & 9 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_4\}$ and the leading variables are $\{v_1, v_2, v_3\}$. Let $v_4 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t, v_2 = t, v_3 = -2t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \\ -2t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \\ -2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \\ -2t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ -2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \\ -2t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -2 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
3	1	1	No	$\begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}$
0	1	1	No	$\begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}$
6	1	1	No	$\begin{bmatrix} -1 \\ 1 \\ -2 \\ 1 \end{bmatrix}$
-3	1	1	No	$\begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 3 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{3t} \\ &= \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix} e^{3t} \end{aligned}$$

Since eigenvalue 0 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{0} \\ &= \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix} e^{0}\end{aligned}$$

Since eigenvalue 6 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_3(t) &= \vec{v}_3 e^{6t} \\ &= \begin{bmatrix} -1 \\ 1 \\ -2 \\ 1 \end{bmatrix} e^{6t}\end{aligned}$$

Since eigenvalue -3 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_4(t) &= \vec{v}_4 e^{-3t} \\ &= \begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \end{bmatrix} e^{-3t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t) + c_4 \vec{x}_4(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = c_1 \begin{bmatrix} 2\,\mathrm{e}^{3t} \\ \mathrm{e}^{3t} \\ \mathrm{e}^{3t} \\ \mathrm{e}^{3t} \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} -\mathrm{e}^{6t} \\ \mathrm{e}^{6t} \\ -2\,\mathrm{e}^{6t} \\ \mathrm{e}^{6t} \end{bmatrix} + c_4 \begin{bmatrix} -\mathrm{e}^{-3t} \\ -\mathrm{e}^{-3t} \\ -\mathrm{e}^{-3t} \\ \mathrm{e}^{-3t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = \begin{bmatrix} 2c_1 e^{3t} + c_2 - c_3 e^{6t} - c_4 e^{-3t} \\ c_1 e^{3t} + 2c_2 + c_3 e^{6t} - c_4 e^{-3t} \\ c_1 e^{3t} - c_2 - 2c_3 e^{6t} - c_4 e^{-3t} \\ c_1 e^{3t} + c_2 + c_3 e^{6t} + c_4 e^{-3t} \end{bmatrix}$$

4.34.3 Maple step by step solution

Let's solve

$[x_1'(t) = 9x_1(t) - 7x_2(t) - 5x_3(t), x_2'(t) = -12x_1(t) + 7x_2(t) + 11x_3(t) + 9x_4(t), x_3'(t) = 24x_1(t) -$

- Define vector

$$\underline{\vec{x}}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{\vec{x}}'(t) = \begin{bmatrix} 9 & -7 & -5 & 0 \\ -12 & 7 & 11 & 9 \\ 24 & -17 & -19 & -9 \\ -18 & 13 & 17 & 9 \end{bmatrix} \cdot \underline{\vec{x}}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{\vec{x}}'(t) = \begin{bmatrix} 9 & -7 & -5 & 0 \\ -12 & 7 & 11 & 9 \\ 24 & -17 & -19 & -9 \\ -18 & 13 & 17 & 9 \end{bmatrix} \cdot \underline{\vec{x}}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 9 & -7 & -5 & 0 \\ -12 & 7 & 11 & 9 \\ 24 & -17 & -19 & -9 \\ -18 & 13 & 17 & 9 \end{bmatrix}$$

- Rewrite the system as

$$\underline{\vec{x}}'(t) = A \cdot \underline{\vec{x}}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-3, \begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \end{bmatrix}\right], \left[0, \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}\right], \left[3, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[6, \begin{bmatrix} -1 \\ 1 \\ -2 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-3, \begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_1 = \mathrm{e}^{-3t} \cdot \begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_{2} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[3, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_{3} = \mathrm{e}^{3t} \cdot \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[6, \begin{bmatrix} -1 \\ 1 \\ -2 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_{4} = \mathrm{e}^{6t} \cdot \begin{bmatrix} -1 \\ 1 \\ -2 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$\underline{\vec{x}} = c_1 \underline{\vec{x}}_{1} + c_2 \underline{\vec{x}}_{2} + c_3 \underline{\vec{x}}_{3} + c_4 \underline{\vec{x}}_{4}$

- Substitute solutions into the general solution

$$\underline{x}^{\rightarrow} = c_1 e^{-3t} \cdot \begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \end{bmatrix} + c_3 e^{3t} \cdot \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix} + c_4 e^{6t} \cdot \begin{bmatrix} -1 \\ 1 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} c_2 \\ 2c_2 \\ -c_2 \\ c_2 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = \begin{bmatrix} -c_1 e^{-3t} + 2c_3 e^{3t} - c_4 e^{6t} + c_2 \\ -c_1 e^{-3t} + c_3 e^{3t} + c_4 e^{6t} + 2c_2 \\ -c_1 e^{-3t} + c_3 e^{3t} - 2c_4 e^{6t} - c_2 \\ c_1 e^{-3t} + c_3 e^{3t} + c_4 e^{6t} + c_2 \end{bmatrix}$$

- Solution to the system of ODEs

$\{x_1(t) = -c_1 e^{-3t} + 2c_3 e^{3t} - c_4 e^{6t} + c_2, x_2(t) = -c_1 e^{-3t} + c_3 e^{3t} + c_4 e^{6t} + 2c_2, x_3(t) = -c_1 e^{-3t} + c_3 e$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 105

```
dsolve([diff(x__1(t),t)=9*x__1(t)-7*x__2(t)-5*x__3(t)+0*x__4(t),diff(x__2(t),t)=-12*x__1(t)+
```

$$x_1(t) = c_1 + c_2 e^{3t} + c_3 e^{6t} + c_4 e^{-3t}$$
$$x_2(t) = \frac{c_2 e^{3t}}{2} - c_3 e^{6t} + c_4 e^{-3t} + 2c_1$$
$$x_3(t) = \frac{c_2 e^{3t}}{2} + 2c_3 e^{6t} + c_4 e^{-3t} - c_1$$
$$x_4(t) = \frac{c_2 e^{3t}}{2} - c_3 e^{6t} - c_4 e^{-3t} + c_1$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 430

```
DSolve[{x1'[t]==9*x1[t]-7*x2[t]-5*x3[t]+0*x4[t],x2'[t]==-12*x1[t]+7*x2[t]+11*x3[t]+9*x4[t],x
```

$$\text{x1}(t) \to \frac{1}{3}e^{-3t}\left(c_1\left(6e^{3t}-6e^{6t}+6e^{9t}-3\right) - \left(e^{3t}-1\right)\left(c_2\left(4e^{6t}+3\right)+c_3\left(-3e^{3t}+5e^{6t}+3\right)+3c_4e^{3t}\left(e^{3t}-1\right)\right)\right)$$

$$\text{x2}(t) \to \frac{1}{3}e^{-3t}\left(-3c_1\left(-4e^{3t}+e^{6t}+2e^{9t}+1\right)+c_2\left(-6e^{3t}+2e^{6t}+4e^{9t}+3\right) + \left(e^{3t}-1\right)\left(c_3\left(9e^{3t}+5e^{6t}-3\right)+3c_4e^{3t}\left(e^{3t}+2\right)\right)\right)$$

$$\text{x3}(t) \to c_1\left(-e^{-3t}-e^{3t}+4e^{6t}-2\right)+c_2\left(e^{-3t}+\frac{2e^{3t}}{3}-\frac{8e^{6t}}{3}+1\right) + c_3e^{-3t}+\frac{4}{3}c_3e^{3t}-\frac{10}{3}c_3e^{6t}+c_4e^{3t}-2c_4e^{6t}+2c_3+c_4$$

$$\text{x4}(t) \to \frac{1}{3}\left(c_1\left(3e^{-3t}-3e^{3t}-6e^{6t}+6\right)+c_2\left(-3e^{-3t}+2e^{3t}+4e^{6t}-3\right)-3c_3e^{-3t}+4c_3e^{3t} + 5c_3e^{6t}+3c_4e^{3t}+3c_4e^{6t}-6c_3-3c_4\right)$$

4.35 problem problem 46

Internal problem ID [349]
Internal file name [`OUTPUT/349_Sunday_June_05_2022_01_39_19_AM_94056782/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.3, The eigenvalue method for linear systems. Page 395
Problem number: problem 46.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= 13x_1(t) - 42x_2(t) + 106x_3(t) + 139x_4(t)\\
x_2'(t) &= 2x_1(t) - 16x_2(t) + 52x_3(t) + 70x_4(t)\\
x_3'(t) &= x_1(t) + 6x_2(t) - 20x_3(t) - 31x_4(t)\\
x_4'(t) &= -x_1(t) - 6x_2(t) + 22x_3(t) + 33x_4(t)
\end{aligned}
$$

4.35.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \\ x_4'(t) \end{bmatrix} = \begin{bmatrix} 13 & -42 & 106 & 139 \\ 2 & -16 & 52 & 70 \\ 1 & 6 & -20 & -31 \\ -1 & -6 & 22 & 33 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} -\frac{3\,e^{-4t}}{4}+\frac{3\,e^{8t}}{4}+e^{4t} & -\frac{3\,e^{8t}}{2}-3\,e^{4t}+\frac{9\,e^{-4t}}{2} & (3\,e^{12t}+8\,e^{8t}+e^{6t}-12)\,e^{-4t} & \frac{(15\,e^{12t}+44\,e^{8t}+4\,e^{6t}-63)\,e}{4} \\ -\frac{e^{8t}}{2}+e^{4t}-\frac{e^{-4t}}{2} & e^{8t}-3\,e^{4t}+3\,e^{-4t} & -2(e^{12t}-4\,e^{8t}-e^{6t}+4)\,e^{-4t} & -\frac{(5\,e^{12t}-22\,e^{8t}-4\,e^{6t}+21)}{2} \\ \frac{3\,e^{8t}}{4}-e^{4t}+\frac{e^{-4t}}{4} & -\frac{3\,e^{8t}}{2}+3\,e^{4t}-\frac{3\,e^{-4t}}{2} & (3\,e^{12t}-8\,e^{8t}+2\,e^{6t}+4)\,e^{-4t} & \frac{(15\,e^{12t}-44\,e^{8t}+8\,e^{6t}+21)\,e}{4} \\ -\frac{3\,e^{8t}}{4}+e^{4t}-\frac{e^{-4t}}{4} & \frac{3\,e^{8t}}{2}-3\,e^{4t}+\frac{3\,e^{-4t}}{2} & -(3\,e^{12t}-8\,e^{8t}+e^{6t}+4)\,e^{-4t} & -\frac{(15\,e^{12t}-44\,e^{8t}+4\,e^{6t}+21)}{4} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = e^{At}\vec{c}$$

$$= \begin{bmatrix} -\frac{3\,e^{-4t}}{4}+\frac{3\,e^{8t}}{4}+e^{4t} & -\frac{3\,e^{8t}}{2}-3\,e^{4t}+\frac{9\,e^{-4t}}{2} & (3\,e^{12t}+8\,e^{8t}+e^{6t}-12)\,e^{-4t} & \frac{(15\,e^{12t}+44\,e^{8t}+4\,e^{6t}-6}{4} \\ -\frac{e^{8t}}{2}+e^{4t}-\frac{e^{-4t}}{2} & e^{8t}-3\,e^{4t}+3\,e^{-4t} & -2(e^{12t}-4\,e^{8t}-e^{6t}+4)\,e^{-4t} & -\frac{(5\,e^{12t}-22\,e^{8t}-4\,e^{6t}+2}{2} \\ \frac{3\,e^{8t}}{4}-e^{4t}+\frac{e^{-4t}}{4} & -\frac{3\,e^{8t}}{2}+3\,e^{4t}-\frac{3\,e^{-4t}}{2} & (3\,e^{12t}-8\,e^{8t}+2\,e^{6t}+4)\,e^{-4t} & \frac{(15\,e^{12t}-44\,e^{8t}+8\,e^{6t}+2}{4} \\ -\frac{3\,e^{8t}}{4}+e^{4t}-\frac{e^{-4t}}{4} & \frac{3\,e^{8t}}{2}-3\,e^{4t}+\frac{3\,e^{-4t}}{2} & -(3\,e^{12t}-8\,e^{8t}+e^{6t}+4)\,e^{-4t} & -\frac{(15\,e^{12t}-44\,e^{8t}+4\,e^{6t}+}{4} \end{bmatrix}$$

$$= \begin{bmatrix} \left(-\frac{3\,e^{-4t}}{4}+\frac{3\,e^{8t}}{4}+e^{4t}\right)c_1+\left(-\frac{3\,e^{8t}}{2}-3\,e^{4t}+\frac{9\,e^{-4t}}{2}\right)c_2+(3\,e^{12t}+8\,e^{8t}+e^{6t}-12)\,e^{-4t}c_3+\frac{(15\,e^{12}}{} \\ \left(-\frac{e^{8t}}{2}+e^{4t}-\frac{e^{-4t}}{2}\right)c_1+(e^{8t}-3\,e^{4t}+3\,e^{-4t})\,c_2-2(e^{12t}-4\,e^{8t}-e^{6t}+4)\,e^{-4t}c_3-\frac{(5\,e^{12t}-22}{} \\ \left(\frac{3\,e^{8t}}{4}-e^{4t}+\frac{e^{-4t}}{4}\right)c_1+\left(-\frac{3\,e^{8t}}{2}+3\,e^{4t}-\frac{3\,e^{-4t}}{2}\right)c_2+(3\,e^{12t}-8\,e^{8t}+2\,e^{6t}+4)\,e^{-4t}c_3+\frac{(15\,e^{12t}}{} \\ \left(-\frac{3\,e^{8t}}{4}+e^{4t}-\frac{e^{-4t}}{4}\right)c_1+\left(\frac{3\,e^{8t}}{2}-3\,e^{4t}+\frac{3\,e^{-4t}}{2}\right)c_2-(3\,e^{12t}-8\,e^{8t}+e^{6t}+4)\,e^{-4t}c_3-\frac{(15\,e^{12t}-}{} \end{bmatrix}$$

$$= \begin{bmatrix} (3\left(\frac{c_1}{4}-\frac{c_2}{2}+c_3+\frac{5c_4}{4}\right)e^{12t}+(c_1-3c_2+8c_3+11c_4)\,e^{8t}+(c_3+c_4)\,e^{6t}-\frac{3c_1}{4}+\frac{9c_2}{2}-12c_3- \\ \left(\left(-\frac{c_1}{2}+c_2-2c_3-\frac{5c_4}{2}\right)e^{12t}+(c_1-3c_2+8c_3+11c_4)\,e^{8t}+(2c_3+2c_4)\,e^{6t}-\frac{c_1}{2}+3c_2-8c_3 \right. \\ -\left(\left(-\frac{3c_1}{4}+\frac{3c_2}{2}-3c_3-\frac{15c_4}{4}\right)e^{12t}+(c_1-3c_2+8c_3+11c_4)\,e^{8t}+(-2c_3-2c_4)\,e^{6t}-\frac{c_1}{4}+\frac{3c_2}{2}-4 \right. \\ \left(\left(-\frac{3c_1}{4}+\frac{3c_2}{2}-3c_3-\frac{15c_4}{4}\right)e^{12t}+(c_1-3c_2+8c_3+11c_4)\,e^{8t}+(-c_3-c_4)\,e^{6t}-\frac{c_1}{4}+\frac{3c_2}{2}-4c_3 \right. \end{bmatrix}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

4.35.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \\ x_4'(t) \end{bmatrix} = \begin{bmatrix} 13 & -42 & 106 & 139 \\ 2 & -16 & 52 & 70 \\ 1 & 6 & -20 & -31 \\ -1 & -6 & 22 & 33 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 13 & -42 & 106 & 139 \\ 2 & -16 & 52 & 70 \\ 1 & 6 & -20 & -31 \\ -1 & -6 & 22 & 33 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 13-\lambda & -42 & 106 & 139 \\ 2 & -16-\lambda & 52 & 70 \\ 1 & 6 & -20-\lambda & -31 \\ -1 & -6 & 22 & 33-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^4 - 10\lambda^3 + 160\lambda - 256 = 0$$

The roots of the above are the eigenvalues.

$$\begin{aligned} \lambda_1 &= 2 \\ \lambda_2 &= 8 \\ \lambda_3 &= 4 \\ \lambda_4 &= -4 \end{aligned}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
2	1	real eigenvalue
−4	1	real eigenvalue
4	1	real eigenvalue
8	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -4$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 13 & -42 & 106 & 139 \\ 2 & -16 & 52 & 70 \\ 1 & 6 & -20 & -31 \\ -1 & -6 & 22 & 33 \end{bmatrix} - (-4)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 17 & -42 & 106 & 139 \\ 2 & -12 & 52 & 70 \\ 1 & 6 & -16 & -31 \\ -1 & -6 & 22 & 37 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} 17 & -42 & 106 & 139 & 0 \\ 2 & -12 & 52 & 70 & 0 \\ 1 & 6 & -16 & -31 & 0 \\ -1 & -6 & 22 & 37 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{2R_1}{17} \Longrightarrow \left[\begin{array}{cccc|c} 17 & -42 & 106 & 139 & 0 \\ 0 & -\frac{120}{17} & \frac{672}{17} & \frac{912}{17} & 0 \\ 1 & 6 & -16 & -31 & 0 \\ -1 & -6 & 22 & 37 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{R_1}{17} \implies \left[\begin{array}{cccc|c} 17 & -42 & 106 & 139 & 0 \\ 0 & -\frac{120}{17} & \frac{672}{17} & \frac{912}{17} & 0 \\ 0 & \frac{144}{17} & -\frac{378}{17} & -\frac{666}{17} & 0 \\ -1 & -6 & 22 & 37 & 0 \end{array}\right]$$

$$R_4 = R_4 + \frac{R_1}{17} \implies \left[\begin{array}{cccc|c} 17 & -42 & 106 & 139 & 0 \\ 0 & -\frac{120}{17} & \frac{672}{17} & \frac{912}{17} & 0 \\ 0 & \frac{144}{17} & -\frac{378}{17} & -\frac{666}{17} & 0 \\ 0 & -\frac{144}{17} & \frac{480}{17} & \frac{768}{17} & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{6R_2}{5} \implies \left[\begin{array}{cccc|c} 17 & -42 & 106 & 139 & 0 \\ 0 & -\frac{120}{17} & \frac{672}{17} & \frac{912}{17} & 0 \\ 0 & 0 & \frac{126}{5} & \frac{126}{5} & 0 \\ 0 & -\frac{144}{17} & \frac{480}{17} & \frac{768}{17} & 0 \end{array}\right]$$

$$R_4 = R_4 - \frac{6R_2}{5} \implies \left[\begin{array}{cccc|c} 17 & -42 & 106 & 139 & 0 \\ 0 & -\frac{120}{17} & \frac{672}{17} & \frac{912}{17} & 0 \\ 0 & 0 & \frac{126}{5} & \frac{126}{5} & 0 \\ 0 & 0 & -\frac{96}{5} & -\frac{96}{5} & 0 \end{array}\right]$$

$$R_4 = R_4 + \frac{16R_3}{21} \implies \left[\begin{array}{cccc|c} 17 & -42 & 106 & 139 & 0 \\ 0 & -\frac{120}{17} & \frac{672}{17} & \frac{912}{17} & 0 \\ 0 & 0 & \frac{126}{5} & \frac{126}{5} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 17 & -42 & 106 & 139 \\ 0 & -\frac{120}{17} & \frac{672}{17} & \frac{912}{17} \\ 0 & 0 & \frac{126}{5} & \frac{126}{5} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_4\}$ and the leading variables are $\{v_1, v_2, v_3\}$. Let $v_4 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 3t, v_2 = 2t, v_3 = -t\}$

Hence the solution is

$$\begin{bmatrix} 3t \\ 2t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} 3t \\ 2t \\ -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 3t \\ 2t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} 3 \\ 2 \\ -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 3t \\ 2t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 13 & -42 & 106 & 139 \\ 2 & -16 & 52 & 70 \\ 1 & 6 & -20 & -31 \\ -1 & -6 & 22 & 33 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 11 & -42 & 106 & 139 \\ 2 & -18 & 52 & 70 \\ 1 & 6 & -22 & -31 \\ -1 & -6 & 22 & 31 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} 11 & -42 & 106 & 139 & 0 \\ 2 & -18 & 52 & 70 & 0 \\ 1 & 6 & -22 & -31 & 0 \\ -1 & -6 & 22 & 31 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{2R_1}{11} \Longrightarrow \left[\begin{array}{cccc|c} 11 & -42 & 106 & 139 & 0 \\ 0 & -\frac{114}{11} & \frac{360}{11} & \frac{492}{11} & 0 \\ 1 & 6 & -22 & -31 & 0 \\ -1 & -6 & 22 & 31 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{R_1}{11} \Longrightarrow \left[\begin{array}{cccc|c} 11 & -42 & 106 & 139 & 0 \\ 0 & -\frac{114}{11} & \frac{360}{11} & \frac{492}{11} & 0 \\ 0 & \frac{108}{11} & -\frac{348}{11} & -\frac{480}{11} & 0 \\ -1 & -6 & 22 & 31 & 0 \end{array}\right]$$

$$R_4 = R_4 + \frac{R_1}{11} \Longrightarrow \left[\begin{array}{cccc|c} 11 & -42 & 106 & 139 & 0 \\ 0 & -\frac{114}{11} & \frac{360}{11} & \frac{492}{11} & 0 \\ 0 & \frac{108}{11} & -\frac{348}{11} & -\frac{480}{11} & 0 \\ 0 & -\frac{108}{11} & \frac{348}{11} & \frac{480}{11} & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{18R_2}{19} \Longrightarrow \left[\begin{array}{cccc|c} 11 & -42 & 106 & 139 & 0 \\ 0 & -\frac{114}{11} & \frac{360}{11} & \frac{492}{11} & 0 \\ 0 & 0 & -\frac{12}{19} & -\frac{24}{19} & 0 \\ 0 & -\frac{108}{11} & \frac{348}{11} & \frac{480}{11} & 0 \end{array}\right]$$

$$R_4 = R_4 - \frac{18R_2}{19} \Longrightarrow \left[\begin{array}{cccc|c} 11 & -42 & 106 & 139 & 0 \\ 0 & -\frac{114}{11} & \frac{360}{11} & \frac{492}{11} & 0 \\ 0 & 0 & -\frac{12}{19} & -\frac{24}{19} & 0 \\ 0 & 0 & \frac{12}{19} & \frac{24}{19} & 0 \end{array}\right]$$

$$R_4 = R_4 + R_3 \Longrightarrow \left[\begin{array}{cccc|c} 11 & -42 & 106 & 139 & 0 \\ 0 & -\frac{114}{11} & \frac{360}{11} & \frac{492}{11} & 0 \\ 0 & 0 & -\frac{12}{19} & -\frac{24}{19} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 11 & -42 & 106 & 139 \\ 0 & -\frac{114}{11} & \frac{360}{11} & \frac{492}{11} \\ 0 & 0 & -\frac{12}{19} & -\frac{24}{19} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_4\}$ and the leading variables are $\{v_1, v_2, v_3\}$. Let $v_4 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t, v_2 = -2t, v_3 = -2t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ -2t \\ -2t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ -2t \\ -2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ -2t \\ -2t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ -2 \\ -2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ -2t \\ -2t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ -2 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = 4$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 13 & -42 & 106 & 139 \\ 2 & -16 & 52 & 70 \\ 1 & 6 & -20 & -31 \\ -1 & -6 & 22 & 33 \end{bmatrix} - (4)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 9 & -42 & 106 & 139 \\ 2 & -20 & 52 & 70 \\ 1 & 6 & -24 & -31 \\ -1 & -6 & 22 & 29 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} 9 & -42 & 106 & 139 & 0 \\ 2 & -20 & 52 & 70 & 0 \\ 1 & 6 & -24 & -31 & 0 \\ -1 & -6 & 22 & 29 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{2R_1}{9} \Longrightarrow \left[\begin{array}{cccc|c} 9 & -42 & 106 & 139 & 0 \\ 0 & -\frac{32}{3} & \frac{256}{9} & \frac{352}{9} & 0 \\ 1 & 6 & -24 & -31 & 0 \\ -1 & -6 & 22 & 29 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{R_1}{9} \Longrightarrow \left[\begin{array}{cccc|c} 9 & -42 & 106 & 139 & 0 \\ 0 & -\frac{32}{3} & \frac{256}{9} & \frac{352}{9} & 0 \\ 0 & \frac{32}{3} & -\frac{322}{9} & -\frac{418}{9} & 0 \\ -1 & -6 & 22 & 29 & 0 \end{array}\right]$$

$$R_4 = R_4 + \frac{R_1}{9} \Longrightarrow \left[\begin{array}{cccc|c} 9 & -42 & 106 & 139 & 0 \\ 0 & -\frac{32}{3} & \frac{256}{9} & \frac{352}{9} & 0 \\ 0 & \frac{32}{3} & -\frac{322}{9} & -\frac{418}{9} & 0 \\ 0 & -\frac{32}{3} & \frac{304}{9} & \frac{400}{9} & 0 \end{array}\right]$$

$$R_3 = R_3 + R_2 \Longrightarrow \left[\begin{array}{cccc|c} 9 & -42 & 106 & 139 & 0 \\ 0 & -\frac{32}{3} & \frac{256}{9} & \frac{352}{9} & 0 \\ 0 & 0 & -\frac{22}{3} & -\frac{22}{3} & 0 \\ 0 & -\frac{32}{3} & \frac{304}{9} & \frac{400}{9} & 0 \end{array}\right]$$

$$R_4 = R_4 - R_2 \Longrightarrow \left[\begin{array}{cccc|c} 9 & -42 & 106 & 139 & 0 \\ 0 & -\frac{32}{3} & \frac{256}{9} & \frac{352}{9} & 0 \\ 0 & 0 & -\frac{22}{3} & -\frac{22}{3} & 0 \\ 0 & 0 & \frac{16}{3} & \frac{16}{3} & 0 \end{array}\right]$$

$$R_4 = R_4 + \frac{8R_3}{11} \Longrightarrow \left[\begin{array}{cccc|c} 9 & -42 & 106 & 139 & 0 \\ 0 & -\frac{32}{3} & \frac{256}{9} & \frac{352}{9} & 0 \\ 0 & 0 & -\frac{22}{3} & -\frac{22}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 9 & -42 & 106 & 139 \\ 0 & -\frac{32}{3} & \frac{256}{9} & \frac{352}{9} \\ 0 & 0 & -\frac{22}{3} & -\frac{22}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_4\}$ and the leading variables are $\{v_1, v_2, v_3\}$. Let $v_4 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = t, v_3 = -t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \\ -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this

eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_4 = 8$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 13 & -42 & 106 & 139 \\ 2 & -16 & 52 & 70 \\ 1 & 6 & -20 & -31 \\ -1 & -6 & 22 & 33 \end{bmatrix} - (8) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 5 & -42 & 106 & 139 \\ 2 & -24 & 52 & 70 \\ 1 & 6 & -28 & -31 \\ -1 & -6 & 22 & 25 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} 5 & -42 & 106 & 139 & 0 \\ 2 & -24 & 52 & 70 & 0 \\ 1 & 6 & -28 & -31 & 0 \\ -1 & -6 & 22 & 25 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{2R_1}{5} \Longrightarrow \left[\begin{array}{cccc|c} 5 & -42 & 106 & 139 & 0 \\ 0 & -\frac{36}{5} & \frac{48}{5} & \frac{72}{5} & 0 \\ 1 & 6 & -28 & -31 & 0 \\ -1 & -6 & 22 & 25 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{R_1}{5} \Longrightarrow \left[\begin{array}{cccc|c} 5 & -42 & 106 & 139 & 0 \\ 0 & -\frac{36}{5} & \frac{48}{5} & \frac{72}{5} & 0 \\ 0 & \frac{72}{5} & -\frac{246}{5} & -\frac{294}{5} & 0 \\ -1 & -6 & 22 & 25 & 0 \end{array}\right]$$

$$R_4 = R_4 + \frac{R_1}{5} \Longrightarrow \left[\begin{array}{cccc|c} 5 & -42 & 106 & 139 & 0 \\ 0 & -\frac{36}{5} & \frac{48}{5} & \frac{72}{5} & 0 \\ 0 & \frac{72}{5} & -\frac{246}{5} & -\frac{294}{5} & 0 \\ 0 & -\frac{72}{5} & \frac{216}{5} & \frac{264}{5} & 0 \end{array}\right]$$

$$R_3 = R_3 + 2R_2 \Longrightarrow \left[\begin{array}{cccc|c} 5 & -42 & 106 & 139 & 0 \\ 0 & -\frac{36}{5} & \frac{48}{5} & \frac{72}{5} & 0 \\ 0 & 0 & -30 & -30 & 0 \\ 0 & -\frac{72}{5} & \frac{216}{5} & \frac{264}{5} & 0 \end{array}\right]$$

$$R_4 = R_4 - 2R_2 \Longrightarrow \left[\begin{array}{cccc|c} 5 & -42 & 106 & 139 & 0 \\ 0 & -\frac{36}{5} & \frac{48}{5} & \frac{72}{5} & 0 \\ 0 & 0 & -30 & -30 & 0 \\ 0 & 0 & 24 & 24 & 0 \end{array}\right]$$

$$R_4 = R_4 + \frac{4R_3}{5} \Longrightarrow \left[\begin{array}{cccc|c} 5 & -42 & 106 & 139 & 0 \\ 0 & -\frac{36}{5} & \frac{48}{5} & \frac{72}{5} & 0 \\ 0 & 0 & -30 & -30 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 5 & -42 & 106 & 139 \\ 0 & -\frac{36}{5} & \frac{48}{5} & \frac{72}{5} \\ 0 & 0 & -30 & -30 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_4\}$ and the leading variables are $\{v_1, v_2, v_3\}$. Let $v_4 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -t, v_2 = \frac{2t}{3}, v_3 = -t\right\}$

Hence the solution is

$$\begin{bmatrix} -t \\ \frac{2t}{3} \\ -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ \frac{2t}{3} \\ -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ \frac{2t}{3} \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ \frac{2}{3} \\ -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ \frac{2t}{3} \\ -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ \frac{2}{3} \\ -1 \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -t \\ \frac{2t}{3} \\ -t \\ t \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \\ -3 \\ 3 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
2	1	1	No	$\begin{bmatrix} -1 \\ -2 \\ -2 \\ 1 \end{bmatrix}$
8	1	1	No	$\begin{bmatrix} -1 \\ \frac{2}{3} \\ -1 \\ 1 \end{bmatrix}$
4	1	1	No	$\begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$
-4	1	1	No	$\begin{bmatrix} 3 \\ 2 \\ -1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 2 is real and distinct then the

corresponding eigenvector solution is

$$\vec{x}_1(t) = \vec{v}_1 e^{2t}$$
$$= \begin{bmatrix} -1 \\ -2 \\ -2 \\ 1 \end{bmatrix} e^{2t}$$

Since eigenvalue 8 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_2(t) = \vec{v}_2 e^{8t}$$
$$= \begin{bmatrix} -1 \\ \frac{2}{3} \\ -1 \\ 1 \end{bmatrix} e^{8t}$$

Since eigenvalue 4 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_3(t) = \vec{v}_3 e^{4t}$$
$$= \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} e^{4t}$$

Since eigenvalue -4 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_4(t) = \vec{v}_4 e^{-4t}$$
$$= \begin{bmatrix} 3 \\ 2 \\ -1 \\ 1 \end{bmatrix} e^{-4t}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t) + c_4\vec{x}_4(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = c_1 \begin{bmatrix} -e^{2t} \\ -2\,e^{2t} \\ -2\,e^{2t} \\ e^{2t} \end{bmatrix} + c_2 \begin{bmatrix} -e^{8t} \\ \frac{2\,e^{8t}}{3} \\ -e^{8t} \\ e^{8t} \end{bmatrix} + c_3 \begin{bmatrix} e^{4t} \\ e^{4t} \\ -e^{4t} \\ e^{4t} \end{bmatrix} + c_4 \begin{bmatrix} 3\,e^{-4t} \\ 2\,e^{-4t} \\ -e^{-4t} \\ e^{-4t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = \begin{bmatrix} -(c_2 e^{12t} - c_3 e^{8t} + c_1 e^{6t} - 3c_4)\,e^{-4t} \\ \frac{(2c_2 e^{12t} + 3c_3 e^{8t} - 6c_1 e^{6t} + 6c_4)\,e^{-4t}}{3} \\ e^{-4t}(-c_2 e^{12t} - c_3 e^{8t} - 2c_1 e^{6t} - c_4) \\ (c_2 e^{12t} + c_3 e^{8t} + c_1 e^{6t} + c_4)\,e^{-4t} \end{bmatrix}$$

4.35.3 Maple step by step solution

Let's solve

$[x_1'(t) = 13x_1(t) - 42x_2(t) + 106x_3(t) + 139x_4(t)\,, x_2'(t) = 2x_1(t) - 16x_2(t) + 52x_3(t) + 70x_4(t)\,, x$

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x}^{\rightarrow}{}'(t) = \begin{bmatrix} 13 & -42 & 106 & 139 \\ 2 & -16 & 52 & 70 \\ 1 & 6 & -20 & -31 \\ -1 & -6 & 22 & 33 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{\vec{x}}'(t) = \begin{bmatrix} 13 & -42 & 106 & 139 \\ 2 & -16 & 52 & 70 \\ 1 & 6 & -20 & -31 \\ -1 & -6 & 22 & 33 \end{bmatrix} \cdot \underline{\vec{x}}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 13 & -42 & 106 & 139 \\ 2 & -16 & 52 & 70 \\ 1 & 6 & -20 & -31 \\ -1 & -6 & 22 & 33 \end{bmatrix}$$

- Rewrite the system as

$$\underline{\vec{x}}'(t) = A \cdot \underline{\vec{x}}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-4, \begin{bmatrix} 3 \\ 2 \\ -1 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} -1 \\ -2 \\ -2 \\ 1 \end{bmatrix}\right], \left[4, \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[8, \begin{bmatrix} -1 \\ \frac{2}{3} \\ -1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-4, \begin{bmatrix} 3 \\ 2 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_1 = \mathrm{e}^{-4t} \cdot \begin{bmatrix} 3 \\ 2 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} -1 \\ -2 \\ -2 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\overrightarrow{x}}_{2} = e^{2t} \cdot \begin{bmatrix} -1 \\ -2 \\ -2 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[4, \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\overrightarrow{x}}_{3} = e^{4t} \cdot \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[8, \begin{bmatrix} -1 \\ \frac{2}{3} \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_4 = e^{8t} \cdot \begin{bmatrix} -1 \\ \frac{2}{3} \\ -1 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs
$$\vec{x} = c_1 \vec{x}_1 + c_2 \vec{x}_2 + c_3 \vec{x}_3 + c_4 \vec{x}_4$$
- Substitute solutions into the general solution
$$\vec{x} = c_1 e^{-4t} \cdot \begin{bmatrix} 3 \\ 2 \\ -1 \\ 1 \end{bmatrix} + c_2 e^{2t} \cdot \begin{bmatrix} -1 \\ -2 \\ -2 \\ 1 \end{bmatrix} + c_3 e^{4t} \cdot \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} + c_4 e^{8t} \cdot \begin{bmatrix} -1 \\ \frac{2}{3} \\ -1 \\ 1 \end{bmatrix}$$
- Substitute in vector of dependent variables
$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = \begin{bmatrix} (-c_4 e^{12t} + c_3 e^{8t} - c_2 e^{6t} + 3c_1) e^{-4t} \\ \frac{(2c_4 e^{12t} + 3c_3 e^{8t} - 6c_2 e^{6t} + 6c_1) e^{-4t}}{3} \\ -(c_4 e^{12t} + c_3 e^{8t} + 2c_2 e^{6t} + c_1) e^{-4t} \\ (c_4 e^{12t} + c_3 e^{8t} + c_2 e^{6t} + c_1) e^{-4t} \end{bmatrix}$$
- Solution to the system of ODEs
$$\left\{x_1(t) = (-c_4 e^{12t} + c_3 e^{8t} - c_2 e^{6t} + 3c_1) e^{-4t}, x_2(t) = \frac{(2c_4 e^{12t} + 3c_3 e^{8t} - 6c_2 e^{6t} + 6c_1) e^{-4t}}{3}, x_3(t) = -(c_4 e^{12t} \right.$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 123

```
dsolve([diff(x__1(t),t)=13*x__1(t)-42*x__2(t)+106*x__3(t)+139*x__4(t),diff(x__2(t),t)=2*x__1
```

$$x_1(t) = c_1 e^{4t} + c_2 e^{-4t} + c_3 e^{2t} + c_4 e^{8t}$$
$$x_2(t) = c_1 e^{4t} + \frac{2c_2 e^{-4t}}{3} + 2c_3 e^{2t} - \frac{2c_4 e^{8t}}{3}$$
$$x_3(t) = -c_1 e^{4t} - \frac{c_2 e^{-4t}}{3} + 2c_3 e^{2t} + c_4 e^{8t}$$
$$x_4(t) = c_1 e^{4t} + \frac{c_2 e^{-4t}}{3} - c_3 e^{2t} - c_4 e^{8t}$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 449

```
DSolve[{x1'[t]==13*x1[t]-42*x2[t]+106*x3[t]+139*x4[t],x2'[t]==2*x1[t]-16*x2[t]+52*x3[t]+70*x
```

$$
\begin{aligned}
\mathrm{x1}(t) \to \frac{1}{4}e^{-4t}\big(&c_1\left(4e^{8t}+3e^{12t}-3\right)-6c_2\left(2e^{8t}+e^{12t}-3\right)+4c_3e^{6t}+32c_3e^{8t}+12c_3e^{12t} \\
&+4c_4e^{6t}+44c_4e^{8t}+15c_4e^{12t}-48c_3-63c_4\big) \\
\mathrm{x2}(t) \to \frac{1}{2}e^{-4t}\big(&-\left(c_1\left(-2e^{8t}+e^{12t}+1\right)\right)+2c_2\left(-3e^{8t}+e^{12t}+3\right)+4c_3e^{6t}+16c_3e^{8t} \\
&-4c_3e^{12t}+4c_4e^{6t}+22c_4e^{8t}-5c_4e^{12t}-16c_3-21c_4\big) \\
\mathrm{x3}(t) \to \frac{1}{4}e^{-4t}\big(&c_1\left(-4e^{8t}+3e^{12t}+1\right)-6c_2\left(-2e^{8t}+e^{12t}+1\right)+8c_3e^{6t}-32c_3e^{8t} \\
&+12c_3e^{12t}+8c_4e^{6t}-44c_4e^{8t}+15c_4e^{12t}+16c_3+21c_4\big) \\
\mathrm{x4}(t) \to \frac{1}{4}e^{-4t}\big(&c_1\left(4e^{8t}-3e^{12t}-1\right)+6c_2\left(-2e^{8t}+e^{12t}+1\right)-4c_3e^{6t}+32c_3e^{8t}-12c_3e^{12t} \\
&-4c_4e^{6t}+44c_4e^{8t}-15c_4e^{12t}-16c_3-21c_4\big)
\end{aligned}
$$

4.36 problem problem 47

Internal problem ID [350]
Internal file name [`OUTPUT/350_Sunday_June_05_2022_01_39_21_AM_71857108/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.3, The eigenvalue method for linear systems. Page 395
Problem number: problem 47.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= 23x_1(t) - 18x_2(t) - 16x_3(t)\\
x_2'(t) &= -8x_1(t) + 6x_2(t) + 7x_3(t) + 9x_4(t)\\
x_3'(t) &= 34x_1(t) - 27x_2(t) - 26x_3(t) - 9x_4(t)\\
x_4'(t) &= -26x_1(t) + 21x_2(t) + 25x_3(t) + 12x_4(t)
\end{aligned}
$$

4.36.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$
\begin{bmatrix} x_1'(t)\\ x_2'(t)\\ x_3'(t)\\ x_4'(t) \end{bmatrix} = \begin{bmatrix} 23 & -18 & -16 & 0\\ -8 & 6 & 7 & 9\\ 34 & -27 & -26 & -9\\ -26 & 21 & 25 & 12 \end{bmatrix} \begin{bmatrix} x_1(t)\\ x_2(t)\\ x_3(t)\\ x_4(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} -2\,\mathrm{e}^{-3t} + 3\,\mathrm{e}^{3t} - \frac{8\,\mathrm{e}^{6t}}{3} + \frac{8\,\mathrm{e}^{9t}}{3} & -2\,\mathrm{e}^{9t} + 2\,\mathrm{e}^{6t} - 2\,\mathrm{e}^{3t} + 2\,\mathrm{e}^{-3t} & -\frac{7\,\mathrm{e}^{9t}}{3} + \frac{10\,\mathrm{e}^{6t}}{3} - 3\,\mathrm{e}^{3t} + 2\,\mathrm{e}^{-3t} & -\mathrm{e}^{9t} \\ 6\,\mathrm{e}^{3t} - 2\,\mathrm{e}^{-3t} - \frac{4\,\mathrm{e}^{6t}}{3} - \frac{8\,\mathrm{e}^{9t}}{3} & 2\,\mathrm{e}^{-3t} + 2\,\mathrm{e}^{9t} + \mathrm{e}^{6t} - 4\,\mathrm{e}^{3t} & \frac{7\,\mathrm{e}^{9t}}{3} + \frac{5\,\mathrm{e}^{6t}}{3} - 6\,\mathrm{e}^{3t} + 2\,\mathrm{e}^{-3t} & \mathrm{e}^{9t} \\ -3\,\mathrm{e}^{3t} - \mathrm{e}^{-3t} - \frac{4\,\mathrm{e}^{6t}}{3} + \frac{16\,\mathrm{e}^{9t}}{3} & -4\,\mathrm{e}^{9t} + \mathrm{e}^{6t} + 2\,\mathrm{e}^{3t} + \mathrm{e}^{-3t} & \mathrm{e}^{-3t} - \frac{14\,\mathrm{e}^{9t}}{3} + \frac{5\,\mathrm{e}^{6t}}{3} + 3\,\mathrm{e}^{3t} & -2\,\mathrm{e}^{9t} \\ 3\,\mathrm{e}^{3t} + \mathrm{e}^{-3t} - \frac{4\,\mathrm{e}^{6t}}{3} - \frac{8\,\mathrm{e}^{9t}}{3} & 2\,\mathrm{e}^{9t} + \mathrm{e}^{6t} - 2\,\mathrm{e}^{3t} - \mathrm{e}^{-3t} & \frac{7\,\mathrm{e}^{9t}}{3} + \frac{5\,\mathrm{e}^{6t}}{3} - 3\,\mathrm{e}^{3t} - \mathrm{e}^{-3t} & \mathrm{e}^{9t} \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\vec{x}_h(t) = e^{At}\vec{c}
$$

$$
= \begin{bmatrix} -2\,\mathrm{e}^{-3t} + 3\,\mathrm{e}^{3t} - \frac{8\,\mathrm{e}^{6t}}{3} + \frac{8\,\mathrm{e}^{9t}}{3} & -2\,\mathrm{e}^{9t} + 2\,\mathrm{e}^{6t} - 2\,\mathrm{e}^{3t} + 2\,\mathrm{e}^{-3t} & -\frac{7\,\mathrm{e}^{9t}}{3} + \frac{10\,\mathrm{e}^{6t}}{3} - 3\,\mathrm{e}^{3t} + 2\,\mathrm{e}^{-3t} & -\mathrm{e}^{9t} \\ 6\,\mathrm{e}^{3t} - 2\,\mathrm{e}^{-3t} - \frac{4\,\mathrm{e}^{6t}}{3} - \frac{8\,\mathrm{e}^{9t}}{3} & 2\,\mathrm{e}^{-3t} + 2\,\mathrm{e}^{9t} + \mathrm{e}^{6t} - 4\,\mathrm{e}^{3t} & \frac{7\,\mathrm{e}^{9t}}{3} + \frac{5\,\mathrm{e}^{6t}}{3} - 6\,\mathrm{e}^{3t} + 2\,\mathrm{e}^{-3t} & \mathrm{e}^{9t} \\ -3\,\mathrm{e}^{3t} - \mathrm{e}^{-3t} - \frac{4\,\mathrm{e}^{6t}}{3} + \frac{16\,\mathrm{e}^{9t}}{3} & -4\,\mathrm{e}^{9t} + \mathrm{e}^{6t} + 2\,\mathrm{e}^{3t} + \mathrm{e}^{-3t} & \mathrm{e}^{-3t} - \frac{14\,\mathrm{e}^{9t}}{3} + \frac{5\,\mathrm{e}^{6t}}{3} + 3\,\mathrm{e}^{3t} & -2 \\ 3\,\mathrm{e}^{3t} + \mathrm{e}^{-3t} - \frac{4\,\mathrm{e}^{6t}}{3} - \frac{8\,\mathrm{e}^{9t}}{3} & 2\,\mathrm{e}^{9t} + \mathrm{e}^{6t} - 2\,\mathrm{e}^{3t} - \mathrm{e}^{-3t} & \frac{7\,\mathrm{e}^{9t}}{3} + \frac{5\,\mathrm{e}^{6t}}{3} - 3\,\mathrm{e}^{3t} - \mathrm{e}^{-3t} & \mathrm{e}^{9} \end{bmatrix}
$$

$$
= \begin{bmatrix} \left(-2\,\mathrm{e}^{-3t} + 3\,\mathrm{e}^{3t} - \frac{8\,\mathrm{e}^{6t}}{3} + \frac{8\,\mathrm{e}^{9t}}{3}\right) c_1 + \left(-2\,\mathrm{e}^{9t} + 2\,\mathrm{e}^{6t} - 2\,\mathrm{e}^{3t} + 2\,\mathrm{e}^{-3t}\right) c_2 + \left(-\frac{7\,\mathrm{e}^{9t}}{3} + \frac{10\,\mathrm{e}^{6t}}{3} - 3\,\mathrm{e}^{3t} \right. \\ \left(6\,\mathrm{e}^{3t} - 2\,\mathrm{e}^{-3t} - \frac{4\,\mathrm{e}^{6t}}{3} - \frac{8\,\mathrm{e}^{9t}}{3}\right) c_1 + \left(2\,\mathrm{e}^{-3t} + 2\,\mathrm{e}^{9t} + \mathrm{e}^{6t} - 4\,\mathrm{e}^{3t}\right) c_2 + \left(\frac{7\,\mathrm{e}^{9t}}{3} + \frac{5\,\mathrm{e}^{6t}}{3} - 6\,\mathrm{e}^{3t} + 2 \right. \\ \left(-3\,\mathrm{e}^{3t} - \mathrm{e}^{-3t} - \frac{4\,\mathrm{e}^{6t}}{3} + \frac{16\,\mathrm{e}^{9t}}{3}\right) c_1 + \left(-4\,\mathrm{e}^{9t} + \mathrm{e}^{6t} + 2\,\mathrm{e}^{3t} + \mathrm{e}^{-3t}\right) c_2 + \left(\mathrm{e}^{-3t} - \frac{14\,\mathrm{e}^{9t}}{3} + \frac{5\,\mathrm{e}^{6t}}{3} + 3 \right. \\ \left(3\,\mathrm{e}^{3t} + \mathrm{e}^{-3t} - \frac{4\,\mathrm{e}^{6t}}{3} - \frac{8\,\mathrm{e}^{9t}}{3}\right) c_1 + \left(2\,\mathrm{e}^{9t} + \mathrm{e}^{6t} - 2\,\mathrm{e}^{3t} - \mathrm{e}^{-3t}\right) c_2 + \left(\frac{7\,\mathrm{e}^{9t}}{3} + \frac{5\,\mathrm{e}^{6t}}{3} - 3\,\mathrm{e}^{3t} - \mathrm{e} \right. \end{bmatrix}
$$

$$
= \begin{bmatrix} (3c_1 - 2c_2 - 3c_3 - c_4)\,\mathrm{e}^{3t} + \frac{2(-4c_1+3c_2+5c_3+3c_4)\mathrm{e}^{6t}}{3} + \frac{(8c_1-6c_2-7c_3-3c_4)\mathrm{e}^{9t}}{3} - 2(c_1 - c_2 - c_3)\,\mathrm{e}^{-3t} \\ 2(3c_1 - 2c_2 - 3c_3 - c_4)\,\mathrm{e}^{3t} + \frac{(-4c_1+3c_2+5c_3+3c_4)\mathrm{e}^{6t}}{3} + \frac{(-8c_1+6c_2+7c_3+3c_4)\mathrm{e}^{9t}}{3} - 2(c_1 - c_2 - c_3)\,\mathrm{e}^{-3t} \\ (-3c_1 + 2c_2 + 3c_3 + c_4)\,\mathrm{e}^{3t} + \frac{(-4c_1+3c_2+5c_3+3c_4)\mathrm{e}^{6t}}{3} + \frac{2(8c_1-6c_2-7c_3-3c_4)\mathrm{e}^{9t}}{3} - (c_1 - c_2 - c_3)\,\mathrm{e}^{-3t} \\ (3c_1 - 2c_2 - 3c_3 - c_4)\,\mathrm{e}^{3t} + \frac{(-4c_1+3c_2+5c_3+3c_4)\mathrm{e}^{6t}}{3} + \frac{(-8c_1+6c_2+7c_3+3c_4)\mathrm{e}^{9t}}{3} + (c_1 - c_2 - c_3)\,\mathrm{e}^{-3t} \end{bmatrix}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

4.36.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \\ x_4'(t) \end{bmatrix} = \begin{bmatrix} 23 & -18 & -16 & 0 \\ -8 & 6 & 7 & 9 \\ 34 & -27 & -26 & -9 \\ -26 & 21 & 25 & 12 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 23 & -18 & -16 & 0 \\ -8 & 6 & 7 & 9 \\ 34 & -27 & -26 & -9 \\ -26 & 21 & 25 & 12 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 23-\lambda & -18 & -16 & 0 \\ -8 & 6-\lambda & 7 & 9 \\ 34 & -27 & -26-\lambda & -9 \\ -26 & 21 & 25 & 12-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^4 - 15\lambda^3 + 45\lambda^2 + 135\lambda - 486 = 0$$

The roots of the above are the eigenvalues.

$$\begin{aligned} \lambda_1 &= 3 \\ \lambda_2 &= -3 \\ \lambda_3 &= 9 \\ \lambda_4 &= 6 \end{aligned}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-3	1	real eigenvalue
3	1	real eigenvalue
6	1	real eigenvalue
9	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 23 & -18 & -16 & 0 \\ -8 & 6 & 7 & 9 \\ 34 & -27 & -26 & -9 \\ -26 & 21 & 25 & 12 \end{bmatrix} - (-3)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 26 & -18 & -16 & 0 \\ -8 & 9 & 7 & 9 \\ 34 & -27 & -23 & -9 \\ -26 & 21 & 25 & 15 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} 26 & -18 & -16 & 0 & 0 \\ -8 & 9 & 7 & 9 & 0 \\ 34 & -27 & -23 & -9 & 0 \\ -26 & 21 & 25 & 15 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{4R_1}{13} \Longrightarrow \left[\begin{array}{cccc|c} 26 & -18 & -16 & 0 & 0 \\ 0 & \frac{45}{13} & \frac{27}{13} & 9 & 0 \\ 34 & -27 & -23 & -9 & 0 \\ -26 & 21 & 25 & 15 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{17R_1}{13} \implies \left[\begin{array}{cccc|c} 26 & -18 & -16 & 0 & 0 \\ 0 & \frac{45}{13} & \frac{27}{13} & 9 & 0 \\ 0 & -\frac{45}{13} & -\frac{27}{13} & -9 & 0 \\ -26 & 21 & 25 & 15 & 0 \end{array}\right]$$

$$R_4 = R_4 + R_1 \implies \left[\begin{array}{cccc|c} 26 & -18 & -16 & 0 & 0 \\ 0 & \frac{45}{13} & \frac{27}{13} & 9 & 0 \\ 0 & -\frac{45}{13} & -\frac{27}{13} & -9 & 0 \\ 0 & 3 & 9 & 15 & 0 \end{array}\right]$$

$$R_3 = R_3 + R_2 \implies \left[\begin{array}{cccc|c} 26 & -18 & -16 & 0 & 0 \\ 0 & \frac{45}{13} & \frac{27}{13} & 9 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 9 & 15 & 0 \end{array}\right]$$

$$R_4 = R_4 - \frac{13R_2}{15} \implies \left[\begin{array}{cccc|c} 26 & -18 & -16 & 0 & 0 \\ 0 & \frac{45}{13} & \frac{27}{13} & 9 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{36}{5} & \frac{36}{5} & 0 \end{array}\right]$$

Since the current pivot $A(3,3)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 3 and row 4 gives

$$\left[\begin{array}{cccc|c} 26 & -18 & -16 & 0 & 0 \\ 0 & \frac{45}{13} & \frac{27}{13} & 9 & 0 \\ 0 & 0 & \frac{36}{5} & \frac{36}{5} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 26 & -18 & -16 & 0 \\ 0 & \frac{45}{13} & \frac{27}{13} & 9 \\ 0 & 0 & \frac{36}{5} & \frac{36}{5} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_4\}$ and the leading variables are $\{v_1, v_2, v_3\}$. Let $v_4 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -2t, v_2 = -2t, v_3 = -t\}$

Hence the solution is

$$\begin{bmatrix} -2t \\ -2t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} -2t \\ -2t \\ -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -2t \\ -2t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} -2 \\ -2 \\ -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -2t \\ -2t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ -1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 23 & -18 & -16 & 0 \\ -8 & 6 & 7 & 9 \\ 34 & -27 & -26 & -9 \\ -26 & 21 & 25 & 12 \end{bmatrix} - (3) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 20 & -18 & -16 & 0 \\ -8 & 3 & 7 & 9 \\ 34 & -27 & -29 & -9 \\ -26 & 21 & 25 & 9 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} 20 & -18 & -16 & 0 & 0 \\ -8 & 3 & 7 & 9 & 0 \\ 34 & -27 & -29 & -9 & 0 \\ -26 & 21 & 25 & 9 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{2R_1}{5} \implies \left[\begin{array}{cccc|c} 20 & -18 & -16 & 0 & 0 \\ 0 & -\frac{21}{5} & \frac{3}{5} & 9 & 0 \\ 34 & -27 & -29 & -9 & 0 \\ -26 & 21 & 25 & 9 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{17R_1}{10} \implies \left[\begin{array}{cccc|c} 20 & -18 & -16 & 0 & 0 \\ 0 & -\frac{21}{5} & \frac{3}{5} & 9 & 0 \\ 0 & \frac{18}{5} & -\frac{9}{5} & -9 & 0 \\ -26 & 21 & 25 & 9 & 0 \end{array}\right]$$

$$R_4 = R_4 + \frac{13R_1}{10} \implies \left[\begin{array}{cccc|c} 20 & -18 & -16 & 0 & 0 \\ 0 & -\frac{21}{5} & \frac{3}{5} & 9 & 0 \\ 0 & \frac{18}{5} & -\frac{9}{5} & -9 & 0 \\ 0 & -\frac{12}{5} & \frac{21}{5} & 9 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{6R_2}{7} \implies \left[\begin{array}{cccc|c} 20 & -18 & -16 & 0 & 0 \\ 0 & -\frac{21}{5} & \frac{3}{5} & 9 & 0 \\ 0 & 0 & -\frac{9}{7} & -\frac{9}{7} & 0 \\ 0 & -\frac{12}{5} & \frac{21}{5} & 9 & 0 \end{array}\right]$$

$$R_4 = R_4 - \frac{4R_2}{7} \implies \left[\begin{array}{cccc|c} 20 & -18 & -16 & 0 & 0 \\ 0 & -\frac{21}{5} & \frac{3}{5} & 9 & 0 \\ 0 & 0 & -\frac{9}{7} & -\frac{9}{7} & 0 \\ 0 & 0 & \frac{27}{7} & \frac{27}{7} & 0 \end{array}\right]$$

$$R_4 = R_4 + 3R_3 \Longrightarrow \left[\begin{array}{cccc|c} 20 & -18 & -16 & 0 & 0 \\ 0 & -\frac{21}{5} & \frac{3}{5} & 9 & 0 \\ 0 & 0 & -\frac{9}{7} & -\frac{9}{7} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 20 & -18 & -16 & 0 \\ 0 & -\frac{21}{5} & \frac{3}{5} & 9 \\ 0 & 0 & -\frac{9}{7} & -\frac{9}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_4\}$ and the leading variables are $\{v_1, v_2, v_3\}$. Let $v_4 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = 2t, v_3 = -t\}$

Hence the solution is

$$\begin{bmatrix} t \\ 2t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} t \\ 2t \\ -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ 2t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ 2t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = 6$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 23 & -18 & -16 & 0 \\ -8 & 6 & 7 & 9 \\ 34 & -27 & -26 & -9 \\ -26 & 21 & 25 & 12 \end{bmatrix} - (6)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 17 & -18 & -16 & 0 \\ -8 & 0 & 7 & 9 \\ 34 & -27 & -32 & -9 \\ -26 & 21 & 25 & 6 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} 17 & -18 & -16 & 0 & 0 \\ -8 & 0 & 7 & 9 & 0 \\ 34 & -27 & -32 & -9 & 0 \\ -26 & 21 & 25 & 6 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{8R_1}{17} \implies \left[\begin{array}{cccc|c} 17 & -18 & -16 & 0 & 0 \\ 0 & -\frac{144}{17} & -\frac{9}{17} & 9 & 0 \\ 34 & -27 & -32 & -9 & 0 \\ -26 & 21 & 25 & 6 & 0 \end{array}\right]$$

$$R_3 = R_3 - 2R_1 \implies \left[\begin{array}{cccc|c} 17 & -18 & -16 & 0 & 0 \\ 0 & -\frac{144}{17} & -\frac{9}{17} & 9 & 0 \\ 0 & 9 & 0 & -9 & 0 \\ -26 & 21 & 25 & 6 & 0 \end{array}\right]$$

$$R_4 = R_4 + \frac{26R_1}{17} \implies \left[\begin{array}{cccc|c} 17 & -18 & -16 & 0 & 0 \\ 0 & -\frac{144}{17} & -\frac{9}{17} & 9 & 0 \\ 0 & 9 & 0 & -9 & 0 \\ 0 & -\frac{111}{17} & \frac{9}{17} & 6 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{17R_2}{16} \Longrightarrow \left[\begin{array}{cccc|c} 17 & -18 & -16 & 0 & 0 \\ 0 & -\frac{144}{17} & -\frac{9}{17} & 9 & 0 \\ 0 & 0 & -\frac{9}{16} & \frac{9}{16} & 0 \\ 0 & -\frac{111}{17} & \frac{9}{17} & 6 & 0 \end{array}\right]$$

$$R_4 = R_4 - \frac{37R_2}{48} \Longrightarrow \left[\begin{array}{cccc|c} 17 & -18 & -16 & 0 & 0 \\ 0 & -\frac{144}{17} & -\frac{9}{17} & 9 & 0 \\ 0 & 0 & -\frac{9}{16} & \frac{9}{16} & 0 \\ 0 & 0 & \frac{15}{16} & -\frac{15}{16} & 0 \end{array}\right]$$

$$R_4 = R_4 + \frac{5R_3}{3} \Longrightarrow \left[\begin{array}{cccc|c} 17 & -18 & -16 & 0 & 0 \\ 0 & -\frac{144}{17} & -\frac{9}{17} & 9 & 0 \\ 0 & 0 & -\frac{9}{16} & \frac{9}{16} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 17 & -18 & -16 & 0 \\ 0 & -\frac{144}{17} & -\frac{9}{17} & 9 \\ 0 & 0 & -\frac{9}{16} & \frac{9}{16} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_4\}$ and the leading variables are $\{v_1, v_2, v_3\}$. Let $v_4 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2t, v_2 = t, v_3 = t\}$

Hence the solution is

$$\begin{bmatrix} 2t \\ t \\ t \\ t \end{bmatrix} = \begin{bmatrix} 2t \\ t \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this

eigenvalue. The above can be written as

$$\begin{bmatrix} 2t \\ t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 2t \\ t \\ t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_4 = 9$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 23 & -18 & -16 & 0 \\ -8 & 6 & 7 & 9 \\ 34 & -27 & -26 & -9 \\ -26 & 21 & 25 & 12 \end{bmatrix} - (9) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 14 & -18 & -16 & 0 \\ -8 & -3 & 7 & 9 \\ 34 & -27 & -35 & -9 \\ -26 & 21 & 25 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} 14 & -18 & -16 & 0 & 0 \\ -8 & -3 & 7 & 9 & 0 \\ 34 & -27 & -35 & -9 & 0 \\ -26 & 21 & 25 & 3 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{4R_1}{7} \implies \left[\begin{array}{cccc|c} 14 & -18 & -16 & 0 & 0 \\ 0 & -\frac{93}{7} & -\frac{15}{7} & 9 & 0 \\ 34 & -27 & -35 & -9 & 0 \\ -26 & 21 & 25 & 3 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{17R_1}{7} \implies \left[\begin{array}{cccc|c} 14 & -18 & -16 & 0 & 0 \\ 0 & -\frac{93}{7} & -\frac{15}{7} & 9 & 0 \\ 0 & \frac{117}{7} & \frac{27}{7} & -9 & 0 \\ -26 & 21 & 25 & 3 & 0 \end{array}\right]$$

$$R_4 = R_4 + \frac{13R_1}{7} \implies \left[\begin{array}{cccc|c} 14 & -18 & -16 & 0 & 0 \\ 0 & -\frac{93}{7} & -\frac{15}{7} & 9 & 0 \\ 0 & \frac{117}{7} & \frac{27}{7} & -9 & 0 \\ 0 & -\frac{87}{7} & -\frac{33}{7} & 3 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{39R_2}{31} \implies \left[\begin{array}{cccc|c} 14 & -18 & -16 & 0 & 0 \\ 0 & -\frac{93}{7} & -\frac{15}{7} & 9 & 0 \\ 0 & 0 & \frac{36}{31} & \frac{72}{31} & 0 \\ 0 & -\frac{87}{7} & -\frac{33}{7} & 3 & 0 \end{array}\right]$$

$$R_4 = R_4 - \frac{29R_2}{31} \implies \left[\begin{array}{cccc|c} 14 & -18 & -16 & 0 & 0 \\ 0 & -\frac{93}{7} & -\frac{15}{7} & 9 & 0 \\ 0 & 0 & \frac{36}{31} & \frac{72}{31} & 0 \\ 0 & 0 & -\frac{84}{31} & -\frac{168}{31} & 0 \end{array}\right]$$

$$R_4 = R_4 + \frac{7R_3}{3} \implies \left[\begin{array}{cccc|c} 14 & -18 & -16 & 0 & 0 \\ 0 & -\frac{93}{7} & -\frac{15}{7} & 9 & 0 \\ 0 & 0 & \frac{36}{31} & \frac{72}{31} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 14 & -18 & -16 & 0 \\ 0 & -\frac{93}{7} & -\frac{15}{7} & 9 \\ 0 & 0 & \frac{36}{31} & \frac{72}{31} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_4\}$ and the leading variables are $\{v_1, v_2, v_3\}$. Let $v_4 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t, v_2 = t, v_3 = -2t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \\ -2t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \\ -2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \\ -2t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ -2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \\ -2t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -2 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
3	1	1	No	$\begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}$
-3	1	1	No	$\begin{bmatrix} -2 \\ -2 \\ -1 \\ 1 \end{bmatrix}$
9	1	1	No	$\begin{bmatrix} -1 \\ 1 \\ -2 \\ 1 \end{bmatrix}$
6	1	1	No	$\begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 3 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{3t} \\ &= \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix} e^{3t}\end{aligned}$$

Since eigenvalue -3 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_2(t) = \vec{v}_2 e^{-3t}$$
$$= \begin{bmatrix} -2 \\ -2 \\ -1 \\ 1 \end{bmatrix} e^{-3t}$$

Since eigenvalue 9 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_3(t) = \vec{v}_3 e^{9t}$$
$$= \begin{bmatrix} -1 \\ 1 \\ -2 \\ 1 \end{bmatrix} e^{9t}$$

Since eigenvalue 6 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_4(t) = \vec{v}_4 e^{6t}$$
$$= \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix} e^{6t}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t) + c_4\vec{x}_4(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = c_1 \begin{bmatrix} e^{3t} \\ 2\,e^{3t} \\ -e^{3t} \\ e^{3t} \end{bmatrix} + c_2 \begin{bmatrix} -2\,e^{-3t} \\ -2\,e^{-3t} \\ -e^{-3t} \\ e^{-3t} \end{bmatrix} + c_3 \begin{bmatrix} -e^{9t} \\ e^{9t} \\ -2\,e^{9t} \\ e^{9t} \end{bmatrix} + c_4 \begin{bmatrix} 2\,e^{6t} \\ e^{6t} \\ e^{6t} \\ e^{6t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{3t} - 2c_2 e^{-3t} - c_3 e^{9t} + 2c_4 e^{6t} \\ 2c_1 e^{3t} - 2c_2 e^{-3t} + c_3 e^{9t} + c_4 e^{6t} \\ -c_1 e^{3t} - c_2 e^{-3t} - 2c_3 e^{9t} + c_4 e^{6t} \\ c_1 e^{3t} + c_2 e^{-3t} + c_3 e^{9t} + c_4 e^{6t} \end{bmatrix}$$

4.36.3 Maple step by step solution

Let's solve

$[x_1'(t) = 23x_1(t) - 18x_2(t) - 16x_3(t), x_2'(t) = -8x_1(t) + 6x_2(t) + 7x_3(t) + 9x_4(t), x_3'(t) = 34x_1(t)$

- Define vector

$$\underline{\vec{x}}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{\vec{x}}'(t) = \begin{bmatrix} 23 & -18 & -16 & 0 \\ -8 & 6 & 7 & 9 \\ 34 & -27 & -26 & -9 \\ -26 & 21 & 25 & 12 \end{bmatrix} \cdot \underline{\vec{x}}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{\vec{x}}'(t) = \begin{bmatrix} 23 & -18 & -16 & 0 \\ -8 & 6 & 7 & 9 \\ 34 & -27 & -26 & -9 \\ -26 & 21 & 25 & 12 \end{bmatrix} \cdot \underline{\vec{x}}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 23 & -18 & -16 & 0 \\ -8 & 6 & 7 & 9 \\ 34 & -27 & -26 & -9 \\ -26 & 21 & 25 & 12 \end{bmatrix}$$

- Rewrite the system as

  $\underline{\vec{x}}'(t) = A \cdot \underline{\vec{x}}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-3, \begin{bmatrix} -2 \\ -2 \\ -1 \\ 1 \end{bmatrix}\right], \left[3, \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}\right], \left[6, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[9, \begin{bmatrix} -1 \\ 1 \\ -2 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-3, \begin{bmatrix} -2 \\ -2 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_1 = \mathrm{e}^{-3t} \cdot \begin{bmatrix} -2 \\ -2 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[3, \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_{2} = e^{3t} \cdot \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix} 6, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_{3} = e^{6t} \cdot \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix} 9, \begin{bmatrix} -1 \\ 1 \\ -2 \\ 1 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_{4} = e^{9t} \cdot \begin{bmatrix} -1 \\ 1 \\ -2 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\underline{\vec{x}} = c_1 \underline{\vec{x}}_{1} + c_2 \underline{\vec{x}}_{2} + c_3 \underline{\vec{x}}_{3} + c_4 \underline{\vec{x}}_{4}$$

- Substitute solutions into the general solution

$$\underline{\overrightarrow{x}} = c_1 e^{-3t} \cdot \begin{bmatrix} -2 \\ -2 \\ -1 \\ 1 \end{bmatrix} + c_2 e^{3t} \cdot \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix} + c_3 e^{6t} \cdot \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix} + c_4 e^{9t} \cdot \begin{bmatrix} -1 \\ 1 \\ -2 \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = \begin{bmatrix} -2c_1 e^{-3t} + c_2 e^{3t} + 2c_3 e^{6t} - c_4 e^{9t} \\ -2c_1 e^{-3t} + 2c_2 e^{3t} + c_3 e^{6t} + c_4 e^{9t} \\ -c_1 e^{-3t} - c_2 e^{3t} + c_3 e^{6t} - 2c_4 e^{9t} \\ c_1 e^{-3t} + c_2 e^{3t} + c_3 e^{6t} + c_4 e^{9t} \end{bmatrix}$$

- Solution to the system of ODEs

$$\{x_1(t) = -2c_1 e^{-3t} + c_2 e^{3t} + 2c_3 e^{6t} - c_4 e^{9t}, x_2(t) = -2c_1 e^{-3t} + 2c_2 e^{3t} + c_3 e^{6t} + c_4 e^{9t}, x_3(t) = -c_1 e^{-}$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 124

```
dsolve([diff(x__1(t),t)=23*x__1(t)-18*x__2(t)-16*x__3(t)+0*x__4(t),diff(x__2(t),t)=-8*x__1(t
```

$$x_1(t) = c_1 e^{3t} + c_2 e^{6t} + c_3 e^{9t} + c_4 e^{-3t}$$
$$x_2(t) = 2c_1 e^{3t} + \frac{c_2 e^{6t}}{2} - c_3 e^{9t} + c_4 e^{-3t}$$
$$x_3(t) = -c_1 e^{3t} + \frac{c_2 e^{6t}}{2} + 2c_3 e^{9t} + \frac{c_4 e^{-3t}}{2}$$
$$x_4(t) = c_1 e^{3t} + \frac{c_2 e^{6t}}{2} - c_3 e^{9t} - \frac{c_4 e^{-3t}}{2}$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 469

```
DSolve[{x1'[t]==23*x1[t]-18*x2[t]-16*x3[t]+0*x4[t],x2'[t]==-8*x1[t]+6*x2[t]+7*x3[t]+9*x4[t],
```

$$\begin{aligned}
\text{x1}(t) \to{} & \frac{1}{3}e^{-3t}\big(c_1\big(9e^{6t}-8e^{9t}+8e^{12t}-6\big) \\
& -\big(e^{3t}-1\big)\big(6c_2\big(e^{3t}+e^{9t}+1\big)+c_3\big(6e^{3t}-3e^{6t}+7e^{9t}+6\big)+3c_4e^{6t}\big(e^{3t}-1\big)\big)\big) \\
\text{x2}(t) \to{} & \frac{1}{3}e^{-3t}\big(-2c_1\big(-9e^{6t}+2e^{9t}+4e^{12t}+3\big)+3c_2\big(-4e^{6t}+e^{9t}+2e^{12t}+2\big) \\
& +\big(e^{3t}-1\big)\big(c_3\big(-6e^{3t}+12e^{6t}+7e^{9t}-6\big)+3c_4e^{6t}\big(e^{3t}+2\big)\big)\big) \\
\text{x3}(t) \to{} & \frac{1}{3}e^{-3t}\big(c_1\big(-9e^{6t}-4e^{9t}+16e^{12t}-3\big)+3c_2\big(2e^{6t}+e^{9t}-4e^{12t}+1\big)+9c_3e^{6t} \\
& +5c_3e^{9t}-14c_3e^{12t}+3c_4e^{6t}+3c_4e^{9t}-6c_4e^{12t}+3c_3\big) \\
\text{x4}(t) \to{} & \frac{1}{3}e^{-3t}\big(c_1\big(9e^{6t}-4e^{9t}-8e^{12t}+3\big)+3c_2\big(-2e^{6t}+e^{9t}+2e^{12t}-1\big)-9c_3e^{6t} \\
& +5c_3e^{9t}+7c_3e^{12t}-3c_4e^{6t}+3c_4e^{9t}+3c_4e^{12t}-3c_3\big)
\end{aligned}$$

4.37 problem problem 48

Internal problem ID [351]
Internal file name [`OUTPUT/351_Sunday_June_05_2022_01_39_23_AM_18060656/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.3, The eigenvalue method for linear systems. Page 395
Problem number: problem 48.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= 47x_1(t) - 8x_2(t) + 5x_3(t) - 5x_4(t) \\
x_2'(t) &= -10x_1(t) + 32x_2(t) + 18x_3(t) - 2x_4(t) \\
x_3'(t) &= 139x_1(t) - 40x_2(t) - 167x_3(t) - 121x_4(t) \\
x_4'(t) &= -232x_1(t) + 64x_2(t) + 360x_3(t) + 248x_4(t)
\end{aligned}
$$

4.37.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \\ x_4'(t) \end{bmatrix} = \begin{bmatrix} 47 & -8 & 5 & -5 \\ -10 & 32 & 18 & -2 \\ 139 & -40 & -167 & -121 \\ -232 & 64 & 360 & 248 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{33\,\mathrm{e}^{16t}}{16} - \frac{19\,\mathrm{e}^{32t}}{8} - \frac{3\,\mathrm{e}^{48t}}{8} + \frac{27\,\mathrm{e}^{64t}}{16} & -\frac{\mathrm{e}^{64t}}{2} + \mathrm{e}^{32t} - \frac{\mathrm{e}^{16t}}{2} & -\frac{39\,\mathrm{e}^{64t}}{16} + \frac{15\,\mathrm{e}^{48t}}{8} + \frac{31\,\mathrm{e}^{32t}}{8} - \frac{53\,\mathrm{e}^{16t}}{16} & -\frac{25\,\mathrm{e}^{64t}}{16} + \cdots \\ -\frac{95\,\mathrm{e}^{32t}}{16} + \frac{33\,\mathrm{e}^{16t}}{8} + \frac{\mathrm{e}^{48t}}{8} + \frac{27\,\mathrm{e}^{64t}}{16} & -\mathrm{e}^{16t} - \frac{\mathrm{e}^{64t}}{2} + \frac{5\,\mathrm{e}^{32t}}{2} & -\frac{39\,\mathrm{e}^{64t}}{16} - \frac{5\,\mathrm{e}^{48t}}{8} + \frac{155\,\mathrm{e}^{32t}}{16} - \frac{53\,\mathrm{e}^{16t}}{8} & -\frac{25\,\mathrm{e}^{64t}}{16} - \cdots \\ -\frac{19\,\mathrm{e}^{32t}}{16} - \frac{33\,\mathrm{e}^{16t}}{16} - \frac{\mathrm{e}^{48t}}{8} + \frac{27\,\mathrm{e}^{64t}}{8} & -\mathrm{e}^{64t} + \frac{\mathrm{e}^{32t}}{2} + \frac{\mathrm{e}^{16t}}{2} & \frac{53\,\mathrm{e}^{16t}}{16} - \frac{39\,\mathrm{e}^{64t}}{8} + \frac{5\,\mathrm{e}^{48t}}{8} + \frac{31\,\mathrm{e}^{32t}}{16} & -\frac{25\,\mathrm{e}^{64t}}{8} + \cdots \\ \frac{19\,\mathrm{e}^{32t}}{16} + \frac{33\,\mathrm{e}^{16t}}{8} - \frac{\mathrm{e}^{48t}}{4} - \frac{81\,\mathrm{e}^{64t}}{16} & \frac{3\,\mathrm{e}^{64t}}{2} - \frac{\mathrm{e}^{32t}}{2} - \mathrm{e}^{16t} & \frac{117\,\mathrm{e}^{64t}}{16} + \frac{5\,\mathrm{e}^{48t}}{4} - \frac{31\,\mathrm{e}^{32t}}{16} - \frac{53\,\mathrm{e}^{16t}}{8} & -\frac{27\,\mathrm{e}^{16t}}{8} + \cdots \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} \frac{33\,\mathrm{e}^{16t}}{16} - \frac{19\,\mathrm{e}^{32t}}{8} - \frac{3\,\mathrm{e}^{48t}}{8} + \frac{27\,\mathrm{e}^{64t}}{16} & -\frac{\mathrm{e}^{64t}}{2} + \mathrm{e}^{32t} - \frac{\mathrm{e}^{16t}}{2} & -\frac{39\,\mathrm{e}^{64t}}{16} + \frac{15\,\mathrm{e}^{48t}}{8} + \frac{31\,\mathrm{e}^{32t}}{8} - \frac{53\,\mathrm{e}^{16t}}{16} & -\frac{25\,\mathrm{e}^{64t}}{16} \cdots \\ -\frac{95\,\mathrm{e}^{32t}}{16} + \frac{33\,\mathrm{e}^{16t}}{8} + \frac{\mathrm{e}^{48t}}{8} + \frac{27\,\mathrm{e}^{64t}}{16} & -\mathrm{e}^{16t} - \frac{\mathrm{e}^{64t}}{2} + \frac{5\,\mathrm{e}^{32t}}{2} & -\frac{39\,\mathrm{e}^{64t}}{16} - \frac{5\,\mathrm{e}^{48t}}{8} + \frac{155\,\mathrm{e}^{32t}}{16} - \frac{53\,\mathrm{e}^{16t}}{8} & -\frac{25\,\mathrm{e}^{64t}}{16} \cdots \\ -\frac{19\,\mathrm{e}^{32t}}{16} - \frac{33\,\mathrm{e}^{16t}}{16} - \frac{\mathrm{e}^{48t}}{8} + \frac{27\,\mathrm{e}^{64t}}{8} & -\mathrm{e}^{64t} + \frac{\mathrm{e}^{32t}}{2} + \frac{\mathrm{e}^{16t}}{2} & \frac{53\,\mathrm{e}^{16t}}{16} - \frac{39\,\mathrm{e}^{64t}}{8} + \frac{5\,\mathrm{e}^{48t}}{8} + \frac{31\,\mathrm{e}^{32t}}{16} & -\frac{25\,\mathrm{e}^{64t}}{8} \cdots \\ \frac{19\,\mathrm{e}^{32t}}{16} + \frac{33\,\mathrm{e}^{16t}}{8} - \frac{\mathrm{e}^{48t}}{4} - \frac{81\,\mathrm{e}^{64t}}{16} & \frac{3\,\mathrm{e}^{64t}}{2} - \frac{\mathrm{e}^{32t}}{2} - \mathrm{e}^{16t} & \frac{117\,\mathrm{e}^{64t}}{16} + \frac{5\,\mathrm{e}^{48t}}{4} - \frac{31\,\mathrm{e}^{32t}}{16} - \frac{53\,\mathrm{e}^{16t}}{8} & -\frac{27\,\mathrm{e}^{16t}}{8} \cdots \end{bmatrix} \\ &= \begin{bmatrix} \left(\frac{33\,\mathrm{e}^{16t}}{16} - \frac{19\,\mathrm{e}^{32t}}{8} - \frac{3\,\mathrm{e}^{48t}}{8} + \frac{27\,\mathrm{e}^{64t}}{16}\right)c_1 + \left(-\frac{\mathrm{e}^{64t}}{2} + \mathrm{e}^{32t} - \frac{\mathrm{e}^{16t}}{2}\right)c_2 + \left(-\frac{39\,\mathrm{e}^{64t}}{16} + \frac{15\,\mathrm{e}^{48t}}{8} + \frac{31\,\mathrm{e}^{32t}}{8} - \cdots\right. \\ \left(-\frac{95\,\mathrm{e}^{32t}}{16} + \frac{33\,\mathrm{e}^{16t}}{8} + \frac{\mathrm{e}^{48t}}{8} + \frac{27\,\mathrm{e}^{64t}}{16}\right)c_1 + \left(-\mathrm{e}^{16t} - \frac{\mathrm{e}^{64t}}{2} + \frac{5\,\mathrm{e}^{32t}}{2}\right)c_2 + \left(-\frac{39\,\mathrm{e}^{64t}}{16} - \frac{5\,\mathrm{e}^{48t}}{8} + \frac{155\,\mathrm{e}^{32t}}{16} - \cdots\right. \\ \left(-\frac{19\,\mathrm{e}^{32t}}{16} - \frac{33\,\mathrm{e}^{16t}}{16} - \frac{\mathrm{e}^{48t}}{8} + \frac{27\,\mathrm{e}^{64t}}{8}\right)c_1 + \left(-\mathrm{e}^{64t} + \frac{\mathrm{e}^{32t}}{2} + \frac{\mathrm{e}^{16t}}{2}\right)c_2 + \left(\frac{53\,\mathrm{e}^{16t}}{16} - \frac{39\,\mathrm{e}^{64t}}{8} + \frac{5\,\mathrm{e}^{48t}}{8} + \cdots\right. \\ \left(\frac{19\,\mathrm{e}^{32t}}{16} + \frac{33\,\mathrm{e}^{16t}}{8} - \frac{\mathrm{e}^{48t}}{4} - \frac{81\,\mathrm{e}^{64t}}{16}\right)c_1 + \left(\frac{3\,\mathrm{e}^{64t}}{2} - \frac{\mathrm{e}^{32t}}{2} - \mathrm{e}^{16t}\right)c_2 + \left(\frac{117\,\mathrm{e}^{64t}}{16} + \frac{5\,\mathrm{e}^{48t}}{4} - \frac{31\,\mathrm{e}^{32t}}{16} - \cdots\right. \end{bmatrix} \\ &= \begin{bmatrix} \frac{(33c_1-8c_2-53c_3-27c_4)\mathrm{e}^{16t}}{16} + \frac{(-19c_1+8c_2+31c_3+17c_4)\mathrm{e}^{32t}}{8} + \frac{(27c_1-8c_2-39c_3-25c_4)\mathrm{e}^{64t}}{16} - \frac{3\,\mathrm{e}^{48t}(c_1-5c_3-3c_4)}{8} \\ \frac{(33c_1-8c_2-53c_3-27c_4)\mathrm{e}^{16t}}{8} + \frac{5(-19c_1+8c_2+31c_3+17c_4)\mathrm{e}^{32t}}{16} + \frac{(27c_1-8c_2-39c_3-25c_4)\mathrm{e}^{64t}}{16} + \frac{\mathrm{e}^{48t}(c_1-5c_3-3c_4)}{8} \\ \frac{(-33c_1+8c_2+53c_3+27c_4)\mathrm{e}^{16t}}{16} + \frac{(-19c_1+8c_2+31c_3+17c_4)\mathrm{e}^{32t}}{16} + \frac{(27c_1-8c_2-39c_3-25c_4)\mathrm{e}^{64t}}{8} - \frac{\mathrm{e}^{48t}(c_1-5c_3-3c_4)}{8} \\ \frac{(33c_1-8c_2-53c_3-27c_4)\mathrm{e}^{16t}}{8} + \frac{(19c_1-8c_2-31c_3-17c_4)\mathrm{e}^{32t}}{16} + \frac{3(-27c_1+8c_2+39c_3+25c_4)\mathrm{e}^{64t}}{16} - \frac{\mathrm{e}^{48t}(c_1-5c_3-3c_4)}{4} \end{bmatrix} \end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

4.37.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \\ x_4'(t) \end{bmatrix} = \begin{bmatrix} 47 & -8 & 5 & -5 \\ -10 & 32 & 18 & -2 \\ 139 & -40 & -167 & -121 \\ -232 & 64 & 360 & 248 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 47 & -8 & 5 & -5 \\ -10 & 32 & 18 & -2 \\ 139 & -40 & -167 & -121 \\ -232 & 64 & 360 & 248 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 47-\lambda & -8 & 5 & -5 \\ -10 & 32-\lambda & 18 & -2 \\ 139 & -40 & -167-\lambda & -121 \\ -232 & 64 & 360 & 248-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^4 - 160\lambda^3 + 8960\lambda^2 - 204800\lambda + 1572864 = 0$$

The roots of the above are the eigenvalues.

$$\begin{aligned} \lambda_1 &= 48 \\ \lambda_2 &= 32 \\ \lambda_3 &= 16 \\ \lambda_4 &= 64 \end{aligned}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
16	1	real eigenvalue
32	1	real eigenvalue
48	1	real eigenvalue
64	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 16$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 47 & -8 & 5 & -5 \\ -10 & 32 & 18 & -2 \\ 139 & -40 & -167 & -121 \\ -232 & 64 & 360 & 248 \end{bmatrix} - (16)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 31 & -8 & 5 & -5 \\ -10 & 16 & 18 & -2 \\ 139 & -40 & -183 & -121 \\ -232 & 64 & 360 & 232 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} 31 & -8 & 5 & -5 & 0 \\ -10 & 16 & 18 & -2 & 0 \\ 139 & -40 & -183 & -121 & 0 \\ -232 & 64 & 360 & 232 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{10R_1}{31} \Longrightarrow \left[\begin{array}{cccc|c} 31 & -8 & 5 & -5 & 0 \\ 0 & \frac{416}{31} & \frac{608}{31} & -\frac{112}{31} & 0 \\ 139 & -40 & -183 & -121 & 0 \\ -232 & 64 & 360 & 232 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{139R_1}{31} \implies \left[\begin{array}{cccc|c} 31 & -8 & 5 & -5 & 0 \\ 0 & \frac{416}{31} & \frac{608}{31} & -\frac{112}{31} & 0 \\ 0 & -\frac{128}{31} & -\frac{6368}{31} & -\frac{3056}{31} & 0 \\ -232 & 64 & 360 & 232 & 0 \end{array}\right]$$

$$R_4 = R_4 + \frac{232R_1}{31} \implies \left[\begin{array}{cccc|c} 31 & -8 & 5 & -5 & 0 \\ 0 & \frac{416}{31} & \frac{608}{31} & -\frac{112}{31} & 0 \\ 0 & -\frac{128}{31} & -\frac{6368}{31} & -\frac{3056}{31} & 0 \\ 0 & \frac{128}{31} & \frac{12320}{31} & \frac{6032}{31} & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{4R_2}{13} \implies \left[\begin{array}{cccc|c} 31 & -8 & 5 & -5 & 0 \\ 0 & \frac{416}{31} & \frac{608}{31} & -\frac{112}{31} & 0 \\ 0 & 0 & -\frac{2592}{13} & -\frac{1296}{13} & 0 \\ 0 & \frac{128}{31} & \frac{12320}{31} & \frac{6032}{31} & 0 \end{array}\right]$$

$$R_4 = R_4 - \frac{4R_2}{13} \implies \left[\begin{array}{cccc|c} 31 & -8 & 5 & -5 & 0 \\ 0 & \frac{416}{31} & \frac{608}{31} & -\frac{112}{31} & 0 \\ 0 & 0 & -\frac{2592}{13} & -\frac{1296}{13} & 0 \\ 0 & 0 & \frac{5088}{13} & \frac{2544}{13} & 0 \end{array}\right]$$

$$R_4 = R_4 + \frac{53R_3}{27} \implies \left[\begin{array}{cccc|c} 31 & -8 & 5 & -5 & 0 \\ 0 & \frac{416}{31} & \frac{608}{31} & -\frac{112}{31} & 0 \\ 0 & 0 & -\frac{2592}{13} & -\frac{1296}{13} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 31 & -8 & 5 & -5 \\ 0 & \frac{416}{31} & \frac{608}{31} & -\frac{112}{31} \\ 0 & 0 & -\frac{2592}{13} & -\frac{1296}{13} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_4\}$ and the leading variables are $\{v_1, v_2, v_3\}$. Let $v_4 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{2}, v_2 = t, v_3 = -\frac{t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{2} \\ t \\ -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{2} \\ t \\ -\frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{2} \\ t \\ -\frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} \\ 1 \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{t}{2} \\ t \\ -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{t}{2} \\ t \\ -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 32$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 47 & -8 & 5 & -5 \\ -10 & 32 & 18 & -2 \\ 139 & -40 & -167 & -121 \\ -232 & 64 & 360 & 248 \end{bmatrix} - (32)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 15 & -8 & 5 & -5 \\ -10 & 0 & 18 & -2 \\ 139 & -40 & -199 & -121 \\ -232 & 64 & 360 & 216 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} 15 & -8 & 5 & -5 & 0 \\ -10 & 0 & 18 & -2 & 0 \\ 139 & -40 & -199 & -121 & 0 \\ -232 & 64 & 360 & 216 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{2R_1}{3} \Longrightarrow \left[\begin{array}{cccc|c} 15 & -8 & 5 & -5 & 0 \\ 0 & -\frac{16}{3} & \frac{64}{3} & -\frac{16}{3} & 0 \\ 139 & -40 & -199 & -121 & 0 \\ -232 & 64 & 360 & 216 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{139R_1}{15} \Longrightarrow \left[\begin{array}{cccc|c} 15 & -8 & 5 & -5 & 0 \\ 0 & -\frac{16}{3} & \frac{64}{3} & -\frac{16}{3} & 0 \\ 0 & \frac{512}{15} & -\frac{736}{3} & -\frac{224}{3} & 0 \\ -232 & 64 & 360 & 216 & 0 \end{array}\right]$$

$$R_4 = R_4 + \frac{232R_1}{15} \Longrightarrow \left[\begin{array}{cccc|c} 15 & -8 & 5 & -5 & 0 \\ 0 & -\frac{16}{3} & \frac{64}{3} & -\frac{16}{3} & 0 \\ 0 & \frac{512}{15} & -\frac{736}{3} & -\frac{224}{3} & 0 \\ 0 & -\frac{896}{15} & \frac{1312}{3} & \frac{416}{3} & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{32R_2}{5} \Longrightarrow \left[\begin{array}{cccc|c} 15 & -8 & 5 & -5 & 0 \\ 0 & -\frac{16}{3} & \frac{64}{3} & -\frac{16}{3} & 0 \\ 0 & 0 & -\frac{544}{5} & -\frac{544}{5} & 0 \\ 0 & -\frac{896}{15} & \frac{1312}{3} & \frac{416}{3} & 0 \end{array}\right]$$

$$R_4 = R_4 - \frac{56R_2}{5} \Longrightarrow \left[\begin{array}{cccc|c} 15 & -8 & 5 & -5 & 0 \\ 0 & -\frac{16}{3} & \frac{64}{3} & -\frac{16}{3} & 0 \\ 0 & 0 & -\frac{544}{5} & -\frac{544}{5} & 0 \\ 0 & 0 & \frac{992}{5} & \frac{992}{5} & 0 \end{array}\right]$$

$$R_4 = R_4 + \frac{31R_3}{17} \Longrightarrow \left[\begin{array}{cccc|c} 15 & -8 & 5 & -5 & 0 \\ 0 & -\frac{16}{3} & \frac{64}{3} & -\frac{16}{3} & 0 \\ 0 & 0 & -\frac{544}{5} & -\frac{544}{5} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 15 & -8 & 5 & -5 \\ 0 & -\frac{16}{3} & \frac{64}{3} & -\frac{16}{3} \\ 0 & 0 & -\frac{544}{5} & -\frac{544}{5} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_4\}$ and the leading variables are $\{v_1, v_2, v_3\}$. Let $v_4 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -2t, v_2 = -5t, v_3 = -t\}$

Hence the solution is

$$\begin{bmatrix} -2t \\ -5t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} -2t \\ -5t \\ -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this

eigenvalue. The above can be written as

$$\begin{bmatrix} -2t \\ -5t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} -2 \\ -5 \\ -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -2t \\ -5t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} -2 \\ -5 \\ -1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = 48$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 47 & -8 & 5 & -5 \\ -10 & 32 & 18 & -2 \\ 139 & -40 & -167 & -121 \\ -232 & 64 & 360 & 248 \end{bmatrix} - (48) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -8 & 5 & -5 \\ -10 & -16 & 18 & -2 \\ 139 & -40 & -215 & -121 \\ -232 & 64 & 360 & 200 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} -1 & -8 & 5 & -5 & 0 \\ -10 & -16 & 18 & -2 & 0 \\ 139 & -40 & -215 & -121 & 0 \\ -232 & 64 & 360 & 200 & 0 \end{array}\right]$$

$$R_2 = R_2 - 10R_1 \Longrightarrow \left[\begin{array}{cccc|c} -1 & -8 & 5 & -5 & 0 \\ 0 & 64 & -32 & 48 & 0 \\ 139 & -40 & -215 & -121 & 0 \\ -232 & 64 & 360 & 200 & 0 \end{array}\right]$$

$$R_3 = R_3 + 139R_1 \Longrightarrow \left[\begin{array}{cccc|c} -1 & -8 & 5 & -5 & 0 \\ 0 & 64 & -32 & 48 & 0 \\ 0 & -1152 & 480 & -816 & 0 \\ -232 & 64 & 360 & 200 & 0 \end{array}\right]$$

$$R_4 = R_4 - 232R_1 \Longrightarrow \left[\begin{array}{cccc|c} -1 & -8 & 5 & -5 & 0 \\ 0 & 64 & -32 & 48 & 0 \\ 0 & -1152 & 480 & -816 & 0 \\ 0 & 1920 & -800 & 1360 & 0 \end{array}\right]$$

$$R_3 = R_3 + 18R_2 \Longrightarrow \left[\begin{array}{cccc|c} -1 & -8 & 5 & -5 & 0 \\ 0 & 64 & -32 & 48 & 0 \\ 0 & 0 & -96 & 48 & 0 \\ 0 & 1920 & -800 & 1360 & 0 \end{array}\right]$$

$$R_4 = R_4 - 30R_2 \Longrightarrow \left[\begin{array}{cccc|c} -1 & -8 & 5 & -5 & 0 \\ 0 & 64 & -32 & 48 & 0 \\ 0 & 0 & -96 & 48 & 0 \\ 0 & 0 & 160 & -80 & 0 \end{array}\right]$$

$$R_4 = R_4 + \frac{5R_3}{3} \Longrightarrow \left[\begin{array}{cccc|c} -1 & -8 & 5 & -5 & 0 \\ 0 & 64 & -32 & 48 & 0 \\ 0 & 0 & -96 & 48 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & -8 & 5 & -5 \\ 0 & 64 & -32 & 48 \\ 0 & 0 & -96 & 48 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_4\}$ and the leading variables are $\{v_1, v_2, v_3\}$. Let $v_4 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{3t}{2}, v_2 = -\frac{t}{2}, v_3 = \frac{t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{3t}{2} \\ -\frac{t}{2} \\ \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{3t}{2} \\ -\frac{t}{2} \\ \frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{3t}{2} \\ -\frac{t}{2} \\ \frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{3}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{3t}{2} \\ -\frac{t}{2} \\ \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{3t}{2} \\ -\frac{t}{2} \\ \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 1 \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_4 = 64$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 47 & -8 & 5 & -5 \\ -10 & 32 & 18 & -2 \\ 139 & -40 & -167 & -121 \\ -232 & 64 & 360 & 248 \end{bmatrix} - (64)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -17 & -8 & 5 & -5 \\ -10 & -32 & 18 & -2 \\ 139 & -40 & -231 & -121 \\ -232 & 64 & 360 & 184 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} -17 & -8 & 5 & -5 & 0 \\ -10 & -32 & 18 & -2 & 0 \\ 139 & -40 & -231 & -121 & 0 \\ -232 & 64 & 360 & 184 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{10R_1}{17} \implies \left[\begin{array}{cccc|c} -17 & -8 & 5 & -5 & 0 \\ 0 & -\frac{464}{17} & \frac{256}{17} & \frac{16}{17} & 0 \\ 139 & -40 & -231 & -121 & 0 \\ -232 & 64 & 360 & 184 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{139R_1}{17} \implies \left[\begin{array}{cccc|c} -17 & -8 & 5 & -5 & 0 \\ 0 & -\frac{464}{17} & \frac{256}{17} & \frac{16}{17} & 0 \\ 0 & -\frac{1792}{17} & -\frac{3232}{17} & -\frac{2752}{17} & 0 \\ -232 & 64 & 360 & 184 & 0 \end{array}\right]$$

$$R_4 = R_4 - \frac{232R_1}{17} \implies \left[\begin{array}{cccc|c} -17 & -8 & 5 & -5 & 0 \\ 0 & -\frac{464}{17} & \frac{256}{17} & \frac{16}{17} & 0 \\ 0 & -\frac{1792}{17} & -\frac{3232}{17} & -\frac{2752}{17} & 0 \\ 0 & \frac{2944}{17} & \frac{4960}{17} & \frac{4288}{17} & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{112R_2}{29} \implies \left[\begin{array}{cccc|c} -17 & -8 & 5 & -5 & 0 \\ 0 & -\frac{464}{17} & \frac{256}{17} & \frac{16}{17} & 0 \\ 0 & 0 & -\frac{7200}{29} & -\frac{4800}{29} & 0 \\ 0 & \frac{2944}{17} & \frac{4960}{17} & \frac{4288}{17} & 0 \end{array}\right]$$

$$R_4 = R_4 + \frac{184R_2}{29} \implies \left[\begin{array}{cccc|c} -17 & -8 & 5 & -5 & 0 \\ 0 & -\frac{464}{17} & \frac{256}{17} & \frac{16}{17} & 0 \\ 0 & 0 & -\frac{7200}{29} & -\frac{4800}{29} & 0 \\ 0 & 0 & \frac{11232}{29} & \frac{7488}{29} & 0 \end{array}\right]$$

$$R_4 = R_4 + \frac{39R_3}{25} \implies \left[\begin{array}{cccc|c} -17 & -8 & 5 & -5 & 0 \\ 0 & -\frac{464}{17} & \frac{256}{17} & \frac{16}{17} & 0 \\ 0 & 0 & -\frac{7200}{29} & -\frac{4800}{29} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -17 & -8 & 5 & -5 \\ 0 & -\frac{464}{17} & \frac{256}{17} & \frac{16}{17} \\ 0 & 0 & -\frac{7200}{29} & -\frac{4800}{29} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_4\}$ and the leading variables are $\{v_1, v_2, v_3\}$. Let $v_4 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{t}{3}, v_2 = -\frac{t}{3}, v_3 = -\frac{2t}{3}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{t}{3} \\ -\frac{t}{3} \\ -\frac{2t}{3} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{t}{3} \\ -\frac{t}{3} \\ -\frac{2t}{3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{t}{3} \\ -\frac{t}{3} \\ -\frac{2t}{3} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ -\frac{2}{3} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{t}{3} \\ -\frac{t}{3} \\ -\frac{2t}{3} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ -\frac{2}{3} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{t}{3} \\ -\frac{t}{3} \\ -\frac{2t}{3} \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -2 \\ 3 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity algebraic m	multiplicity geometric k	defective?	eigenvectors
48	1	1	No	$\begin{bmatrix} \frac{3}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$
32	1	1	No	$\begin{bmatrix} -2 \\ -5 \\ -1 \\ 1 \end{bmatrix}$
16	1	1	No	$\begin{bmatrix} \frac{1}{2} \\ 1 \\ -\frac{1}{2} \\ 1 \end{bmatrix}$
64	1	1	No	$\begin{bmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ -\frac{2}{3} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 48 is real and distinct then the

corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{48t}\\ &= \begin{bmatrix} \frac{3}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix} e^{48t}\end{aligned}$$

Since eigenvalue 32 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{32t}\\ &= \begin{bmatrix} -2 \\ -5 \\ -1 \\ 1 \end{bmatrix} e^{32t}\end{aligned}$$

Since eigenvalue 16 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_3(t) &= \vec{v}_3 e^{16t}\\ &= \begin{bmatrix} \frac{1}{2} \\ 1 \\ -\frac{1}{2} \\ 1 \end{bmatrix} e^{16t}\end{aligned}$$

Since eigenvalue 64 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_4(t) &= \vec{v}_4 e^{64t}\\ &= \begin{bmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ -\frac{2}{3} \\ 1 \end{bmatrix} e^{64t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t) + c_4\vec{x}_4(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = c_1 \begin{bmatrix} \frac{3\,\mathrm{e}^{48t}}{2} \\ -\frac{\mathrm{e}^{48t}}{2} \\ \frac{\mathrm{e}^{48t}}{2} \\ \mathrm{e}^{48t} \end{bmatrix} + c_2 \begin{bmatrix} -2\,\mathrm{e}^{32t} \\ -5\,\mathrm{e}^{32t} \\ -\mathrm{e}^{32t} \\ \mathrm{e}^{32t} \end{bmatrix} + c_3 \begin{bmatrix} \frac{\mathrm{e}^{16t}}{2} \\ \mathrm{e}^{16t} \\ -\frac{\mathrm{e}^{16t}}{2} \\ \mathrm{e}^{16t} \end{bmatrix} + c_4 \begin{bmatrix} -\frac{\mathrm{e}^{64t}}{3} \\ -\frac{\mathrm{e}^{64t}}{3} \\ -\frac{2\,\mathrm{e}^{64t}}{3} \\ \mathrm{e}^{64t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = \begin{bmatrix} \frac{3c_1\mathrm{e}^{48t}}{2} - 2c_2\mathrm{e}^{32t} + \frac{c_3\mathrm{e}^{16t}}{2} - \frac{c_4\mathrm{e}^{64t}}{3} \\ -\frac{c_1\mathrm{e}^{48t}}{2} - 5c_2\mathrm{e}^{32t} + c_3\mathrm{e}^{16t} - \frac{c_4\mathrm{e}^{64t}}{3} \\ \frac{c_1\mathrm{e}^{48t}}{2} - c_2\mathrm{e}^{32t} - \frac{c_3\mathrm{e}^{16t}}{2} - \frac{2c_4\mathrm{e}^{64t}}{3} \\ c_1\mathrm{e}^{48t} + c_2\mathrm{e}^{32t} + c_3\mathrm{e}^{16t} + c_4\mathrm{e}^{64t} \end{bmatrix}$$

4.37.3 Maple step by step solution

Let's solve

$[x_1'(t) = 47x_1(t) - 8x_2(t) + 5x_3(t) - 5x_4(t)\,, x_2'(t) = -10x_1(t) + 32x_2(t) + 18x_3(t) - 2x_4(t)\,, x_3'(t)$

- Define vector

$$\underline{x^{\rightarrow}}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x^{\rightarrow}}'(t) = \begin{bmatrix} 47 & -8 & 5 & -5 \\ -10 & 32 & 18 & -2 \\ 139 & -40 & -167 & -121 \\ -232 & 64 & 360 & 248 \end{bmatrix} \cdot \underline{x^{\rightarrow}}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{x}^{\rightarrow\prime}(t) = \begin{bmatrix} 47 & -8 & 5 & -5 \\ -10 & 32 & 18 & -2 \\ 139 & -40 & -167 & -121 \\ -232 & 64 & 360 & 248 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 47 & -8 & 5 & -5 \\ -10 & 32 & 18 & -2 \\ 139 & -40 & -167 & -121 \\ -232 & 64 & 360 & 248 \end{bmatrix}$$

- Rewrite the system as

$$\underline{x}^{\rightarrow\prime}(t) = A \cdot \underline{x}^{\rightarrow}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[16, \begin{bmatrix} \frac{1}{2} \\ 1 \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right], \left[32, \begin{bmatrix} -2 \\ -5 \\ -1 \\ 1 \end{bmatrix}\right], \left[48, \begin{bmatrix} \frac{3}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right], \left[64, \begin{bmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ -\frac{2}{3} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[16, \begin{bmatrix} \frac{1}{2} \\ 1 \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{x}^{\rightarrow}_{1} = \mathrm{e}^{16t} \cdot \begin{bmatrix} \frac{1}{2} \\ 1 \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[32, \begin{bmatrix} -2 \\ -5 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_2 = e^{32t} \cdot \begin{bmatrix} -2 \\ -5 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[48, \begin{bmatrix} \frac{3}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_3 = e^{48t} \cdot \begin{bmatrix} \frac{3}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[64, \begin{bmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ -\frac{2}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{x}^{\rightarrow}{}_{4} = e^{64t} \cdot \begin{bmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ -\frac{2}{3} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\underline{x}^{\rightarrow} = c_1 \underline{x}^{\rightarrow}{}_{1} + c_2 \underline{x}^{\rightarrow}{}_{2} + c_3 \underline{x}^{\rightarrow}{}_{3} + c_4 \underline{x}^{\rightarrow}{}_{4}$$

- Substitute solutions into the general solution

$$\underline{x}^{\rightarrow} = c_1 e^{16t} \cdot \begin{bmatrix} \frac{1}{2} \\ 1 \\ -\frac{1}{2} \\ 1 \end{bmatrix} + c_2 e^{32t} \cdot \begin{bmatrix} -2 \\ -5 \\ -1 \\ 1 \end{bmatrix} + c_3 e^{48t} \cdot \begin{bmatrix} \frac{3}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix} + c_4 e^{64t} \cdot \begin{bmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ -\frac{2}{3} \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = \begin{bmatrix} \frac{c_1 e^{16t}}{2} - 2c_2 e^{32t} + \frac{3c_3 e^{48t}}{2} - \frac{c_4 e^{64t}}{3} \\ c_1 e^{16t} - 5c_2 e^{32t} - \frac{c_3 e^{48t}}{2} - \frac{c_4 e^{64t}}{3} \\ -\frac{c_1 e^{16t}}{2} - c_2 e^{32t} + \frac{c_3 e^{48t}}{2} - \frac{2c_4 e^{64t}}{3} \\ c_1 e^{16t} + c_2 e^{32t} + c_3 e^{48t} + c_4 e^{64t} \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x_1(t) = \frac{c_1 e^{16t}}{2} - 2c_2 e^{32t} + \frac{3c_3 e^{48t}}{2} - \frac{c_4 e^{64t}}{3}, x_2(t) = c_1 e^{16t} - 5c_2 e^{32t} - \frac{c_3 e^{48t}}{2} - \frac{c_4 e^{64t}}{3}, x_3(t) = -\frac{c_1 e^{16t}}{2} -\right.$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 125

```
dsolve([diff(x__1(t),t)=47*x__1(t)-8*x__2(t)+5*x__3(t)-5*x__4(t),diff(x__2(t),t)=-10*x__1(t)
```

$$x_1(t) = c_1 e^{48t} + c_2 e^{16t} + c_3 e^{32t} + c_4 e^{64t}$$
$$x_2(t) = -\frac{c_1 e^{48t}}{3} + 2c_2 e^{16t} + \frac{5c_3 e^{32t}}{2} + c_4 e^{64t}$$
$$x_3(t) = \frac{c_1 e^{48t}}{3} - c_2 e^{16t} + \frac{c_3 e^{32t}}{2} + 2c_4 e^{64t}$$
$$x_4(t) = \frac{2c_1 e^{48t}}{3} + 2c_2 e^{16t} - \frac{c_3 e^{32t}}{2} - 3c_4 e^{64t}$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 448

```
DSolve[{x1'[t]==47*x1[t]-8*x2[t]+5*x3[t]-5*x4[t],x2'[t]==-10*x1[t]+32*x2[t]+18*x3[t]-2*x4[t]
```

$$x1(t) \to \frac{1}{16}e^{16t}\big(c_1(-38e^{16t}-6e^{32t}+27e^{48t}+33) - (e^{16t}-1)\left(8c_2(e^{16t}+e^{32t}-1)+c_3(9e^{16t}+39e^{32t}-53)+c_4(7e^{16t}+25e^{32t}-27)\right)\big)$$

$$x2(t) \to \frac{1}{16}e^{16t}\big(c_1(-95e^{16t}+2e^{32t}+27e^{48t}+66) - 8c_2(-5e^{16t}+e^{48t}+2) - (e^{16t}-1)\left(c_3(49e^{16t}+39e^{32t}-106)+c_4(31e^{16t}+25e^{32t}-54)\right)\big)$$

$$x3(t) \to \frac{1}{16}e^{16t}\big(c_1(-19e^{16t}-2e^{32t}+54e^{48t}-33)+8c_2(e^{16t}-2e^{48t}+1)+31c_3e^{16t} + 10c_3e^{32t}-78c_3e^{48t}+17c_4e^{16t}+6c_4e^{32t}-50c_4e^{48t}+53c_3+27c_4\big)$$

$$x4(t) \to -\frac{1}{16}e^{16t}\big(c_1(-19e^{16t}+4e^{32t}+81e^{48t}-66)+8c_2(e^{16t}-3e^{48t}+2)+31c_3e^{16t} - 20c_3e^{32t}-117c_3e^{48t}+17c_4e^{16t}-12c_4e^{32t}-75c_4e^{48t}+106c_3+54c_4\big)$$

4.38 problem problem 49

Internal problem ID [352]
Internal file name [`OUTPUT/352_Sunday_June_05_2022_01_39_25_AM_55030875/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.3, The eigenvalue method for linear systems. Page 395
Problem number: problem 49.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= 139x_1(t) - 14x_2(t) - 52x_3(t) - 14x_4(t) + 28x_5(t)\\
x_2'(t) &= -22x_1(t) + 5x_2(t) + 7x_3(t) + 8x_4(t) - 7x_5(t)\\
x_3'(t) &= 370x_1(t) - 38x_2(t) - 139x_3(t) - 38x_4(t) + 76x_5(t)\\
x_4'(t) &= 152x_1(t) - 16x_2(t) - 59x_3(t) - 13x_4(t) + 35x_5(t)\\
x_5'(t) &= 95x_1(t) - 10x_2(t) - 38x_3(t) - 7x_4(t) + 23x_5(t)
\end{aligned}
$$

4.38.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$
\begin{bmatrix} x_1'(t)\\ x_2'(t)\\ x_3'(t)\\ x_4'(t)\\ x_5'(t) \end{bmatrix}
=
\begin{bmatrix}
139 & -14 & -52 & -14 & 28\\
-22 & 5 & 7 & 8 & -7\\
370 & -38 & -139 & -38 & 76\\
152 & -16 & -59 & -13 & 35\\
95 & -10 & -38 & -7 & 23
\end{bmatrix}
\begin{bmatrix} x_1(t)\\ x_2(t)\\ x_3(t)\\ x_4(t)\\ x_5(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} -10\,\mathrm{e}^{-3t} - \frac{5\,\mathrm{e}^{3t}}{3} + \frac{38\,\mathrm{e}^{9t}}{3} & -\frac{4\,\mathrm{e}^{9t}}{3} + \frac{\mathrm{e}^{3t}}{3} + \mathrm{e}^{-3t} & \frac{2\,\mathrm{e}^{3t}}{3} + 4\,\mathrm{e}^{-3t} - \frac{14\,\mathrm{e}^{9t}}{3} \\ \frac{19}{2} - \frac{35\,\mathrm{e}^{3t}}{3} + \frac{13\,\mathrm{e}^{6t}}{6} & -\frac{\mathrm{e}^{6t}}{3} + \frac{7\,\mathrm{e}^{3t}}{3} - 1 & -\frac{7\,\mathrm{e}^{6t}}{6} + \frac{14\,\mathrm{e}^{3t}}{3} - \frac{7}{2} \\ -30\,\mathrm{e}^{-3t} - \frac{5\,\mathrm{e}^{3t}}{3} + \frac{95\,\mathrm{e}^{9t}}{3} & -\frac{10\,\mathrm{e}^{9t}}{3} + \frac{\mathrm{e}^{3t}}{3} + 3\,\mathrm{e}^{-3t} & 12\,\mathrm{e}^{-3t} - \frac{35\,\mathrm{e}^{9t}}{3} + \frac{2\,\mathrm{e}^{3t}}{3} \\ -\frac{19}{6} - 10\,\mathrm{e}^{-3t} - \frac{5\,\mathrm{e}^{3t}}{3} + \frac{13\,\mathrm{e}^{6t}}{6} + \frac{38\,\mathrm{e}^{9t}}{3} & -\frac{4\,\mathrm{e}^{9t}}{3} - \frac{\mathrm{e}^{6t}}{3} + \frac{\mathrm{e}^{3t}}{3} + \frac{1}{3} + \mathrm{e}^{-3t} & -\frac{14\,\mathrm{e}^{9t}}{3} - \frac{7\,\mathrm{e}^{6t}}{6} + \frac{2\,\mathrm{e}^{3t}}{3} + \frac{7}{6} + 4\,\mathrm{e} \\ \frac{19}{6} - 10\,\mathrm{e}^{-3t} - \frac{5\,\mathrm{e}^{3t}}{3} + \frac{13\,\mathrm{e}^{6t}}{6} + \frac{19\,\mathrm{e}^{9t}}{3} & -\frac{2\,\mathrm{e}^{9t}}{3} - \frac{\mathrm{e}^{6t}}{3} + \frac{\mathrm{e}^{3t}}{3} - \frac{1}{3} + \mathrm{e}^{-3t} & -\frac{7\,\mathrm{e}^{9t}}{3} - \frac{7\,\mathrm{e}^{6t}}{6} + \frac{2\,\mathrm{e}^{3t}}{3} - \frac{7}{6} + 4\,\mathrm{e} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = e^{At}\vec{c}$$

$$= \begin{bmatrix} -10\,\mathrm{e}^{-3t} - \frac{5\,\mathrm{e}^{3t}}{3} + \frac{38\,\mathrm{e}^{9t}}{3} & -\frac{4\,\mathrm{e}^{9t}}{3} + \frac{\mathrm{e}^{3t}}{3} + \mathrm{e}^{-3t} & \frac{2\,\mathrm{e}^{3t}}{3} + 4\,\mathrm{e}^{-3t} - \frac{14\,\mathrm{e}^{9t}}{3} \\ \frac{19}{2} - \frac{35\,\mathrm{e}^{3t}}{3} + \frac{13\,\mathrm{e}^{6t}}{6} & -\frac{\mathrm{e}^{6t}}{3} + \frac{7\,\mathrm{e}^{3t}}{3} - 1 & -\frac{7\,\mathrm{e}^{6t}}{6} + \frac{14\,\mathrm{e}^{3t}}{3} - \frac{7}{2} \\ -30\,\mathrm{e}^{-3t} - \frac{5\,\mathrm{e}^{3t}}{3} + \frac{95\,\mathrm{e}^{9t}}{3} & -\frac{10\,\mathrm{e}^{9t}}{3} + \frac{\mathrm{e}^{3t}}{3} + 3\,\mathrm{e}^{-3t} & 12\,\mathrm{e}^{-3t} - \frac{35\,\mathrm{e}^{9t}}{3} + \frac{2\,\mathrm{e}^{3t}}{3} \\ -\frac{19}{6} - 10\,\mathrm{e}^{-3t} - \frac{5\,\mathrm{e}^{3t}}{3} + \frac{13\,\mathrm{e}^{6t}}{6} + \frac{38\,\mathrm{e}^{9t}}{3} & -\frac{4\,\mathrm{e}^{9t}}{3} - \frac{\mathrm{e}^{6t}}{3} + \frac{\mathrm{e}^{3t}}{3} + \frac{1}{3} + \mathrm{e}^{-3t} & -\frac{14\,\mathrm{e}^{9t}}{3} - \frac{7\,\mathrm{e}^{6t}}{6} + \frac{2\,\mathrm{e}^{3t}}{3} + \frac{7}{6} + \\ \frac{19}{6} - 10\,\mathrm{e}^{-3t} - \frac{5\,\mathrm{e}^{3t}}{3} + \frac{13\,\mathrm{e}^{6t}}{6} + \frac{19\,\mathrm{e}^{9t}}{3} & -\frac{2\,\mathrm{e}^{9t}}{3} - \frac{\mathrm{e}^{6t}}{3} + \frac{\mathrm{e}^{3t}}{3} - \frac{1}{3} + \mathrm{e}^{-3t} & -\frac{7\,\mathrm{e}^{9t}}{3} - \frac{7\,\mathrm{e}^{6t}}{6} + \frac{2\,\mathrm{e}^{3t}}{3} - \frac{7}{6} + \end{bmatrix}$$

= Expression too large to display

$$= \begin{bmatrix} (-10c_1 + c_2 + 4c_3 + c_4 - 2c_5)\,\mathrm{e}^{-3t} + \frac{(-5c_1+c_2+2c_3+c_4-2c_5)\mathrm{e}^{3t}}{3} + \frac{38\left(c_1 - \cdots\right.}{} \\ \frac{7(-5c_1+c_2+2c_3+c_4-2c_5)\mathrm{e}^{3t}}{3} + \frac{(13c_1-2c_2-7c_3+c_4+7c_5)\mathrm{e}^{6t}}{6} + \frac{19c_1}{2} - c_2 - \frac{7c_3}{2} \\ 3(-10c_1 + c_2 + 4c_3 + c_4 - 2c_5)\,\mathrm{e}^{-3t} + \frac{(-5c_1+c_2+2c_3+c_4-2c_5)\mathrm{e}^{3t}}{3} + \frac{95\left(c_1 - \cdots\right.}{} \\ (-10c_1 + c_2 + 4c_3 + c_4 - 2c_5)\,\mathrm{e}^{-3t} + \frac{(-5c_1+c_2+2c_3+c_4-2c_5)\mathrm{e}^{3t}}{3} + \frac{(13c_1-2c_2-7c_3+c_4+7c_5)\mathrm{e}^{6t}}{6} + \frac{2(19c_1-2c_2-\cdots}{} \\ (-10c_1 + c_2 + 4c_3 + c_4 - 2c_5)\,\mathrm{e}^{-3t} + \frac{(-5c_1+c_2+2c_3+c_4-2c_5)\mathrm{e}^{3t}}{3} + \frac{(13c_1-2c_2-7c_3+c_4+7c_5)\mathrm{e}^{6t}}{6} + \frac{(19c_1-2c_2-\cdots}{} \end{bmatrix}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

4.38.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \\ x_4'(t) \\ x_5'(t) \end{bmatrix} = \begin{bmatrix} 139 & -14 & -52 & -14 & 28 \\ -22 & 5 & 7 & 8 & -7 \\ 370 & -38 & -139 & -38 & 76 \\ 152 & -16 & -59 & -13 & 35 \\ 95 & -10 & -38 & -7 & 23 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ x_5(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 139 & -14 & -52 & -14 & 28 \\ -22 & 5 & 7 & 8 & -7 \\ 370 & -38 & -139 & -38 & 76 \\ 152 & -16 & -59 & -13 & 35 \\ 95 & -10 & -38 & -7 & 23 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 139-\lambda & -14 & -52 & -14 & 28 \\ -22 & 5-\lambda & 7 & 8 & -7 \\ 370 & -38 & -139-\lambda & -38 & 76 \\ 152 & -16 & -59 & -13-\lambda & 35 \\ 95 & -10 & -38 & -7 & 23-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^5 - 15\lambda^4 + 45\lambda^3 + 135\lambda^2 - 486\lambda = 0$$

The roots of the above are the eigenvalues.

$$\begin{aligned} \lambda_1 &= 0 \\ \lambda_2 &= 6 \\ \lambda_3 &= 9 \\ \lambda_4 &= 3 \\ \lambda_5 &= -3 \end{aligned}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
0	1	real eigenvalue
-3	1	real eigenvalue
3	1	real eigenvalue
6	1	real eigenvalue
9	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 139 & -14 & -52 & -14 & 28 \\ -22 & 5 & 7 & 8 & -7 \\ 370 & -38 & -139 & -38 & 76 \\ 152 & -16 & -59 & -13 & 35 \\ 95 & -10 & -38 & -7 & 23 \end{bmatrix} - (-3)\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 142 & -14 & -52 & -14 & 28 \\ -22 & 8 & 7 & 8 & -7 \\ 370 & -38 & -136 & -38 & 76 \\ 152 & -16 & -59 & -10 & 35 \\ 95 & -10 & -38 & -7 & 26 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccccc|c} 142 & -14 & -52 & -14 & 28 & 0 \\ -22 & 8 & 7 & 8 & -7 & 0 \\ 370 & -38 & -136 & -38 & 76 & 0 \\ 152 & -16 & -59 & -10 & 35 & 0 \\ 95 & -10 & -38 & -7 & 26 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{11R_1}{71} \implies \left[\begin{array}{ccccc|c} 142 & -14 & -52 & -14 & 28 & 0 \\ 0 & \frac{414}{71} & -\frac{75}{71} & \frac{414}{71} & -\frac{189}{71} & 0 \\ 370 & -38 & -136 & -38 & 76 & 0 \\ 152 & -16 & -59 & -10 & 35 & 0 \\ 95 & -10 & -38 & -7 & 26 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{185R_1}{71} \implies \left[\begin{array}{ccccc|c} 142 & -14 & -52 & -14 & 28 & 0 \\ 0 & \frac{414}{71} & -\frac{75}{71} & \frac{414}{71} & -\frac{189}{71} & 0 \\ 0 & -\frac{108}{71} & -\frac{36}{71} & -\frac{108}{71} & \frac{216}{71} & 0 \\ 152 & -16 & -59 & -10 & 35 & 0 \\ 95 & -10 & -38 & -7 & 26 & 0 \end{array}\right]$$

$$R_4 = R_4 - \frac{76R_1}{71} \implies \left[\begin{array}{ccccc|c} 142 & -14 & -52 & -14 & 28 & 0 \\ 0 & \frac{414}{71} & -\frac{75}{71} & \frac{414}{71} & -\frac{189}{71} & 0 \\ 0 & -\frac{108}{71} & -\frac{36}{71} & -\frac{108}{71} & \frac{216}{71} & 0 \\ 0 & -\frac{72}{71} & -\frac{237}{71} & \frac{354}{71} & \frac{357}{71} & 0 \\ 95 & -10 & -38 & -7 & 26 & 0 \end{array}\right]$$

$$R_5 = R_5 - \frac{95R_1}{142} \implies \left[\begin{array}{ccccc|c} 142 & -14 & -52 & -14 & 28 & 0 \\ 0 & \frac{414}{71} & -\frac{75}{71} & \frac{414}{71} & -\frac{189}{71} & 0 \\ 0 & -\frac{108}{71} & -\frac{36}{71} & -\frac{108}{71} & \frac{216}{71} & 0 \\ 0 & -\frac{72}{71} & -\frac{237}{71} & \frac{354}{71} & \frac{357}{71} & 0 \\ 0 & -\frac{45}{71} & -\frac{228}{71} & \frac{168}{71} & \frac{516}{71} & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{6R_2}{23} \implies \left[\begin{array}{ccccc|c} 142 & -14 & -52 & -14 & 28 & 0 \\ 0 & \frac{414}{71} & -\frac{75}{71} & \frac{414}{71} & -\frac{189}{71} & 0 \\ 0 & 0 & -\frac{18}{23} & 0 & \frac{54}{23} & 0 \\ 0 & -\frac{72}{71} & -\frac{237}{71} & \frac{354}{71} & \frac{357}{71} & 0 \\ 0 & -\frac{45}{71} & -\frac{228}{71} & \frac{168}{71} & \frac{516}{71} & 0 \end{array}\right]$$

$$R_4 = R_4 + \frac{4R_2}{23} \Longrightarrow \left[\begin{array}{ccccc|c} 142 & -14 & -52 & -14 & 28 & 0 \\ 0 & \frac{414}{71} & -\frac{75}{71} & \frac{414}{71} & -\frac{189}{71} & 0 \\ 0 & 0 & -\frac{18}{23} & 0 & \frac{54}{23} & 0 \\ 0 & 0 & -\frac{81}{23} & 6 & \frac{105}{23} & 0 \\ 0 & -\frac{45}{71} & -\frac{228}{71} & \frac{168}{71} & \frac{516}{71} & 0 \end{array}\right]$$

$$R_5 = R_5 + \frac{5R_2}{46} \Longrightarrow \left[\begin{array}{ccccc|c} 142 & -14 & -52 & -14 & 28 & 0 \\ 0 & \frac{414}{71} & -\frac{75}{71} & \frac{414}{71} & -\frac{189}{71} & 0 \\ 0 & 0 & -\frac{18}{23} & 0 & \frac{54}{23} & 0 \\ 0 & 0 & -\frac{81}{23} & 6 & \frac{105}{23} & 0 \\ 0 & 0 & -\frac{153}{46} & 3 & \frac{321}{46} & 0 \end{array}\right]$$

$$R_4 = R_4 - \frac{9R_3}{2} \Longrightarrow \left[\begin{array}{ccccc|c} 142 & -14 & -52 & -14 & 28 & 0 \\ 0 & \frac{414}{71} & -\frac{75}{71} & \frac{414}{71} & -\frac{189}{71} & 0 \\ 0 & 0 & -\frac{18}{23} & 0 & \frac{54}{23} & 0 \\ 0 & 0 & 0 & 6 & -6 & 0 \\ 0 & 0 & -\frac{153}{46} & 3 & \frac{321}{46} & 0 \end{array}\right]$$

$$R_5 = R_5 - \frac{17R_3}{4} \Longrightarrow \left[\begin{array}{ccccc|c} 142 & -14 & -52 & -14 & 28 & 0 \\ 0 & \frac{414}{71} & -\frac{75}{71} & \frac{414}{71} & -\frac{189}{71} & 0 \\ 0 & 0 & -\frac{18}{23} & 0 & \frac{54}{23} & 0 \\ 0 & 0 & 0 & 6 & -6 & 0 \\ 0 & 0 & 0 & 3 & -3 & 0 \end{array}\right]$$

$$R_5 = R_5 - \frac{R_4}{2} \Longrightarrow \left[\begin{array}{ccccc|c} 142 & -14 & -52 & -14 & 28 & 0 \\ 0 & \frac{414}{71} & -\frac{75}{71} & \frac{414}{71} & -\frac{189}{71} & 0 \\ 0 & 0 & -\frac{18}{23} & 0 & \frac{54}{23} & 0 \\ 0 & 0 & 0 & 6 & -6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 142 & -14 & -52 & -14 & 28 \\ 0 & \frac{414}{71} & -\frac{75}{71} & \frac{414}{71} & -\frac{189}{71} \\ 0 & 0 & -\frac{18}{23} & 0 & \frac{54}{23} \\ 0 & 0 & 0 & 6 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_5\}$ and the leading variables are $\{v_1, v_2, v_3, v_4\}$. Let $v_5 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = 0, v_3 = 3t, v_4 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ 0 \\ 3t \\ t \\ t \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 3t \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ 0 \\ 3t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ 0 \\ 3t \\ t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 0$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 139 & -14 & -52 & -14 & 28 \\ -22 & 5 & 7 & 8 & -7 \\ 370 & -38 & -139 & -38 & 76 \\ 152 & -16 & -59 & -13 & 35 \\ 95 & -10 & -38 & -7 & 23 \end{bmatrix} - (0)\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 139 & -14 & -52 & -14 & 28 \\ -22 & 5 & 7 & 8 & -7 \\ 370 & -38 & -139 & -38 & 76 \\ 152 & -16 & -59 & -13 & 35 \\ 95 & -10 & -38 & -7 & 23 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccccc|c} 139 & -14 & -52 & -14 & 28 & 0 \\ -22 & 5 & 7 & 8 & -7 & 0 \\ 370 & -38 & -139 & -38 & 76 & 0 \\ 152 & -16 & -59 & -13 & 35 & 0 \\ 95 & -10 & -38 & -7 & 23 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{22R_1}{139} \Longrightarrow \left[\begin{array}{ccccc|c} 139 & -14 & -52 & -14 & 28 & 0 \\ 0 & \frac{387}{139} & -\frac{171}{139} & \frac{804}{139} & -\frac{357}{139} & 0 \\ 370 & -38 & -139 & -38 & 76 & 0 \\ 152 & -16 & -59 & -13 & 35 & 0 \\ 95 & -10 & -38 & -7 & 23 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{370R_1}{139} \Longrightarrow \left[\begin{array}{ccccc|c} 139 & -14 & -52 & -14 & 28 & 0 \\ 0 & \frac{387}{139} & -\frac{171}{139} & \frac{804}{139} & -\frac{357}{139} & 0 \\ 0 & -\frac{102}{139} & -\frac{81}{139} & -\frac{102}{139} & \frac{204}{139} & 0 \\ 152 & -16 & -59 & -13 & 35 & 0 \\ 95 & -10 & -38 & -7 & 23 & 0 \end{array}\right]$$

$$R_4 = R_4 - \frac{152R_1}{139} \Longrightarrow \left[\begin{array}{ccccc|c} 139 & -14 & -52 & -14 & 28 & 0 \\ 0 & \frac{387}{139} & -\frac{171}{139} & \frac{804}{139} & -\frac{357}{139} & 0 \\ 0 & -\frac{102}{139} & -\frac{81}{139} & -\frac{102}{139} & \frac{204}{139} & 0 \\ 0 & -\frac{96}{139} & -\frac{297}{139} & \frac{321}{139} & \frac{609}{139} & 0 \\ 95 & -10 & -38 & -7 & 23 & 0 \end{array}\right]$$

$$R_5 = R_5 - \frac{95R_1}{139} \Longrightarrow \left[\begin{array}{ccccc|c} 139 & -14 & -52 & -14 & 28 & 0 \\ 0 & \frac{387}{139} & -\frac{171}{139} & \frac{804}{139} & -\frac{357}{139} & 0 \\ 0 & -\frac{102}{139} & -\frac{81}{139} & -\frac{102}{139} & \frac{204}{139} & 0 \\ 0 & -\frac{96}{139} & -\frac{297}{139} & \frac{321}{139} & \frac{609}{139} & 0 \\ 0 & -\frac{60}{139} & -\frac{342}{139} & \frac{357}{139} & \frac{537}{139} & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{34R_2}{129} \Longrightarrow \left[\begin{array}{ccccc|c} 139 & -14 & -52 & -14 & 28 & 0 \\ 0 & \frac{387}{139} & -\frac{171}{139} & \frac{804}{139} & -\frac{357}{139} & 0 \\ 0 & 0 & -\frac{39}{43} & \frac{34}{43} & \frac{34}{43} & 0 \\ 0 & -\frac{96}{139} & -\frac{297}{139} & \frac{321}{139} & \frac{609}{139} & 0 \\ 0 & -\frac{60}{139} & -\frac{342}{139} & \frac{357}{139} & \frac{537}{139} & 0 \end{array}\right]$$

$$R_4 = R_4 + \frac{32R_2}{129} \Longrightarrow \left[\begin{array}{ccccc|c} 139 & -14 & -52 & -14 & 28 & 0 \\ 0 & \frac{387}{139} & -\frac{171}{139} & \frac{804}{139} & -\frac{357}{139} & 0 \\ 0 & 0 & -\frac{39}{43} & \frac{34}{43} & \frac{34}{43} & 0 \\ 0 & 0 & -\frac{105}{43} & \frac{161}{43} & \frac{161}{43} & 0 \\ 0 & -\frac{60}{139} & -\frac{342}{139} & \frac{357}{139} & \frac{537}{139} & 0 \end{array}\right]$$

$$R_5 = R_5 + \frac{20R_2}{129} \Longrightarrow \left[\begin{array}{ccccc|c} 139 & -14 & -52 & -14 & 28 & 0 \\ 0 & \frac{387}{139} & -\frac{171}{139} & \frac{804}{139} & -\frac{357}{139} & 0 \\ 0 & 0 & -\frac{39}{43} & \frac{34}{43} & \frac{34}{43} & 0 \\ 0 & 0 & -\frac{105}{43} & \frac{161}{43} & \frac{161}{43} & 0 \\ 0 & 0 & -\frac{114}{43} & \frac{149}{43} & \frac{149}{43} & 0 \end{array}\right]$$

$$R_4 = R_4 - \frac{35R_3}{13} \Longrightarrow \left[\begin{array}{ccccc|c} 139 & -14 & -52 & -14 & 28 & 0 \\ 0 & \frac{387}{139} & -\frac{171}{139} & \frac{804}{139} & -\frac{357}{139} & 0 \\ 0 & 0 & -\frac{39}{43} & \frac{34}{43} & \frac{34}{43} & 0 \\ 0 & 0 & 0 & \frac{21}{13} & \frac{21}{13} & 0 \\ 0 & 0 & -\frac{114}{43} & \frac{149}{43} & \frac{149}{43} & 0 \end{array}\right]$$

$$R_5 = R_5 - \frac{38R_3}{13} \Longrightarrow \left[\begin{array}{ccccc|c} 139 & -14 & -52 & -14 & 28 & 0 \\ 0 & \frac{387}{139} & -\frac{171}{139} & \frac{804}{139} & -\frac{357}{139} & 0 \\ 0 & 0 & -\frac{39}{43} & \frac{34}{43} & \frac{34}{43} & 0 \\ 0 & 0 & 0 & \frac{21}{13} & \frac{21}{13} & 0 \\ 0 & 0 & 0 & \frac{15}{13} & \frac{15}{13} & 0 \end{array}\right]$$

$$R_5 = R_5 - \frac{5R_4}{7} \Longrightarrow \left[\begin{array}{ccccc|c} 139 & -14 & -52 & -14 & 28 & 0 \\ 0 & \frac{387}{139} & -\frac{171}{139} & \frac{804}{139} & -\frac{357}{139} & 0 \\ 0 & 0 & -\frac{39}{43} & \frac{34}{43} & \frac{34}{43} & 0 \\ 0 & 0 & 0 & \frac{21}{13} & \frac{21}{13} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 139 & -14 & -52 & -14 & 28 \\ 0 & \frac{387}{139} & -\frac{171}{139} & \frac{804}{139} & -\frac{357}{139} \\ 0 & 0 & -\frac{39}{43} & \frac{34}{43} & \frac{34}{43} \\ 0 & 0 & 0 & \frac{21}{13} & \frac{21}{13} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_5\}$ and the leading variables are $\{v_1, v_2, v_3, v_4\}$. Let $v_5 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_2 = 3t, v_3 = 0, v_4 = -t\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ 3t \\ 0 \\ -t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 3t \\ 0 \\ -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ 3t \\ 0 \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 3 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ 3t \\ 0 \\ -t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = 3$

We need to solve $A\vec{v} = \lambda \vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 139 & -14 & -52 & -14 & 28 \\ -22 & 5 & 7 & 8 & -7 \\ 370 & -38 & -139 & -38 & 76 \\ 152 & -16 & -59 & -13 & 35 \\ 95 & -10 & -38 & -7 & 23 \end{bmatrix} - (3) \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 136 & -14 & -52 & -14 & 28 \\ -22 & 2 & 7 & 8 & -7 \\ 370 & -38 & -142 & -38 & 76 \\ 152 & -16 & -59 & -16 & 35 \\ 95 & -10 & -38 & -7 & 20 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccccc|c} 136 & -14 & -52 & -14 & 28 & 0 \\ -22 & 2 & 7 & 8 & -7 & 0 \\ 370 & -38 & -142 & -38 & 76 & 0 \\ 152 & -16 & -59 & -16 & 35 & 0 \\ 95 & -10 & -38 & -7 & 20 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{11R_1}{68} \implies \left[\begin{array}{ccccc|c} 136 & -14 & -52 & -14 & 28 & 0 \\ 0 & -\frac{9}{34} & -\frac{24}{17} & \frac{195}{34} & -\frac{42}{17} & 0 \\ 370 & -38 & -142 & -38 & 76 & 0 \\ 152 & -16 & -59 & -16 & 35 & 0 \\ 95 & -10 & -38 & -7 & 20 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{185R_1}{68} \implies \left[\begin{array}{ccccc|c} 136 & -14 & -52 & -14 & 28 & 0 \\ 0 & -\frac{9}{34} & -\frac{24}{17} & \frac{195}{34} & -\frac{42}{17} & 0 \\ 0 & \frac{3}{34} & -\frac{9}{17} & \frac{3}{34} & -\frac{3}{17} & 0 \\ 152 & -16 & -59 & -16 & 35 & 0 \\ 95 & -10 & -38 & -7 & 20 & 0 \end{array}\right]$$

$$R_4 = R_4 - \frac{19R_1}{17} \Longrightarrow \left[\begin{array}{ccccc|c} 136 & -14 & -52 & -14 & 28 & 0 \\ 0 & -\frac{9}{34} & -\frac{24}{17} & \frac{195}{34} & -\frac{42}{17} & 0 \\ 0 & \frac{3}{34} & -\frac{9}{17} & \frac{3}{34} & -\frac{3}{17} & 0 \\ 0 & -\frac{6}{17} & -\frac{15}{17} & -\frac{6}{17} & \frac{63}{17} & 0 \\ 95 & -10 & -38 & -7 & 20 & 0 \end{array}\right]$$

$$R_5 = R_5 - \frac{95R_1}{136} \Longrightarrow \left[\begin{array}{ccccc|c} 136 & -14 & -52 & -14 & 28 & 0 \\ 0 & -\frac{9}{34} & -\frac{24}{17} & \frac{195}{34} & -\frac{42}{17} & 0 \\ 0 & \frac{3}{34} & -\frac{9}{17} & \frac{3}{34} & -\frac{3}{17} & 0 \\ 0 & -\frac{6}{17} & -\frac{15}{17} & -\frac{6}{17} & \frac{63}{17} & 0 \\ 0 & -\frac{15}{68} & -\frac{57}{34} & \frac{189}{68} & \frac{15}{34} & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{R_2}{3} \Longrightarrow \left[\begin{array}{ccccc|c} 136 & -14 & -52 & -14 & 28 & 0 \\ 0 & -\frac{9}{34} & -\frac{24}{17} & \frac{195}{34} & -\frac{42}{17} & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & -\frac{6}{17} & -\frac{15}{17} & -\frac{6}{17} & \frac{63}{17} & 0 \\ 0 & -\frac{15}{68} & -\frac{57}{34} & \frac{189}{68} & \frac{15}{34} & 0 \end{array}\right]$$

$$R_4 = R_4 - \frac{4R_2}{3} \Longrightarrow \left[\begin{array}{ccccc|c} 136 & -14 & -52 & -14 & 28 & 0 \\ 0 & -\frac{9}{34} & -\frac{24}{17} & \frac{195}{34} & -\frac{42}{17} & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 1 & -8 & 7 & 0 \\ 0 & -\frac{15}{68} & -\frac{57}{34} & \frac{189}{68} & \frac{15}{34} & 0 \end{array}\right]$$

$$R_5 = R_5 - \frac{5R_2}{6} \Longrightarrow \left[\begin{array}{ccccc|c} 136 & -14 & -52 & -14 & 28 & 0 \\ 0 & -\frac{9}{34} & -\frac{24}{17} & \frac{195}{34} & -\frac{42}{17} & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 1 & -8 & 7 & 0 \\ 0 & 0 & -\frac{1}{2} & -2 & \frac{5}{2} & 0 \end{array}\right]$$

$$R_4 = R_4 + R_3 \Longrightarrow \left[\begin{array}{ccccc|c} 136 & -14 & -52 & -14 & 28 & 0 \\ 0 & -\frac{9}{34} & -\frac{24}{17} & \frac{195}{34} & -\frac{42}{17} & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -6 & 6 & 0 \\ 0 & 0 & -\frac{1}{2} & -2 & \frac{5}{2} & 0 \end{array}\right]$$

$$R_5 = R_5 - \frac{R_3}{2} \Longrightarrow \left[\begin{array}{ccccc|c} 136 & -14 & -52 & -14 & 28 & 0 \\ 0 & -\frac{9}{34} & -\frac{24}{17} & \frac{195}{34} & -\frac{42}{17} & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -6 & 6 & 0 \\ 0 & 0 & 0 & -3 & 3 & 0 \end{array}\right]$$

$$R_5 = R_5 - \frac{R_4}{2} \Longrightarrow \left[\begin{array}{ccccc|c} 136 & -14 & -52 & -14 & 28 & 0 \\ 0 & -\frac{9}{34} & -\frac{24}{17} & \frac{195}{34} & -\frac{42}{17} & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -6 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 136 & -14 & -52 & -14 & 28 \\ 0 & -\frac{9}{34} & -\frac{24}{17} & \frac{195}{34} & -\frac{42}{17} \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -6 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_5\}$ and the leading variables are $\{v_1, v_2, v_3, v_4\}$. Let $v_5 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = 7t, v_3 = t, v_4 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ 7t \\ t \\ t \\ t \end{bmatrix} = \begin{bmatrix} t \\ 7t \\ t \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ 7t \\ t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 7 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ 7t \\ t \\ t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_4 = 6$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 139 & -14 & -52 & -14 & 28 \\ -22 & 5 & 7 & 8 & -7 \\ 370 & -38 & -139 & -38 & 76 \\ 152 & -16 & -59 & -13 & 35 \\ 95 & -10 & -38 & -7 & 23 \end{bmatrix} - (6)\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 133 & -14 & -52 & -14 & 28 \\ -22 & -1 & 7 & 8 & -7 \\ 370 & -38 & -145 & -38 & 76 \\ 152 & -16 & -59 & -19 & 35 \\ 95 & -10 & -38 & -7 & 17 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccccc|c} 133 & -14 & -52 & -14 & 28 & 0 \\ -22 & -1 & 7 & 8 & -7 & 0 \\ 370 & -38 & -145 & -38 & 76 & 0 \\ 152 & -16 & -59 & -19 & 35 & 0 \\ 95 & -10 & -38 & -7 & 17 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{22R_1}{133} \implies \left[\begin{array}{ccccc|c} 133 & -14 & -52 & -14 & 28 & 0 \\ 0 & -\frac{63}{19} & -\frac{213}{133} & \frac{108}{19} & -\frac{45}{19} & 0 \\ 370 & -38 & -145 & -38 & 76 & 0 \\ 152 & -16 & -59 & -19 & 35 & 0 \\ 95 & -10 & -38 & -7 & 17 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{370R_1}{133} \implies \left[\begin{array}{ccccc|c} 133 & -14 & -52 & -14 & 28 & 0 \\ 0 & -\frac{63}{19} & -\frac{213}{133} & \frac{108}{19} & -\frac{45}{19} & 0 \\ 0 & \frac{18}{19} & -\frac{45}{133} & \frac{18}{19} & -\frac{36}{19} & 0 \\ 152 & -16 & -59 & -19 & 35 & 0 \\ 95 & -10 & -38 & -7 & 17 & 0 \end{array}\right]$$

$$R_4 = R_4 - \frac{8R_1}{7} \implies \left[\begin{array}{ccccc|c} 133 & -14 & -52 & -14 & 28 & 0 \\ 0 & -\frac{63}{19} & -\frac{213}{133} & \frac{108}{19} & -\frac{45}{19} & 0 \\ 0 & \frac{18}{19} & -\frac{45}{133} & \frac{18}{19} & -\frac{36}{19} & 0 \\ 0 & 0 & \frac{3}{7} & -3 & 3 & 0 \\ 95 & -10 & -38 & -7 & 17 & 0 \end{array}\right]$$

$$R_5 = R_5 - \frac{5R_1}{7} \implies \left[\begin{array}{ccccc|c} 133 & -14 & -52 & -14 & 28 & 0 \\ 0 & -\frac{63}{19} & -\frac{213}{133} & \frac{108}{19} & -\frac{45}{19} & 0 \\ 0 & \frac{18}{19} & -\frac{45}{133} & \frac{18}{19} & -\frac{36}{19} & 0 \\ 0 & 0 & \frac{3}{7} & -3 & 3 & 0 \\ 0 & 0 & -\frac{6}{7} & 3 & -3 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{2R_2}{7} \implies \left[\begin{array}{ccccc|c} 133 & -14 & -52 & -14 & 28 & 0 \\ 0 & -\frac{63}{19} & -\frac{213}{133} & \frac{108}{19} & -\frac{45}{19} & 0 \\ 0 & 0 & -\frac{39}{49} & \frac{18}{7} & -\frac{18}{7} & 0 \\ 0 & 0 & \frac{3}{7} & -3 & 3 & 0 \\ 0 & 0 & -\frac{6}{7} & 3 & -3 & 0 \end{array}\right]$$

$$R_4 = R_4 + \frac{7R_3}{13} \implies \left[\begin{array}{ccccc|c} 133 & -14 & -52 & -14 & 28 & 0 \\ 0 & -\frac{63}{19} & -\frac{213}{133} & \frac{108}{19} & -\frac{45}{19} & 0 \\ 0 & 0 & -\frac{39}{49} & \frac{18}{7} & -\frac{18}{7} & 0 \\ 0 & 0 & 0 & -\frac{21}{13} & \frac{21}{13} & 0 \\ 0 & 0 & -\frac{6}{7} & 3 & -3 & 0 \end{array}\right]$$

$$R_5 = R_5 - \frac{14R_3}{13} \implies \left[\begin{array}{ccccc|c} 133 & -14 & -52 & -14 & 28 & 0 \\ 0 & -\frac{63}{19} & -\frac{213}{133} & \frac{108}{19} & -\frac{45}{19} & 0 \\ 0 & 0 & -\frac{39}{49} & \frac{18}{7} & -\frac{18}{7} & 0 \\ 0 & 0 & 0 & -\frac{21}{13} & \frac{21}{13} & 0 \\ 0 & 0 & 0 & \frac{3}{13} & -\frac{3}{13} & 0 \end{array}\right]$$

$$R_5 = R_5 + \frac{R_4}{7} \Longrightarrow \left[\begin{array}{ccccc|c} 133 & -14 & -52 & -14 & 28 & 0 \\ 0 & -\frac{63}{19} & -\frac{213}{133} & \frac{108}{19} & -\frac{45}{19} & 0 \\ 0 & 0 & -\frac{39}{49} & \frac{18}{7} & -\frac{18}{7} & 0 \\ 0 & 0 & 0 & -\frac{21}{13} & \frac{21}{13} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 133 & -14 & -52 & -14 & 28 \\ 0 & -\frac{63}{19} & -\frac{213}{133} & \frac{108}{19} & -\frac{45}{19} \\ 0 & 0 & -\frac{39}{49} & \frac{18}{7} & -\frac{18}{7} \\ 0 & 0 & 0 & -\frac{21}{13} & \frac{21}{13} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_5\}$ and the leading variables are $\{v_1, v_2, v_3, v_4\}$. Let $v_5 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_2 = t, v_3 = 0, v_4 = t\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ t \\ 0 \\ t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ t \\ 0 \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ t \\ 0 \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ t \\ 0 \\ t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_5 = 9$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 139 & -14 & -52 & -14 & 28 \\ -22 & 5 & 7 & 8 & -7 \\ 370 & -38 & -139 & -38 & 76 \\ 152 & -16 & -59 & -13 & 35 \\ 95 & -10 & -38 & -7 & 23 \end{bmatrix} - (9) \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 130 & -14 & -52 & -14 & 28 \\ -22 & -4 & 7 & 8 & -7 \\ 370 & -38 & -148 & -38 & 76 \\ 152 & -16 & -59 & -22 & 35 \\ 95 & -10 & -38 & -7 & 14 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccccc|c} 130 & -14 & -52 & -14 & 28 & 0 \\ -22 & -4 & 7 & 8 & -7 & 0 \\ 370 & -38 & -148 & -38 & 76 & 0 \\ 152 & -16 & -59 & -22 & 35 & 0 \\ 95 & -10 & -38 & -7 & 14 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{11R_1}{65} \Longrightarrow \left[\begin{array}{ccccc|c} 130 & -14 & -52 & -14 & 28 & 0 \\ 0 & -\frac{414}{65} & -\frac{9}{5} & \frac{366}{65} & -\frac{147}{65} & 0 \\ 370 & -38 & -148 & -38 & 76 & 0 \\ 152 & -16 & -59 & -22 & 35 & 0 \\ 95 & -10 & -38 & -7 & 14 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{37R_1}{13} \Longrightarrow \left[\begin{array}{ccccc|c} 130 & -14 & -52 & -14 & 28 & 0 \\ 0 & -\frac{414}{65} & -\frac{9}{5} & \frac{366}{65} & -\frac{147}{65} & 0 \\ 0 & \frac{24}{13} & 0 & \frac{24}{13} & -\frac{48}{13} & 0 \\ 152 & -16 & -59 & -22 & 35 & 0 \\ 95 & -10 & -38 & -7 & 14 & 0 \end{array}\right]$$

$$R_4 = R_4 - \frac{76R_1}{65} \Longrightarrow \left[\begin{array}{ccccc|c} 130 & -14 & -52 & -14 & 28 & 0 \\ 0 & -\frac{414}{65} & -\frac{9}{5} & \frac{366}{65} & -\frac{147}{65} & 0 \\ 0 & \frac{24}{13} & 0 & \frac{24}{13} & -\frac{48}{13} & 0 \\ 0 & \frac{24}{65} & \frac{9}{5} & -\frac{366}{65} & \frac{147}{65} & 0 \\ 95 & -10 & -38 & -7 & 14 & 0 \end{array}\right]$$

$$R_5 = R_5 - \frac{19R_1}{26} \Longrightarrow \left[\begin{array}{ccccc|c} 130 & -14 & -52 & -14 & 28 & 0 \\ 0 & -\frac{414}{65} & -\frac{9}{5} & \frac{366}{65} & -\frac{147}{65} & 0 \\ 0 & \frac{24}{13} & 0 & \frac{24}{13} & -\frac{48}{13} & 0 \\ 0 & \frac{24}{65} & \frac{9}{5} & -\frac{366}{65} & \frac{147}{65} & 0 \\ 0 & \frac{3}{13} & 0 & \frac{42}{13} & -\frac{84}{13} & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{20R_2}{69} \Longrightarrow \left[\begin{array}{ccccc|c} 130 & -14 & -52 & -14 & 28 & 0 \\ 0 & -\frac{414}{65} & -\frac{9}{5} & \frac{366}{65} & -\frac{147}{65} & 0 \\ 0 & 0 & -\frac{12}{23} & \frac{80}{23} & -\frac{100}{23} & 0 \\ 0 & \frac{24}{65} & \frac{9}{5} & -\frac{366}{65} & \frac{147}{65} & 0 \\ 0 & \frac{3}{13} & 0 & \frac{42}{13} & -\frac{84}{13} & 0 \end{array}\right]$$

$$R_4 = R_4 + \frac{4R_2}{69} \Longrightarrow \left[\begin{array}{ccccc|c} 130 & -14 & -52 & -14 & 28 & 0 \\ 0 & -\frac{414}{65} & -\frac{9}{5} & \frac{366}{65} & -\frac{147}{65} & 0 \\ 0 & 0 & -\frac{12}{23} & \frac{80}{23} & -\frac{100}{23} & 0 \\ 0 & 0 & \frac{39}{23} & -\frac{122}{23} & \frac{49}{23} & 0 \\ 0 & \frac{3}{13} & 0 & \frac{42}{13} & -\frac{84}{13} & 0 \end{array}\right]$$

$$R_5 = R_5 + \frac{5R_2}{138} \Longrightarrow \left[\begin{array}{ccccc|c} 130 & -14 & -52 & -14 & 28 & 0 \\ 0 & -\frac{414}{65} & -\frac{9}{5} & \frac{366}{65} & -\frac{147}{65} & 0 \\ 0 & 0 & -\frac{12}{23} & \frac{80}{23} & -\frac{100}{23} & 0 \\ 0 & 0 & \frac{39}{23} & -\frac{122}{23} & \frac{49}{23} & 0 \\ 0 & 0 & -\frac{3}{46} & \frac{79}{23} & -\frac{301}{46} & 0 \end{array}\right]$$

$$R_4 = R_4 + \frac{13R_3}{4} \Longrightarrow \left[\begin{array}{ccccc|c} 130 & -14 & -52 & -14 & 28 & 0 \\ 0 & -\frac{414}{65} & -\frac{9}{5} & \frac{366}{65} & -\frac{147}{65} & 0 \\ 0 & 0 & -\frac{12}{23} & \frac{80}{23} & -\frac{100}{23} & 0 \\ 0 & 0 & 0 & 6 & -12 & 0 \\ 0 & 0 & -\frac{3}{46} & \frac{79}{23} & -\frac{301}{46} & 0 \end{array}\right]$$

$$R_5 = R_5 - \frac{R_3}{8} \Longrightarrow \left[\begin{array}{ccccc|c} 130 & -14 & -52 & -14 & 28 & 0 \\ 0 & -\frac{414}{65} & -\frac{9}{5} & \frac{366}{65} & -\frac{147}{65} & 0 \\ 0 & 0 & -\frac{12}{23} & \frac{80}{23} & -\frac{100}{23} & 0 \\ 0 & 0 & 0 & 6 & -12 & 0 \\ 0 & 0 & 0 & 3 & -6 & 0 \end{array}\right]$$

$$R_5 = R_5 - \frac{R_4}{2} \Longrightarrow \left[\begin{array}{ccccc|c} 130 & -14 & -52 & -14 & 28 & 0 \\ 0 & -\frac{414}{65} & -\frac{9}{5} & \frac{366}{65} & -\frac{147}{65} & 0 \\ 0 & 0 & -\frac{12}{23} & \frac{80}{23} & -\frac{100}{23} & 0 \\ 0 & 0 & 0 & 6 & -12 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 130 & -14 & -52 & -14 & 28 \\ 0 & -\frac{414}{65} & -\frac{9}{5} & \frac{366}{65} & -\frac{147}{65} \\ 0 & 0 & -\frac{12}{23} & \frac{80}{23} & -\frac{100}{23} \\ 0 & 0 & 0 & 6 & -12 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_5\}$ and the leading variables are $\{v_1, v_2, v_3, v_4\}$. Let $v_5 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2t, v_2 = 0, v_3 = 5t, v_4 = 2t\}$

Hence the solution is

$$\begin{bmatrix} 2t \\ 0 \\ 5t \\ 2t \\ t \end{bmatrix} = \begin{bmatrix} 2t \\ 0 \\ 5t \\ 2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 2t \\ 0 \\ 5t \\ 2t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 0 \\ 5 \\ 2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 2t \\ 0 \\ 5t \\ 2t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 5 \\ 2 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated

with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
0	1	1	No	$\begin{bmatrix} 0 \\ 3 \\ 0 \\ -1 \\ 1 \end{bmatrix}$
6	1	1	No	$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$
9	1	1	No	$\begin{bmatrix} 2 \\ 0 \\ 5 \\ 2 \\ 1 \end{bmatrix}$
3	1	1	No	$\begin{bmatrix} 1 \\ 7 \\ 1 \\ 1 \\ 1 \end{bmatrix}$
-3	1	1	No	$\begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 0 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{0} \\
&= \begin{bmatrix} 0 \\ 3 \\ 0 \\ -1 \\ 1 \end{bmatrix} e^{0}
\end{aligned}
$$

Since eigenvalue 6 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{6t} \\
&= \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} e^{6t}
\end{aligned}
$$

Since eigenvalue 9 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_3(t) &= \vec{v}_3 e^{9t} \\
&= \begin{bmatrix} 2 \\ 0 \\ 5 \\ 2 \\ 1 \end{bmatrix} e^{9t}
\end{aligned}
$$

Since eigenvalue 3 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_4(t) &= \vec{v}_4 e^{3t} \\ &= \begin{bmatrix} 1 \\ 7 \\ 1 \\ 1 \\ 1 \end{bmatrix} e^{3t}\end{aligned}$$

Since eigenvalue -3 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_5(t) &= \vec{v}_5 e^{-3t} \\ &= \begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \\ 1 \end{bmatrix} e^{-3t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t) + c_4 \vec{x}_4(t) + c_5 \vec{x}_5(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ x_5(t) \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ 3 \\ 0 \\ -1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ e^{6t} \\ 0 \\ e^{6t} \\ e^{6t} \end{bmatrix} + c_3 \begin{bmatrix} 2\,e^{9t} \\ 0 \\ 5\,e^{9t} \\ 2\,e^{9t} \\ e^{9t} \end{bmatrix} + c_4 \begin{bmatrix} e^{3t} \\ 7\,e^{3t} \\ e^{3t} \\ e^{3t} \\ e^{3t} \end{bmatrix} + c_5 \begin{bmatrix} e^{-3t} \\ 0 \\ 3\,e^{-3t} \\ e^{-3t} \\ e^{-3t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ x_5(t) \end{bmatrix} = \begin{bmatrix} 2c_3 e^{9t} + c_4 e^{3t} + c_5 e^{-3t} \\ 3c_1 + c_2 e^{6t} + 7c_4 e^{3t} \\ 5c_3 e^{9t} + c_4 e^{3t} + 3c_5 e^{-3t} \\ -c_1 + c_2 e^{6t} + 2c_3 e^{9t} + c_4 e^{3t} + c_5 e^{-3t} \\ c_1 + c_2 e^{6t} + c_3 e^{9t} + c_4 e^{3t} + c_5 e^{-3t} \end{bmatrix}$$

4.38.3 Maple step by step solution

Let's solve

$[x_1'(t) = 139x_1(t) - 14x_2(t) - 52x_3(t) - 14x_4(t) + 28x_5(t), x_2'(t) = -22x_1(t) + 5x_2(t) + 7x_3(t) + 8$

- Define vector

$$\underline{x^{\rightarrow}}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ x_5(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x^{\rightarrow}}'(t) = \begin{bmatrix} 139 & -14 & -52 & -14 & 28 \\ -22 & 5 & 7 & 8 & -7 \\ 370 & -38 & -139 & -38 & 76 \\ 152 & -16 & -59 & -13 & 35 \\ 95 & -10 & -38 & -7 & 23 \end{bmatrix} \cdot \underline{x^{\rightarrow}}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{x^{\rightarrow}}'(t) = \begin{bmatrix} 139 & -14 & -52 & -14 & 28 \\ -22 & 5 & 7 & 8 & -7 \\ 370 & -38 & -139 & -38 & 76 \\ 152 & -16 & -59 & -13 & 35 \\ 95 & -10 & -38 & -7 & 23 \end{bmatrix} \cdot \underline{x^{\rightarrow}}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 139 & -14 & -52 & -14 & 28 \\ -22 & 5 & 7 & 8 & -7 \\ 370 & -38 & -139 & -38 & 76 \\ 152 & -16 & -59 & -13 & 35 \\ 95 & -10 & -38 & -7 & 23 \end{bmatrix}$$

- Rewrite the system as

$$\underline{\vec{x}}'(t) = A \cdot \underline{\vec{x}}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-3, \begin{bmatrix}1\\0\\3\\1\\1\end{bmatrix}\right], \left[0, \begin{bmatrix}0\\3\\0\\-1\\1\end{bmatrix}\right], \left[3, \begin{bmatrix}1\\7\\1\\1\\1\end{bmatrix}\right], \left[6, \begin{bmatrix}0\\1\\0\\1\\1\end{bmatrix}\right], \left[9, \begin{bmatrix}2\\0\\5\\2\\1\end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-3, \begin{bmatrix}1\\0\\3\\1\\1\end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_1 = \mathrm{e}^{-3t} \cdot \begin{bmatrix}1\\0\\3\\1\\1\end{bmatrix}$$

- Consider eigenpair

$$\left[0, \begin{bmatrix}0\\3\\0\\-1\\1\end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_2 = \begin{bmatrix} 0 \\ 3 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[3, \begin{bmatrix} 1 \\ 7 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_3 = e^{3t} \cdot \begin{bmatrix} 1 \\ 7 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[6, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_{4} = \mathrm{e}^{6t} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[9, \begin{bmatrix} 2 \\ 0 \\ 5 \\ 2 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_{5} = \mathrm{e}^{9t} \cdot \begin{bmatrix} 2 \\ 0 \\ 5 \\ 2 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\underline{\vec{x}} = c_1 \underline{\vec{x}}_{1} + c_2 \underline{\vec{x}}_{2} + c_3 \underline{\vec{x}}_{3} + c_4 \underline{\vec{x}}_{4} + c_5 \underline{\vec{x}}_{5}$$

- Substitute solutions into the general solution

$$\underline{\vec{x}} = c_1 \mathrm{e}^{-3t} \cdot \begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \\ 1 \end{bmatrix} + c_3 \mathrm{e}^{3t} \cdot \begin{bmatrix} 1 \\ 7 \\ 1 \\ 1 \\ 1 \end{bmatrix} + c_4 \mathrm{e}^{6t} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + c_5 \mathrm{e}^{9t} \cdot \begin{bmatrix} 2 \\ 0 \\ 5 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 3c_2 \\ 0 \\ -c_2 \\ c_2 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ x_5(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{-3t} + c_3 e^{3t} + 2c_5 e^{9t} \\ 7c_3 e^{3t} + c_4 e^{6t} + 3c_2 \\ 3c_1 e^{-3t} + c_3 e^{3t} + 5c_5 e^{9t} \\ c_1 e^{-3t} + c_3 e^{3t} + c_4 e^{6t} + 2c_5 e^{9t} - c_2 \\ c_1 e^{-3t} + c_3 e^{3t} + c_4 e^{6t} + c_5 e^{9t} + c_2 \end{bmatrix}$$

- Solution to the system of ODEs

  $\{x_1(t) = c_1 e^{-3t} + c_3 e^{3t} + 2c_5 e^{9t}, x_2(t) = 7c_3 e^{3t} + c_4 e^{6t} + 3c_2, x_3(t) = 3c_1 e^{-3t} + c_3 e^{3t} + 5c_5 e^{9t}, x_4(t)$

✓ Solution by Maple

Time used: 0.109 (sec). Leaf size: 132

```
dsolve([diff(x__1(t),t)=139*x__1(t)-14*x__2(t)-52*x__3(t)-14*x__4(t)+28*x__5(t),diff(x__2(t)
```

$$x_1(t) = c_3 e^{3t} + c_4 e^{9t} + c_5 e^{-3t}$$
$$x_2(t) = \frac{e^{6t} c_1}{6} + 7c_3 e^{3t} + c_2$$
$$x_3(t) = c_3 e^{3t} + \frac{5c_4 e^{9t}}{2} + 3c_5 e^{-3t}$$
$$x_4(t) = c_3 e^{3t} + c_4 e^{9t} + c_5 e^{-3t} + \frac{e^{6t} c_1}{6} - \frac{c_2}{3}$$
$$x_5(t) = c_3 e^{3t} + \frac{e^{6t} c_1}{6} + \frac{c_4 e^{9t}}{2} + c_5 e^{-3t} + \frac{c_2}{3}$$

✓ Solution by Mathematica

Time used: 0.052 (sec). Leaf size: 2676

```
DSolve[{x1'[t]==139*x1[t]-14*x2[t]-52*x3[t]-14*x4[t]+28*x5[t],x2'[t]==-22*x1[t]+5*x2[t]+7*x3
```

Too large to display

4.39 problem problem 50

Internal problem ID [353]
Internal file name [`OUTPUT/353_Sunday_June_05_2022_01_39_28_AM_35276350/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.3, The eigenvalue method for linear systems. Page 395
Problem number: problem 50.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= 9x_1(t) + 13x_2(t) - 13x_6(t) \\
x_2'(t) &= -14x_1(t) + 19x_2(t) - 10x_3(t) - 20x_4(t) + 10x_5(t) + 4x_6(t) \\
x_3'(t) &= -30x_1(t) + 12x_2(t) - 7x_3(t) - 30x_4(t) + 12x_5(t) + 18x_6(t) \\
x_4'(t) &= -12x_1(t) + 10x_2(t) - 10x_3(t) - 9x_4(t) + 10x_5(t) + 2x_6(t) \\
x_5'(t) &= 6x_1(t) + 9x_2(t) + 6x_4(t) + 5x_5(t) - 15x_6(t) \\
x_6'(t) &= -14x_1(t) + 23x_2(t) - 10x_3(t) - 20x_4(t) + 10x_5(t)
\end{aligned}
$$

4.39.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \\ x_4'(t) \\ x_5'(t) \\ x_6'(t) \end{bmatrix} = \begin{bmatrix} 9 & 13 & 0 & 0 & 0 & -13 \\ -14 & 19 & -10 & -20 & 10 & 4 \\ -30 & 12 & -7 & -30 & 12 & 18 \\ -12 & 10 & -10 & -9 & 10 & 2 \\ 6 & 9 & 0 & 6 & 5 & -15 \\ -14 & 23 & -10 & -20 & 10 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ x_5(t) \\ x_6(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \mathrm{e}^{9t} & (\mathrm{e}^{13t}-1)\,\mathrm{e}^{-4t} & 0 & 0 & \\ (\mathrm{e}^{16t}-3\,\mathrm{e}^{10t}+2)\,\mathrm{e}^{-7t} & (\mathrm{e}^{16t}+\mathrm{e}^{10t}-1)\,\mathrm{e}^{-7t} & -(\mathrm{e}^{10t}-1)\,\mathrm{e}^{-7t} & -2(\mathrm{e}^{10t}-1)\,\mathrm{e}^{-7t} & (\mathrm{e}^{10\cdots} \\ -(\mathrm{e}^{18t}+\mathrm{e}^{12t}-2)\,\mathrm{e}^{-7t} & (\mathrm{e}^{12t}-1)\,\mathrm{e}^{-7t} & \mathrm{e}^{-7t} & -(\mathrm{e}^{18t}+\mathrm{e}^{12t}-2)\,\mathrm{e}^{-7t} & (\mathrm{e}^{12\cdots} \\ (\mathrm{e}^{18t}-3\,\mathrm{e}^{10t}+2)\,\mathrm{e}^{-7t} & (\mathrm{e}^{10t}-1)\,\mathrm{e}^{-7t} & -(\mathrm{e}^{10t}-1)\,\mathrm{e}^{-7t} & (\mathrm{e}^{18t}-2\,\mathrm{e}^{10t}+2)\,\mathrm{e}^{-7t} & (\mathrm{e}^{10\cdots} \\ \mathrm{e}^{11t}-\mathrm{e}^{5t} & (\mathrm{e}^{9t}-1)\,\mathrm{e}^{-4t} & 0 & \mathrm{e}^{11t}-\mathrm{e}^{5t} & \\ (\mathrm{e}^{16t}-3\,\mathrm{e}^{10t}+2)\,\mathrm{e}^{-7t} & (\mathrm{e}^{16t}+\mathrm{e}^{10t}-\mathrm{e}^{3t}-1)\,\mathrm{e}^{-7t} & -(\mathrm{e}^{10t}-1)\,\mathrm{e}^{-7t} & -2(\mathrm{e}^{10t}-1)\,\mathrm{e}^{-7t} & (\mathrm{e}^{10\cdots} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\vec{x}_h(t) = e^{At}\vec{c}
$$

$$
= \begin{bmatrix}
\mathrm{e}^{9t} & (\mathrm{e}^{13t}-1)\,\mathrm{e}^{-4t} & 0 & 0 & \\
(\mathrm{e}^{16t}-3\,\mathrm{e}^{10t}+2)\,\mathrm{e}^{-7t} & (\mathrm{e}^{16t}+\mathrm{e}^{10t}-1)\,\mathrm{e}^{-7t} & -(\mathrm{e}^{10t}-1)\,\mathrm{e}^{-7t} & -2(\mathrm{e}^{10t}-1)\,\mathrm{e}^{-7t} & (\mathrm{e} \\
-(\mathrm{e}^{18t}+\mathrm{e}^{12t}-2)\,\mathrm{e}^{-7t} & (\mathrm{e}^{12t}-1)\,\mathrm{e}^{-7t} & \mathrm{e}^{-7t} & -(\mathrm{e}^{18t}+\mathrm{e}^{12t}-2)\,\mathrm{e}^{-7t} & (\mathrm{e} \\
(\mathrm{e}^{18t}-3\,\mathrm{e}^{10t}+2)\,\mathrm{e}^{-7t} & (\mathrm{e}^{10t}-1)\,\mathrm{e}^{-7t} & -(\mathrm{e}^{10t}-1)\,\mathrm{e}^{-7t} & (\mathrm{e}^{18t}-2\,\mathrm{e}^{10t}+2)\,\mathrm{e}^{-7t} & (\mathrm{e} \\
\mathrm{e}^{11t}-\mathrm{e}^{5t} & (\mathrm{e}^{9t}-1)\,\mathrm{e}^{-4t} & 0 & \mathrm{e}^{11t}-\mathrm{e}^{5t} & \\
(\mathrm{e}^{16t}-3\,\mathrm{e}^{10t}+2)\,\mathrm{e}^{-7t} & (\mathrm{e}^{16t}+\mathrm{e}^{10t}-\mathrm{e}^{3t}-1)\,\mathrm{e}^{-7t} & -(\mathrm{e}^{10t}-1)\,\mathrm{e}^{-7t} & -2(\mathrm{e}^{10t}-1)\,\mathrm{e}^{-7t} & (\mathrm{e}
\end{bmatrix}
$$

$$
= \begin{bmatrix}
\mathrm{e}^{9t}c_1 + (\mathrm{e}^{13t}-1)\,\mathrm{e}^{-4t}c_2 - (\mathrm{e}^{13t}-1)\,\mathrm{e}^{-4t}c_6 \\
(\mathrm{e}^{16t}-3\,\mathrm{e}^{10t}+2)\,\mathrm{e}^{-7t}c_1 + (\mathrm{e}^{16t}+\mathrm{e}^{10t}-1)\,\mathrm{e}^{-7t}c_2 - (\mathrm{e}^{10t}-1)\,\mathrm{e}^{-7t}c_3 - 2(\mathrm{e}^{10t}-1)\,\mathrm{e}^{-7t}c_4 + \\
-(\mathrm{e}^{18t}+\mathrm{e}^{12t}-2)\,\mathrm{e}^{-7t}c_1 + (\mathrm{e}^{12t}-1)\,\mathrm{e}^{-7t}c_2 + \mathrm{e}^{-7t}c_3 - (\mathrm{e}^{18t}+\mathrm{e}^{12t}-2)\,\mathrm{e}^{-7t}c_4 + \\
(\mathrm{e}^{18t}-3\,\mathrm{e}^{10t}+2)\,\mathrm{e}^{-7t}c_1 + (\mathrm{e}^{10t}-1)\,\mathrm{e}^{-7t}c_2 - (\mathrm{e}^{10t}-1)\,\mathrm{e}^{-7t}c_3 + (\mathrm{e}^{18t}-2\,\mathrm{e}^{10t}+2)\,\mathrm{e}^{-7t}c_4 + \\
(\mathrm{e}^{11t}-\mathrm{e}^{5t})\,c_1 + (\mathrm{e}^{9t}-1)\,\mathrm{e}^{-4t}c_2 + (\mathrm{e}^{11t}-\mathrm{e}^{5t})\,c_4 + \mathrm{e}^{5t}c_5 - (\mathrm{e} \\
(\mathrm{e}^{16t}-3\,\mathrm{e}^{10t}+2)\,\mathrm{e}^{-7t}c_1 + (\mathrm{e}^{16t}+\mathrm{e}^{10t}-\mathrm{e}^{3t}-1)\,\mathrm{e}^{-7t}c_2 - (\mathrm{e}^{10t}-1)\,\mathrm{e}^{-7t}c_3 - 2(\mathrm{e}^{10t}-1)\,\mathrm{e}^{-7t}c_4 +
\end{bmatrix}
$$

$$
= \begin{bmatrix}
((c_1+c_2-c_6)\,\mathrm{e}^{13t} - c_2 + c_6)\,\mathrm{e}^{-4t} \\
\mathrm{e}^{-7t}((-3c_1+c_2-c_3-2c_4+c_5+2c_6)\,\mathrm{e}^{10t} + (c_1+c_2-c_6)\,\mathrm{e}^{16t} + 2c_1 - c_2 + c_3 + 2c_4 - c_5 \\
-((c_1-c_2+c_4-c_5)\,\mathrm{e}^{12t} + (c_1+c_4-c_6)\,\mathrm{e}^{18t} - 2c_1 + c_2 - c_3 - 2c_4 + c_5 + c_6)\,\mathrm{e}^{-7t} \\
((-3c_1+c_2-c_3-2c_4+c_5+2c_6)\,\mathrm{e}^{10t} + (c_1+c_4-c_6)\,\mathrm{e}^{18t} + 2c_1 - c_2 + c_3 + 2c_4 - c_5 - c_6 \\
-\mathrm{e}^{-4t}((c_1-c_2+c_4-c_5)\,\mathrm{e}^{9t} + (-c_1-c_4+c_6)\,\mathrm{e}^{15t} + c_2 - c_6) \\
-3\,\mathrm{e}^{-7t}\Big(\big(c_1 - \frac{c_2}{3} + \frac{c_3}{3} + \frac{2c_4}{3} - \frac{c_5}{3} - \frac{2c_6}{3}\big)\,\mathrm{e}^{10t} + \frac{(-c_1-c_2+c_6)\mathrm{e}^{16t}}{3} + \frac{(c_2-c_6)\mathrm{e}^{3t}}{3} - \frac{2c_1}{3} + \frac{c_2}{3} - \frac{c_3}{3} - \frac{2c_4}{3} +
\end{bmatrix}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

4.39.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \\ x_4'(t) \\ x_5'(t) \\ x_6'(t) \end{bmatrix} = \begin{bmatrix} 9 & 13 & 0 & 0 & 0 & -13 \\ -14 & 19 & -10 & -20 & 10 & 4 \\ -30 & 12 & -7 & -30 & 12 & 18 \\ -12 & 10 & -10 & -9 & 10 & 2 \\ 6 & 9 & 0 & 6 & 5 & -15 \\ -14 & 23 & -10 & -20 & 10 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ x_5(t) \\ x_6(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 9 & 13 & 0 & 0 & 0 & -13 \\ -14 & 19 & -10 & -20 & 10 & 4 \\ -30 & 12 & -7 & -30 & 12 & 18 \\ -12 & 10 & -10 & -9 & 10 & 2 \\ 6 & 9 & 0 & 6 & 5 & -15 \\ -14 & 23 & -10 & -20 & 10 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 9-\lambda & 13 & 0 & 0 & 0 & -13 \\ -14 & 19-\lambda & -10 & -20 & 10 & 4 \\ -30 & 12 & -7-\lambda & -30 & 12 & 18 \\ -12 & 10 & -10 & -9-\lambda & 10 & 2 \\ 6 & 9 & 0 & 6 & 5-\lambda & -15 \\ -14 & 23 & -10 & -20 & 10 & -\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^6 - 17\lambda^5 - 6\lambda^4 + 1138\lambda^3 - 2855\lambda^2 - 14241\lambda + 41580 = 0$$

The roots of the above are the eigenvalues.

$$\begin{aligned}
\lambda_1 &= 11\\
\lambda_2 &= 9\\
\lambda_3 &= -4\\
\lambda_4 &= 5\\
\lambda_5 &= 3\\
\lambda_6 &= -7
\end{aligned}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-4	1	real eigenvalue
3	1	real eigenvalue
5	1	real eigenvalue
-7	1	real eigenvalue
9	1	real eigenvalue
11	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -7$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 9 & 13 & 0 & 0 & 0 & -13 \\ -14 & 19 & -10 & -20 & 10 & 4 \\ -30 & 12 & -7 & -30 & 12 & 18 \\ -12 & 10 & -10 & -9 & 10 & 2 \\ 6 & 9 & 0 & 6 & 5 & -15 \\ -14 & 23 & -10 & -20 & 10 & 0 \end{bmatrix} - (-7)\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 16 & 13 & 0 & 0 & 0 & -13 \\ -14 & 26 & -10 & -20 & 10 & 4 \\ -30 & 12 & 0 & -30 & 12 & 18 \\ -12 & 10 & -10 & -2 & 10 & 2 \\ 6 & 9 & 0 & 6 & 12 & -15 \\ -14 & 23 & -10 & -20 & 10 & 7 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccccc|c} 16 & 13 & 0 & 0 & 0 & -13 & 0 \\ -14 & 26 & -10 & -20 & 10 & 4 & 0 \\ -30 & 12 & 0 & -30 & 12 & 18 & 0 \\ -12 & 10 & -10 & -2 & 10 & 2 & 0 \\ 6 & 9 & 0 & 6 & 12 & -15 & 0 \\ -14 & 23 & -10 & -20 & 10 & 7 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{7R_1}{8} \Longrightarrow \left[\begin{array}{cccccc|c} 16 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & \frac{299}{8} & -10 & -20 & 10 & -\frac{59}{8} & 0 \\ -30 & 12 & 0 & -30 & 12 & 18 & 0 \\ -12 & 10 & -10 & -2 & 10 & 2 & 0 \\ 6 & 9 & 0 & 6 & 12 & -15 & 0 \\ -14 & 23 & -10 & -20 & 10 & 7 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{15R_1}{8} \implies \left[\begin{array}{cccccc|c} 16 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & \frac{299}{8} & -10 & -20 & 10 & -\frac{59}{8} & 0 \\ 0 & \frac{291}{8} & 0 & -30 & 12 & -\frac{51}{8} & 0 \\ -12 & 10 & -10 & -2 & 10 & 2 & 0 \\ 6 & 9 & 0 & 6 & 12 & -15 & 0 \\ -14 & 23 & -10 & -20 & 10 & 7 & 0 \end{array}\right]$$

$$R_4 = R_4 + \frac{3R_1}{4} \implies \left[\begin{array}{cccccc|c} 16 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & \frac{299}{8} & -10 & -20 & 10 & -\frac{59}{8} & 0 \\ 0 & \frac{291}{8} & 0 & -30 & 12 & -\frac{51}{8} & 0 \\ 0 & \frac{79}{4} & -10 & -2 & 10 & -\frac{31}{4} & 0 \\ 6 & 9 & 0 & 6 & 12 & -15 & 0 \\ -14 & 23 & -10 & -20 & 10 & 7 & 0 \end{array}\right]$$

$$R_5 = R_5 - \frac{3R_1}{8} \implies \left[\begin{array}{cccccc|c} 16 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & \frac{299}{8} & -10 & -20 & 10 & -\frac{59}{8} & 0 \\ 0 & \frac{291}{8} & 0 & -30 & 12 & -\frac{51}{8} & 0 \\ 0 & \frac{79}{4} & -10 & -2 & 10 & -\frac{31}{4} & 0 \\ 0 & \frac{33}{8} & 0 & 6 & 12 & -\frac{81}{8} & 0 \\ -14 & 23 & -10 & -20 & 10 & 7 & 0 \end{array}\right]$$

$$R_6 = R_6 + \frac{7R_1}{8} \implies \left[\begin{array}{cccccc|c} 16 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & \frac{299}{8} & -10 & -20 & 10 & -\frac{59}{8} & 0 \\ 0 & \frac{291}{8} & 0 & -30 & 12 & -\frac{51}{8} & 0 \\ 0 & \frac{79}{4} & -10 & -2 & 10 & -\frac{31}{4} & 0 \\ 0 & \frac{33}{8} & 0 & 6 & 12 & -\frac{81}{8} & 0 \\ 0 & \frac{275}{8} & -10 & -20 & 10 & -\frac{35}{8} & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{291R_2}{299} \Longrightarrow \left[\begin{array}{cccccc|c} 16 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & \frac{299}{8} & -10 & -20 & 10 & -\frac{59}{8} & 0 \\ 0 & 0 & \frac{2910}{299} & -\frac{3150}{299} & \frac{678}{299} & \frac{240}{299} & 0 \\ 0 & \frac{79}{4} & -10 & -2 & 10 & -\frac{31}{4} & 0 \\ 0 & \frac{33}{8} & 0 & 6 & 12 & -\frac{81}{8} & 0 \\ 0 & \frac{275}{8} & -10 & -20 & 10 & -\frac{35}{8} & 0 \end{array}\right]$$

$$R_4 = R_4 - \frac{158R_2}{299} \Longrightarrow \left[\begin{array}{cccccc|c} 16 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & \frac{299}{8} & -10 & -20 & 10 & -\frac{59}{8} & 0 \\ 0 & 0 & \frac{2910}{299} & -\frac{3150}{299} & \frac{678}{299} & \frac{240}{299} & 0 \\ 0 & 0 & -\frac{1410}{299} & \frac{2562}{299} & \frac{1410}{299} & -\frac{1152}{299} & 0 \\ 0 & \frac{33}{8} & 0 & 6 & 12 & -\frac{81}{8} & 0 \\ 0 & \frac{275}{8} & -10 & -20 & 10 & -\frac{35}{8} & 0 \end{array}\right]$$

$$R_5 = R_5 - \frac{33R_2}{299} \Longrightarrow \left[\begin{array}{cccccc|c} 16 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & \frac{299}{8} & -10 & -20 & 10 & -\frac{59}{8} & 0 \\ 0 & 0 & \frac{2910}{299} & -\frac{3150}{299} & \frac{678}{299} & \frac{240}{299} & 0 \\ 0 & 0 & -\frac{1410}{299} & \frac{2562}{299} & \frac{1410}{299} & -\frac{1152}{299} & 0 \\ 0 & 0 & \frac{330}{299} & \frac{2454}{299} & \frac{3258}{299} & -\frac{2784}{299} & 0 \\ 0 & \frac{275}{8} & -10 & -20 & 10 & -\frac{35}{8} & 0 \end{array}\right]$$

$$R_6 = R_6 - \frac{275R_2}{299} \Longrightarrow \left[\begin{array}{cccccc|c} 16 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & \frac{299}{8} & -10 & -20 & 10 & -\frac{59}{8} & 0 \\ 0 & 0 & \frac{2910}{299} & -\frac{3150}{299} & \frac{678}{299} & \frac{240}{299} & 0 \\ 0 & 0 & -\frac{1410}{299} & \frac{2562}{299} & \frac{1410}{299} & -\frac{1152}{299} & 0 \\ 0 & 0 & \frac{330}{299} & \frac{2454}{299} & \frac{3258}{299} & -\frac{2784}{299} & 0 \\ 0 & 0 & -\frac{240}{299} & -\frac{480}{299} & \frac{240}{299} & \frac{720}{299} & 0 \end{array}\right]$$

$$R_4 = R_4 + \frac{47R_3}{97} \Longrightarrow \left[\begin{array}{cccccc|c} 16 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & \frac{299}{8} & -10 & -20 & 10 & -\frac{59}{8} & 0 \\ 0 & 0 & \frac{2910}{299} & -\frac{3150}{299} & \frac{678}{299} & \frac{240}{299} & 0 \\ 0 & 0 & 0 & \frac{336}{97} & \frac{564}{97} & -\frac{336}{97} & 0 \\ 0 & 0 & \frac{330}{299} & \frac{2454}{299} & \frac{3258}{299} & -\frac{2784}{299} & 0 \\ 0 & 0 & -\frac{240}{299} & -\frac{480}{299} & \frac{240}{299} & \frac{720}{299} & 0 \end{array}\right]$$

$$R_5 = R_5 - \frac{11R_3}{97} \Longrightarrow \left[\begin{array}{cccccc|c} 16 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & \frac{299}{8} & -10 & -20 & 10 & -\frac{59}{8} & 0 \\ 0 & 0 & \frac{2910}{299} & -\frac{3150}{299} & \frac{678}{299} & \frac{240}{299} & 0 \\ 0 & 0 & 0 & \frac{336}{97} & \frac{564}{97} & -\frac{336}{97} & 0 \\ 0 & 0 & 0 & \frac{912}{97} & \frac{1032}{97} & -\frac{912}{97} & 0 \\ 0 & 0 & -\frac{240}{299} & -\frac{480}{299} & \frac{240}{299} & \frac{720}{299} & 0 \end{array}\right]$$

$$R_6 = R_6 + \frac{8R_3}{97} \Longrightarrow \left[\begin{array}{cccccc|c} 16 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & \frac{299}{8} & -10 & -20 & 10 & -\frac{59}{8} & 0 \\ 0 & 0 & \frac{2910}{299} & -\frac{3150}{299} & \frac{678}{299} & \frac{240}{299} & 0 \\ 0 & 0 & 0 & \frac{336}{97} & \frac{564}{97} & -\frac{336}{97} & 0 \\ 0 & 0 & 0 & \frac{912}{97} & \frac{1032}{97} & -\frac{912}{97} & 0 \\ 0 & 0 & 0 & -\frac{240}{97} & \frac{96}{97} & \frac{240}{97} & 0 \end{array}\right]$$

$$R_5 = R_5 - \frac{19R_4}{7} \Longrightarrow \left[\begin{array}{cccccc|c} 16 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & \frac{299}{8} & -10 & -20 & 10 & -\frac{59}{8} & 0 \\ 0 & 0 & \frac{2910}{299} & -\frac{3150}{299} & \frac{678}{299} & \frac{240}{299} & 0 \\ 0 & 0 & 0 & \frac{336}{97} & \frac{564}{97} & -\frac{336}{97} & 0 \\ 0 & 0 & 0 & 0 & -\frac{36}{7} & 0 & 0 \\ 0 & 0 & 0 & -\frac{240}{97} & \frac{96}{97} & \frac{240}{97} & 0 \end{array}\right]$$

$$R_6 = R_6 + \frac{5R_4}{7} \Longrightarrow \left[\begin{array}{cccccc|c} 16 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & \frac{299}{8} & -10 & -20 & 10 & -\frac{59}{8} & 0 \\ 0 & 0 & \frac{2910}{299} & -\frac{3150}{299} & \frac{678}{299} & \frac{240}{299} & 0 \\ 0 & 0 & 0 & \frac{336}{97} & \frac{564}{97} & -\frac{336}{97} & 0 \\ 0 & 0 & 0 & 0 & -\frac{36}{7} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{36}{7} & 0 & 0 \end{array}\right]$$

$$R_6 = R_6 + R_5 \Longrightarrow \left[\begin{array}{cccccc|c} 16 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & \frac{299}{8} & -10 & -20 & 10 & -\frac{59}{8} & 0 \\ 0 & 0 & \frac{2910}{299} & -\frac{3150}{299} & \frac{678}{299} & \frac{240}{299} & 0 \\ 0 & 0 & 0 & \frac{336}{97} & \frac{564}{97} & -\frac{336}{97} & 0 \\ 0 & 0 & 0 & 0 & -\frac{36}{7} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 16 & 13 & 0 & 0 & 0 & -13 \\ 0 & \frac{299}{8} & -10 & -20 & 10 & -\frac{59}{8} \\ 0 & 0 & \frac{2910}{299} & -\frac{3150}{299} & \frac{678}{299} & \frac{240}{299} \\ 0 & 0 & 0 & \frac{336}{97} & \frac{564}{97} & -\frac{336}{97} \\ 0 & 0 & 0 & 0 & -\frac{36}{7} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_6\}$ and the leading variables are $\{v_1, v_2, v_3, v_4, v_5\}$. Let $v_6 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_2 = t, v_3 = t, v_4 = t, v_5 = 0\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ t \\ t \\ t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ t \\ t \\ t \\ 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ t \\ t \\ t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ t \\ t \\ t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -4$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 9 & 13 & 0 & 0 & 0 & -13 \\ -14 & 19 & -10 & -20 & 10 & 4 \\ -30 & 12 & -7 & -30 & 12 & 18 \\ -12 & 10 & -10 & -9 & 10 & 2 \\ 6 & 9 & 0 & 6 & 5 & -15 \\ -14 & 23 & -10 & -20 & 10 & 0 \end{bmatrix} - (-4) \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 13 & 13 & 0 & 0 & 0 & -13 \\ -14 & 23 & -10 & -20 & 10 & 4 \\ -30 & 12 & -3 & -30 & 12 & 18 \\ -12 & 10 & -10 & -5 & 10 & 2 \\ 6 & 9 & 0 & 6 & 9 & -15 \\ -14 & 23 & -10 & -20 & 10 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccccc|c} 13 & 13 & 0 & 0 & 0 & -13 & 0 \\ -14 & 23 & -10 & -20 & 10 & 4 & 0 \\ -30 & 12 & -3 & -30 & 12 & 18 & 0 \\ -12 & 10 & -10 & -5 & 10 & 2 & 0 \\ 6 & 9 & 0 & 6 & 9 & -15 & 0 \\ -14 & 23 & -10 & -20 & 10 & 4 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{14R_1}{13} \Longrightarrow \left[\begin{array}{cccccc|c} 13 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & 37 & -10 & -20 & 10 & -10 & 0 \\ -30 & 12 & -3 & -30 & 12 & 18 & 0 \\ -12 & 10 & -10 & -5 & 10 & 2 & 0 \\ 6 & 9 & 0 & 6 & 9 & -15 & 0 \\ -14 & 23 & -10 & -20 & 10 & 4 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{30R_1}{13} \Longrightarrow \left[\begin{array}{cccccc|c} 13 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & 37 & -10 & -20 & 10 & -10 & 0 \\ 0 & 42 & -3 & -30 & 12 & -12 & 0 \\ -12 & 10 & -10 & -5 & 10 & 2 & 0 \\ 6 & 9 & 0 & 6 & 9 & -15 & 0 \\ -14 & 23 & -10 & -20 & 10 & 4 & 0 \end{array}\right]$$

$$R_4 = R_4 + \frac{12R_1}{13} \Longrightarrow \left[\begin{array}{cccccc|c} 13 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & 37 & -10 & -20 & 10 & -10 & 0 \\ 0 & 42 & -3 & -30 & 12 & -12 & 0 \\ 0 & 22 & -10 & -5 & 10 & -10 & 0 \\ 6 & 9 & 0 & 6 & 9 & -15 & 0 \\ -14 & 23 & -10 & -20 & 10 & 4 & 0 \end{array}\right]$$

$$R_5 = R_5 - \frac{6R_1}{13} \implies \left[\begin{array}{cccccc|c} 13 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & 37 & -10 & -20 & 10 & -10 & 0 \\ 0 & 42 & -3 & -30 & 12 & -12 & 0 \\ 0 & 22 & -10 & -5 & 10 & -10 & 0 \\ 0 & 3 & 0 & 6 & 9 & -9 & 0 \\ -14 & 23 & -10 & -20 & 10 & 4 & 0 \end{array}\right]$$

$$R_6 = R_6 + \frac{14R_1}{13} \implies \left[\begin{array}{cccccc|c} 13 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & 37 & -10 & -20 & 10 & -10 & 0 \\ 0 & 42 & -3 & -30 & 12 & -12 & 0 \\ 0 & 22 & -10 & -5 & 10 & -10 & 0 \\ 0 & 3 & 0 & 6 & 9 & -9 & 0 \\ 0 & 37 & -10 & -20 & 10 & -10 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{42R_2}{37} \implies \left[\begin{array}{cccccc|c} 13 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & 37 & -10 & -20 & 10 & -10 & 0 \\ 0 & 0 & \frac{309}{37} & -\frac{270}{37} & \frac{24}{37} & -\frac{24}{37} & 0 \\ 0 & 22 & -10 & -5 & 10 & -10 & 0 \\ 0 & 3 & 0 & 6 & 9 & -9 & 0 \\ 0 & 37 & -10 & -20 & 10 & -10 & 0 \end{array}\right]$$

$$R_4 = R_4 - \frac{22R_2}{37} \implies \left[\begin{array}{cccccc|c} 13 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & 37 & -10 & -20 & 10 & -10 & 0 \\ 0 & 0 & \frac{309}{37} & -\frac{270}{37} & \frac{24}{37} & -\frac{24}{37} & 0 \\ 0 & 0 & -\frac{150}{37} & \frac{255}{37} & \frac{150}{37} & -\frac{150}{37} & 0 \\ 0 & 3 & 0 & 6 & 9 & -9 & 0 \\ 0 & 37 & -10 & -20 & 10 & -10 & 0 \end{array}\right]$$

$$R_5 = R_5 - \frac{3R_2}{37} \Longrightarrow \left[\begin{array}{cccccc|c} 13 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & 37 & -10 & -20 & 10 & -10 & 0 \\ 0 & 0 & \frac{309}{37} & -\frac{270}{37} & \frac{24}{37} & -\frac{24}{37} & 0 \\ 0 & 0 & -\frac{150}{37} & \frac{255}{37} & \frac{150}{37} & -\frac{150}{37} & 0 \\ 0 & 0 & \frac{30}{37} & \frac{282}{37} & \frac{303}{37} & -\frac{303}{37} & 0 \\ 0 & 37 & -10 & -20 & 10 & -10 & 0 \end{array}\right]$$

$$R_6 = R_6 - R_2 \Longrightarrow \left[\begin{array}{cccccc|c} 13 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & 37 & -10 & -20 & 10 & -10 & 0 \\ 0 & 0 & \frac{309}{37} & -\frac{270}{37} & \frac{24}{37} & -\frac{24}{37} & 0 \\ 0 & 0 & -\frac{150}{37} & \frac{255}{37} & \frac{150}{37} & -\frac{150}{37} & 0 \\ 0 & 0 & \frac{30}{37} & \frac{282}{37} & \frac{303}{37} & -\frac{303}{37} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_4 = R_4 + \frac{50R_3}{103} \Longrightarrow \left[\begin{array}{cccccc|c} 13 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & 37 & -10 & -20 & 10 & -10 & 0 \\ 0 & 0 & \frac{309}{37} & -\frac{270}{37} & \frac{24}{37} & -\frac{24}{37} & 0 \\ 0 & 0 & 0 & \frac{345}{103} & \frac{450}{103} & -\frac{450}{103} & 0 \\ 0 & 0 & \frac{30}{37} & \frac{282}{37} & \frac{303}{37} & -\frac{303}{37} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_5 = R_5 - \frac{10R_3}{103} \Longrightarrow \left[\begin{array}{cccccc|c} 13 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & 37 & -10 & -20 & 10 & -10 & 0 \\ 0 & 0 & \frac{309}{37} & -\frac{270}{37} & \frac{24}{37} & -\frac{24}{37} & 0 \\ 0 & 0 & 0 & \frac{345}{103} & \frac{450}{103} & -\frac{450}{103} & 0 \\ 0 & 0 & 0 & \frac{858}{103} & \frac{837}{103} & -\frac{837}{103} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_5 = R_5 - \frac{286R_4}{115} \Longrightarrow \left[\begin{array}{cccccc|c} 13 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & 37 & -10 & -20 & 10 & -10 & 0 \\ 0 & 0 & \frac{309}{37} & -\frac{270}{37} & \frac{24}{37} & -\frac{24}{37} & 0 \\ 0 & 0 & 0 & \frac{345}{103} & \frac{450}{103} & -\frac{450}{103} & 0 \\ 0 & 0 & 0 & 0 & -\frac{63}{23} & \frac{63}{23} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 13 & 13 & 0 & 0 & 0 & -13 \\ 0 & 37 & -10 & -20 & 10 & -10 \\ 0 & 0 & \frac{309}{37} & -\frac{270}{37} & \frac{24}{37} & -\frac{24}{37} \\ 0 & 0 & 0 & \frac{345}{103} & \frac{450}{103} & -\frac{450}{103} \\ 0 & 0 & 0 & 0 & -\frac{63}{23} & \frac{63}{23} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_6\}$ and the leading variables are $\{v_1, v_2, v_3, v_4, v_5\}$. Let $v_6 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = 0, v_3 = 0, v_4 = 0, v_5 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ 0 \\ 0 \\ 0 \\ t \\ t \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \\ 0 \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this

eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ 0 \\ 0 \\ 0 \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ 0 \\ 0 \\ 0 \\ t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = 3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 9 & 13 & 0 & 0 & 0 & -13 \\ -14 & 19 & -10 & -20 & 10 & 4 \\ -30 & 12 & -7 & -30 & 12 & 18 \\ -12 & 10 & -10 & -9 & 10 & 2 \\ 6 & 9 & 0 & 6 & 5 & -15 \\ -14 & 23 & -10 & -20 & 10 & 0 \end{bmatrix} - (3) \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 13 & 0 & 0 & 0 & -13 \\ -14 & 16 & -10 & -20 & 10 & 4 \\ -30 & 12 & -10 & -30 & 12 & 18 \\ -12 & 10 & -10 & -12 & 10 & 2 \\ 6 & 9 & 0 & 6 & 2 & -15 \\ -14 & 23 & -10 & -20 & 10 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccccc|c} 6 & 13 & 0 & 0 & 0 & -13 & 0 \\ -14 & 16 & -10 & -20 & 10 & 4 & 0 \\ -30 & 12 & -10 & -30 & 12 & 18 & 0 \\ -12 & 10 & -10 & -12 & 10 & 2 & 0 \\ 6 & 9 & 0 & 6 & 2 & -15 & 0 \\ -14 & 23 & -10 & -20 & 10 & -3 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{7R_1}{3} \Longrightarrow \left[\begin{array}{cccccc|c} 6 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & \frac{139}{3} & -10 & -20 & 10 & -\frac{79}{3} & 0 \\ -30 & 12 & -10 & -30 & 12 & 18 & 0 \\ -12 & 10 & -10 & -12 & 10 & 2 & 0 \\ 6 & 9 & 0 & 6 & 2 & -15 & 0 \\ -14 & 23 & -10 & -20 & 10 & -3 & 0 \end{array}\right]$$

$$R_3 = R_3 + 5R_1 \Longrightarrow \left[\begin{array}{cccccc|c} 6 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & \frac{139}{3} & -10 & -20 & 10 & -\frac{79}{3} & 0 \\ 0 & 77 & -10 & -30 & 12 & -47 & 0 \\ -12 & 10 & -10 & -12 & 10 & 2 & 0 \\ 6 & 9 & 0 & 6 & 2 & -15 & 0 \\ -14 & 23 & -10 & -20 & 10 & -3 & 0 \end{array}\right]$$

$$R_4 = R_4 + 2R_1 \Longrightarrow \left[\begin{array}{cccccc|c} 6 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & \frac{139}{3} & -10 & -20 & 10 & -\frac{79}{3} & 0 \\ 0 & 77 & -10 & -30 & 12 & -47 & 0 \\ 0 & 36 & -10 & -12 & 10 & -24 & 0 \\ 6 & 9 & 0 & 6 & 2 & -15 & 0 \\ -14 & 23 & -10 & -20 & 10 & -3 & 0 \end{array}\right]$$

$$R_5 = R_5 - R_1 \Longrightarrow \left[\begin{array}{cccccc|c} 6 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & \frac{139}{3} & -10 & -20 & 10 & -\frac{79}{3} & 0 \\ 0 & 77 & -10 & -30 & 12 & -47 & 0 \\ 0 & 36 & -10 & -12 & 10 & -24 & 0 \\ 0 & -4 & 0 & 6 & 2 & -2 & 0 \\ -14 & 23 & -10 & -20 & 10 & -3 & 0 \end{array}\right]$$

$$R_6 = R_6 + \frac{7R_1}{3} \Longrightarrow \left[\begin{array}{cccccc|c} 6 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & \frac{139}{3} & -10 & -20 & 10 & -\frac{79}{3} & 0 \\ 0 & 77 & -10 & -30 & 12 & -47 & 0 \\ 0 & 36 & -10 & -12 & 10 & -24 & 0 \\ 0 & -4 & 0 & 6 & 2 & -2 & 0 \\ 0 & \frac{160}{3} & -10 & -20 & 10 & -\frac{100}{3} & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{231R_2}{139} \Longrightarrow \left[\begin{array}{cccccc|c} 6 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & \frac{139}{3} & -10 & -20 & 10 & -\frac{79}{3} & 0 \\ 0 & 0 & \frac{920}{139} & \frac{450}{139} & -\frac{642}{139} & -\frac{450}{139} & 0 \\ 0 & 36 & -10 & -12 & 10 & -24 & 0 \\ 0 & -4 & 0 & 6 & 2 & -2 & 0 \\ 0 & \frac{160}{3} & -10 & -20 & 10 & -\frac{100}{3} & 0 \end{array}\right]$$

$$R_4 = R_4 - \frac{108R_2}{139} \Longrightarrow \left[\begin{array}{cccccc|c} 6 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & \frac{139}{3} & -10 & -20 & 10 & -\frac{79}{3} & 0 \\ 0 & 0 & \frac{920}{139} & \frac{450}{139} & -\frac{642}{139} & -\frac{450}{139} & 0 \\ 0 & 0 & -\frac{310}{139} & \frac{492}{139} & \frac{310}{139} & -\frac{492}{139} & 0 \\ 0 & -4 & 0 & 6 & 2 & -2 & 0 \\ 0 & \frac{160}{3} & -10 & -20 & 10 & -\frac{100}{3} & 0 \end{array}\right]$$

$$R_5 = R_5 + \frac{12R_2}{139} \Longrightarrow \left[\begin{array}{cccccc|c} 6 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & \frac{139}{3} & -10 & -20 & 10 & -\frac{79}{3} & 0 \\ 0 & 0 & \frac{920}{139} & \frac{450}{139} & -\frac{642}{139} & -\frac{450}{139} & 0 \\ 0 & 0 & -\frac{310}{139} & \frac{492}{139} & \frac{310}{139} & -\frac{492}{139} & 0 \\ 0 & 0 & -\frac{120}{139} & \frac{594}{139} & \frac{398}{139} & -\frac{594}{139} & 0 \\ 0 & \frac{160}{3} & -10 & -20 & 10 & -\frac{100}{3} & 0 \end{array}\right]$$

$$R_6 = R_6 - \frac{160R_2}{139} \Longrightarrow \left[\begin{array}{cccccc|c} 6 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & \frac{139}{3} & -10 & -20 & 10 & -\frac{79}{3} & 0 \\ 0 & 0 & \frac{920}{139} & \frac{450}{139} & -\frac{642}{139} & -\frac{450}{139} & 0 \\ 0 & 0 & -\frac{310}{139} & \frac{492}{139} & \frac{310}{139} & -\frac{492}{139} & 0 \\ 0 & 0 & -\frac{120}{139} & \frac{594}{139} & \frac{398}{139} & -\frac{594}{139} & 0 \\ 0 & 0 & \frac{210}{139} & \frac{420}{139} & -\frac{210}{139} & -\frac{420}{139} & 0 \end{array}\right]$$

$$R_4 = R_4 + \frac{31R_3}{92} \Longrightarrow \left[\begin{array}{cccccc|c} 6 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & \frac{139}{3} & -10 & -20 & 10 & -\frac{79}{3} & 0 \\ 0 & 0 & \frac{920}{139} & \frac{450}{139} & -\frac{642}{139} & -\frac{450}{139} & 0 \\ 0 & 0 & 0 & \frac{213}{46} & \frac{31}{46} & -\frac{213}{46} & 0 \\ 0 & 0 & -\frac{120}{139} & \frac{594}{139} & \frac{398}{139} & -\frac{594}{139} & 0 \\ 0 & 0 & \frac{210}{139} & \frac{420}{139} & -\frac{210}{139} & -\frac{420}{139} & 0 \end{array}\right]$$

$$R_5 = R_5 + \frac{3R_3}{23} \Longrightarrow \left[\begin{array}{cccccc|c} 6 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & \frac{139}{3} & -10 & -20 & 10 & -\frac{79}{3} & 0 \\ 0 & 0 & \frac{920}{139} & \frac{450}{139} & -\frac{642}{139} & -\frac{450}{139} & 0 \\ 0 & 0 & 0 & \frac{213}{46} & \frac{31}{46} & -\frac{213}{46} & 0 \\ 0 & 0 & 0 & \frac{108}{23} & \frac{52}{23} & -\frac{108}{23} & 0 \\ 0 & 0 & \frac{210}{139} & \frac{420}{139} & -\frac{210}{139} & -\frac{420}{139} & 0 \end{array}\right]$$

$$R_6 = R_6 - \frac{21R_3}{92} \Longrightarrow \left[\begin{array}{cccccc|c} 6 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & \frac{139}{3} & -10 & -20 & 10 & -\frac{79}{3} & 0 \\ 0 & 0 & \frac{920}{139} & \frac{450}{139} & -\frac{642}{139} & -\frac{450}{139} & 0 \\ 0 & 0 & 0 & \frac{213}{46} & \frac{31}{46} & -\frac{213}{46} & 0 \\ 0 & 0 & 0 & \frac{108}{23} & \frac{52}{23} & -\frac{108}{23} & 0 \\ 0 & 0 & 0 & \frac{105}{46} & -\frac{21}{46} & -\frac{105}{46} & 0 \end{array}\right]$$

$$R_5 = R_5 - \frac{72R_4}{71} \Longrightarrow \left[\begin{array}{cccccc|c} 6 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & \frac{139}{3} & -10 & -20 & 10 & -\frac{79}{3} & 0 \\ 0 & 0 & \frac{920}{139} & \frac{450}{139} & -\frac{642}{139} & -\frac{450}{139} & 0 \\ 0 & 0 & 0 & \frac{213}{46} & \frac{31}{46} & -\frac{213}{46} & 0 \\ 0 & 0 & 0 & 0 & \frac{112}{71} & 0 & 0 \\ 0 & 0 & 0 & \frac{105}{46} & -\frac{21}{46} & -\frac{105}{46} & 0 \end{array}\right]$$

$$R_6 = R_6 - \frac{35R_4}{71} \Longrightarrow \left[\begin{array}{cccccc|c} 6 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & \frac{139}{3} & -10 & -20 & 10 & -\frac{79}{3} & 0 \\ 0 & 0 & \frac{920}{139} & \frac{450}{139} & -\frac{642}{139} & -\frac{450}{139} & 0 \\ 0 & 0 & 0 & \frac{213}{46} & \frac{31}{46} & -\frac{213}{46} & 0 \\ 0 & 0 & 0 & 0 & \frac{112}{71} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{56}{71} & 0 & 0 \end{array}\right]$$

$$R_6 = R_6 + \frac{R_5}{2} \Longrightarrow \left[\begin{array}{cccccc|c} 6 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & \frac{139}{3} & -10 & -20 & 10 & -\frac{79}{3} & 0 \\ 0 & 0 & \frac{920}{139} & \frac{450}{139} & -\frac{642}{139} & -\frac{450}{139} & 0 \\ 0 & 0 & 0 & \frac{213}{46} & \frac{31}{46} & -\frac{213}{46} & 0 \\ 0 & 0 & 0 & 0 & \frac{112}{71} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 6 & 13 & 0 & 0 & 0 & -13 \\ 0 & \frac{139}{3} & -10 & -20 & 10 & -\frac{79}{3} \\ 0 & 0 & \frac{920}{139} & \frac{450}{139} & -\frac{642}{139} & -\frac{450}{139} \\ 0 & 0 & 0 & \frac{213}{46} & \frac{31}{46} & -\frac{213}{46} \\ 0 & 0 & 0 & 0 & \frac{112}{71} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_6\}$ and the leading variables are $\{v_1, v_2, v_3, v_4, v_5\}$. Let $v_6 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_2 = t, v_3 = 0, v_4 = t, v_5 = 0\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ t \\ 0 \\ t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ t \\ 0 \\ t \\ 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ t \\ 0 \\ t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ t \\ 0 \\ t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_4 = 5$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 9 & 13 & 0 & 0 & 0 & -13 \\ -14 & 19 & -10 & -20 & 10 & 4 \\ -30 & 12 & -7 & -30 & 12 & 18 \\ -12 & 10 & -10 & -9 & 10 & 2 \\ 6 & 9 & 0 & 6 & 5 & -15 \\ -14 & 23 & -10 & -20 & 10 & 0 \end{bmatrix} - (5) \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 13 & 0 & 0 & 0 & -13 \\ -14 & 14 & -10 & -20 & 10 & 4 \\ -30 & 12 & -12 & -30 & 12 & 18 \\ -12 & 10 & -10 & -14 & 10 & 2 \\ 6 & 9 & 0 & 6 & 0 & -15 \\ -14 & 23 & -10 & -20 & 10 & -5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccccc|c} 4 & 13 & 0 & 0 & 0 & -13 & 0 \\ -14 & 14 & -10 & -20 & 10 & 4 & 0 \\ -30 & 12 & -12 & -30 & 12 & 18 & 0 \\ -12 & 10 & -10 & -14 & 10 & 2 & 0 \\ 6 & 9 & 0 & 6 & 0 & -15 & 0 \\ -14 & 23 & -10 & -20 & 10 & -5 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{7R_1}{2} \Longrightarrow \left[\begin{array}{cccccc|c} 4 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & \frac{119}{2} & -10 & -20 & 10 & -\frac{83}{2} & 0 \\ -30 & 12 & -12 & -30 & 12 & 18 & 0 \\ -12 & 10 & -10 & -14 & 10 & 2 & 0 \\ 6 & 9 & 0 & 6 & 0 & -15 & 0 \\ -14 & 23 & -10 & -20 & 10 & -5 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{15R_1}{2} \Longrightarrow \left[\begin{array}{cccccc|c} 4 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & \frac{119}{2} & -10 & -20 & 10 & -\frac{83}{2} & 0 \\ 0 & \frac{219}{2} & -12 & -30 & 12 & -\frac{159}{2} & 0 \\ -12 & 10 & -10 & -14 & 10 & 2 & 0 \\ 6 & 9 & 0 & 6 & 0 & -15 & 0 \\ -14 & 23 & -10 & -20 & 10 & -5 & 0 \end{array}\right]$$

$$R_4 = R_4 + 3R_1 \Longrightarrow \left[\begin{array}{cccccc|c} 4 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & \frac{119}{2} & -10 & -20 & 10 & -\frac{83}{2} & 0 \\ 0 & \frac{219}{2} & -12 & -30 & 12 & -\frac{159}{2} & 0 \\ 0 & 49 & -10 & -14 & 10 & -37 & 0 \\ 6 & 9 & 0 & 6 & 0 & -15 & 0 \\ -14 & 23 & -10 & -20 & 10 & -5 & 0 \end{array}\right]$$

$$R_5 = R_5 - \frac{3R_1}{2} \Longrightarrow \left[\begin{array}{cccccc|c} 4 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & \frac{119}{2} & -10 & -20 & 10 & -\frac{83}{2} & 0 \\ 0 & \frac{219}{2} & -12 & -30 & 12 & -\frac{159}{2} & 0 \\ 0 & 49 & -10 & -14 & 10 & -37 & 0 \\ 0 & -\frac{21}{2} & 0 & 6 & 0 & \frac{9}{2} & 0 \\ -14 & 23 & -10 & -20 & 10 & -5 & 0 \end{array}\right]$$

$$R_6 = R_6 + \frac{7R_1}{2} \Longrightarrow \left[\begin{array}{cccccc|c} 4 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & \frac{119}{2} & -10 & -20 & 10 & -\frac{83}{2} & 0 \\ 0 & \frac{219}{2} & -12 & -30 & 12 & -\frac{159}{2} & 0 \\ 0 & 49 & -10 & -14 & 10 & -37 & 0 \\ 0 & -\frac{21}{2} & 0 & 6 & 0 & \frac{9}{2} & 0 \\ 0 & \frac{137}{2} & -10 & -20 & 10 & -\frac{101}{2} & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{219R_2}{119} \Longrightarrow \left[\begin{array}{cccccc|c} 4 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & \frac{119}{2} & -10 & -20 & 10 & -\frac{83}{2} & 0 \\ 0 & 0 & \frac{762}{119} & \frac{810}{119} & -\frac{762}{119} & -\frac{372}{119} & 0 \\ 0 & 49 & -10 & -14 & 10 & -37 & 0 \\ 0 & -\frac{21}{2} & 0 & 6 & 0 & \frac{9}{2} & 0 \\ 0 & \frac{137}{2} & -10 & -20 & 10 & -\frac{101}{2} & 0 \end{array}\right]$$

$$R_4 = R_4 - \frac{14R_2}{17} \Longrightarrow \left[\begin{array}{cccccc|c} 4 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & \frac{119}{2} & -10 & -20 & 10 & -\frac{83}{2} & 0 \\ 0 & 0 & \frac{762}{119} & \frac{810}{119} & -\frac{762}{119} & -\frac{372}{119} & 0 \\ 0 & 0 & -\frac{30}{17} & \frac{42}{17} & \frac{30}{17} & -\frac{48}{17} & 0 \\ 0 & -\frac{21}{2} & 0 & 6 & 0 & \frac{9}{2} & 0 \\ 0 & \frac{137}{2} & -10 & -20 & 10 & -\frac{101}{2} & 0 \end{array}\right]$$

$$R_5 = R_5 + \frac{3R_2}{17} \Longrightarrow \left[\begin{array}{cccccc|c} 4 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & \frac{119}{2} & -10 & -20 & 10 & -\frac{83}{2} & 0 \\ 0 & 0 & \frac{762}{119} & \frac{810}{119} & -\frac{762}{119} & -\frac{372}{119} & 0 \\ 0 & 0 & -\frac{30}{17} & \frac{42}{17} & \frac{30}{17} & -\frac{48}{17} & 0 \\ 0 & 0 & -\frac{30}{17} & \frac{42}{17} & \frac{30}{17} & -\frac{48}{17} & 0 \\ 0 & \frac{137}{2} & -10 & -20 & 10 & -\frac{101}{2} & 0 \end{array}\right]$$

$$R_6 = R_6 - \frac{137R_2}{119} \Longrightarrow \left[\begin{array}{cccccc|c} 4 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & \frac{119}{2} & -10 & -20 & 10 & -\frac{83}{2} & 0 \\ 0 & 0 & \frac{762}{119} & \frac{810}{119} & -\frac{762}{119} & -\frac{372}{119} & 0 \\ 0 & 0 & -\frac{30}{17} & \frac{42}{17} & \frac{30}{17} & -\frac{48}{17} & 0 \\ 0 & 0 & -\frac{30}{17} & \frac{42}{17} & \frac{30}{17} & -\frac{48}{17} & 0 \\ 0 & 0 & \frac{180}{119} & \frac{360}{119} & -\frac{180}{119} & -\frac{324}{119} & 0 \end{array}\right]$$

$$R_4 = R_4 + \frac{35R_3}{127} \Longrightarrow \left[\begin{array}{cccccc|c} 4 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & \frac{119}{2} & -10 & -20 & 10 & -\frac{83}{2} & 0 \\ 0 & 0 & \frac{762}{119} & \frac{810}{119} & -\frac{762}{119} & -\frac{372}{119} & 0 \\ 0 & 0 & 0 & \frac{552}{127} & 0 & -\frac{468}{127} & 0 \\ 0 & 0 & -\frac{30}{17} & \frac{42}{17} & \frac{30}{17} & -\frac{48}{17} & 0 \\ 0 & 0 & \frac{180}{119} & \frac{360}{119} & -\frac{180}{119} & -\frac{324}{119} & 0 \end{array}\right]$$

$$R_5 = R_5 + \frac{35R_3}{127} \Longrightarrow \left[\begin{array}{cccccc|c} 4 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & \frac{119}{2} & -10 & -20 & 10 & -\frac{83}{2} & 0 \\ 0 & 0 & \frac{762}{119} & \frac{810}{119} & -\frac{762}{119} & -\frac{372}{119} & 0 \\ 0 & 0 & 0 & \frac{552}{127} & 0 & -\frac{468}{127} & 0 \\ 0 & 0 & 0 & \frac{552}{127} & 0 & -\frac{468}{127} & 0 \\ 0 & 0 & \frac{180}{119} & \frac{360}{119} & -\frac{180}{119} & -\frac{324}{119} & 0 \end{array}\right]$$

$$R_6 = R_6 - \frac{30R_3}{127} \Longrightarrow \left[\begin{array}{cccccc|c} 4 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & \frac{119}{2} & -10 & -20 & 10 & -\frac{83}{2} & 0 \\ 0 & 0 & \frac{762}{119} & \frac{810}{119} & -\frac{762}{119} & -\frac{372}{119} & 0 \\ 0 & 0 & 0 & \frac{552}{127} & 0 & -\frac{468}{127} & 0 \\ 0 & 0 & 0 & \frac{552}{127} & 0 & -\frac{468}{127} & 0 \\ 0 & 0 & 0 & \frac{180}{127} & 0 & -\frac{252}{127} & 0 \end{array}\right]$$

$$R_5 = R_5 - R_4 \Longrightarrow \left[\begin{array}{cccccc|c} 4 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & \frac{119}{2} & -10 & -20 & 10 & -\frac{83}{2} & 0 \\ 0 & 0 & \frac{762}{119} & \frac{810}{119} & -\frac{762}{119} & -\frac{372}{119} & 0 \\ 0 & 0 & 0 & \frac{552}{127} & 0 & -\frac{468}{127} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{180}{127} & 0 & -\frac{252}{127} & 0 \end{array}\right]$$

$$R_6 = R_6 - \frac{15R_4}{46} \Longrightarrow \left[\begin{array}{cccccc|c} 4 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & \frac{119}{2} & -10 & -20 & 10 & -\frac{83}{2} & 0 \\ 0 & 0 & \frac{762}{119} & \frac{810}{119} & -\frac{762}{119} & -\frac{372}{119} & 0 \\ 0 & 0 & 0 & \frac{552}{127} & 0 & -\frac{468}{127} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{18}{23} & 0 \end{array}\right]$$

Since the current pivot $A(5,6)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 5 and row 6 gives

$$\left[\begin{array}{cccccc|c} 4 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & \frac{119}{2} & -10 & -20 & 10 & -\frac{83}{2} & 0 \\ 0 & 0 & \frac{762}{119} & \frac{810}{119} & -\frac{762}{119} & -\frac{372}{119} & 0 \\ 0 & 0 & 0 & \frac{552}{127} & 0 & -\frac{468}{127} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{18}{23} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 4 & 13 & 0 & 0 & 0 & -13 \\ 0 & \frac{119}{2} & -10 & -20 & 10 & -\frac{83}{2} \\ 0 & 0 & \frac{762}{119} & \frac{810}{119} & -\frac{762}{119} & -\frac{372}{119} \\ 0 & 0 & 0 & \frac{552}{127} & 0 & -\frac{468}{127} \\ 0 & 0 & 0 & 0 & 0 & -\frac{18}{23} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_5\}$ and the leading variables are $\{v_1, v_2, v_3, v_4, v_6\}$. Let $v_5 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_2 = 0, v_3 = t, v_4 = 0, v_6 = 0\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ 0 \\ t \\ 0 \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ t \\ 0 \\ t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ 0 \\ t \\ 0 \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ 0 \\ t \\ 0 \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Considering the eigenvalue $\lambda_5 = 9$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 9 & 13 & 0 & 0 & 0 & -13 \\ -14 & 19 & -10 & -20 & 10 & 4 \\ -30 & 12 & -7 & -30 & 12 & 18 \\ -12 & 10 & -10 & -9 & 10 & 2 \\ 6 & 9 & 0 & 6 & 5 & -15 \\ -14 & 23 & -10 & -20 & 10 & 0 \end{bmatrix} - (9) \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 13 & 0 & 0 & 0 & -13 \\ -14 & 10 & -10 & -20 & 10 & 4 \\ -30 & 12 & -16 & -30 & 12 & 18 \\ -12 & 10 & -10 & -18 & 10 & 2 \\ 6 & 9 & 0 & 6 & -4 & -15 \\ -14 & 23 & -10 & -20 & 10 & -9 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccccc|c} 0 & 13 & 0 & 0 & 0 & -13 & 0 \\ -14 & 10 & -10 & -20 & 10 & 4 & 0 \\ -30 & 12 & -16 & -30 & 12 & 18 & 0 \\ -12 & 10 & -10 & -18 & 10 & 2 & 0 \\ 6 & 9 & 0 & 6 & -4 & -15 & 0 \\ -14 & 23 & -10 & -20 & 10 & -9 & 0 \end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{cccccc|c} -14 & 10 & -10 & -20 & 10 & 4 & 0 \\ 0 & 13 & 0 & 0 & 0 & -13 & 0 \\ -30 & 12 & -16 & -30 & 12 & 18 & 0 \\ -12 & 10 & -10 & -18 & 10 & 2 & 0 \\ 6 & 9 & 0 & 6 & -4 & -15 & 0 \\ -14 & 23 & -10 & -20 & 10 & -9 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{15R_1}{7} \implies \left[\begin{array}{cccccc|c} -14 & 10 & -10 & -20 & 10 & 4 & 0 \\ 0 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & -\frac{66}{7} & \frac{38}{7} & \frac{90}{7} & -\frac{66}{7} & \frac{66}{7} & 0 \\ -12 & 10 & -10 & -18 & 10 & 2 & 0 \\ 6 & 9 & 0 & 6 & -4 & -15 & 0 \\ -14 & 23 & -10 & -20 & 10 & -9 & 0 \end{array}\right]$$

$$R_4 = R_4 - \frac{6R_1}{7} \implies \left[\begin{array}{cccccc|c} -14 & 10 & -10 & -20 & 10 & 4 & 0 \\ 0 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & -\frac{66}{7} & \frac{38}{7} & \frac{90}{7} & -\frac{66}{7} & \frac{66}{7} & 0 \\ 0 & \frac{10}{7} & -\frac{10}{7} & -\frac{6}{7} & \frac{10}{7} & -\frac{10}{7} & 0 \\ 6 & 9 & 0 & 6 & -4 & -15 & 0 \\ -14 & 23 & -10 & -20 & 10 & -9 & 0 \end{array}\right]$$

$$R_5 = R_5 + \frac{3R_1}{7} \implies \left[\begin{array}{cccccc|c} -14 & 10 & -10 & -20 & 10 & 4 & 0 \\ 0 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & -\frac{66}{7} & \frac{38}{7} & \frac{90}{7} & -\frac{66}{7} & \frac{66}{7} & 0 \\ 0 & \frac{10}{7} & -\frac{10}{7} & -\frac{6}{7} & \frac{10}{7} & -\frac{10}{7} & 0 \\ 0 & \frac{93}{7} & -\frac{30}{7} & -\frac{18}{7} & \frac{2}{7} & -\frac{93}{7} & 0 \\ -14 & 23 & -10 & -20 & 10 & -9 & 0 \end{array}\right]$$

$$R_6 = R_6 - R_1 \implies \left[\begin{array}{cccccc|c} -14 & 10 & -10 & -20 & 10 & 4 & 0 \\ 0 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & -\frac{66}{7} & \frac{38}{7} & \frac{90}{7} & -\frac{66}{7} & \frac{66}{7} & 0 \\ 0 & \frac{10}{7} & -\frac{10}{7} & -\frac{6}{7} & \frac{10}{7} & -\frac{10}{7} & 0 \\ 0 & \frac{93}{7} & -\frac{30}{7} & -\frac{18}{7} & \frac{2}{7} & -\frac{93}{7} & 0 \\ 0 & 13 & 0 & 0 & 0 & -13 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{66R_2}{91} \Longrightarrow \left[\begin{array}{cccccc|c} -14 & 10 & -10 & -20 & 10 & 4 & 0 \\ 0 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & 0 & \frac{38}{7} & \frac{90}{7} & -\frac{66}{7} & 0 & 0 \\ 0 & \frac{10}{7} & -\frac{10}{7} & -\frac{6}{7} & \frac{10}{7} & -\frac{10}{7} & 0 \\ 0 & \frac{93}{7} & -\frac{30}{7} & -\frac{18}{7} & \frac{2}{7} & -\frac{93}{7} & 0 \\ 0 & 13 & 0 & 0 & 0 & -13 & 0 \end{array}\right]$$

$$R_4 = R_4 - \frac{10R_2}{91} \Longrightarrow \left[\begin{array}{cccccc|c} -14 & 10 & -10 & -20 & 10 & 4 & 0 \\ 0 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & 0 & \frac{38}{7} & \frac{90}{7} & -\frac{66}{7} & 0 & 0 \\ 0 & 0 & -\frac{10}{7} & -\frac{6}{7} & \frac{10}{7} & 0 & 0 \\ 0 & \frac{93}{7} & -\frac{30}{7} & -\frac{18}{7} & \frac{2}{7} & -\frac{93}{7} & 0 \\ 0 & 13 & 0 & 0 & 0 & -13 & 0 \end{array}\right]$$

$$R_5 = R_5 - \frac{93R_2}{91} \Longrightarrow \left[\begin{array}{cccccc|c} -14 & 10 & -10 & -20 & 10 & 4 & 0 \\ 0 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & 0 & \frac{38}{7} & \frac{90}{7} & -\frac{66}{7} & 0 & 0 \\ 0 & 0 & -\frac{10}{7} & -\frac{6}{7} & \frac{10}{7} & 0 & 0 \\ 0 & 0 & -\frac{30}{7} & -\frac{18}{7} & \frac{2}{7} & 0 & 0 \\ 0 & 13 & 0 & 0 & 0 & -13 & 0 \end{array}\right]$$

$$R_6 = R_6 - R_2 \Longrightarrow \left[\begin{array}{cccccc|c} -14 & 10 & -10 & -20 & 10 & 4 & 0 \\ 0 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & 0 & \frac{38}{7} & \frac{90}{7} & -\frac{66}{7} & 0 & 0 \\ 0 & 0 & -\frac{10}{7} & -\frac{6}{7} & \frac{10}{7} & 0 & 0 \\ 0 & 0 & -\frac{30}{7} & -\frac{18}{7} & \frac{2}{7} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_4 = R_4 + \frac{5R_3}{19} \Longrightarrow \left[\begin{array}{cccccc|c} -14 & 10 & -10 & -20 & 10 & 4 & 0 \\ 0 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & 0 & \frac{38}{7} & \frac{90}{7} & -\frac{66}{7} & 0 & 0 \\ 0 & 0 & 0 & \frac{48}{19} & -\frac{20}{19} & 0 & 0 \\ 0 & 0 & -\frac{30}{7} & -\frac{18}{7} & \frac{2}{7} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_5 = R_5 + \frac{15R_3}{19} \Longrightarrow \left[\begin{array}{cccccc|c} -14 & 10 & -10 & -20 & 10 & 4 & 0 \\ 0 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & 0 & \frac{38}{7} & \frac{90}{7} & -\frac{66}{7} & 0 & 0 \\ 0 & 0 & 0 & \frac{48}{19} & -\frac{20}{19} & 0 & 0 \\ 0 & 0 & 0 & \frac{144}{19} & -\frac{136}{19} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_5 = R_5 - 3R_4 \Longrightarrow \left[\begin{array}{cccccc|c} -14 & 10 & -10 & -20 & 10 & 4 & 0 \\ 0 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & 0 & \frac{38}{7} & \frac{90}{7} & -\frac{66}{7} & 0 & 0 \\ 0 & 0 & 0 & \frac{48}{19} & -\frac{20}{19} & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -14 & 10 & -10 & -20 & 10 & 4 \\ 0 & 13 & 0 & 0 & 0 & -13 \\ 0 & 0 & \frac{38}{7} & \frac{90}{7} & -\frac{66}{7} & 0 \\ 0 & 0 & 0 & \frac{48}{19} & -\frac{20}{19} & 0 \\ 0 & 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_6\}$ and the leading variables are $\{v_1, v_2, v_3, v_4, v_5\}$. Let $v_6 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = t, v_3 = 0, v_4 = 0, v_5 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \\ 0 \\ 0 \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \\ 0 \\ 0 \\ 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \\ 0 \\ 0 \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \\ 0 \\ 0 \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_6 = 11$

We need to solve $A\vec{v}=\lambda\vec{v}$ or $(A-\lambda I)\vec{v}=\vec{0}$ which becomes

$$\left(\begin{bmatrix} 9 & 13 & 0 & 0 & 0 & -13 \\ -14 & 19 & -10 & -20 & 10 & 4 \\ -30 & 12 & -7 & -30 & 12 & 18 \\ -12 & 10 & -10 & -9 & 10 & 2 \\ 6 & 9 & 0 & 6 & 5 & -15 \\ -14 & 23 & -10 & -20 & 10 & 0 \end{bmatrix} - (11)\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 13 & 0 & 0 & 0 & -13 \\ -14 & 8 & -10 & -20 & 10 & 4 \\ -30 & 12 & -18 & -30 & 12 & 18 \\ -12 & 10 & -10 & -20 & 10 & 2 \\ 6 & 9 & 0 & 6 & -6 & -15 \\ -14 & 23 & -10 & -20 & 10 & -11 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccccc|c} -2 & 13 & 0 & 0 & 0 & -13 & 0 \\ -14 & 8 & -10 & -20 & 10 & 4 & 0 \\ -30 & 12 & -18 & -30 & 12 & 18 & 0 \\ -12 & 10 & -10 & -20 & 10 & 2 & 0 \\ 6 & 9 & 0 & 6 & -6 & -15 & 0 \\ -14 & 23 & -10 & -20 & 10 & -11 & 0 \end{array}\right]$$

$$R_2 = R_2 - 7R_1 \Longrightarrow \left[\begin{array}{cccccc|c} -2 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & -83 & -10 & -20 & 10 & 95 & 0 \\ -30 & 12 & -18 & -30 & 12 & 18 & 0 \\ -12 & 10 & -10 & -20 & 10 & 2 & 0 \\ 6 & 9 & 0 & 6 & -6 & -15 & 0 \\ -14 & 23 & -10 & -20 & 10 & -11 & 0 \end{array}\right]$$

$$R_3 = R_3 - 15R_1 \Longrightarrow \left[\begin{array}{cccccc|c} -2 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & -83 & -10 & -20 & 10 & 95 & 0 \\ 0 & -183 & -18 & -30 & 12 & 213 & 0 \\ -12 & 10 & -10 & -20 & 10 & 2 & 0 \\ 6 & 9 & 0 & 6 & -6 & -15 & 0 \\ -14 & 23 & -10 & -20 & 10 & -11 & 0 \end{array}\right]$$

$$R_4 = R_4 - 6R_1 \Longrightarrow \left[\begin{array}{cccccc|c} -2 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & -83 & -10 & -20 & 10 & 95 & 0 \\ 0 & -183 & -18 & -30 & 12 & 213 & 0 \\ 0 & -68 & -10 & -20 & 10 & 80 & 0 \\ 6 & 9 & 0 & 6 & -6 & -15 & 0 \\ -14 & 23 & -10 & -20 & 10 & -11 & 0 \end{array}\right]$$

$$R_5 = R_5 + 3R_1 \Longrightarrow \left[\begin{array}{cccccc|c} -2 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & -83 & -10 & -20 & 10 & 95 & 0 \\ 0 & -183 & -18 & -30 & 12 & 213 & 0 \\ 0 & -68 & -10 & -20 & 10 & 80 & 0 \\ 0 & 48 & 0 & 6 & -6 & -54 & 0 \\ -14 & 23 & -10 & -20 & 10 & -11 & 0 \end{array}\right]$$

$$R_6 = R_6 - 7R_1 \Longrightarrow \left[\begin{array}{cccccc|c} -2 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & -83 & -10 & -20 & 10 & 95 & 0 \\ 0 & -183 & -18 & -30 & 12 & 213 & 0 \\ 0 & -68 & -10 & -20 & 10 & 80 & 0 \\ 0 & 48 & 0 & 6 & -6 & -54 & 0 \\ 0 & -68 & -10 & -20 & 10 & 80 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{183R_2}{83} \Longrightarrow \left[\begin{array}{cccccc|c} -2 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & -83 & -10 & -20 & 10 & 95 & 0 \\ 0 & 0 & \frac{336}{83} & \frac{1170}{83} & -\frac{834}{83} & \frac{294}{83} & 0 \\ 0 & -68 & -10 & -20 & 10 & 80 & 0 \\ 0 & 48 & 0 & 6 & -6 & -54 & 0 \\ 0 & -68 & -10 & -20 & 10 & 80 & 0 \end{array}\right]$$

$$R_4 = R_4 - \frac{68R_2}{83} \Longrightarrow \left[\begin{array}{cccccc|c} -2 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & -83 & -10 & -20 & 10 & 95 & 0 \\ 0 & 0 & \frac{336}{83} & \frac{1170}{83} & -\frac{834}{83} & \frac{294}{83} & 0 \\ 0 & 0 & -\frac{150}{83} & -\frac{300}{83} & \frac{150}{83} & \frac{180}{83} & 0 \\ 0 & 48 & 0 & 6 & -6 & -54 & 0 \\ 0 & -68 & -10 & -20 & 10 & 80 & 0 \end{array}\right]$$

$$R_5 = R_5 + \frac{48R_2}{83} \Longrightarrow \left[\begin{array}{cccccc|c} -2 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & -83 & -10 & -20 & 10 & 95 & 0 \\ 0 & 0 & \frac{336}{83} & \frac{1170}{83} & -\frac{834}{83} & \frac{294}{83} & 0 \\ 0 & 0 & -\frac{150}{83} & -\frac{300}{83} & \frac{150}{83} & \frac{180}{83} & 0 \\ 0 & 0 & -\frac{480}{83} & -\frac{462}{83} & -\frac{18}{83} & \frac{78}{83} & 0 \\ 0 & -68 & -10 & -20 & 10 & 80 & 0 \end{array}\right]$$

$$R_6 = R_6 - \frac{68R_2}{83} \Longrightarrow \left[\begin{array}{cccccc|c} -2 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & -83 & -10 & -20 & 10 & 95 & 0 \\ 0 & 0 & \frac{336}{83} & \frac{1170}{83} & -\frac{834}{83} & \frac{294}{83} & 0 \\ 0 & 0 & -\frac{150}{83} & -\frac{300}{83} & \frac{150}{83} & \frac{180}{83} & 0 \\ 0 & 0 & -\frac{480}{83} & -\frac{462}{83} & -\frac{18}{83} & \frac{78}{83} & 0 \\ 0 & 0 & -\frac{150}{83} & -\frac{300}{83} & \frac{150}{83} & \frac{180}{83} & 0 \end{array}\right]$$

$$R_4 = R_4 + \frac{25R_3}{56} \Longrightarrow \left[\begin{array}{cccccc|c} -2 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & -83 & -10 & -20 & 10 & 95 & 0 \\ 0 & 0 & \frac{336}{83} & \frac{1170}{83} & -\frac{834}{83} & \frac{294}{83} & 0 \\ 0 & 0 & 0 & \frac{75}{28} & -\frac{75}{28} & \frac{15}{4} & 0 \\ 0 & 0 & -\frac{480}{83} & -\frac{462}{83} & -\frac{18}{83} & \frac{78}{83} & 0 \\ 0 & 0 & -\frac{150}{83} & -\frac{300}{83} & \frac{150}{83} & \frac{180}{83} & 0 \end{array}\right]$$

$$R_5 = R_5 + \frac{10R_3}{7} \Longrightarrow \left[\begin{array}{cccccc|c} -2 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & -83 & -10 & -20 & 10 & 95 & 0 \\ 0 & 0 & \frac{336}{83} & \frac{1170}{83} & -\frac{834}{83} & \frac{294}{83} & 0 \\ 0 & 0 & 0 & \frac{75}{28} & -\frac{75}{28} & \frac{15}{4} & 0 \\ 0 & 0 & 0 & \frac{102}{7} & -\frac{102}{7} & 6 & 0 \\ 0 & 0 & -\frac{150}{83} & -\frac{300}{83} & \frac{150}{83} & \frac{180}{83} & 0 \end{array}\right]$$

$$R_6 = R_6 + \frac{25R_3}{56} \Longrightarrow \left[\begin{array}{cccccc|c} -2 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & -83 & -10 & -20 & 10 & 95 & 0 \\ 0 & 0 & \frac{336}{83} & \frac{1170}{83} & -\frac{834}{83} & \frac{294}{83} & 0 \\ 0 & 0 & 0 & \frac{75}{28} & -\frac{75}{28} & \frac{15}{4} & 0 \\ 0 & 0 & 0 & \frac{102}{7} & -\frac{102}{7} & 6 & 0 \\ 0 & 0 & 0 & \frac{75}{28} & -\frac{75}{28} & \frac{15}{4} & 0 \end{array}\right]$$

$$R_5 = R_5 - \frac{136R_4}{25} \Longrightarrow \left[\begin{array}{cccccc|c} -2 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & -83 & -10 & -20 & 10 & 95 & 0 \\ 0 & 0 & \frac{336}{83} & \frac{1170}{83} & -\frac{834}{83} & \frac{294}{83} & 0 \\ 0 & 0 & 0 & \frac{75}{28} & -\frac{75}{28} & \frac{15}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{72}{5} & 0 \\ 0 & 0 & 0 & \frac{75}{28} & -\frac{75}{28} & \frac{15}{4} & 0 \end{array}\right]$$

$$R_6 = R_6 - R_4 \Longrightarrow \left[\begin{array}{cccccc|c} -2 & 13 & 0 & 0 & 0 & -13 & 0 \\ 0 & -83 & -10 & -20 & 10 & 95 & 0 \\ 0 & 0 & \frac{336}{83} & \frac{1170}{83} & -\frac{834}{83} & \frac{294}{83} & 0 \\ 0 & 0 & 0 & \frac{75}{28} & -\frac{75}{28} & \frac{15}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{72}{5} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & 13 & 0 & 0 & 0 & -13 \\ 0 & -83 & -10 & -20 & 10 & 95 \\ 0 & 0 & \frac{336}{83} & \frac{1170}{83} & -\frac{834}{83} & \frac{294}{83} \\ 0 & 0 & 0 & \frac{75}{28} & -\frac{75}{28} & \frac{15}{4} \\ 0 & 0 & 0 & 0 & 0 & -\frac{72}{5} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_5\}$ and the leading variables are $\{v_1, v_2, v_3, v_4, v_6\}$. Let $v_5 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_2 = 0, v_3 = -t, v_4 = t, v_6 = 0\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ 0 \\ -t \\ t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -t \\ t \\ t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this

eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ 0 \\ -t \\ t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ 0 \\ -t \\ t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
11	1	1	No	$\begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$
9	1	1	No	$\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$
-4	1	1	No	$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$
5	1	1	No	$\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$
3	1	1	No	$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 11 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{11t} \\
&= \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} e^{11t}
\end{aligned}
$$

Since eigenvalue 9 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{9t} \\
&= \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} e^{9t}
\end{aligned}
$$

Since eigenvalue -4 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_3(t) &= \vec{v}_3 e^{-4t} \\
&= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} e^{-4t}
\end{aligned}
$$

Since eigenvalue 5 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_4(t) &= \vec{v}_4 e^{5t} \\ &= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} e^{5t}\end{aligned}$$

Since eigenvalue 3 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_5(t) &= \vec{v}_5 e^{3t} \\ &= \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} e^{3t}\end{aligned}$$

Since eigenvalue -7 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_6(t) &= \vec{v}_6 e^{-7t} \\ &= \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} e^{-7t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t) + c_4\vec{x}_4(t) + c_5\vec{x}_5(t) + c_6\vec{x}_6(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ x_5(t) \\ x_6(t) \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ 0 \\ -\mathrm{e}^{11t} \\ \mathrm{e}^{11t} \\ \mathrm{e}^{11t} \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{9t} \\ \mathrm{e}^{9t} \\ 0 \\ 0 \\ 0 \\ \mathrm{e}^{9t} \end{bmatrix} + c_3 \begin{bmatrix} \mathrm{e}^{-4t} \\ 0 \\ 0 \\ 0 \\ \mathrm{e}^{-4t} \\ \mathrm{e}^{-4t} \end{bmatrix} + c_4 \begin{bmatrix} 0 \\ 0 \\ \mathrm{e}^{5t} \\ 0 \\ \mathrm{e}^{5t} \\ 0 \end{bmatrix} + c_5 \begin{bmatrix} 0 \\ \mathrm{e}^{3t} \\ 0 \\ \mathrm{e}^{3t} \\ 0 \\ \mathrm{e}^{3t} \end{bmatrix} + c_6 \begin{bmatrix} 0 \\ \mathrm{e}^{-7t} \\ \mathrm{e}^{-7t} \\ \mathrm{e}^{-7t} \\ 0 \\ \mathrm{e}^{-7t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ x_5(t) \\ x_6(t) \end{bmatrix} = \begin{bmatrix} (c_2\mathrm{e}^{13t} + c_3)\,\mathrm{e}^{-4t} \\ (c_2\mathrm{e}^{16t} + c_5\mathrm{e}^{10t} + c_6)\,\mathrm{e}^{-7t} \\ -(c_1\mathrm{e}^{18t} - c_4\mathrm{e}^{12t} - c_6)\,\mathrm{e}^{-7t} \\ (c_1\mathrm{e}^{18t} + c_5\mathrm{e}^{10t} + c_6)\,\mathrm{e}^{-7t} \\ (c_1\mathrm{e}^{15t} + c_4\mathrm{e}^{9t} + c_3)\,\mathrm{e}^{-4t} \\ (c_2\mathrm{e}^{16t} + c_5\mathrm{e}^{10t} + c_3\mathrm{e}^{3t} + c_6)\,\mathrm{e}^{-7t} \end{bmatrix}$$

4.39.3 Maple step by step solution

Let's solve

$[x_1'(t) = 9x_1(t) + 13x_2(t) - 13x_6(t)\,, x_2'(t) = -14x_1(t) + 19x_2(t) - 10x_3(t) - 20x_4(t) + 10x_5(t) +$

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ x_5(t) \\ x_6(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{\vec{x}}'(t) = \begin{bmatrix} 9 & 13 & 0 & 0 & 0 & -13 \\ -14 & 19 & -10 & -20 & 10 & 4 \\ -30 & 12 & -7 & -30 & 12 & 18 \\ -12 & 10 & -10 & -9 & 10 & 2 \\ 6 & 9 & 0 & 6 & 5 & -15 \\ -14 & 23 & -10 & -20 & 10 & 0 \end{bmatrix} \cdot \underline{\vec{x}}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{\vec{x}}'(t) = \begin{bmatrix} 9 & 13 & 0 & 0 & 0 & -13 \\ -14 & 19 & -10 & -20 & 10 & 4 \\ -30 & 12 & -7 & -30 & 12 & 18 \\ -12 & 10 & -10 & -9 & 10 & 2 \\ 6 & 9 & 0 & 6 & 5 & -15 \\ -14 & 23 & -10 & -20 & 10 & 0 \end{bmatrix} \cdot \underline{\vec{x}}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 9 & 13 & 0 & 0 & 0 & -13 \\ -14 & 19 & -10 & -20 & 10 & 4 \\ -30 & 12 & -7 & -30 & 12 & 18 \\ -12 & 10 & -10 & -9 & 10 & 2 \\ 6 & 9 & 0 & 6 & 5 & -15 \\ -14 & 23 & -10 & -20 & 10 & 0 \end{bmatrix}$$

- Rewrite the system as

  $\underline{\vec{x}}'(t) = A \cdot \underline{\vec{x}}(t)$
- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-7, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}\right], \left[-4, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}\right], \left[3, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}\right], \left[5, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right], \left[9, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right], \left[11, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-7, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_{1} = \mathrm{e}^{-7t} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[-4, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_2 = e^{-4t} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix} 3, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_3 = e^{3t} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix} 5, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\underrightarrow{x}_4 = e^{5t} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\left[9, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underrightarrow{x}_5 = e^{9t} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[11, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{x^{\rightarrow}}_6 = e^{11t} \cdot \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

- General solution to the system of ODEs

$\underline{x^{\rightarrow}} = c_1 \underline{x^{\rightarrow}}_1 + c_2 \underline{x^{\rightarrow}}_2 + c_3 \underline{x^{\rightarrow}}_3 + c_4 \underline{x^{\rightarrow}}_4 + c_5 \underline{x^{\rightarrow}}_5 + c_6 \underline{x^{\rightarrow}}_6$

- Substitute solutions into the general solution

$$\underline{x^{\rightarrow}} = c_1 e^{-7t} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + c_2 e^{-4t} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} + c_3 e^{3t} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + c_4 e^{5t} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c_5 e^{9t} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + c_6 e^{11t} \cdot \Bigg[$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ x_5(t) \\ x_6(t) \end{bmatrix} = \begin{bmatrix} (c_5 e^{13t} + c_2) e^{-4t} \\ (c_5 e^{16t} + c_3 e^{10t} + c_1) e^{-7t} \\ (-c_6 e^{18t} + c_4 e^{12t} + c_1) e^{-7t} \\ (c_6 e^{18t} + c_3 e^{10t} + c_1) e^{-7t} \\ (c_6 e^{15t} + c_4 e^{9t} + c_2) e^{-4t} \\ (c_5 e^{16t} + c_3 e^{10t} + c_2 e^{3t} + c_1) e^{-7t} \end{bmatrix}$$

- Solution to the system of ODEs

$\{x_1(t) = (c_5 e^{13t} + c_2) e^{-4t}, x_2(t) = (c_5 e^{16t} + c_3 e^{10t} + c_1) e^{-7t}, x_3(t) = (-c_6 e^{18t} + c_4 e^{12t} + c_1) e^{-7t}, x_4$

✓ Solution by Maple

Time used: 0.141 (sec). Leaf size: 135

```
dsolve([diff(x__1(t),t)=9*x__1(t)+13*x__2(t)+0*x__3(t)+0*x__4(t)+0*x__5(t)-13*x__6(t),diff(x
```

$$
\begin{aligned}
x_1(t) &= c_5 e^{-4t} + c_6 e^{9t} \\
x_2(t) &= c_6 e^{9t} + c_4 e^{3t} + e^{-7t} c_3 \\
x_3(t) &= e^{-7t} c_3 - e^{11t} c_2 + e^{5t} c_1 \\
x_4(t) &= e^{11t} c_2 + c_4 e^{3t} + e^{-7t} c_3 \\
x_5(t) &= e^{11t} c_2 + e^{5t} c_1 + c_5 e^{-4t} \\
x_6(t) &= c_6 e^{9t} + c_5 e^{-4t} + c_4 e^{3t} + e^{-7t} c_3
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.107 (sec). Leaf size: 1882

```
DSolve[{x1'[t]==9*x1[t]+13*x2[t]-13*x6[t],x2'[t]==-14*x1[t]+19*x2[t]-10*x3[t]-20*x4[t]+10*x5
```

Too large to display

5 Section 7.6, Multiple Eigenvalue Solutions. Examples. Page 437

5.1 problem Example 1

Internal problem ID [354]
Internal file name [`OUTPUT/354_Sunday_June_05_2022_01_39_32_AM_36099955/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.6, Multiple Eigenvalue Solutions. Examples. Page 437
Problem number: Example 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned}
x_1'(t) &= 9x_1(t) + 4x_2(t) \\
x_2'(t) &= -6x_1(t) - x_2(t) \\
x_3'(t) &= 6x_1(t) + 4x_2(t) + 3x_3(t)
\end{aligned}$$

5.1.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 9 & 4 & 0 \\ -6 & -1 & 0 \\ 6 & 4 & 3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} -2\,\mathrm{e}^{3t} + 3\,\mathrm{e}^{5t} & -2\,\mathrm{e}^{3t} + 2\,\mathrm{e}^{5t} & 0 \\ -3\,\mathrm{e}^{5t} + 3\,\mathrm{e}^{3t} & 3\,\mathrm{e}^{3t} - 2\,\mathrm{e}^{5t} & 0 \\ 3\,\mathrm{e}^{5t} - 3\,\mathrm{e}^{3t} & -2\,\mathrm{e}^{3t} + 2\,\mathrm{e}^{5t} & \mathrm{e}^{3t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} -2\,e^{3t}+3\,e^{5t} & -2\,e^{3t}+2\,e^{5t} & 0 \\ -3\,e^{5t}+3\,e^{3t} & 3\,e^{3t}-2\,e^{5t} & 0 \\ 3\,e^{5t}-3\,e^{3t} & -2\,e^{3t}+2\,e^{5t} & e^{3t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} (-2\,e^{3t}+3\,e^{5t})\,c_1+(-2\,e^{3t}+2\,e^{5t})\,c_2 \\ (-3\,e^{5t}+3\,e^{3t})\,c_1+(3\,e^{3t}-2\,e^{5t})\,c_2 \\ (3\,e^{5t}-3\,e^{3t})\,c_1+(-2\,e^{3t}+2\,e^{5t})\,c_2+e^{3t}c_3 \end{bmatrix} \\
&= \begin{bmatrix} (-2c_1-2c_2)\,e^{3t}+3\left(c_1+\frac{2c_2}{3}\right)e^{5t} \\ (3c_1+3c_2)\,e^{3t}-3\left(c_1+\frac{2c_2}{3}\right)e^{5t} \\ (-3c_1-2c_2+c_3)\,e^{3t}+3\left(c_1+\frac{2c_2}{3}\right)e^{5t} \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

5.1.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 9 & 4 & 0 \\ -6 & -1 & 0 \\ 6 & 4 & 3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A-\lambda I\right)=0$$

Expanding gives

$$\det\left(\begin{bmatrix} 9 & 4 & 0 \\ -6 & -1 & 0 \\ 6 & 4 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)=0$$

Therefore

$$\det\left(\begin{bmatrix} 9-\lambda & 4 & 0 \\ -6 & -1-\lambda & 0 \\ 6 & 4 & 3-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 11\lambda^2 + 39\lambda - 45 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 3$$
$$\lambda_2 = 5$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
3	1	real eigenvalue
5	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 9 & 4 & 0 \\ -6 & -1 & 0 \\ 6 & 4 & 3 \end{bmatrix} - (3)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 6 & 4 & 0 \\ -6 & -4 & 0 \\ 6 & 4 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 6 & 4 & 0 & 0 \\ -6 & -4 & 0 & 0 \\ 6 & 4 & 0 & 0 \end{array}\right]$$

$$
R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{ccc|c} 6 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 6 & 4 & 0 & 0 \end{array}\right]
$$

$$
R_3 = R_3 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} 6 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]
$$

Therefore the system in Echelon form is

$$
\begin{bmatrix} 6 & 4 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

The free variables are $\{v_2, v_3\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Let $v_3 = s$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{2t}{3}\right\}$

Hence the solution is

$$
\begin{bmatrix} -\frac{2t}{3} \\ t \\ s \end{bmatrix} = \begin{bmatrix} -\frac{2t}{3} \\ t \\ s \end{bmatrix}
$$

Since there are two free Variable, we have found two eigenvectors associated with this eigenvalue. The above can be written as

$$
\begin{aligned}
\begin{bmatrix} -\frac{2t}{3} \\ t \\ s \end{bmatrix} &= \begin{bmatrix} -\frac{2t}{3} \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ s \end{bmatrix} \\
&= t \begin{bmatrix} -\frac{2}{3} \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\end{aligned}
$$

By letting $t = 1$ and $s = 1$ then the above becomes

$$
\begin{bmatrix} -\frac{2t}{3} \\ t \\ s \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
$$

Hence the two eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix}-\frac{2}{3}\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right)$$

Which are normalized to

$$\left(\begin{bmatrix}-2\\3\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right)$$

Considering the eigenvalue $\lambda_2 = 5$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix}9 & 4 & 0\\-6 & -1 & 0\\6 & 4 & 3\end{bmatrix} - (5)\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}4 & 4 & 0\\-6 & -6 & 0\\6 & 4 & -2\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c}4 & 4 & 0 & 0\\-6 & -6 & 0 & 0\\6 & 4 & -2 & 0\end{array}\right]$$

$$R_2 = R_2 + \frac{3R_1}{2} \implies \left[\begin{array}{ccc|c}4 & 4 & 0 & 0\\0 & 0 & 0 & 0\\6 & 4 & -2 & 0\end{array}\right]$$

$$R_3 = R_3 - \frac{3R_1}{2} \implies \left[\begin{array}{ccc|c}4 & 4 & 0 & 0\\0 & 0 & 0 & 0\\0 & -2 & -2 & 0\end{array}\right]$$

Since the current pivot $A(2,2)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{ccc|c} 4 & 4 & 0 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 4 & 4 & 0 \\ 0 & -2 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = -t\}$

Hence the solution is

$$\begin{bmatrix} t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} t \\ -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
3	2	2	No	$\begin{bmatrix} 0 & -\frac{2}{3} \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$
5	1	1	No	$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 3 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

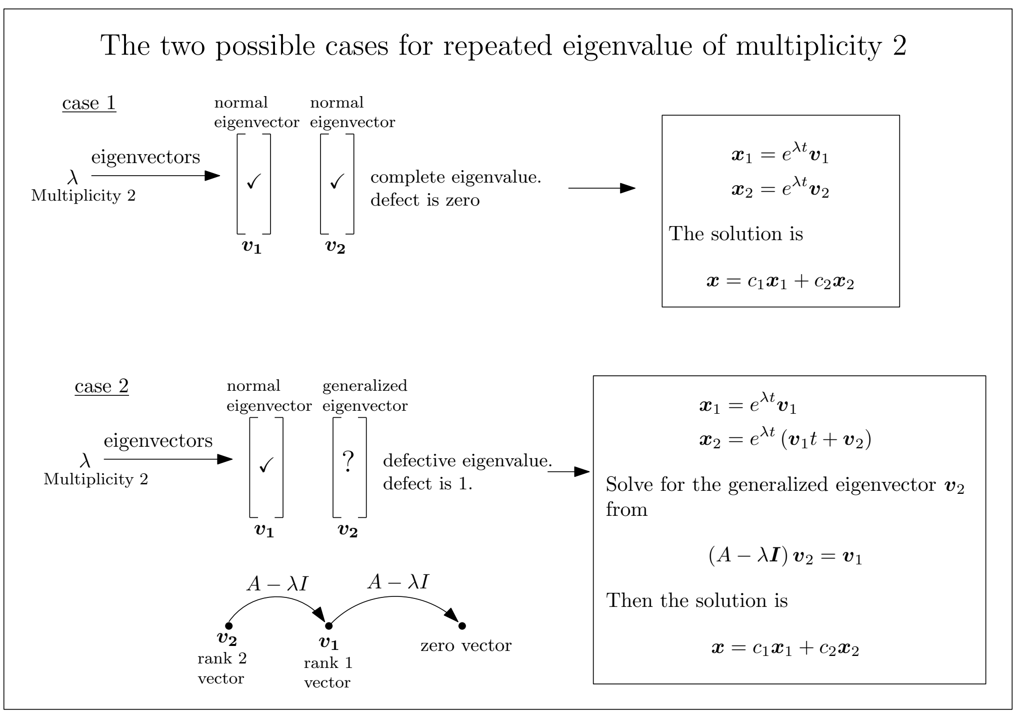


Figure 26: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2 which is the same as its geometric

multiplicity 2, then it is complete eigenvalue and this falls into case 1 shown above. Hence the corresponding eigenvector basis are

$$\vec{x}_1(t) = \vec{v}_1 e^{3t}$$
$$= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{3t}$$

$$\vec{x}_2(t) = \vec{v}_2 e^{3t}$$
$$= \begin{bmatrix} -\frac{2}{3} \\ 1 \\ 0 \end{bmatrix} e^{3t}$$

Since eigenvalue 5 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_3(t) = \vec{v}_3 e^{5t}$$
$$= \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^{5t}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ 0 \\ \mathrm{e}^{3t} \end{bmatrix} + c_2 \begin{bmatrix} -\frac{2\,\mathrm{e}^{3t}}{3} \\ \mathrm{e}^{3t} \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} \mathrm{e}^{5t} \\ -\mathrm{e}^{5t} \\ \mathrm{e}^{5t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} -\frac{2c_2\mathrm{e}^{3t}}{3} + c_3\mathrm{e}^{5t} \\ c_2\mathrm{e}^{3t} - c_3\mathrm{e}^{5t} \\ c_1\mathrm{e}^{3t} + c_3\mathrm{e}^{5t} \end{bmatrix}$$

5.1.3 Maple step by step solution

Let's solve

$[x_1'(t) = 9x_1(t) + 4x_2(t), x_2'(t) = -6x_1(t) - x_2(t), x_3'(t) = 6x_1(t) + 4x_2(t) + 3x_3(t)]$

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x}^{\rightarrow\prime}(t) = \begin{bmatrix} 9 & 4 & 0 \\ -6 & -1 & 0 \\ 6 & 4 & 3 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{x}^{\rightarrow\prime}(t) = \begin{bmatrix} 9 & 4 & 0 \\ -6 & -1 & 0 \\ 6 & 4 & 3 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 9 & 4 & 0 \\ -6 & -1 & 0 \\ 6 & 4 & 3 \end{bmatrix}$$

- Rewrite the system as

$\underline{x}^{\rightarrow\prime}(t) = A \cdot \underline{x}^{\rightarrow}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[3, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right], \left[3, \begin{bmatrix} -\frac{2}{3} \\ 1 \\ 0 \end{bmatrix}\right], \left[5, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[3, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue 3

$$\vec{x}\_\_\_1(t) = e^{3t} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = 3$ is the eigenvalue, and

$$\vec{x}\_\_\_2(t) = e^{\lambda t}\left(t\vec{v} + \vec{p}\right)$$

- Note that the t multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\vec{x}\_\_\_2(t)$ into the homogeneous system

$$\lambda e^{\lambda t}\left(t\vec{v} + \vec{p}\right) + e^{\lambda t}\vec{v} = \left(e^{\lambda t}A\right) \cdot \left(t\vec{v} + \vec{p}\right)$$

- Use the fact that $\vec{v}$ is an eigenvector of A

$$\lambda e^{\lambda t}\left(t\vec{v} + \vec{p}\right) + e^{\lambda t}\vec{v} = e^{\lambda t}\left(\lambda t\vec{v} + A \cdot \vec{p}\right)$$

- Simplify equation

$$\lambda\vec{p} + \vec{v} = A \cdot \vec{p}$$

- Make use of the identity matrix I

$$(\lambda \cdot I) \cdot \vec{p} + \vec{v} = A \cdot \vec{p}$$

- Condition $\vec{p}$ must meet for $\vec{x}\_\_\_2(t)$ to be a solution to the homogeneous system

$$(A - \lambda \cdot I) \cdot \vec{p} = \vec{v}$$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue 3

$$\left(\begin{bmatrix} 9 & 4 & 0 \\ -6 & -1 & 0 \\ 6 & 4 & 3 \end{bmatrix} - 3 \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \cdot \vec{p} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

- Choice of $\vec{p}$

$$\vec{p} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue 3

$$\underline{x^{\rightarrow}}_2(t) = e^{3t} \cdot \left(t \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right)$$

- Consider eigenpair

$$\left[5, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{x^{\rightarrow}}_3 = e^{5t} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$\underline{x^{\rightarrow}} = c_1 \underline{x^{\rightarrow}}_1(t) + c_2 \underline{x^{\rightarrow}}_2(t) + c_3 \underline{x^{\rightarrow}}_3$

- Substitute solutions into the general solution

$$\underline{x^{\rightarrow}} = c_1 e^{3t} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + c_2 e^{3t} \cdot \left(t \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right) + c_3 e^{5t} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} c_3 e^{5t} \\ -c_3 e^{5t} \\ (c_2 t + c_1) e^{3t} + c_3 e^{5t} \end{bmatrix}$$

- Solution to the system of ODEs

$\{x_1(t) = c_3 e^{5t}, x_2(t) = -c_3 e^{5t}, x_3(t) = (c_2 t + c_1) e^{3t} + c_3 e^{5t}\}$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 58

```
dsolve([diff(x__1(t),t)=9*x__1(t)+4*x__2(t)+0*x__3(t),diff(x__2(t),t)=-6*x__1(t)-1*x__2(t)+0
```

$$x_1(t) = c_2 e^{3t} + c_3 e^{5t}$$
$$x_2(t) = -\frac{3c_2 e^{3t}}{2} - c_3 e^{5t}$$
$$x_3(t) = c_2 e^{3t} + c_3 e^{5t} + c_1 e^{3t}$$

✓ Solution by Mathematica

Time used: 0.009 (sec). Leaf size: 113

```
DSolve[{x1'[t]==9*x1[t]+4*x2[t]+0*x3[t],x2'[t]==-6*x1[t]-1*x2[t]+0*x3[t],x3'[t]==6*x1[t]+4*x
```

$$\text{x1}(t) \to e^{3t}\left(c_1\left(3e^{2t} - 2\right) + 2c_2\left(e^{2t} - 1\right)\right)$$
$$\text{x2}(t) \to -e^{3t}\left(3c_1\left(e^{2t} - 1\right) + c_2\left(2e^{2t} - 3\right)\right)$$
$$\text{x3}(t) \to \int_1^t 3x(K[1])dK[1] + \frac{6}{5}c_1\left(e^{5t} - 1\right) + \frac{4}{5}c_2\left(e^{5t} - 1\right) + c_3$$

5.2 problem Example 3

Internal problem ID [355]
Internal file name [`OUTPUT/355_Sunday_June_05_2022_01_39_33_AM_56987758/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.6, Multiple Eigenvalue Solutions. Examples. Page 437
Problem number: Example 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x_1'(t) = x_1(t) - 3x_2(t)$$
$$x_2'(t) = 3x_1(t) + 7x_2(t)$$

5.2.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \mathrm{e}^{4t}(1-3t) & -3t\,\mathrm{e}^{4t} \\ 3t\,\mathrm{e}^{4t} & \mathrm{e}^{4t}(1+3t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \mathrm{e}^{4t}(1-3t) & -3t\,\mathrm{e}^{4t} \\ 3t\,\mathrm{e}^{4t} & \mathrm{e}^{4t}(1+3t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^{4t}(1-3t)\,c_1 - 3t\,\mathrm{e}^{4t}c_2 \\ 3t\,\mathrm{e}^{4t}c_1 + \mathrm{e}^{4t}(1+3t)\,c_2 \end{bmatrix} \\
&= \begin{bmatrix} (c_1(1-3t) - 3c_2t)\,\mathrm{e}^{4t} \\ \mathrm{e}^{4t}(3tc_1 + 3c_2t + c_2) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

5.2.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det(A - \lambda I) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} 1 & -3 \\ 3 & 7 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} 1-\lambda & -3 \\ 3 & 7-\lambda \end{bmatrix}\right) = 0
$$

Which gives the characteristic equation

$$\lambda^2 - 8\lambda + 16 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 4$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
4	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 4$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -3 \\ 3 & 7 \end{bmatrix} - (4)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -3 & -3 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -3 & -3 & 0 \\ 3 & 3 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cc|c} -3 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -3 & -3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
4	2	1	Yes	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 4 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

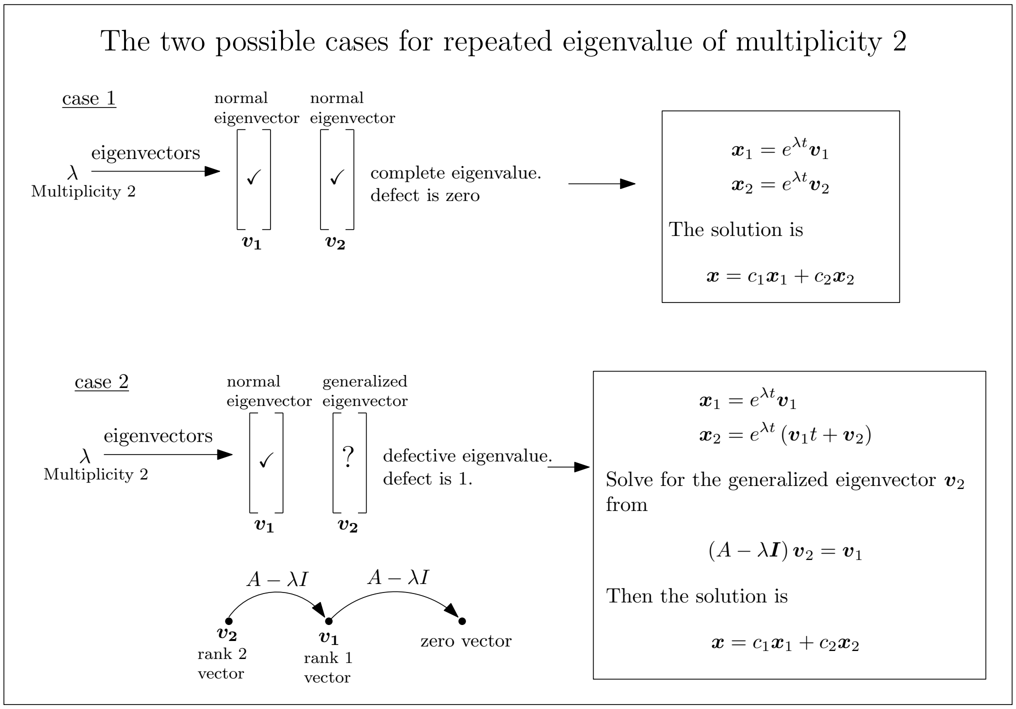


Figure 27: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} 1 & -3 \\ 3 & 7 \end{bmatrix} - (4)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} -3 & -3 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue 4. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t}\\ &= \begin{bmatrix} -1 \\ 1 \end{bmatrix} \mathrm{e}^{4t}\\ &= \begin{bmatrix} -\mathrm{e}^{4t} \\ \mathrm{e}^{4t} \end{bmatrix}\end{aligned}$$

And

$$\begin{aligned}\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t}\\ &= \left(\begin{bmatrix} -1 \\ 1 \end{bmatrix} t + \begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix}\right) \mathrm{e}^{4t}\\ &= \begin{bmatrix} -\frac{\mathrm{e}^{4t}(3t+2)}{3} \\ \mathrm{e}^{4t}(t+1) \end{bmatrix}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} -\mathrm{e}^{4t} \\ \mathrm{e}^{4t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{4t}\left(-t - \frac{2}{3}\right) \\ \mathrm{e}^{4t}(t+1) \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{4t}\left(-c_1 - c_2 t - \frac{2}{3} c_2\right) \\ \mathrm{e}^{4t}(c_2 t + c_1 + c_2) \end{bmatrix}$$

The following is the phase plot of the system.

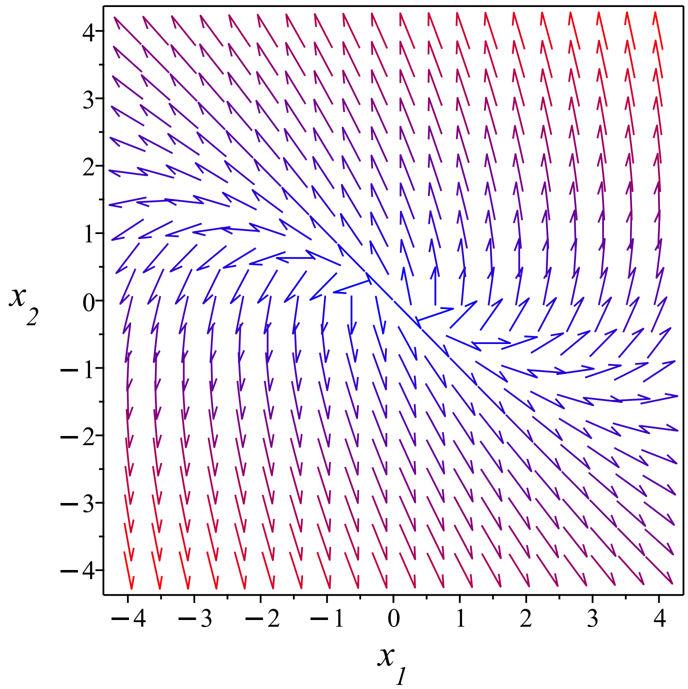


Figure 28: Phase plot

5.2.3 Maple step by step solution

Let's solve

$[x_1'(t) = x_1(t) - 3x_2(t), x_2'(t) = 3x_1(t) + 7x_2(t)]$

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x}^{\rightarrow\prime}(t) = \begin{bmatrix} 1 & -3 \\ 3 & 7 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{x}^{\rightarrow\prime}(t) = \begin{bmatrix} 1 & -3 \\ 3 & 7 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & -3 \\ 3 & 7 \end{bmatrix}$$

- Rewrite the system as

  $\underline{x^{\rightarrow}}'(t) = A \cdot \underline{x^{\rightarrow}}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[4, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right], \left[4, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

  $$\left[4, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue 4

  $$\underline{x^{\rightarrow}}_1(t) = e^{4t} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = 4$ is the eigenvalue, and

  $$\underline{x^{\rightarrow}}_2(t) = e^{\lambda t}\left(t\vec{v} + \vec{p}\right)$$

- Note that the t multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\underline{x^{\rightarrow}}_2(t)$ into the homogeneous system

  $$\lambda e^{\lambda t}\left(t\vec{v} + \vec{p}\right) + e^{\lambda t}\vec{v} = \left(e^{\lambda t} A\right) \cdot \left(t\vec{v} + \vec{p}\right)$$

- Use the fact that $\vec{v}$ is an eigenvector of A

  $$\lambda e^{\lambda t}\left(t\vec{v} + \vec{p}\right) + e^{\lambda t}\vec{v} = e^{\lambda t}\left(\lambda t\vec{v} + A \cdot \vec{p}\right)$$

- Simplify equation

  $$\lambda\vec{p} + \vec{v} = A \cdot \vec{p}$$

- Make use of the identity matrix I

  $$(\lambda \cdot I) \cdot \vec{p} + \vec{v} = A \cdot \vec{p}$$

- Condition $\vec{p}$ must meet for $\underline{x^{\rightarrow}}_2(t)$ to be a solution to the homogeneous system

  $$(A - \lambda \cdot I) \cdot \vec{p} = \vec{v}$$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue 4

$$\left(\begin{bmatrix} 1 & -3 \\ 3 & 7 \end{bmatrix} - 4 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \cdot \vec{p} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- Choice of $\vec{p}$

$$\vec{p} = \begin{bmatrix} \frac{1}{3} \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue 4

$$\underline{x^{\rightarrow}}_2(t) = e^{4t} \cdot \left(t \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{3} \\ 0 \end{bmatrix}\right)$$

- General solution to the system of ODEs

$$\underline{x^{\rightarrow}} = c_1 \underline{x^{\rightarrow}}_1(t) + c_2 \underline{x^{\rightarrow}}_2(t)$$

- Substitute solutions into the general solution

$$\underline{x^{\rightarrow}} = c_1 e^{4t} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{4t} \cdot \left(t \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{3} \\ 0 \end{bmatrix}\right)$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} e^{4t}\left(-c_1 - c_2 t + \frac{1}{3} c_2\right) \\ e^{4t}(c_2 t + c_1) \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x_1(t) = e^{4t}\left(-c_1 - c_2 t + \frac{1}{3} c_2\right), x_2(t) = e^{4t}(c_2 t + c_1)\right\}$$

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 33

```
dsolve([diff(x__1(t),t)=1*x__1(t)-3*x__2(t),diff(x__2(t),t)=3*x__1(t)+7*x__2(t)],singsol=all
```

$$x_1(t) = e^{4t}(c_2 t + c_1)$$
$$x_2(t) = -\frac{e^{4t}(3c_2 t + 3c_1 + c_2)}{3}$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 46

```
DSolve[{x1'[t]==1*x1[t]-3*x2[t],x2'[t]==3*x1[t]+7*x2[t]},{x1[t],x2[t]},t,IncludeSingularSolu
```

$$
\begin{aligned}
\text{x1}(t) &\to -e^{4t}(c_1(3t-1)+3c_2t) \\
\text{x2}(t) &\to e^{4t}(3(c_1+c_2)t+c_2)
\end{aligned}
$$

5.3 problem Example 4

Internal problem ID [356]
Internal file name [`OUTPUT/356_Sunday_June_05_2022_01_39_34_AM_40841151/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.6, Multiple Eigenvalue Solutions. Examples. Page 437
Problem number: Example 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= x_2(t) + 2x_3(t) \\
x_2'(t) &= -5x_1(t) - 3x_2(t) - 7x_3(t) \\
x_3'(t) &= x_1(t)
\end{aligned}
$$

5.3.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ -5 & -3 & -7 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \mathrm{e}^{-t}(-t^2+t+1) & -\frac{\mathrm{e}^{-t}t(t-2)}{2} & -\frac{\mathrm{e}^{-t}t(3t-4)}{2} \\ -\mathrm{e}^{-t}t(t+5) & \mathrm{e}^{-t}\left(1-\frac{1}{2}t^2-2t\right) & -\frac{\mathrm{e}^{-t}t(3t+14)}{2} \\ \mathrm{e}^{-t}t(t+1) & \frac{t^2\mathrm{e}^{-t}}{2} & \mathrm{e}^{-t}\left(1+\frac{3}{2}t^2+t\right) \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{-t}(-t^2+t+1) & -\frac{e^{-t}t(t-2)}{2} & -\frac{e^{-t}t(3t-4)}{2} \\ -e^{-t}t(t+5) & e^{-t}\left(1-\frac{1}{2}t^2-2t\right) & -\frac{e^{-t}t(3t+14)}{2} \\ e^{-t}t(t+1) & \frac{t^2e^{-t}}{2} & e^{-t}\left(1+\frac{3}{2}t^2+t\right) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} e^{-t}(-t^2+t+1)\,c_1 - \frac{e^{-t}t(t-2)c_2}{2} - \frac{e^{-t}t(3t-4)c_3}{2} \\ -e^{-t}t(t+5)\,c_1 + e^{-t}\left(1-\frac{1}{2}t^2-2t\right)c_2 - \frac{e^{-t}t(3t+14)c_3}{2} \\ e^{-t}t(t+1)\,c_1 + \frac{t^2e^{-t}c_2}{2} + e^{-t}\left(1+\frac{3}{2}t^2+t\right)c_3 \end{bmatrix} \\
&= \begin{bmatrix} -\left(\left(c_1+\frac{c_2}{2}+\frac{3c_3}{2}\right)t^2 + (-c_1-c_2-2c_3)\,t - c_1\right)e^{-t} \\ -\left(\left(c_1+\frac{c_2}{2}+\frac{3c_3}{2}\right)t^2 + (5c_1+2c_2+7c_3)\,t - c_2\right)e^{-t} \\ \left(\left(c_1+\frac{c_2}{2}+\frac{3c_3}{2}\right)t^2 + (c_1+c_3)\,t + c_3\right)e^{-t} \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

5.3.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ -5 & -3 & -7 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 0 & 1 & 2 \\ -5 & -3 & -7 \\ 1 & 0 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -\lambda & 1 & 2 \\ -5 & -3-\lambda & -7 \\ 1 & 0 & -\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 1 & 2 \\ -5 & -3 & -7 \\ 1 & 0 & 0 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 \\ -5 & -2 & -7 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ -5 & -2 & -7 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 + 5R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 3 & 3 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right]$$

$$
R_3 = R_3 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right]
$$

$$
R_3 = R_3 + \frac{R_2}{3} \Longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]
$$

Therefore the system in Echelon form is

$$
\begin{bmatrix} 1 & 1 & 2 \\ 0 & 3 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t, v_2 = -t\}$

Hence the solution is

$$
\begin{bmatrix} -t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ -t \\ t \end{bmatrix}
$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$
\begin{bmatrix} -t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}
$$

Let $t = 1$ the eigenvector becomes

$$
\begin{bmatrix} -t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}
$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated

with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
-1	3	1	Yes	$\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue -1 is real and repated eigenvalue of multiplicity 3.There are three possible cases that can happen. This is illustrated in this diagram

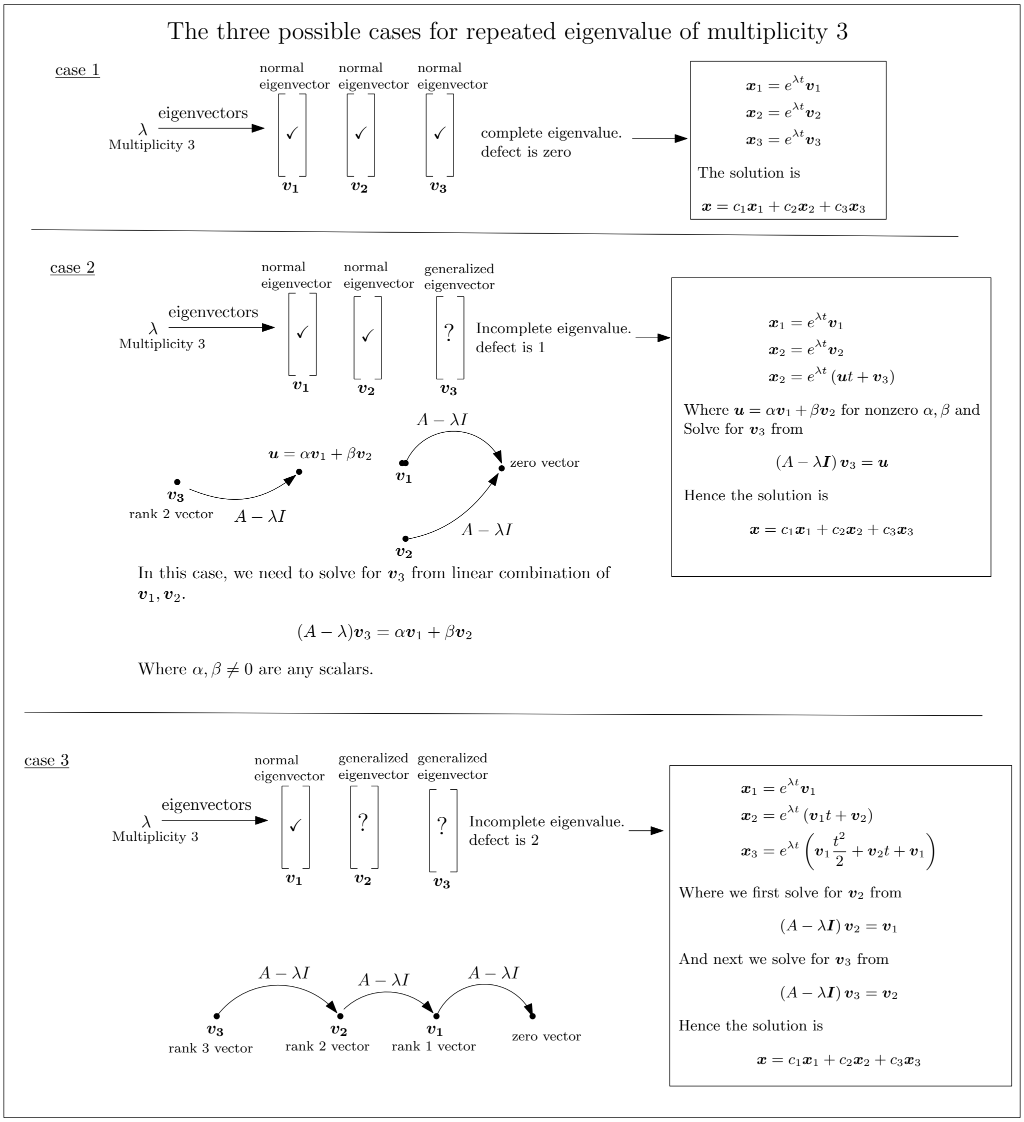


Figure 29: Possible case for repeated λ of multiplicity 3

This eigenvalue has algebraic multiplicity of 3, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 2. This falls into case 3 shown above. First we find generalized eigenvector $\vec{v}_2$ of rank 2 and then use this to find generalized eigenvector $\vec{v}_3$ of rank 3.$\vec{v}_2$ is found by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence

$$\left(\begin{bmatrix}0 & 1 & 2\\ -5 & -3 & -7\\ 1 & 0 & 0\end{bmatrix} - (-1)\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}-1\\ -1\\ 1\end{bmatrix}$$

$$\begin{bmatrix}1 & 1 & 2\\ -5 & -2 & -7\\ 1 & 0 & 1\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}-1\\ -1\\ 1\end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix}0\\ -3\\ 1\end{bmatrix}$$

Now $\vec{v}_3$ is found by solving

$$(A - \lambda I)\,\vec{v}_3 = \vec{v}_2$$

Where $\vec{v}_2$ is the (rank 2) generalized eigenvector found above. Hence

$$\left(\begin{bmatrix}0 & 1 & 2\\ -5 & -3 & -7\\ 1 & 0 & 0\end{bmatrix} - (-1)\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}0\\ -3\\ 1\end{bmatrix}$$

$$\begin{bmatrix}1 & 1 & 2\\ -5 & -2 & -7\\ 1 & 0 & 1\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}0\\ -3\\ 1\end{bmatrix}$$

Solving for $\vec{v}_3$ gives

$$\vec{v}_3 = \begin{bmatrix}0\\ -2\\ 1\end{bmatrix}$$

We have found three generalized eigenvectors for eigenvalue -1. Therefore the three

basis solutions associated with this eigenvalue are

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\
&= \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \mathrm{e}^{-t} \\
&= \begin{bmatrix} -\mathrm{e}^{-t} \\ -\mathrm{e}^{-t} \\ \mathrm{e}^{-t} \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_2(t) &= e^{\lambda t}(\vec{v}_1 t + \vec{v}_2) \\
&= \mathrm{e}^{-t} \left(\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} t + \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix} \right) \\
&= \begin{bmatrix} -t\,\mathrm{e}^{-t} \\ -\mathrm{e}^{-t}(t+3) \\ \mathrm{e}^{-t}(t+1) \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_3(t) &= \left(\vec{v}_1 \frac{t^2}{2} + \vec{v}_2 t + \vec{v}_3 \right) e^{\lambda t} \\
&= \left(\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \frac{t^2}{2} + \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix} t + \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \right) \mathrm{e}^{-t} \\
&= \begin{bmatrix} -\frac{t^2 \mathrm{e}^{-t}}{2} \\ -\frac{\mathrm{e}^{-t}(t^2+6t+4)}{2} \\ \frac{\mathrm{e}^{-t}(t^2+2t+2)}{2} \end{bmatrix}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)
$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 \begin{bmatrix} -\mathrm{e}^{-t} \\ -\mathrm{e}^{-t} \\ \mathrm{e}^{-t} \end{bmatrix} + c_2 \begin{bmatrix} -t\,\mathrm{e}^{-t} \\ \mathrm{e}^{-t}(-t-3) \\ \mathrm{e}^{-t}(t+1) \end{bmatrix} + c_3 \begin{bmatrix} -\frac{t^2 \mathrm{e}^{-t}}{2} \\ \mathrm{e}^{-t}\left(-\frac{1}{2}t^2 - 3t - 2\right) \\ \mathrm{e}^{-t}\left(t + \frac{1}{2}t^2 + 1\right) \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{-t}\left(-c_1 - tc_2 - \frac{1}{2}t^2 c_3\right) \\ -\frac{\mathrm{e}^{-t}\left((t^2+6t+4)c_3+2tc_2+2c_1+6c_2\right)}{2} \\ \frac{\mathrm{e}^{-t}\left((t^2+2t+2)c_3+2tc_2+2c_1+2c_2\right)}{2} \end{bmatrix}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 75

```
dsolve([diff(x__1(t),t)=0*x__1(t)+1*x__2(t)+2*x__3(t),diff(x__2(t),t)=-5*x__1(t)-3*x__2(t)-7
```

$$x_1(t) = -\mathrm{e}^{-t}\left(c_3 t^2 + c_2 t - 2c_3 t + c_1 - c_2\right)$$
$$x_2(t) = -\mathrm{e}^{-t}\left(c_3 t^2 + c_2 t + 4c_3 t + c_1 + 2c_2 - 2c_3\right)$$
$$x_3(t) = \mathrm{e}^{-t}\left(c_3 t^2 + c_2 t + c_1\right)$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 134

```
DSolve[{x1'[t]==0*x1[t]+1*x2[t]+2*x3[t],x2'[t]==-5*x1[t]-3*x2[t]-7*x3[t],x3'[t]==1*x1[t]+0*x
```

$$\mathrm{x1}(t) \to \frac{1}{2}e^{-t}\left(c_1\left(-2t^2+2t+2\right) - c_2(t-2)t + c_3(4-3t)t\right)$$
$$\mathrm{x2}(t) \to \frac{1}{2}e^{-t}\left(-\left((2c_1+c_2+3c_3)t^2\right) - 2(5c_1+2c_2+7c_3)t + 2c_2\right)$$
$$\mathrm{x3}(t) \to \frac{1}{2}e^{-t}\left((2c_1+c_2+3c_3)t^2 + 2(c_1+c_3)t + 2c_3\right)$$

5.4 problem Example 6

Internal problem ID [357]
Internal file name [OUTPUT/357_Sunday_June_05_2022_01_39_35_AM_28085054/index.tex]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.6, Multiple Eigenvalue Solutions. Examples. Page 437
Problem number: Example 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= x_3(t) \\
x_2'(t) &= x_4(t) \\
x_3'(t) &= -2x_1(t) + 2x_2(t) - 3x_3(t) + x_4(t) \\
x_4'(t) &= 2x_1(t) - 2x_2(t) + x_3(t) - 3x_4(t)
\end{aligned}
$$

5.4.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \\ x_4'(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 2 & -3 & 1 \\ 2 & -2 & 1 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{e^{-2t}}{2}+\frac{1}{2}+t\,e^{-2t} & \frac{1}{2}-\frac{e^{-2t}}{2}-t\,e^{-2t} & \frac{1}{4}+\frac{(2t-1)e^{-2t}}{4} & \frac{1}{4}+\frac{(-2t-1)e^{-2t}}{4} \\ \frac{1}{2}-\frac{e^{-2t}}{2}-t\,e^{-2t} & \frac{e^{-2t}}{2}+\frac{1}{2}+t\,e^{-2t} & \frac{1}{4}+\frac{(-2t-1)e^{-2t}}{4} & \frac{1}{4}+\frac{(2t-1)e^{-2t}}{4} \\ -2t\,e^{-2t} & 2t\,e^{-2t} & e^{-2t}(1-t) & t\,e^{-2t} \\ 2t\,e^{-2t} & -2t\,e^{-2t} & t\,e^{-2t} & e^{-2t}(1-t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{e^{-2t}}{2}+\frac{1}{2}+t\,e^{-2t} & \frac{1}{2}-\frac{e^{-2t}}{2}-t\,e^{-2t} & \frac{1}{4}+\frac{(2t-1)e^{-2t}}{4} & \frac{1}{4}+\frac{(-2t-1)e^{-2t}}{4} \\ \frac{1}{2}-\frac{e^{-2t}}{2}-t\,e^{-2t} & \frac{e^{-2t}}{2}+\frac{1}{2}+t\,e^{-2t} & \frac{1}{4}+\frac{(-2t-1)e^{-2t}}{4} & \frac{1}{4}+\frac{(2t-1)e^{-2t}}{4} \\ -2t\,e^{-2t} & 2t\,e^{-2t} & e^{-2t}(1-t) & t\,e^{-2t} \\ 2t\,e^{-2t} & -2t\,e^{-2t} & t\,e^{-2t} & e^{-2t}(1-t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} \\
&= \begin{bmatrix} \left(\frac{e^{-2t}}{2}+\frac{1}{2}+t\,e^{-2t}\right)c_1+\left(\frac{1}{2}-\frac{e^{-2t}}{2}-t\,e^{-2t}\right)c_2+\left(\frac{1}{4}+\frac{(2t-1)e^{-2t}}{4}\right)c_3+\left(\frac{1}{4}+\frac{(-2t-1)e^{-2t}}{4}\right)c_4 \\ \left(\frac{1}{2}-\frac{e^{-2t}}{2}-t\,e^{-2t}\right)c_1+\left(\frac{e^{-2t}}{2}+\frac{1}{2}+t\,e^{-2t}\right)c_2+\left(\frac{1}{4}+\frac{(-2t-1)e^{-2t}}{4}\right)c_3+\left(\frac{1}{4}+\frac{(2t-1)e^{-2t}}{4}\right)c_4 \\ -2t\,e^{-2t}c_1+2t\,e^{-2t}c_2+e^{-2t}(1-t)\,c_3+t\,e^{-2t}c_4 \\ 2t\,e^{-2t}c_1-2t\,e^{-2t}c_2+t\,e^{-2t}c_3+e^{-2t}(1-t)\,c_4 \end{bmatrix} \\
&= \begin{bmatrix} \frac{((4c_1-4c_2+2c_3-2c_4)t+2c_1-2c_2-c_3-c_4)e^{-2t}}{4}+\frac{c_1}{2}+\frac{c_2}{2}+\frac{c_3}{4}+\frac{c_4}{4} \\ \frac{((-4c_1+4c_2-2c_3+2c_4)t-2c_1+2c_2-c_3-c_4)e^{-2t}}{4}+\frac{c_1}{2}+\frac{c_2}{2}+\frac{c_3}{4}+\frac{c_4}{4} \\ -2\left(\left(c_1-c_2+\frac{c_3}{2}-\frac{c_4}{2}\right)t-\frac{c_3}{2}\right)e^{-2t} \\ 2\left(\left(c_1-c_2+\frac{c_3}{2}-\frac{c_4}{2}\right)t+\frac{c_4}{2}\right)e^{-2t} \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

5.4.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \\ x_4'(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 2 & -3 & 1 \\ 2 & -2 & 1 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 2 & -3 & 1 \\ 2 & -2 & 1 & -3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -\lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 1 \\ -2 & 2 & -3-\lambda & 1 \\ 2 & -2 & 1 & -3-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^4 + 6\lambda^3 + 12\lambda^2 + 8\lambda = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -2$$
$$\lambda_2 = 0$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
0	1	real eigenvalue
-2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 2 & -3 & 1 \\ 2 & -2 & 1 & -3 \end{bmatrix} - (-2)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ -2 & 2 & -1 & 1 \\ 2 & -2 & 1 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ -2 & 2 & -1 & 1 & 0 \\ 2 & -2 & 1 & -1 & 0 \end{array}\right]$$

$$R_3 = R_3 + R_1 \Longrightarrow \left[\begin{array}{cccc|c} 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 2 & -2 & 1 & -1 & 0 \end{array}\right]$$

$$R_4 = R_4 - R_1 \Longrightarrow \left[\begin{array}{cccc|c} 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 0 & -2 & 0 & -1 & 0 \end{array}\right]$$

$$
R_3 = R_3 - R_2 \Longrightarrow \left[\begin{array}{cccc|c} 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & -1 & 0 \end{array}\right]
$$

$$
R_4 = R_4 + R_2 \Longrightarrow \left[\begin{array}{cccc|c} 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]
$$

Therefore the system in Echelon form is

$$
\begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

The free variables are $\{v_3, v_4\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Let $v_4 = s$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{t}{2}, v_2 = -\frac{s}{2}\right\}$

Hence the solution is

$$
\begin{bmatrix} -\frac{t}{2} \\ -\frac{s}{2} \\ t \\ s \end{bmatrix} = \begin{bmatrix} -\frac{t}{2} \\ -\frac{s}{2} \\ t \\ s \end{bmatrix}
$$

Since there are two free Variable, we have found two eigenvectors associated with this

eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{t}{2} \\ -\frac{s}{2} \\ t \\ s \end{bmatrix} = \begin{bmatrix} -\frac{t}{2} \\ 0 \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{s}{2} \\ 0 \\ s \end{bmatrix}$$
$$= t\begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

By letting $t = 1$ and $s = 1$ then the above becomes

$$\begin{bmatrix} -\frac{t}{2} \\ -\frac{s}{2} \\ t \\ s \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

Hence the two eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \right)$$

Which are normalized to

$$\left(\begin{bmatrix} -1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 2 \end{bmatrix} \right)$$

Considering the eigenvalue $\lambda_2 = 0$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 2 & -3 & 1 \\ 2 & -2 & 1 & -3 \end{bmatrix} - (0)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 2 & -3 & 1 \\ 2 & -2 & 1 & -3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -2 & 2 & -3 & 1 & 0 \\ 2 & -2 & 1 & -3 & 0 \end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 3 gives

$$\left[\begin{array}{cccc|c} -2 & 2 & -3 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 2 & -2 & 1 & -3 & 0 \end{array}\right]$$

$$R_4 = R_4 + R_1 \Longrightarrow \left[\begin{array}{cccc|c} -2 & 2 & -3 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & -2 & 0 \end{array}\right]$$

Since the current pivot $A(2,3)$ is zero, then the current pivot row is replaced with a

row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{cccc|c} -2 & 2 & -3 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & -2 & 0 \end{array}\right]$$

$$R_4 = R_4 + 2R_2 \Longrightarrow \left[\begin{array}{cccc|c} -2 & 2 & -3 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 & 0 \end{array}\right]$$

$$R_4 = R_4 + 2R_3 \Longrightarrow \left[\begin{array}{cccc|c} -2 & 2 & -3 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & 2 & -3 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1, v_3, v_4\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_3 = 0, v_4 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ t \\ 0 \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \\ 0 \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-2	3	2	Yes	$\begin{bmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$
0	1	1	No	$\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue -2 is real and repated eigenvalue of multiplicity 3.There are three possible cases that can happen. This is illustrated in this diagram

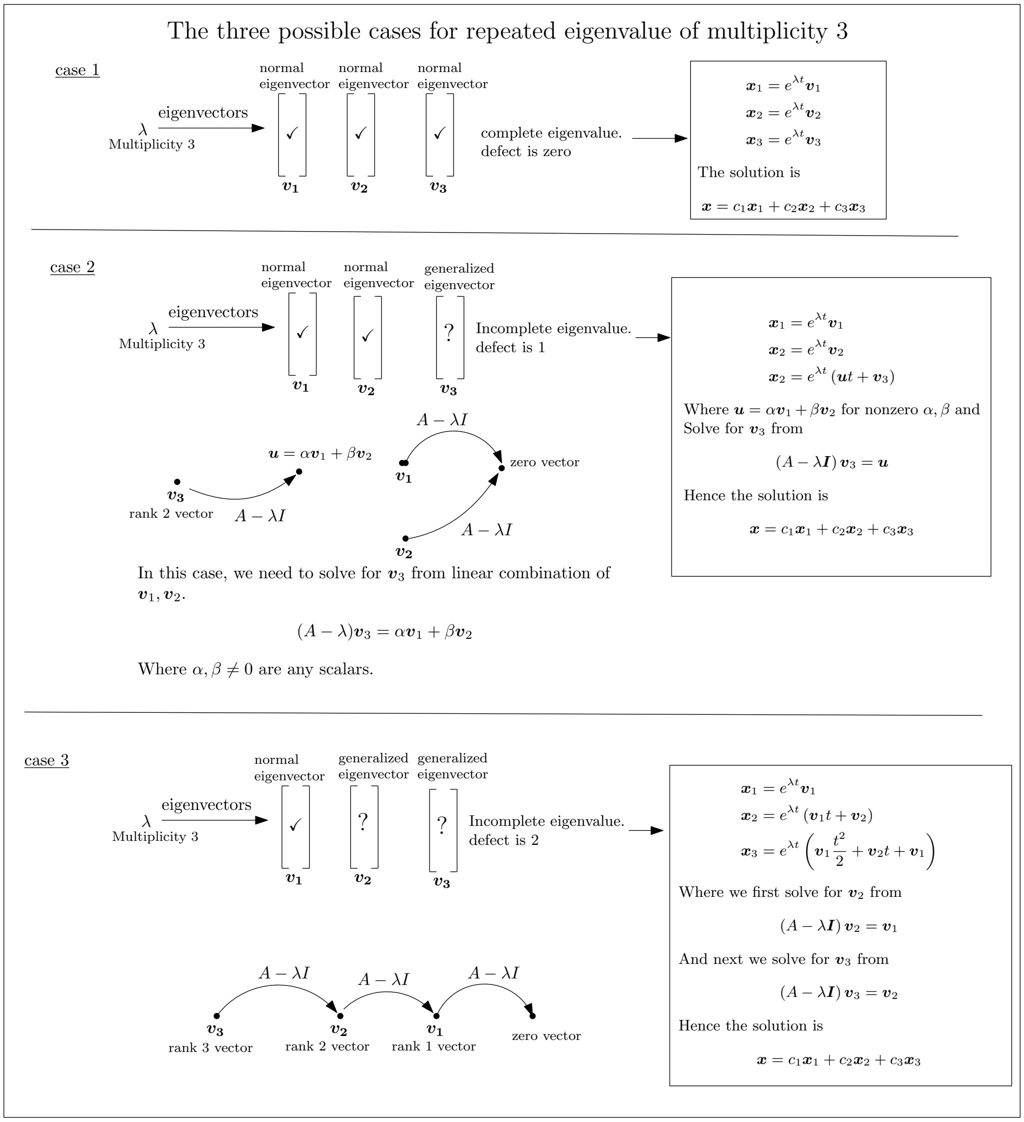


Figure 30: Possible case for repeated λ of multiplicity 3

This eigenvalue has algebraic multiplicity of 3, and geometric multiplicity 2, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to find rank-2 eigenvector $\vec{v}_3$. This eigenvector must therefore satisfy $(A-\lambda I)^2\,\vec{v}_3=\vec{0}$.

$$
\begin{aligned}
(A-\lambda I)^2\,\vec{v}_3 &= \vec{0}\\
\left(\begin{bmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -2 & 2 & -3 & 1\\ 2 & -2 & 1 & -3 \end{bmatrix} - -2\begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{bmatrix}\right)^2 \vec{v}_3 &= \vec{0}\\
\begin{bmatrix} 2 & 2 & 1 & 1\\ 2 & 2 & 1 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{bmatrix} \vec{v}_3 &= \vec{0}
\end{aligned}
$$

Solving for $\vec{v}_3$ from above gives

$$
\begin{bmatrix} 0\\ 1\\ -4\\ 2 \end{bmatrix}
$$

We need now to enforce the condition that $\vec{v}_3$ satisfies

$$
(A-\lambda I)\,\vec{v}_3=\vec{u} \tag{1}
$$

Where $\vec{u}$ is linear combination of $\vec{v}_1,\vec{v}_2$. Hence

$$
\vec{u}=\alpha\vec{v}_1+\beta\vec{v}_2
$$

Where α, β are arbitrary constants (not both zero). Eq. (1) becomes

$$(A-\lambda I)\begin{bmatrix}0\\1\\-4\\2\end{bmatrix}=\alpha\begin{bmatrix}0\\-\frac{1}{2}\\0\\1\end{bmatrix}+\beta\begin{bmatrix}-\frac{1}{2}\\0\\1\\0\end{bmatrix}$$

$$\begin{bmatrix}2&0&1&0\\0&2&0&1\\-2&2&-1&1\\2&-2&1&-1\end{bmatrix}\begin{bmatrix}0\\1\\-4\\2\end{bmatrix}=\alpha\begin{bmatrix}0\\-\frac{1}{2}\\0\\1\end{bmatrix}+\beta\begin{bmatrix}-\frac{1}{2}\\0\\1\\0\end{bmatrix}$$

$$\begin{bmatrix}-4\\4\\8\\-8\end{bmatrix}=\begin{bmatrix}-\frac{\beta}{2}\\-\frac{\alpha}{2}\\\beta\\\alpha\end{bmatrix}$$

Expanding the above gives the following equations equations

$$\begin{aligned}-4&=-\frac{\beta}{2}\\4&=-\frac{\alpha}{2}\\8&=\beta\\-8&=\alpha\end{aligned}$$

solving for α, β from the above gives

$$\begin{aligned}-4&=-\frac{\beta}{2}\\4&=-\frac{\alpha}{2}\end{aligned}$$

Hence

$$\begin{aligned}\alpha&=-8\\\beta&=8\end{aligned}$$

Therefore

$$
\begin{aligned}
\vec{u} &= \alpha \vec{v}_1 + \beta \vec{v}_2 \\
&= -8 \begin{bmatrix} 0 \\ -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix} + (8) \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} -4 \\ 4 \\ 8 \\ -8 \end{bmatrix}
\end{aligned}
$$

Therefore the missing generalized eigenvector is now found. We have found three generalized eigenvectors for eigenvalue -2. Therefore the three basis solutions associated with this eigenvalue are

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\
&= \begin{bmatrix} 0 \\ -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \mathrm{e}^{-2t} \\
&= \begin{bmatrix} 0 \\ -\frac{\mathrm{e}^{-2t}}{2} \\ 0 \\ \mathrm{e}^{-2t} \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{\lambda t} \\
&= \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \\ 0 \end{bmatrix} \mathrm{e}^{-2t} \\
&= \begin{bmatrix} -\frac{\mathrm{e}^{-2t}}{2} \\ 0 \\ \mathrm{e}^{-2t} \\ 0 \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_3(t) &= (\vec{u}t + \vec{v}_3)\, e^{\lambda t} \\
&= \left(\begin{bmatrix} -4 \\ 4 \\ 8 \\ -8 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \\ -4 \\ 2 \end{bmatrix} \right) \mathrm{e}^{-2t}
\end{aligned}
$$

Since eigenvalue 0 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_4(t) &= \vec{v}_4 e^{0} \\
&= \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} e^{0}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t) + c_4 \vec{x}_4(t)
$$

Which is written as

$$
\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ -\frac{\mathrm{e}^{-2t}}{2} \\ 0 \\ \mathrm{e}^{-2t} \end{bmatrix} + c_2 \begin{bmatrix} -\frac{\mathrm{e}^{-2t}}{2} \\ 0 \\ \mathrm{e}^{-2t} \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} -4t\, \mathrm{e}^{-2t} \\ \mathrm{e}^{-2t}(4t+1) \\ \mathrm{e}^{-2t}(-4+8t) \\ \mathrm{e}^{-2t}(2-8t) \end{bmatrix} + c_4 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}
$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = \begin{bmatrix} \frac{(-8tc_3-c_2)e^{-2t}}{2} + c_4 \\ \frac{((8t+2)c_3-c_1)e^{-2t}}{2} + c_4 \\ ((-4+8t)c_3 + c_2)e^{-2t} \\ e^{-2t}(-8tc_3 + c_1 + 2c_3) \end{bmatrix}$$

✓ Solution by Maple

Time used: 0.046 (sec). Leaf size: 95

```
dsolve([diff(x__1(t),t)=0*x__1(t)+0*x__2(t)+1*x__3(t)+0*x__4(t),diff(x__2(t),t)=0*x__1(t)+0*
```

$$x_1(t) = c_2 + c_3 e^{-2t} + c_4 e^{-2t} t$$
$$x_2(t) = -c_3 e^{-2t} - c_4 e^{-2t} t + c_4 e^{-2t} + c_2 + c_1 e^{-2t}$$
$$x_3(t) = -e^{-2t}(2c_4 t + 2c_3 - c_4)$$
$$x_4(t) = -e^{-2t}(-2c_4 t + 2c_1 - 2c_3 + 3c_4)$$

✓ Solution by Mathematica

Time used: 0.06 (sec). Leaf size: 210

```
DSolve[{x1'[t]==0*x1[t]+0*x2[t]+1*x3[t]+0*x4[t],x2'[t]==0*x1[t]+0*x2[t]+0*x3[t]+1*x4[t],x3'[
```

$$\text{x1}(t) \to \frac{1}{4}e^{-2t}\left(2c_1\left(2t+e^{2t}+1\right)+2c_2\left(-2t+e^{2t}-1\right)+c_3e^{2t}+2c_3t+c_4e^{2t}-2c_4t-c_3-c_4\right)$$
$$\text{x2}(t) \to \frac{1}{4}e^{-2t}\left(2c_1\left(-2t+e^{2t}-1\right)+2c_2\left(2t+e^{2t}+1\right)+c_3e^{2t}-2c_3t+c_4e^{2t}+2c_4t-c_3-c_4\right)$$
$$\text{x3}(t) \to e^{-2t}((-2c_1 + 2c_2 - c_3 + c_4)t + c_3)$$
$$\text{x4}(t) \to e^{-2t}((2c_1 - 2c_2 + c_3 - c_4)t + c_4)$$

6 Section 7.6, Multiple Eigenvalue Solutions. Page 451

6.1 problem problem 1

Internal problem ID [358]
Internal file name [`OUTPUT/358_Sunday_June_05_2022_01_39_38_AM_36062462/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.6, Multiple Eigenvalue Solutions. Page 451
Problem number: problem 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= -2x_1(t) + x_2(t) \\
x_2'(t) &= -x_1(t) - 4x_2(t)
\end{aligned}
$$

6.1.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \mathrm{e}^{-3t}(t+1) & t\,\mathrm{e}^{-3t} \\ -t\,\mathrm{e}^{-3t} & \mathrm{e}^{-3t}(1-t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \mathrm{e}^{-3t}(t+1) & t\,\mathrm{e}^{-3t} \\ -t\,\mathrm{e}^{-3t} & \mathrm{e}^{-3t}(1-t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^{-3t}(t+1)\,c_1 + t\,\mathrm{e}^{-3t}c_2 \\ -t\,\mathrm{e}^{-3t}c_1 + \mathrm{e}^{-3t}(1-t)\,c_2 \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^{-3t}(tc_1 + c_2 t + c_1) \\ -((-1+t)\,c_2 + tc_1)\,\mathrm{e}^{-3t} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

6.1.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -2 & 1 \\ -1 & -4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -2-\lambda & 1 \\ -1 & -4-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 6\lambda + 9 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -3$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -2 & 1 \\ -1 & -4 \end{bmatrix} - (-3) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ -1 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-3	2	1	Yes	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue -3 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

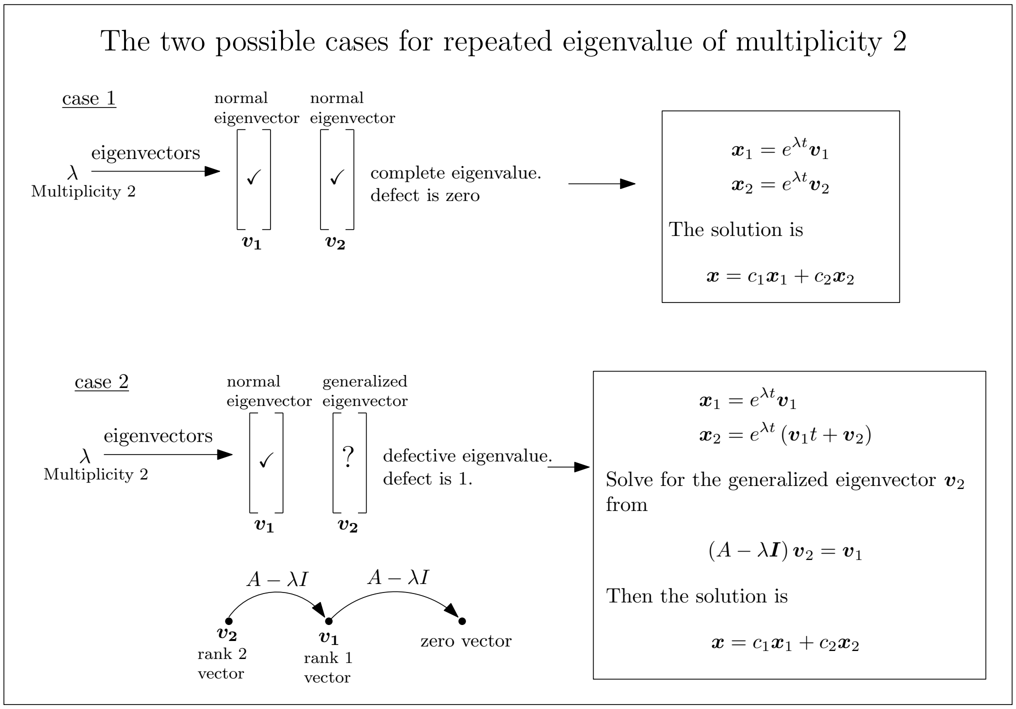


Figure 31: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} -2 & 1 \\ -1 & -4 \end{bmatrix} - (-3)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue -3. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t}\\ &= \begin{bmatrix} -1 \\ 1 \end{bmatrix} \mathrm{e}^{-3t}\\ &= \begin{bmatrix} -\mathrm{e}^{-3t} \\ \mathrm{e}^{-3t} \end{bmatrix}\end{aligned}$$

And

$$\begin{aligned}\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t}\\ &= \left(\begin{bmatrix} -1 \\ 1 \end{bmatrix} t + \begin{bmatrix} -2 \\ 1 \end{bmatrix}\right) \mathrm{e}^{-3t}\\ &= \begin{bmatrix} -\mathrm{e}^{-3t}(2+t) \\ \mathrm{e}^{-3t}(t+1) \end{bmatrix}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} -\mathrm{e}^{-3t} \\ \mathrm{e}^{-3t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{-3t}(-t-2) \\ \mathrm{e}^{-3t}(t+1) \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -((2+t)\, c_2 + c_1)\, \mathrm{e}^{-3t} \\ \mathrm{e}^{-3t}(c_2 t + c_1 + c_2) \end{bmatrix}$$

The following is the phase plot of the system.

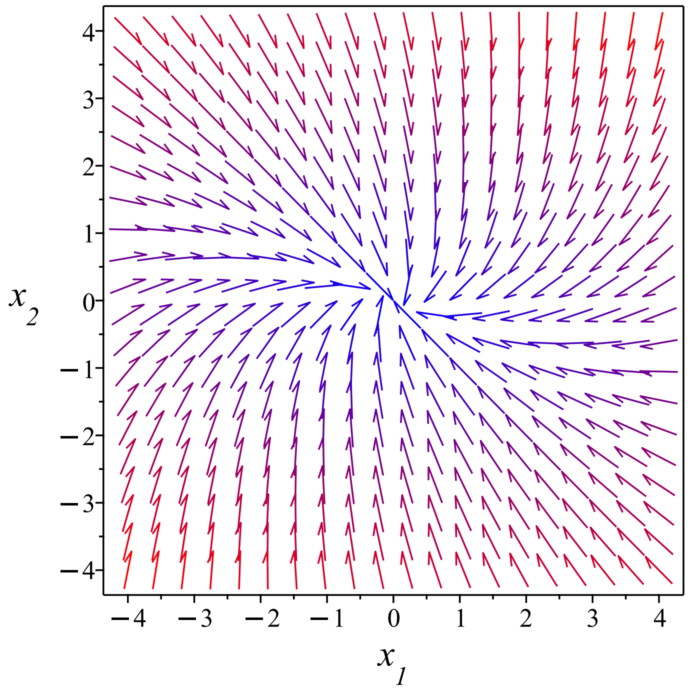


Figure 32: Phase plot

6.1.3 Maple step by step solution

Let's solve

$[x_1'(t) = -2x_1(t) + x_2(t), x_2'(t) = -x_1(t) - 4x_2(t)]$

- Define vector

$$\underline{x^{\rightarrow}}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x^{\rightarrow}}'(t) = \begin{bmatrix} -2 & 1 \\ -1 & -4 \end{bmatrix} \cdot \underline{x^{\rightarrow}}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{x^{\rightarrow}}'(t) = \begin{bmatrix} -2 & 1 \\ -1 & -4 \end{bmatrix} \cdot \underline{x^{\rightarrow}}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -2 & 1 \\ -1 & -4 \end{bmatrix}$$

- Rewrite the system as

  $\underline{x}^{\rightarrow\prime}(t) = A \cdot \underline{x}^{\rightarrow}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[-3, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right], \left[-3, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

  $$\left[-3, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue -3

  $$\underline{x}^{\rightarrow}{}_1(t) = e^{-3t} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = -3$ is the eigenvalue, an

  $$\underline{x}^{\rightarrow}{}_2(t) = e^{\lambda t}\left(t\vec{v} + \vec{p}\right)$$

- Note that the t multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\underline{x}^{\rightarrow}{}_2(t)$ into the homogeneous system

  $$\lambda e^{\lambda t}\left(t\vec{v} + \vec{p}\right) + e^{\lambda t}\vec{v} = \left(e^{\lambda t}A\right) \cdot \left(t\vec{v} + \vec{p}\right)$$

- Use the fact that $\vec{v}$ is an eigenvector of A

  $$\lambda e^{\lambda t}\left(t\vec{v} + \vec{p}\right) + e^{\lambda t}\vec{v} = e^{\lambda t}\left(\lambda t\vec{v} + A \cdot \vec{p}\right)$$

- Simplify equation

  $$\lambda\vec{p} + \vec{v} = A \cdot \vec{p}$$

- Make use of the identity matrix I

  $$(\lambda \cdot I) \cdot \vec{p} + \vec{v} = A \cdot \vec{p}$$

- Condition $\vec{p}$ must meet for $\underline{x}^{\rightarrow}{}_2(t)$ to be a solution to the homogeneous system

  $$(A - \lambda \cdot I) \cdot \vec{p} = \vec{v}$$

- Choose $\overrightarrow{p}$ to use in the second solution to the homogeneous system from eigenvalue -3

$$\left(\left[\begin{array}{cc}-2 & 1 \\ -1 & -4\end{array}\right]-(-3)\cdot\left[\begin{array}{cc}1 & 0 \\ 0 & 1\end{array}\right]\right)\cdot\overrightarrow{p}=\left[\begin{array}{c}-1 \\ 1\end{array}\right]$$

- Choice of $\overrightarrow{p}$

$$\overrightarrow{p}=\left[\begin{array}{c}-1 \\ 0\end{array}\right]$$

- Second solution from eigenvalue -3

$$\overrightarrow{x}_{2}(t)=\mathrm{e}^{-3t}\cdot\left(t\cdot\left[\begin{array}{c}-1 \\ 1\end{array}\right]+\left[\begin{array}{c}-1 \\ 0\end{array}\right]\right)$$

- General solution to the system of ODEs

$$\overrightarrow{x}=c_1\overrightarrow{x}_{1}(t)+c_2\overrightarrow{x}_{2}(t)$$

- Substitute solutions into the general solution

$$\overrightarrow{x}=c_1\mathrm{e}^{-3t}\cdot\left[\begin{array}{c}-1 \\ 1\end{array}\right]+c_2\mathrm{e}^{-3t}\cdot\left(t\cdot\left[\begin{array}{c}-1 \\ 1\end{array}\right]+\left[\begin{array}{c}-1 \\ 0\end{array}\right]\right)$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c}x_1(t) \\ x_2(t)\end{array}\right]=\left[\begin{array}{c}\mathrm{e}^{-3t}(-c_2t-c_1-c_2) \\ \mathrm{e}^{-3t}(c_2t+c_1)\end{array}\right]$$

- Solution to the system of ODEs

$$\{x_1(t)=\mathrm{e}^{-3t}(-c_2t-c_1-c_2)\,,x_2(t)=\mathrm{e}^{-3t}(c_2t+c_1)\}$$

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 32

```
dsolve([diff(x__1(t),t)=-2*x__1(t)+1*x__2(t),diff(x__2(t),t)=-1*x__1(t)-4*x__2(t)],singsol=a
```

$$x_1(t)=\mathrm{e}^{-3t}(c_2t+c_1)$$
$$x_2(t)=-\mathrm{e}^{-3t}(c_2t+c_1-c_2)$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 42

```
DSolve[{x1'[t]==-2*x1[t]+1*x2[t],x2'[t]==-1*x1[t]-4*x2[t]},{x1[t],x2[t]},t,IncludeSingularSo
```

$$\text{x1}(t) \to e^{-3t}(c_1(t+1) + c_2 t)$$
$$\text{x2}(t) \to e^{-3t}(c_2 - (c_1 + c_2)t)$$

6.2 problem problem 2

Internal problem ID [359]
Internal file name [`OUTPUT/359_Sunday_June_05_2022_01_39_38_AM_79876116/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.6, Multiple Eigenvalue Solutions. Page 451
Problem number: problem 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x_1'(t) = 3x_1(t) - x_2(t)$$
$$x_2'(t) = x_1(t) + x_2(t)$$

6.2.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \mathrm{e}^{2t}(t+1) & -\mathrm{e}^{2t}t \\ \mathrm{e}^{2t}t & \mathrm{e}^{2t}(1-t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{2t}(t+1) & -e^{2t}t \\ e^{2t}t & e^{2t}(1-t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{2t}(t+1)\, c_1 - e^{2t}tc_2 \\ e^{2t}tc_1 + e^{2t}(1-t)\, c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{2t}(tc_1 - c_2t + c_1) \\ e^{2t}(tc_1 - c_2t + c_2) \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

6.2.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 3-\lambda & -1 \\ 1 & 1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 4\lambda + 4 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
2	2	1	Yes	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 2 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

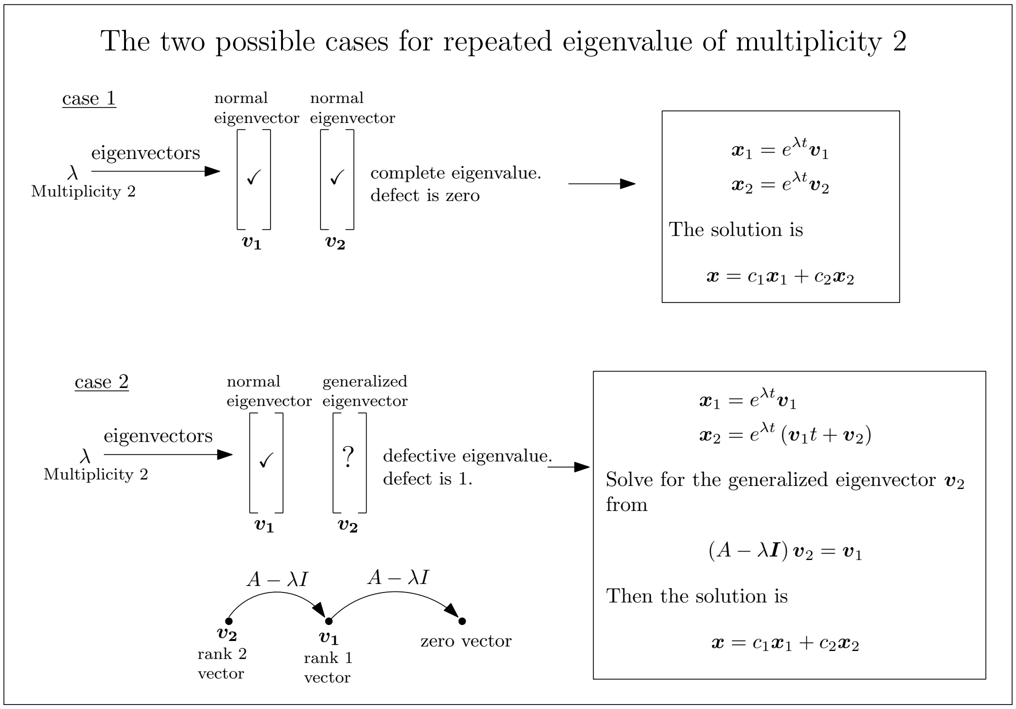


Figure 33: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue 2. Therefore the two basis solution associated with this eigenvalue are

$$\vec{x}_1(t) = \vec{v}_1 e^{\lambda t} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} = \begin{bmatrix} e^{2t} \\ e^{2t} \end{bmatrix}$$

And

$$\vec{x}_2(t) = (\vec{v}_1 t + \vec{v}_2) e^{\lambda t} = \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} t + \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right) e^{2t} = \begin{bmatrix} e^{2t}(2+t) \\ e^{2t}(t+1) \end{bmatrix}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} e^{2t} \\ e^{2t} \end{bmatrix} + c_2 \begin{bmatrix} e^{2t}(2+t) \\ e^{2t}(t+1) \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} ((2+t) c_2 + c_1) e^{2t} \\ e^{2t}(c_2 t + c_1 + c_2) \end{bmatrix}$$

The following is the phase plot of the system.

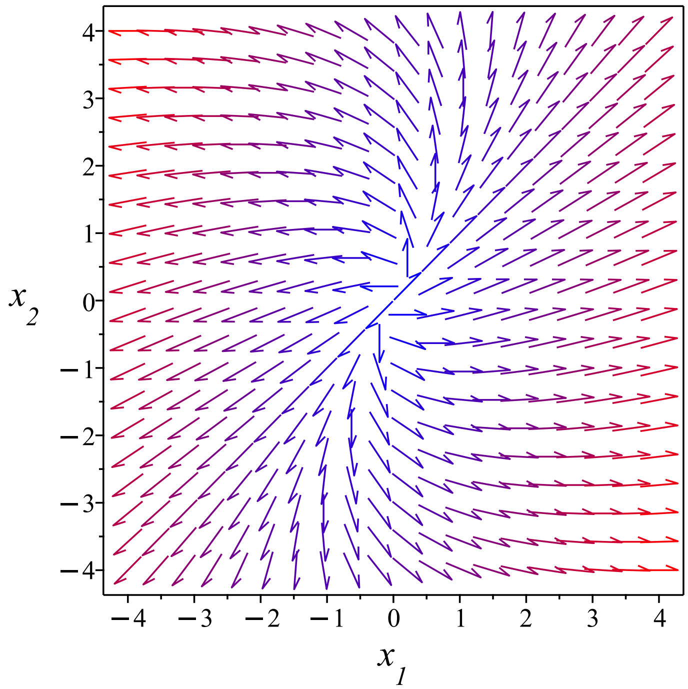


Figure 34: Phase plot

6.2.3 Maple step by step solution

Let's solve

$[x_1'(t) = 3x_1(t) - x_2(t), x_2'(t) = x_1(t) + x_2(t)]$

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x}^{\rightarrow}{}'(t) = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{x}^{\rightarrow}{}'(t) = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$$

- Rewrite the system as

  $\underline{\vec{x}}'(t) = A \cdot \underline{\vec{x}}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[2, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

  $$\left[2, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue 2

  $$\underline{\vec{x}}_1(t) = \mathrm{e}^{2t} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = 2$ is the eigenvalue, and

  $\underline{\vec{x}}_2(t) = \mathrm{e}^{\lambda t}\left(t\vec{v} + \vec{p}\right)$

- Note that the t multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\underline{\vec{x}}_2(t)$ into the homogeneous system

  $\lambda \,\mathrm{e}^{\lambda t}\left(t\vec{v} + \vec{p}\right) + \mathrm{e}^{\lambda t}\vec{v} = \left(\mathrm{e}^{\lambda t}A\right) \cdot \left(t\vec{v} + \vec{p}\right)$

- Use the fact that $\vec{v}$ is an eigenvector of A

  $\lambda \,\mathrm{e}^{\lambda t}\left(t\vec{v} + \vec{p}\right) + \mathrm{e}^{\lambda t}\vec{v} = \mathrm{e}^{\lambda t}\left(\lambda t\vec{v} + A \cdot \vec{p}\right)$

- Simplify equation

  $\lambda\vec{p} + \vec{v} = A \cdot \vec{p}$

- Make use of the identity matrix I

  $(\lambda \cdot I) \cdot \vec{p} + \vec{v} = A \cdot \vec{p}$

- Condition $\vec{p}$ must meet for $\underline{\vec{x}}_2(t)$ to be a solution to the homogeneous system

  $(A - \lambda \cdot I) \cdot \vec{p} = \vec{v}$

- Choose $\overrightarrow{p}$ to use in the second solution to the homogeneous system from eigenvalue 2

$$\left(\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} - 2 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \cdot \overrightarrow{p} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- Choice of $\overrightarrow{p}$

$$\overrightarrow{p} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue 2

$$\overrightarrow{x}_2(t) = e^{2t} \cdot \left(t \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

- General solution to the system of ODEs

$$\overrightarrow{x} = c_1 \overrightarrow{x}_1(t) + c_2 \overrightarrow{x}_2(t)$$

- Substitute solutions into the general solution

$$\overrightarrow{x} = c_1 e^{2t} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{2t} \cdot \left(t \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} e^{2t}(c_2 t + c_1 + c_2) \\ e^{2t}(c_2 t + c_1) \end{bmatrix}$$

- Solution to the system of ODEs

$$\{x_1(t) = e^{2t}(c_2 t + c_1 + c_2), x_2(t) = e^{2t}(c_2 t + c_1)\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 31

```
dsolve([diff(x__1(t),t)=3*x__1(t)-1*x__2(t),diff(x__2(t),t)=1*x__1(t)+1*x__2(t)],singsol=all
```

$$x_1(t) = e^{2t}(c_2 t + c_1)$$
$$x_2(t) = e^{2t}(c_2 t + c_1 - c_2)$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 44

```
DSolve[{x1'[t]==3*x1[t]-1*x2[t],x2'[t]==1*x1[t]+1*x2[t]},{x1[t],x2[t]},t,IncludeSingularSolu
```

$$\text{x1}(t) \to e^{2t}(c_1(t+1) - c_2 t)$$
$$\text{x2}(t) \to e^{2t}((c_1 - c_2)t + c_2)$$

6.3 problem problem 3

Internal problem ID [360]
Internal file name [`OUTPUT/360_Sunday_June_05_2022_01_39_39_AM_72268151/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.6, Multiple Eigenvalue Solutions. Page 451
Problem number: problem 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= x_1(t) - 2x_2(t) \\
x_2'(t) &= 2x_1(t) + 5x_2(t)
\end{aligned}
$$

6.3.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{3t}(1-2t) & -2\,e^{3t}t \\ 2\,e^{3t}t & e^{3t}(1+2t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{3t}(1-2t) & -2\,e^{3t}t \\ 2\,e^{3t}t & e^{3t}(1+2t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{3t}(1-2t)\,c_1 - 2\,e^{3t}tc_2 \\ 2\,e^{3t}tc_1 + e^{3t}(1+2t)\,c_2 \end{bmatrix} \\
&= \begin{bmatrix} (c_1(1-2t) - 2c_2t)\,e^{3t} \\ e^{3t}(2tc_1 + 2c_2t + c_2) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

6.3.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & -2 \\ 2 & 5-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 6\lambda + 9 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 3$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix} - (3)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & -2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -2 & -2 & 0 \\ 2 & 2 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cc|c} -2 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
3	2	1	Yes	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 3 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

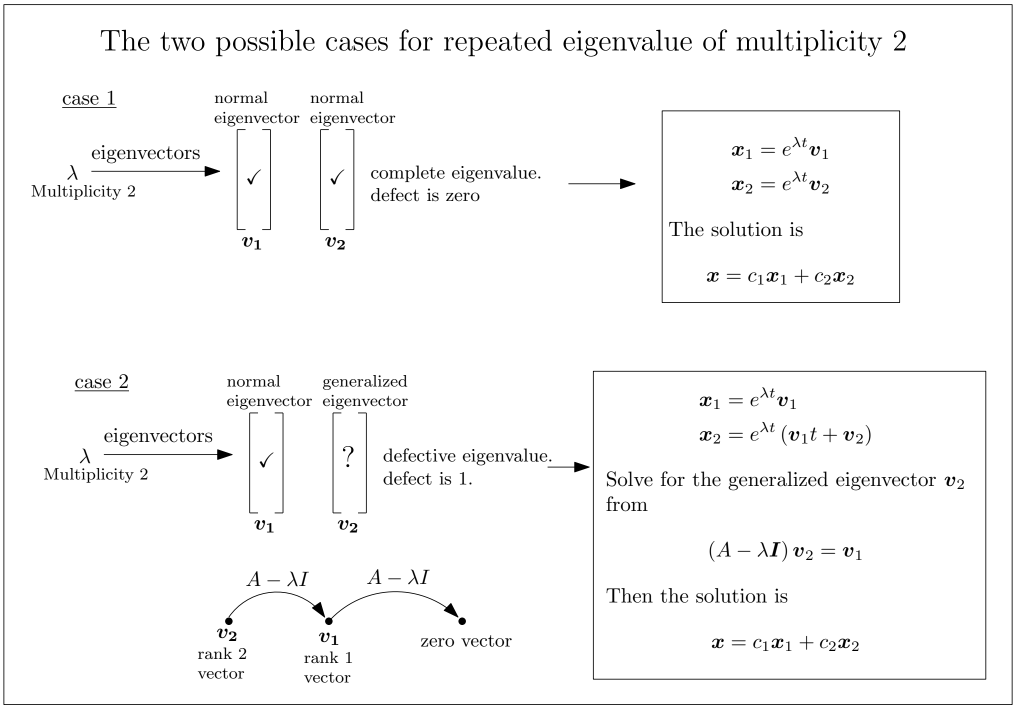


Figure 35: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix} - (3)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} -2 & -2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue 3. Therefore the two basis solution associated with this eigenvalue are

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\
&= \begin{bmatrix} -1 \\ 1 \end{bmatrix} \mathrm{e}^{3t} \\
&= \begin{bmatrix} -\mathrm{e}^{3t} \\ \mathrm{e}^{3t} \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\
&= \left(\begin{bmatrix} -1 \\ 1 \end{bmatrix} t + \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} \right) \mathrm{e}^{3t} \\
&= \begin{bmatrix} -\frac{\mathrm{e}^{3t}(1+2t)}{2} \\ \mathrm{e}^{3t}(t+1) \end{bmatrix}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} -\mathrm{e}^{3t} \\ \mathrm{e}^{3t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{3t}\left(-t - \frac{1}{2}\right) \\ \mathrm{e}^{3t}(t+1) \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{3t}\left(-c_1 - c_2 t - \frac{1}{2} c_2\right) \\ \mathrm{e}^{3t}(c_2 t + c_1 + c_2) \end{bmatrix}
$$

The following is the phase plot of the system.

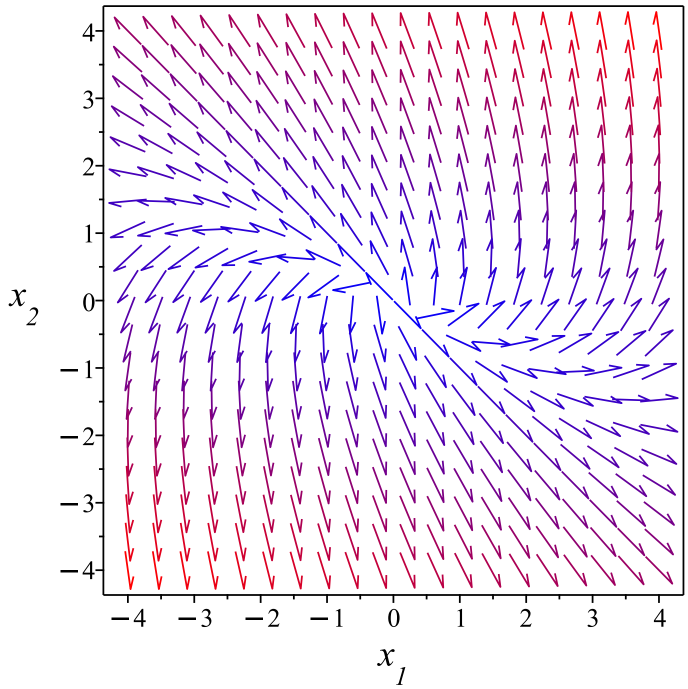


Figure 36: Phase plot

6.3.3 Maple step by step solution

Let's solve

$[x_1'(t) = x_1(t) - 2x_2(t), x_2'(t) = 2x_1(t) + 5x_2(t)]$

- Define vector

$$\underline{\vec{x}}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{\vec{x}}'(t) = \begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix} \cdot \underline{\vec{x}}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{\vec{x}}'(t) = \begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix} \cdot \underline{\vec{x}}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix}$$

- Rewrite the system as

  $\underline{x^{\rightarrow}}'(t) = A \cdot \underline{x^{\rightarrow}}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[3, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right], \left[3, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

  $$\left[3, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue 3

  $$\underline{x^{\rightarrow}}_1(t) = e^{3t} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = 3$ is the eigenvalue, and

  $\underline{x^{\rightarrow}}_2(t) = e^{\lambda t}\left(t\vec{v} + \vec{p}\right)$

- Note that the t multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\underline{x^{\rightarrow}}_2(t)$ into the homogeneous system

  $\lambda e^{\lambda t}\left(t\vec{v} + \vec{p}\right) + e^{\lambda t}\vec{v} = \left(e^{\lambda t}A\right) \cdot \left(t\vec{v} + \vec{p}\right)$

- Use the fact that $\vec{v}$ is an eigenvector of A

  $\lambda e^{\lambda t}\left(t\vec{v} + \vec{p}\right) + e^{\lambda t}\vec{v} = e^{\lambda t}\left(\lambda t\vec{v} + A \cdot \vec{p}\right)$

- Simplify equation

  $\lambda\vec{p} + \vec{v} = A \cdot \vec{p}$

- Make use of the identity matrix I

  $(\lambda \cdot I) \cdot \vec{p} + \vec{v} = A \cdot \vec{p}$

- Condition $\vec{p}$ must meet for $\underline{x^{\rightarrow}}_2(t)$ to be a solution to the homogeneous system

  $(A - \lambda \cdot I) \cdot \vec{p} = \vec{v}$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue 3

$$\left(\begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix} - 3 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \cdot \vec{p} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- Choice of $\vec{p}$

$$\vec{p} = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue 3

$$\underline{x^{\rightarrow}}_2(t) = e^{3t} \cdot \left(t \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}\right)$$

- General solution to the system of ODEs

$$\underline{x^{\rightarrow}} = c_1 \underline{x^{\rightarrow}}_1(t) + c_2 \underline{x^{\rightarrow}}_2(t)$$

- Substitute solutions into the general solution

$$\underline{x^{\rightarrow}} = c_1 e^{3t} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{3t} \cdot \left(t \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}\right)$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} e^{3t}\left(-c_1 - c_2 t + \frac{1}{2}c_2\right) \\ (c_2 t + c_1) e^{3t} \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x_1(t) = e^{3t}\left(-c_1 - c_2 t + \frac{1}{2}c_2\right), x_2(t) = (c_2 t + c_1) e^{3t}\right\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 33

```
dsolve([diff(x__1(t),t)=1*x__1(t)-2*x__2(t),diff(x__2(t),t)=2*x__1(t)+5*x__2(t)],singsol=all
```

$$x_1(t) = e^{3t}(c_2 t + c_1)$$
$$x_2(t) = -\frac{e^{3t}(2c_2 t + 2c_1 + c_2)}{2}$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 46

```
DSolve[{x1'[t]==1*x1[t]-2*x2[t],x2'[t]==2*x1[t]+5*x2[t]},{x1[t],x2[t]},t,IncludeSingularSolu
```

$$\begin{aligned} \text{x1}(t) &\to -e^{3t}(c_1(2t-1)+2c_2t) \\ \text{x2}(t) &\to e^{3t}(2(c_1+c_2)t+c_2) \end{aligned}$$

6.4 problem problem 4

Internal problem ID [361]
Internal file name [`OUTPUT/361_Sunday_June_05_2022_01_39_40_AM_44541264/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.6, Multiple Eigenvalue Solutions. Page 451
Problem number: problem 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x_1'(t) = 3x_1(t) - x_2(t)$$
$$x_2'(t) = x_1(t) + 5x_2(t)$$

6.4.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \mathrm{e}^{4t}(1-t) & -t\,\mathrm{e}^{4t} \\ t\,\mathrm{e}^{4t} & \mathrm{e}^{4t}(t+1) \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{4t}(1-t) & -t\,e^{4t} \\ t\,e^{4t} & e^{4t}(t+1) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{4t}(1-t)\,c_1 - t\,e^{4t}c_2 \\ t\,e^{4t}c_1 + e^{4t}(t+1)\,c_2 \end{bmatrix} \\
&= \begin{bmatrix} -(c_1(-1+t) + c_2 t)\,e^{4t} \\ e^{4t}(tc_1 + c_2 t + c_2) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

6.4.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det(A - \lambda I) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} 3 & -1 \\ 1 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} 3-\lambda & -1 \\ 1 & 5-\lambda \end{bmatrix}\right) = 0
$$

Which gives the characteristic equation

$$\lambda^2 - 8\lambda + 16 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 4$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
4	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 4$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & -1 \\ 1 & 5 \end{bmatrix} - (4)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -1 & -1 & 0 \\ 1 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cc|c} -1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
4	2	1	Yes	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 4 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

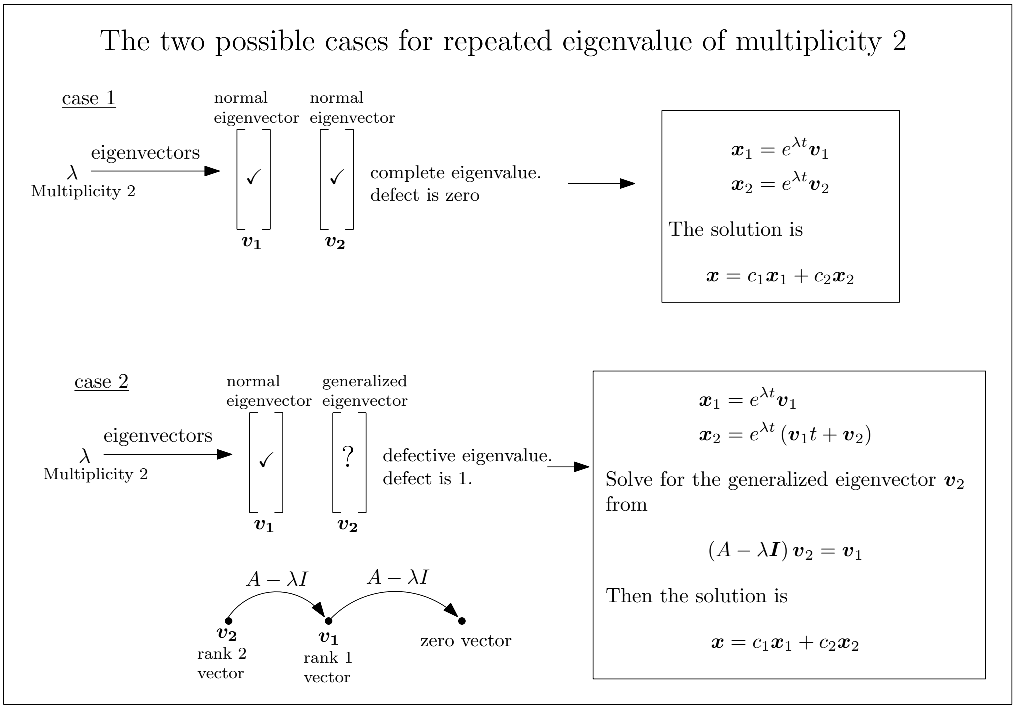


Figure 37: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I) \vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} 3 & -1 \\ 1 & 5 \end{bmatrix} - (4) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue 4. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\ &= \begin{bmatrix} -1 \\ 1 \end{bmatrix} \mathrm{e}^{4t} \\ &= \begin{bmatrix} -\mathrm{e}^{4t} \\ \mathrm{e}^{4t} \end{bmatrix}\end{aligned}$$

And

$$\begin{aligned}\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\ &= \left(\begin{bmatrix} -1 \\ 1 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \mathrm{e}^{4t} \\ &= \begin{bmatrix} -t\,\mathrm{e}^{4t} \\ \mathrm{e}^{4t}(t+1) \end{bmatrix}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} -\mathrm{e}^{4t} \\ \mathrm{e}^{4t} \end{bmatrix} + c_2 \begin{bmatrix} -t\,\mathrm{e}^{4t} \\ \mathrm{e}^{4t}(t+1) \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{4t}(-t c_2 - c_1) \\ \mathrm{e}^{4t}(t c_2 + c_1 + c_2) \end{bmatrix}$$

The following is the phase plot of the system.

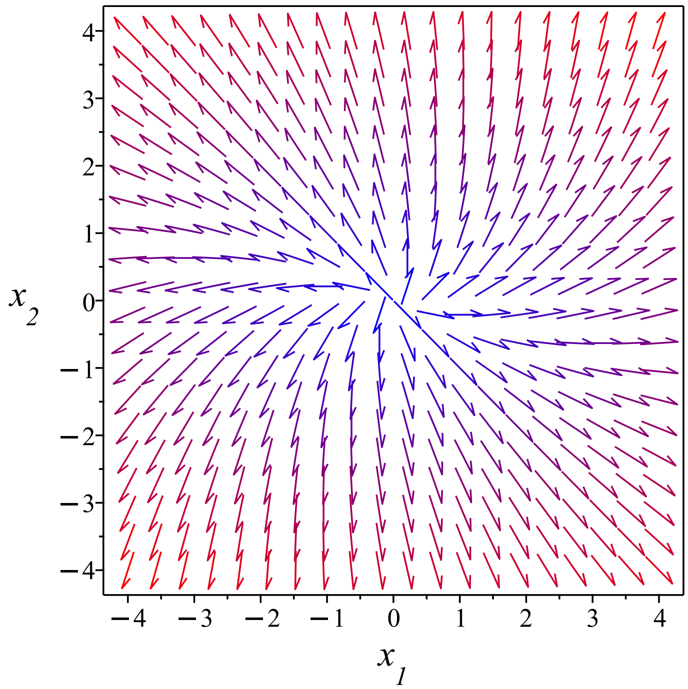


Figure 38: Phase plot

6.4.3 Maple step by step solution

Let's solve

$[x_1'(t) = 3x_1(t) - x_2(t), x_2'(t) = x_1(t) + 5x_2(t)]$

- Define vector

$$\underline{\vec{x}}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{\vec{x}}'(t) = \begin{bmatrix} 3 & -1 \\ 1 & 5 \end{bmatrix} \cdot \underline{\vec{x}}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{\vec{x}}'(t) = \begin{bmatrix} 3 & -1 \\ 1 & 5 \end{bmatrix} \cdot \underline{\vec{x}}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 3 & -1 \\ 1 & 5 \end{bmatrix}$$

- Rewrite the system as

  $\underline{x^{\rightarrow}}'(t) = A \cdot \underline{x^{\rightarrow}}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[4, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right], \left[4, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

  $$\left[4, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue 4

  $$\underline{x^{\rightarrow}}_1(t) = e^{4t} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = 4$ is the eigenvalue, and

  $\underline{x^{\rightarrow}}_2(t) = e^{\lambda t}\left(t\vec{v} + \vec{p}\right)$

- Note that the t multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\underline{x^{\rightarrow}}_2(t)$ into the homogeneous system

  $\lambda e^{\lambda t}\left(t\vec{v} + \vec{p}\right) + e^{\lambda t}\vec{v} = \left(e^{\lambda t} A\right) \cdot \left(t\vec{v} + \vec{p}\right)$

- Use the fact that $\vec{v}$ is an eigenvector of A

  $\lambda e^{\lambda t}\left(t\vec{v} + \vec{p}\right) + e^{\lambda t}\vec{v} = e^{\lambda t}\left(\lambda t\vec{v} + A \cdot \vec{p}\right)$

- Simplify equation

  $\lambda\vec{p} + \vec{v} = A \cdot \vec{p}$

- Make use of the identity matrix I

  $(\lambda \cdot I) \cdot \vec{p} + \vec{v} = A \cdot \vec{p}$

- Condition $\vec{p}$ must meet for $\underline{x^{\rightarrow}}_2(t)$ to be a solution to the homogeneous system

  $(A - \lambda \cdot I) \cdot \vec{p} = \vec{v}$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue 4

$$\left(\begin{bmatrix} 3 & -1 \\ 1 & 5 \end{bmatrix} - 4 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \cdot \vec{p} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- Choice of $\vec{p}$

$$\vec{p} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue 4

$$\underline{\vec{x}}_2(t) = e^{4t} \cdot \left(t \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

- General solution to the system of ODEs

$$\underline{\vec{x}} = c_1 \underline{\vec{x}}_1(t) + c_2 \underline{\vec{x}}_2(t)$$

- Substitute solutions into the general solution

$$\underline{\vec{x}} = c_1 e^{4t} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{4t} \cdot \left(t \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -((-1+t)\, c_2 + c_1)\, e^{4t} \\ e^{4t}(c_2 t + c_1) \end{bmatrix}$$

- Solution to the system of ODEs

$$\{x_1(t) = -((-1+t)\, c_2 + c_1)\, e^{4t}, x_2(t) = e^{4t}(c_2 t + c_1)\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 30

```
dsolve([diff(x__1(t),t)=3*x__1(t)-1*x__2(t),diff(x__2(t),t)=1*x__1(t)+5*x__2(t)],singsol=all
```

$$x_1(t) = e^{4t}(c_2 t + c_1)$$
$$x_2(t) = -e^{4t}(c_2 t + c_1 + c_2)$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 42

```
DSolve[{x1'[t]==3*x1[t]-1*x2[t],x2'[t]==1*x1[t]+5*x2[t]},{x1[t],x2[t]},t,IncludeSingularSolu
```

$$
\begin{aligned}
\mathrm{x1}(t) &\to -e^{4t}(c_1(t-1)+c_2t) \\
\mathrm{x2}(t) &\to e^{4t}((c_1+c_2)t+c_2)
\end{aligned}
$$

6.5 problem problem 5

Internal problem ID [362]
Internal file name [`OUTPUT/362_Sunday_June_05_2022_01_39_41_AM_85394000/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.6, Multiple Eigenvalue Solutions. Page 451
Problem number: problem 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x_1'(t) = 7x_1(t) + x_2(t)$$
$$x_2'(t) = -4x_1(t) + 3x_2(t)$$

6.5.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 7 & 1 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{5t}(1+2t) & t\,e^{5t} \\ -4t\,e^{5t} & e^{5t}(1-2t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \mathrm{e}^{5t}(1+2t) & t\,\mathrm{e}^{5t} \\ -4t\,\mathrm{e}^{5t} & \mathrm{e}^{5t}(1-2t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^{5t}(1+2t)\,c_1 + t\,\mathrm{e}^{5t}c_2 \\ -4t\,\mathrm{e}^{5t}c_1 + \mathrm{e}^{5t}(1-2t)\,c_2 \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^{5t}(2tc_1 + c_2t + c_1) \\ (c_2(1-2t) - 4tc_1)\,\mathrm{e}^{5t} \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

6.5.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 7 & 1 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 7 & 1 \\ -4 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 7-\lambda & 1 \\ -4 & 3-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 10\lambda + 25 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 5$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
5	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 5$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 7 & 1 \\ -4 & 3 \end{bmatrix} - (5)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2 & 1 & 0 \\ -4 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 + 2R_1 \Longrightarrow \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
5	2	1	Yes	$\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 5 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

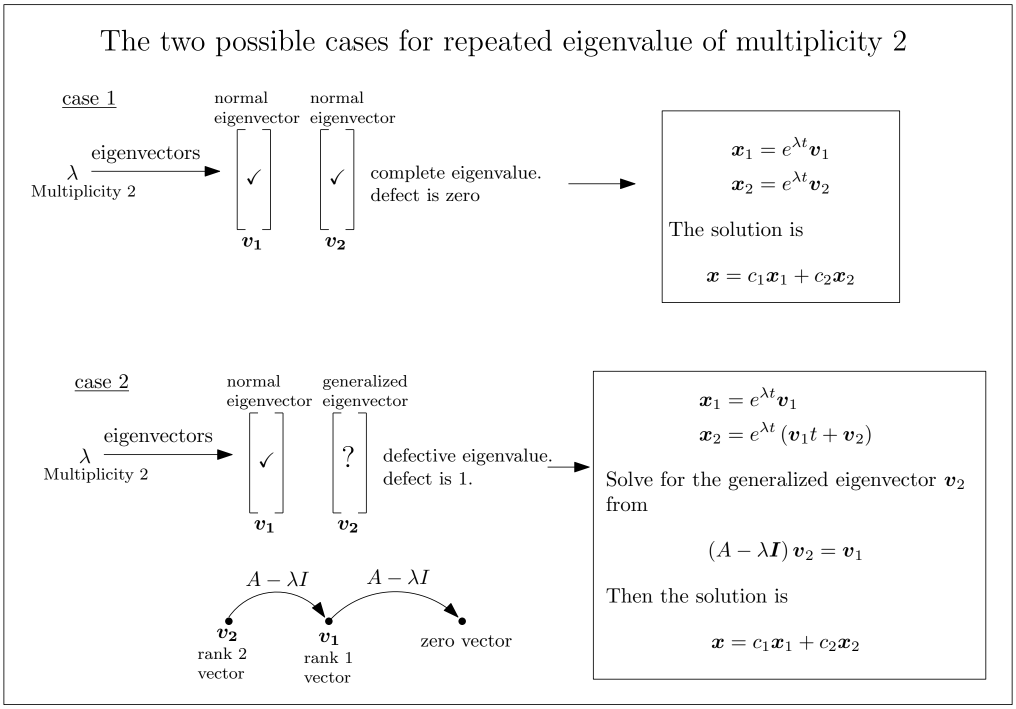


Figure 39: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} 7 & 1 \\ -4 & 3 \end{bmatrix} - (5)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} 1 \\ -\frac{5}{2} \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue 5. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\ &= \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} \mathrm{e}^{5t} \\ &= \begin{bmatrix} -\frac{\mathrm{e}^{5t}}{2} \\ \mathrm{e}^{5t} \end{bmatrix}\end{aligned}$$

And

$$\begin{aligned}\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\ &= \left(\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} t + \begin{bmatrix} 1 \\ -\frac{5}{2} \end{bmatrix} \right) \mathrm{e}^{5t} \\ &= \begin{bmatrix} -\frac{\mathrm{e}^{5t}(t-2)}{2} \\ \frac{\mathrm{e}^{5t}(2t-5)}{2} \end{bmatrix}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} -\frac{\mathrm{e}^{5t}}{2} \\ \mathrm{e}^{5t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{5t}\left(-\frac{t}{2} + 1\right) \\ \mathrm{e}^{5t}\left(t - \frac{5}{2}\right) \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -\frac{((t-2)c_2 + c_1)\mathrm{e}^{5t}}{2} \\ \mathrm{e}^{5t}\left(c_1 + c_2 t - \frac{5}{2} c_2\right) \end{bmatrix}$$

The following is the phase plot of the system.

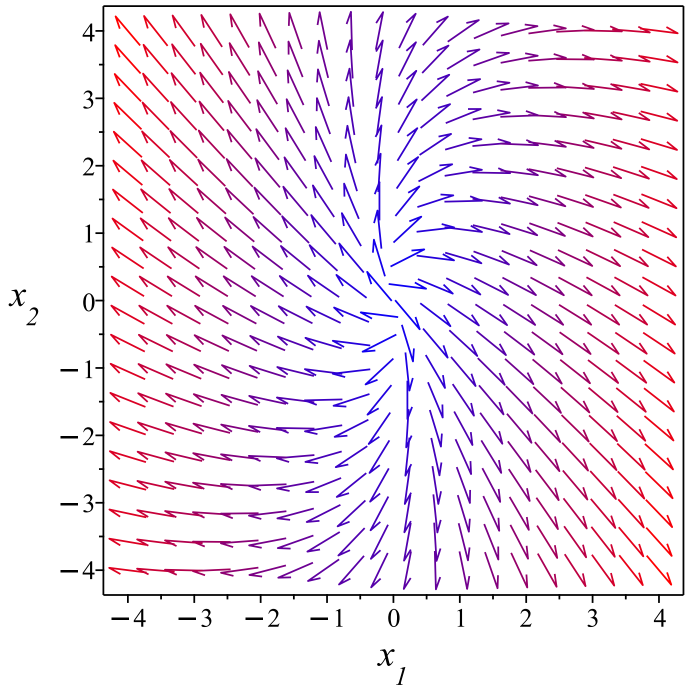


Figure 40: Phase plot

6.5.3 Maple step by step solution

Let's solve

$[x_1'(t) = 7x_1(t) + x_2(t), x_2'(t) = -4x_1(t) + 3x_2(t)]$

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x}^{\rightarrow\prime}(t) = \begin{bmatrix} 7 & 1 \\ -4 & 3 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{x}^{\rightarrow\prime}(t) = \begin{bmatrix} 7 & 1 \\ -4 & 3 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 7 & 1 \\ -4 & 3 \end{bmatrix}$$

- Rewrite the system as

  $\underline{\vec{x}}'(t) = A \cdot \underline{\vec{x}}(t)$
- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[5, \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}\right], \left[5, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right]\right]$$
- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

  $$\left[5, \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}\right]$$
- First solution from eigenvalue 5

  $$\underline{\vec{x}}_1(t) = e^{5t} \cdot \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$
- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = 5$ is the eigenvalue, and

  $\underline{\vec{x}}_2(t) = e^{\lambda t}\left(t\vec{v} + \vec{p}\right)$
- Note that the t multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\underline{\vec{x}}_2(t)$ into the homogeneous system

  $\lambda e^{\lambda t}\left(t\vec{v} + \vec{p}\right) + e^{\lambda t}\vec{v} = \left(e^{\lambda t} A\right) \cdot \left(t\vec{v} + \vec{p}\right)$
- Use the fact that $\vec{v}$ is an eigenvector of A

  $\lambda e^{\lambda t}\left(t\vec{v} + \vec{p}\right) + e^{\lambda t}\vec{v} = e^{\lambda t}\left(\lambda t\vec{v} + A \cdot \vec{p}\right)$
- Simplify equation

  $\lambda\vec{p} + \vec{v} = A \cdot \vec{p}$
- Make use of the identity matrix I

  $(\lambda \cdot I) \cdot \vec{p} + \vec{v} = A \cdot \vec{p}$
- Condition $\vec{p}$ must meet for $\underline{\vec{x}}_2(t)$ to be a solution to the homogeneous system

  $(A - \lambda \cdot I) \cdot \vec{p} = \vec{v}$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue 5

$$\left(\begin{bmatrix} 7 & 1 \\ -4 & 3 \end{bmatrix} - 5 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \cdot \vec{p} = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

- Choice of $\vec{p}$

$$\vec{p} = \begin{bmatrix} -\frac{1}{4} \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue 5

$$\underline{x}^{\rightarrow}{}_{2}(t) = \mathrm{e}^{5t} \cdot \left(t \cdot \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{1}{4} \\ 0 \end{bmatrix}\right)$$

- General solution to the system of ODEs

$$\underline{x}^{\rightarrow} = c_1 \underline{x}^{\rightarrow}{}_{1}(t) + c_2 \underline{x}^{\rightarrow}{}_{2}(t)$$

- Substitute solutions into the general solution

$$\underline{x}^{\rightarrow} = c_1 \mathrm{e}^{5t} \cdot \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{5t} \cdot \left(t \cdot \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{1}{4} \\ 0 \end{bmatrix}\right)$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -\frac{\mathrm{e}^{5t}(2c_2t+2c_1+c_2)}{4} \\ \mathrm{e}^{5t}(c_2t + c_1) \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x_1(t) = -\frac{\mathrm{e}^{5t}(2c_2t+2c_1+c_2)}{4}, x_2(t) = \mathrm{e}^{5t}(c_2t + c_1)\right\}$$

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 35

```
dsolve([diff(x__1(t),t)=7*x__1(t)+1*x__2(t),diff(x__2(t),t)=-4*x__1(t)+3*x__2(t)],singsol=al
```

$$x_1(t) = \mathrm{e}^{5t}(c_2t + c_1)$$
$$x_2(t) = -\mathrm{e}^{5t}(2c_2t + 2c_1 - c_2)$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 45

```
DSolve[{x1'[t]==7*x1[t]+1*x2[t],x2'[t]==-4*x1[t]+3*x2[t]},{x1[t],x2[t]},t,IncludeSingularSol
```

$$\begin{aligned} \mathrm{x1}(t) &\to e^{5t}(2c_1t + c_2t + c_1) \\ \mathrm{x2}(t) &\to e^{5t}(c_2 - 2(2c_1 + c_2)t) \end{aligned}$$

6.6 problem problem 6

Internal problem ID [363]
Internal file name [`OUTPUT/363_Sunday_June_05_2022_01_39_42_AM_53229074/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.6, Multiple Eigenvalue Solutions. Page 451
Problem number: problem 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= x_1(t) - 4x_2(t) \\
x_2'(t) &= 4x_1(t) + 9x_2(t)
\end{aligned}
$$

6.6.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ 4 & 9 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{5t}(1-4t) & -4t\,e^{5t} \\ 4t\,e^{5t} & e^{5t}(4t+1) \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{5t}(1-4t) & -4t\,e^{5t} \\ 4t\,e^{5t} & e^{5t}(4t+1) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{5t}(1-4t)\,c_1 - 4t\,e^{5t}c_2 \\ 4t\,e^{5t}c_1 + e^{5t}(4t+1)\,c_2 \end{bmatrix} \\
&= \begin{bmatrix} (c_1(1-4t) - 4c_2t)\,e^{5t} \\ e^{5t}(4tc_1 + 4c_2t + c_2) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

6.6.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ 4 & 9 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & -4 \\ 4 & 9 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & -4 \\ 4 & 9-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 10\lambda + 25 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 5$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
5	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 5$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -4 \\ 4 & 9 \end{bmatrix} - (5)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & -4 \\ 4 & 4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -4 & -4 & 0 \\ 4 & 4 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cc|c} -4 & -4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -4 & -4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
5	2	1	Yes	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 5 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

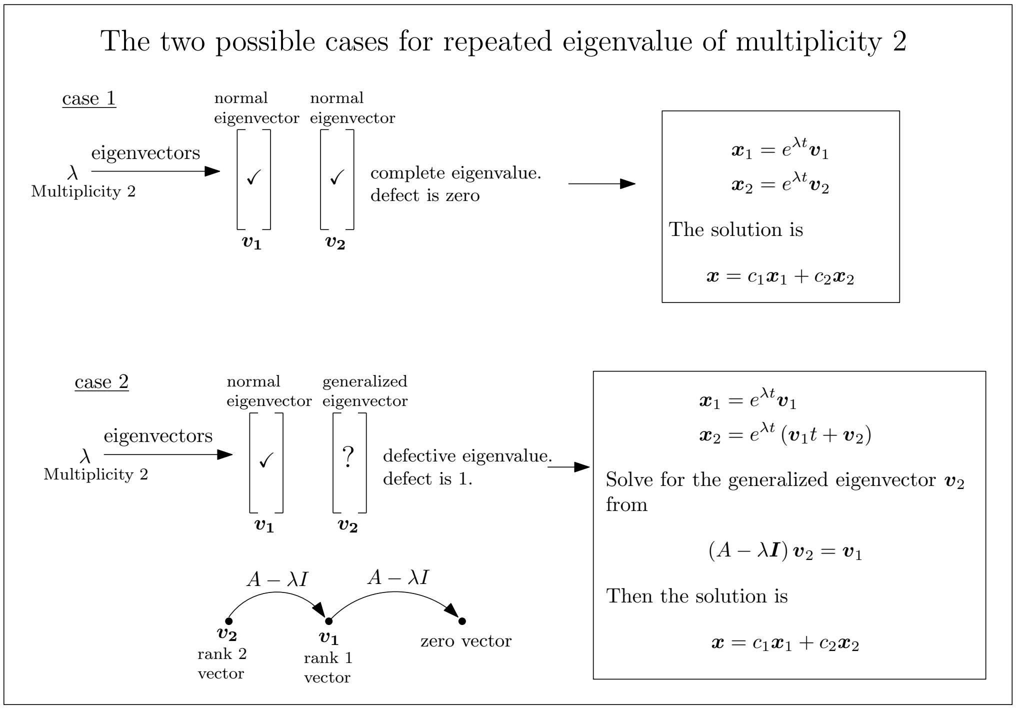


Figure 41: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\begin{aligned}\left(\begin{bmatrix}1 & -4\\ 4 & 9\end{bmatrix} - (5)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\end{bmatrix} &= \begin{bmatrix}-1\\ 1\end{bmatrix}\\ \begin{bmatrix}-4 & -4\\ 4 & 4\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix} &= \begin{bmatrix}-1\\ 1\end{bmatrix}\end{aligned}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix}-\frac{3}{4}\\ 1\end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue 5. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\ &= \begin{bmatrix} -1 \\ 1 \end{bmatrix} \mathrm{e}^{5t} \\ &= \begin{bmatrix} -\mathrm{e}^{5t} \\ \mathrm{e}^{5t} \end{bmatrix}\end{aligned}$$

And

$$\begin{aligned}\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\ &= \left(\begin{bmatrix} -1 \\ 1 \end{bmatrix} t + \begin{bmatrix} -\frac{3}{4} \\ 1 \end{bmatrix}\right) \mathrm{e}^{5t} \\ &= \begin{bmatrix} -\frac{\mathrm{e}^{5t}(4t+3)}{4} \\ \mathrm{e}^{5t}(t+1) \end{bmatrix}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} -\mathrm{e}^{5t} \\ \mathrm{e}^{5t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{5t}\left(-t - \frac{3}{4}\right) \\ \mathrm{e}^{5t}(t+1) \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{5t}\left(-c_1 - c_2 t - \frac{3}{4} c_2\right) \\ \mathrm{e}^{5t}(c_2 t + c_1 + c_2) \end{bmatrix}$$

The following is the phase plot of the system.

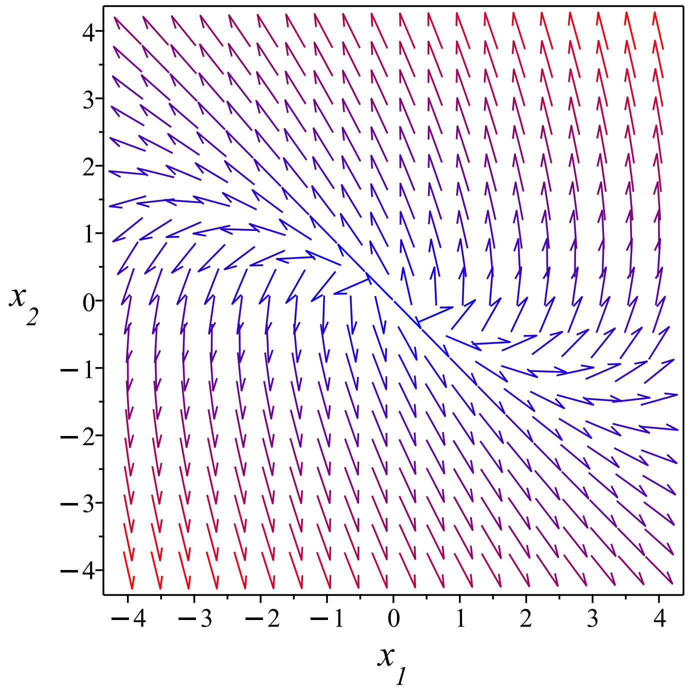


Figure 42: Phase plot

6.6.3 Maple step by step solution

Let's solve

$[x_1'(t) = x_1(t) - 4x_2(t), x_2'(t) = 4x_1(t) + 9x_2(t)]$

- Define vector

$$\underline{x^{\rightarrow}}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x^{\rightarrow}}'(t) = \begin{bmatrix} 1 & -4 \\ 4 & 9 \end{bmatrix} \cdot \underline{x^{\rightarrow}}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{x^{\rightarrow}}'(t) = \begin{bmatrix} 1 & -4 \\ 4 & 9 \end{bmatrix} \cdot \underline{x^{\rightarrow}}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & -4 \\ 4 & 9 \end{bmatrix}$$

- Rewrite the system as

  $\underline{x}^{\rightarrow\prime}(t) = A \cdot \underline{x}^{\rightarrow}(t)$
- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[5, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right], \left[5, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right]\right]$$
- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

  $$\left[5, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right]$$
- First solution from eigenvalue 5

  $$\underline{x}^{\rightarrow}{}_1(t) = e^{5t} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$
- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = 5$ is the eigenvalue, and

  $\underline{x}^{\rightarrow}{}_2(t) = e^{\lambda t}\left(t\vec{v} + \vec{p}\right)$
- Note that the t multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\underline{x}^{\rightarrow}{}_2(t)$ into the homogeneous system

  $\lambda e^{\lambda t}\left(t\vec{v} + \vec{p}\right) + e^{\lambda t}\vec{v} = \left(e^{\lambda t}A\right) \cdot \left(t\vec{v} + \vec{p}\right)$
- Use the fact that $\vec{v}$ is an eigenvector of A

  $\lambda e^{\lambda t}\left(t\vec{v} + \vec{p}\right) + e^{\lambda t}\vec{v} = e^{\lambda t}\left(\lambda t\vec{v} + A \cdot \vec{p}\right)$
- Simplify equation

  $\lambda\vec{p} + \vec{v} = A \cdot \vec{p}$
- Make use of the identity matrix I

  $(\lambda \cdot I) \cdot \vec{p} + \vec{v} = A \cdot \vec{p}$
- Condition $\vec{p}$ must meet for $\underline{x}^{\rightarrow}{}_2(t)$ to be a solution to the homogeneous system

  $(A - \lambda \cdot I) \cdot \vec{p} = \vec{v}$

- Choose $\overrightarrow{p}$ to use in the second solution to the homogeneous system from eigenvalue 5

$$\left(\left[\begin{array}{cc} 1 & -4 \\ 4 & 9 \end{array}\right]-5\cdot\left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right]\right)\cdot\overrightarrow{p}=\left[\begin{array}{c} -1 \\ 1 \end{array}\right]$$

- Choice of $\overrightarrow{p}$

$$\overrightarrow{p}=\left[\begin{array}{c} \frac{1}{4} \\ 0 \end{array}\right]$$

- Second solution from eigenvalue 5

$$\underline{x^{\rightarrow}}_{2}(t)=\mathrm{e}^{5t}\cdot\left(t\cdot\left[\begin{array}{c} -1 \\ 1 \end{array}\right]+\left[\begin{array}{c} \frac{1}{4} \\ 0 \end{array}\right]\right)$$

- General solution to the system of ODEs

$$\underline{x^{\rightarrow}}=c_{1}\underline{x^{\rightarrow}}_{1}(t)+c_{2}\underline{x^{\rightarrow}}_{2}(t)$$

- Substitute solutions into the general solution

$$\underline{x^{\rightarrow}}=c_{1}\mathrm{e}^{5t}\cdot\left[\begin{array}{c} -1 \\ 1 \end{array}\right]+c_{2}\mathrm{e}^{5t}\cdot\left(t\cdot\left[\begin{array}{c} -1 \\ 1 \end{array}\right]+\left[\begin{array}{c} \frac{1}{4} \\ 0 \end{array}\right]\right)$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c} x_{1}(t) \\ x_{2}(t) \end{array}\right]=\left[\begin{array}{c} \mathrm{e}^{5t}\left(-c_{1}-c_{2}t+\frac{1}{4}c_{2}\right) \\ \mathrm{e}^{5t}(c_{2}t+c_{1}) \end{array}\right]$$

- Solution to the system of ODEs

$$\left\{x_{1}(t)=\mathrm{e}^{5t}\left(-c_{1}-c_{2}t+\frac{1}{4}c_{2}\right),x_{2}(t)=\mathrm{e}^{5t}(c_{2}t+c_{1})\right\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 33

```
dsolve([diff(x__1(t),t)=1*x__1(t)-4*x__2(t),diff(x__2(t),t)=4*x__1(t)+9*x__2(t)],singsol=all
```

$$x_{1}(t)=\mathrm{e}^{5t}(c_{2}t+c_{1})$$
$$x_{2}(t)=-\frac{\mathrm{e}^{5t}(4c_{2}t+4c_{1}+c_{2})}{4}$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 46

```
DSolve[{x1'[t]==1*x1[t]-4*x2[t],x2'[t]==4*x1[t]+9*x2[t]},{x1[t],x2[t]},t,IncludeSingularSolu
```

$$\begin{aligned}
\mathrm{x1}(t) &\to -e^{5t}(c_1(4t-1)+4c_2t) \\
\mathrm{x2}(t) &\to e^{5t}(4(c_1+c_2)t+c_2)
\end{aligned}$$

6.7 problem problem 7

Internal problem ID [364]
Internal file name [OUTPUT/364_Sunday_June_05_2022_01_39_43_AM_17766894/index.tex]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.6, Multiple Eigenvalue Solutions. Page 451
Problem number: problem 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= 2x_1(t) \\
x_2'(t) &= -7x_1(t) + 9x_2(t) + 7x_3(t) \\
x_3'(t) &= 2x_3(t)
\end{aligned}
$$

6.7.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ -7 & 9 & 7 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{2t} & 0 & 0 \\ -e^{9t} + e^{2t} & e^{9t} & e^{9t} - e^{2t} \\ 0 & 0 & e^{2t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{2t} & 0 & 0 \\ -e^{9t}+e^{2t} & e^{9t} & e^{9t}-e^{2t} \\ 0 & 0 & e^{2t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} e^{2t}c_1 \\ \left(-e^{9t}+e^{2t}\right)c_1 + e^{9t}c_2 + \left(e^{9t}-e^{2t}\right)c_3 \\ e^{2t}c_3 \end{bmatrix} \\
&= \begin{bmatrix} e^{2t}c_1 \\ \left(-c_1+c_2+c_3\right)e^{9t} + e^{2t}(c_1-c_3) \\ e^{2t}c_3 \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

6.7.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ -7 & 9 & 7 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det\left(A - \lambda I\right) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} 2 & 0 & 0 \\ -7 & 9 & 7 \\ 0 & 0 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$\det\left(\begin{bmatrix} 2-\lambda & 0 & 0 \\ -7 & 9-\lambda & 7 \\ 0 & 0 & 2-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 13\lambda^2 + 40\lambda - 36 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2$$
$$\lambda_2 = 9$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
2	1	real eigenvalue
9	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 0 & 0 \\ -7 & 9 & 7 \\ 0 & 0 & 2 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 0 & 0 \\ -7 & 7 & 7 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -7 & 7 & 7 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{ccc|c} -7 & 7 & 7 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -7 & 7 & 7 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2, v_3\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Let $v_3 = s$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t + s\}$

Hence the solution is

$$\begin{bmatrix} t+s \\ t \\ s \end{bmatrix} = \begin{bmatrix} t+s \\ t \\ s \end{bmatrix}$$

Since there are two free Variable, we have found two eigenvectors associated with this eigenvalue. The above can be written as

$$\begin{aligned} \begin{bmatrix} t+s \\ t \\ s \end{bmatrix} &= \begin{bmatrix} t \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} s \\ 0 \\ s \end{bmatrix} \\ &= t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

By letting $t = 1$ and $s = 1$ then the above becomes

$$\begin{bmatrix} t+s \\ t \\ s \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Hence the two eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix}1\\1\\0\end{bmatrix},\begin{bmatrix}1\\0\\1\end{bmatrix}\right)$$

Considering the eigenvalue $\lambda_2 = 9$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix}2 & 0 & 0\\-7 & 9 & 7\\0 & 0 & 2\end{bmatrix} - (9)\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}-7 & 0 & 0\\-7 & 0 & 7\\0 & 0 & -7\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c}-7 & 0 & 0 & 0\\-7 & 0 & 7 & 0\\0 & 0 & -7 & 0\end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{ccc|c}-7 & 0 & 0 & 0\\0 & 0 & 7 & 0\\0 & 0 & -7 & 0\end{array}\right]$$

$$R_3 = R_3 + R_2 \Longrightarrow \left[\begin{array}{ccc|c}-7 & 0 & 0 & 0\\0 & 0 & 7 & 0\\0 & 0 & 0 & 0\end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix}-7 & 0 & 0\\0 & 0 & 7\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1, v_3\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
2	2	2	No	$\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$
9	1	1	No	$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 2 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

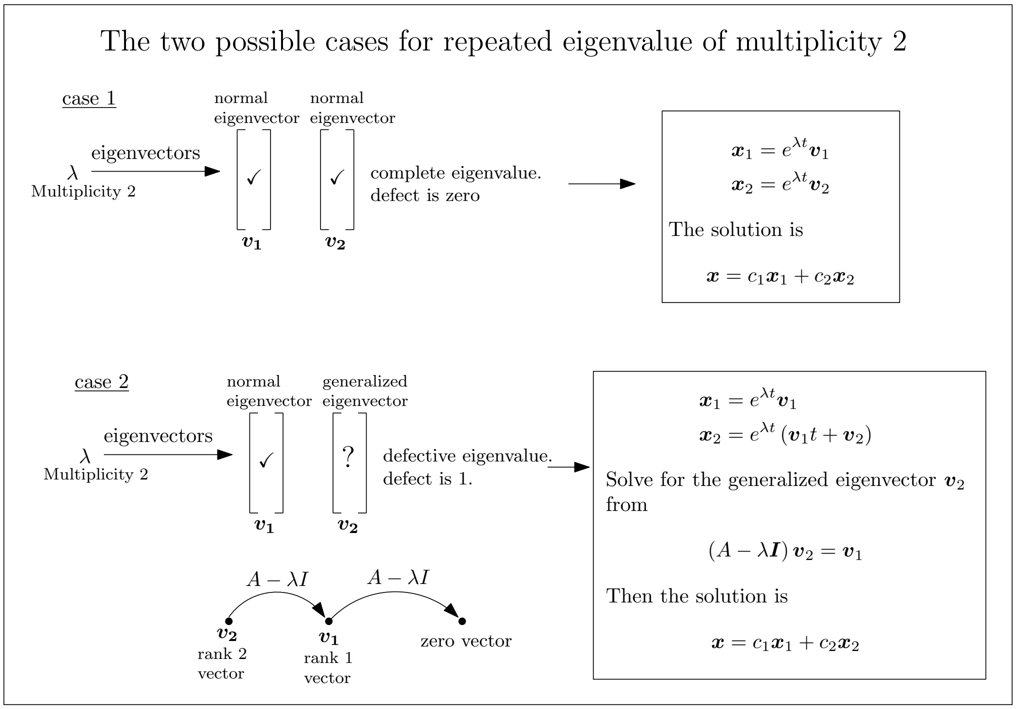


Figure 43: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2 which is the same as its geometric multiplicity 2, then it is complete eigenvalue and this falls into case 1 shown above. Hence the corresponding eigenvector basis are

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{2t} \\
&= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{2t}
\end{aligned}
$$

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{2t} \\
&= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{2t}
\end{aligned}
$$

Since eigenvalue 9 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_3(t) &= \vec{v}_3 e^{9t} \\
&= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{9t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)
$$

Which is written as

$$
\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 \begin{bmatrix} \mathrm{e}^{2t} \\ 0 \\ \mathrm{e}^{2t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{2t} \\ \mathrm{e}^{2t} \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ \mathrm{e}^{9t} \\ 0 \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{2t}(c_1 + c_2) \\ c_2 \mathrm{e}^{2t} + c_3 \mathrm{e}^{9t} \\ c_1 \mathrm{e}^{2t} \end{bmatrix}
$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 43

```
dsolve([diff(x__1(t),t)=2*x__1(t)+0*x__2(t)+0*x__3(t),diff(x__2(t),t)=-7*x__1(t)+9*x__2(t)+7
```

$$
\begin{aligned}
x_1(t) &= c_3 \mathrm{e}^{2t} \\
x_2(t) &= -c_2 \mathrm{e}^{2t} + c_3 \mathrm{e}^{2t} + c_1 \mathrm{e}^{9t} \\
x_3(t) &= c_2 \mathrm{e}^{2t}
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 60

```
DSolve[{x1'[t]==2*x1[t]+0*x2[t]+0*x3[t],x2'[t]==-7*x1[t]+9*x2[t]+7*x3[t],x3'[t]==0*x1[t]+0*x
```

$$
\begin{aligned}
\text{x1}(t) &\to c_1 e^{2t} \\
\text{x2}(t) &\to e^{2t}\left(-\left(c_1\left(e^{7t}-1\right)\right)+(c_2+c_3)e^{7t}-c_3\right) \\
\text{x3}(t) &\to c_3 e^{2t}
\end{aligned}
$$

6.8 problem problem 8

Internal problem ID [365]
Internal file name [`OUTPUT/365_Sunday_June_05_2022_01_39_44_AM_77594711/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.6, Multiple Eigenvalue Solutions. Page 451
Problem number: problem 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned}
x_1'(t) &= 25x_1(t) + 12x_2(t) \\
x_2'(t) &= -18x_1(t) - 5x_2(t) \\
x_3'(t) &= 6x_1(t) + 6x_2(t) + 13x_3(t)
\end{aligned}$$

6.8.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 25 & 12 & 0 \\ -18 & -5 & 0 \\ 6 & 6 & 13 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} -2\,e^{7t} + 3\,e^{13t} & 2\,e^{13t} - 2\,e^{7t} & 0 \\ -3\,e^{13t} + 3\,e^{7t} & 3\,e^{7t} - 2\,e^{13t} & 0 \\ e^{13t} - e^{7t} & e^{13t} - e^{7t} & e^{13t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} -2\,e^{7t}+3\,e^{13t} & 2\,e^{13t}-2\,e^{7t} & 0 \\ -3\,e^{13t}+3\,e^{7t} & 3\,e^{7t}-2\,e^{13t} & 0 \\ e^{13t}-e^{7t} & e^{13t}-e^{7t} & e^{13t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} \left(-2\,e^{7t}+3\,e^{13t}\right)c_1+\left(2\,e^{13t}-2\,e^{7t}\right)c_2 \\ \left(-3\,e^{13t}+3\,e^{7t}\right)c_1+\left(3\,e^{7t}-2\,e^{13t}\right)c_2 \\ \left(e^{13t}-e^{7t}\right)c_1+\left(e^{13t}-e^{7t}\right)c_2+e^{13t}c_3 \end{bmatrix} \\
&= \begin{bmatrix} e^{13t}(3c_1+2c_2)-2\,e^{7t}(c_1+c_2) \\ (-3c_1-2c_2)\,e^{13t}+3\,e^{7t}(c_1+c_2) \\ (c_1+c_2+c_3)\,e^{13t}-e^{7t}(c_1+c_2) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

6.8.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 25 & 12 & 0 \\ -18 & -5 & 0 \\ 6 & 6 & 13 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A-\lambda I\right)=0$$

Expanding gives

$$\det\left(\begin{bmatrix} 25 & 12 & 0 \\ -18 & -5 & 0 \\ 6 & 6 & 13 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 25-\lambda & 12 & 0 \\ -18 & -5-\lambda & 0 \\ 6 & 6 & 13-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 33\lambda^2 + 351\lambda - 1183 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 13$$
$$\lambda_2 = 7$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
7	1	real eigenvalue
13	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 7$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 25 & 12 & 0 \\ -18 & -5 & 0 \\ 6 & 6 & 13 \end{bmatrix} - (7)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 18 & 12 & 0 \\ -18 & -12 & 0 \\ 6 & 6 & 6 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 18 & 12 & 0 & 0 \\ -18 & -12 & 0 & 0 \\ 6 & 6 & 6 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{ccc|c} 18 & 12 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 6 & 6 & 6 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{R_1}{3} \Longrightarrow \left[\begin{array}{ccc|c} 18 & 12 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 6 & 0 \end{array}\right]$$

Since the current pivot $A(2,2)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{ccc|c} 18 & 12 & 0 & 0 \\ 0 & 2 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 18 & 12 & 0 \\ 0 & 2 & 6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2t, v_2 = -3t\}$

Hence the solution is

$$\begin{bmatrix} 2t \\ -3t \\ t \end{bmatrix} = \begin{bmatrix} 2t \\ -3t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 2t \\ -3t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 2t \\ -3t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 13$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 25 & 12 & 0 \\ -18 & -5 & 0 \\ 6 & 6 & 13 \end{bmatrix} - (13) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 12 & 12 & 0 \\ -18 & -18 & 0 \\ 6 & 6 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 12 & 12 & 0 & 0 \\ -18 & -18 & 0 & 0 \\ 6 & 6 & 0 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{3R_1}{2} \implies \left[\begin{array}{ccc|c} 12 & 12 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 6 & 6 & 0 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{R_1}{2} \implies \left[\begin{array}{ccc|c} 12 & 12 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 12 & 12 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2, v_3\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Let $v_3 = s$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \\ s \end{bmatrix} = \begin{bmatrix} -t \\ t \\ s \end{bmatrix}$$

Since there are two free Variable, we have found two eigenvectors associated with this eigenvalue. The above can be written as

$$\begin{aligned} \begin{bmatrix} -t \\ t \\ s \end{bmatrix} &= \begin{bmatrix} -t \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ s \end{bmatrix} \\ &= t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

By letting $t = 1$ and $s = 1$ then the above becomes

$$\begin{bmatrix} -t \\ t \\ s \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Hence the two eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
13	2	2	No	$\begin{bmatrix} 0 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$
7	1	1	No	$\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 13 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

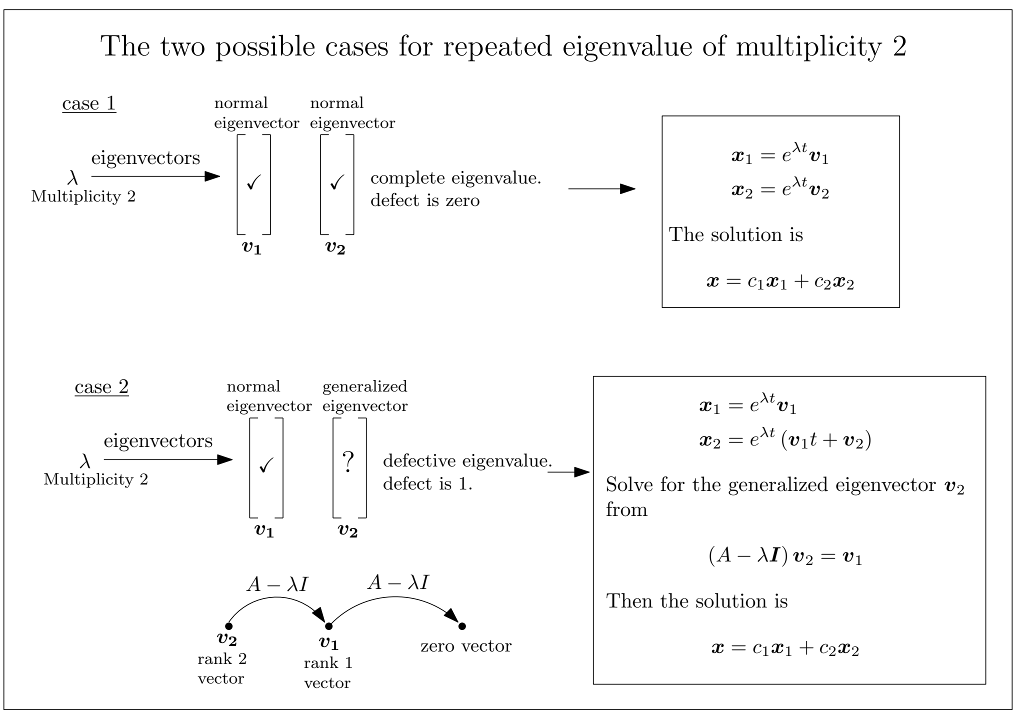


Figure 44: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2 which is the same as its geometric

multiplicity 2, then it is complete eigenvalue and this falls into case 1 shown above. Hence the corresponding eigenvector basis are

$$\vec{x}_1(t) = \vec{v}_1 e^{13t}$$
$$= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{13t}$$

$$\vec{x}_2(t) = \vec{v}_2 e^{13t}$$
$$= \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{13t}$$

Since eigenvalue 7 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_3(t) = \vec{v}_3 e^{7t}$$
$$= \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} e^{7t}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ 0 \\ \mathrm{e}^{13t} \end{bmatrix} + c_2 \begin{bmatrix} -\mathrm{e}^{13t} \\ \mathrm{e}^{13t} \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 2\,\mathrm{e}^{7t} \\ -3\,\mathrm{e}^{7t} \\ \mathrm{e}^{7t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} -c_2\mathrm{e}^{13t} + 2c_3\mathrm{e}^{7t} \\ c_2\mathrm{e}^{13t} - 3c_3\mathrm{e}^{7t} \\ c_1\mathrm{e}^{13t} + c_3\mathrm{e}^{7t} \end{bmatrix}$$

6.8.3 Maple step by step solution

Let's solve

$[x_1'(t) = 25x_1(t) + 12x_2(t), x_2'(t) = -18x_1(t) - 5x_2(t), x_3'(t) = 6x_1(t) + 6x_2(t) + 13x_3(t)]$

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x}^{\rightarrow\prime}(t) = \begin{bmatrix} 25 & 12 & 0 \\ -18 & -5 & 0 \\ 6 & 6 & 13 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{x}^{\rightarrow\prime}(t) = \begin{bmatrix} 25 & 12 & 0 \\ -18 & -5 & 0 \\ 6 & 6 & 13 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 25 & 12 & 0 \\ -18 & -5 & 0 \\ 6 & 6 & 13 \end{bmatrix}$$

- Rewrite the system as

$\underline{x}^{\rightarrow\prime}(t) = A \cdot \underline{x}^{\rightarrow}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[7, \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}\right], \left[13, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right], \left[13, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[7, \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_{1} = e^{7t} \cdot \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[13, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue 13

$$\underline{\vec{x}}_{2}(t) = e^{13t} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = 13$ is the eigenvalue, and

$$\underline{\vec{x}}_{3}(t) = e^{\lambda t}\left(t\vec{v} + \vec{p}\right)$$

- Note that the t multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obtain
- Substitute $\underline{\vec{x}}_{3}(t)$ into the homogeneous system

$$\lambda e^{\lambda t}\left(t\vec{v} + \vec{p}\right) + e^{\lambda t}\vec{v} = \left(e^{\lambda t}A\right) \cdot \left(t\vec{v} + \vec{p}\right)$$

- Use the fact that $\vec{v}$ is an eigenvector of A

$$\lambda e^{\lambda t}\left(t\vec{v} + \vec{p}\right) + e^{\lambda t}\vec{v} = e^{\lambda t}\left(\lambda t\vec{v} + A \cdot \vec{p}\right)$$

- Simplify equation

$$\lambda\vec{p} + \vec{v} = A \cdot \vec{p}$$

- Make use of the identity matrix I

$$(\lambda \cdot I) \cdot \vec{p} + \vec{v} = A \cdot \vec{p}$$

- Condition $\vec{p}$ must meet for $\underline{\vec{x}}_{3}(t)$ to be a solution to the homogeneous system

$$(A - \lambda \cdot I) \cdot \vec{p} = \vec{v}$$

- Choose $\overrightarrow{p}$ to use in the second solution to the homogeneous system from eigenvalue 13

$$\left(\left[\begin{array}{ccc} 25 & 12 & 0 \\ -18 & -5 & 0 \\ 6 & 6 & 13 \end{array}\right]-13\cdot\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]\right)\cdot\overrightarrow{p}=\left[\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right]$$

- Choice of $\overrightarrow{p}$

$$\overrightarrow{p}=\left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right]$$

- Second solution from eigenvalue 13

$$\underline{x}^{\rightarrow}{}_{3}(t)=\mathrm{e}^{13t}\cdot\left(t\cdot\left[\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right]+\left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right]\right)$$

- General solution to the system of ODEs

$$\underline{x}^{\rightarrow}=c_{1}\underline{x}^{\rightarrow}{}_{1}+c_{2}\underline{x}^{\rightarrow}{}_{2}(t)+c_{3}\underline{x}^{\rightarrow}{}_{3}(t)$$

- Substitute solutions into the general solution

$$\underline{x}^{\rightarrow}=c_{1}\mathrm{e}^{7t}\cdot\left[\begin{array}{c} 2 \\ -3 \\ 1 \end{array}\right]+c_{2}\mathrm{e}^{13t}\cdot\left[\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right]+c_{3}\mathrm{e}^{13t}\cdot\left(t\cdot\left[\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right]+\left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right]\right)$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c} x_{1}(t) \\ x_{2}(t) \\ x_{3}(t) \end{array}\right]=\left[\begin{array}{c} 2c_{1}\mathrm{e}^{7t} \\ -3c_{1}\mathrm{e}^{7t} \\ \left(tc_{3}+c_{2}\right)\mathrm{e}^{13t}+c_{1}\mathrm{e}^{7t} \end{array}\right]$$

- Solution to the system of ODEs

$$\left\{x_{1}(t)=2c_{1}\mathrm{e}^{7t}, x_{2}(t)=-3c_{1}\mathrm{e}^{7t}, x_{3}(t)=\left(tc_{3}+c_{2}\right)\mathrm{e}^{13t}+c_{1}\mathrm{e}^{7t}\right\}$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 60

```
dsolve([diff(x__1(t),t)=25*x__1(t)+12*x__2(t)+0*x__3(t),diff(x__2(t),t)=-18*x__1(t)-5*x__2(t
```

$$x_1(t) = c_2 e^{7t} + c_3 e^{13t}$$
$$x_2(t) = -\frac{3c_2 e^{7t}}{2} - c_3 e^{13t}$$
$$x_3(t) = \frac{c_2 e^{7t}}{2} + \frac{c_3 e^{13t}}{2} + e^{13t} c_1$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 107

```
DSolve[{x1'[t]==25*x1[t]+12*x2[t]+0*x3[t],x2'[t]==-18*x1[t]-5*x2[t]+0*x3[t],x3'[t]==6*x1[t]+
```

$$\text{x1}(t) \to e^{7t}\left(c_1\left(3e^{6t} - 2\right) + 2c_2\left(e^{6t} - 1\right)\right)$$
$$\text{x2}(t) \to -e^{7t}\left(3c_1\left(e^{6t} - 1\right) + c_2\left(2e^{6t} - 3\right)\right)$$
$$\text{x3}(t) \to e^{7t}\left(c_1\left(e^{6t} - 1\right) + c_2\left(e^{6t} - 1\right) + c_3 e^{6t}\right)$$

6.9 problem problem 9

Internal problem ID [366]
Internal file name [`OUTPUT/366_Sunday_June_05_2022_01_39_46_AM_72759233/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.6, Multiple Eigenvalue Solutions. Page 451
Problem number: problem 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= -19x_1(t) + 12x_2(t) + 84x_3(t) \\
x_2'(t) &= 5x_2(t) \\
x_3'(t) &= -8x_1(t) + 4x_2(t) + 33x_3(t)
\end{aligned}
$$

6.9.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} -19 & 12 & 84 \\ 0 & 5 & 0 \\ -8 & 4 & 33 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} 7\,e^{5t} - 6\,e^{9t} & 3\,e^{9t} - 3\,e^{5t} & 21\,e^{9t} - 21\,e^{5t} \\ 0 & e^{5t} & 0 \\ -2\,e^{9t} + 2\,e^{5t} & e^{9t} - e^{5t} & -6\,e^{5t} + 7\,e^{9t} \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} 7\,\mathrm{e}^{5t}-6\,\mathrm{e}^{9t} & 3\,\mathrm{e}^{9t}-3\,\mathrm{e}^{5t} & 21\,\mathrm{e}^{9t}-21\,\mathrm{e}^{5t} \\ 0 & \mathrm{e}^{5t} & 0 \\ -2\,\mathrm{e}^{9t}+2\,\mathrm{e}^{5t} & \mathrm{e}^{9t}-\mathrm{e}^{5t} & -6\,\mathrm{e}^{5t}+7\,\mathrm{e}^{9t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} (7\,\mathrm{e}^{5t}-6\,\mathrm{e}^{9t})\,c_1 + (3\,\mathrm{e}^{9t}-3\,\mathrm{e}^{5t})\,c_2 + (21\,\mathrm{e}^{9t}-21\,\mathrm{e}^{5t})\,c_3 \\ \mathrm{e}^{5t}c_2 \\ (-2\,\mathrm{e}^{9t}+2\,\mathrm{e}^{5t})\,c_1 + (\mathrm{e}^{9t}-\mathrm{e}^{5t})\,c_2 + (-6\,\mathrm{e}^{5t}+7\,\mathrm{e}^{9t})\,c_3 \end{bmatrix} \\
&= \begin{bmatrix} (7c_1-3c_2-21c_3)\,\mathrm{e}^{5t} - 6\left(c_1-\frac{c_2}{2}-\frac{7c_3}{2}\right)\mathrm{e}^{9t} \\ \mathrm{e}^{5t}c_2 \\ (2c_1-c_2-6c_3)\,\mathrm{e}^{5t} - 2\left(c_1-\frac{c_2}{2}-\frac{7c_3}{2}\right)\mathrm{e}^{9t} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

6.9.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} -19 & 12 & 84 \\ 0 & 5 & 0 \\ -8 & 4 & 33 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A-\lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -19 & 12 & 84 \\ 0 & 5 & 0 \\ -8 & 4 & 33 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -19-\lambda & 12 & 84 \\ 0 & 5-\lambda & 0 \\ -8 & 4 & 33-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 19\lambda^2 + 115\lambda - 225 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 9$$
$$\lambda_2 = 5$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
5	1	real eigenvalue
9	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 5$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -19 & 12 & 84 \\ 0 & 5 & 0 \\ -8 & 4 & 33 \end{bmatrix} - (5)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -24 & 12 & 84 \\ 0 & 0 & 0 \\ -8 & 4 & 28 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -24 & 12 & 84 & 0 \\ 0 & 0 & 0 & 0 \\ -8 & 4 & 28 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{R_1}{3} \Longrightarrow \left[\begin{array}{ccc|c} -24 & 12 & 84 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -24 & 12 & 84 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2, v_3\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Let $v_3 = s$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{2} + \frac{7s}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{2} + \frac{7s}{2} \\ t \\ s \end{bmatrix} = \begin{bmatrix} \frac{t}{2} + \frac{7s}{2} \\ t \\ s \end{bmatrix}$$

Since there are two free Variable, we have found two eigenvectors associated with this eigenvalue. The above can be written as

$$\begin{aligned} \begin{bmatrix} \frac{t}{2} + \frac{7s}{2} \\ t \\ s \end{bmatrix} &= \begin{bmatrix} \frac{t}{2} \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{7s}{2} \\ 0 \\ s \end{bmatrix} \\ &= t \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} \frac{7}{2} \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

By letting $t = 1$ and $s = 1$ then the above becomes

$$\begin{bmatrix} \frac{t}{2} + \frac{7s}{2} \\ t \\ s \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{7}{2} \\ 0 \\ 1 \end{bmatrix}$$

Hence the two eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix}\frac{1}{2}\\1\\0\end{bmatrix},\begin{bmatrix}\frac{7}{2}\\0\\1\end{bmatrix}\right)$$

Which are normalized to

$$\left(\begin{bmatrix}1\\2\\0\end{bmatrix},\begin{bmatrix}7\\0\\2\end{bmatrix}\right)$$

Considering the eigenvalue $\lambda_2 = 9$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix}-19 & 12 & 84\\0 & 5 & 0\\-8 & 4 & 33\end{bmatrix} - (9)\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}-28 & 12 & 84\\0 & -4 & 0\\-8 & 4 & 24\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c}-28 & 12 & 84 & 0\\0 & -4 & 0 & 0\\-8 & 4 & 24 & 0\end{array}\right]$$

$$R_3 = R_3 - \frac{2R_1}{7} \implies \left[\begin{array}{ccc|c}-28 & 12 & 84 & 0\\0 & -4 & 0 & 0\\0 & \frac{4}{7} & 0 & 0\end{array}\right]$$

$$R_3 = R_3 + \frac{R_2}{7} \implies \left[\begin{array}{ccc|c}-28 & 12 & 84 & 0\\0 & -4 & 0 & 0\\0 & 0 & 0 & 0\end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -28 & 12 & 84 \\ 0 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 3t, v_2 = 0\}$

Hence the solution is

$$\begin{bmatrix} 3t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 3t \\ 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 3t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 3t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
9	1	1	No	$\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$
5	2	2	No	$\begin{bmatrix} \frac{7}{2} & \frac{1}{2} \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 9 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{9t} \\ &= \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} e^{9t}\end{aligned}$$

eigenvalue 5 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

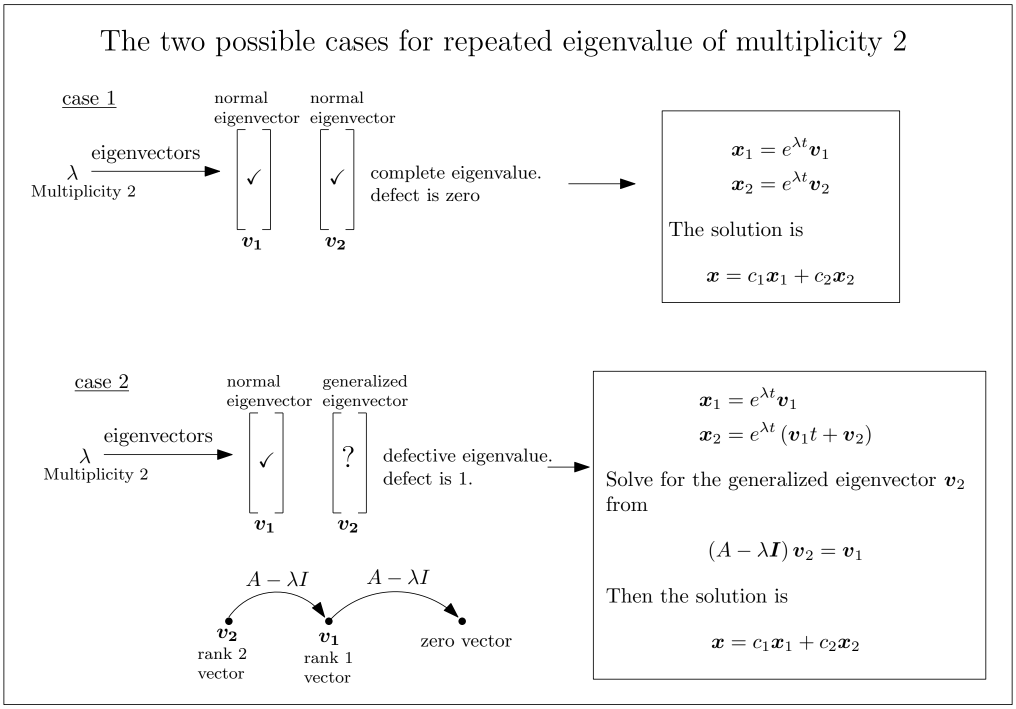


Figure 45: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2 which is the same as its geometric multiplicity 2, then it is complete eigenvalue and this falls into case 1 shown above. Hence the corresponding eigenvector basis are

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{5t} \\
&= \begin{bmatrix} \frac{7}{2} \\ 0 \\ 1 \end{bmatrix} e^{5t}
\end{aligned}
$$

$$
\begin{aligned}
\vec{x}_3(t) &= \vec{v}_3 e^{5t} \\
&= \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} e^{5t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)
$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 \begin{bmatrix} 3\,e^{9t} \\ 0 \\ e^{9t} \end{bmatrix} + c_2 \begin{bmatrix} \frac{7\,e^{5t}}{2} \\ 0 \\ e^{5t} \end{bmatrix} + c_3 \begin{bmatrix} \frac{e^{5t}}{2} \\ e^{5t} \\ 0 \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} \frac{(7c_2+c_3)e^{5t}}{2} + 3c_1e^{9t} \\ c_3e^{5t} \\ c_1e^{9t} + c_2e^{5t} \end{bmatrix}$$

6.9.3 Maple step by step solution

Let's solve

$[x_1'(t) = -19x_1(t) + 12x_2(t) + 84x_3(t)\,, x_2'(t) = 5x_2(t)\,, x_3'(t) = -8x_1(t) + 4x_2(t) + 33x_3(t)]$

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x}^{\rightarrow\prime}(t) = \begin{bmatrix} -19 & 12 & 84 \\ 0 & 5 & 0 \\ -8 & 4 & 33 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{x}^{\rightarrow\prime}(t) = \begin{bmatrix} -19 & 12 & 84 \\ 0 & 5 & 0 \\ -8 & 4 & 33 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -19 & 12 & 84 \\ 0 & 5 & 0 \\ -8 & 4 & 33 \end{bmatrix}$$

- Rewrite the system as
  $\underline{\vec{x}}'(t) = A \cdot \underline{\vec{x}}(t)$
- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A
  $$\left[\left[5, \begin{bmatrix} \frac{7}{2} \\ 0 \\ 1 \end{bmatrix}\right], \left[5, \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}\right], \left[9, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}\right]\right]$$
- Consider eigenpair, with eigenvalue of algebraic multiplicity 2
  $$\left[5, \begin{bmatrix} \frac{7}{2} \\ 0 \\ 1 \end{bmatrix}\right]$$
- First solution from eigenvalue 5
  $$\underline{\vec{x}}_1(t) = e^{5t} \cdot \begin{bmatrix} \frac{7}{2} \\ 0 \\ 1 \end{bmatrix}$$
- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = 5$ is the eigenvalue, and
  $\underline{\vec{x}}_2(t) = e^{\lambda t}\left(t\vec{v} + \vec{p}\right)$
- Note that the t multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\underline{\vec{x}}_2(t)$ into the homogeneous system
  $\lambda e^{\lambda t}\left(t\vec{v} + \vec{p}\right) + e^{\lambda t}\vec{v} = \left(e^{\lambda t} A\right) \cdot \left(t\vec{v} + \vec{p}\right)$
- Use the fact that $\vec{v}$ is an eigenvector of A
  $\lambda e^{\lambda t}\left(t\vec{v} + \vec{p}\right) + e^{\lambda t}\vec{v} = e^{\lambda t}\left(\lambda t\vec{v} + A \cdot \vec{p}\right)$
- Simplify equation
  $\lambda\vec{p} + \vec{v} = A \cdot \vec{p}$

- Make use of the identity matrix I

  $(\lambda \cdot I) \cdot \overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$

- Condition $\overrightarrow{p}$ must meet for $\overrightarrow{x\_\_}_2(t)$ to be a solution to the homogeneous system

  $(A - \lambda \cdot I) \cdot \overrightarrow{p} = \overrightarrow{v}$

- Choose $\overrightarrow{p}$ to use in the second solution to the homogeneous system from eigenvalue 5

$$\left(\begin{bmatrix} -19 & 12 & 84 \\ 0 & 5 & 0 \\ -8 & 4 & 33 \end{bmatrix} - 5 \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \cdot \overrightarrow{p} = \begin{bmatrix} \frac{7}{2} \\ 0 \\ 1 \end{bmatrix}$$

- Choice of $\overrightarrow{p}$

$$\overrightarrow{p} = \begin{bmatrix} -\frac{7}{48} \\ 0 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue 5

$$\overrightarrow{x\_\_}_2(t) = \mathrm{e}^{5t} \cdot \left(t \cdot \begin{bmatrix} \frac{7}{2} \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{7}{48} \\ 0 \\ 0 \end{bmatrix}\right)$$

- Consider eigenpair

$$\left[9, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x\_\_}_3 = \mathrm{e}^{9t} \cdot \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

  $\overrightarrow{x\_\_} = c_1 \overrightarrow{x\_\_}_1(t) + c_2 \overrightarrow{x\_\_}_2(t) + c_3 \overrightarrow{x\_\_}_3$

- Substitute solutions into the general solution

$$\underline{\vec{x}} = c_1 e^{5t} \cdot \begin{bmatrix} \frac{7}{2} \\ 0 \\ 1 \end{bmatrix} + c_2 e^{5t} \cdot \left(t \cdot \begin{bmatrix} \frac{7}{2} \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{7}{48} \\ 0 \\ 0 \end{bmatrix} \right) + c_3 e^{9t} \cdot \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} \frac{7((-1+24t)c_2+24c_1)e^{5t}}{48} + 3c_3 e^{9t} \\ 0 \\ e^{5t}(c_2 t + c_1) + c_3 e^{9t} \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x_1(t) = \frac{7((-1+24t)c_2+24c_1)e^{5t}}{48} + 3c_3 e^{9t}, x_2(t) = 0, x_3(t) = e^{5t}(c_2 t + c_1) + c_3 e^{9t}\right\}$$

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 52

```
dsolve([diff(x__1(t),t)=-19*x__1(t)+12*x__2(t)+84*x__3(t),diff(x__2(t),t)=0*x__1(t)+5*x__2(t
```

$$x_1(t) = c_1 e^{9t} + c_2 e^{5t}$$
$$x_2(t) = c_3 e^{5t}$$
$$x_3(t) = \frac{c_1 e^{9t}}{3} + \frac{2c_2 e^{5t}}{7} - \frac{c_3 e^{5t}}{7}$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 94

```
DSolve[{x1'[t]==-19*x1[t]+12*x2[t]+84*x3[t],x2'[t]==0*x1[t]+5*x2[t]+0*x3[t],x3'[t]==-8*x1[t]
```

$$\text{x1}(t) \to e^{5t}\left(c_1\left(7 - 6e^{4t}\right) + 3(c_2 + 7c_3)\left(e^{4t} - 1\right)\right)$$
$$\text{x2}(t) \to c_2 e^{5t}$$
$$\text{x3}(t) \to e^{5t}\left(-2c_1\left(e^{4t} - 1\right) + c_2\left(e^{4t} - 1\right) + c_3\left(7e^{4t} - 6\right)\right)$$

6.10 problem problem 10

Internal problem ID [367]
Internal file name [`OUTPUT/367_Sunday_June_05_2022_01_39_47_AM_31168890/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.6, Multiple Eigenvalue Solutions. Page 451
Problem number: problem 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= -13x_1(t) + 40x_2(t) - 48x_3(t)\\
x_2'(t) &= -8x_1(t) + 23x_2(t) - 24x_3(t)\\
x_3'(t) &= 3x_3(t)
\end{aligned}
$$

6.10.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} -13 & 40 & -48 \\ -8 & 23 & -24 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} 5\,\mathrm{e}^{3t} - 4\,\mathrm{e}^{7t} & 10\,\mathrm{e}^{7t} - 10\,\mathrm{e}^{3t} & 12\,\mathrm{e}^{3t} - 12\,\mathrm{e}^{7t} \\ -2\,\mathrm{e}^{7t} + 2\,\mathrm{e}^{3t} & -4\,\mathrm{e}^{3t} + 5\,\mathrm{e}^{7t} & 6\,\mathrm{e}^{3t} - 6\,\mathrm{e}^{7t} \\ 0 & 0 & \mathrm{e}^{3t} \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} 5\,\mathrm{e}^{3t}-4\,\mathrm{e}^{7t} & 10\,\mathrm{e}^{7t}-10\,\mathrm{e}^{3t} & 12\,\mathrm{e}^{3t}-12\,\mathrm{e}^{7t} \\ -2\,\mathrm{e}^{7t}+2\,\mathrm{e}^{3t} & -4\,\mathrm{e}^{3t}+5\,\mathrm{e}^{7t} & 6\,\mathrm{e}^{3t}-6\,\mathrm{e}^{7t} \\ 0 & 0 & \mathrm{e}^{3t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} (5\,\mathrm{e}^{3t}-4\,\mathrm{e}^{7t})\,c_1+(10\,\mathrm{e}^{7t}-10\,\mathrm{e}^{3t})\,c_2+(12\,\mathrm{e}^{3t}-12\,\mathrm{e}^{7t})\,c_3 \\ (-2\,\mathrm{e}^{7t}+2\,\mathrm{e}^{3t})\,c_1+(-4\,\mathrm{e}^{3t}+5\,\mathrm{e}^{7t})\,c_2+(6\,\mathrm{e}^{3t}-6\,\mathrm{e}^{7t})\,c_3 \\ \mathrm{e}^{3t}c_3 \end{bmatrix} \\
&= \begin{bmatrix} (5c_1-10c_2+12c_3)\,\mathrm{e}^{3t}-4\left(c_1-\frac{5c_2}{2}+3c_3\right)\mathrm{e}^{7t} \\ (2c_1-4c_2+6c_3)\,\mathrm{e}^{3t}-2\left(c_1-\frac{5c_2}{2}+3c_3\right)\mathrm{e}^{7t} \\ \mathrm{e}^{3t}c_3 \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

6.10.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} -13 & 40 & -48 \\ -8 & 23 & -24 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A-\lambda I)=0$$

Expanding gives

$$\det\left(\begin{bmatrix} -13 & 40 & -48 \\ -8 & 23 & -24 \\ 0 & 0 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$
\det\left(\begin{bmatrix} -13-\lambda & 40 & -48 \\ -8 & 23-\lambda & -24 \\ 0 & 0 & 3-\lambda \end{bmatrix}\right) = 0
$$

Which gives the characteristic equation

$$
\lambda^3 - 13\lambda^2 + 51\lambda - 63 = 0
$$

The roots of the above are the eigenvalues.

$$
\begin{aligned}
\lambda_1 &= 7 \\
\lambda_2 &= 3
\end{aligned}
$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
3	1	real eigenvalue
7	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$
\left(\begin{bmatrix} -13 & 40 & -48 \\ -8 & 23 & -24 \\ 0 & 0 & 3 \end{bmatrix} - (3)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$
\begin{bmatrix} -16 & 40 & -48 \\ -8 & 20 & -24 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$
\left[\begin{array}{ccc|c} -16 & 40 & -48 & 0 \\ -8 & 20 & -24 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]
$$

$$R_2 = R_2 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} -16 & 40 & -48 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -16 & 40 & -48 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2, v_3\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Let $v_3 = s$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{5t}{2} - 3s\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{5t}{2} - 3s \\ t \\ s \end{bmatrix} = \begin{bmatrix} \frac{5t}{2} - 3s \\ t \\ s \end{bmatrix}$$

Since there are two free Variable, we have found two eigenvectors associated with this eigenvalue. The above can be written as

$$\begin{aligned} \begin{bmatrix} \frac{5t}{2} - 3s \\ t \\ s \end{bmatrix} &= \begin{bmatrix} \frac{5t}{2} \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} -3s \\ 0 \\ s \end{bmatrix} \\ &= t \begin{bmatrix} \frac{5}{2} \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

By letting $t = 1$ and $s = 1$ then the above becomes

$$\begin{bmatrix} \frac{5t}{2} - 3s \\ t \\ s \end{bmatrix} = \begin{bmatrix} \frac{5}{2} \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

Hence the two eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix} \frac{5}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right)$$

Which are normalized to

$$\left(\begin{bmatrix} 5 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right)$$

Considering the eigenvalue $\lambda_2 = 7$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -13 & 40 & -48 \\ -8 & 23 & -24 \\ 0 & 0 & 3 \end{bmatrix} - (7) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -20 & 40 & -48 \\ -8 & 16 & -24 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -20 & 40 & -48 & 0 \\ -8 & 16 & -24 & 0 \\ 0 & 0 & -4 & 0 \end{array} \right]$$

$$R_2 = R_2 - \frac{2R_1}{5} \implies \left[\begin{array}{ccc|c} -20 & 40 & -48 & 0 \\ 0 & 0 & -\frac{24}{5} & 0 \\ 0 & 0 & -4 & 0 \end{array} \right]$$

$$R_3 = R_3 - \frac{5R_2}{6} \implies \left[\begin{array}{ccc|c} -20 & 40 & -48 & 0 \\ 0 & 0 & -\frac{24}{5} & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -20 & 40 & -48 \\ 0 & 0 & -\frac{24}{5} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1, v_3\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2t, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} 2t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 2t \\ t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 2t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 2t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
7	1	1	No	$\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$
3	2	2	No	$\begin{bmatrix} -3 & \frac{5}{2} \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 7 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{7t} \\ &= \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} e^{7t} \end{aligned}$$

eigenvalue 3 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

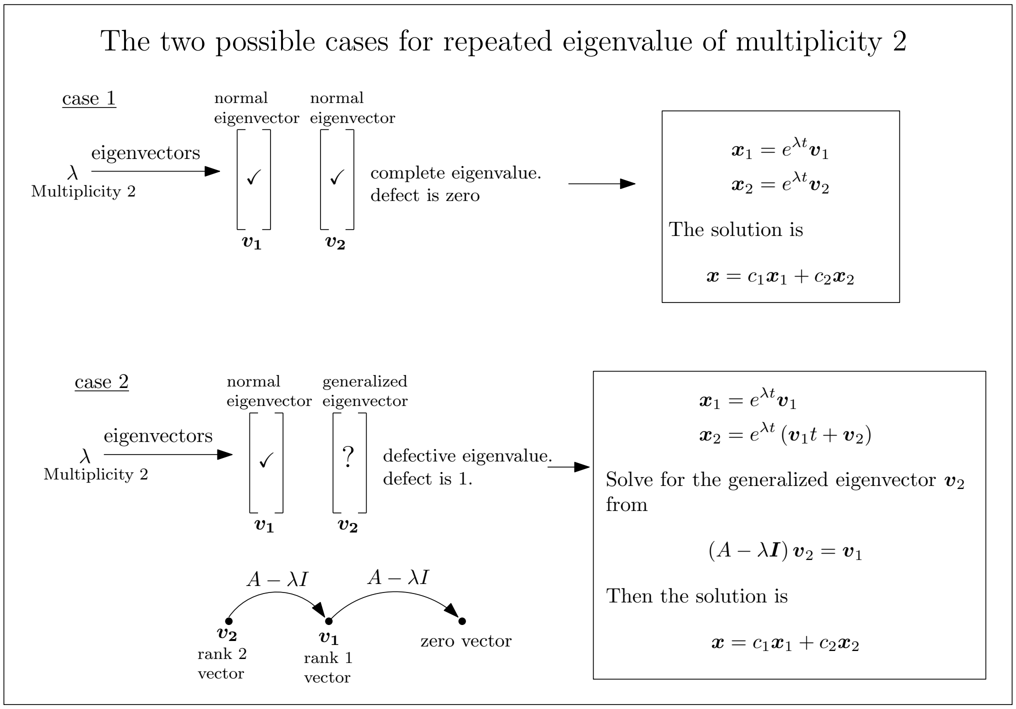


Figure 46: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2 which is the same as its geometric multiplicity 2, then it is complete eigenvalue and this falls into case 1 shown above. Hence the corresponding eigenvector basis are

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{3t}\\ &= \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} e^{3t}\end{aligned}$$

$$\begin{aligned}\vec{x}_3(t) &= \vec{v}_3 e^{3t}\\ &= \begin{bmatrix} \frac{5}{2} \\ 1 \\ 0 \end{bmatrix} e^{3t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 \begin{bmatrix} 2\,\mathrm{e}^{7t} \\ \mathrm{e}^{7t} \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -3\,\mathrm{e}^{3t} \\ 0 \\ \mathrm{e}^{3t} \end{bmatrix} + c_3 \begin{bmatrix} \frac{5\,\mathrm{e}^{3t}}{2} \\ \mathrm{e}^{3t} \\ 0 \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} \frac{(-6c_2+5c_3)\mathrm{e}^{3t}}{2} + 2c_1\mathrm{e}^{7t} \\ c_1\mathrm{e}^{7t} + c_3\mathrm{e}^{3t} \\ c_2\mathrm{e}^{3t} \end{bmatrix}$$

6.10.3 Maple step by step solution

Let's solve

$[x_1'(t) = -13x_1(t) + 40x_2(t) - 48x_3(t)\,, x_2'(t) = -8x_1(t) + 23x_2(t) - 24x_3(t)\,, x_3'(t) = 3x_3(t)]$

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x}^{\rightarrow\prime}(t) = \begin{bmatrix} -13 & 40 & -48 \\ -8 & 23 & -24 \\ 0 & 0 & 3 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{x}^{\rightarrow\prime}(t) = \begin{bmatrix} -13 & 40 & -48 \\ -8 & 23 & -24 \\ 0 & 0 & 3 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -13 & 40 & -48 \\ -8 & 23 & -24 \\ 0 & 0 & 3 \end{bmatrix}$$

- Rewrite the system as

  $\underline{\vec{x}}'(t) = A \cdot \underline{\vec{x}}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[3, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}\right], \left[3, \begin{bmatrix} \frac{5}{2} \\ 1 \\ 0 \end{bmatrix}\right], \left[7, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

  $$\left[3, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue 3

  $$\underline{\vec{x}}_1(t) = e^{3t} \cdot \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = 3$ is the eigenvalue, and

  $\underline{\vec{x}}_2(t) = e^{\lambda t}\left(t\vec{v} + \vec{p}\right)$

- Note that the t multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\underline{\vec{x}}_2(t)$ into the homogeneous system

  $\lambda e^{\lambda t}\left(t\vec{v} + \vec{p}\right) + e^{\lambda t}\vec{v} = \left(e^{\lambda t}A\right) \cdot \left(t\vec{v} + \vec{p}\right)$

- Use the fact that $\vec{v}$ is an eigenvector of A

  $\lambda e^{\lambda t}\left(t\vec{v} + \vec{p}\right) + e^{\lambda t}\vec{v} = e^{\lambda t}\left(\lambda t\vec{v} + A \cdot \vec{p}\right)$

- Simplify equation

  $\lambda\vec{p} + \vec{v} = A \cdot \vec{p}$

- Make use of the identity matrix I

  $(\lambda \cdot I) \cdot \overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$

- Condition $\overrightarrow{p}$ must meet for $\overrightarrow{x\_\_\_}_2(t)$ to be a solution to the homogeneous system

  $(A - \lambda \cdot I) \cdot \overrightarrow{p} = \overrightarrow{v}$

- Choose $\overrightarrow{p}$ to use in the second solution to the homogeneous system from eigenvalue 3

  $$\left(\begin{bmatrix} -13 & 40 & -48 \\ -8 & 23 & -24 \\ 0 & 0 & 3 \end{bmatrix} - 3 \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \cdot \overrightarrow{p} = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

- Choice of $\overrightarrow{p}$

  $$\overrightarrow{p} = \begin{bmatrix} \frac{3}{16} \\ 0 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue 3

  $$\overrightarrow{x\_\_\_}_2(t) = \mathrm{e}^{3t} \cdot \left(t \cdot \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{3}{16} \\ 0 \\ 0 \end{bmatrix}\right)$$

- Consider eigenpair

  $$\left[7, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

  $$\overrightarrow{x\_\_\_}_3 = \mathrm{e}^{7t} \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

- General solution to the system of ODEs

  $\overrightarrow{x\_\_\_} = c_1 \overrightarrow{x\_\_\_}_1(t) + c_2 \overrightarrow{x\_\_\_}_2(t) + c_3 \overrightarrow{x\_\_\_}_3$

- Substitute solutions into the general solution

$$\underline{\overrightarrow{x}} = c_1 \mathrm{e}^{3t} \cdot \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{3t} \cdot \left(t \cdot \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{3}{16} \\ 0 \\ 0 \end{bmatrix} \right) + c_3 \mathrm{e}^{7t} \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} \frac{((-48t+3)c_2-48c_1)\mathrm{e}^{3t}}{16} + 2c_3 \mathrm{e}^{7t} \\ c_3 \mathrm{e}^{7t} \\ (c_2 t + c_1) \mathrm{e}^{3t} \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x_1(t) = \frac{((-48t+3)c_2-48c_1)\mathrm{e}^{3t}}{16} + 2c_3 \mathrm{e}^{7t}, x_2(t) = c_3 \mathrm{e}^{7t}, x_3(t) = (c_2 t + c_1) \mathrm{e}^{3t}\right\}$$

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 52

```
dsolve([diff(x__1(t),t)=-13*x__1(t)+40*x__2(t)-48*x__3(t),diff(x__2(t),t)=-8*x__1(t)+23*x__2
```

$$x_1(t) = c_1 \mathrm{e}^{3t} + c_2 \mathrm{e}^{7t}$$
$$x_2(t) = \frac{2c_1 \mathrm{e}^{3t}}{5} + \frac{c_2 \mathrm{e}^{7t}}{2} + \frac{6c_3 \mathrm{e}^{3t}}{5}$$
$$x_3(t) = c_3 \mathrm{e}^{3t}$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 98

```
DSolve[{x1'[t]==-13*x1[t]+40*x2[t]-48*x3[t],x2'[t]==-8*x1[t]+23*x2[t]-24*x3[t],x3'[t]==0*x1[
```

$$\mathrm{x1}(t) \to e^{3t}\left(c_1\left(5 - 4e^{4t}\right) + 2(5c_2 - 6c_3)\left(e^{4t} - 1\right)\right)$$
$$\mathrm{x2}(t) \to -e^{3t}\left(2c_1\left(e^{4t} - 1\right) + c_2\left(4 - 5e^{4t}\right) + 6c_3\left(e^{4t} - 1\right)\right)$$
$$\mathrm{x3}(t) \to c_3 e^{3t}$$

6.11 problem problem 11

Internal problem ID [368]
Internal file name [`OUTPUT/368_Sunday_June_05_2022_01_39_49_AM_70097848/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.6, Multiple Eigenvalue Solutions. Page 451
Problem number: problem 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= -3x_1(t) - 4x_3(t) \\
x_2'(t) &= -x_1(t) - x_2(t) - x_3(t) \\
x_3'(t) &= x_1(t) + x_3(t)
\end{aligned}
$$

6.11.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} -3 & 0 & -4 \\ -1 & -1 & -1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{-t}(1-2t) & 0 & -4t\,e^{-t} \\ \frac{e^{-t}t(t-2)}{2} & e^{-t} & e^{-t}t(-1+t) \\ t\,e^{-t} & 0 & e^{-t}(1+2t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \mathrm{e}^{-t}(1-2t) & 0 & -4t\,\mathrm{e}^{-t} \\ \frac{\mathrm{e}^{-t}t(t-2)}{2} & \mathrm{e}^{-t} & \mathrm{e}^{-t}t(-1+t) \\ t\,\mathrm{e}^{-t} & 0 & \mathrm{e}^{-t}(1+2t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^{-t}(1-2t)\,c_1 - 4t\,\mathrm{e}^{-t}c_3 \\ \frac{\mathrm{e}^{-t}t(t-2)c_1}{2} + \mathrm{e}^{-t}c_2 + \mathrm{e}^{-t}t(-1+t)\,c_3 \\ t\,\mathrm{e}^{-t}c_1 + \mathrm{e}^{-t}(1+2t)\,c_3 \end{bmatrix} \\
&= \begin{bmatrix} (c_1(1-2t) - 4c_3t)\,\mathrm{e}^{-t} \\ \frac{((c_1+2c_3)t^2+(-2c_1-2c_3)t+2c_2)\mathrm{e}^{-t}}{2} \\ \mathrm{e}^{-t}(tc_1 + 2c_3t + c_3) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

6.11.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} -3 & 0 & -4 \\ -1 & -1 & -1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det\left(A - \lambda I\right) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} -3 & 0 & -4 \\ -1 & -1 & -1 \\ 1 & 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$\det\left(\begin{bmatrix} -3-\lambda & 0 & -4 \\ -1 & -1-\lambda & -1 \\ 1 & 0 & 1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -3 & 0 & -4 \\ -1 & -1 & -1 \\ 1 & 0 & 1 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 0 & -4 \\ -1 & 0 & -1 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -2 & 0 & -4 & 0 \\ -1 & 0 & -1 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} -2 & 0 & -4 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} -2 & 0 & -4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & 0 & -4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1, v_3\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
-1	3	1	Yes	$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue -1 is real and repated eigenvalue of multiplicity 3.There are three possible cases that can happen. This is illustrated in this diagram

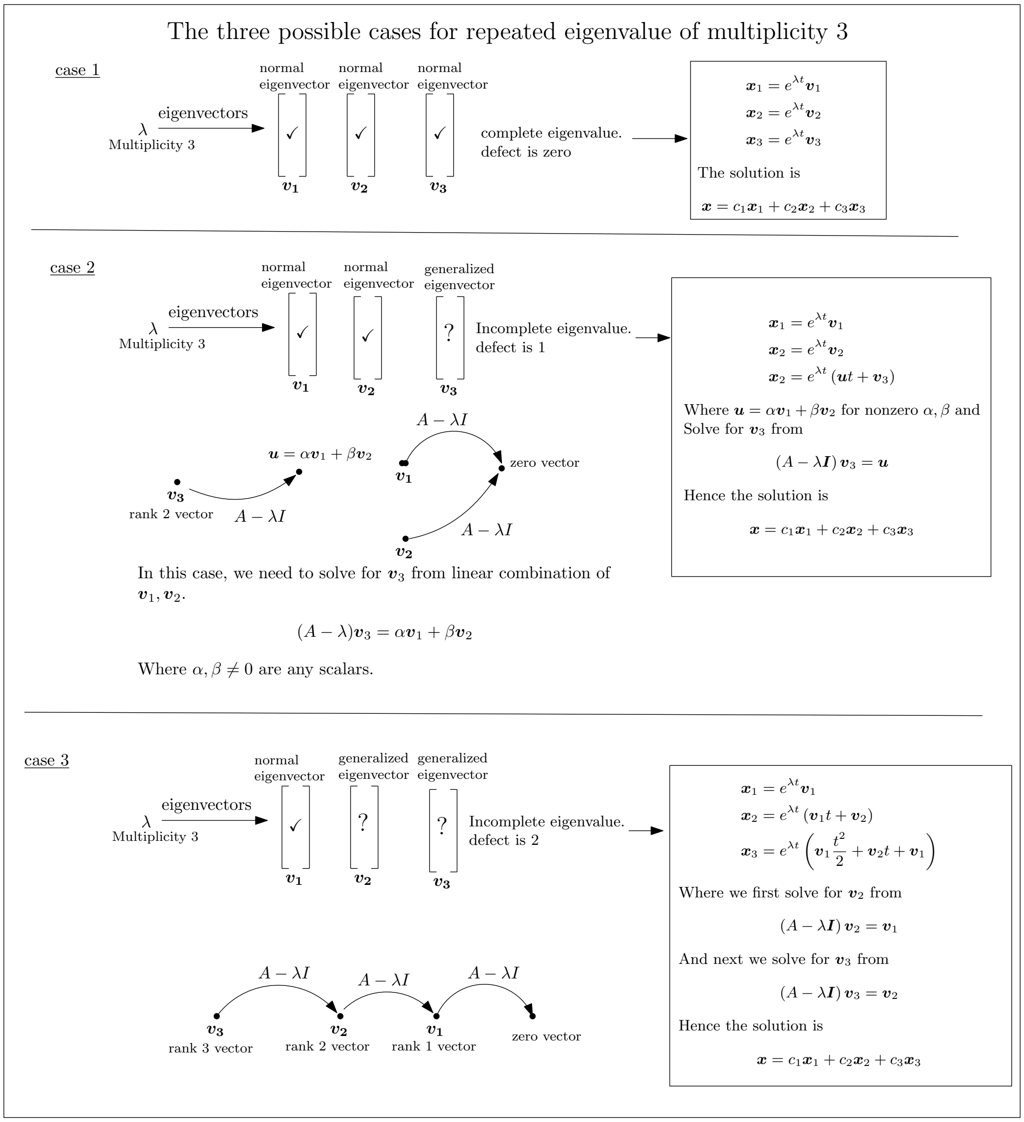


Figure 47: Possible case for repeated λ of multiplicity 3

This eigenvalue has algebraic multiplicity of 3, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 2. This falls into case 3 shown above. First we find generalized eigenvector $\vec{v}_2$ of rank 2 and then use this to find generalized eigenvector $\vec{v}_3$ of rank 3.$\vec{v}_2$ is found by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence

$$\left(\begin{bmatrix}-3 & 0 & -4\\ -1 & -1 & -1\\ 1 & 0 & 1\end{bmatrix} - (-1)\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}0\\ 1\\ 0\end{bmatrix}$$
$$\begin{bmatrix}-2 & 0 & -4\\ -1 & 0 & -1\\ 1 & 0 & 2\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}0\\ 1\\ 0\end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix}-2\\ 1\\ 1\end{bmatrix}$$

Now $\vec{v}_3$ is found by solving

$$(A - \lambda I)\,\vec{v}_3 = \vec{v}_2$$

Where $\vec{v}_2$ is the (rank 2) generalized eigenvector found above. Hence

$$\left(\begin{bmatrix}-3 & 0 & -4\\ -1 & -1 & -1\\ 1 & 0 & 1\end{bmatrix} - (-1)\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}-2\\ 1\\ 1\end{bmatrix}$$
$$\begin{bmatrix}-2 & 0 & -4\\ -1 & 0 & -1\\ 1 & 0 & 2\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}-2\\ 1\\ 1\end{bmatrix}$$

Solving for $\vec{v}_3$ gives

$$\vec{v}_3 = \begin{bmatrix}-3\\ 1\\ 2\end{bmatrix}$$

We have found three generalized eigenvectors for eigenvalue -1. Therefore the three

basis solutions associated with this eigenvalue are

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\
&= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \mathrm{e}^{-t} \\
&= \begin{bmatrix} 0 \\ \mathrm{e}^{-t} \\ 0 \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_2(t) &= e^{\lambda t}(\vec{v}_1 t + \vec{v}_2) \\
&= \mathrm{e}^{-t} \left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right) \\
&= \begin{bmatrix} -2\,\mathrm{e}^{-t} \\ \mathrm{e}^{-t}(t+1) \\ \mathrm{e}^{-t} \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_3(t) &= \left(\vec{v}_1 \frac{t^2}{2} + \vec{v}_2 t + \vec{v}_3 \right) e^{\lambda t} \\
&= \left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \frac{t^2}{2} + \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} t + \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix} \right) \mathrm{e}^{-t} \\
&= \begin{bmatrix} \mathrm{e}^{-t}(-2t-3) \\ \frac{\mathrm{e}^{-t}(t^2+2t+2)}{2} \\ \mathrm{e}^{-t}(2+t) \end{bmatrix}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)
$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ e^{-t} \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -2\,e^{-t} \\ e^{-t}(t+1) \\ e^{-t} \end{bmatrix} + c_3 \begin{bmatrix} e^{-t}(-2t-3) \\ e^{-t}\left(t+\frac{1}{2}t^2+1\right) \\ e^{-t}(2+t) \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} e^{-t}(-2c_3t-2c_2-3c_3) \\ \frac{e^{-t}\left((t^2+2t+2)c_3+2c_2t+2c_1+2c_2\right)}{2} \\ ((2+t)\,c_3+c_2)\,e^{-t} \end{bmatrix}$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 59

```
dsolve([diff(x__1(t),t)=-3*x__1(t)+0*x__2(t)-4*x__3(t),diff(x__2(t),t)=-1*x__1(t)-1*x__2(t)-
```

$$x_1(t) = e^{-t}(c_3t+c_2)$$
$$x_2(t) = \frac{(-c_3t^2-2c_2t+c_3t+4c_1)\,e^{-t}}{4}$$
$$x_3(t) = -\frac{e^{-t}(2c_3t+2c_2+c_3)}{4}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 81

```
DSolve[{x1'[t]==-3*x1[t]+0*x2[t]-4*x3[t],x2'[t]==-1*x1[t]-1*x2[t]-1*x3[t],x3'[t]==1*x1[t]+0*
```

$$\text{x1}(t) \to e^{-t}(-2c_1t-4c_3t+c_1)$$
$$\text{x2}(t) \to \frac{1}{2}e^{-t}\left((c_1+2c_3)t^2-2(c_1+c_3)t+2c_2\right)$$
$$\text{x3}(t) \to e^{-t}((c_1+2c_3)t+c_3)$$

6.12 problem problem 12

Internal problem ID [369]
Internal file name [OUTPUT/369_Sunday_June_05_2022_01_39_50_AM_49271009/index.tex]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.6, Multiple Eigenvalue Solutions. Page 451
Problem number: problem 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= -x_1(t) + x_3(t) \\
x_2'(t) &= -x_2(t) + x_3(t) \\
x_3'(t) &= x_1(t) - x_2(t) - x_3(t)
\end{aligned}
$$

6.12.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \mathrm{e}^{-t}\left(1+\frac{t^2}{2}\right) & -\frac{t^2 \mathrm{e}^{-t}}{2} & t\,\mathrm{e}^{-t} \\ \frac{t^2 \mathrm{e}^{-t}}{2} & \mathrm{e}^{-t}\left(1-\frac{t^2}{2}\right) & t\,\mathrm{e}^{-t} \\ t\,\mathrm{e}^{-t} & -t\,\mathrm{e}^{-t} & \mathrm{e}^{-t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{-t}\left(1+\frac{t^2}{2}\right) & -\frac{t^2 e^{-t}}{2} & t\,e^{-t} \\ \frac{t^2 e^{-t}}{2} & e^{-t}\left(1-\frac{t^2}{2}\right) & t\,e^{-t} \\ t\,e^{-t} & -t\,e^{-t} & e^{-t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} e^{-t}\left(1+\frac{t^2}{2}\right)c_1 - \frac{t^2 e^{-t} c_2}{2} + t\,e^{-t} c_3 \\ \frac{t^2 e^{-t} c_1}{2} + e^{-t}\left(1-\frac{t^2}{2}\right)c_2 + t\,e^{-t} c_3 \\ t\,e^{-t} c_1 - t\,e^{-t} c_2 + e^{-t} c_3 \end{bmatrix} \\
&= \begin{bmatrix} \frac{((-c_2+c_1)t^2+2c_3t+2c_1)e^{-t}}{2} \\ \frac{((-c_2+c_1)t^2+2c_3t+2c_2)e^{-t}}{2} \\ ((-c_2+c_1)\,t+c_3)\,e^{-t} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

6.12.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & -1 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -1-\lambda & 0 & 1 \\ 0 & -1-\lambda & 1 \\ 1 & -1 & -1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & -1 & -1 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 3 gives

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$R_3 = R_3 - R_2 \Longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1, v_3\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
-1	3	1	Yes	$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue -1 is real and repated eigenvalue of multiplicity 3.There are three possible cases that can happen. This is illustrated in this diagram

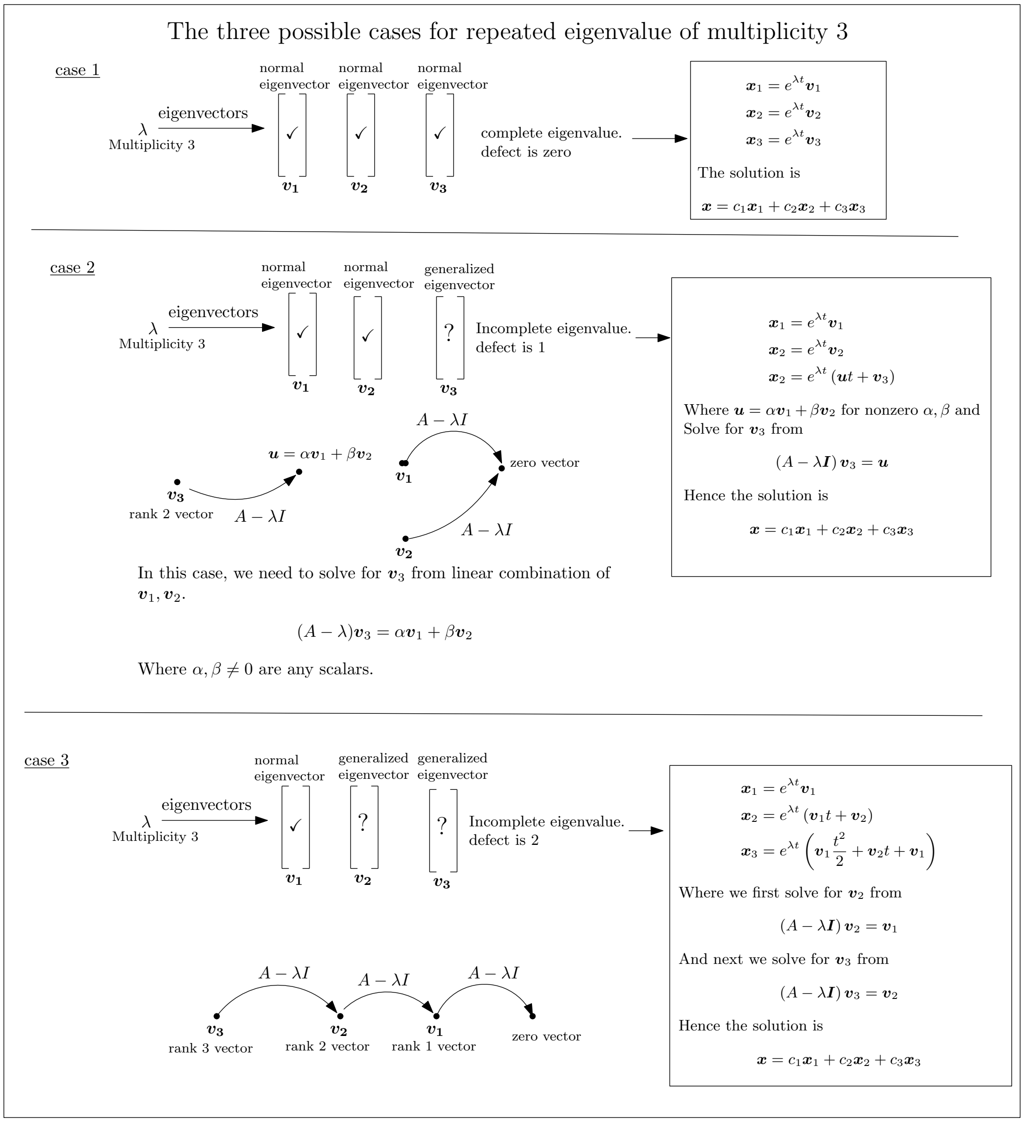


Figure 48: Possible case for repeated λ of multiplicity 3

This eigenvalue has algebraic multiplicity of 3, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 2. This falls into case 3 shown above. First we find generalized eigenvector $\vec{v}_2$ of rank 2 and then use this to find generalized eigenvector $\vec{v}_3$ of rank 3.$\vec{v}_2$ is found by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence

$$\left(\begin{bmatrix}-1 & 0 & 1\\ 0 & -1 & 1\\ 1 & -1 & -1\end{bmatrix} - (-1)\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}1\\ 1\\ 0\end{bmatrix}$$

$$\begin{bmatrix}0 & 0 & 1\\ 0 & 0 & 1\\ 1 & -1 & 0\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}1\\ 1\\ 0\end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix}1\\ 1\\ 1\end{bmatrix}$$

Now $\vec{v}_3$ is found by solving

$$(A - \lambda I)\,\vec{v}_3 = \vec{v}_2$$

Where $\vec{v}_2$ is the (rank 2) generalized eigenvector found above. Hence

$$\left(\begin{bmatrix}-1 & 0 & 1\\ 0 & -1 & 1\\ 1 & -1 & -1\end{bmatrix} - (-1)\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}1\\ 1\\ 1\end{bmatrix}$$

$$\begin{bmatrix}0 & 0 & 1\\ 0 & 0 & 1\\ 1 & -1 & 0\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}1\\ 1\\ 1\end{bmatrix}$$

Solving for $\vec{v}_3$ gives

$$\vec{v}_3 = \begin{bmatrix}2\\ 1\\ 1\end{bmatrix}$$

We have found three generalized eigenvectors for eigenvalue -1. Therefore the three

basis solutions associated with this eigenvalue are

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\
&= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \mathrm{e}^{-t} \\
&= \begin{bmatrix} \mathrm{e}^{-t} \\ \mathrm{e}^{-t} \\ 0 \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_2(t) &= e^{\lambda t}(\vec{v}_1 t + \vec{v}_2) \\
&= \mathrm{e}^{-t}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) \\
&= \begin{bmatrix} \mathrm{e}^{-t}(t+1) \\ \mathrm{e}^{-t}(t+1) \\ \mathrm{e}^{-t} \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_3(t) &= \left(\vec{v}_1 \frac{t^2}{2} + \vec{v}_2 t + \vec{v}_3\right) e^{\lambda t} \\
&= \left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \frac{t^2}{2} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} t + \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}\right) \mathrm{e}^{-t} \\
&= \begin{bmatrix} \frac{\mathrm{e}^{-t}(t^2+2t+4)}{2} \\ \frac{\mathrm{e}^{-t}(t^2+2t+2)}{2} \\ \mathrm{e}^{-t}(t+1) \end{bmatrix}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)
$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 \begin{bmatrix} e^{-t} \\ e^{-t} \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} e^{-t}(t+1) \\ e^{-t}(t+1) \\ e^{-t} \end{bmatrix} + c_3 \begin{bmatrix} e^{-t}\left(\frac{1}{2}t^2 + t + 2\right) \\ e^{-t}\left(t + \frac{1}{2}t^2 + 1\right) \\ e^{-t}(t+1) \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} \frac{((t^2+2t+4)c_3+2c_2t+2c_1+2c_2)e^{-t}}{2} \\ \frac{e^{-t}((t^2+2t+2)c_3+2c_2t+2c_1+2c_2)}{2} \\ e^{-t}(c_3t + c_2 + c_3) \end{bmatrix}$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 62

```
dsolve([diff(x__1(t),t)=-1*x__1(t)+0*x__2(t)+1*x__3(t),diff(x__2(t),t)=0*x__1(t)-1*x__2(t)+1
```

$$x_1(t) = \frac{(c_3t^2 + 2c_2t + 2c_1)e^{-t}}{2}$$
$$x_2(t) = \frac{e^{-t}(c_3t^2 + 2c_2t + 2c_1 - 2c_3)}{2}$$
$$x_3(t) = e^{-t}(c_3t + c_2)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 89

```
DSolve[{x1'[t]==-1*x1[t]+0*x2[t]+1*x3[t],x2'[t]==0*x1[t]-1*x2[t]+1*x3[t],x3'[t]==1*x1[t]-1*x
```

$$\text{x1}(t) \to \frac{1}{2}e^{-t}\left(c_1\left(t^2+2\right) + t(2c_3 - c_2t)\right)$$
$$\text{x2}(t) \to \frac{1}{2}e^{-t}\left((c_1 - c_2)t^2 + 2c_3t + 2c_2\right)$$
$$\text{x3}(t) \to e^{-t}((c_1 - c_2)t + c_3)$$

6.13 problem problem 13

Internal problem ID [370]
Internal file name [`OUTPUT/370_Sunday_June_05_2022_01_39_51_AM_96807373/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.6, Multiple Eigenvalue Solutions. Page 451
Problem number: problem 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= -x_1(t) + x_3(t) \\
x_2'(t) &= x_2(t) - 4x_3(t) \\
x_3'(t) &= x_2(t) - 3x_3(t)
\end{aligned}
$$

6.13.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & -4 \\ 0 & 1 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} \mathrm{e}^{-t} & \frac{t^2\mathrm{e}^{-t}}{2} & -\mathrm{e}^{-t}t(-1+t) \\ 0 & \mathrm{e}^{-t}(1+2t) & -4t\,\mathrm{e}^{-t} \\ 0 & t\,\mathrm{e}^{-t} & \mathrm{e}^{-t}(1-2t) \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{-t} & \frac{t^2 e^{-t}}{2} & -e^{-t}t(-1+t) \\ 0 & e^{-t}(1+2t) & -4t\,e^{-t} \\ 0 & t\,e^{-t} & e^{-t}(1-2t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} e^{-t}c_1 + \frac{t^2 e^{-t} c_2}{2} - e^{-t}t(-1+t)\,c_3 \\ e^{-t}(1+2t)\,c_2 - 4t\,e^{-t}c_3 \\ t\,e^{-t}c_2 + e^{-t}(1-2t)\,c_3 \end{bmatrix} \\
&= \begin{bmatrix} \frac{e^{-t}((c_2-2c_3)t^2+2tc_3+2c_1)}{2} \\ e^{-t}(2tc_2 - 4tc_3 + c_2) \\ e^{-t}(tc_2 - 2tc_3 + c_3) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

6.13.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & -4 \\ 0 & 1 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det(A - \lambda I) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & -4 \\ 0 & 1 & -3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) = 0
$$

Therefore

$$\det\left(\begin{bmatrix} -1-\lambda & 0 & 1 \\ 0 & 1-\lambda & -4 \\ 0 & 1 & -3-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & -4 \\ 0 & 1 & -3 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & -4 \\ 0 & 1 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 2 & -4 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right]$$

Since the current pivot $A(1,2)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{ccc|c} 0 & 2 & -4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right]$$

$$
R_3 = R_3 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} 0 & 2 & -4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]
$$

Therefore the system in Echelon form is

$$
\begin{bmatrix} 0 & 2 & -4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

The free variables are $\{v_1\}$ and the leading variables are $\{v_2, v_3\}$. Let $v_1 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_2 = 0, v_3 = 0\}$

Hence the solution is

$$
\begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix}
$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$
\begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}
$$

Let $t = 1$ the eigenvector becomes

$$
\begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}
$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
-1	3	1	Yes	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue -1 is real and repated eigenvalue of multiplicity 3.There are three possible cases that can happen. This is illustrated in this diagram

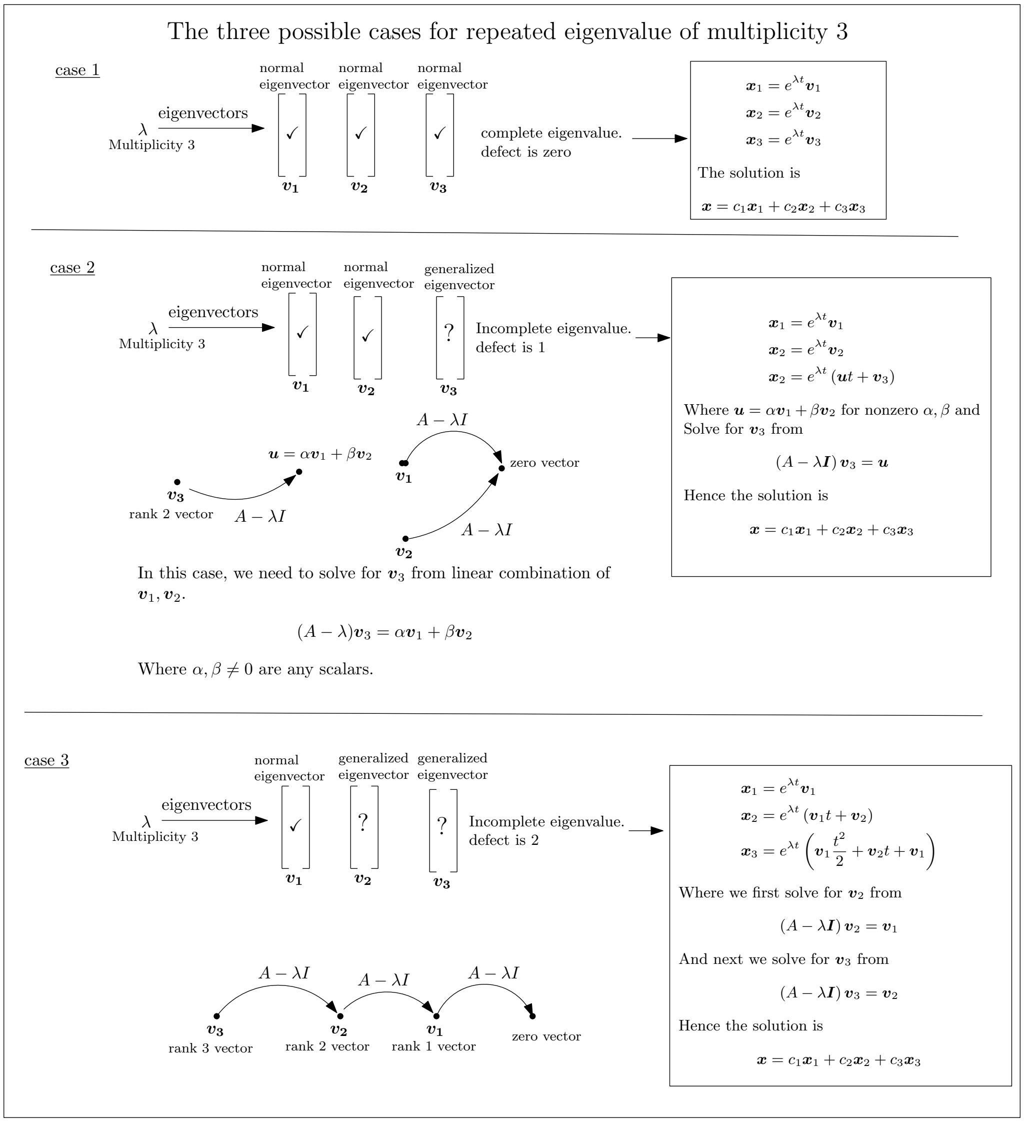


Figure 49: Possible case for repeated λ of multiplicity 3

This eigenvalue has algebraic multiplicity of 3, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 2. This falls into case 3 shown above. First we find generalized eigenvector $\vec{v}_2$ of rank 2 and then use this to find generalized eigenvector $\vec{v}_3$ of rank 3.$\vec{v}_2$ is found by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence

$$\left(\begin{bmatrix}-1 & 0 & 1\\ 0 & 1 & -4\\ 0 & 1 & -3\end{bmatrix} - (-1)\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}1\\ 0\\ 0\end{bmatrix}$$

$$\begin{bmatrix}0 & 0 & 1\\ 0 & 2 & -4\\ 0 & 1 & -2\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}1\\ 0\\ 0\end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix}1\\ 2\\ 1\end{bmatrix}$$

Now $\vec{v}_3$ is found by solving

$$(A - \lambda I)\,\vec{v}_3 = \vec{v}_2$$

Where $\vec{v}_2$ is the (rank 2) generalized eigenvector found above. Hence

$$\left(\begin{bmatrix}-1 & 0 & 1\\ 0 & 1 & -4\\ 0 & 1 & -3\end{bmatrix} - (-1)\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}1\\ 2\\ 1\end{bmatrix}$$

$$\begin{bmatrix}0 & 0 & 1\\ 0 & 2 & -4\\ 0 & 1 & -2\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}1\\ 2\\ 1\end{bmatrix}$$

Solving for $\vec{v}_3$ gives

$$\vec{v}_3 = \begin{bmatrix}1\\ 3\\ 1\end{bmatrix}$$

We have found three generalized eigenvectors for eigenvalue -1. Therefore the three

basis solutions associated with this eigenvalue are

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\
&= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \mathrm{e}^{-t} \\
&= \begin{bmatrix} \mathrm{e}^{-t} \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_2(t) &= e^{\lambda t}(\vec{v}_1 t + \vec{v}_2) \\
&= \mathrm{e}^{-t}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t + \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right) \\
&= \begin{bmatrix} \mathrm{e}^{-t}(t+1) \\ 2\,\mathrm{e}^{-t} \\ \mathrm{e}^{-t} \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_3(t) &= \left(\vec{v}_1 \frac{t^2}{2} + \vec{v}_2 t + \vec{v}_3\right) e^{\lambda t} \\
&= \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \frac{t^2}{2} + \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}\right) \mathrm{e}^{-t} \\
&= \begin{bmatrix} \frac{\mathrm{e}^{-t}(t^2+2t+2)}{2} \\ \mathrm{e}^{-t}(2t+3) \\ \mathrm{e}^{-t}(t+1) \end{bmatrix}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)
$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 \begin{bmatrix} e^{-t} \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} e^{-t}(t+1) \\ 2\,e^{-t} \\ e^{-t} \end{bmatrix} + c_3 \begin{bmatrix} e^{-t}\left(t+\frac{1}{2}t^2+1\right) \\ e^{-t}(2t+3) \\ e^{-t}(t+1) \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} \frac{e^{-t}\left((t^2+2t+2)c_3+2c_2t+2c_1+2c_2\right)}{2} \\ \left((2t+3)\,c_3+2c_2\right)e^{-t} \\ e^{-t}(c_3t+c_2+c_3) \end{bmatrix}$$

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 54

```
dsolve([diff(x__1(t),t)=-1*x__1(t)+0*x__2(t)+1*x__3(t),diff(x__2(t),t)=0*x__1(t)+1*x__2(t)-4
```

$$x_1(t) = \frac{(c_3t^2+2c_2t+2c_1)\,e^{-t}}{2}$$
$$x_2(t) = e^{-t}(2c_3t+2c_2+c_3)$$
$$x_3(t) = e^{-t}(c_3t+c_2)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 78

```
DSolve[{x1'[t]==-1*x1[t]+0*x2[t]+1*x3[t],x2'[t]==0*x1[t]+1*x2[t]-4*x3[t],x3'[t]==0*x1[t]+1*x
```

$$\text{x1}(t) \to \frac{1}{2}e^{-t}(t((c_2-2c_3)t+2c_3)+2c_1)$$
$$\text{x2}(t) \to e^{-t}(2c_2t-4c_3t+c_2)$$
$$\text{x3}(t) \to e^{-t}((c_2-2c_3)t+c_3)$$

6.14 problem problem 14

Internal problem ID [371]
Internal file name [OUTPUT/371_Sunday_June_05_2022_01_39_52_AM_79064891/index.tex]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.6, Multiple Eigenvalue Solutions. Page 451
Problem number: problem 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= x_3(t) \\
x_2'(t) &= -5x_1(t) - x_2(t) - 5x_3(t) \\
x_3'(t) &= 4x_1(t) + x_2(t) - 2x_3(t)
\end{aligned}
$$

6.14.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -5 & -1 & -5 \\ 4 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \mathrm{e}^{-t}\left(1 + \frac{5}{2}t^2 + t\right) & \frac{t^2 \mathrm{e}^{-t}}{2} & t\,\mathrm{e}^{-t} \\ -\frac{5\,\mathrm{e}^{-t} t(5t+2)}{2} & \mathrm{e}^{-t}\left(1 - \frac{5t^2}{2}\right) & -5t\,\mathrm{e}^{-t} \\ -\frac{\mathrm{e}^{-t} t(5t-8)}{2} & -\frac{\mathrm{e}^{-t} t(t-2)}{2} & \mathrm{e}^{-t}(1-t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{-t}\left(1+\frac{5}{2}t^2+t\right) & \frac{t^2 e^{-t}}{2} & t\,e^{-t} \\ -\frac{5\,e^{-t}t(5t+2)}{2} & e^{-t}\left(1-\frac{5t^2}{2}\right) & -5t\,e^{-t} \\ -\frac{e^{-t}t(5t-8)}{2} & -\frac{e^{-t}t(t-2)}{2} & e^{-t}(1-t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} e^{-t}\left(1+\frac{5}{2}t^2+t\right)c_1 + \frac{t^2 e^{-t} c_2}{2} + t\,e^{-t}c_3 \\ -\frac{5\,e^{-t}t(5t+2)c_1}{2} + e^{-t}\left(1-\frac{5t^2}{2}\right)c_2 - 5t\,e^{-t}c_3 \\ -\frac{e^{-t}t(5t-8)c_1}{2} - \frac{e^{-t}t(t-2)c_2}{2} + e^{-t}(1-t)\,c_3 \end{bmatrix} \\
&= \begin{bmatrix} \frac{5\,e^{-t}\left(\left(c_1+\frac{c_2}{5}\right)t^2+\frac{2(c_1+c_3)t}{5}+\frac{2c_1}{5}\right)}{2} \\ -\frac{25\left(\left(c_1+\frac{c_2}{5}\right)t^2+\frac{2(c_1+c_3)t}{5}-\frac{2c_2}{25}\right)e^{-t}}{2} \\ -\frac{5\left(\left(c_1+\frac{c_2}{5}\right)t^2+\frac{2(-4c_1-c_2+c_3)t}{5}-\frac{2c_3}{5}\right)e^{-t}}{2} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

6.14.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -5 & -1 & -5 \\ 4 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A-\lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 0 & 0 & 1 \\ -5 & -1 & -5 \\ 4 & 1 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -\lambda & 0 & 1 \\ -5 & -1-\lambda & -5 \\ 4 & 1 & -2-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 0 & 1 \\ -5 & -1 & -5 \\ 4 & 1 & -2 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ -5 & 0 & -5 \\ 4 & 1 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ -5 & 0 & -5 & 0 \\ 4 & 1 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 + 5R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 4 & 1 & -1 & 0 \end{array}\right]$$

$$R_3 = R_3 - 4R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -5 & 0 \end{array}\right]$$

Since the current pivot $A(2,2)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t, v_2 = 5t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ 5t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ 5t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ 5t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ 5t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-1	3	1	Yes	$\begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue -1 is real and repated eigenvalue of multiplicity 3.There are three possible cases that can happen. This is illustrated in this diagram

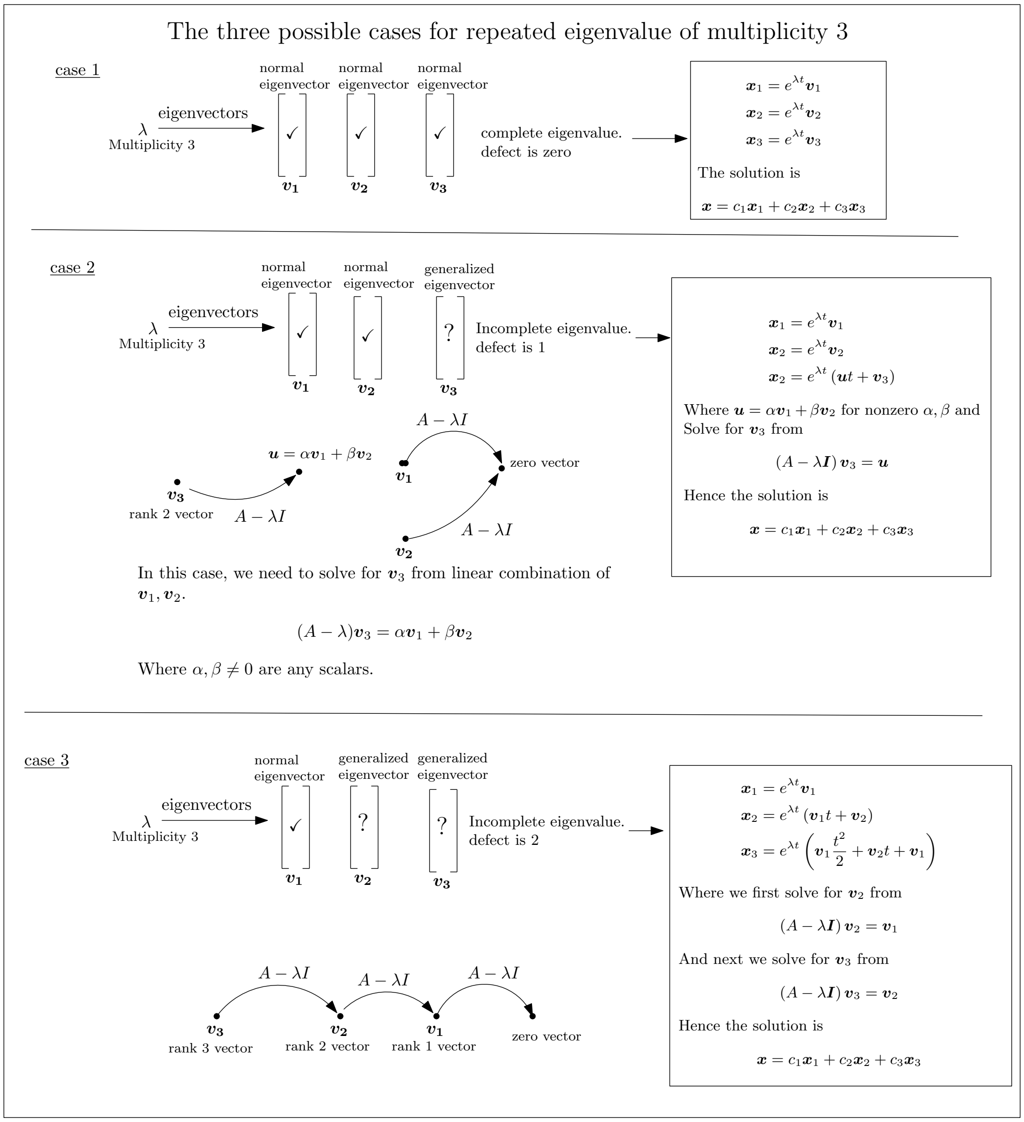


Figure 50: Possible case for repeated λ of multiplicity 3

This eigenvalue has algebraic multiplicity of 3, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 2. This falls into case 3 shown above. First we find generalized eigenvector $\vec{v}_2$ of rank 2 and then use this to find generalized eigenvector $\vec{v}_3$ of rank 3.$\vec{v}_2$ is found by solving

$$(A - \lambda I) \vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence

$$\left(\begin{bmatrix} 0 & 0 & 1 \\ -5 & -1 & -5 \\ 4 & 1 & -2 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ -5 & 0 & -5 \\ 4 & 1 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} -2 \\ 10 \\ 1 \end{bmatrix}$$

Now $\vec{v}_3$ is found by solving

$$(A - \lambda I)\,\vec{v}_3 = \vec{v}_2$$

Where $\vec{v}_2$ is the (rank 2) generalized eigenvector found above. Hence

$$\left(\begin{bmatrix} 0 & 0 & 1 \\ -5 & -1 & -5 \\ 4 & 1 & -2 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 10 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ -5 & 0 & -5 \\ 4 & 1 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 10 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_3$ gives

$$\vec{v}_3 = \begin{bmatrix} -3 \\ 14 \\ 1 \end{bmatrix}$$

We have found three generalized eigenvectors for eigenvalue -1. Therefore the three

basis solutions associated with this eigenvalue are

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\ &= \begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix} \mathrm{e}^{-t} \\ &= \begin{bmatrix} -\mathrm{e}^{-t} \\ 5\,\mathrm{e}^{-t} \\ \mathrm{e}^{-t} \end{bmatrix}\end{aligned}$$

And

$$\begin{aligned}\vec{x}_2(t) &= e^{\lambda t}(\vec{v}_1 t + \vec{v}_2) \\ &= \mathrm{e}^{-t}\left(\begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix} t + \begin{bmatrix} -2 \\ 10 \\ 1 \end{bmatrix}\right) \\ &= \begin{bmatrix} -\mathrm{e}^{-t}(2+t) \\ 5\,\mathrm{e}^{-t}(2+t) \\ \mathrm{e}^{-t}(t+1) \end{bmatrix}\end{aligned}$$

And

$$\begin{aligned}\vec{x}_3(t) &= \left(\vec{v}_1 \frac{t^2}{2} + \vec{v}_2 t + \vec{v}_3\right) e^{\lambda t} \\ &= \left(\begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix} \frac{t^2}{2} + \begin{bmatrix} -2 \\ 10 \\ 1 \end{bmatrix} t + \begin{bmatrix} -3 \\ 14 \\ 1 \end{bmatrix}\right) \mathrm{e}^{-t} \\ &= \begin{bmatrix} -\frac{\mathrm{e}^{-t}(t^2+4t+6)}{2} \\ \frac{\mathrm{e}^{-t}(5t^2+20t+28)}{2} \\ \frac{\mathrm{e}^{-t}(t^2+2t+2)}{2} \end{bmatrix}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 \begin{bmatrix} -e^{-t} \\ 5\,e^{-t} \\ e^{-t} \end{bmatrix} + c_2 \begin{bmatrix} e^{-t}(-t-2) \\ e^{-t}(5t+10) \\ e^{-t}(t+1) \end{bmatrix} + c_3 \begin{bmatrix} e^{-t}\left(-\frac{1}{2}t^2 - 2t - 3\right) \\ e^{-t}\left(\frac{5}{2}t^2 + 10t + 14\right) \\ e^{-t}\left(t + \frac{1}{2}t^2 + 1\right) \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} -\frac{((t^2+4t+6)c_3+2c_2t+2c_1+4c_2)e^{-t}}{2} \\ \frac{5\left(\left(t^2+4t+\frac{28}{5}\right)c_3+2c_2t+2c_1+4c_2\right)e^{-t}}{2} \\ \frac{e^{-t}((t^2+2t+2)c_3+2c_2t+2c_1+2c_2)}{2} \end{bmatrix}$$

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 72

```
dsolve([diff(x__1(t),t)=0*x__1(t)+0*x__2(t)+1*x__3(t),diff(x__2(t),t)=-5*x__1(t)-1*x__2(t)-5
```

$$x_1(t) = e^{-t}\left(c_3t^2 + c_2t + c_1\right)$$
$$x_2(t) = -e^{-t}\left(5c_3t^2 + 5c_2t + 5c_1 - 2c_3\right)$$
$$x_3(t) = -e^{-t}\left(c_3t^2 + c_2t - 2c_3t + c_1 - c_2\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 119

```
DSolve[{x1'[t]==0*x1[t]+0*x2[t]+1*x3[t],x2'[t]==-5*x1[t]-1*x2[t]-5*x3[t],x3'[t]==4*x1[t]+1*x
```

$$\text{x1}(t) \to \frac{1}{2}e^{-t}\left(c_1\left(5t^2 + 2t + 2\right) + t(c_2t + 2c_3)\right)$$
$$\text{x2}(t) \to \frac{1}{2}e^{-t}\left(-5(5c_1 + c_2)t^2 - 10(c_1 + c_3)t + 2c_2\right)$$
$$\text{x3}(t) \to \frac{1}{2}e^{-t}\left(-\left((5c_1 + c_2)t^2\right) + 2(4c_1 + c_2 - c_3)t + 2c_3\right)$$

6.15 problem problem 15

Internal problem ID [372]
Internal file name [`OUTPUT/372_Sunday_June_05_2022_01_39_54_AM_26166692/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.6, Multiple Eigenvalue Solutions. Page 451
Problem number: problem 15.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= -2x_1(t) - 9x_2(t)\\
x_2'(t) &= x_1(t) + 4x_2(t)\\
x_3'(t) &= x_1(t) + 3x_2(t) + x_3(t)
\end{aligned}
$$

6.15.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} -2 & -9 & 0 \\ 1 & 4 & 0 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} \mathrm{e}^{t}(1-3t) & -9t\,\mathrm{e}^{t} & 0 \\ t\,\mathrm{e}^{t} & \mathrm{e}^{t}(1+3t) & 0 \\ t\,\mathrm{e}^{t} & 3t\,\mathrm{e}^{t} & \mathrm{e}^{t} \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^t(1-3t) & -9t\,e^t & 0 \\ t\,e^t & e^t(1+3t) & 0 \\ t\,e^t & 3t\,e^t & e^t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} e^t(1-3t)\,c_1 - 9t\,e^t c_2 \\ t\,e^t c_1 + e^t(1+3t)\,c_2 \\ t\,e^t c_1 + 3t\,e^t c_2 + e^t c_3 \end{bmatrix} \\
&= \begin{bmatrix} (c_1(1-3t) - 9c_2 t)\,e^t \\ e^t(tc_1 + 3c_2 t + c_2) \\ ((c_1+3c_2)\,t + c_3)\,e^t \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

6.15.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} -2 & -9 & 0 \\ 1 & 4 & 0 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det(A - \lambda I) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} -2 & -9 & 0 \\ 1 & 4 & 0 \\ 1 & 3 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$\det\left(\begin{bmatrix} -2-\lambda & -9 & 0 \\ 1 & 4-\lambda & 0 \\ 1 & 3 & 1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -2 & -9 & 0 \\ 1 & 4 & 0 \\ 1 & 3 & 1 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & -9 & 0 \\ 1 & 3 & 0 \\ 1 & 3 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -3 & -9 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 3 & 0 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{3} \Longrightarrow \left[\begin{array}{ccc|c} -3 & -9 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{R_1}{3} \Longrightarrow \left[\begin{array}{ccc|c} -3 & -9 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -3 & -9 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2, v_3\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Let $v_3 = s$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -3t\}$

Hence the solution is

$$\begin{bmatrix} -3t \\ t \\ s \end{bmatrix} = \begin{bmatrix} -3t \\ t \\ s \end{bmatrix}$$

Since there are two free Variable, we have found two eigenvectors associated with this eigenvalue. The above can be written as

$$\begin{aligned} \begin{bmatrix} -3t \\ t \\ s \end{bmatrix} &= \begin{bmatrix} -3t \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ s \end{bmatrix} \\ &= t \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

By letting $t = 1$ and $s = 1$ then the above becomes

$$\begin{bmatrix} -3t \\ t \\ s \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Hence the two eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
1	3	2	Yes	$\begin{bmatrix} -3 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 1 is real and repated eigenvalue of multiplicity 3.There are three possible cases that can happen. This is illustrated in this diagram

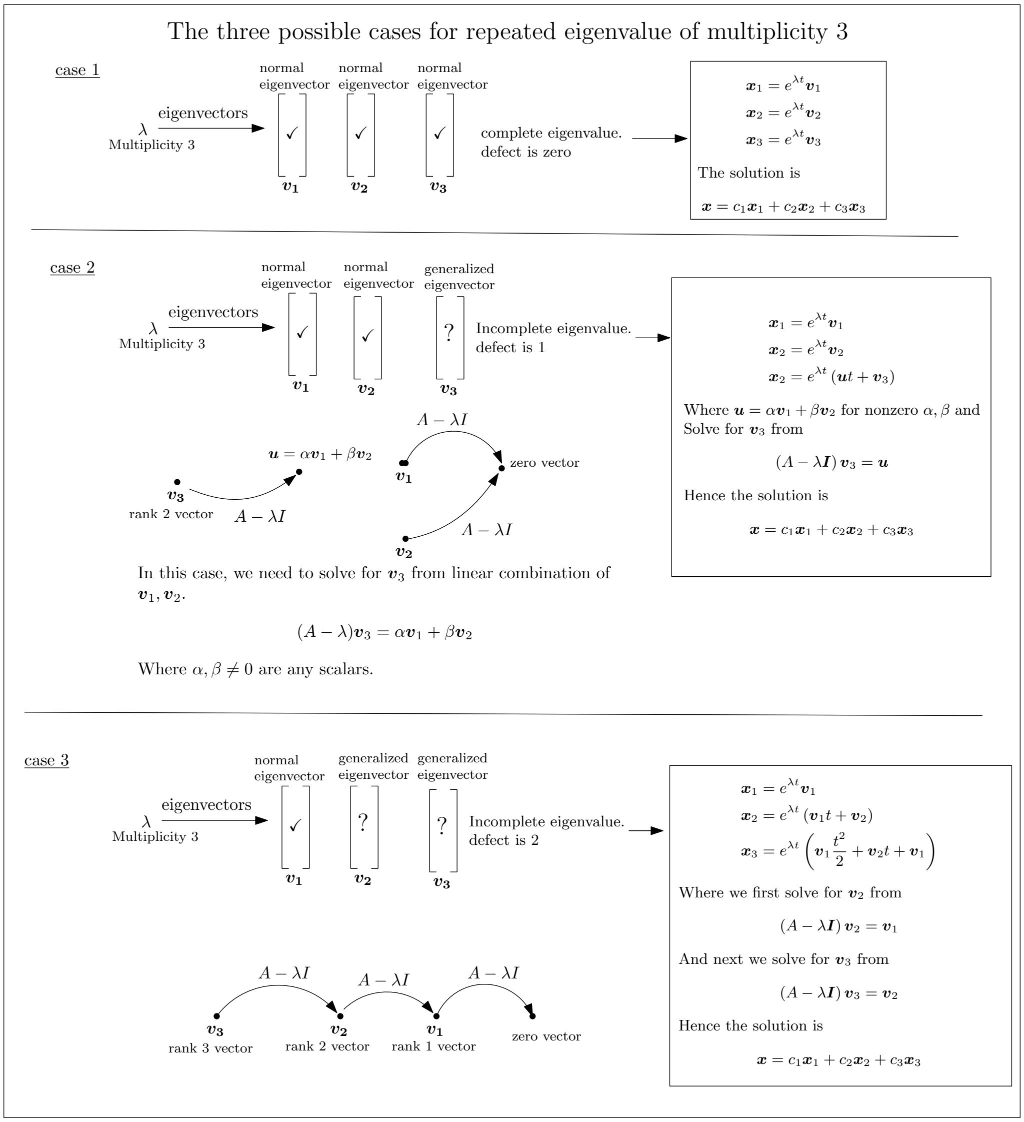


Figure 51: Possible case for repeated λ of multiplicity 3

This eigenvalue has algebraic multiplicity of 3, and geometric multiplicity 2, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to find rank-2 eigenvector $\vec{v}_3$. This eigenvector must therefore satisfy $(A-\lambda I)^2\,\vec{v}_3=\vec{0}$.

But

$$(A-\lambda I)^2 = \left(\begin{bmatrix} -2 & -9 & 0 \\ 1 & 4 & 0 \\ 1 & 3 & 1 \end{bmatrix} - 1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)^2$$
$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore $\vec{v}_3$ could be any eigenvector vector we want (but not the zero vector). Let

$$\vec{v}_3 = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix}$$

To determine the actual $\vec{v}_3$ we need now to enforce the condition that $\vec{v}_3$ satisfies

$$(A-\lambda I)\,\vec{v}_3 = \vec{u} \tag{1}$$

Where $\vec{u}$ is linear combination of $\vec{v}_1, \vec{v}_2$. Hence

$$\vec{u} = \alpha\vec{v}_1 + \beta\vec{v}_2$$

Where α, β are arbitrary constants (not both zero). Eq. (1) becomes

$$(A-\lambda I) \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} = \alpha \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} -3 & -9 & 0 \\ 1 & 3 & 0 \\ 1 & 3 & 0 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} = \alpha \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} -3\eta_1 - 9\eta_2 \\ \eta_1 + 3\eta_2 \\ \eta_1 + 3\eta_2 \end{bmatrix} = \begin{bmatrix} -3\alpha \\ \alpha \\ \beta \end{bmatrix}$$

Expanding the above gives the following equations equations

$$
\begin{aligned}
-3\eta_1 - 9\eta_2 &= -3\alpha \\
\eta_1 + 3\eta_2 &= \alpha \\
\eta_1 + 3\eta_2 &= \beta
\end{aligned}
$$

solving for α, β from the above gives

$$
\begin{aligned}
-3\eta_1 - 9\eta_2 &= -3\alpha \\
\eta_1 + 3\eta_2 &= \alpha
\end{aligned}
$$

Since α, β are not both zero, then we just need to determine η_i values, not all zero, which satisfy the above equations for α, β not both zero. By inspection we see that the following values satisfy this condition

$$
[\eta_1 = -1, \eta_2 = 0]
$$

Hence we found the missing generalized eigenvector

$$
\vec{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}
$$

Which implies that

$$
\begin{aligned}
\alpha &= -1 \\
\beta &= -1
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\vec{u} &= \alpha\vec{v}_1 + \beta\vec{v}_2 \\
&= -1\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + (-1)\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix}
\end{aligned}
$$

Therefore the missing generalized eigenvector is now found. We have found three generalized eigenvectors for eigenvalue 1. Therefore the three basis solutions associated with this eigenvalue are

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t}\\ &= \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} \mathrm{e}^{t}\\ &= \begin{bmatrix} -3\,\mathrm{e}^{t} \\ \mathrm{e}^{t} \\ 0 \end{bmatrix}\end{aligned}$$

And

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{\lambda t}\\ &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \mathrm{e}^{t}\\ &= \begin{bmatrix} 0 \\ 0 \\ \mathrm{e}^{t} \end{bmatrix}\end{aligned}$$

And

$$\begin{aligned}\vec{x}_3(t) &= (\vec{u}t + \vec{v}_3)\, e^{\lambda t}\\ &= \left(\begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix} t + \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}\right) \mathrm{e}^{t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 \begin{bmatrix} -3\,\mathrm{e}^{t} \\ \mathrm{e}^{t} \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 0 \\ \mathrm{e}^{t} \end{bmatrix} + c_3 \begin{bmatrix} \mathrm{e}^{t}(-1+3t) \\ -t\,\mathrm{e}^{t} \\ -t\,\mathrm{e}^{t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} ((-1+3t)\,c_3 - 3c_1)\,\mathrm{e}^t \\ \mathrm{e}^t(-c_3 t + c_1) \\ \mathrm{e}^t(-c_3 t + c_2) \end{bmatrix}$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 47

```
dsolve([diff(x__1(t),t)=-2*x__1(t)-9*x__2(t)-0*x__3(t),diff(x__2(t),t)=1*x__1(t)+4*x__2(t)-0
```

$$x_1(t) = \mathrm{e}^t(c_3 t + c_2)$$
$$x_2(t) = -\frac{\mathrm{e}^t(3c_3 t + 3c_2 + c_3)}{9}$$
$$x_3(t) = \frac{\mathrm{e}^t(-c_3 t + 3c_1 - c_2)}{3}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 62

```
DSolve[{x1'[t]==-2*x1[t]-9*x2[t]-0*x3[t],x2'[t]==1*x1[t]+4*x2[t]-0*x3[t],x3'[t]==1*x1[t]+3*x
```

$$\text{x1}(t) \to -e^t(c_1(3t-1) + 9c_2 t)$$
$$\text{x2}(t) \to e^t((c_1 + 3c_2)t + c_2)$$
$$\text{x3}(t) \to e^t((c_1 + 3c_2)t + c_3)$$

6.16 problem problem 16

Internal problem ID [373]
Internal file name [`OUTPUT/373_Sunday_June_05_2022_01_39_55_AM_98654921/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.6, Multiple Eigenvalue Solutions. Page 451
Problem number: problem 16.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= x_1(t) \\
x_2'(t) &= -2x_1(t) - 2x_2(t) - 3x_3(t) \\
x_3'(t) &= 2x_1(t) + 3x_2(t) + 4x_3(t)
\end{aligned}
$$

6.16.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & -2 & -3 \\ 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \mathrm{e}^t & 0 & 0 \\ -2t\,\mathrm{e}^t & \mathrm{e}^t(1-3t) & -3t\,\mathrm{e}^t \\ 2t\,\mathrm{e}^t & 3t\,\mathrm{e}^t & \mathrm{e}^t(1+3t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \mathrm{e}^{t} & 0 & 0 \\ -2t\,\mathrm{e}^{t} & \mathrm{e}^{t}(1-3t) & -3t\,\mathrm{e}^{t} \\ 2t\,\mathrm{e}^{t} & 3t\,\mathrm{e}^{t} & \mathrm{e}^{t}(1+3t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^{t}c_1 \\ -2t\,\mathrm{e}^{t}c_1 + \mathrm{e}^{t}(1-3t)\,c_2 - 3t\,\mathrm{e}^{t}c_3 \\ 2t\,\mathrm{e}^{t}c_1 + 3t\,\mathrm{e}^{t}c_2 + \mathrm{e}^{t}(1+3t)\,c_3 \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^{t}c_1 \\ -2\,\mathrm{e}^{t}\left(\left(c_1 + \frac{3c_2}{2} + \frac{3c_3}{2}\right)t - \frac{c_2}{2}\right) \\ \mathrm{e}^{t}(2c_1t + 3tc_2 + 3c_3t + c_3) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

6.16.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & -2 & -3 \\ 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & 0 & 0 \\ -2 & -2 & -3 \\ 2 & 3 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & 0 & 0 \\ -2 & -2-\lambda & -3 \\ 2 & 3 & 4-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ -2 & -2 & -3 \\ 2 & 3 & 4 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ -2 & -3 & -3 \\ 2 & 3 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -2 & -3 & -3 & 0 \\ 2 & 3 & 3 & 0 \end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{ccc|c} -2 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 3 & 3 & 0 \end{array}\right]$$

$$R_3 = R_3 + R_1 \implies \left[\begin{array}{ccc|c} -2 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & -3 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2, v_3\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Let $v_3 = s$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{3t}{2} - \frac{3s}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{3t}{2} - \frac{3s}{2} \\ t \\ s \end{bmatrix} = \begin{bmatrix} -\frac{3t}{2} - \frac{3s}{2} \\ t \\ s \end{bmatrix}$$

Since there are two free Variable, we have found two eigenvectors associated with this eigenvalue. The above can be written as

$$\begin{aligned} \begin{bmatrix} -\frac{3t}{2} - \frac{3s}{2} \\ t \\ s \end{bmatrix} &= \begin{bmatrix} -\frac{3t}{2} \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{3s}{2} \\ 0 \\ s \end{bmatrix} \\ &= t \begin{bmatrix} -\frac{3}{2} \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -\frac{3}{2} \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

By letting $t = 1$ and $s = 1$ then the above becomes

$$\begin{bmatrix} -\frac{3t}{2} - \frac{3s}{2} \\ t \\ s \end{bmatrix} = \begin{bmatrix} -\frac{3}{2} \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{3}{2} \\ 0 \\ 1 \end{bmatrix}$$

Hence the two eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix} -\frac{3}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{3}{2} \\ 0 \\ 1 \end{bmatrix} \right)$$

Which are normalized to

$$\left(\begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix} \right)$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
1	3	2	Yes	$\begin{bmatrix} -\frac{3}{2} & -\frac{3}{2} \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 1 is real and repated eigenvalue of multiplicity 3.There are three possible cases that can happen. This is illustrated in this diagram

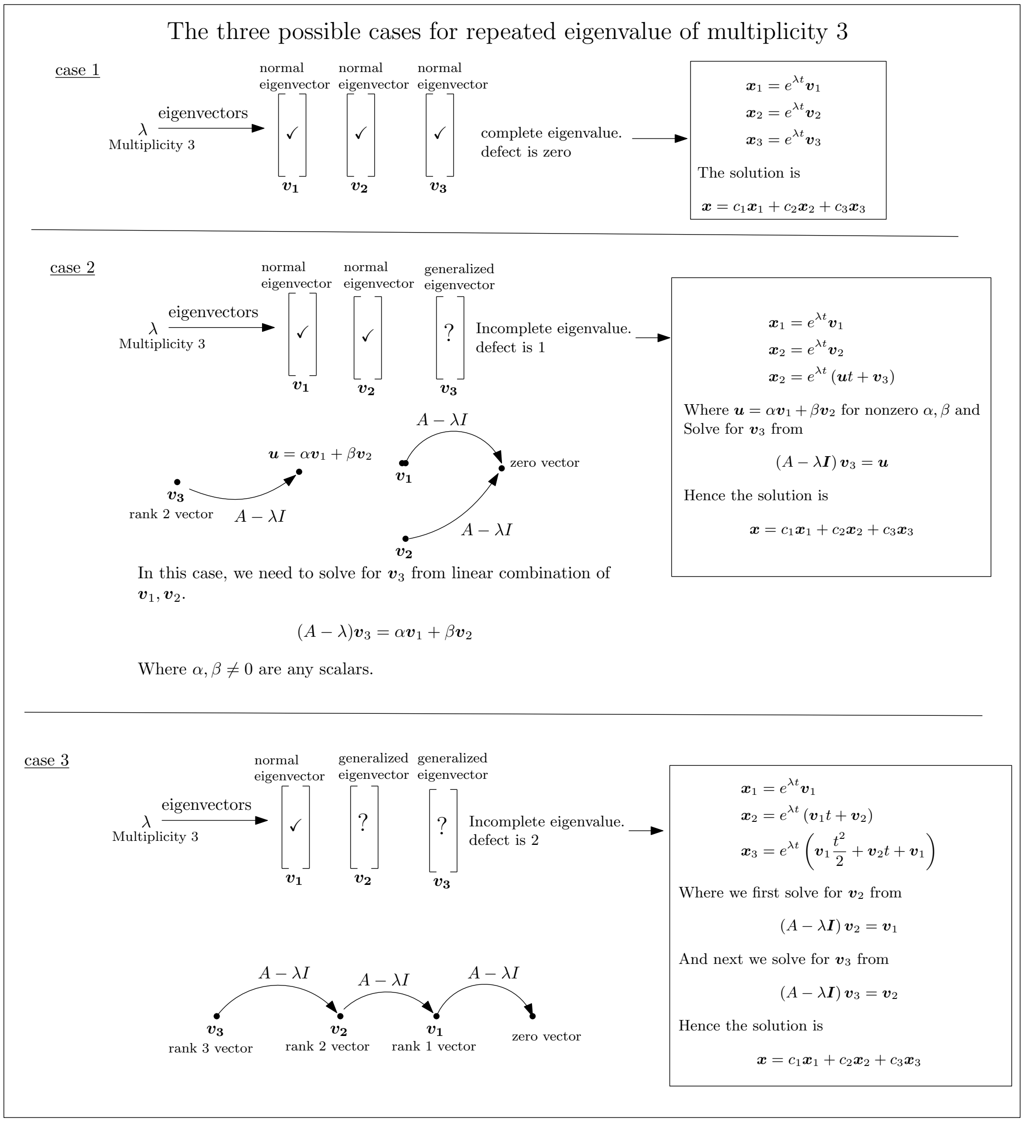


Figure 52: Possible case for repeated λ of multiplicity 3

This eigenvalue has algebraic multiplicity of 3, and geometric multiplicity 2, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to find rank-2 eigenvector $\vec{v}_3$. This eigenvector must therefore satisfy $(A - \lambda I)^2 \vec{v}_3 = \vec{0}$.

But

$$(A-\lambda I)^2 = \left(\begin{bmatrix} 1 & 0 & 0 \\ -2 & -2 & -3 \\ 2 & 3 & 4 \end{bmatrix} - 1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)^2$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore $\vec{v}_3$ could be any eigenvector vector we want (but not the zero vector). Let

$$\vec{v}_3 = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix}$$

To determine the actual $\vec{v}_3$ we need now to enforce the condition that $\vec{v}_3$ satisfies

$$(A-\lambda I)\,\vec{v}_3 = \vec{u} \tag{1}$$

Where $\vec{u}$ is linear combination of $\vec{v}_1, \vec{v}_2$. Hence

$$\vec{u} = \alpha \vec{v}_1 + \beta \vec{v}_2$$

Where α, β are arbitrary constants (not both zero). Eq. (1) becomes

$$(A-\lambda I) \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} = \alpha \begin{bmatrix} -\frac{3}{2} \\ 0 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} -\frac{3}{2} \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ -2 & -3 & -3 \\ 2 & 3 & 3 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} = \alpha \begin{bmatrix} -\frac{3}{2} \\ 0 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} -\frac{3}{2} \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ -2\eta_1 - 3\eta_2 - 3\eta_3 \\ 2\eta_1 + 3\eta_2 + 3\eta_3 \end{bmatrix} = \begin{bmatrix} -\frac{3\alpha}{2} - \frac{3\beta}{2} \\ \beta \\ \alpha \end{bmatrix}$$

Expanding the above gives the following equations equations

$$\begin{aligned}0 &= -\frac{3\alpha}{2} - \frac{3\beta}{2}\\ -2\eta_1 - 3\eta_2 - 3\eta_3 &= \beta\\ 2\eta_1 + 3\eta_2 + 3\eta_3 &= \alpha\end{aligned}$$

solving for α, β from the above gives

$$\begin{aligned}0 &= -\frac{3\alpha}{2} - \frac{3\beta}{2}\\ -2\eta_1 - 3\eta_2 - 3\eta_3 &= \beta\end{aligned}$$

Since α, β are not both zero, then we just need to determine η_i values, not all zero, which satisfy the above equations for α, β not both zero. By inspection we see that the following values satisfy this condition

$$[\eta_1 = 0, \eta_2 = -1, \eta_3 = 0]$$

Hence we found the missing generalized eigenvector

$$\vec{v}_3 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

Which implies that

$$\begin{aligned}\alpha &= -3\\ \beta &= 3\end{aligned}$$

Therefore

$$\begin{aligned}\vec{u} &= \alpha\vec{v}_1 + \beta\vec{v}_2\\ &= -3\begin{bmatrix} -\frac{3}{2} \\ 0 \\ 1 \end{bmatrix} + (3)\begin{bmatrix} -\frac{3}{2} \\ 1 \\ 0 \end{bmatrix}\\ &= \begin{bmatrix} 0 \\ 3 \\ -3 \end{bmatrix}\end{aligned}$$

Therefore the missing generalized eigenvector is now found. We have found three generalized eigenvectors for eigenvalue 1. Therefore the three basis solutions associated with this eigenvalue are

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\ &= \begin{bmatrix} -\frac{3}{2} \\ 0 \\ 1 \end{bmatrix} \mathrm{e}^{t} \\ &= \begin{bmatrix} -\frac{3\,\mathrm{e}^{t}}{2} \\ 0 \\ \mathrm{e}^{t} \end{bmatrix}\end{aligned}$$

And

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{\lambda t} \\ &= \begin{bmatrix} -\frac{3}{2} \\ 1 \\ 0 \end{bmatrix} \mathrm{e}^{t} \\ &= \begin{bmatrix} -\frac{3\,\mathrm{e}^{t}}{2} \\ \mathrm{e}^{t} \\ 0 \end{bmatrix}\end{aligned}$$

And

$$\begin{aligned}\vec{x}_3(t) &= (\vec{u}t + \vec{v}_3)\, e^{\lambda t} \\ &= \left(\begin{bmatrix} 0 \\ 3 \\ -3 \end{bmatrix} t + \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}\right) \mathrm{e}^{t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 \begin{bmatrix} -\frac{3\,\mathrm{e}^{t}}{2} \\ 0 \\ \mathrm{e}^{t} \end{bmatrix} + c_2 \begin{bmatrix} -\frac{3\,\mathrm{e}^{t}}{2} \\ \mathrm{e}^{t} \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ \mathrm{e}^{t}(-1+3t) \\ -3t\,\mathrm{e}^{t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} -\frac{3\,e^t(c_1+c_2)}{2} \\ ((-1+3t)\,c_3+c_2)\,e^t \\ e^t(-3c_3t+c_1) \end{bmatrix}$$

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 39

```
dsolve([diff(x__1(t),t)=1*x__1(t)+0*x__2(t)-0*x__3(t),diff(x__2(t),t)=-2*x__1(t)-2*x__2(t)-3
```

$$x_1(t) = c_3e^t$$
$$x_2(t) = e^t(c_2t + c_1)$$
$$x_3(t) = -\frac{e^t(3c_2t + 3c_1 + c_2 + 2c_3)}{3}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 57

```
DSolve[{x1'[t]==1*x1[t]+0*x2[t]-0*x3[t],x2'[t]==-2*x1[t]-2*x2[t]-3*x3[t],x3'[t]==2*x1[t]+3*x
```

$$\text{x1}(t) \to c_1e^t$$
$$\text{x2}(t) \to e^t(-2c_1t - 3(c_2 + c_3)t + c_2)$$
$$\text{x3}(t) \to e^t(2c_1t + 3(c_2 + c_3)t + c_3)$$

6.17 problem problem 17

Internal problem ID [374]
Internal file name [`OUTPUT/374_Sunday_June_05_2022_01_39_56_AM_59265710/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.6, Multiple Eigenvalue Solutions. Page 451
Problem number: problem 17.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= x_1(t) \\
x_2'(t) &= 18x_1(t) + 7x_2(t) + 4x_3(t) \\
x_3'(t) &= -27x_1(t) - 9x_2(t) - 5x_3(t)
\end{aligned}
$$

6.17.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 18 & 7 & 4 \\ -27 & -9 & -5 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{t} & 0 & 0 \\ 18t\,e^{t} & e^{t}(1+6t) & 4t\,e^{t} \\ -27t\,e^{t} & -9t\,e^{t} & e^{t}(1-6t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^t & 0 & 0 \\ 18t\,e^t & e^t(1+6t) & 4t\,e^t \\ -27t\,e^t & -9t\,e^t & e^t(1-6t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} e^t c_1 \\ 18t\,e^t c_1 + e^t(1+6t)\,c_2 + 4t\,e^t c_3 \\ -27t\,e^t c_1 - 9t\,e^t c_2 + e^t(1-6t)\,c_3 \end{bmatrix} \\
&= \begin{bmatrix} e^t c_1 \\ e^t(18c_1 t + 6tc_2 + 4c_3 t + c_2) \\ -27\left(\left(c_1 + \frac{c_2}{3} + \frac{2c_3}{9}\right)t - \frac{c_3}{27}\right)e^t \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

6.17.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 18 & 7 & 4 \\ -27 & -9 & -5 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det(A - \lambda I) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} 1 & 0 & 0 \\ 18 & 7 & 4 \\ -27 & -9 & -5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & 0 & 0 \\ 18 & 7-\lambda & 4 \\ -27 & -9 & -5-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 18 & 7 & 4 \\ -27 & -9 & -5 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 18 & 6 & 4 \\ -27 & -9 & -6 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 18 & 6 & 4 & 0 \\ -27 & -9 & -6 & 0 \end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{ccc|c} 18 & 6 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ -27 & -9 & -6 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{3R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} 18 & 6 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 18 & 6 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2, v_3\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Let $v_3 = s$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{t}{3} - \frac{2s}{9}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{t}{3} - \frac{2s}{9} \\ t \\ s \end{bmatrix} = \begin{bmatrix} -\frac{t}{3} - \frac{2s}{9} \\ t \\ s \end{bmatrix}$$

Since there are two free Variable, we have found two eigenvectors associated with this eigenvalue. The above can be written as

$$\begin{aligned} \begin{bmatrix} -\frac{t}{3} - \frac{2s}{9} \\ t \\ s \end{bmatrix} &= \begin{bmatrix} -\frac{t}{3} \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{2s}{9} \\ 0 \\ s \end{bmatrix} \\ &= t \begin{bmatrix} -\frac{1}{3} \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -\frac{2}{9} \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

By letting $t = 1$ and $s = 1$ then the above becomes

$$\begin{bmatrix} -\frac{t}{3} - \frac{2s}{9} \\ t \\ s \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{2}{9} \\ 0 \\ 1 \end{bmatrix}$$

Hence the two eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix} -\frac{1}{3} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{2}{9} \\ 0 \\ 1 \end{bmatrix}\right)$$

Which are normalized to

$$\left(\begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 9 \end{bmatrix}\right)$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
1	3	2	Yes	$\begin{bmatrix} -\frac{2}{9} & -\frac{1}{3} \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 1 is real and repated eigenvalue of multiplicity 3.There are three possible cases that can happen. This is illustrated in this diagram

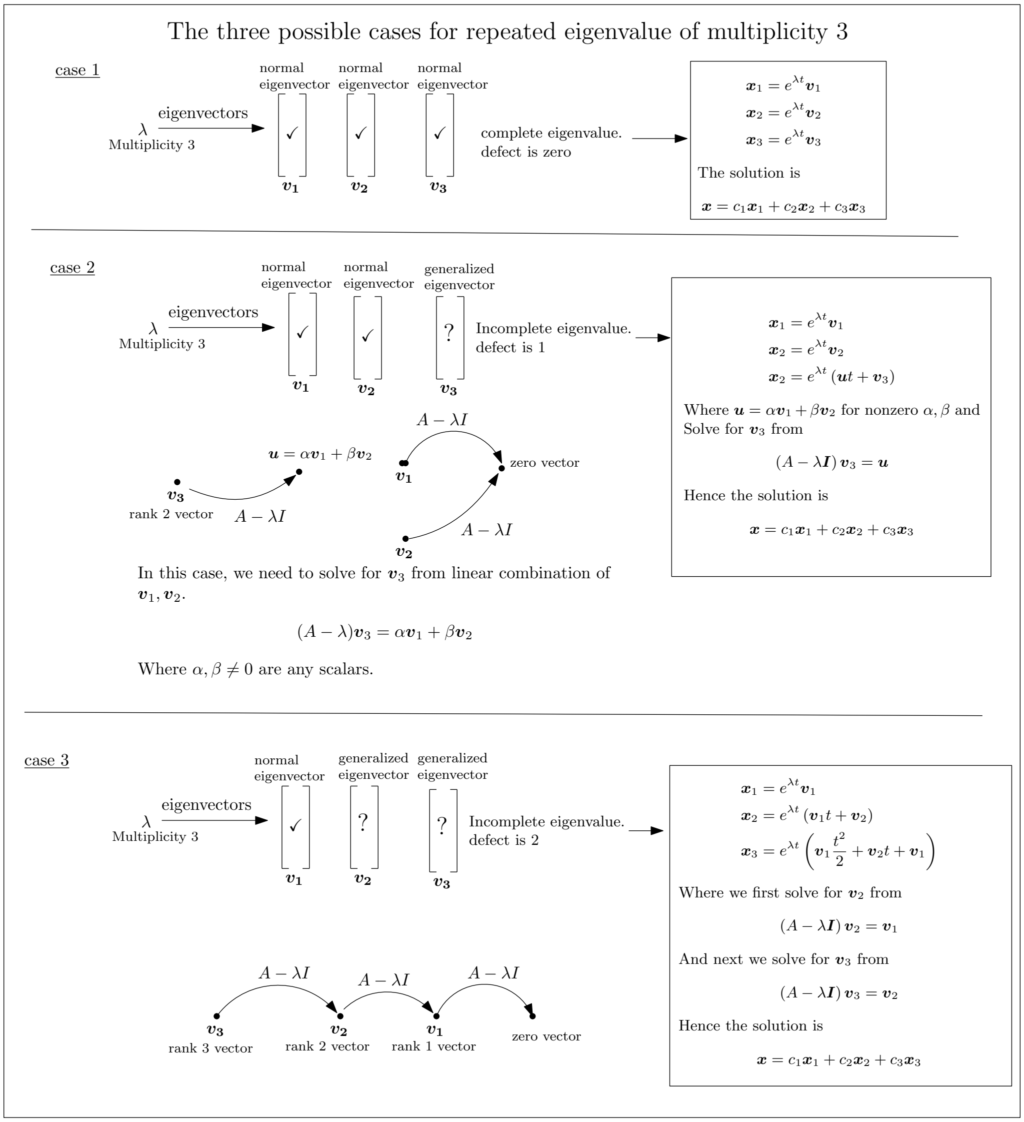


Figure 53: Possible case for repeated λ of multiplicity 3

This eigenvalue has algebraic multiplicity of 3, and geometric multiplicity 2, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to find rank-2 eigenvector $\vec{v}_3$. This eigenvector must therefore satisfy $(A - \lambda I)^2 \vec{v}_3 = \vec{0}$.

But

$$(A-\lambda I)^2 = \left(\begin{bmatrix} 1 & 0 & 0 \\ 18 & 7 & 4 \\ -27 & -9 & -5 \end{bmatrix} - 1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)^2$$
$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore $\vec{v}_3$ could be any eigenvector vector we want (but not the zero vector). Let

$$\vec{v}_3 = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix}$$

To determine the actual $\vec{v}_3$ we need now to enforce the condition that $\vec{v}_3$ satisfies

$$(A-\lambda I)\,\vec{v}_3 = \vec{u} \tag{1}$$

Where $\vec{u}$ is linear combination of $\vec{v}_1, \vec{v}_2$. Hence

$$\vec{u} = \alpha\vec{v}_1 + \beta\vec{v}_2$$

Where α, β are arbitrary constants (not both zero). Eq. (1) becomes

$$(A-\lambda I) \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} = \alpha \begin{bmatrix} -\frac{2}{9} \\ 0 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} -\frac{1}{3} \\ 1 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 0 & 0 \\ 18 & 6 & 4 \\ -27 & -9 & -6 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} = \alpha \begin{bmatrix} -\frac{2}{9} \\ 0 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} -\frac{1}{3} \\ 1 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 \\ 18\eta_1 + 6\eta_2 + 4\eta_3 \\ -27\eta_1 - 9\eta_2 - 6\eta_3 \end{bmatrix} = \begin{bmatrix} -\frac{2\alpha}{9} - \frac{\beta}{3} \\ \beta \\ \alpha \end{bmatrix}$$

Expanding the above gives the following equations equations

$$
\begin{aligned}
0 &= -\frac{2\alpha}{9} - \frac{\beta}{3} \\
18\eta_1 + 6\eta_2 + 4\eta_3 &= \beta \\
-27\eta_1 - 9\eta_2 - 6\eta_3 &= \alpha
\end{aligned}
$$

solving for α, β from the above gives

$$
\begin{aligned}
0 &= -\frac{2\alpha}{9} - \frac{\beta}{3} \\
18\eta_1 + 6\eta_2 + 4\eta_3 &= \beta
\end{aligned}
$$

Since α, β are not both zero, then we just need to determine η_i values, not all zero, which satisfy the above equations for α, β not both zero. By inspection we see that the following values satisfy this condition

$$
[\eta_1 = -1, \eta_2 = 0, \eta_3 = 0]
$$

Hence we found the missing generalized eigenvector

$$
\vec{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}
$$

Which implies that

$$
\begin{aligned}
\alpha &= 27 \\
\beta &= -18
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\vec{u} &= \alpha\vec{v}_1 + \beta\vec{v}_2 \\
&= 27\begin{bmatrix} -\frac{2}{9} \\ 0 \\ 1 \end{bmatrix} + (-18)\begin{bmatrix} -\frac{1}{3} \\ 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ -18 \\ 27 \end{bmatrix}
\end{aligned}
$$

Therefore the missing generalized eigenvector is now found. We have found three generalized eigenvectors for eigenvalue 1. Therefore the three basis solutions associated with this eigenvalue are

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\
&= \begin{bmatrix} -\frac{2}{9} \\ 0 \\ 1 \end{bmatrix} \mathrm{e}^{t} \\
&= \begin{bmatrix} -\frac{2\,\mathrm{e}^{t}}{9} \\ 0 \\ \mathrm{e}^{t} \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{\lambda t} \\
&= \begin{bmatrix} -\frac{1}{3} \\ 1 \\ 0 \end{bmatrix} \mathrm{e}^{t} \\
&= \begin{bmatrix} -\frac{\mathrm{e}^{t}}{3} \\ \mathrm{e}^{t} \\ 0 \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_3(t) &= (\vec{u}t + \vec{v}_3)\, e^{\lambda t} \\
&= \left(\begin{bmatrix} 0 \\ -18 \\ 27 \end{bmatrix} t + \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \right) \mathrm{e}^{t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)
$$

Which is written as

$$
\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 \begin{bmatrix} -\frac{2\,\mathrm{e}^{t}}{9} \\ 0 \\ \mathrm{e}^{t} \end{bmatrix} + c_2 \begin{bmatrix} -\frac{\mathrm{e}^{t}}{3} \\ \mathrm{e}^{t} \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} -\mathrm{e}^{t} \\ -18t\,\mathrm{e}^{t} \\ 27t\,\mathrm{e}^{t} \end{bmatrix}
$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} -\frac{e^t(2c_1+3c_2+9c_3)}{9} \\ e^t(-18tc_3 + c_2) \\ e^t(27tc_3 + c_1) \end{bmatrix}$$

✓ Solution by Maple

Time used: 0.109 (sec). Leaf size: 41

```
dsolve([diff(x__1(t),t)=1*x__1(t)+0*x__2(t)-0*x__3(t),diff(x__2(t),t)=18*x__1(t)+7*x__2(t)+4
```

$$x_1(t) = c_3 e^t$$
$$x_2(t) = e^t(c_2 t + c_1)$$
$$x_3(t) = -\frac{e^t(6c_2 t + 6c_1 - c_2 + 18c_3)}{4}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 63

```
DSolve[{x1'[t]==1*x1[t]+0*x2[t]-0*x3[t],x2'[t]==18*x1[t]+7*x2[t]+4*x3[t],x3'[t]==-27*x1[t]-9
```

$$\text{x1}(t) \to c_1 e^t$$
$$\text{x2}(t) \to e^t(2(9c_1 + 3c_2 + 2c_3)t + c_2)$$
$$\text{x3}(t) \to e^t(c_3 - 3(9c_1 + 3c_2 + 2c_3)t)$$

6.18 problem problem 18

Internal problem ID [375]
Internal file name [`OUTPUT/375_Sunday_June_05_2022_01_39_58_AM_71350364/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.6, Multiple Eigenvalue Solutions. Page 451
Problem number: problem 18.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned}
x_1'(t) &= x_1(t) \\
x_2'(t) &= x_1(t) + 3x_2(t) + x_3(t) \\
x_3'(t) &= -2x_1(t) - 4x_2(t) - x_3(t)
\end{aligned}$$

6.18.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & 1 \\ -2 & -4 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{t} & 0 & 0 \\ t\,e^{t} & e^{t}(1+2t) & t\,e^{t} \\ -2t\,e^{t} & -4t\,e^{t} & e^{t}(1-2t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^t & 0 & 0 \\ t\,e^t & e^t(1+2t) & t\,e^t \\ -2t\,e^t & -4t\,e^t & e^t(1-2t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} e^t c_1 \\ t\,e^t c_1 + e^t(1+2t)\,c_2 + t\,e^t c_3 \\ -2t\,e^t c_1 - 4t\,e^t c_2 + e^t(1-2t)\,c_3 \end{bmatrix} \\
&= \begin{bmatrix} e^t c_1 \\ ((c_1+2c_2+c_3)\,t + c_2)\,e^t \\ -2\,e^t\left((c_1+2c_2+c_3)\,t - \frac{c_3}{2}\right) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

6.18.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & 1 \\ -2 & -4 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det(A - \lambda I) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & 1 \\ -2 & -4 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) = 0
$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & 0 & 0 \\ 1 & 3-\lambda & 1 \\ -2 & -4 & -1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & 1 \\ -2 & -4 & -1 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 1 \\ -2 & -4 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ -2 & -4 & -2 & 0 \end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & -4 & -2 & 0 \end{array}\right]$$

$$R_3 = R_3 + 2R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2, v_3\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Let $v_3 = s$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -2t - s\}$

Hence the solution is

$$\begin{bmatrix} -2t - s \\ t \\ s \end{bmatrix} = \begin{bmatrix} -2t - s \\ t \\ s \end{bmatrix}$$

Since there are two free Variable, we have found two eigenvectors associated with this eigenvalue. The above can be written as

$$\begin{aligned} \begin{bmatrix} -2t - s \\ t \\ s \end{bmatrix} &= \begin{bmatrix} -2t \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} -s \\ 0 \\ s \end{bmatrix} \\ &= t \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

By letting $t = 1$ and $s = 1$ then the above becomes

$$\begin{bmatrix} -2t - s \\ t \\ s \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Hence the two eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right)$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m-k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
1	3	2	Yes	$\begin{bmatrix} -1 & -2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 1 is real and repated eigenvalue of multiplicity 3.There are three possible cases that can happen. This is illustrated in this diagram

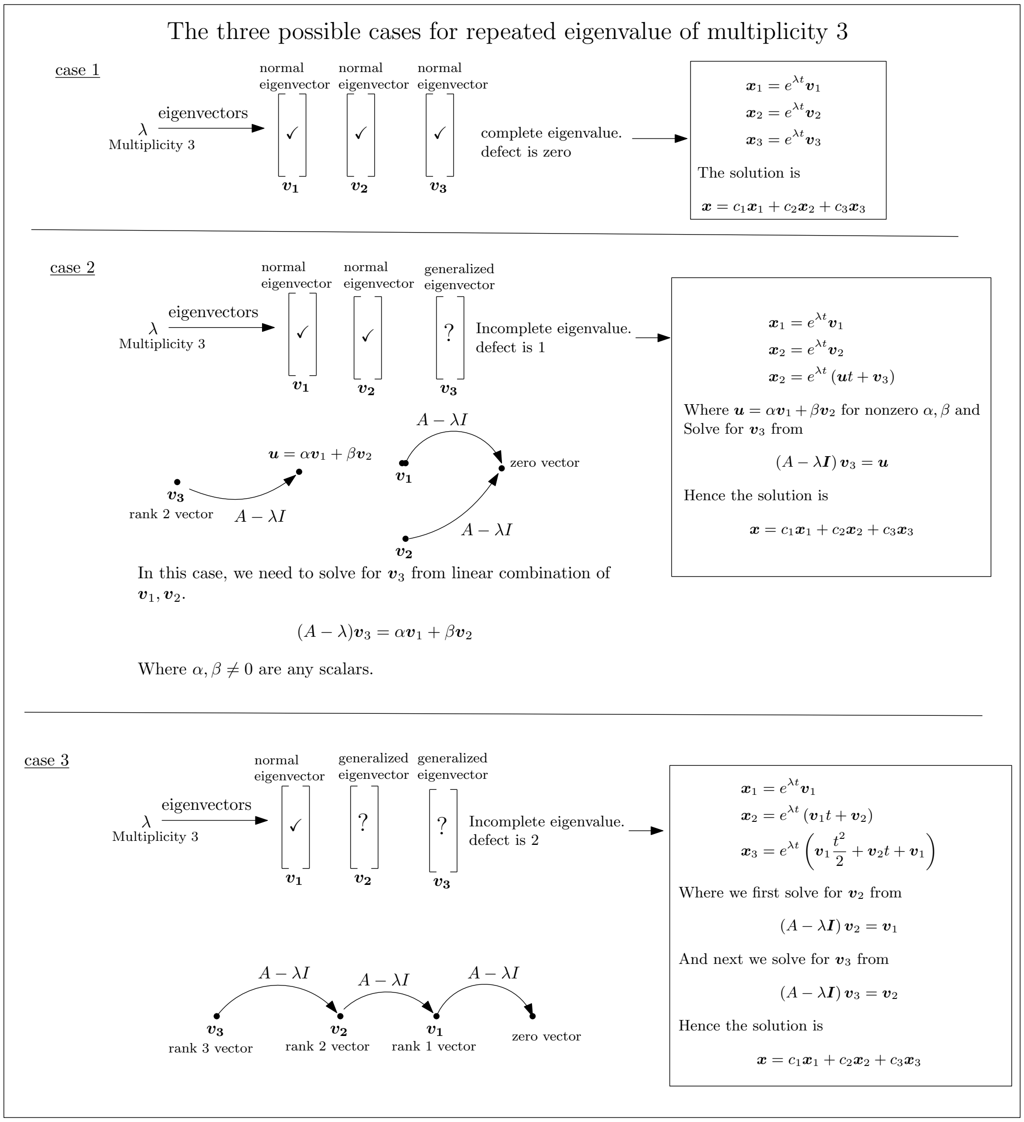


Figure 54: Possible case for repeated λ of multiplicity 3

This eigenvalue has algebraic multiplicity of 3, and geometric multiplicity 2, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to find rank-2 eigenvector $\vec{v}_3$. This eigenvector must therefore satisfy $(A - \lambda I)^2 \vec{v}_3 = \vec{0}$.

But

$$
\begin{aligned}
(A-\lambda I)^2 &= \left(\begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & 1 \\ -2 & -4 & -1 \end{bmatrix} - 1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)^2 \\
&= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\end{aligned}
$$

Therefore $\vec{v}_3$ could be any eigenvector vector we want (but not the zero vector). Let

$$
\vec{v}_3 = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix}
$$

To determine the actual $\vec{v}_3$ we need now to enforce the condition that $\vec{v}_3$ satisfies

$$
(A-\lambda I)\,\vec{v}_3 = \vec{u} \tag{1}
$$

Where $\vec{u}$ is linear combination of $\vec{v}_1, \vec{v}_2$. Hence

$$
\vec{u} = \alpha \vec{v}_1 + \beta \vec{v}_2
$$

Where α, β are arbitrary constants (not both zero). Eq. (1) becomes

$$
\begin{aligned}
(A-\lambda I) \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} &= \alpha \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 1 \\ -2 & -4 & -2 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} &= \alpha \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 0 \\ \eta_1 + 2\eta_2 + \eta_3 \\ -2\eta_1 - 4\eta_2 - 2\eta_3 \end{bmatrix} &= \begin{bmatrix} -\alpha - 2\beta \\ \beta \\ \alpha \end{bmatrix}
\end{aligned}
$$

Expanding the above gives the following equations equations

$$
\begin{aligned}
0 &= -\alpha - 2\beta \\
\eta_1 + 2\eta_2 + \eta_3 &= \beta \\
-2\eta_1 - 4\eta_2 - 2\eta_3 &= \alpha
\end{aligned}
$$

solving for α, β from the above gives

$$
\begin{aligned}
0 &= -\alpha - 2\beta \\
\eta_1 + 2\eta_2 + \eta_3 &= \beta
\end{aligned}
$$

Since α, β are not both zero, then we just need to determine η_i values, not all zero, which satisfy the above equations for α, β not both zero. By inspection we see that the following values satisfy this condition

$$
[\eta_1 = -1, \eta_2 = 0, \eta_3 = 0]
$$

Hence we found the missing generalized eigenvector

$$
\vec{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}
$$

Which implies that

$$
\begin{aligned}
\alpha &= 2 \\
\beta &= -1
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\vec{u} &= \alpha \vec{v}_1 + \beta \vec{v}_2 \\
&= 2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + (-1) \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}
\end{aligned}
$$

Therefore the missing generalized eigenvector is now found. We have found three generalized eigenvectors for eigenvalue 1. Therefore the three basis solutions associated with this eigenvalue are

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\
&= \left[\begin{array}{c} -1 \\ 0 \\ 1 \end{array}\right] \mathrm{e}^{t} \\
&= \left[\begin{array}{c} -\mathrm{e}^{t} \\ 0 \\ \mathrm{e}^{t} \end{array}\right]
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{\lambda t} \\
&= \left[\begin{array}{c} -2 \\ 1 \\ 0 \end{array}\right] \mathrm{e}^{t} \\
&= \left[\begin{array}{c} -2\,\mathrm{e}^{t} \\ \mathrm{e}^{t} \\ 0 \end{array}\right]
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_3(t) &= (\vec{u}t + \vec{v}_3)\, e^{\lambda t} \\
&= \left(\left[\begin{array}{c} 0 \\ -1 \\ 2 \end{array}\right] t + \left[\begin{array}{c} -1 \\ 0 \\ 0 \end{array}\right]\right) \mathrm{e}^{t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)
$$

Which is written as

$$
\left[\begin{array}{c} x_1(t) \\ x_2(t) \\ x_3(t) \end{array}\right] = c_1 \left[\begin{array}{c} -\mathrm{e}^{t} \\ 0 \\ \mathrm{e}^{t} \end{array}\right] + c_2 \left[\begin{array}{c} -2\,\mathrm{e}^{t} \\ \mathrm{e}^{t} \\ 0 \end{array}\right] + c_3 \left[\begin{array}{c} -\mathrm{e}^{t} \\ -t\,\mathrm{e}^{t} \\ 2t\,\mathrm{e}^{t} \end{array}\right]
$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} e^t(-c_1 - 2c_2 - c_3) \\ e^t(-tc_3 + c_2) \\ e^t(2tc_3 + c_1) \end{bmatrix}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 39

```
dsolve([diff(x__1(t),t)=1*x__1(t)+0*x__2(t)-0*x__3(t),diff(x__2(t),t)=1*x__1(t)+3*x__2(t)+1*
```

$$x_1(t) = c_3 e^t$$
$$x_2(t) = e^t(c_2 t + c_1)$$
$$x_3(t) = -e^t(2c_2 t + 2c_1 - c_2 + c_3)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 54

```
DSolve[{x1'[t]==1*x1[t]+0*x2[t]-0*x3[t],x2'[t]==1*x1[t]+3*x2[t]+1*x3[t],x3'[t]==-2*x1[t]-4*x
```

$$\text{x1}(t) \to c_1 e^t$$
$$\text{x2}(t) \to e^t((c_1 + 2c_2 + c_3)t + c_2)$$
$$\text{x3}(t) \to e^t(c_3 - 2(c_1 + 2c_2 + c_3)t)$$

6.19 problem problem 19

Internal problem ID [376]
Internal file name [`OUTPUT/376_Sunday_June_05_2022_01_39_59_AM_35326549/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.6, Multiple Eigenvalue Solutions. Page 451
Problem number: problem 19.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= x_1(t) - 4x_2(t) - 2x_4(t) \\
x_2'(t) &= x_2(t) \\
x_3'(t) &= 6x_1(t) - 12x_2(t) - x_3(t) - 6x_4(t) \\
x_4'(t) &= -4x_2(t) - x_4(t)
\end{aligned}
$$

6.19.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \\ x_4'(t) \end{bmatrix} = \begin{bmatrix} 1 & -4 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 6 & -12 & -1 & -6 \\ 0 & -4 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^t & -2e^t + 2e^{-t} & 0 & -e^t + e^{-t} \\ 0 & e^t & 0 & 0 \\ 3e^t - 3e^{-t} & -6e^t + 6e^{-t} & e^{-t} & -3e^t + 3e^{-t} \\ 0 & -2e^t + 2e^{-t} & 0 & e^{-t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^t & -2e^t + 2e^{-t} & 0 & -e^t + e^{-t} \\ 0 & e^t & 0 & 0 \\ 3e^t - 3e^{-t} & -6e^t + 6e^{-t} & e^{-t} & -3e^t + 3e^{-t} \\ 0 & -2e^t + 2e^{-t} & 0 & e^{-t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} \\
&= \begin{bmatrix} e^t c_1 + (-2e^t + 2e^{-t})\, c_2 + (-e^t + e^{-t})\, c_4 \\ e^t c_2 \\ (3e^t - 3e^{-t})\, c_1 + (-6e^t + 6e^{-t})\, c_2 + e^{-t} c_3 + (-3e^t + 3e^{-t})\, c_4 \\ (-2e^t + 2e^{-t})\, c_2 + e^{-t} c_4 \end{bmatrix} \\
&= \begin{bmatrix} (2c_2 + c_4)\, e^{-t} + e^t(c_1 - 2c_2 - c_4) \\ e^t c_2 \\ (-3c_1 + 6c_2 + c_3 + 3c_4)\, e^{-t} + 3e^t(c_1 - 2c_2 - c_4) \\ (2c_2 + c_4)\, e^{-t} - 2e^t c_2 \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

6.19.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \\ x_4'(t) \end{bmatrix} = \begin{bmatrix} 1 & -4 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 6 & -12 & -1 & -6 \\ 0 & -4 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & -4 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 6 & -12 & -1 & -6 \\ 0 & -4 & 0 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & -4 & 0 & -2 \\ 0 & 1-\lambda & 0 & 0 \\ 6 & -12 & -1-\lambda & -6 \\ 0 & -4 & 0 & -1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^4 - 2\lambda^2 + 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 1$$
$$\lambda_2 = -1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
1	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -4 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 6 & -12 & -1 & -6 \\ 0 & -4 & 0 & -1 \end{bmatrix} - (-1) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -4 & 0 & -2 \\ 0 & 2 & 0 & 0 \\ 6 & -12 & 0 & -6 \\ 0 & -4 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} 2 & -4 & 0 & -2 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 6 & -12 & 0 & -6 & 0 \\ 0 & -4 & 0 & 0 & 0 \end{array}\right]$$

$$R_3 = R_3 - 3R_1 \Longrightarrow \left[\begin{array}{cccc|c} 2 & -4 & 0 & -2 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 & 0 \end{array}\right]$$

$$R_4 = R_4 + 2R_2 \Longrightarrow \left[\begin{array}{cccc|c} 2 & -4 & 0 & -2 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & -4 & 0 & -2 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3, v_4\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Let $v_4 = s$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = s, v_2 = 0\}$

Hence the solution is

$$\begin{bmatrix} s \\ 0 \\ t \\ s \end{bmatrix} = \begin{bmatrix} s \\ 0 \\ t \\ s \end{bmatrix}$$

Since there are two free Variable, we have found two eigenvectors associated with this eigenvalue. The above can be written as

$$\begin{aligned}\begin{bmatrix} s \\ 0 \\ t \\ s \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} s \\ 0 \\ 0 \\ s \end{bmatrix} \\ &= t \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}\end{aligned}$$

By letting $t = 1$ and $s = 1$ then the above becomes

$$\begin{bmatrix} s \\ 0 \\ t \\ s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Hence the two eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

Considering the eigenvalue $\lambda_2 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -4 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 6 & -12 & -1 & -6 \\ 0 & -4 & 0 & -1 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -4 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 6 & -12 & -2 & -6 \\ 0 & -4 & 0 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} 0 & -4 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 6 & -12 & -2 & -6 & 0 \\ 0 & -4 & 0 & -2 & 0 \end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 3 gives

$$\left[\begin{array}{cccc|c} 6 & -12 & -2 & -6 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & -2 & 0 \\ 0 & -4 & 0 & -2 & 0 \end{array}\right]$$

Since the current pivot $A(2,2)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{cccc|c} 6 & -12 & -2 & -6 & 0 \\ 0 & -4 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & -2 & 0 \end{array}\right]$$

$$
R_4 = R_4 - R_2 \Longrightarrow \left[\begin{array}{cccc|c} 6 & -12 & -2 & -6 & 0 \\ 0 & -4 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]
$$

Therefore the system in Echelon form is

$$
\begin{bmatrix} 6 & -12 & -2 & -6 \\ 0 & -4 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

The free variables are $\{v_3, v_4\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Let $v_4 = s$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{3}, v_2 = -\frac{s}{2}\right\}$

Hence the solution is

$$
\begin{bmatrix} \frac{t}{3} \\ -\frac{s}{2} \\ t \\ s \end{bmatrix} = \begin{bmatrix} \frac{t}{3} \\ -\frac{s}{2} \\ t \\ s \end{bmatrix}
$$

Since there are two free Variable, we have found two eigenvectors associated with this eigenvalue. The above can be written as

$$
\begin{aligned}
\begin{bmatrix} \frac{t}{3} \\ -\frac{s}{2} \\ t \\ s \end{bmatrix} &= \begin{bmatrix} \frac{t}{3} \\ 0 \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{s}{2} \\ 0 \\ s \end{bmatrix} \\
&= t \begin{bmatrix} \frac{1}{3} \\ 0 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}
\end{aligned}
$$

By letting $t = 1$ and $s = 1$ then the above becomes

$$\begin{bmatrix} \frac{t}{3} \\ -\frac{s}{2} \\ t \\ s \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

Hence the two eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix} \frac{1}{3} \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \right)$$

Which are normalized to

$$\left(\begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 2 \end{bmatrix} \right)$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
1	2	2	No	$\begin{bmatrix} 0 & \frac{1}{3} \\ -\frac{1}{2} & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$
-1	2	2	No	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 1 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

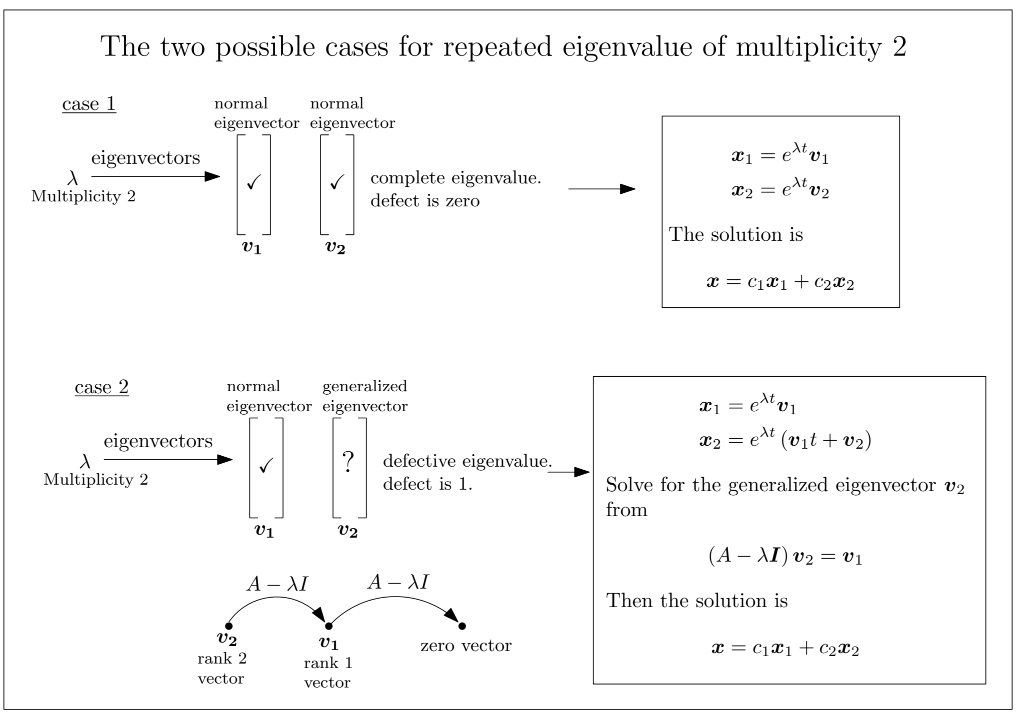


Figure 55: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2 which is the same as its geometric multiplicity 2, then it is complete eigenvalue and this falls into case 1 shown above. Hence the corresponding eigenvector basis are

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^t \\
&= \begin{bmatrix} 0 \\ -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix} e^t
\end{aligned}
$$

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^t \\
&= \begin{bmatrix} \frac{1}{3} \\ 0 \\ 1 \\ 0 \end{bmatrix} e^t
\end{aligned}
$$

eigenvalue -1 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

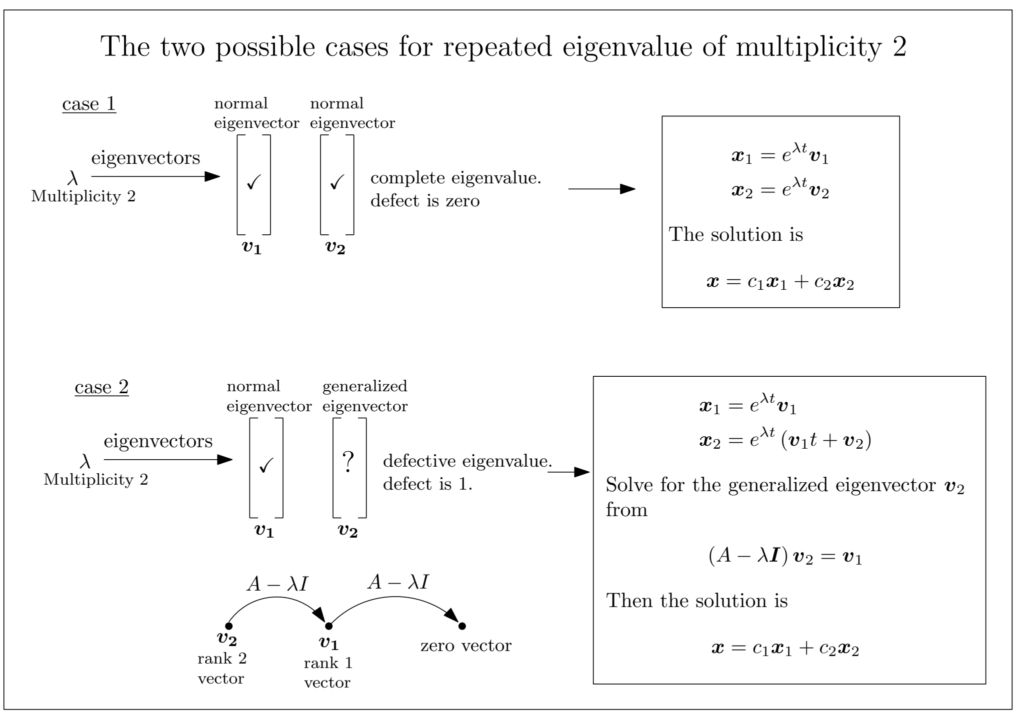


Figure 56: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2 which is the same as its geometric multiplicity 2, then it is complete eigenvalue and this falls into case 1 shown above. Hence the corresponding eigenvector basis are

$$\begin{aligned}\vec{x}_3(t) &= \vec{v}_3 e^{-t} \\ &= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} e^{-t}\end{aligned}$$

$$\begin{aligned}\vec{x}_4(t) &= \vec{v}_4 e^{-t} \\ &= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} e^{-t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t) + c_4 \vec{x}_4(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ -\frac{e^t}{2} \\ 0 \\ e^t \end{bmatrix} + c_2 \begin{bmatrix} \frac{e^t}{3} \\ 0 \\ e^t \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} e^{-t} \\ 0 \\ 0 \\ e^{-t} \end{bmatrix} + c_4 \begin{bmatrix} 0 \\ 0 \\ e^{-t} \\ 0 \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = \begin{bmatrix} \frac{c_2 e^t}{3} + c_3 e^{-t} \\ -\frac{c_1 e^t}{2} \\ c_2 e^t + c_4 e^{-t} \\ c_1 e^t + c_3 e^{-t} \end{bmatrix}$$

6.19.3 Maple step by step solution

Let's solve

$[x_1'(t) = x_1(t) - 4x_2(t) - 2x_4(t), x_2'(t) = x_2(t), x_3'(t) = 6x_1(t) - 12x_2(t) - x_3(t) - 6x_4(t), x_4'(t) =$

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x}^{\rightarrow\prime}(t) = \begin{bmatrix} 1 & -4 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 6 & -12 & -1 & -6 \\ 0 & -4 & 0 & -1 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{x}^{\rightarrow\prime}(t) = \begin{bmatrix} 1 & -4 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 6 & -12 & -1 & -6 \\ 0 & -4 & 0 & -1 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & -4 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 6 & -12 & -1 & -6 \\ 0 & -4 & 0 & -1 \end{bmatrix}$$

- Rewrite the system as

$\underline{x}^{\rightarrow\prime}(t) = A \cdot \underline{x}^{\rightarrow}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right], \left[-1, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right], \left[1, \begin{bmatrix} 0 \\ -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} \frac{1}{3} \\ 0 \\ 1 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[-1, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue -1

$$\underline{\vec{x}}_1(t) = \mathrm{e}^{-t} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = -1$ is the eigenvalue, an

$$\underline{\vec{x}}_2(t) = \mathrm{e}^{\lambda t}\left(t\vec{v} + \vec{p}\right)$$

- Note that the t multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\underline{\vec{x}}_2(t)$ into the homogeneous system

$$\lambda\,\mathrm{e}^{\lambda t}\left(t\vec{v} + \vec{p}\right) + \mathrm{e}^{\lambda t}\vec{v} = \left(\mathrm{e}^{\lambda t} A\right) \cdot \left(t\vec{v} + \vec{p}\right)$$

- Use the fact that $\vec{v}$ is an eigenvector of A

$$\lambda\,\mathrm{e}^{\lambda t}\left(t\vec{v} + \vec{p}\right) + \mathrm{e}^{\lambda t}\vec{v} = \mathrm{e}^{\lambda t}\left(\lambda t\vec{v} + A \cdot \vec{p}\right)$$

- Simplify equation

$$\lambda\vec{p} + \vec{v} = A \cdot \vec{p}$$

- Make use of the identity matrix I

$$(\lambda \cdot I) \cdot \vec{p} + \vec{v} = A \cdot \vec{p}$$

- Condition $\vec{p}$ must meet for $\underline{\vec{x}}_2(t)$ to be a solution to the homogeneous system

$$(A - \lambda \cdot I) \cdot \vec{p} = \vec{v}$$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue -1

$$\left(\begin{bmatrix} 1 & -4 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 6 & -12 & -1 & -6 \\ 0 & -4 & 0 & -1 \end{bmatrix} - (-1) \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) \cdot \vec{p} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

- Choice of $\vec{p}$

$$\vec{p} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue -1

$$\vec{x}_2(t) = e^{-t} \cdot \left(t \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{bmatrix}\right)$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[1, \begin{bmatrix} 0 \\ -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue 1

$$\vec{x}_3(t) = e^{t} \cdot \begin{bmatrix} 0 \\ -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = 1$ is the eigenvalue, and

$$\vec{x}_4(t) = e^{\lambda t}\left(t\vec{v} + \vec{p}\right)$$

- Note that the t multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\vec{x}_4(t)$ into the homogeneous system

$$\lambda e^{\lambda t}\left(t\vec{v}+\vec{p}\right)+e^{\lambda t}\vec{v}=\left(e^{\lambda t}A\right)\cdot\left(t\vec{v}+\vec{p}\right)$$

- Use the fact that $\vec{v}$ is an eigenvector of A

$$\lambda e^{\lambda t}\left(t\vec{v}+\vec{p}\right)+e^{\lambda t}\vec{v}=e^{\lambda t}\left(\lambda t\vec{v}+A\cdot\vec{p}\right)$$

- Simplify equation

$$\lambda\vec{p}+\vec{v}=A\cdot\vec{p}$$

- Make use of the identity matrix I

$$(\lambda\cdot I)\cdot\vec{p}+\vec{v}=A\cdot\vec{p}$$

- Condition $\vec{p}$ must meet for $\vec{x}_4(t)$ to be a solution to the homogeneous system

$$(A-\lambda\cdot I)\cdot\vec{p}=\vec{v}$$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue 1

$$\left(\begin{bmatrix}1 & -4 & 0 & -2\\ 0 & 1 & 0 & 0\\ 6 & -12 & -1 & -6\\ 0 & -4 & 0 & -1\end{bmatrix}-1\cdot\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\right)\cdot\vec{p}=\begin{bmatrix}0\\ -\frac{1}{2}\\ 0\\ 1\end{bmatrix}$$

- Choice of $\vec{p}$

$$\vec{p}=\begin{bmatrix}0\\ 0\\ 0\\ 0\end{bmatrix}$$

- Second solution from eigenvalue 1

$$\vec{x}_4(t)=e^{t}\cdot\left(t\cdot\begin{bmatrix}0\\ -\frac{1}{2}\\ 0\\ 1\end{bmatrix}+\begin{bmatrix}0\\ 0\\ 0\\ 0\end{bmatrix}\right)$$

- General solution to the system of ODEs

$$\vec{x}=c_1\vec{x}_1(t)+c_2\vec{x}_2(t)+c_3\vec{x}_3(t)+c_4\vec{x}_4(t)$$

- Substitute solutions into the general solution

$$\vec{x} = c_1 e^{-t} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + c_2 e^{-t} \cdot \left(t \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{bmatrix} \right) + c_3 e^{t} \cdot \begin{bmatrix} 0 \\ -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix} + c_4 e^{t} \cdot \left(t \cdot \begin{bmatrix} 0 \\ -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = \begin{bmatrix} e^{-t}\left(c_1 + c_2 t + \frac{1}{2}c_2\right) \\ -\frac{e^{t}(tc_4+c_3)}{2} \\ 0 \\ (c_2 t + c_1)\,e^{-t} + e^{t}(tc_4 + c_3) \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x_1(t) = e^{-t}\left(c_1 + c_2 t + \frac{1}{2}c_2\right), x_2(t) = -\frac{e^{t}(tc_4+c_3)}{2}, x_3(t) = 0, x_4(t) = (c_2 t + c_1)\,e^{-t} + e^{t}(tc_4 + c_3)\right\}$$

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 53

```
dsolve([diff(x__1(t),t)=1*x__1(t)-4*x__2(t)+0*x__3(t)-2*x__4(t),diff(x__2(t),t)=0*x__1(t)+1*
```

$$x_1(t) = c_2 e^{t} + c_3 e^{-t}$$
$$x_2(t) = c_4 e^{t}$$
$$x_3(t) = 3c_2 e^{t} + e^{-t}c_1$$
$$x_4(t) = -2c_4 e^{t} + c_3 e^{-t}$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 114

```
DSolve[{x1'[t]==1*x1[t]-4*x2[t]+0*x3[t]-2*x4[t],x2'[t]==0*x1[t]+1*x2[t]+0*x3[t]+0*x4[t],x3'[
```

$$\text{x1}(t) \to e^{-t}\left((c_1 - 2c_2 - c_4)e^{2t} + 2c_2 + c_4\right)$$
$$\text{x2}(t) \to c_2 e^{t}$$
$$\text{x3}(t) \to e^{-t}\left(3c_1\left(e^{2t} - 1\right) - 6c_2\left(e^{2t} - 1\right) - 3c_4 e^{2t} + c_3 + 3c_4\right)$$
$$\text{x4}(t) \to e^{-t}\left(c_4 - 2c_2\left(e^{2t} - 1\right)\right)$$

6.20 problem problem 20

Internal problem ID [377]
Internal file name [OUTPUT/377_Sunday_June_05_2022_01_40_01_AM_9212484/index.tex]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.6, Multiple Eigenvalue Solutions. Page 451
Problem number: problem 20.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned}
x_1'(t) &= 2x_1(t) + x_2(t) + x_4(t) \\
x_2'(t) &= 2x_2(t) + x_3(t) \\
x_3'(t) &= 2x_3(t) + x_4(t) \\
x_4'(t) &= 2x_4(t)
\end{aligned}$$

6.20.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \\ x_4'(t) \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} e^{2t} & e^{2t}t & \frac{e^{2t}t^2}{2} & e^{2t}\left(\frac{1}{6}t^3 + t\right) \\ 0 & e^{2t} & e^{2t}t & \frac{e^{2t}t^2}{2} \\ 0 & 0 & e^{2t} & e^{2t}t \\ 0 & 0 & 0 & e^{2t} \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{2t} & e^{2t}t & \frac{e^{2t}t^2}{2} & e^{2t}\left(\frac{1}{6}t^3 + t\right) \\ 0 & e^{2t} & e^{2t}t & \frac{e^{2t}t^2}{2} \\ 0 & 0 & e^{2t} & e^{2t}t \\ 0 & 0 & 0 & e^{2t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} \\
&= \begin{bmatrix} e^{2t}c_1 + e^{2t}tc_2 + \frac{e^{2t}t^2c_3}{2} + e^{2t}\left(\frac{1}{6}t^3 + t\right)c_4 \\ e^{2t}c_2 + e^{2t}tc_3 + \frac{e^{2t}t^2c_4}{2} \\ e^{2t}c_3 + e^{2t}tc_4 \\ e^{2t}c_4 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(c_4t^3 + 3c_3t^2 + (6c_2 + 6c_4)t + 6c_1)e^{2t}}{6} \\ e^{2t}\left(c_2 + c_3t + \frac{1}{2}c_4t^2\right) \\ e^{2t}(tc_4 + c_3) \\ e^{2t}c_4 \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

6.20.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \\ x_4'(t) \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 2-\lambda & 1 & 0 & 1 \\ 0 & 2-\lambda & 1 & 0 \\ 0 & 0 & 2-\lambda & 1 \\ 0 & 0 & 0 & 2-\lambda \end{bmatrix}\right) = 0$$

Since the matrix A is triangular matrix, then the determinant is the product of the elements along the diagonal. Therefore the above becomes

$$(2-\lambda)(2-\lambda)(2-\lambda)(2-\lambda) = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_1\}$ and the leading variables are $\{v_2, v_3, v_4\}$. Let $v_1 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_2 = 0, v_3 = 0, v_4 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this

eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ 0 \\ 0 \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
2	4	1	Yes	$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 2 is real and repated eigenvalue of multiplicity 4.There are four possible cases that can happen. This is illustrated in this diagram

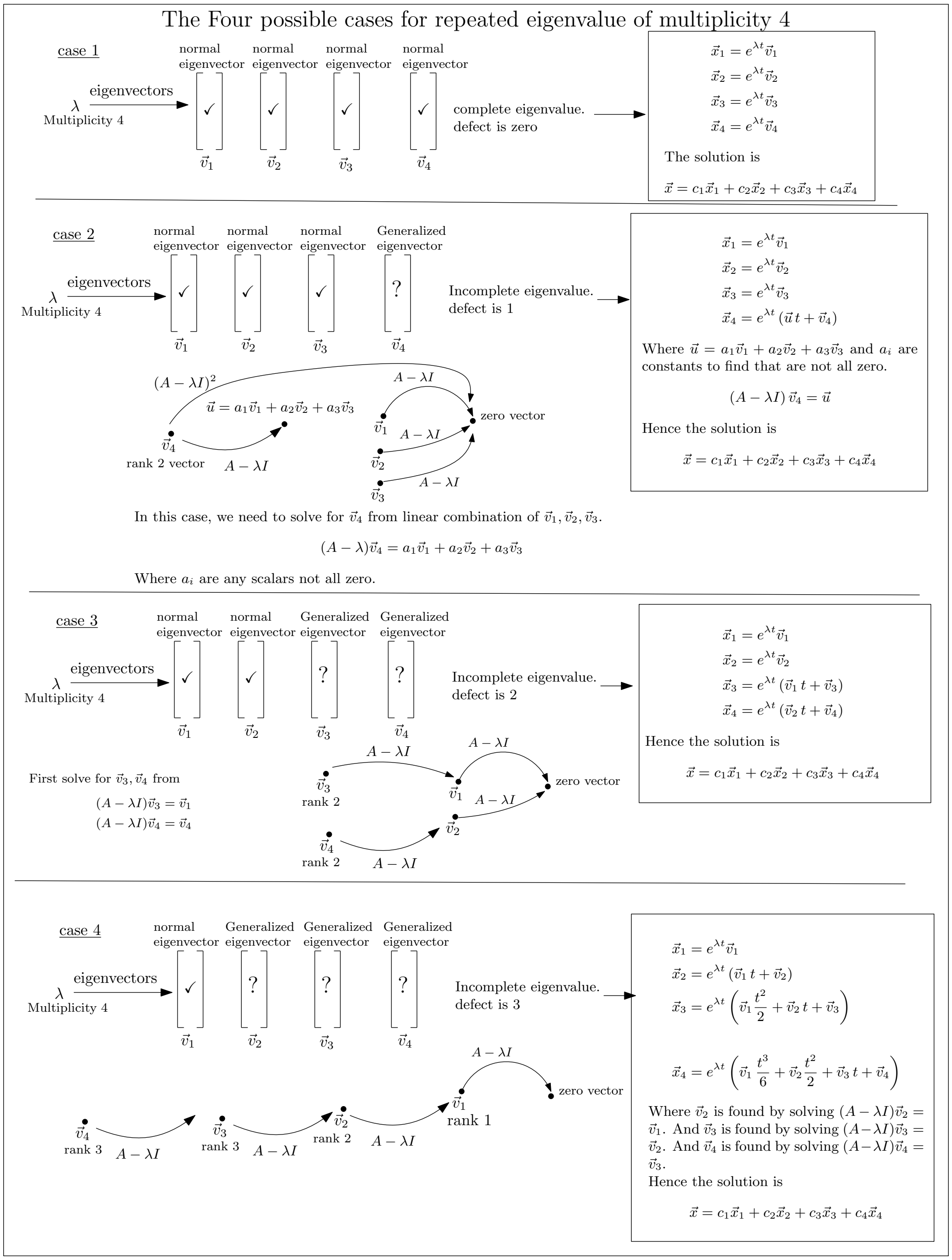


Figure 57: Possible case for repeated λ of multiplicity 4

This eigenvalue has algebraic multiplicity of 4, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 3. This falls into case 4 shown above. First we find generalized eigenvector $\vec{v}_2$ of rank 2 and then use this to find generalized eigenvector

$\vec{v}_3$ of rank 3.$\vec{v}_2$ is found by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found earlier. Hence

$$\left(\begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Now $\vec{v}_3$ is found by solving

$$(A - \lambda I)\,\vec{v}_3 = \vec{v}_2$$

Where $\vec{v}_2$ is the (rank 2) generalized eigenvector found above. Hence

$$\left(\begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Solving for $\vec{v}_3$ gives

$$\vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

Now $\vec{v}_4$ is found by solving

$$(A - \lambda I)\,\vec{v}_4 = \vec{v}_3$$

Where $\vec{v}_3$ is the (rank 3) generalized eigenvector found above. Hence

$$\left(\begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

Solving for $\vec{v}_4$ gives

$$\vec{v}_4 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

We have found three generalized eigenvectors for eigenvalue 2. Therefore the three basis

solutions associated with this eigenvalue are

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\
&= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \mathrm{e}^{2t} \\
&= \begin{bmatrix} \mathrm{e}^{2t} \\ 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_2(t) &= e^{\lambda t}(\vec{v}_1 t + \vec{v}_2) \\
&= \mathrm{e}^{2t} \left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} t + \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right) \\
&= \begin{bmatrix} \mathrm{e}^{2t}(t+1) \\ \mathrm{e}^{2t} \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

And

$$\vec{x}_3(t) = \left(\vec{v}_1 \frac{t^2}{2} + \vec{v}_2 t + \vec{v}_3\right) e^{\lambda t}$$
$$= \left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \frac{t^2}{2} + \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} t + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}\right) \mathrm{e}^{2t}$$
$$= \begin{bmatrix} \frac{\mathrm{e}^{2t}(t^2+2t+2)}{2} \\ \mathrm{e}^{2t}(t+1) \\ \mathrm{e}^{2t} \\ 0 \end{bmatrix}$$

And

$$\vec{x}_4(t) = \left(\vec{v}_1 \frac{t^3}{6} + \vec{v}_2 \frac{t^2}{2} + \vec{v}_3 t + \vec{v}_4\right) e^{\lambda t}$$
$$= \left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \frac{t^3}{6} + \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \frac{t^2}{2} + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}\right) \mathrm{e}^{2t}$$
$$= \begin{bmatrix} \frac{\mathrm{e}^{2t}(t^3+3t^2+6t+6)}{6} \\ \frac{\mathrm{e}^{2t}t(2+t)}{2} \\ \mathrm{e}^{2t}(t+1) \\ \mathrm{e}^{2t} \end{bmatrix}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t) + c_4\vec{x}_4(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = c_1 \begin{bmatrix} \mathrm{e}^{2t} \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{2t}(t+1) \\ \mathrm{e}^{2t} \\ 0 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} \mathrm{e}^{2t}\left(t + \frac{1}{2}t^2 + 1\right) \\ \mathrm{e}^{2t}(t+1) \\ \mathrm{e}^{2t} \\ 0 \end{bmatrix} + c_4 \begin{bmatrix} \mathrm{e}^{2t}\left(\frac{1}{6}t^3 + \frac{1}{2}t^2 + t + 1\right) \\ \mathrm{e}^{2t}\left(\frac{1}{2}t^2 + t\right) \\ \mathrm{e}^{2t}(t+1) \\ \mathrm{e}^{2t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = \begin{bmatrix} \frac{(c_4t^3+(3c_3+3c_4)t^2+(6c_2+6c_3+6c_4)t+6c_1+6c_2+6c_3+6c_4)\,e^{2t}}{6} \\ \frac{(c_4t^2+(2c_3+2c_4)t+2c_2+2c_3)\,e^{2t}}{2} \\ e^{2t}(c_4t+c_3+c_4) \\ c_4e^{2t} \end{bmatrix}$$

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 78

```
dsolve([diff(x__1(t),t)=2*x__1(t)+1*x__2(t)+0*x__3(t)+1*x__4(t),diff(x__2(t),t)=0*x__1(t)+2*
```

$$x_1(t) = \frac{(c_4t^3 + 3c_3t^2 + 6c_2t + 6c_4t + 6c_1)\,e^{2t}}{6}$$
$$x_2(t) = \frac{(c_4t^2 + 2c_3t + 2c_2)\,e^{2t}}{2}$$
$$x_3(t) = (c_4t + c_3)\,e^{2t}$$
$$x_4(t) = c_4e^{2t}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 96

```
DSolve[{x1'[t]==2*x1[t]+1*x2[t]+0*x3[t]+1*x4[t],x2'[t]==0*x1[t]+2*x2[t]+1*x3[t]+0*x4[t],x3'[
```

$$\text{x1}(t) \to \frac{1}{6}e^{2t}\left(t\left(c_4t^2 + 3c_3t + 6c_2 + 6c_4\right) + 6c_1\right)$$
$$\text{x2}(t) \to \frac{1}{2}e^{2t}(t(c_4t + 2c_3) + 2c_2)$$
$$\text{x3}(t) \to e^{2t}(c_4t + c_3)$$
$$\text{x4}(t) \to c_4e^{2t}$$

6.21 problem problem 21

Internal problem ID [378]
Internal file name [OUTPUT/378_Sunday_June_05_2022_01_40_03_AM_15647572/index.tex]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.6, Multiple Eigenvalue Solutions. Page 451
Problem number: problem 21.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned}
x_1'(t) &= -x_1(t) - 4x_2(t) \\
x_2'(t) &= x_1(t) + 3x_2(t) \\
x_3'(t) &= x_1(t) + 2x_2(t) + x_3(t) \\
x_4'(t) &= x_2(t) + x_4(t)
\end{aligned}$$

6.21.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \\ x_4'(t) \end{bmatrix} = \begin{bmatrix} -1 & -4 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^t(1-2t) & -4t\,e^t & 0 & 0 \\ t\,e^t & e^t(1+2t) & 0 & 0 \\ t\,e^t & 2t\,e^t & e^t & 0 \\ \frac{t^2 e^t}{2} & e^t t(t+1) & 0 & e^t \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^t(1-2t) & -4t\,e^t & 0 & 0 \\ t\,e^t & e^t(1+2t) & 0 & 0 \\ t\,e^t & 2t\,e^t & e^t & 0 \\ \frac{t^2 e^t}{2} & e^t t(t+1) & 0 & e^t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} \\
&= \begin{bmatrix} e^t(1-2t)\,c_1 - 4t\,e^t c_2 \\ t\,e^t c_1 + e^t(1+2t)\,c_2 \\ t\,e^t c_1 + 2t\,e^t c_2 + e^t c_3 \\ \frac{t^2 e^t c_1}{2} + e^t t(t+1)\,c_2 + e^t c_4 \end{bmatrix} \\
&= \begin{bmatrix} (c_1(1-2t) - 4c_2 t)\,e^t \\ e^t(tc_1 + 2c_2 t + c_2) \\ ((c_1 + 2c_2)\,t + c_3)\,e^t \\ \frac{e^t((c_1+2c_2)t^2 + 2c_2 t + 2c_4)}{2} \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

6.21.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \\ x_4'(t) \end{bmatrix} = \begin{bmatrix} -1 & -4 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -1 & -4 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -1-\lambda & -4 & 0 & 0 \\ 1 & 3-\lambda & 0 & 0 \\ 1 & 2 & 1-\lambda & 0 \\ 0 & 1 & 0 & 1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^4 - 4\lambda^3 + 6\lambda^2 - 4\lambda + 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & -4 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & -4 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} -2 & -4 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{2} \Longrightarrow \left[\begin{array}{cccc|c} -2 & -4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{R_1}{2} \Longrightarrow \left[\begin{array}{cccc|c} -2 & -4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{array}\right]$$

Since the current pivot $A(2,2)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 4 gives

$$\left[\begin{array}{cccc|c} -2 & -4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & -4 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3, v_4\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Let $v_4 = s$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_2 = 0\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ 0 \\ t \\ s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ t \\ s \end{bmatrix}$$

Since there are two free Variable, we have found two eigenvectors associated with this eigenvalue. The above can be written as

$$\begin{aligned} \begin{bmatrix} 0 \\ 0 \\ t \\ s \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ s \end{bmatrix} \\ &= t \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

By letting $t = 1$ and $s = 1$ then the above becomes

$$\begin{bmatrix} 0 \\ 0 \\ t \\ s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Hence the two eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right)$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
1	4	2	Yes	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

This case will be solved using the Jordan form of the matrix A. The Jordan form diagonalization is

$$A = PJP^{-1}$$

Which can be found to be

$$\begin{bmatrix} -1 & -4 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}^{-1}$$

Looking at the P matrix above, we see there are 2 chains. Therefore, we now construct the basis solution by following these chains as follows.

$$\vec{x}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ e^t \end{bmatrix}$$

$$\vec{x}_2 = \begin{bmatrix} -2\,e^t \\ e^t \\ e^t \\ t\,e^t \end{bmatrix}$$

$$\vec{x}_3 = \begin{bmatrix} -2t\,e^t + e^t \\ t\,e^t \\ t\,e^t + e^t \\ \frac{t^2 e^t}{2} \end{bmatrix}$$

$$\vec{x}_4 = \begin{bmatrix} 0 \\ 0 \\ e^t \\ 0 \end{bmatrix}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t) + c_4\vec{x}_4(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ 0 \\ 0 \\ e^t \end{bmatrix} + c_2 \begin{bmatrix} -2\,e^t \\ e^t \\ e^t \\ t\,e^t \end{bmatrix} + c_3 \begin{bmatrix} -2t\,e^t + e^t \\ t\,e^t \\ t\,e^t + e^t \\ \frac{t^2 e^t}{2} \end{bmatrix} + c_4 \begin{bmatrix} 0 \\ 0 \\ e^t \\ 0 \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = \begin{bmatrix} e^t(-2c_3t - 2c_2 + c_3) \\ e^t(c_3t + c_2) \\ e^t(c_3t + c_2 + c_3 + c_4) \\ e^t\left(c_1 + tc_2 + \frac{1}{2}t^2c_3\right) \end{bmatrix}$$

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 63

```
dsolve([diff(x__1(t),t)=-1*x__1(t)-4*x__2(t)+0*x__3(t)+0*x__4(t),diff(x__2(t),t)=1*x__1(t)+3
```

$$x_1(t) = -e^t(2c_4t + 2c_3 - c_4)$$
$$x_2(t) = e^t(c_4t + c_3)$$
$$x_3(t) = e^t(c_4t + c_1 + c_3)$$
$$x_4(t) = \frac{(c_4t^2 + 2c_3t + 2c_2)e^t}{2}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 91

```
DSolve[{x1'[t]==-1*x1[t]-4*x2[t]+0*x3[t]+0*x4[t],x2'[t]==1*x1[t]+3*x2[t]+0*x3[t]+0*x4[t],x3'
```

$$\text{x1}(t) \to -e^t(c_1(2t - 1) + 4c_2t)$$
$$\text{x2}(t) \to e^t((c_1 + 2c_2)t + c_2)$$
$$\text{x3}(t) \to e^t((c_1 + 2c_2)t + c_3)$$
$$\text{x4}(t) \to \frac{1}{2}e^t\left(c_1t^2 + 2c_2(t + 1)t + 2c_4\right)$$

6.22 problem problem 22

Internal problem ID [379]
Internal file name [`OUTPUT/379_Sunday_June_05_2022_01_40_04_AM_84937060/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.6, Multiple Eigenvalue Solutions. Page 451
Problem number: problem 22.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= x_1(t) + 3x_2(t) + 7x_3(t) \\
x_2'(t) &= -x_2(t) - 4x_3(t) \\
x_3'(t) &= x_2(t) + 3x_3(t) \\
x_4'(t) &= -6x_2(t) - 14x_3(t) + x_4(t)
\end{aligned}
$$

6.22.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \\ x_4'(t) \end{bmatrix} = \begin{bmatrix} 1 & 3 & 7 & 0 \\ 0 & -1 & -4 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & -6 & -14 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^t & \frac{e^t t(t+6)}{2} & e^t t(t+7) & 0 \\ 0 & e^t(1-2t) & -4t\,e^t & 0 \\ 0 & t\,e^t & e^t(1+2t) & 0 \\ 0 & -e^t t(t+6) & -2\,e^t t(t+7) & e^t \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} e^t & \frac{e^t t(t+6)}{2} & e^t t(t+7) & 0 \\ 0 & e^t(1-2t) & -4t\,e^t & 0 \\ 0 & t\,e^t & e^t(1+2t) & 0 \\ 0 & -e^t t(t+6) & -2\,e^t t(t+7) & e^t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} \\ &= \begin{bmatrix} e^t c_1 + \frac{e^t t(t+6)c_2}{2} + e^t t(t+7)\,c_3 \\ e^t(1-2t)\,c_2 - 4t\,e^t c_3 \\ t\,e^t c_2 + e^t(1+2t)\,c_3 \\ -e^t t(t+6)\,c_2 - 2\,e^t t(t+7)\,c_3 + e^t c_4 \end{bmatrix} \\ &= \begin{bmatrix} \frac{((c_2+2c_3)t^2+(6c_2+14c_3)t+2c_1)e^t}{2} \\ (c_2(1-2t) - 4tc_3)\,e^t \\ e^t(tc_2 + 2tc_3 + c_3) \\ -((c_2+2c_3)\,t^2 + (6c_2+14c_3)\,t - c_4)\,e^t \end{bmatrix} \end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

6.22.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \\ x_4'(t) \end{bmatrix} = \begin{bmatrix} 1 & 3 & 7 & 0 \\ 0 & -1 & -4 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & -6 & -14 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & 3 & 7 & 0 \\ 0 & -1 & -4 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & -6 & -14 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & 3 & 7 & 0 \\ 0 & -1-\lambda & -4 & 0 \\ 0 & 1 & 3-\lambda & 0 \\ 0 & -6 & -14 & 1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^4 - 4\lambda^3 + 6\lambda^2 - 4\lambda + 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 3 & 7 & 0 \\ 0 & -1 & -4 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & -6 & -14 & 1 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 3 & 7 & 0 \\ 0 & -2 & -4 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -6 & -14 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} 0 & 3 & 7 & 0 & 0 \\ 0 & -2 & -4 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & -6 & -14 & 0 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{2R_1}{3} \Longrightarrow \left[\begin{array}{cccc|c} 0 & 3 & 7 & 0 & 0 \\ 0 & 0 & \frac{2}{3} & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & -6 & -14 & 0 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{R_1}{3} \Longrightarrow \left[\begin{array}{cccc|c} 0 & 3 & 7 & 0 & 0 \\ 0 & 0 & \frac{2}{3} & 0 & 0 \\ 0 & 0 & -\frac{1}{3} & 0 & 0 \\ 0 & -6 & -14 & 0 & 0 \end{array}\right]$$

$$R_4 = R_4 + 2R_1 \Longrightarrow \left[\begin{array}{cccc|c} 0 & 3 & 7 & 0 & 0 \\ 0 & 0 & \frac{2}{3} & 0 & 0 \\ 0 & 0 & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{R_2}{2} \Longrightarrow \left[\begin{array}{cccc|c} 0 & 3 & 7 & 0 & 0 \\ 0 & 0 & \frac{2}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 0 & 3 & 7 & 0 \\ 0 & 0 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_1, v_4\}$ and the leading variables are $\{v_2, v_3\}$. Let $v_1 = t$. Let $v_4 = s$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_2 = 0, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ 0 \\ 0 \\ s \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \\ s \end{bmatrix}$$

Since there are two free Variable, we have found two eigenvectors associated with this eigenvalue. The above can be written as

$$\begin{aligned} \begin{bmatrix} t \\ 0 \\ 0 \\ s \end{bmatrix} &= \begin{bmatrix} t \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ s \end{bmatrix} \\ &= t \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

By letting $t = 1$ and $s = 1$ then the above becomes

$$\begin{bmatrix} t \\ 0 \\ 0 \\ s \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Hence the two eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
1	4	2	Yes	$\begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$

This case will be solved using the Jordan form of the matrix A. The Jordan form diagonalization is

$$A = PJP^{-1}$$

Which can be found to be

$$\begin{bmatrix} 1 & 3 & 7 & 0 \\ 0 & -1 & -4 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & -6 & -14 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & -6 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & -6 & 1 & 1 \end{bmatrix}^{-1}$$

Looking at the P matrix above, we see there are 2 chains. Therefore, we now construct the basis solution by following these chains as follows.

$$\vec{x}_1 = \begin{bmatrix} e^t \\ 0 \\ 0 \\ -2\,e^t \end{bmatrix}$$

$$\vec{x}_2 = \begin{bmatrix} t\,e^t + 3\,e^t \\ -2\,e^t \\ e^t \\ -2t\,e^t - 6\,e^t \end{bmatrix}$$

$$\vec{x}_3 = \begin{bmatrix} \frac{t^2 e^t}{2} + 3t\,e^t \\ -2t\,e^t + e^t \\ t\,e^t \\ -t^2 e^t - 6t\,e^t + e^t \end{bmatrix}$$

$$\vec{x}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ e^t \end{bmatrix}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t) + c_4\vec{x}_4(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = c_1 \begin{bmatrix} e^t \\ 0 \\ 0 \\ -2\,e^t \end{bmatrix} + c_2 \begin{bmatrix} t\,e^t + 3\,e^t \\ -2\,e^t \\ e^t \\ -2t\,e^t - 6\,e^t \end{bmatrix} + c_3 \begin{bmatrix} \frac{t^2 e^t}{2} + 3t\,e^t \\ -2t\,e^t + e^t \\ t\,e^t \\ -t^2 e^t - 6t\,e^t + e^t \end{bmatrix} + c_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ e^t \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = \begin{bmatrix} \frac{(c_3t^2+(2c_2+6c_3)t+2c_1+6c_2)\,\mathrm{e}^t}{2} \\ \mathrm{e}^t(-2c_3t-2c_2+c_3) \\ \mathrm{e}^t(c_3t+c_2) \\ -\mathrm{e}^t((t^2+6t-1)\,c_3+2tc_2+2c_1+6c_2-c_4) \end{bmatrix}$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 78

```
dsolve([diff(x__1(t),t)=1*x__1(t)+3*x__2(t)+7*x__3(t)+0*x__4(t),diff(x__2(t),t)=0*x__1(t)-1*
```

$$x_1(t) = \frac{(-c_4t^2-2c_3t-7c_4t+4c_2)\,\mathrm{e}^t}{4}$$
$$x_2(t) = \mathrm{e}^t(c_4t+c_3)$$
$$x_3(t) = -\frac{\mathrm{e}^t(2c_4t+2c_3+c_4)}{4}$$
$$x_4(t) = \frac{(c_4t^2+2c_3t+7c_4t+2c_1)\,\mathrm{e}^t}{2}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 99

```
DSolve[{x1'[t]==1*x1[t]+3*x2[t]+7*x3[t]+0*x4[t],x2'[t]==0*x1[t]-1*x2[t]-4*x3[t]+0*x4[t],x3'[
```

$$\text{x1}(t) \to \frac{1}{2}e^t(c_2t(t+6)+2c_3t(t+7)+2c_1)$$
$$\text{x2}(t) \to -e^t(c_2(2t-1)+4c_3t)$$
$$\text{x3}(t) \to e^t((c_2+2c_3)t+c_3)$$
$$\text{x4}(t) \to e^t(c_2(-t)(t+6)-2c_3t(t+7)+c_4)$$

6.23 problem problem 23

Internal problem ID [380]
Internal file name [`OUTPUT/380_Sunday_June_05_2022_01_40_06_AM_22289725/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.6, Multiple Eigenvalue Solutions. Page 451
Problem number: problem 23.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= 39x_1(t) + 8x_2(t) - 16x_3(t) \\
x_2'(t) &= -36x_1(t) - 5x_2(t) + 16x_3(t) \\
x_3'(t) &= 72x_1(t) + 16x_2(t) - 29x_3(t)
\end{aligned}
$$

6.23.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 39 & 8 & -16 \\ -36 & -5 & 16 \\ 72 & 16 & -29 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} 10\,\mathrm{e}^{3t} - 9\,\mathrm{e}^{-t} & 2\,\mathrm{e}^{3t} - 2\,\mathrm{e}^{-t} & -4\,\mathrm{e}^{3t} + 4\,\mathrm{e}^{-t} \\ -9\,\mathrm{e}^{3t} + 9\,\mathrm{e}^{-t} & -\mathrm{e}^{3t} + 2\,\mathrm{e}^{-t} & 4\,\mathrm{e}^{3t} - 4\,\mathrm{e}^{-t} \\ 18\,\mathrm{e}^{3t} - 18\,\mathrm{e}^{-t} & 4\,\mathrm{e}^{3t} - 4\,\mathrm{e}^{-t} & -7\,\mathrm{e}^{3t} + 8\,\mathrm{e}^{-t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} 10\,\mathrm{e}^{3t}-9\,\mathrm{e}^{-t} & 2\,\mathrm{e}^{3t}-2\,\mathrm{e}^{-t} & -4\,\mathrm{e}^{3t}+4\,\mathrm{e}^{-t} \\ -9\,\mathrm{e}^{3t}+9\,\mathrm{e}^{-t} & -\mathrm{e}^{3t}+2\,\mathrm{e}^{-t} & 4\,\mathrm{e}^{3t}-4\,\mathrm{e}^{-t} \\ 18\,\mathrm{e}^{3t}-18\,\mathrm{e}^{-t} & 4\,\mathrm{e}^{3t}-4\,\mathrm{e}^{-t} & -7\,\mathrm{e}^{3t}+8\,\mathrm{e}^{-t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} (10\,\mathrm{e}^{3t}-9\,\mathrm{e}^{-t})\,c_1+(2\,\mathrm{e}^{3t}-2\,\mathrm{e}^{-t})\,c_2+(-4\,\mathrm{e}^{3t}+4\,\mathrm{e}^{-t})\,c_3 \\ (-9\,\mathrm{e}^{3t}+9\,\mathrm{e}^{-t})\,c_1+(-\mathrm{e}^{3t}+2\,\mathrm{e}^{-t})\,c_2+(4\,\mathrm{e}^{3t}-4\,\mathrm{e}^{-t})\,c_3 \\ (18\,\mathrm{e}^{3t}-18\,\mathrm{e}^{-t})\,c_1+(4\,\mathrm{e}^{3t}-4\,\mathrm{e}^{-t})\,c_2+(-7\,\mathrm{e}^{3t}+8\,\mathrm{e}^{-t})\,c_3 \end{bmatrix} \\
&= \begin{bmatrix} (-9c_1-2c_2+4c_3)\,\mathrm{e}^{-t}+10\left(c_1+\frac{c_2}{5}-\frac{2c_3}{5}\right)\mathrm{e}^{3t} \\ (9c_1+2c_2-4c_3)\,\mathrm{e}^{-t}-9\left(c_1+\frac{c_2}{9}-\frac{4c_3}{9}\right)\mathrm{e}^{3t} \\ (-18c_1-4c_2+8c_3)\,\mathrm{e}^{-t}+18\left(c_1+\frac{2c_2}{9}-\frac{7c_3}{18}\right)\mathrm{e}^{3t} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

6.23.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 39 & 8 & -16 \\ -36 & -5 & 16 \\ 72 & 16 & -29 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det(A-\lambda I) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} 39 & 8 & -16 \\ -36 & -5 & 16 \\ 72 & 16 & -29 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$\det\left(\begin{bmatrix} 39-\lambda & 8 & -16 \\ -36 & -5-\lambda & 16 \\ 72 & 16 & -29-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 5\lambda^2 + 3\lambda + 9 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1$$
$$\lambda_2 = 3$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 39 & 8 & -16 \\ -36 & -5 & 16 \\ 72 & 16 & -29 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 40 & 8 & -16 \\ -36 & -4 & 16 \\ 72 & 16 & -28 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 40 & 8 & -16 & 0 \\ -36 & -4 & 16 & 0 \\ 72 & 16 & -28 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{9R_1}{10} \Longrightarrow \left[\begin{array}{ccc|c} 40 & 8 & -16 & 0 \\ 0 & \frac{16}{5} & \frac{8}{5} & 0 \\ 72 & 16 & -28 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{9R_1}{5} \Longrightarrow \left[\begin{array}{ccc|c} 40 & 8 & -16 & 0 \\ 0 & \frac{16}{5} & \frac{8}{5} & 0 \\ 0 & \frac{8}{5} & \frac{4}{5} & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{R_2}{2} \Longrightarrow \left[\begin{array}{ccc|c} 40 & 8 & -16 & 0 \\ 0 & \frac{16}{5} & \frac{8}{5} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 40 & 8 & -16 \\ 0 & \frac{16}{5} & \frac{8}{5} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{2}, v_2 = -\frac{t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{2} \\ -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{2} \\ -\frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{2} \\ -\frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

Let $t=1$ the eigenvector becomes

$$\begin{bmatrix} \frac{t}{2} \\ -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{t}{2} \\ -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 39 & 8 & -16 \\ -36 & -5 & 16 \\ 72 & 16 & -29 \end{bmatrix} - (3) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 36 & 8 & -16 \\ -36 & -8 & 16 \\ 72 & 16 & -32 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 36 & 8 & -16 & 0 \\ -36 & -8 & 16 & 0 \\ 72 & 16 & -32 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \implies \left[\begin{array}{ccc|c} 36 & 8 & -16 & 0 \\ 0 & 0 & 0 & 0 \\ 72 & 16 & -32 & 0 \end{array}\right]$$

$$R_3 = R_3 - 2R_1 \implies \left[\begin{array}{ccc|c} 36 & 8 & -16 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 36 & 8 & -16 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2, v_3\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Let $v_3 = s$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -\frac{2t}{9} + \frac{4s}{9}\}$

Hence the solution is

$$\begin{bmatrix} -\frac{2t}{9} + \frac{4s}{9} \\ t \\ s \end{bmatrix} = \begin{bmatrix} -\frac{2t}{9} + \frac{4s}{9} \\ t \\ s \end{bmatrix}$$

Since there are two free Variable, we have found two eigenvectors associated with this eigenvalue. The above can be written as

$$\begin{aligned} \begin{bmatrix} -\frac{2t}{9} + \frac{4s}{9} \\ t \\ s \end{bmatrix} &= \begin{bmatrix} -\frac{2t}{9} \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{4s}{9} \\ 0 \\ s \end{bmatrix} \\ &= t \begin{bmatrix} -\frac{2}{9} \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} \frac{4}{9} \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

By letting $t = 1$ and $s = 1$ then the above becomes

$$\begin{bmatrix} -\frac{2t}{9} + \frac{4s}{9} \\ t \\ s \end{bmatrix} = \begin{bmatrix} -\frac{2}{9} \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{4}{9} \\ 0 \\ 1 \end{bmatrix}$$

Hence the two eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix} -\frac{2}{9} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{4}{9} \\ 0 \\ 1 \end{bmatrix} \right)$$

Which are normalized to

$$\left(\begin{bmatrix} -2 \\ 9 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 9 \end{bmatrix} \right)$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-1	1	1	No	$\begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$
3	2	2	No	$\begin{bmatrix} \frac{4}{9} & -\frac{2}{9} \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{-t} \\ &= \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix} e^{-t} \end{aligned}$$

eigenvalue 3 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

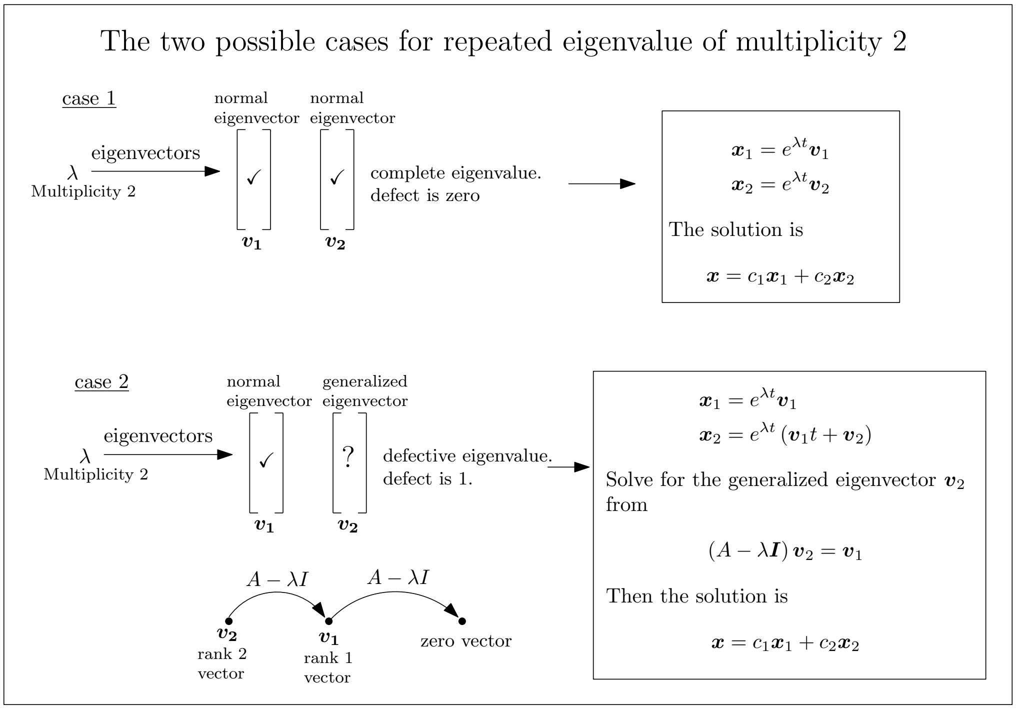


Figure 58: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2 which is the same as its geometric multiplicity 2, then it is complete eigenvalue and this falls into case 1 shown above. Hence the corresponding eigenvector basis are

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{3t}\\ &= \begin{bmatrix}\frac{4}{9}\\ 0\\ 1\end{bmatrix} e^{3t}\end{aligned}$$

$$\begin{aligned}\vec{x}_3(t) &= \vec{v}_3 e^{3t}\\ &= \begin{bmatrix}-\frac{2}{9}\\ 1\\ 0\end{bmatrix} e^{3t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 \begin{bmatrix} \frac{e^{-t}}{2} \\ -\frac{e^{-t}}{2} \\ e^{-t} \end{bmatrix} + c_2 \begin{bmatrix} \frac{4\,e^{3t}}{9} \\ 0 \\ e^{3t} \end{bmatrix} + c_3 \begin{bmatrix} -\frac{2\,e^{3t}}{9} \\ e^{3t} \\ 0 \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} \frac{2(2c_2-c_3)e^{3t}}{9} + \frac{c_1 e^{-t}}{2} \\ -\frac{c_1 e^{-t}}{2} + c_3 e^{3t} \\ c_1 e^{-t} + c_2 e^{3t} \end{bmatrix}$$

6.23.3 Maple step by step solution

Let's solve

$[x_1'(t) = 39x_1(t) + 8x_2(t) - 16x_3(t)\,, x_2'(t) = -36x_1(t) - 5x_2(t) + 16x_3(t)\,, x_3'(t) = 72x_1(t) + 16x_2($

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x}^{\rightarrow}{}'(t) = \begin{bmatrix} 39 & 8 & -16 \\ -36 & -5 & 16 \\ 72 & 16 & -29 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{x}^{\rightarrow}{}'(t) = \begin{bmatrix} 39 & 8 & -16 \\ -36 & -5 & 16 \\ 72 & 16 & -29 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 39 & 8 & -16 \\ -36 & -5 & 16 \\ 72 & 16 & -29 \end{bmatrix}$$

- Rewrite the system as

  $\underline{x}^{\rightarrow}{}'(t) = A \cdot \underline{x}^{\rightarrow}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[-1, \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right], \left[3, \begin{bmatrix} \frac{4}{9} \\ 0 \\ 1 \end{bmatrix}\right], \left[3, \begin{bmatrix} -\frac{2}{9} \\ 1 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair

  $$\left[-1, \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

  $$\underline{x}^{\rightarrow}{}_1 = e^{-t} \cdot \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

  $$\left[3, \begin{bmatrix} \frac{4}{9} \\ 0 \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue 3

  $$\underline{x}^{\rightarrow}{}_2(t) = e^{3t} \cdot \begin{bmatrix} \frac{4}{9} \\ 0 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = 3$ is the eigenvalue, and

$\overrightarrow{x\_\_}_3(t) = e^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right)$

- Note that the t multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\overrightarrow{x\_\_}_3(t)$ into the homogeneous system

  $\lambda e^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right) + e^{\lambda t}\overrightarrow{v} = \left(e^{\lambda t}A\right) \cdot \left(t\overrightarrow{v} + \overrightarrow{p}\right)$
- Use the fact that $\overrightarrow{v}$ is an eigenvector of A

  $\lambda e^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right) + e^{\lambda t}\overrightarrow{v} = e^{\lambda t}\left(\lambda t\overrightarrow{v} + A \cdot \overrightarrow{p}\right)$
- Simplify equation

  $\lambda\overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$
- Make use of the identity matrix I

  $(\lambda \cdot I) \cdot \overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$
- Condition $\overrightarrow{p}$ must meet for $\overrightarrow{x\_\_}_3(t)$ to be a solution to the homogeneous system

  $(A - \lambda \cdot I) \cdot \overrightarrow{p} = \overrightarrow{v}$
- Choose $\overrightarrow{p}$ to use in the second solution to the homogeneous system from eigenvalue 3

  $$\left(\begin{bmatrix} 39 & 8 & -16 \\ -36 & -5 & 16 \\ 72 & 16 & -29 \end{bmatrix} - 3 \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \cdot \overrightarrow{p} = \begin{bmatrix} \frac{4}{9} \\ 0 \\ 1 \end{bmatrix}$$
- Choice of $\overrightarrow{p}$

  $$\overrightarrow{p} = \begin{bmatrix} \frac{1}{81} \\ 0 \\ 0 \end{bmatrix}$$
- Second solution from eigenvalue 3

  $$\overrightarrow{x\_\_}_3(t) = e^{3t} \cdot \left(t \cdot \begin{bmatrix} \frac{4}{9} \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{81} \\ 0 \\ 0 \end{bmatrix}\right)$$
- General solution to the system of ODEs

  $\overrightarrow{x\_\_} = c_1\overrightarrow{x\_\_}_1 + c_2\overrightarrow{x\_\_}_2(t) + c_3\overrightarrow{x\_\_}_3(t)$
- Substitute solutions into the general solution

$$\underline{\overrightarrow{x}} = c_1 e^{-t} \cdot \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix} + c_2 e^{3t} \cdot \begin{bmatrix} \frac{4}{9} \\ 0 \\ 1 \end{bmatrix} + c_3 e^{3t} \cdot \left(t \cdot \begin{bmatrix} \frac{4}{9} \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{81} \\ 0 \\ 0 \end{bmatrix} \right)$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} \frac{((36t+1)c_3+36c_2)e^{3t}}{81} + \frac{c_1 e^{-t}}{2} \\ -\frac{c_1 e^{-t}}{2} \\ (c_3 t + c_2) e^{3t} + c_1 e^{-t} \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x_1(t) = \frac{((36t+1)c_3+36c_2)e^{3t}}{81} + \frac{c_1 e^{-t}}{2}, x_2(t) = -\frac{c_1 e^{-t}}{2}, x_3(t) = (c_3 t + c_2) e^{3t} + c_1 e^{-t}\right\}$$

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 67

```
dsolve([diff(x__1(t),t)=39*x__1(t)+8*x__2(t)-16*x__3(t),diff(x__2(t),t)=-36*x__1(t)-5*x__2(t
```

$$x_1(t) = c_2 e^{3t} + c_3 e^{-t}$$
$$x_2(t) = -c_2 e^{3t} - c_3 e^{-t} + c_1 e^{3t}$$
$$x_3(t) = \frac{7c_2 e^{3t}}{4} + 2c_3 e^{-t} + \frac{c_1 e^{3t}}{2}$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 127

```
DSolve[{x1'[t]==39*x1[t]+8*x2[t]-16*x3[t],x2'[t]==-36*x1[t]-5*x2[t]+16*x3[t],x3'[t]==72*x1[t
```

$$\text{x1}(t) \to e^{-t}\left(c_1\left(10e^{4t} - 9\right) + 2(c_2 - 2c_3)\left(e^{4t} - 1\right)\right)$$
$$\text{x2}(t) \to e^{-t}\left(-9c_1\left(e^{4t} - 1\right) - c_2\left(e^{4t} - 2\right) + 4c_3\left(e^{4t} - 1\right)\right)$$
$$\text{x3}(t) \to e^{-t}\left(18c_1\left(e^{4t} - 1\right) + 4c_2\left(e^{4t} - 1\right) + c_3\left(8 - 7e^{4t}\right)\right)$$

6.24 problem problem 24

Internal problem ID [381]
Internal file name [`OUTPUT/381_Sunday_June_05_2022_01_40_07_AM_40772406/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.6, Multiple Eigenvalue Solutions. Page 451
Problem number: problem 24.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= 28x_1(t) + 50x_2(t) + 100x_3(t) \\
x_2'(t) &= 15x_1(t) + 33x_2(t) + 60x_3(t) \\
x_3'(t) &= -15x_1(t) - 30x_2(t) - 57x_3(t)
\end{aligned}
$$

6.24.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 28 & 50 & 100 \\ 15 & 33 & 60 \\ -15 & -30 & -57 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} (6\,e^{5t}-5)\,e^{-2t} & 10(e^{5t}-1)\,e^{-2t} & 20(e^{5t}-1)\,e^{-2t} \\ 3(e^{5t}-1)\,e^{-2t} & (7\,e^{5t}-6)\,e^{-2t} & 12(e^{5t}-1)\,e^{-2t} \\ -3(e^{5t}-1)\,e^{-2t} & -6(e^{5t}-1)\,e^{-2t} & (-11\,e^{5t}+12)\,e^{-2t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} (6\,\mathrm{e}^{5t}-5)\,\mathrm{e}^{-2t} & 10(\mathrm{e}^{5t}-1)\,\mathrm{e}^{-2t} & 20(\mathrm{e}^{5t}-1)\,\mathrm{e}^{-2t} \\ 3(\mathrm{e}^{5t}-1)\,\mathrm{e}^{-2t} & (7\,\mathrm{e}^{5t}-6)\,\mathrm{e}^{-2t} & 12(\mathrm{e}^{5t}-1)\,\mathrm{e}^{-2t} \\ -3(\mathrm{e}^{5t}-1)\,\mathrm{e}^{-2t} & -6(\mathrm{e}^{5t}-1)\,\mathrm{e}^{-2t} & (-11\,\mathrm{e}^{5t}+12)\,\mathrm{e}^{-2t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} (6\,\mathrm{e}^{5t}-5)\,\mathrm{e}^{-2t}c_1 + 10(\mathrm{e}^{5t}-1)\,\mathrm{e}^{-2t}c_2 + 20(\mathrm{e}^{5t}-1)\,\mathrm{e}^{-2t}c_3 \\ 3(\mathrm{e}^{5t}-1)\,\mathrm{e}^{-2t}c_1 + (7\,\mathrm{e}^{5t}-6)\,\mathrm{e}^{-2t}c_2 + 12(\mathrm{e}^{5t}-1)\,\mathrm{e}^{-2t}c_3 \\ -3(\mathrm{e}^{5t}-1)\,\mathrm{e}^{-2t}c_1 - 6(\mathrm{e}^{5t}-1)\,\mathrm{e}^{-2t}c_2 + (-11\,\mathrm{e}^{5t}+12)\,\mathrm{e}^{-2t}c_3 \end{bmatrix} \\
&= \begin{bmatrix} 6\left(\left(c_1+\frac{5c_2}{3}+\frac{10c_3}{3}\right)\mathrm{e}^{5t}-\frac{5c_1}{6}-\frac{5c_2}{3}-\frac{10c_3}{3}\right)\mathrm{e}^{-2t} \\ 3\left(\left(c_1+\frac{7c_2}{3}+4c_3\right)\mathrm{e}^{5t}-c_1-2c_2-4c_3\right)\mathrm{e}^{-2t} \\ -3\,\mathrm{e}^{-2t}\left(\left(c_1+2c_2+\frac{11c_3}{3}\right)\mathrm{e}^{5t}-c_1-2c_2-4c_3\right) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

6.24.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 28 & 50 & 100 \\ 15 & 33 & 60 \\ -15 & -30 & -57 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 28 & 50 & 100 \\ 15 & 33 & 60 \\ -15 & -30 & -57 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 28-\lambda & 50 & 100 \\ 15 & 33-\lambda & 60 \\ -15 & -30 & -57-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 4\lambda^2 - 3\lambda + 18 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -2$$
$$\lambda_2 = 3$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-2	1	real eigenvalue
3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 28 & 50 & 100 \\ 15 & 33 & 60 \\ -15 & -30 & -57 \end{bmatrix} - (-2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 30 & 50 & 100 \\ 15 & 35 & 60 \\ -15 & -30 & -55 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 30 & 50 & 100 & 0 \\ 15 & 35 & 60 & 0 \\ -15 & -30 & -55 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} 30 & 50 & 100 & 0 \\ 0 & 10 & 10 & 0 \\ -15 & -30 & -55 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} 30 & 50 & 100 & 0 \\ 0 & 10 & 10 & 0 \\ 0 & -5 & -5 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{R_2}{2} \Longrightarrow \left[\begin{array}{ccc|c} 30 & 50 & 100 & 0 \\ 0 & 10 & 10 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 30 & 50 & 100 \\ 0 & 10 & 10 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{5t}{3}, v_2 = -t\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{5t}{3} \\ -t \\ t \end{bmatrix} = \begin{bmatrix} -\frac{5t}{3} \\ -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{5t}{3} \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{5}{3} \\ -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{5t}{3} \\ -t \\ t \end{bmatrix} = \begin{bmatrix} -\frac{5}{3} \\ -1 \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{5t}{3} \\ -t \\ t \end{bmatrix} = \begin{bmatrix} -5 \\ -3 \\ 3 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 28 & 50 & 100 \\ 15 & 33 & 60 \\ -15 & -30 & -57 \end{bmatrix} - (3) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 25 & 50 & 100 \\ 15 & 30 & 60 \\ -15 & -30 & -60 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 25 & 50 & 100 & 0 \\ 15 & 30 & 60 & 0 \\ -15 & -30 & -60 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{3R_1}{5} \implies \left[\begin{array}{ccc|c} 25 & 50 & 100 & 0 \\ 0 & 0 & 0 & 0 \\ -15 & -30 & -60 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{3R_1}{5} \implies \left[\begin{array}{ccc|c} 25 & 50 & 100 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 25 & 50 & 100 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2, v_3\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Let $v_3 = s$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -2t - 4s\}$

Hence the solution is

$$\begin{bmatrix} -2t - 4s \\ t \\ s \end{bmatrix} = \begin{bmatrix} -2t - 4s \\ t \\ s \end{bmatrix}$$

Since there are two free Variable, we have found two eigenvectors associated with this eigenvalue. The above can be written as

$$\begin{aligned} \begin{bmatrix} -2t - 4s \\ t \\ s \end{bmatrix} &= \begin{bmatrix} -2t \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} -4s \\ 0 \\ s \end{bmatrix} \\ &= t \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

By letting $t = 1$ and $s = 1$ then the above becomes

$$\begin{bmatrix} -2t - 4s \\ t \\ s \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$$

Hence the two eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix} \right)$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
-2	1	1	No	$\begin{bmatrix} -\frac{5}{3} \\ -1 \\ 1 \end{bmatrix}$
3	2	2	No	$\begin{bmatrix} -4 & -2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -2 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{-2t} \\ &= \begin{bmatrix} -\frac{5}{3} \\ -1 \\ 1 \end{bmatrix} e^{-2t} \end{aligned}$$

eigenvalue 3 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

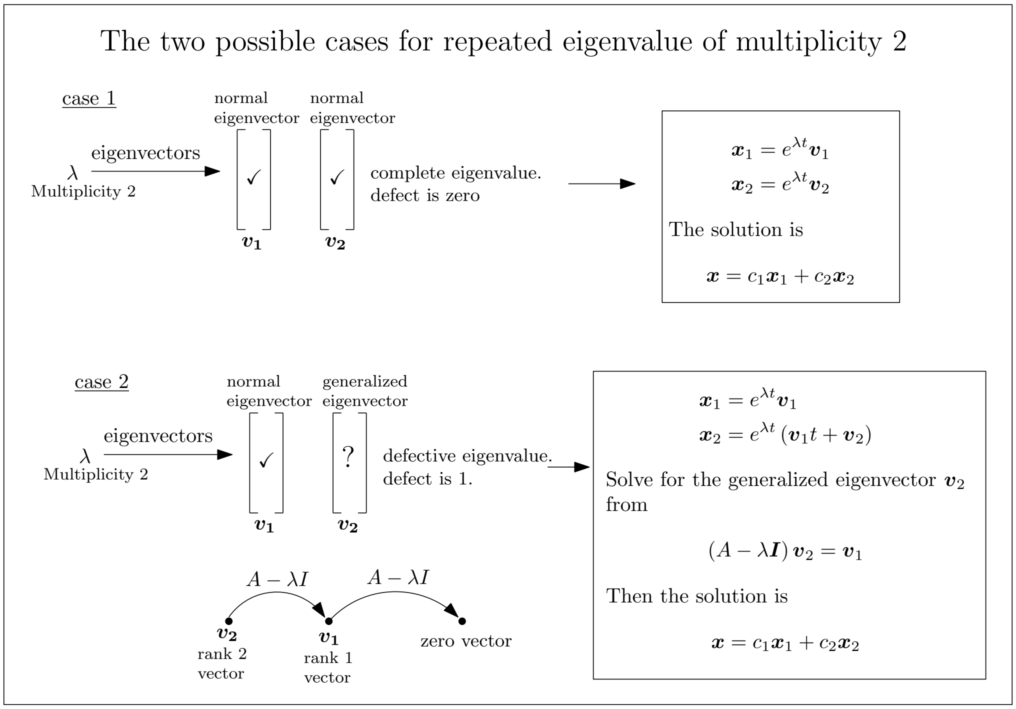


Figure 59: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2 which is the same as its geometric multiplicity 2, then it is complete eigenvalue and this falls into case 1 shown above. Hence the corresponding eigenvector basis are

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{3t}\\ &= \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix} e^{3t}\end{aligned}$$

$$\begin{aligned}\vec{x}_3(t) &= \vec{v}_3 e^{3t}\\ &= \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} e^{3t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 \begin{bmatrix} -\frac{5\,e^{-2t}}{3} \\ -e^{-2t} \\ e^{-2t} \end{bmatrix} + c_2 \begin{bmatrix} -4\,e^{3t} \\ 0 \\ e^{3t} \end{bmatrix} + c_3 \begin{bmatrix} -2\,e^{3t} \\ e^{3t} \\ 0 \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} -\frac{5\left(\frac{6(2c_2+c_3)e^{5t}}{5}+c_1\right)e^{-2t}}{3} \\ -(-c_3e^{5t}+c_1)\,e^{-2t} \\ (c_2e^{5t}+c_1)\,e^{-2t} \end{bmatrix}$$

6.24.3 Maple step by step solution

Let's solve

$[x_1'(t) = 28x_1(t) + 50x_2(t) + 100x_3(t)\,, x_2'(t) = 15x_1(t) + 33x_2(t) + 60x_3(t)\,, x_3'(t) = -15x_1(t) - 30$

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x}^{\rightarrow\prime}(t) = \begin{bmatrix} 28 & 50 & 100 \\ 15 & 33 & 60 \\ -15 & -30 & -57 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{x}^{\rightarrow\prime}(t) = \begin{bmatrix} 28 & 50 & 100 \\ 15 & 33 & 60 \\ -15 & -30 & -57 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 28 & 50 & 100 \\ 15 & 33 & 60 \\ -15 & -30 & -57 \end{bmatrix}$$

- Rewrite the system as

  $\underline{\vec{x}}'(t) = A \cdot \underline{\vec{x}}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[-2, \begin{bmatrix} -\frac{5}{3} \\ -1 \\ 1 \end{bmatrix}\right], \left[3, \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}\right], \left[3, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair

  $$\left[-2, \begin{bmatrix} -\frac{5}{3} \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

  $$\underline{\vec{x}}_1 = e^{-2t} \cdot \begin{bmatrix} -\frac{5}{3} \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

  $$\left[3, \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue 3

  $$\underline{\vec{x}}_2(t) = e^{3t} \cdot \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = 3$ is the eigenvalue, and

$\overrightarrow{x}_3(t) = e^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right)$

- Note that the t multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\overrightarrow{x}_3(t)$ into the homogeneous system

  $\lambda e^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right) + e^{\lambda t}\overrightarrow{v} = \left(e^{\lambda t}A\right)\cdot\left(t\overrightarrow{v} + \overrightarrow{p}\right)$
- Use the fact that $\overrightarrow{v}$ is an eigenvector of A

  $\lambda e^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right) + e^{\lambda t}\overrightarrow{v} = e^{\lambda t}\left(\lambda t\overrightarrow{v} + A\cdot\overrightarrow{p}\right)$
- Simplify equation

  $\lambda\overrightarrow{p} + \overrightarrow{v} = A\cdot\overrightarrow{p}$
- Make use of the identity matrix I

  $(\lambda\cdot I)\cdot\overrightarrow{p} + \overrightarrow{v} = A\cdot\overrightarrow{p}$
- Condition $\overrightarrow{p}$ must meet for $\overrightarrow{x}_3(t)$ to be a solution to the homogeneous system

  $(A - \lambda\cdot I)\cdot\overrightarrow{p} = \overrightarrow{v}$
- Choose $\overrightarrow{p}$ to use in the second solution to the homogeneous system from eigenvalue 3

  $$\left(\begin{bmatrix} 28 & 50 & 100 \\ 15 & 33 & 60 \\ -15 & -30 & -57 \end{bmatrix} - 3\cdot\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\cdot\overrightarrow{p} = \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$$
- Choice of $\overrightarrow{p}$

  $$\overrightarrow{p} = \begin{bmatrix} -\frac{4}{25} \\ 0 \\ 0 \end{bmatrix}$$
- Second solution from eigenvalue 3

  $$\overrightarrow{x}_3(t) = e^{3t}\cdot\left(t\cdot\begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{4}{25} \\ 0 \\ 0 \end{bmatrix}\right)$$
- General solution to the system of ODEs

  $\overrightarrow{x} = c_1\overrightarrow{x}_1 + c_2\overrightarrow{x}_2(t) + c_3\overrightarrow{x}_3(t)$
- Substitute solutions into the general solution

$$\underline{\overrightarrow{x}} = c_1 e^{-2t} \cdot \begin{bmatrix} -\frac{5}{3} \\ -1 \\ 1 \end{bmatrix} + c_2 e^{3t} \cdot \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix} + c_3 e^{3t} \cdot \left(t \cdot \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{4}{25} \\ 0 \\ 0 \end{bmatrix} \right)$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} -4\left(\left(\left(t + \frac{1}{25}\right) c_3 + c_2\right) e^{5t} + \frac{5c_1}{12}\right) e^{-2t} \\ -c_1 e^{-2t} \\ e^{-2t}\left((c_3 t + c_2) e^{5t} + c_1\right) \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x_1(t) = -4\left(\left(\left(t + \frac{1}{25}\right) c_3 + c_2\right) e^{5t} + \frac{5c_1}{12}\right) e^{-2t}, x_2(t) = -c_1 e^{-2t}, x_3(t) = e^{-2t}\left((c_3 t + c_2) e^{5t} + c_1\right)\right\}$$

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 67

```
dsolve([diff(x__1(t),t)=28*x__1(t)+50*x__2(t)+100*x__3(t),diff(x__2(t),t)=15*x__1(t)+33*x__2
```

$$x_1(t) = c_2 e^{3t} + c_3 e^{-2t}$$
$$x_2(t) = \frac{3c_2 e^{3t}}{5} + \frac{3c_3 e^{-2t}}{5} + c_1 e^{3t}$$
$$x_3(t) = -\frac{11c_2 e^{3t}}{20} - \frac{3c_3 e^{-2t}}{5} - \frac{c_1 e^{3t}}{2}$$

✓ Solution by Mathematica

Time used: 0.046 (sec). Leaf size: 229

```
DSolve[{x1'[t]==28*x1[t]+50*x2[t]+100*x3[t],x2'[t]==15*x1[t]+33*x2[t]+60*x3[t],x3'[t]==-15*x
```

$$x1(t) \to \frac{1}{57}e^{t/2}\left(19(3c_1-5c_2)e^{5t/2} + 95c_2\cos\left(\frac{5\sqrt{95}t}{2}\right) + \sqrt{95}(6c_1+13c_2+24c_3)\sin\left(\frac{5\sqrt{95}t}{2}\right)\right)$$

$$x2(t) \to \frac{1}{95}e^{t/2}\left(95c_2\cos\left(\frac{5\sqrt{95}t}{2}\right) + \sqrt{95}(6c_1+13c_2+24c_3)\sin\left(\frac{5\sqrt{95}t}{2}\right)\right)$$

$$x3(t) \to -\frac{e^{t/2}\left(95(3c_1-5c_2)e^{5t/2} - 95(3c_1-5c_2+12c_3)\cos\left(\frac{5\sqrt{95}t}{2}\right) + \sqrt{95}(69c_1+197c_2+276c_3)\sin\left(\frac{5\sqrt{95}t}{2}\right)\right)}{1140}$$

6.25 problem problem 25

Internal problem ID [382]
Internal file name [`OUTPUT/382_Sunday_June_05_2022_01_40_09_AM_77665157/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.6, Multiple Eigenvalue Solutions. Page 451
Problem number: problem 25.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= -2x_1(t) + 17x_2(t) + 4x_3(t) \\
x_2'(t) &= -x_1(t) + 6x_2(t) + x_3(t) \\
x_3'(t) &= x_2(t) + 2x_3(t)
\end{aligned}
$$

6.25.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} -2 & 17 & 4 \\ -1 & 6 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \mathrm{e}^{2t}\left(1 - \frac{1}{2}t^2 - 4t\right) & \mathrm{e}^{2t}t(2t+17) & \frac{\mathrm{e}^{2t}t(t+8)}{2} \\ -\mathrm{e}^{2t}t & \mathrm{e}^{2t}(4t+1) & \mathrm{e}^{2t}t \\ -\frac{\mathrm{e}^{2t}t^2}{2} & \mathrm{e}^{2t}(2t^2+t) & \mathrm{e}^{2t}\left(1 + \frac{t^2}{2}\right) \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c}\\
&= \begin{bmatrix} e^{2t}\left(1-\frac{1}{2}t^2-4t\right) & e^{2t}t(2t+17) & \frac{e^{2t}t(t+8)}{2} \\ -e^{2t}t & e^{2t}(4t+1) & e^{2t}t \\ -\frac{e^{2t}t^2}{2} & e^{2t}(2t^2+t) & e^{2t}\left(1+\frac{t^2}{2}\right) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}\\
&= \begin{bmatrix} e^{2t}\left(1-\frac{1}{2}t^2-4t\right)c_1 + e^{2t}t(2t+17)\,c_2 + \frac{e^{2t}t(t+8)c_3}{2} \\ -e^{2t}tc_1 + e^{2t}(4t+1)\,c_2 + e^{2t}tc_3 \\ -\frac{e^{2t}t^2c_1}{2} + e^{2t}(2t^2+t)\,c_2 + e^{2t}\left(1+\frac{t^2}{2}\right)c_3 \end{bmatrix}\\
&= \begin{bmatrix} -\frac{((c_1-4c_2-c_3)t^2+(8c_1-34c_2-8c_3)t-2c_1)e^{2t}}{2} \\ -((c_1-4c_2-c_3)\,t-c_2)\,e^{2t} \\ -\frac{((c_1-4c_2-c_3)t^2-2tc_2-2c_3)e^{2t}}{2} \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

6.25.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} -2 & 17 & 4 \\ -1 & 6 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A-\lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -2 & 17 & 4 \\ -1 & 6 & 1 \\ 0 & 1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix}-2-\lambda & 17 & 4\\ -1 & 6-\lambda & 1\\ 0 & 1 & 2-\lambda\end{bmatrix}\right)=0$$

Which gives the characteristic equation

$$\lambda^3-6\lambda^2+12\lambda-8=0$$

The roots of the above are the eigenvalues.

$$\lambda_1=2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1=2$

We need to solve $A\vec{v}=\lambda\vec{v}$ or $(A-\lambda I)\vec{v}=\vec{0}$ which becomes

$$\left(\begin{bmatrix}-2 & 17 & 4\\ -1 & 6 & 1\\ 0 & 1 & 2\end{bmatrix}-(2)\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$\begin{bmatrix}-4 & 17 & 4\\ -1 & 4 & 1\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c}-4 & 17 & 4 & 0\\ -1 & 4 & 1 & 0\\ 0 & 1 & 0 & 0\end{array}\right]$$

$$R_2=R_2-\frac{R_1}{4}\Longrightarrow\left[\begin{array}{ccc|c}-4 & 17 & 4 & 0\\ 0 & -\frac{1}{4} & 0 & 0\\ 0 & 1 & 0 & 0\end{array}\right]$$

$$R_3 = R_3 + 4R_2 \Longrightarrow \left[\begin{array}{ccc|c} -4 & 17 & 4 & 0 \\ 0 & -\frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -4 & 17 & 4 \\ 0 & -\frac{1}{4} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
2	3	1	Yes	$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 2 is real and repated eigenvalue of multiplicity 3.There are three possible cases that can happen. This is illustrated in this diagram

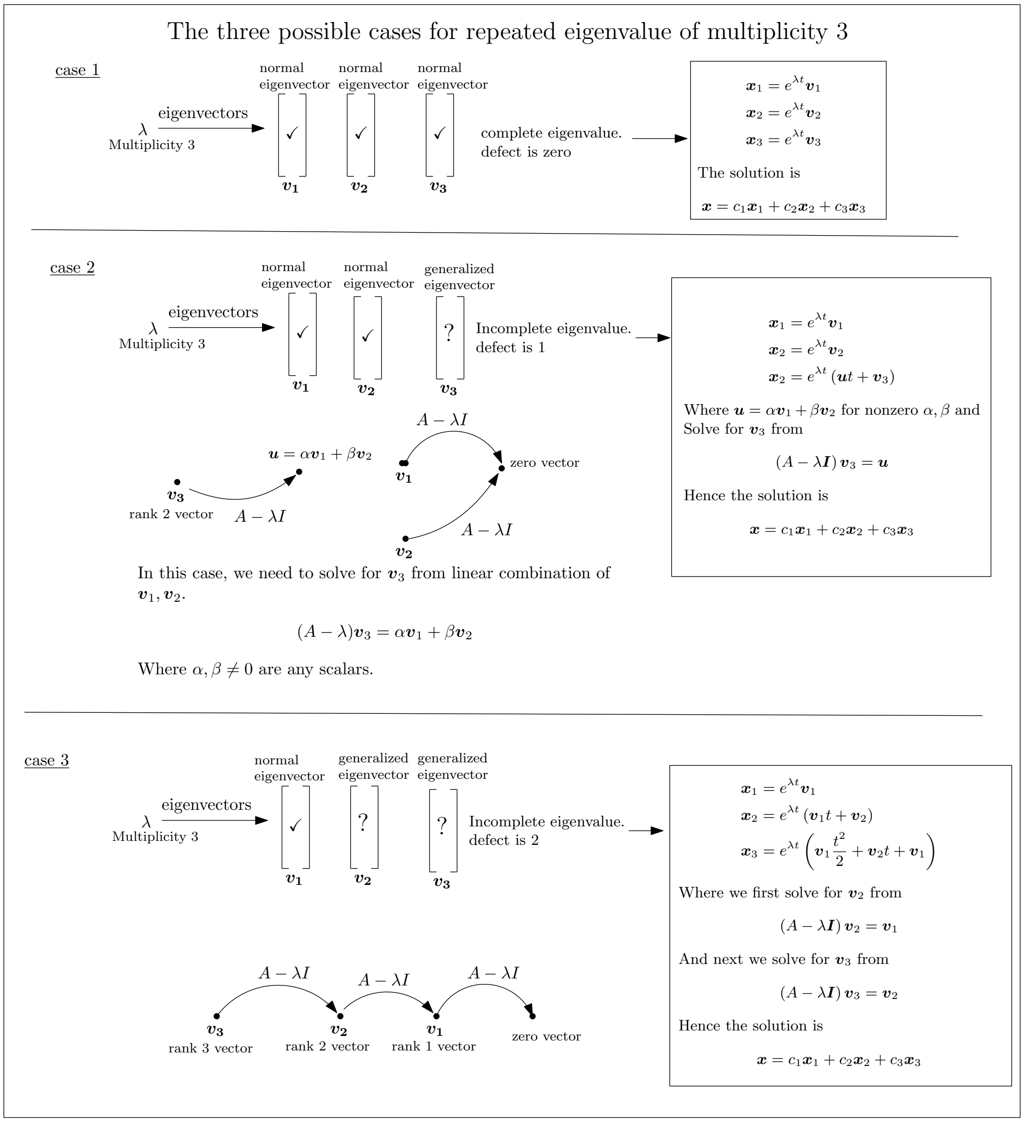


Figure 60: Possible case for repeated λ of multiplicity 3

This eigenvalue has algebraic multiplicity of 3, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 2. This falls into case 3 shown above. First we find generalized eigenvector $\vec{v}_2$ of rank 2 and then use this to find generalized eigenvector $\vec{v}_3$ of rank 3.$\vec{v}_2$ is found by solving

$$(A - \lambda I) \vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence

$$\left(\begin{bmatrix}-2 & 17 & 4\\ -1 & 6 & 1\\ 0 & 1 & 2\end{bmatrix} - (2)\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}1\\ 0\\ 1\end{bmatrix}$$

$$\begin{bmatrix}-4 & 17 & 4\\ -1 & 4 & 1\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}1\\ 0\\ 1\end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix}5\\ 1\\ 1\end{bmatrix}$$

Now $\vec{v}_3$ is found by solving

$$(A - \lambda I)\,\vec{v}_3 = \vec{v}_2$$

Where $\vec{v}_2$ is the (rank 2) generalized eigenvector found above. Hence

$$\left(\begin{bmatrix}-2 & 17 & 4\\ -1 & 6 & 1\\ 0 & 1 & 2\end{bmatrix} - (2)\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}5\\ 1\\ 1\end{bmatrix}$$

$$\begin{bmatrix}-4 & 17 & 4\\ -1 & 4 & 1\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}5\\ 1\\ 1\end{bmatrix}$$

Solving for $\vec{v}_3$ gives

$$\vec{v}_3 = \begin{bmatrix}4\\ 1\\ 1\end{bmatrix}$$

We have found three generalized eigenvectors for eigenvalue 2. Therefore the three basis

solutions associated with this eigenvalue are

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\
&= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \mathrm{e}^{2t} \\
&= \begin{bmatrix} \mathrm{e}^{2t} \\ 0 \\ \mathrm{e}^{2t} \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_2(t) &= e^{\lambda t}(\vec{v}_1 t + \vec{v}_2) \\
&= \mathrm{e}^{2t} \left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} t + \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix} \right) \\
&= \begin{bmatrix} \mathrm{e}^{2t}(t+5) \\ \mathrm{e}^{2t} \\ \mathrm{e}^{2t}(t+1) \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_3(t) &= \left(\vec{v}_1 \frac{t^2}{2} + \vec{v}_2 t + \vec{v}_3 \right) e^{\lambda t} \\
&= \left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \frac{t^2}{2} + \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix} t + \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} \right) \mathrm{e}^{2t} \\
&= \begin{bmatrix} \frac{\mathrm{e}^{2t}(t^2+10t+8)}{2} \\ \mathrm{e}^{2t}(t+1) \\ \frac{\mathrm{e}^{2t}(t^2+2t+2)}{2} \end{bmatrix}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)
$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 \begin{bmatrix} e^{2t} \\ 0 \\ e^{2t} \end{bmatrix} + c_2 \begin{bmatrix} e^{2t}(t+5) \\ e^{2t} \\ e^{2t}(t+1) \end{bmatrix} + c_3 \begin{bmatrix} e^{2t}\left(\frac{1}{2}t^2 + 5t + 4\right) \\ e^{2t}(t+1) \\ e^{2t}\left(t + \frac{1}{2}t^2 + 1\right) \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} \frac{e^{2t}\left((t^2+10t+8)c_3+2c_2t+2c_1+10c_2\right)}{2} \\ e^{2t}(c_3t + c_2 + c_3) \\ \frac{e^{2t}\left((t^2+2t+2)c_3+2c_2t+2c_1+2c_2\right)}{2} \end{bmatrix}$$

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 62

```
dsolve([diff(x__1(t),t)=-2*x__1(t)+17*x__2(t)+4*x__3(t),diff(x__2(t),t)=-1*x__1(t)+6*x__2(t)
```

$$x_1(t) = e^{2t}\left(c_3t^2 + c_2t + 8c_3t + c_1 + 4c_2 - 2c_3\right)$$
$$x_2(t) = e^{2t}(2c_3t + c_2)$$
$$x_3(t) = e^{2t}\left(c_3t^2 + c_2t + c_1\right)$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 105

```
DSolve[{x1'[t]==-2*x1[t]+17*x2[t]+4*x3[t],x2'[t]==-1*x1[t]+6*x2[t]+1*x3[t],x3'[t]==0*x1[t]+1
```

$$\text{x1}(t) \to \frac{1}{2}e^{2t}\left(-\left(c_1\left(t^2+8t-2\right)\right) + c_2t(4t+34) + c_3t(t+8)\right)$$
$$\text{x2}(t) \to e^{2t}((-c_1+4c_2+c_3)t + c_2)$$
$$\text{x3}(t) \to \frac{1}{2}e^{2t}\left((-c_1+4c_2+c_3)t^2 + 2c_2t + 2c_3\right)$$

6.26 problem problem 26

Internal problem ID [383]
Internal file name [`OUTPUT/383_Sunday_June_05_2022_01_40_10_AM_15558559/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.6, Multiple Eigenvalue Solutions. Page 451
Problem number: problem 26.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= 5x_1(t) - x_2(t) + x_3(t) \\
x_2'(t) &= x_1(t) + 3x_2(t) \\
x_3'(t) &= -3x_1(t) + 2x_2(t) + x_3(t)
\end{aligned}
$$

6.26.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 5 & -1 & 1 \\ 1 & 3 & 0 \\ -3 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \mathrm{e}^{3t}(1+2t) & -\mathrm{e}^{3t}t & \mathrm{e}^{3t}t \\ \mathrm{e}^{3t}t(t+1) & \mathrm{e}^{3t}\left(1-\frac{t^2}{2}\right) & \frac{\mathrm{e}^{3t}t^2}{2} \\ \mathrm{e}^{3t}t(t-3) & -\frac{\mathrm{e}^{3t}t(t-4)}{2} & \mathrm{e}^{3t}\left(1+\frac{1}{2}t^2-2t\right) \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{3t}(1+2t) & -e^{3t}t & e^{3t}t \\ e^{3t}t(t+1) & e^{3t}\left(1-\frac{t^2}{2}\right) & \frac{e^{3t}t^2}{2} \\ e^{3t}t(t-3) & -\frac{e^{3t}t(t-4)}{2} & e^{3t}\left(1+\frac{1}{2}t^2-2t\right) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} e^{3t}(1+2t)\,c_1 - e^{3t}tc_2 + e^{3t}tc_3 \\ e^{3t}t(t+1)\,c_1 + e^{3t}\left(1-\frac{t^2}{2}\right)c_2 + \frac{e^{3t}t^2c_3}{2} \\ e^{3t}t(t-3)\,c_1 - \frac{e^{3t}t(t-4)c_2}{2} + e^{3t}\left(1+\frac{1}{2}t^2-2t\right)c_3 \end{bmatrix} \\
&= \begin{bmatrix} e^{3t}(2c_1t - c_2t + c_3t + c_1) \\ \left(\left(c_1 - \frac{c_2}{2} + \frac{c_3}{2}\right)t^2 + c_1t + c_2\right)e^{3t} \\ \left(\left(c_1 - \frac{c_2}{2} + \frac{c_3}{2}\right)t^2 + (-3c_1 + 2c_2 - 2c_3)\,t + c_3\right)e^{3t} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

6.26.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 5 & -1 & 1 \\ 1 & 3 & 0 \\ -3 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 5 & -1 & 1 \\ 1 & 3 & 0 \\ -3 & 2 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 5-\lambda & -1 & 1 \\ 1 & 3-\lambda & 0 \\ -3 & 2 & 1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 9\lambda^2 + 27\lambda - 27 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 3$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 5 & -1 & 1 \\ 1 & 3 & 0 \\ -3 & 2 & 1 \end{bmatrix} - (3)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & 0 \\ -3 & 2 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ -3 & 2 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} 2 & -1 & 1 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ -3 & 2 & -2 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{3R_1}{2} \implies \left[\begin{array}{ccc|c} 2 & -1 & 1 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \end{array}\right]$$

$$R_3 = R_3 - R_2 \implies \left[\begin{array}{ccc|c} 2 & -1 & 1 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & -1 & 1 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_2 = t\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated

with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
3	3	1	Yes	$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 3 is real and repated eigenvalue of multiplicity 3.There are three possible cases that can happen. This is illustrated in this diagram

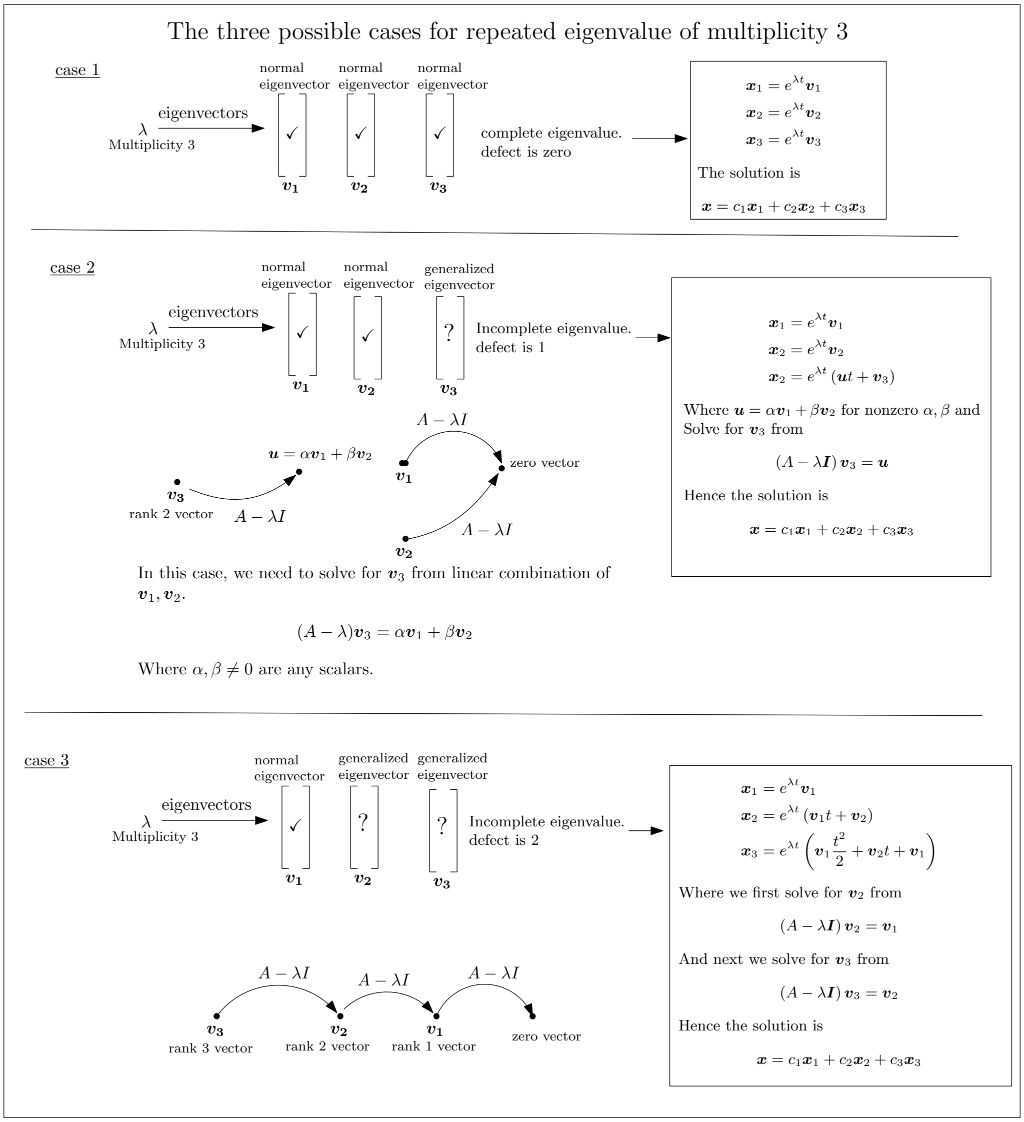


Figure 61: Possible case for repeated λ of multiplicity 3

This eigenvalue has algebraic multiplicity of 3, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 2. This falls into case 3 shown above. First we find generalized eigenvector $\vec{v}_2$ of rank 2 and then use this to find generalized eigenvector $\vec{v}_3$ of rank 3.$\vec{v}_2$ is found by solving

$$(A - \lambda I) \vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence

$$\left(\begin{bmatrix} 5 & -1 & 1 \\ 1 & 3 & 0 \\ -3 & 2 & 1 \end{bmatrix} - (3) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & 0 \\ -3 & 2 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

Now $\vec{v}_3$ is found by solving

$$(A - \lambda I)\, \vec{v}_3 = \vec{v}_2$$

Where $\vec{v}_2$ is the (rank 2) generalized eigenvector found above. Hence

$$\left(\begin{bmatrix} 5 & -1 & 1 \\ 1 & 3 & 0 \\ -3 & 2 & 1 \end{bmatrix} - (3) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & 0 \\ -3 & 2 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_3$ gives

$$\vec{v}_3 = \begin{bmatrix} 3 \\ 6 \\ 1 \end{bmatrix}$$

We have found three generalized eigenvectors for eigenvalue 3. Therefore the three basis

solutions associated with this eigenvalue are

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\
&= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \mathrm{e}^{3t} \\
&= \begin{bmatrix} 0 \\ \mathrm{e}^{3t} \\ \mathrm{e}^{3t} \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_2(t) &= e^{\lambda t}(\vec{v}_1 t + \vec{v}_2) \\
&= \mathrm{e}^{3t}\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} t + \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}\right) \\
&= \begin{bmatrix} \mathrm{e}^{3t} \\ \mathrm{e}^{3t}(t+3) \\ \mathrm{e}^{3t}(t+1) \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_3(t) &= \left(\vec{v}_1 \frac{t^2}{2} + \vec{v}_2 t + \vec{v}_3\right) e^{\lambda t} \\
&= \left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \frac{t^2}{2} + \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} t + \begin{bmatrix} 3 \\ 6 \\ 1 \end{bmatrix}\right) \mathrm{e}^{3t} \\
&= \begin{bmatrix} \mathrm{e}^{3t}(t+3) \\ \frac{\mathrm{e}^{3t}(t^2+6t+12)}{2} \\ \frac{\mathrm{e}^{3t}(t^2+2t+2)}{2} \end{bmatrix}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)
$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ e^{3t} \\ e^{3t} \end{bmatrix} + c_2 \begin{bmatrix} e^{3t} \\ e^{3t}(t+3) \\ e^{3t}(t+1) \end{bmatrix} + c_3 \begin{bmatrix} e^{3t}(t+3) \\ e^{3t}\left(\frac{1}{2}t^2+3t+6\right) \\ e^{3t}\left(t+\frac{1}{2}t^2+1\right) \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} ((t+3)\,c_3+c_2)\,e^{3t} \\ \frac{((t^2+6t+12)c_3+2c_2t+2c_1+6c_2)e^{3t}}{2} \\ \frac{e^{3t}((t^2+2t+2)c_3+2c_2t+2c_1+2c_2)}{2} \end{bmatrix}$$

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 62

```
dsolve([diff(x__1(t),t)=5*x__1(t)-1*x__2(t)+1*x__3(t),diff(x__2(t),t)=1*x__1(t)+3*x__2(t)+0*
```

$$x_1(t) = e^{3t}(2c_3t + c_2)$$
$$x_2(t) = e^{3t}\left(c_3t^2 + c_2t + c_1\right)$$
$$x_3(t) = e^{3t}\left(c_3t^2 + c_2t - 4c_3t + c_1 - 2c_2 + 2c_3\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 105

```
DSolve[{x1'[t]==5*x1[t]-1*x2[t]+1*x3[t],x2'[t]==1*x1[t]+3*x2[t]+0*x3[t],x3'[t]==-3*x1[t]+2*x
```

$$\text{x1}(t) \to e^{3t}(2c_1t - c_2t + c_3t + c_1)$$
$$\text{x2}(t) \to \frac{1}{2}e^{3t}\left((2c_1 - c_2 + c_3)t^2 + 2c_1t + 2c_2\right)$$
$$\text{x3}(t) \to \frac{1}{2}e^{3t}\left(c_3\left(t^2 - 4t + 2\right) + 2c_1(t-3)t - c_2(t-4)t\right)$$

6.27 problem problem 27

Internal problem ID [384]
Internal file name [`OUTPUT/384_Sunday_June_05_2022_01_40_11_AM_49369200/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.6, Multiple Eigenvalue Solutions. Page 451
Problem number: problem 27.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= -3x_1(t) + 5x_2(t) - 5x_3(t) \\
x_2'(t) &= 3x_1(t) - x_2(t) + 3x_3(t) \\
x_3'(t) &= 8x_1(t) - 8x_2(t) + 10x_3(t)
\end{aligned}
$$

6.27.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} -3 & 5 & -5 \\ 3 & -1 & 3 \\ 8 & -8 & 10 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} e^{2t}(1-5t) & 5\,e^{2t}t & -5\,e^{2t}t \\ 3\,e^{2t}t & e^{2t}(1-3t) & 3\,e^{2t}t \\ 8\,e^{2t}t & -8\,e^{2t}t & e^{2t}(1+8t) \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{2t}(1-5t) & 5\,e^{2t}t & -5\,e^{2t}t \\ 3\,e^{2t}t & e^{2t}(1-3t) & 3\,e^{2t}t \\ 8\,e^{2t}t & -8\,e^{2t}t & e^{2t}(1+8t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} e^{2t}(1-5t)\,c_1 + 5\,e^{2t}tc_2 - 5\,e^{2t}tc_3 \\ 3\,e^{2t}tc_1 + e^{2t}(1-3t)\,c_2 + 3\,e^{2t}tc_3 \\ 8\,e^{2t}tc_1 - 8\,e^{2t}tc_2 + e^{2t}(1+8t)\,c_3 \end{bmatrix} \\
&= \begin{bmatrix} -5\left((c_1-c_2+c_3)\,t - \frac{c_1}{5}\right) e^{2t} \\ 3\left((c_1-c_2+c_3)\,t + \frac{c_2}{3}\right) e^{2t} \\ 8\,e^{2t}\left((c_1-c_2+c_3)\,t + \frac{c_3}{8}\right) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

6.27.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} -3 & 5 & -5 \\ 3 & -1 & 3 \\ 8 & -8 & 10 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det\left(A - \lambda I\right) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} -3 & 5 & -5 \\ 3 & -1 & 3 \\ 8 & -8 & 10 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$\det\left(\begin{bmatrix} -3-\lambda & 5 & -5 \\ 3 & -1-\lambda & 3 \\ 8 & -8 & 10-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 6\lambda^2 + 12\lambda - 8 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -3 & 5 & -5 \\ 3 & -1 & 3 \\ 8 & -8 & 10 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -5 & 5 & -5 \\ 3 & -3 & 3 \\ 8 & -8 & 8 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -5 & 5 & -5 & 0 \\ 3 & -3 & 3 & 0 \\ 8 & -8 & 8 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{3R_1}{5} \Longrightarrow \left[\begin{array}{ccc|c} -5 & 5 & -5 & 0 \\ 0 & 0 & 0 & 0 \\ 8 & -8 & 8 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{8R_1}{5} \Longrightarrow \left[\begin{array}{ccc|c} -5 & 5 & -5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -5 & 5 & -5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2, v_3\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Let $v_3 = s$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t - s\}$

Hence the solution is

$$\begin{bmatrix} t-s \\ t \\ s \end{bmatrix} = \begin{bmatrix} t-s \\ t \\ s \end{bmatrix}$$

Since there are two free Variable, we have found two eigenvectors associated with this eigenvalue. The above can be written as

$$\begin{aligned} \begin{bmatrix} t-s \\ t \\ s \end{bmatrix} &= \begin{bmatrix} t \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} -s \\ 0 \\ s \end{bmatrix} \\ &= t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

By letting $t = 1$ and $s = 1$ then the above becomes

$$\begin{bmatrix} t-s \\ t \\ s \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Hence the two eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right)$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
2	3	2	Yes	$\begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 2 is real and repated eigenvalue of multiplicity 3.There are three possible cases that can happen. This is illustrated in this diagram

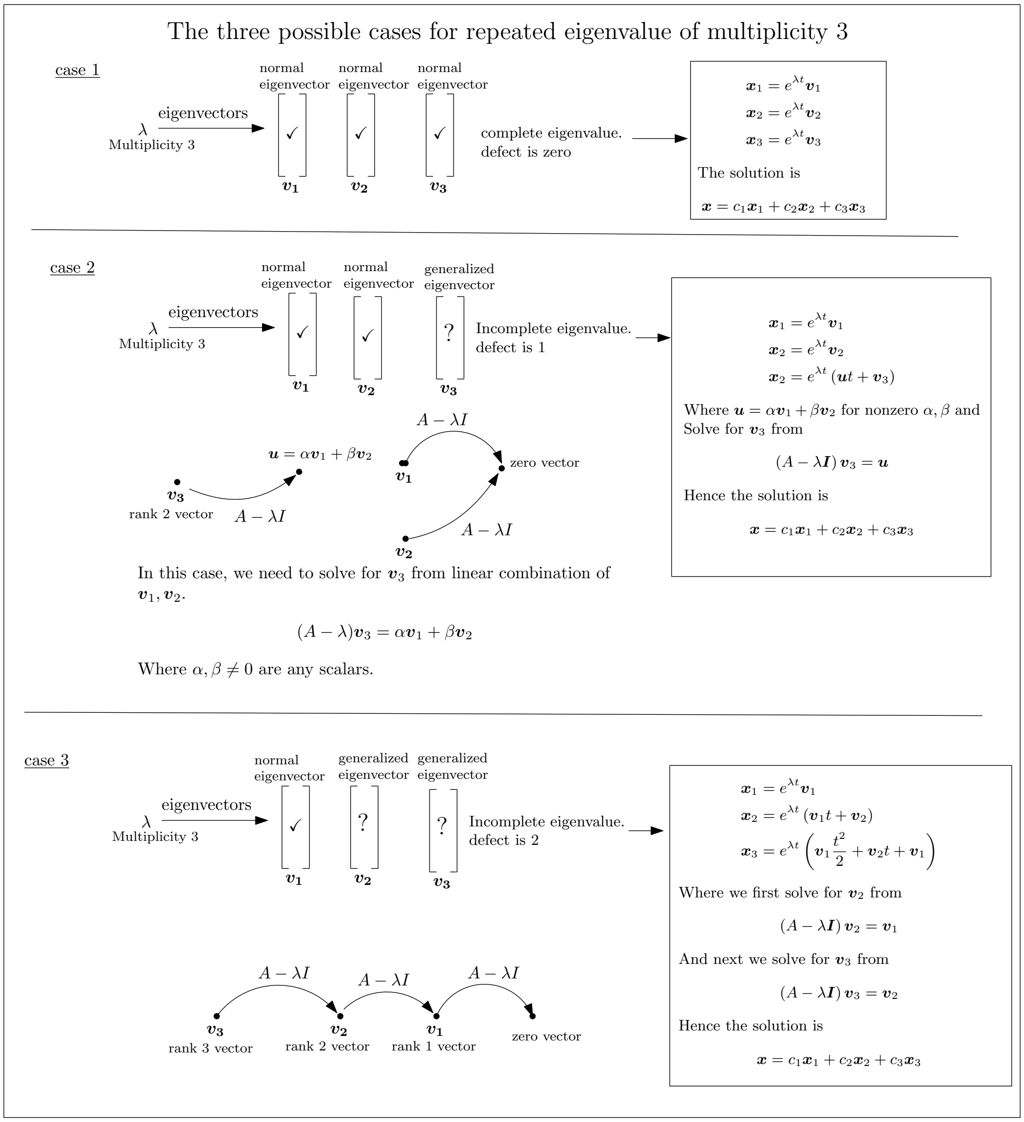


Figure 62: Possible case for repeated λ of multiplicity 3

This eigenvalue has algebraic multiplicity of 3, and geometric multiplicity 2, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to find rank-2 eigenvector $\vec{v}_3$. This eigenvector must therefore satisfy $(A - \lambda I)^2 \vec{v}_3 = \vec{0}$.

But

$$
(A-\lambda I)^2 = \left(\begin{bmatrix} -3 & 5 & -5 \\ 3 & -1 & 3 \\ 8 & -8 & 10 \end{bmatrix} - 2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)^2
$$

$$
= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
$$

Therefore $\vec{v}_3$ could be any eigenvector vector we want (but not the zero vector). Let

$$
\vec{v}_3 = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix}
$$

To determine the actual $\vec{v}_3$ we need now to enforce the condition that $\vec{v}_3$ satisfies

$$
(A - \lambda I)\,\vec{v}_3 = \vec{u} \tag{1}
$$

Where $\vec{u}$ is linear combination of $\vec{v}_1, \vec{v}_2$. Hence

$$
\vec{u} = \alpha \vec{v}_1 + \beta \vec{v}_2
$$

Where α, β are arbitrary constants (not both zero). Eq. (1) becomes

$$
(A - \lambda I) \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} = \alpha \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}
$$

$$
\begin{bmatrix} -5 & 5 & -5 \\ 3 & -3 & 3 \\ 8 & -8 & 8 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} = \alpha \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}
$$

$$
\begin{bmatrix} -5\eta_1 + 5\eta_2 - 5\eta_3 \\ 3\eta_1 - 3\eta_2 + 3\eta_3 \\ 8\eta_1 - 8\eta_2 + 8\eta_3 \end{bmatrix} = \begin{bmatrix} -\alpha + \beta \\ \beta \\ \alpha \end{bmatrix}
$$

Expanding the above gives the following equations equations

$$
\begin{aligned}
-5\eta_1 + 5\eta_2 - 5\eta_3 &= -\alpha + \beta \\
3\eta_1 - 3\eta_2 + 3\eta_3 &= \beta \\
8\eta_1 - 8\eta_2 + 8\eta_3 &= \alpha
\end{aligned}
$$

solving for α, β from the above gives

$$
\begin{aligned}
-5\eta_1 + 5\eta_2 - 5\eta_3 &= -\alpha + \beta \\
3\eta_1 - 3\eta_2 + 3\eta_3 &= \beta
\end{aligned}
$$

Since α, β are not both zero, then we just need to determine η_i values, not all zero, which satisfy the above equations for α, β not both zero. By inspection we see that the following values satisfy this condition

$$
[\eta_1 = -1, \eta_2 = 0, \eta_3 = 0]
$$

Hence we found the missing generalized eigenvector

$$
\vec{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}
$$

Which implies that

$$
\begin{aligned}
\alpha &= -8 \\
\beta &= -3
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\vec{u} &= \alpha \vec{v}_1 + \beta \vec{v}_2 \\
&= -8 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + (-3) \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 5 \\ -3 \\ -8 \end{bmatrix}
\end{aligned}
$$

Therefore the missing generalized eigenvector is now found. We have found three generalized eigenvectors for eigenvalue 2. Therefore the three basis solutions associated with this eigenvalue are

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\ &= \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \mathrm{e}^{2t} \\ &= \begin{bmatrix} -\mathrm{e}^{2t} \\ 0 \\ \mathrm{e}^{2t} \end{bmatrix}\end{aligned}$$

And

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{\lambda t} \\ &= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \mathrm{e}^{2t} \\ &= \begin{bmatrix} \mathrm{e}^{2t} \\ \mathrm{e}^{2t} \\ 0 \end{bmatrix}\end{aligned}$$

And

$$\begin{aligned}\vec{x}_3(t) &= (\vec{u}t + \vec{v}_3)\, e^{\lambda t} \\ &= \left(\begin{bmatrix} 5 \\ -3 \\ -8 \end{bmatrix} t + \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}\right) \mathrm{e}^{2t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 \begin{bmatrix} -\mathrm{e}^{2t} \\ 0 \\ \mathrm{e}^{2t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{2t} \\ \mathrm{e}^{2t} \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} \mathrm{e}^{2t}(-1 + 5t) \\ -3\,\mathrm{e}^{2t} t \\ -8\,\mathrm{e}^{2t} t \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} ((-1+5t)\,c_3 - c_1 + c_2)\,\mathrm{e}^{2t} \\ \mathrm{e}^{2t}(-3c_3 t + c_2) \\ \mathrm{e}^{2t}(-8c_3 t + c_1) \end{bmatrix}$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 58

```
dsolve([diff(x__1(t),t)=-3*x__1(t)+5*x__2(t)-5*x__3(t),diff(x__2(t),t)=3*x__1(t)-1*x__2(t)+3
```

$$x_1(t) = \mathrm{e}^{2t}(c_3 t + c_2)$$
$$x_2(t) = \frac{\mathrm{e}^{2t}(-3c_3 t + 5c_1 - 3c_2)}{5}$$
$$x_3(t) = \frac{\mathrm{e}^{2t}(-8c_3 t + 5c_1 - 8c_2 - c_3)}{5}$$

✓ Solution by Mathematica

Time used: 0.034 (sec). Leaf size: 174

```
DSolve[{x1'[t]==-3*x1[t]+5*x2[t]-5*x3[t],x2'[t]==4*x1[t]-1*x2[t]+4*x3[t],x3'[t]==8*x1[t]-8*x
```

$$\mathrm{x1}(t) \to \frac{1}{3}e^{2t}\left(-5(c_1+c_3)\cos\left(\sqrt{3}t\right) - 5\sqrt{3}(c_1-c_2+c_3)\sin\left(\sqrt{3}t\right) + 8c_1 + 5c_3\right)$$
$$\mathrm{x2}(t) \to \frac{1}{3}e^{2t}\left(3c_2\cos\left(\sqrt{3}t\right) + \sqrt{3}(4c_1-3c_2+4c_3)\sin\left(\sqrt{3}t\right)\right)$$
$$\mathrm{x3}(t) \to \frac{1}{3}e^{2t}\left(8(c_1+c_3)\cos\left(\sqrt{3}t\right) + 8\sqrt{3}(c_1-c_2+c_3)\sin\left(\sqrt{3}t\right) - 8c_1 - 5c_3\right)$$

6.28 problem problem 28

Internal problem ID [385]
Internal file name [`OUTPUT/385_Sunday_June_05_2022_01_40_13_AM_90500381/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.6, Multiple Eigenvalue Solutions. Page 451
Problem number: problem 28.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= -15x_1(t) - 7x_2(t) + 4x_3(t) \\
x_2'(t) &= 34x_1(t) + 16x_2(t) - 11x_3(t) \\
x_3'(t) &= 17x_1(t) + 7x_2(t) + 5x_3(t)
\end{aligned}
$$

6.28.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} -15 & -7 & 4 \\ 34 & 16 & -11 \\ 17 & 7 & 5 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} e^{2t}\left(1 + \frac{119}{2}t^2 - 17t\right) & \frac{7\,e^{2t}t(7t-2)}{2} & \frac{e^{2t}t(21t+8)}{2} \\ -\frac{17\,e^{2t}t(17t-4)}{2} & e^{2t}\left(1 - \frac{119}{2}t^2 + 14t\right) & -\frac{e^{2t}t(51t+22)}{2} \\ 17\,e^{2t}t & 7\,e^{2t}t & e^{2t}(1+3t) \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{2t}\left(1+\frac{119}{2}t^2-17t\right) & \frac{7\,e^{2t}t(7t-2)}{2} & \frac{e^{2t}t(21t+8)}{2} \\ -\frac{17\,e^{2t}t(17t-4)}{2} & e^{2t}\left(1-\frac{119}{2}t^2+14t\right) & -\frac{e^{2t}t(51t+22)}{2} \\ 17\,e^{2t}t & 7\,e^{2t}t & e^{2t}(1+3t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} e^{2t}\left(1+\frac{119}{2}t^2-17t\right)c_1+\frac{7\,e^{2t}t(7t-2)c_2}{2}+\frac{e^{2t}t(21t+8)c_3}{2} \\ -\frac{17\,e^{2t}t(17t-4)c_1}{2}+e^{2t}\left(1-\frac{119}{2}t^2+14t\right)c_2-\frac{e^{2t}t(51t+22)c_3}{2} \\ 17\,e^{2t}tc_1+7\,e^{2t}tc_2+e^{2t}(1+3t)\,c_3 \end{bmatrix} \\
&= \begin{bmatrix} \frac{119\left(\left(c_1+\frac{7c_2}{17}+\frac{3c_3}{17}\right)t^2+\left(-\frac{2c_1}{7}-\frac{2c_2}{17}+\frac{8c_3}{119}\right)t+\frac{2c_1}{119}\right)e^{2t}}{2} \\ -\frac{289\left(\left(c_1+\frac{7c_2}{17}+\frac{3c_3}{17}\right)t^2+\left(-\frac{4c_1}{17}-\frac{28c_2}{289}+\frac{22c_3}{289}\right)t-\frac{2c_2}{289}\right)e^{2t}}{2} \\ e^{2t}(17tc_1+7tc_2+3tc_3+c_3) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

6.28.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} -15 & -7 & 4 \\ 34 & 16 & -11 \\ 17 & 7 & 5 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A-\lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -15 & -7 & 4 \\ 34 & 16 & -11 \\ 17 & 7 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -15-\lambda & -7 & 4 \\ 34 & 16-\lambda & -11 \\ 17 & 7 & 5-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 6\lambda^2 + 12\lambda - 8 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -15 & -7 & 4 \\ 34 & 16 & -11 \\ 17 & 7 & 5 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -17 & -7 & 4 \\ 34 & 14 & -11 \\ 17 & 7 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -17 & -7 & 4 & 0 \\ 34 & 14 & -11 & 0 \\ 17 & 7 & 3 & 0 \end{array}\right]$$

$$R_2 = R_2 + 2R_1 \Longrightarrow \left[\begin{array}{ccc|c} -17 & -7 & 4 & 0 \\ 0 & 0 & -3 & 0 \\ 17 & 7 & 3 & 0 \end{array}\right]$$

$$R_3 = R_3 + R_1 \Longrightarrow \left[\begin{array}{ccc|c} -17 & -7 & 4 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 7 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{7R_2}{3} \Longrightarrow \left[\begin{array}{ccc|c} -17 & -7 & 4 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -17 & -7 & 4 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1, v_3\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{7t}{17}, v_3 = 0\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{7t}{17} \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{7t}{17} \\ t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{7t}{17} \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} -\frac{7}{17} \\ 1 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{7t}{17} \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{7}{17} \\ 1 \\ 0 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{7t}{17} \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} -7 \\ 17 \\ 0 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
2	3	1	Yes	$\begin{bmatrix} -\frac{7}{17} \\ 1 \\ 0 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 2 is real and repated eigenvalue of multiplicity 3.There are three possible cases that can happen. This is illustrated in this diagram

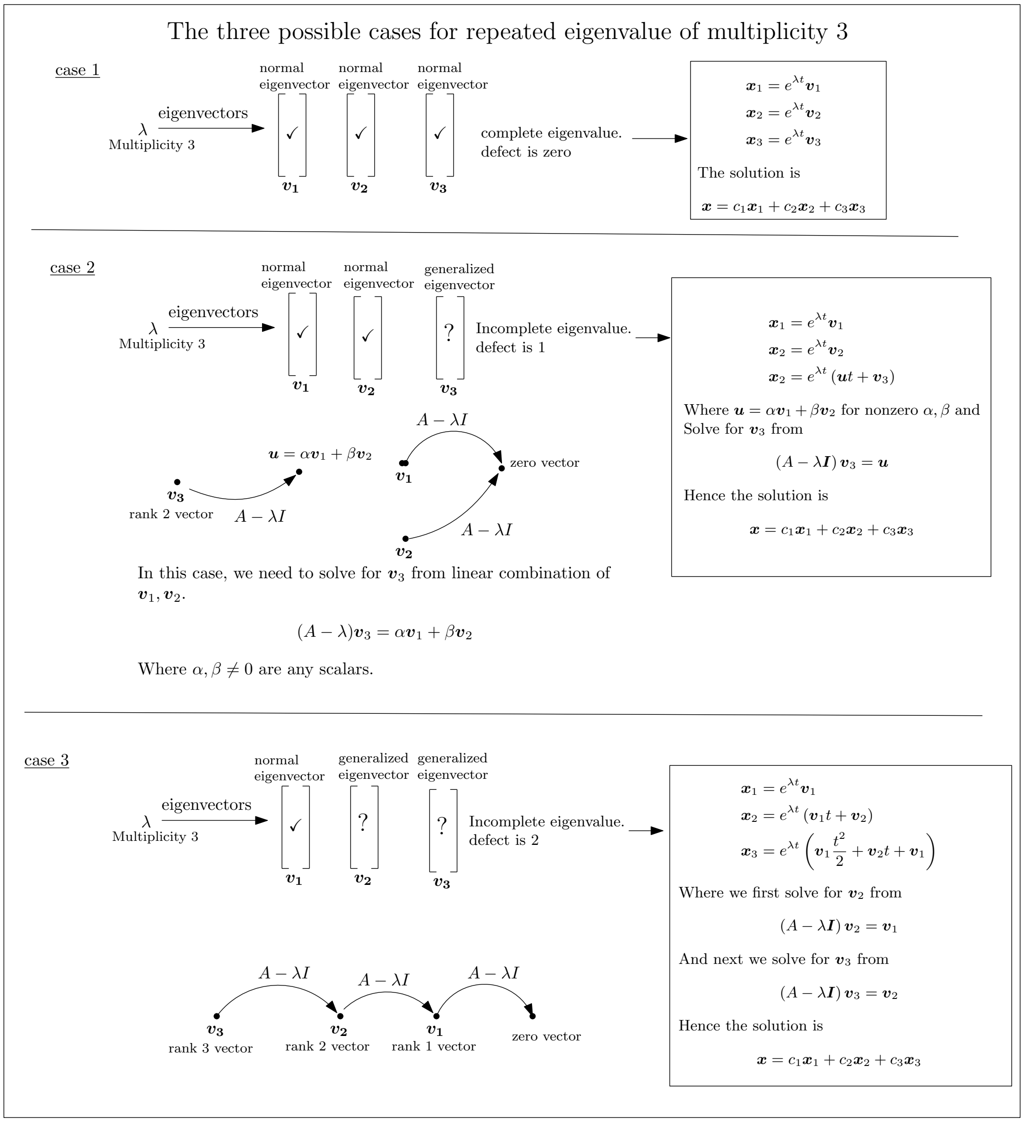


Figure 63: Possible case for repeated λ of multiplicity 3

This eigenvalue has algebraic multiplicity of 3, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 2. This falls into case 3 shown above. First we find generalized eigenvector $\vec{v}_2$ of rank 2 and then use this to find generalized eigenvector $\vec{v}_3$ of rank 3.$\vec{v}_2$ is found by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence

$$\left(\begin{bmatrix}-15 & -7 & 4\\ 34 & 16 & -11\\ 17 & 7 & 5\end{bmatrix} - (2)\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}-\frac{7}{17}\\ 1\\ 0\end{bmatrix}$$

$$\begin{bmatrix}-17 & -7 & 4\\ 34 & 14 & -11\\ 17 & 7 & 3\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}-\frac{7}{17}\\ 1\\ 0\end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix}1\\ -\frac{286}{119}\\ -\frac{1}{17}\end{bmatrix}$$

Now $\vec{v}_3$ is found by solving

$$(A - \lambda I)\,\vec{v}_3 = \vec{v}_2$$

Where $\vec{v}_2$ is the (rank 2) generalized eigenvector found above. Hence

$$\left(\begin{bmatrix}-15 & -7 & 4\\ 34 & 16 & -11\\ 17 & 7 & 5\end{bmatrix} - (2)\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}1\\ -\frac{286}{119}\\ -\frac{1}{17}\end{bmatrix}$$

$$\begin{bmatrix}-17 & -7 & 4\\ 34 & 14 & -11\\ 17 & 7 & 3\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}1\\ -\frac{286}{119}\\ -\frac{1}{17}\end{bmatrix}$$

Solving for $\vec{v}_3$ gives

$$\vec{v}_3 = \begin{bmatrix}1\\ -\frac{2078}{833}\\ \frac{16}{119}\end{bmatrix}$$

We have found three generalized eigenvectors for eigenvalue 2. Therefore the three basis

solutions associated with this eigenvalue are

$$\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\
&= \begin{bmatrix} -\frac{7}{17} \\ 1 \\ 0 \end{bmatrix} \mathrm{e}^{2t} \\
&= \begin{bmatrix} -\frac{7\,\mathrm{e}^{2t}}{17} \\ \mathrm{e}^{2t} \\ 0 \end{bmatrix}
\end{aligned}$$

And

$$\begin{aligned}
\vec{x}_2(t) &= e^{\lambda t}(\vec{v}_1 t + \vec{v}_2) \\
&= \mathrm{e}^{2t} \left(\begin{bmatrix} -\frac{7}{17} \\ 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} 1 \\ -\frac{286}{119} \\ -\frac{1}{17} \end{bmatrix} \right) \\
&= \begin{bmatrix} -\frac{\mathrm{e}^{2t}(7t-17)}{17} \\ \frac{\mathrm{e}^{2t}(119t-286)}{119} \\ -\frac{\mathrm{e}^{2t}}{17} \end{bmatrix}
\end{aligned}$$

And

$$\begin{aligned}
\vec{x}_3(t) &= \left(\vec{v}_1 \frac{t^2}{2} + \vec{v}_2 t + \vec{v}_3 \right) e^{\lambda t} \\
&= \left(\begin{bmatrix} -\frac{7}{17} \\ 1 \\ 0 \end{bmatrix} \frac{t^2}{2} + \begin{bmatrix} 1 \\ -\frac{286}{119} \\ -\frac{1}{17} \end{bmatrix} t + \begin{bmatrix} 1 \\ -\frac{2078}{833} \\ \frac{16}{119} \end{bmatrix} \right) \mathrm{e}^{2t} \\
&= \begin{bmatrix} -\frac{\mathrm{e}^{2t}(7t^2-34t-34)}{34} \\ \frac{\mathrm{e}^{2t}(833t^2-4004t-4156)}{1666} \\ -\frac{\mathrm{e}^{2t}(7t-16)}{119} \end{bmatrix}
\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 \begin{bmatrix} -\frac{7\,\mathrm{e}^{2t}}{17} \\ \mathrm{e}^{2t} \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{2t}\left(-\frac{7t}{17}+1\right) \\ \mathrm{e}^{2t}\left(t-\frac{286}{119}\right) \\ -\frac{\mathrm{e}^{2t}}{17} \end{bmatrix} + c_3 \begin{bmatrix} \mathrm{e}^{2t}\left(-\frac{7}{34}t^2+t+1\right) \\ \mathrm{e}^{2t}\left(\frac{1}{2}t^2-\frac{286}{119}t-\frac{2078}{833}\right) \\ \mathrm{e}^{2t}\left(-\frac{t}{17}+\frac{16}{119}\right) \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} -\frac{7\,\mathrm{e}^{2t}\left(\left(t^2-\frac{34}{7}t-\frac{34}{7}\right)c_3+2c_2t+2c_1-\frac{34c_2}{7}\right)}{34} \\ \frac{\left(\left(833t^2-4004t-4156\right)c_3+1666c_2t+1666c_1-4004c_2\right)\mathrm{e}^{2t}}{1666} \\ \frac{\left(\left(-7t+16\right)c_3-7c_2\right)\mathrm{e}^{2t}}{119} \end{bmatrix}$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 73

```
dsolve([diff(x__1(t),t)=-15*x__1(t)-7*x__2(t)+4*x__3(t),diff(x__2(t),t)=34*x__1(t)+16*x__2(t
```

$$x_1(t) = \mathrm{e}^{2t}\left(c_3t^2+c_2t+c_1\right)$$

$$x_2(t) = -\frac{\mathrm{e}^{2t}(833c_3t^2+833c_2t+42c_3t+833c_1+21c_2-8c_3)}{343}$$

$$x_3(t) = \frac{\mathrm{e}^{2t}(14c_3t+7c_2+2c_3)}{49}$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 124

```
DSolve[{x1'[t]==-15*x1[t]-7*x2[t]+4*x3[t],x2'[t]==34*x1[t]+16*x2[t]-11*x3[t],x3'[t]==17*x1[t
```

$$\mathrm{x1}(t) \to \frac{1}{2}e^{2t}\left(c_1\left(119t^2-34t+2\right)+7c_2t(7t-2)+c_3t(21t+8)\right)$$

$$\mathrm{x2}(t) \to -\frac{1}{2}e^{2t}\left(17(17c_1+7c_2+3c_3)t^2+(-68c_1-28c_2+22c_3)t-2c_2\right)$$

$$\mathrm{x3}(t) \to e^{2t}((17c_1+7c_2+3c_3)t+c_3)$$

6.29 problem problem 29

Internal problem ID [386]
Internal file name [`OUTPUT/386_Sunday_June_05_2022_01_40_14_AM_3084983/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.6, Multiple Eigenvalue Solutions. Page 451
Problem number: problem 29.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= -x_1(t) + x_2(t) + x_3(t) - 2x_4(t) \\
x_2'(t) &= 7x_1(t) - 4x_2(t) - 6x_3(t) + 11x_4(t) \\
x_3'(t) &= 5x_1(t) - x_2(t) + x_3(t) + 3x_4(t) \\
x_4'(t) &= 6x_1(t) - 2x_2(t) - 2x_3(t) + 6x_4(t)
\end{aligned}
$$

6.29.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \\ x_4'(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 & 1 & -2 \\ 7 & -4 & -6 & 11 \\ 5 & -1 & 1 & 3 \\ 6 & -2 & -2 & 6 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{-t} & t\,e^{-t} & t\,e^{-t} & -2t\,e^{-t} \\ (-2t+3)\,e^{2t} - 3\,e^{-t} & e^{-t}(1-3t) & -3t\,e^{-t} - e^{2t} + e^{-t} & (6t-2)\,e^{-t} - e^{2t}(t-2) \\ 2\,e^{2t}t + e^{2t} - e^{-t} & -t\,e^{-t} & -t\,e^{-t} + e^{2t} & t(e^{2t} + 2\,e^{-t}) \\ -2\,e^{-t} + 2\,e^{2t} & -2t\,e^{-t} & -2t\,e^{-t} & 4t\,e^{-t} + e^{2t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{-t} & t\,e^{-t} & t\,e^{-t} & -2t\,e^{-t} \\ (-2t+3)\,e^{2t} - 3\,e^{-t} & e^{-t}(1-3t) & -3t\,e^{-t} - e^{2t} + e^{-t} & (6t-2)\,e^{-t} - e^{2t}(t-2) \\ 2\,e^{2t}t + e^{2t} - e^{-t} & -t\,e^{-t} & -t\,e^{-t} + e^{2t} & t(e^{2t} + 2\,e^{-t}) \\ -2\,e^{-t} + 2\,e^{2t} & -2t\,e^{-t} & -2t\,e^{-t} & 4t\,e^{-t} + e^{2t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} \\
&= \begin{bmatrix} e^{-t}c_1 + t\,e^{-t}c_2 + t\,e^{-t}c_3 - 2t\,e^{-t}c_4 \\ ((-2t+3)\,e^{2t} - 3\,e^{-t})\,c_1 + e^{-t}(1-3t)\,c_2 + (-3t\,e^{-t} - e^{2t} + e^{-t})\,c_3 + ((6t-2)\,e^{-t} - e^{2t}(t-2) \\ (2\,e^{2t}t + e^{2t} - e^{-t})\,c_1 - t\,e^{-t}c_2 + (-t\,e^{-t} + e^{2t})\,c_3 + t(e^{2t} + 2\,e^{-t})\,c_4 \\ (-2\,e^{-t} + 2\,e^{2t})\,c_1 - 2t\,e^{-t}c_2 - 2t\,e^{-t}c_3 + (4t\,e^{-t} + e^{2t})\,c_4 \end{bmatrix} \\
&= \begin{bmatrix} ((c_2 + c_3 - 2c_4)\,t + c_1)\,e^{-t} \\ ((-3c_2 - 3c_3 + 6c_4)\,t - 3c_1 + c_2 + c_3 - 2c_4)\,e^{-t} - 2\,e^{2t}\left(\left(c_1 + \frac{c_4}{2}\right)t - \frac{3c_1}{2} + \frac{c_3}{2} - c_4\right) \\ ((-c_2 - c_3 + 2c_4)\,t - c_1)\,e^{-t} + 2\left(\left(c_1 + \frac{c_4}{2}\right)t + \frac{c_1}{2} + \frac{c_3}{2}\right)e^{2t} \\ ((-2c_2 - 2c_3 + 4c_4)\,t - 2c_1)\,e^{-t} + 2\left(c_1 + \frac{c_4}{2}\right)e^{2t} \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

6.29.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \\ x_4'(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 & 1 & -2 \\ 7 & -4 & -6 & 11 \\ 5 & -1 & 1 & 3 \\ 6 & -2 & -2 & 6 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -1 & 1 & 1 & -2 \\ 7 & -4 & -6 & 11 \\ 5 & -1 & 1 & 3 \\ 6 & -2 & -2 & 6 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -1-\lambda & 1 & 1 & -2 \\ 7 & -4-\lambda & -6 & 11 \\ 5 & -1 & 1-\lambda & 3 \\ 6 & -2 & -2 & 6-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^4 - 2\lambda^3 - 3\lambda^2 + 4\lambda + 4 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1$$
$$\lambda_2 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & 1 & 1 & -2 \\ 7 & -4 & -6 & 11 \\ 5 & -1 & 1 & 3 \\ 6 & -2 & -2 & 6 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 1 & -2 \\ 7 & -3 & -6 & 11 \\ 5 & -1 & 2 & 3 \\ 6 & -2 & -2 & 7 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} 0 & 1 & 1 & -2 & 0 \\ 7 & -3 & -6 & 11 & 0 \\ 5 & -1 & 2 & 3 & 0 \\ 6 & -2 & -2 & 7 & 0 \end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{cccc|c} 7 & -3 & -6 & 11 & 0 \\ 0 & 1 & 1 & -2 & 0 \\ 5 & -1 & 2 & 3 & 0 \\ 6 & -2 & -2 & 7 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{5R_1}{7} \Longrightarrow \left[\begin{array}{cccc|c} 7 & -3 & -6 & 11 & 0 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & \frac{8}{7} & \frac{44}{7} & -\frac{34}{7} & 0 \\ 6 & -2 & -2 & 7 & 0 \end{array}\right]$$

$$R_4 = R_4 - \frac{6R_1}{7} \Longrightarrow \left[\begin{array}{cccc|c} 7 & -3 & -6 & 11 & 0 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & \frac{8}{7} & \frac{44}{7} & -\frac{34}{7} & 0 \\ 0 & \frac{4}{7} & \frac{22}{7} & -\frac{17}{7} & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{8R_2}{7} \Longrightarrow \left[\begin{array}{cccc|c} 7 & -3 & -6 & 11 & 0 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & \frac{36}{7} & -\frac{18}{7} & 0 \\ 0 & \frac{4}{7} & \frac{22}{7} & -\frac{17}{7} & 0 \end{array}\right]$$

$$R_4 = R_4 - \frac{4R_2}{7} \Longrightarrow \left[\begin{array}{cccc|c} 7 & -3 & -6 & 11 & 0 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & \frac{36}{7} & -\frac{18}{7} & 0 \\ 0 & 0 & \frac{18}{7} & -\frac{9}{7} & 0 \end{array}\right]$$

$$R_4 = R_4 - \frac{R_3}{2} \Longrightarrow \left[\begin{array}{cccc|c} 7 & -3 & -6 & 11 & 0 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & \frac{36}{7} & -\frac{18}{7} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 7 & -3 & -6 & 11 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & \frac{36}{7} & -\frac{18}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_4\}$ and the leading variables are $\{v_1, v_2, v_3\}$. Let $v_4 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{t}{2}, v_2 = \frac{3t}{2}, v_3 = \frac{t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{t}{2} \\ \frac{3t}{2} \\ \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{t}{2} \\ \frac{3t}{2} \\ \frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this

eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{t}{2} \\ \frac{3t}{2} \\ \frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{2} \\ \frac{3}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{t}{2} \\ \frac{3t}{2} \\ \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{3}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{t}{2} \\ \frac{3t}{2} \\ \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & 1 & 1 & -2 \\ 7 & -4 & -6 & 11 \\ 5 & -1 & 1 & 3 \\ 6 & -2 & -2 & 6 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 1 & 1 & -2 \\ 7 & -6 & -6 & 11 \\ 5 & -1 & -1 & 3 \\ 6 & -2 & -2 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} -3 & 1 & 1 & -2 & 0 \\ 7 & -6 & -6 & 11 & 0 \\ 5 & -1 & -1 & 3 & 0 \\ 6 & -2 & -2 & 4 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{7R_1}{3} \Longrightarrow \left[\begin{array}{cccc|c} -3 & 1 & 1 & -2 & 0 \\ 0 & -\frac{11}{3} & -\frac{11}{3} & \frac{19}{3} & 0 \\ 5 & -1 & -1 & 3 & 0 \\ 6 & -2 & -2 & 4 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{5R_1}{3} \Longrightarrow \left[\begin{array}{cccc|c} -3 & 1 & 1 & -2 & 0 \\ 0 & -\frac{11}{3} & -\frac{11}{3} & \frac{19}{3} & 0 \\ 0 & \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} & 0 \\ 6 & -2 & -2 & 4 & 0 \end{array}\right]$$

$$R_4 = R_4 + 2R_1 \Longrightarrow \left[\begin{array}{cccc|c} -3 & 1 & 1 & -2 & 0 \\ 0 & -\frac{11}{3} & -\frac{11}{3} & \frac{19}{3} & 0 \\ 0 & \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{2R_2}{11} \Longrightarrow \left[\begin{array}{cccc|c} -3 & 1 & 1 & -2 & 0 \\ 0 & -\frac{11}{3} & -\frac{11}{3} & \frac{19}{3} & 0 \\ 0 & 0 & 0 & \frac{9}{11} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -3 & 1 & 1 & -2 \\ 0 & -\frac{11}{3} & -\frac{11}{3} & \frac{19}{3} \\ 0 & 0 & 0 & \frac{9}{11} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2, v_4\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_2 = -t, v_4 = 0\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ -t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -t \\ t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ -t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ -t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity algebraic m	multiplicity geometric k	defective?	eigenvectors
-1	2	1	Yes	$\begin{bmatrix} -\frac{1}{2} \\ \frac{3}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$
2	2	1	Yes	$\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue -1 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

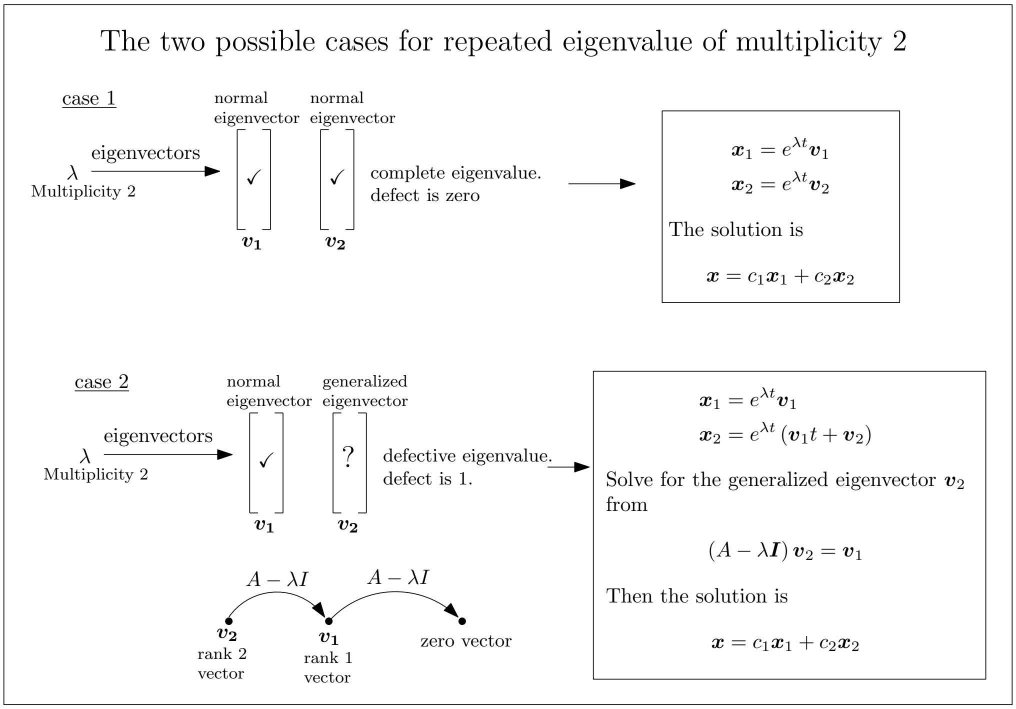


Figure 64: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} -1 & 1 & 1 & -2 \\ 7 & -4 & -6 & 11 \\ 5 & -1 & 1 & 3 \\ 6 & -2 & -2 & 6 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{3}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 1 & -2 \\ 7 & -3 & -6 & 11 \\ 5 & -1 & 2 & 3 \\ 6 & -2 & -2 & 7 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{3}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} -1 \\ \frac{5}{2} \\ 1 \\ 2 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue -1. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\
&= \begin{bmatrix} -\frac{1}{2} \\ \frac{3}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix} \mathrm{e}^{-t} \\
&= \begin{bmatrix} -\frac{\mathrm{e}^{-t}}{2} \\ \frac{3\,\mathrm{e}^{-t}}{2} \\ \frac{\mathrm{e}^{-t}}{2} \\ \mathrm{e}^{-t} \end{bmatrix}
\end{aligned}$$

And

$$\begin{aligned}
\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\
&= \left(\begin{bmatrix} -\frac{1}{2} \\ \frac{3}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix} t + \begin{bmatrix} -1 \\ \frac{5}{2} \\ 1 \\ 2 \end{bmatrix} \right) \mathrm{e}^{-t} \\
&= \begin{bmatrix} -\frac{\mathrm{e}^{-t}(2+t)}{2} \\ \frac{(3t+5)\mathrm{e}^{-t}}{2} \\ \frac{\mathrm{e}^{-t}(2+t)}{2} \\ \mathrm{e}^{-t}(2+t) \end{bmatrix}
\end{aligned}$$

eigenvalue 2 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

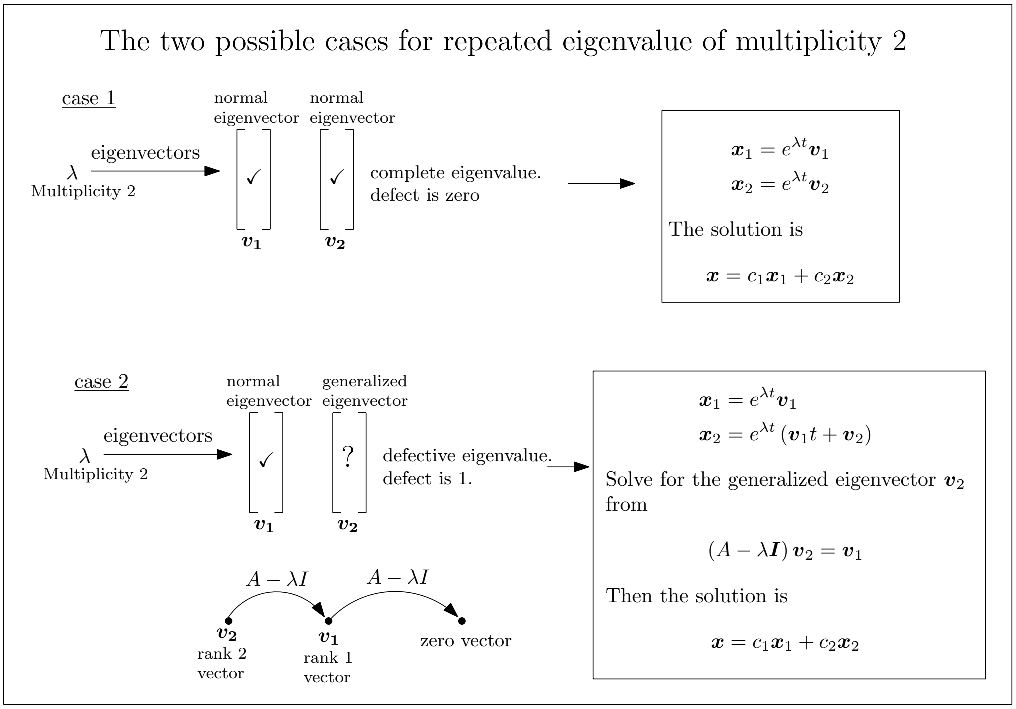


Figure 65: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A-\lambda I)\,\vec{v}_2=\vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix}-1 & 1 & 1 & -2\\ 7 & -4 & -6 & 11\\ 5 & -1 & 1 & 3\\ 6 & -2 & -2 & 6\end{bmatrix}-(2)\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\\ v_3\\ v_4\end{bmatrix}=\begin{bmatrix}0\\ -1\\ 1\\ 0\end{bmatrix}$$

$$\begin{bmatrix}-3 & 1 & 1 & -2\\ 7 & -6 & -6 & 11\\ 5 & -1 & -1 & 3\\ 6 & -2 & -2 & 4\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\\ v_4\end{bmatrix}=\begin{bmatrix}0\\ -1\\ 1\\ 0\end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue 2. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned} \vec{x}_3(t) &= \vec{v}_1 e^{\lambda t} \\ &= \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \mathrm{e}^{2t} \\ &= \begin{bmatrix} 0 \\ -\mathrm{e}^{2t} \\ \mathrm{e}^{2t} \\ 0 \end{bmatrix} \end{aligned}$$

And

$$\begin{aligned} \vec{x}_4(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\ &= \left(\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right) \mathrm{e}^{2t} \\ &= \begin{bmatrix} 0 \\ -\mathrm{e}^{2t}(-1+t) \\ \mathrm{e}^{2t}(t+1) \\ \mathrm{e}^{2t} \end{bmatrix} \end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t) + c_4 \vec{x}_4(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = c_1 \begin{bmatrix} -\frac{e^{-t}}{2} \\ \frac{3\,e^{-t}}{2} \\ \frac{e^{-t}}{2} \\ e^{-t} \end{bmatrix} + c_2 \begin{bmatrix} e^{-t}\left(-\frac{t}{2}-1\right) \\ e^{-t}\left(\frac{3t}{2}+\frac{5}{2}\right) \\ e^{-t}\left(\frac{t}{2}+1\right) \\ e^{-t}(2+t) \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ -e^{2t} \\ e^{2t} \\ 0 \end{bmatrix} + c_4 \begin{bmatrix} 0 \\ e^{2t}(1-t) \\ e^{2t}(t+1) \\ e^{2t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = \begin{bmatrix} -\frac{(c_2(2+t)+c_1)e^{-t}}{2} \\ \frac{(c_2(3t+5)+3c_1)e^{-t}}{2} - e^{2t}((-1+t)\,c_4 + c_3) \\ \frac{(c_2(2+t)+c_1)e^{-t}}{2} + ((t+1)\,c_4 + c_3)\,e^{2t} \\ (c_2(2+t)+c_1)\,e^{-t} + c_4 e^{2t} \end{bmatrix}$$

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 120

```
dsolve([diff(x__1(t),t)=-1*x__1(t)+1*x__2(t)+1*x__3(t)-2*x__4(t),diff(x__2(t),t)=7*x__1(t)-4
```

$$x_1(t) = e^{-t}(c_4 t + c_3)$$
$$x_2(t) = -3c_4 e^{-t} t - 3c_3 e^{-t} + c_4 e^{-t} + e^{2t} t c_1 + c_2 e^{2t}$$
$$x_3(t) = -c_4 e^{-t} t - c_3 e^{-t} - e^{2t} t c_1 - 2c_1 e^{2t} - c_2 e^{2t}$$
$$x_4(t) = -2c_4 e^{-t} t - 2c_3 e^{-t} - c_1 e^{2t}$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 196

```
DSolve[{x1'[t]==-1*x1[t]+1*x2[t]+1*x3[t]-2*x4[t],x2'[t]==7*x1[t]-4*x2[t]-6*x3[t]+11*x4[t],x3
```

$$\text{x1}(t) \to e^{-t}((c_2 + c_3 - 2c_4)t + c_1)$$
$$\text{x2}(t) \to e^{-t}\left(c_1\left(e^{3t}(3-2t)-3\right) - 3c_2 t - c_3 e^{3t} - 3c_3 t + 2c_4 e^{3t} - c_4 e^{3t} t + 6c_4 t + c_2 + c_3 - 2c_4\right)$$
$$\text{x3}(t) \to e^{-t}\left(c_1\left(e^{3t}(2t+1)-1\right) + c_3 e^{3t} - t\left(-c_4\left(e^{3t}+2\right) + c_2 + c_3\right)\right)$$
$$\text{x4}(t) \to e^{-t}\left(2c_1\left(e^{3t}-1\right) - 2(c_2 + c_3 - 2c_4)t + c_4 e^{3t}\right)$$

6.30 problem problem 30

Internal problem ID [387]
Internal file name [`OUTPUT/387_Sunday_June_05_2022_01_40_16_AM_81676938/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.6, Multiple Eigenvalue Solutions. Page 451
Problem number: problem 30.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= 2x_1(t) + x_2(t) - 2x_3(t) + x_4(t) \\
x_2'(t) &= 3x_2(t) - 5x_3(t) + 3x_4(t) \\
x_3'(t) &= -13x_2(t) + 22x_3(t) - 12x_4(t) \\
x_4'(t) &= -27x_2(t) + 45x_3(t) - 25x_4(t)
\end{aligned}
$$

6.30.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \\ x_4'(t) \end{bmatrix} = \begin{bmatrix} 2 & 1 & -2 & 1 \\ 0 & 3 & -5 & 3 \\ 0 & -13 & 22 & -12 \\ 0 & -27 & 45 & -25 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{2t} & e^{2t}t & -2\,e^{2t}t & e^{2t}t \\ 0 & (4t+1)\,e^{-t} & -5t\,e^{-t} & 3t\,e^{-t} \\ 0 & (-4t+3)\,e^{-t}-3\,e^{2t} & (5t-5)\,e^{-t}+6\,e^{2t} & (-3t+3)\,e^{-t}-3\,e^{2t} \\ 0 & (-12t+5)\,e^{-t}-5\,e^{2t} & (15t-10)\,e^{-t}+10\,e^{2t} & (-9t+6)\,e^{-t}-5\,e^{2t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{2t} & e^{2t}t & -2\,e^{2t}t & e^{2t}t \\ 0 & (4t+1)\,e^{-t} & -5t\,e^{-t} & 3t\,e^{-t} \\ 0 & (-4t+3)\,e^{-t}-3\,e^{2t} & (5t-5)\,e^{-t}+6\,e^{2t} & (-3t+3)\,e^{-t}-3\,e^{2t} \\ 0 & (-12t+5)\,e^{-t}-5\,e^{2t} & (15t-10)\,e^{-t}+10\,e^{2t} & (-9t+6)\,e^{-t}-5\,e^{2t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} \\
&= \begin{bmatrix} e^{2t}c_1 + e^{2t}tc_2 - 2\,e^{2t}tc_3 + e^{2t}tc_4 \\ (4t+1)\,e^{-t}c_2 - 5t\,e^{-t}c_3 + 3t\,e^{-t}c_4 \\ ((-4t+3)\,e^{-t}-3\,e^{2t})\,c_2 + ((5t-5)\,e^{-t}+6\,e^{2t})\,c_3 + ((-3t+3)\,e^{-t}-3\,e^{2t})\,c_4 \\ ((-12t+5)\,e^{-t}-5\,e^{2t})\,c_2 + ((15t-10)\,e^{-t}+10\,e^{2t})\,c_3 + ((-9t+6)\,e^{-t}-5\,e^{2t})\,c_4 \end{bmatrix} \\
&= \begin{bmatrix} ((c_2-2c_3+c_4)\,t+c_1)\,e^{2t} \\ e^{-t}(4c_2t-5c_3t+3c_4t+c_2) \\ \left((-4t+3)\,c_2+5(-1+t)\left(c_3-\frac{3c_4}{5}\right)\right)e^{-t}-3\,e^{2t}(c_2-2c_3+c_4) \\ \left((-12t+5)\,c_2+15\left(c_3-\frac{3c_4}{5}\right)\left(-\frac{2}{3}+t\right)\right)e^{-t}-5\,e^{2t}(c_2-2c_3+c_4) \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

6.30.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \\ x_4'(t) \end{bmatrix} = \begin{bmatrix} 2 & 1 & -2 & 1 \\ 0 & 3 & -5 & 3 \\ 0 & -13 & 22 & -12 \\ 0 & -27 & 45 & -25 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 2 & 1 & -2 & 1 \\ 0 & 3 & -5 & 3 \\ 0 & -13 & 22 & -12 \\ 0 & -27 & 45 & -25 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 2-\lambda & 1 & -2 & 1 \\ 0 & 3-\lambda & -5 & 3 \\ 0 & -13 & 22-\lambda & -12 \\ 0 & -27 & 45 & -25-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^4 - 2\lambda^3 - 3\lambda^2 + 4\lambda + 4 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1$$
$$\lambda_2 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 1 & -2 & 1 \\ 0 & 3 & -5 & 3 \\ 0 & -13 & 22 & -12 \\ 0 & -27 & 45 & -25 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 & -2 & 1 \\ 0 & 4 & -5 & 3 \\ 0 & -13 & 23 & -12 \\ 0 & -27 & 45 & -24 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} 3 & 1 & -2 & 1 & 0 \\ 0 & 4 & -5 & 3 & 0 \\ 0 & -13 & 23 & -12 & 0 \\ 0 & -27 & 45 & -24 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{13R_2}{4} \Longrightarrow \left[\begin{array}{cccc|c} 3 & 1 & -2 & 1 & 0 \\ 0 & 4 & -5 & 3 & 0 \\ 0 & 0 & \frac{27}{4} & -\frac{9}{4} & 0 \\ 0 & -27 & 45 & -24 & 0 \end{array}\right]$$

$$R_4 = R_4 + \frac{27R_2}{4} \Longrightarrow \left[\begin{array}{cccc|c} 3 & 1 & -2 & 1 & 0 \\ 0 & 4 & -5 & 3 & 0 \\ 0 & 0 & \frac{27}{4} & -\frac{9}{4} & 0 \\ 0 & 0 & \frac{45}{4} & -\frac{15}{4} & 0 \end{array}\right]$$

$$R_4 = R_4 - \frac{5R_3}{3} \Longrightarrow \left[\begin{array}{cccc|c} 3 & 1 & -2 & 1 & 0 \\ 0 & 4 & -5 & 3 & 0 \\ 0 & 0 & \frac{27}{4} & -\frac{9}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3 & 1 & -2 & 1 \\ 0 & 4 & -5 & 3 \\ 0 & 0 & \frac{27}{4} & -\frac{9}{4} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_4\}$ and the leading variables are $\{v_1, v_2, v_3\}$. Let $v_4 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = 0, v_2 = -\frac{t}{3}, v_3 = \frac{t}{3}\right\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ -\frac{t}{3} \\ \frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{t}{3} \\ \frac{t}{3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ -\frac{t}{3} \\ \frac{t}{3} \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ -\frac{1}{3} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ -\frac{t}{3} \\ \frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{3} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} 0 \\ -\frac{t}{3} \\ \frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 3 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 1 & -2 & 1 \\ 0 & 3 & -5 & 3 \\ 0 & -13 & 22 & -12 \\ 0 & -27 & 45 & -25 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & -2 & 1 \\ 0 & 1 & -5 & 3 \\ 0 & -13 & 20 & -12 \\ 0 & -27 & 45 & -27 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} 0 & 1 & -2 & 1 & 0 \\ 0 & 1 & -5 & 3 & 0 \\ 0 & -13 & 20 & -12 & 0 \\ 0 & -27 & 45 & -27 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cccc|c} 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & -3 & 2 & 0 \\ 0 & -13 & 20 & -12 & 0 \\ 0 & -27 & 45 & -27 & 0 \end{array}\right]$$

$$R_3 = R_3 + 13R_1 \Longrightarrow \left[\begin{array}{cccc|c} 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & -3 & 2 & 0 \\ 0 & 0 & -6 & 1 & 0 \\ 0 & -27 & 45 & -27 & 0 \end{array}\right]$$

$$R_4 = R_4 + 27R_1 \Longrightarrow \left[\begin{array}{cccc|c} 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & -3 & 2 & 0 \\ 0 & 0 & -6 & 1 & 0 \\ 0 & 0 & -9 & 0 & 0 \end{array}\right]$$

$$R_3 = R_3 - 2R_2 \Longrightarrow \left[\begin{array}{cccc|c} 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & -3 & 2 & 0 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & -9 & 0 & 0 \end{array}\right]$$

$$R_4 = R_4 - 3R_2 \Longrightarrow \left[\begin{array}{cccc|c} 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & -3 & 2 & 0 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -6 & 0 \end{array}\right]$$

$$R_4 = R_4 - 2R_3 \Longrightarrow \left[\begin{array}{cccc|c} 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & -3 & 2 & 0 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 0 & 1 & -2 & 1 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_1\}$ and the leading variables are $\{v_2, v_3, v_4\}$. Let $v_1 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_2 = 0, v_3 = 0, v_4 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this

eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ 0 \\ 0 \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-1	2	1	Yes	$\begin{bmatrix} 0 \\ -\frac{1}{3} \\ \frac{1}{3} \\ 1 \end{bmatrix}$
2	2	1	Yes	$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue -1 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

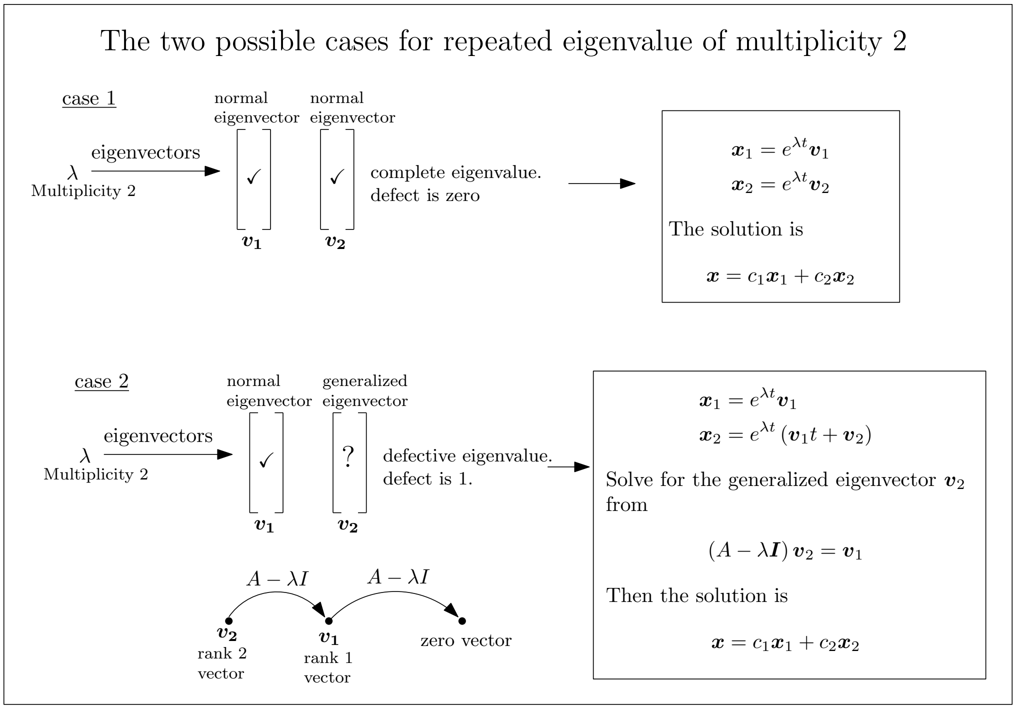


Figure 66: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} 2 & 1 & -2 & 1 \\ 0 & 3 & -5 & 3 \\ 0 & -13 & 22 & -12 \\ 0 & -27 & 45 & -25 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{3} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 & -2 & 1 \\ 0 & 4 & -5 & 3 \\ 0 & -13 & 23 & -12 \\ 0 & -27 & 45 & -24 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{3} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} 0 \\ -\frac{4}{3} \\ 1 \\ \frac{10}{3} \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue -1. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\ &= \begin{bmatrix} 0 \\ -\frac{1}{3} \\ \frac{1}{3} \\ 1 \end{bmatrix} \mathrm{e}^{-t} \\ &= \begin{bmatrix} 0 \\ -\frac{\mathrm{e}^{-t}}{3} \\ \frac{\mathrm{e}^{-t}}{3} \\ \mathrm{e}^{-t} \end{bmatrix} \end{aligned}$$

And

$$
\begin{aligned}
\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\
&= \left(\begin{bmatrix} 0 \\ -\frac{1}{3} \\ \frac{1}{3} \\ 1 \end{bmatrix} t + \begin{bmatrix} 0 \\ -\frac{4}{3} \\ 1 \\ \frac{10}{3} \end{bmatrix} \right) \mathrm{e}^{-t} \\
&= \begin{bmatrix} 0 \\ -\frac{\mathrm{e}^{-t}(t+4)}{3} \\ \frac{\mathrm{e}^{-t}(t+3)}{3} \\ \frac{\mathrm{e}^{-t}(3t+10)}{3} \end{bmatrix}
\end{aligned}
$$

eigenvalue 2 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

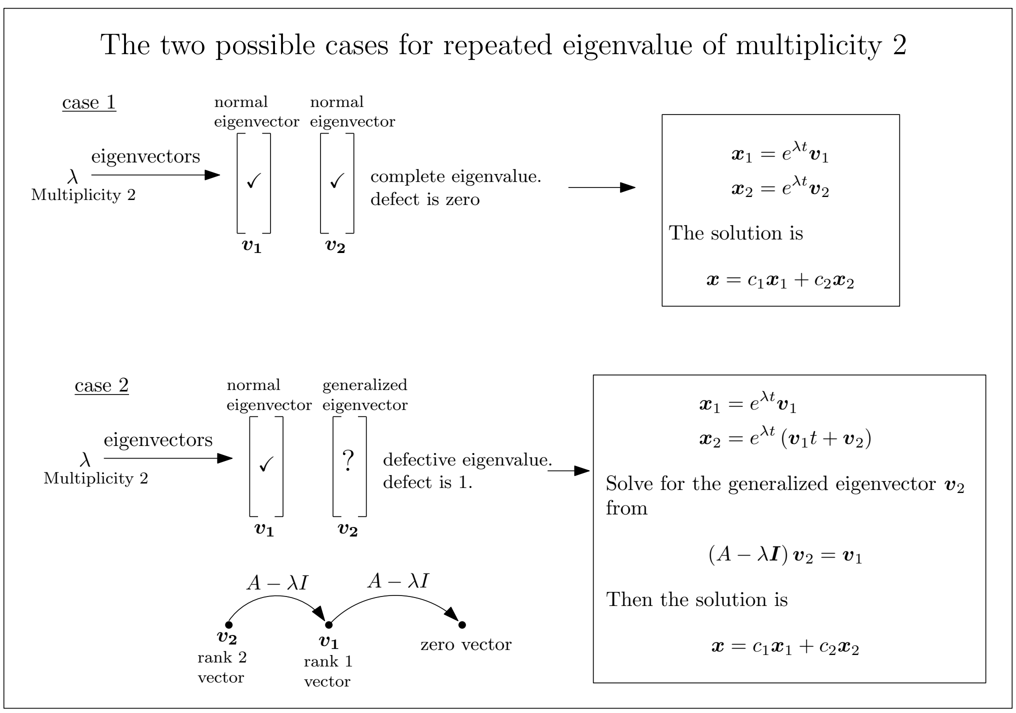


Figure 67: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} 2 & 1 & -2 & 1 \\ 0 & 3 & -5 & 3 \\ 0 & -13 & 22 & -12 \\ 0 & -27 & 45 & -25 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & -2 & 1 \\ 0 & 1 & -5 & 3 \\ 0 & -13 & 20 & -12 \\ 0 & -27 & 45 & -27 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -3 \\ -5 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue 2. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned} \vec{x}_3(t) &= \vec{v}_1 e^{\lambda t} \\ &= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \mathrm{e}^{2t} \\ &= \begin{bmatrix} \mathrm{e}^{2t} \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

And

$$
\begin{aligned}
\vec{x}_4(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\
&= \left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} t + \begin{bmatrix} 1 \\ 0 \\ -3 \\ -5 \end{bmatrix} \right) \mathrm{e}^{2t} \\
&= \begin{bmatrix} \mathrm{e}^{2t}(t+1) \\ 0 \\ -3\,\mathrm{e}^{2t} \\ -5\,\mathrm{e}^{2t} \end{bmatrix}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t) + c_4 \vec{x}_4(t)
$$

Which is written as

$$
\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ -\frac{\mathrm{e}^{-t}}{3} \\ \frac{\mathrm{e}^{-t}}{3} \\ \mathrm{e}^{-t} \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ \mathrm{e}^{-t}\left(-\frac{t}{3} - \frac{4}{3}\right) \\ \mathrm{e}^{-t}\left(\frac{t}{3} + 1\right) \\ \mathrm{e}^{-t}\left(t + \frac{10}{3}\right) \end{bmatrix} + c_3 \begin{bmatrix} \mathrm{e}^{2t} \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_4 \begin{bmatrix} \mathrm{e}^{2t}(t+1) \\ 0 \\ -3\,\mathrm{e}^{2t} \\ -5\,\mathrm{e}^{2t} \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{2t}(c_4 t + c_3 + c_4) \\ -\frac{((t+4)c_2 + c_1)\mathrm{e}^{-t}}{3} \\ \frac{(c_2(t+3)+c_1)\mathrm{e}^{-t}}{3} - 3c_4 \mathrm{e}^{2t} \\ \frac{(c_2(3t+10)+3c_1)\mathrm{e}^{-t}}{3} - 5c_4 \mathrm{e}^{2t} \end{bmatrix}
$$

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 89

```
dsolve([diff(x__1(t),t)=2*x__1(t)+1*x__2(t)-2*x__3(t)+1*x__4(t),diff(x__2(t),t)=0*x__1(t)+3*
```

$$x_1(t) = \frac{(-c_2t + 3c_1)\,e^{2t}}{3}$$
$$x_2(t) = e^{-t}(c_4t + c_3)$$
$$x_3(t) = \left(-e^{-3t}(c_4t + c_3 - c_4) + c_2\right)e^{2t}$$
$$x_4(t) = -3c_3e^{-t} - 3c_4e^{-t}t + 2c_4e^{-t} + \frac{5c_2e^{2t}}{3}$$

✓ Solution by Mathematica

Time used: 0.008 (sec). Leaf size: 161

```
DSolve[{x1'[t]==2*x1[t]+1*x2[t]-2*x3[t]+1*x4[t],x2'[t]==0*x1[t]+3*x2[t]-5*x3[t]+3*x4[t],x3'[
```

$$\text{x1}(t) \to e^{2t}((c_2 - 2c_3 + c_4)t + c_1)$$
$$\text{x2}(t) \to e^{-t}(4c_2t - 5c_3t + 3c_4t + c_2)$$
$$\text{x3}(t) \to e^{-t}\left(c_2\left(-4t - 3e^{3t} + 3\right) + c_3\left(5t + 6e^{3t} - 5\right) - 3c_4\left(t + e^{3t} - 1\right)\right)$$
$$\text{x4}(t) \to e^{-t}\left(c_2\left(-12t - 5e^{3t} + 5\right) + 5c_3\left(3t + 2e^{3t} - 2\right) - c_4\left(9t + 5e^{3t} - 6\right)\right)$$

6.31 problem problem 31

Internal problem ID [388]
Internal file name [OUTPUT/388_Sunday_June_05_2022_01_40_19_AM_72207861/index.tex]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.6, Multiple Eigenvalue Solutions. Page 451
Problem number: problem 31.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= 35x_1(t) - 12x_2(t) + 4x_3(t) + 30x_4(t) \\
x_2'(t) &= 22x_1(t) - 8x_2(t) + 3x_3(t) + 19x_4(t) \\
x_3'(t) &= -10x_1(t) + 3x_2(t) - 9x_4(t) \\
x_4'(t) &= -27x_1(t) + 9x_2(t) - 3x_3(t) - 23x_4(t)
\end{aligned}
$$

6.31.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \\ x_4'(t) \end{bmatrix} = \begin{bmatrix} 35 & -12 & 4 & 30 \\ 22 & -8 & 3 & 19 \\ -10 & 3 & 0 & -9 \\ -27 & 9 & -3 & -23 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^t(21t^2+34t+1) & (-9t^2-12t)\,e^t & e^t t(3t+4) & (18t^2+30t)\,e^t \\ \frac{e^t t(7t+44)}{2} & -\frac{3\left(t^2+6t-\frac{2}{3}\right)e^t}{2} & \frac{e^t t(t+6)}{2} & e^t t(3t+19) \\ -\frac{e^t t(21t+20)}{2} & \frac{3\,e^t t(3t+2)}{2} & e^t\left(1-t-\frac{3}{2}t^2\right) & -9\,e^t t(t+1) \\ (-21t^2-27t)\,e^t & 9\,e^t t(t+1) & -3\,e^t t(t+1) & e^t(-18t^2-24t+1) \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} e^t(21t^2+34t+1) & (-9t^2-12t)\,e^t & e^t t(3t+4) & (18t^2+30t)\,e^t \\ \frac{e^t t(7t+44)}{2} & -\frac{3\left(t^2+6t-\frac{2}{3}\right)e^t}{2} & \frac{e^t t(t+6)}{2} & e^t t(3t+19) \\ -\frac{e^t t(21t+20)}{2} & \frac{3\,e^t t(3t+2)}{2} & e^t\left(1-t-\frac{3}{2}t^2\right) & -9\,e^t t(t+1) \\ (-21t^2-27t)\,e^t & 9\,e^t t(t+1) & -3\,e^t t(t+1) & e^t(-18t^2-24t+1) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} \\ &= \begin{bmatrix} e^t(21t^2+34t+1)\,c_1 + (-9t^2-12t)\,e^t c_2 + e^t t(3t+4)\,c_3 + (18t^2+30t)\,e^t c_4 \\ \frac{e^t t(7t+44)c_1}{2} - \frac{3\left(t^2+6t-\frac{2}{3}\right)e^t c_2}{2} + \frac{e^t t(t+6)c_3}{2} + e^t t(3t+19)\,c_4 \\ -\frac{e^t t(21t+20)c_1}{2} + \frac{3\,e^t t(3t+2)c_2}{2} + e^t\left(1-t-\frac{3}{2}t^2\right)c_3 - 9\,e^t t(t+1)\,c_4 \\ (-21t^2-27t)\,e^t c_1 + 9\,e^t t(t+1)\,c_2 - 3\,e^t t(t+1)\,c_3 + e^t(-18t^2-24t+1)\,c_4 \end{bmatrix} \\ &= \begin{bmatrix} 21\left(\left(c_1-\frac{3c_2}{7}+\frac{c_3}{7}+\frac{6c_4}{7}\right)t^2 + \frac{2\left(\frac{17c_1}{3}-2c_2+\frac{2c_3}{3}+5c_4\right)t}{7} + \frac{c_1}{21}\right)e^t \\ \frac{7\left(\left(c_1-\frac{3c_2}{7}+\frac{c_3}{7}+\frac{6c_4}{7}\right)t^2+\frac{2(22c_1-9c_2+3c_3+19c_4)t}{7}+\frac{2c_2}{7}\right)e^t}{2} \\ -\frac{21\left(\left(c_1-\frac{3c_2}{7}+\frac{c_3}{7}+\frac{6c_4}{7}\right)t^2+\frac{2\left(\frac{10c_1}{3}-c_2+\frac{c_3}{3}+3c_4\right)t}{7}-\frac{2c_3}{21}\right)e^t}{2} \\ -21\left(\left(c_1-\frac{3c_2}{7}+\frac{c_3}{7}+\frac{6c_4}{7}\right)t^2 + \frac{(9c_1-3c_2+c_3+8c_4)t}{7} - \frac{c_4}{21}\right)e^t \end{bmatrix} \end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

6.31.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \\ x_4'(t) \end{bmatrix} = \begin{bmatrix} 35 & -12 & 4 & 30 \\ 22 & -8 & 3 & 19 \\ -10 & 3 & 0 & -9 \\ -27 & 9 & -3 & -23 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 35 & -12 & 4 & 30 \\ 22 & -8 & 3 & 19 \\ -10 & 3 & 0 & -9 \\ -27 & 9 & -3 & -23 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 35-\lambda & -12 & 4 & 30 \\ 22 & -8-\lambda & 3 & 19 \\ -10 & 3 & -\lambda & -9 \\ -27 & 9 & -3 & -23-\lambda \end{bmatrix} \right) = 0$$

Which gives the characteristic equation

$$\lambda^4 - 4\lambda^3 + 6\lambda^2 - 4\lambda + 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 35 & -12 & 4 & 30 \\ 22 & -8 & 3 & 19 \\ -10 & 3 & 0 & -9 \\ -27 & 9 & -3 & -23 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 34 & -12 & 4 & 30 \\ 22 & -9 & 3 & 19 \\ -10 & 3 & -1 & -9 \\ -27 & 9 & -3 & -24 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} 34 & -12 & 4 & 30 & 0 \\ 22 & -9 & 3 & 19 & 0 \\ -10 & 3 & -1 & -9 & 0 \\ -27 & 9 & -3 & -24 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{11R_1}{17} \Longrightarrow \left[\begin{array}{cccc|c} 34 & -12 & 4 & 30 & 0 \\ 0 & -\frac{21}{17} & \frac{7}{17} & -\frac{7}{17} & 0 \\ -10 & 3 & -1 & -9 & 0 \\ -27 & 9 & -3 & -24 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{5R_1}{17} \Longrightarrow \left[\begin{array}{cccc|c} 34 & -12 & 4 & 30 & 0 \\ 0 & -\frac{21}{17} & \frac{7}{17} & -\frac{7}{17} & 0 \\ 0 & -\frac{9}{17} & \frac{3}{17} & -\frac{3}{17} & 0 \\ -27 & 9 & -3 & -24 & 0 \end{array}\right]$$

$$R_4 = R_4 + \frac{27R_1}{34} \Longrightarrow \left[\begin{array}{cccc|c} 34 & -12 & 4 & 30 & 0 \\ 0 & -\frac{21}{17} & \frac{7}{17} & -\frac{7}{17} & 0 \\ 0 & -\frac{9}{17} & \frac{3}{17} & -\frac{3}{17} & 0 \\ 0 & -\frac{9}{17} & \frac{3}{17} & -\frac{3}{17} & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{3R_2}{7} \Longrightarrow \left[\begin{array}{cccc|c} 34 & -12 & 4 & 30 & 0 \\ 0 & -\frac{21}{17} & \frac{7}{17} & -\frac{7}{17} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{9}{17} & \frac{3}{17} & -\frac{3}{17} & 0 \end{array}\right]$$

$$R_4 = R_4 - \frac{3R_2}{7} \Longrightarrow \left[\begin{array}{cccc|c} 34 & -12 & 4 & 30 & 0 \\ 0 & -\frac{21}{17} & \frac{7}{17} & -\frac{7}{17} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 34 & -12 & 4 & 30 \\ 0 & -\frac{21}{17} & \frac{7}{17} & -\frac{7}{17} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3, v_4\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Let $v_4 = s$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -s, v_2 = \frac{t}{3} - \frac{s}{3}\right\}$

Hence the solution is

$$\begin{bmatrix} -s \\ \frac{t}{3} - \frac{s}{3} \\ t \\ s \end{bmatrix} = \begin{bmatrix} -s \\ \frac{t}{3} - \frac{s}{3} \\ t \\ s \end{bmatrix}$$

Since there are two free Variable, we have found two eigenvectors associated with this

eigenvalue. The above can be written as

$$\begin{bmatrix} -s \\ \frac{t}{3} - \frac{s}{3} \\ t \\ s \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{t}{3} \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} -s \\ -\frac{s}{3} \\ 0 \\ s \end{bmatrix}$$

$$= t \begin{bmatrix} 0 \\ \frac{1}{3} \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ -\frac{1}{3} \\ 0 \\ 1 \end{bmatrix}$$

By letting $t = 1$ and $s = 1$ then the above becomes

$$\begin{bmatrix} -s \\ \frac{t}{3} - \frac{s}{3} \\ t \\ s \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{3} \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ -\frac{1}{3} \\ 0 \\ 1 \end{bmatrix}$$

Hence the two eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix} 0 \\ \frac{1}{3} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -\frac{1}{3} \\ 0 \\ 1 \end{bmatrix} \right)$$

Which are normalized to

$$\left(\begin{bmatrix} 0 \\ 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -1 \\ 0 \\ 3 \end{bmatrix} \right)$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
1	4	2	Yes	$\begin{bmatrix} -1 & 0 \\ -\frac{1}{3} & \frac{1}{3} \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

This case will be solved using the Jordan form of the matrix A. The Jordan form diagonalization is

$$A = PJP^{-1}$$

Which can be found to be

$$\begin{bmatrix} 35 & -12 & 4 & 30 \\ 22 & -8 & 3 & 19 \\ -10 & 3 & 0 & -9 \\ -27 & 9 & -3 & -23 \end{bmatrix} = \begin{bmatrix} 42 & 34 & -\frac{27}{49} & -\frac{76}{49} \\ 7 & 22 & -\frac{22}{49} & -\frac{22}{49} \\ -21 & -10 & \frac{10}{49} & \frac{10}{49} \\ -42 & -27 & \frac{76}{49} & \frac{76}{49} \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 42 & 34 & -\frac{27}{49} & -\frac{76}{49} \\ 7 & 22 & -\frac{22}{49} & -\frac{22}{49} \\ -21 & -10 & \frac{10}{49} & \frac{10}{49} \\ -42 & -27 & \frac{76}{49} & \frac{76}{49} \end{bmatrix}^{-1}$$

Looking at the P matrix above, we see there are 2 chains. Therefore, we now construct the basis solution by following these chains as follows.

$$\vec{x}_1 = \begin{bmatrix} 42\,\mathrm{e}^{t} \\ 7\,\mathrm{e}^{t} \\ -21\,\mathrm{e}^{t} \\ -42\,\mathrm{e}^{t} \end{bmatrix}$$

$$\vec{x}_2 = \begin{bmatrix} 42t\,\mathrm{e}^{t}+34\,\mathrm{e}^{t} \\ 7t\,\mathrm{e}^{t}+22\,\mathrm{e}^{t} \\ -21t\,\mathrm{e}^{t}-10\,\mathrm{e}^{t} \\ -42t\,\mathrm{e}^{t}-27\,\mathrm{e}^{t} \end{bmatrix}$$

$$\vec{x}_3 = \begin{bmatrix} 21t^{2}\mathrm{e}^{t}+34t\,\mathrm{e}^{t}-\frac{27\,\mathrm{e}^{t}}{49} \\ \frac{7t^{2}\mathrm{e}^{t}}{2}+22t\,\mathrm{e}^{t}-\frac{22\,\mathrm{e}^{t}}{49} \\ -\frac{21t^{2}\mathrm{e}^{t}}{2}-10t\,\mathrm{e}^{t}+\frac{10\,\mathrm{e}^{t}}{49} \\ -21t^{2}\mathrm{e}^{t}-27t\,\mathrm{e}^{t}+\frac{76\,\mathrm{e}^{t}}{49} \end{bmatrix}$$

$$\vec{x}_4 = \begin{bmatrix} -\frac{76\,\mathrm{e}^{t}}{49} \\ -\frac{22\,\mathrm{e}^{t}}{49} \\ \frac{10\,\mathrm{e}^{t}}{49} \\ \frac{76\,\mathrm{e}^{t}}{49} \end{bmatrix}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t) + c_4\vec{x}_4(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = c_1 \begin{bmatrix} 42\,\mathrm{e}^{t} \\ 7\,\mathrm{e}^{t} \\ -21\,\mathrm{e}^{t} \\ -42\,\mathrm{e}^{t} \end{bmatrix} + c_2 \begin{bmatrix} 42t\,\mathrm{e}^{t}+34\,\mathrm{e}^{t} \\ 7t\,\mathrm{e}^{t}+22\,\mathrm{e}^{t} \\ -21t\,\mathrm{e}^{t}-10\,\mathrm{e}^{t} \\ -42t\,\mathrm{e}^{t}-27\,\mathrm{e}^{t} \end{bmatrix} + c_3 \begin{bmatrix} 21t^{2}\mathrm{e}^{t}+34t\,\mathrm{e}^{t}-\frac{27\,\mathrm{e}^{t}}{49} \\ \frac{7t^{2}\mathrm{e}^{t}}{2}+22t\,\mathrm{e}^{t}-\frac{22\,\mathrm{e}^{t}}{49} \\ -\frac{21t^{2}\mathrm{e}^{t}}{2}-10t\,\mathrm{e}^{t}+\frac{10\,\mathrm{e}^{t}}{49} \\ -21t^{2}\mathrm{e}^{t}-27t\,\mathrm{e}^{t}+\frac{76\,\mathrm{e}^{t}}{49} \end{bmatrix} + c_4 \begin{bmatrix} -\frac{76\,\mathrm{e}^{t}}{49} \\ -\frac{22\,\mathrm{e}^{t}}{49} \\ \frac{10\,\mathrm{e}^{t}}{49} \\ \frac{76\,\mathrm{e}^{t}}{49} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = \begin{bmatrix} e^t\left(42c_1 + 42tc_2 + 34c_2 + 21c_3t^2 + 34c_3t - \frac{27}{49}c_3 - \frac{76}{49}c_4\right) \\ \frac{((343t^2+2156t-44)c_3+686tc_2+686c_1+2156c_2-44c_4)e^t}{98} \\ \frac{((-1029t^2-980t+20)c_3-2058tc_2-2058c_1-980c_2+20c_4)e^t}{98} \\ e^t\left(-42c_1 - 42tc_2 - 27c_2 - 21c_3t^2 - 27c_3t + \frac{76}{49}c_3 + \frac{76}{49}c_4\right) \end{bmatrix}$$

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 117

```
dsolve([diff(x__1(t),t)=35*x__1(t)-12*x__2(t)+4*x__3(t)+30*x__4(t),diff(x__2(t),t)=22*x__1(t
```

$$x_1(t) = \frac{e^t(-6c_4t^2 - 6c_3t - 4c_4t + 3c_1 - 6c_2 - 2c_3)}{3}$$

$$x_2(t) = \frac{e^t(-3c_4t^2 - 3c_3t - 16c_4t + 3c_1 - 3c_2 - 8c_3 + 6c_4)}{9}$$

$$x_3(t) = e^t\left(c_4t^2 + c_3t + c_2\right)$$

$$x_4(t) = -\frac{e^t(-18c_4t^2 - 18c_3t - 6c_4t + 9c_1 - 18c_2 - 3c_3 - 2c_4)}{9}$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 207

```
DSolve[{x1'[t]==35*x1[t]-12*x2[t]+4*x3[t]+30*x4[t],x2'[t]==22*x1[t]-8*x2[t]+3*x3[t]+19*x4[t]
```

$$\text{x1}(t) \to e^t\left(c_1\left(21t^2 + 34t + 1\right) - 3c_2t(3t + 4) + c_3t(3t + 4) + 6c_4t(3t + 5)\right)$$

$$\text{x2}(t) \to \frac{1}{2}e^t\left((7c_1 - 3c_2 + c_3 + 6c_4)t^2 + 2(22c_1 - 9c_2 + 3c_3 + 19c_4)t + 2c_2\right)$$

$$\text{x3}(t) \to \frac{1}{2}e^t\left(-3(7c_1 - 3c_2 + c_3 + 6c_4)t^2 - 2(10c_1 - 3c_2 + c_3 + 9c_4)t + 2c_3\right)$$

$$\text{x4}(t) \to e^t\left(-3(7c_1 - 3c_2 + c_3 + 6c_4)t^2 - 3(9c_1 - 3c_2 + c_3 + 8c_4)t + c_4\right)$$

6.32 problem problem 32

Internal problem ID [389]
Internal file name [`OUTPUT/389_Sunday_June_05_2022_01_40_21_AM_25067120/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.6, Multiple Eigenvalue Solutions. Page 451
Problem number: problem 32.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= 11x_1(t) - x_2(t) + 26x_3(t) + 6x_4(t) - 3x_5(t)\\
x_2'(t) &= 3x_2(t)\\
x_3'(t) &= -9x_1(t) - 24x_3(t) - 6x_4(t) + 3x_5(t)\\
x_4'(t) &= 3x_1(t) + 9x_3(t) + 5x_4(t) - x_5(t)\\
x_5'(t) &= -48x_1(t) - 3x_2(t) - 138x_3(t) - 30x_4(t) + 18x_5(t)
\end{aligned}
$$

6.32.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$
\begin{bmatrix} x_1'(t)\\ x_2'(t)\\ x_3'(t)\\ x_4'(t)\\ x_5'(t) \end{bmatrix} = \begin{bmatrix} 11 & -1 & 26 & 6 & -3\\ 0 & 3 & 0 & 0 & 0\\ -9 & 0 & -24 & -6 & 3\\ 3 & 0 & 9 & 5 & -1\\ -48 & -3 & -138 & -30 & 18 \end{bmatrix} \begin{bmatrix} x_1(t)\\ x_2(t)\\ x_3(t)\\ x_4(t)\\ x_5(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} 9\,e^{3t} - 8\,e^{2t} & -e^{3t} + e^{2t} & 26\,e^{3t} - 26\,e^{2t} & 6\,e^{3t} - 6\,e^{2t} & -3\,e^{3t} + 3\,e^{2t} \\ 0 & e^{3t} & 0 & 0 & 0 \\ -9\,e^{3t} + 9\,e^{2t} & 0 & -26\,e^{3t} + 27\,e^{2t} & -6\,e^{3t} + 6\,e^{2t} & -3\,e^{2t} + 3\,e^{3t} \\ -3\,e^{2t} + 3\,e^{3t} & 0 & 9\,e^{3t} - 9\,e^{2t} & 3\,e^{3t} - 2\,e^{2t} & -e^{3t} + e^{2t} \\ -48\,e^{3t} + 48\,e^{2t} & -3\,e^{3t} + 3\,e^{2t} & -138\,e^{3t} + 138\,e^{2t} & -30\,e^{3t} + 30\,e^{2t} & 16\,e^{3t} - 15\,e^{2t} \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} 9\,e^{3t} - 8\,e^{2t} & -e^{3t} + e^{2t} & 26\,e^{3t} - 26\,e^{2t} & 6\,e^{3t} - 6\,e^{2t} & -3\,e^{3t} + 3\,e^{2t} \\ 0 & e^{3t} & 0 & 0 & 0 \\ -9\,e^{3t} + 9\,e^{2t} & 0 & -26\,e^{3t} + 27\,e^{2t} & -6\,e^{3t} + 6\,e^{2t} & -3\,e^{2t} + 3\,e^{3t} \\ -3\,e^{2t} + 3\,e^{3t} & 0 & 9\,e^{3t} - 9\,e^{2t} & 3\,e^{3t} - 2\,e^{2t} & -e^{3t} + e^{2t} \\ -48\,e^{3t} + 48\,e^{2t} & -3\,e^{3t} + 3\,e^{2t} & -138\,e^{3t} + 138\,e^{2t} & -30\,e^{3t} + 30\,e^{2t} & 16\,e^{3t} - 15\,e^{2t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{bmatrix} \\
&= \begin{bmatrix} \left(9\,e^{3t} - 8\,e^{2t}\right) c_1 + \left(-e^{3t} + e^{2t}\right) c_2 + \left(26\,e^{3t} - 26\,e^{2t}\right) c_3 + \left(6\,e^{3t} - 6\,e^{2t}\right) c_4 + (-3\,e^{3t} + 3\,e \\ e^{3t} c_2 \\ \left(-9\,e^{3t} + 9\,e^{2t}\right) c_1 + \left(-26\,e^{3t} + 27\,e^{2t}\right) c_3 + \left(-6\,e^{3t} + 6\,e^{2t}\right) c_4 + \left(-3\,e^{2t} + 3\,e^{3t}\right) c_5 \\ \left(-3\,e^{2t} + 3\,e^{3t}\right) c_1 + \left(9\,e^{3t} - 9\,e^{2t}\right) c_3 + \left(3\,e^{3t} - 2\,e^{2t}\right) c_4 + \left(-e^{3t} + e^{2t}\right) c_5 \\ \left(-48\,e^{3t} + 48\,e^{2t}\right) c_1 + \left(-3\,e^{3t} + 3\,e^{2t}\right) c_2 + \left(-138\,e^{3t} + 138\,e^{2t}\right) c_3 + \left(-30\,e^{3t} + 30\,e^{2t}\right) c_4 + (16\,e \end{bmatrix} \\
&= \begin{bmatrix} \left(-8c_1 + c_2 - 26c_3 - 6c_4 + 3c_5\right) e^{2t} + 9\left(c_1 - \frac{c_2}{9} + \frac{26c_3}{9} + \frac{2c_4}{3} - \frac{c_5}{3}\right) e^{3t} \\ e^{3t} c_2 \\ \left(9c_1 + 27c_3 + 6c_4 - 3c_5\right) e^{2t} - 9\left(c_1 + \frac{26c_3}{9} + \frac{2c_4}{3} - \frac{c_5}{3}\right) e^{3t} \\ \left(-3c_1 - 9c_3 - 2c_4 + c_5\right) e^{2t} + 3\,e^{3t}\left(c_1 + 3c_3 + c_4 - \frac{c_5}{3}\right) \\ 3(16c_1 + c_2 + 46c_3 + 10c_4 - 5c_5)\,e^{2t} - 48\left(c_1 + \frac{c_2}{16} + \frac{23c_3}{8} + \frac{5c_4}{8} - \frac{c_5}{3}\right) e^{3t} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

6.32.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \\ x_4'(t) \\ x_5'(t) \end{bmatrix} = \begin{bmatrix} 11 & -1 & 26 & 6 & -3 \\ 0 & 3 & 0 & 0 & 0 \\ -9 & 0 & -24 & -6 & 3 \\ 3 & 0 & 9 & 5 & -1 \\ -48 & -3 & -138 & -30 & 18 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ x_5(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 11 & -1 & 26 & 6 & -3 \\ 0 & 3 & 0 & 0 & 0 \\ -9 & 0 & -24 & -6 & 3 \\ 3 & 0 & 9 & 5 & -1 \\ -48 & -3 & -138 & -30 & 18 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 11-\lambda & -1 & 26 & 6 & -3 \\ 0 & 3-\lambda & 0 & 0 & 0 \\ -9 & 0 & -24-\lambda & -6 & 3 \\ 3 & 0 & 9 & 5-\lambda & -1 \\ -48 & -3 & -138 & -30 & 18-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^5 - 13\lambda^4 + 67\lambda^3 - 171\lambda^2 + 216\lambda - 108 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 3$$
$$\lambda_2 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
2	1	real eigenvalue
3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 11 & -1 & 26 & 6 & -3 \\ 0 & 3 & 0 & 0 & 0 \\ -9 & 0 & -24 & -6 & 3 \\ 3 & 0 & 9 & 5 & -1 \\ -48 & -3 & -138 & -30 & 18 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 9 & -1 & 26 & 6 & -3 \\ 0 & 1 & 0 & 0 & 0 \\ -9 & 0 & -26 & -6 & 3 \\ 3 & 0 & 9 & 3 & -1 \\ -48 & -3 & -138 & -30 & 16 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccccc|c} 9 & -1 & 26 & 6 & -3 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -9 & 0 & -26 & -6 & 3 & 0 \\ 3 & 0 & 9 & 3 & -1 & 0 \\ -48 & -3 & -138 & -30 & 16 & 0 \end{array}\right]$$

$$R_3 = R_3 + R_1 \Longrightarrow \left[\begin{array}{ccccc|c} 9 & -1 & 26 & 6 & -3 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 3 & 0 & 9 & 3 & -1 & 0 \\ -48 & -3 & -138 & -30 & 16 & 0 \end{array}\right]$$

$$R_4 = R_4 - \frac{R_1}{3} \Longrightarrow \left[\begin{array}{ccccc|c} 9 & -1 & 26 & 6 & -3 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & 1 & 0 & 0 \\ -48 & -3 & -138 & -30 & 16 & 0 \end{array}\right]$$

$$R_5 = R_5 + \frac{16R_1}{3} \Longrightarrow \left[\begin{array}{ccccc|c} 9 & -1 & 26 & 6 & -3 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & 1 & 0 & 0 \\ 0 & -\frac{25}{3} & \frac{2}{3} & 2 & 0 & 0 \end{array}\right]$$

$$R_3 = R_3 + R_2 \Longrightarrow \left[\begin{array}{ccccc|c} 9 & -1 & 26 & 6 & -3 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & 1 & 0 & 0 \\ 0 & -\frac{25}{3} & \frac{2}{3} & 2 & 0 & 0 \end{array}\right]$$

$$R_4 = R_4 - \frac{R_2}{3} \Longrightarrow \left[\begin{array}{ccccc|c} 9 & -1 & 26 & 6 & -3 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 1 & 0 & 0 \\ 0 & -\frac{25}{3} & \frac{2}{3} & 2 & 0 & 0 \end{array}\right]$$

$$R_5 = R_5 + \frac{25R_2}{3} \Longrightarrow \left[\begin{array}{ccccc|c} 9 & -1 & 26 & 6 & -3 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 1 & 0 & 0 \\ 0 & 0 & \frac{2}{3} & 2 & 0 & 0 \end{array}\right]$$

Since the current pivot $A(3,3)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 3 and row 4 gives

$$\left[\begin{array}{ccccc|c} 9 & -1 & 26 & 6 & -3 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{3} & 2 & 0 & 0 \end{array}\right]$$

$$R_5 = R_5 - 2R_3 \Longrightarrow \left[\begin{array}{ccccc|c} 9 & -1 & 26 & 6 & -3 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 9 & -1 & 26 & 6 & -3 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_4, v_5\}$ and the leading variables are $\{v_1, v_2, v_3\}$. Let $v_4 = t$. Let $v_5 = s$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = 8t + \frac{s}{3}, v_2 = 0, v_3 = -3t\right\}$

Hence the solution is

$$\begin{bmatrix} 8t+\frac{s}{3} \\ 0 \\ -3t \\ t \\ s \end{bmatrix} = \begin{bmatrix} 8t+\frac{s}{3} \\ 0 \\ -3t \\ t \\ s \end{bmatrix}$$

Since there are two free Variable, we have found two eigenvectors associated with this eigenvalue. The above can be written as

$$\begin{aligned} \begin{bmatrix} 8t+\frac{s}{3} \\ 0 \\ -3t \\ t \\ s \end{bmatrix} &= \begin{bmatrix} 8t \\ 0 \\ -3t \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{s}{3} \\ 0 \\ 0 \\ 0 \\ s \end{bmatrix} \\ &= t\begin{bmatrix} 8 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} \frac{1}{3} \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

By letting $t=1$ and $s=1$ then the above becomes

$$\begin{bmatrix} 8t+\frac{s}{3} \\ 0 \\ -3t \\ t \\ s \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{3} \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Hence the two eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix} 8 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{3} \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

Which are normalized to

$$\left(\begin{bmatrix} 8 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 3 \end{bmatrix} \right)$$

Considering the eigenvalue $\lambda_2 = 3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 11 & -1 & 26 & 6 & -3 \\ 0 & 3 & 0 & 0 & 0 \\ -9 & 0 & -24 & -6 & 3 \\ 3 & 0 & 9 & 5 & -1 \\ -48 & -3 & -138 & -30 & 18 \end{bmatrix} - (3) \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 8 & -1 & 26 & 6 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ -9 & 0 & -27 & -6 & 3 \\ 3 & 0 & 9 & 2 & -1 \\ -48 & -3 & -138 & -30 & 15 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccccc|c} 8 & -1 & 26 & 6 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -9 & 0 & -27 & -6 & 3 & 0 \\ 3 & 0 & 9 & 2 & -1 & 0 \\ -48 & -3 & -138 & -30 & 15 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{9R_1}{8} \Longrightarrow \left[\begin{array}{ccccc|c} 8 & -1 & 26 & 6 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{9}{8} & \frac{9}{4} & \frac{3}{4} & -\frac{3}{8} & 0 \\ 3 & 0 & 9 & 2 & -1 & 0 \\ -48 & -3 & -138 & -30 & 15 & 0 \end{array}\right]$$

$$R_4 = R_4 - \frac{3R_1}{8} \Longrightarrow \left[\begin{array}{ccccc|c} 8 & -1 & 26 & 6 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{9}{8} & \frac{9}{4} & \frac{3}{4} & -\frac{3}{8} & 0 \\ 0 & \frac{3}{8} & -\frac{3}{4} & -\frac{1}{4} & \frac{1}{8} & 0 \\ -48 & -3 & -138 & -30 & 15 & 0 \end{array}\right]$$

$$R_5 = R_5 + 6R_1 \Longrightarrow \left[\begin{array}{ccccc|c} 8 & -1 & 26 & 6 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{9}{8} & \frac{9}{4} & \frac{3}{4} & -\frac{3}{8} & 0 \\ 0 & \frac{3}{8} & -\frac{3}{4} & -\frac{1}{4} & \frac{1}{8} & 0 \\ 0 & -9 & 18 & 6 & -3 & 0 \end{array}\right]$$

Since the current pivot $A(2,2)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{ccccc|c} 8 & -1 & 26 & 6 & -3 & 0 \\ 0 & -\frac{9}{8} & \frac{9}{4} & \frac{3}{4} & -\frac{3}{8} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{3}{8} & -\frac{3}{4} & -\frac{1}{4} & \frac{1}{8} & 0 \\ 0 & -9 & 18 & 6 & -3 & 0 \end{array}\right]$$

$$R_4 = R_4 + \frac{R_2}{3} \Longrightarrow \left[\begin{array}{ccccc|c} 8 & -1 & 26 & 6 & -3 & 0 \\ 0 & -\frac{9}{8} & \frac{9}{4} & \frac{3}{4} & -\frac{3}{8} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -9 & 18 & 6 & -3 & 0 \end{array}\right]$$

$$R_5 = R_5 - 8R_2 \Longrightarrow \left[\begin{array}{ccccc|c} 8 & -1 & 26 & 6 & -3 & 0 \\ 0 & -\frac{9}{8} & \frac{9}{4} & \frac{3}{4} & -\frac{3}{8} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 8 & -1 & 26 & 6 & -3 \\ 0 & -\frac{9}{8} & \frac{9}{4} & \frac{3}{4} & -\frac{3}{8} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3, v_4, v_5\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Let $v_4 = s$. Let $v_5 = r$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -3t - \frac{2s}{3} + \frac{r}{3}, v_2 = 2t + \frac{2s}{3} - \frac{r}{3}\right\}$

Hence the solution is

$$\begin{bmatrix} -3t - \frac{2s}{3} + \frac{r}{3} \\ 2t + \frac{2s}{3} - \frac{r}{3} \\ t \\ s \\ r \end{bmatrix} = \begin{bmatrix} -3t - \frac{2s}{3} + \frac{r}{3} \\ 2t + \frac{2s}{3} - \frac{r}{3} \\ t \\ s \\ r \end{bmatrix}$$

Since there are three free Variable, we have found three eigenvectors associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -3t - \frac{2s}{3} + \frac{r}{3} \\ 2t + \frac{2s}{3} - \frac{r}{3} \\ t \\ s \\ r \end{bmatrix} = \begin{bmatrix} -3t \\ 2t \\ t \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{2s}{3} \\ \frac{2s}{3} \\ 0 \\ s \\ 0 \end{bmatrix}$$

$$= t \begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -\frac{2}{3} \\ \frac{2}{3} \\ 0 \\ 1 \\ 0 \end{bmatrix} + r \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

By letting $t = 1$ and $s = 1$ and $r = 1$ then the above becomes

$$\begin{bmatrix} -3t - \frac{2s}{3} + \frac{r}{3} \\ 2t + \frac{2s}{3} - \frac{r}{3} \\ t \\ s \\ r \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{2}{3} \\ \frac{2}{3} \\ 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Hence the three eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{2}{3} \\ \frac{2}{3} \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

Which are normalized to

$$\left(\begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 3 \end{bmatrix} \right)$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
3	3	3	No	$\begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & -3 \\ -\frac{1}{3} & \frac{2}{3} & 2 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$
2	2	2	No	$\begin{bmatrix} \frac{1}{3} & 8 \\ 0 & 0 \\ 0 & -3 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 3 is real and repated eigenvalue of multiplicity 3.There are three possible cases that can happen. This is illustrated in this diagram

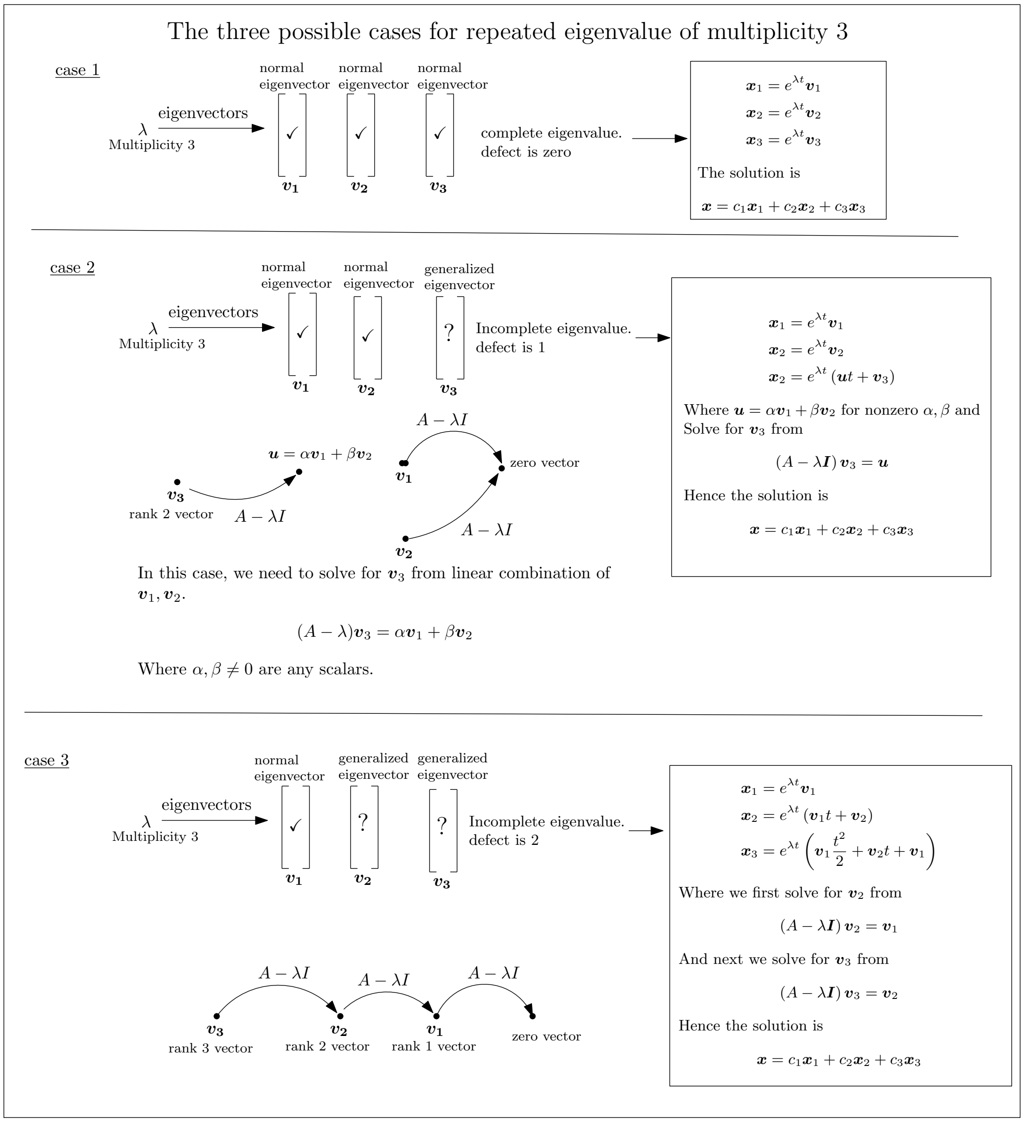


Figure 68: Possible case for repeated λ of multiplicity 3

This eigenvalue has algebraic multiplicity of 3 which is the same as its geometric multiplicity 3, then it is complete eigenvalue and this falls into case 1 shown above.

Hence the corresponding eigenvector basis are

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{3t}\\ &= \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ 0 \\ 0 \\ 1 \end{bmatrix} e^{3t}\end{aligned}$$

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{3t}\\ &= \begin{bmatrix} -\frac{2}{3} \\ \frac{2}{3} \\ 0 \\ 1 \\ 0 \end{bmatrix} e^{3t}\end{aligned}$$

$$\begin{aligned}\vec{x}_3(t) &= \vec{v}_3 e^{3t}\\ &= \begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} e^{3t}\end{aligned}$$

eigenvalue 2 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

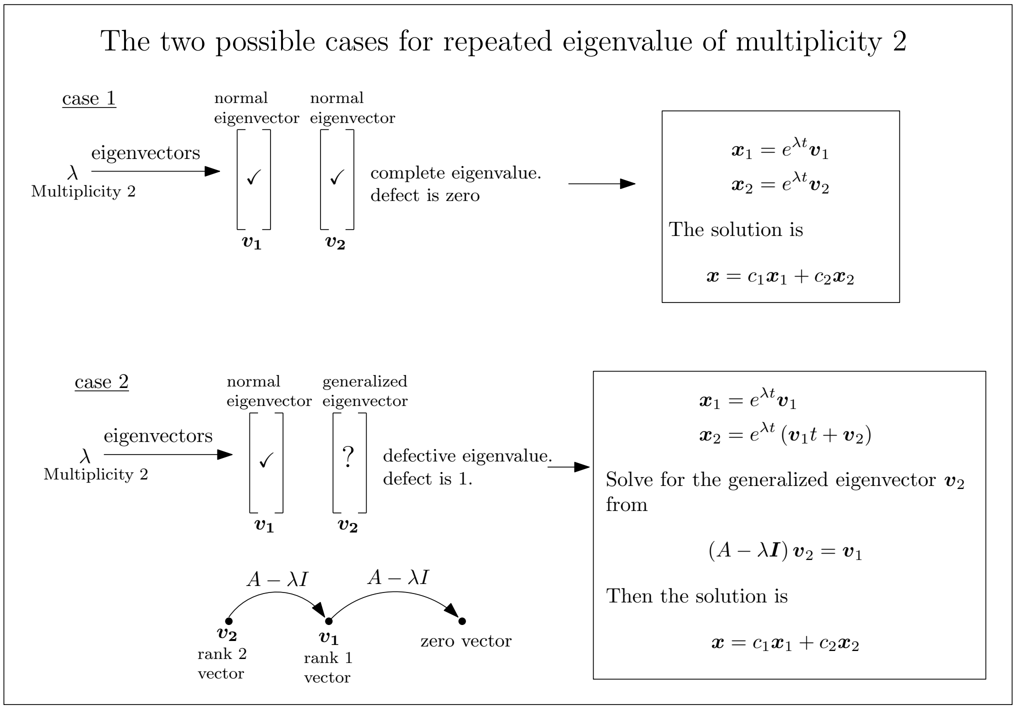


Figure 69: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2 which is the same as its geometric multiplicity 2, then it is complete eigenvalue and this falls into case 1 shown above. Hence the corresponding eigenvector basis are

$$
\begin{aligned}
\vec{x}_4(t) &= \vec{v}_4 e^{2t} \\
&= \begin{bmatrix} \frac{1}{3} \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} e^{2t}
\end{aligned}
$$

$$\vec{x}_5(t) = \vec{v}_5 e^{2t}$$
$$= \begin{bmatrix} 8 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix} e^{2t}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t) + c_4\vec{x}_4(t) + c_5\vec{x}_5(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ x_5(t) \end{bmatrix} = c_1 \begin{bmatrix} \frac{e^{3t}}{3} \\ -\frac{e^{3t}}{3} \\ 0 \\ 0 \\ e^{3t} \end{bmatrix} + c_2 \begin{bmatrix} -\frac{2\,e^{3t}}{3} \\ \frac{2\,e^{3t}}{3} \\ 0 \\ e^{3t} \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} -3\,e^{3t} \\ 2\,e^{3t} \\ e^{3t} \\ 0 \\ 0 \end{bmatrix} + c_4 \begin{bmatrix} \frac{e^{2t}}{3} \\ 0 \\ 0 \\ 0 \\ e^{2t} \end{bmatrix} + c_5 \begin{bmatrix} 8\,e^{2t} \\ 0 \\ -3\,e^{2t} \\ e^{2t} \\ 0 \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ x_5(t) \end{bmatrix} = \begin{bmatrix} \frac{(c_1-2c_2-9c_3)e^{3t}}{3} + \frac{e^{2t}(c_4+24c_5)}{3} \\ -\frac{e^{3t}(c_1-2c_2-6c_3)}{3} \\ c_3 e^{3t} - 3c_5 e^{2t} \\ c_2 e^{3t} + c_5 e^{2t} \\ c_1 e^{3t} + c_4 e^{2t} \end{bmatrix}$$

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 107

```
dsolve([diff(x__1(t),t)=11*x__1(t)-1*x__2(t)+26*x__3(t)+6*x__4(t)-3*x__5(t),diff(x__2(t),t)=
```

$$x_1(t) = \left(-(c_3 + c_5)\,e^{t} + c_1\right)e^{2t}$$
$$x_2(t) = c_5 e^{3t}$$
$$x_3(t) = c_3 e^{3t} + c_4 e^{2t}$$
$$x_4(t) = -\frac{c_3 e^{3t}}{3} - \frac{c_4 e^{2t}}{3} + c_2 e^{3t}$$
$$x_5(t) = \frac{16 c_3 e^{3t}}{3} + 8 c_4 e^{2t} + 2 c_2 e^{3t} - 3 c_5 e^{3t} + 3 c_1 e^{2t}$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 211

```
DSolve[{x1'[t]==11*x1[t]-1*x2[t]+26*x3[t]+6*x4[t]-3*x5[t],x2'[t]==0*x1[t]+3*x2[t],x3'[t]==-9
```

$$x1(t) \to e^{2t}\left(c_1\left(9e^t-8\right)-\left(c_2-26c_3-6c_4+3c_5\right)\left(e^t-1\right)\right)$$
$$x2(t) \to c_2e^{3t}$$
$$x3(t) \to -e^{2t}\left(9c_1\left(e^t-1\right)+c_3\left(26e^t-27\right)+3(2c_4-c_5)\left(e^t-1\right)\right)$$
$$x4(t) \to e^{2t}\left(3c_1\left(e^t-1\right)+9c_3\left(e^t-1\right)+3c_4e^t-c_5e^t-2c_4+c_5\right)$$
$$x5(t) \to -e^{2t}\left(48c_1\left(e^t-1\right)+3c_2\left(e^t-1\right)+138c_3e^t+30c_4e^t-16c_5e^t-138c_3-30c_4+15c_5\right)$$

6.33 problem problem 33

Internal problem ID [390]
Internal file name [`OUTPUT/390_Sunday_June_05_2022_01_40_23_AM_25426689/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.6, Multiple Eigenvalue Solutions. Page 451
Problem number: problem 33.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= 3x_1(t) - 4x_2(t) + x_3(t) \\
x_2'(t) &= 4x_1(t) + 3x_2(t) + x_4(t) \\
x_3'(t) &= 3x_3(t) - 4x_4(t) \\
x_4'(t) &= 4x_3(t) + 3x_4(t)
\end{aligned}
$$

6.33.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \\ x_4'(t) \end{bmatrix} = \begin{bmatrix} 3 & -4 & 1 & 0 \\ 4 & 3 & 0 & 1 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 4 & 3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{3t}\cos(4t) & -e^{3t}\sin(4t) & t\,e^{3t}\cos(4t) & -t\,e^{3t}\sin(4t) \\ e^{3t}\sin(4t) & e^{3t}\cos(4t) & t\,e^{3t}\sin(4t) & t\,e^{3t}\cos(4t) \\ 0 & 0 & e^{3t}\cos(4t) & -e^{3t}\sin(4t) \\ 0 & 0 & e^{3t}\sin(4t) & e^{3t}\cos(4t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{3t}\cos(4t) & -e^{3t}\sin(4t) & t\,e^{3t}\cos(4t) & -t\,e^{3t}\sin(4t) \\ e^{3t}\sin(4t) & e^{3t}\cos(4t) & t\,e^{3t}\sin(4t) & t\,e^{3t}\cos(4t) \\ 0 & 0 & e^{3t}\cos(4t) & -e^{3t}\sin(4t) \\ 0 & 0 & e^{3t}\sin(4t) & e^{3t}\cos(4t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} \\
&= \begin{bmatrix} e^{3t}\cos(4t)\,c_1 - e^{3t}\sin(4t)\,c_2 + t\,e^{3t}\cos(4t)\,c_3 - t\,e^{3t}\sin(4t)\,c_4 \\ e^{3t}\sin(4t)\,c_1 + e^{3t}\cos(4t)\,c_2 + t\,e^{3t}\sin(4t)\,c_3 + t\,e^{3t}\cos(4t)\,c_4 \\ e^{3t}\cos(4t)\,c_3 - e^{3t}\sin(4t)\,c_4 \\ e^{3t}\sin(4t)\,c_3 + e^{3t}\cos(4t)\,c_4 \end{bmatrix} \\
&= \begin{bmatrix} ((tc_3 + c_1)\cos(4t) - \sin(4t)\,(tc_4 + c_2))\,e^{3t} \\ ((tc_4 + c_2)\cos(4t) + \sin(4t)\,(tc_3 + c_1))\,e^{3t} \\ e^{3t}(\cos(4t)\,c_3 - \sin(4t)\,c_4) \\ e^{3t}(\sin(4t)\,c_3 + \cos(4t)\,c_4) \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

6.33.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \\ x_4'(t) \end{bmatrix} = \begin{bmatrix} 3 & -4 & 1 & 0 \\ 4 & 3 & 0 & 1 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 4 & 3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 3 & -4 & 1 & 0 \\ 4 & 3 & 0 & 1 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 4 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 3-\lambda & -4 & 1 & 0 \\ 4 & 3-\lambda & 0 & 1 \\ 0 & 0 & 3-\lambda & -4 \\ 0 & 0 & 4 & 3-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^4 - 12\lambda^3 + 86\lambda^2 - 300\lambda + 625 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 3 + 4i$$
$$\lambda_2 = 3 - 4i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$3 + 4i$	1	complex eigenvalue
$3 - 4i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 3 - 4i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & -4 & 1 & 0 \\ 4 & 3 & 0 & 1 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 4 & 3 \end{bmatrix} - (3-4i)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4i & -4 & 1 & 0 \\ 4 & 4i & 0 & 1 \\ 0 & 0 & 4i & -4 \\ 0 & 0 & 4 & 4i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} 4i & -4 & 1 & 0 & 0 \\ 4 & 4i & 0 & 1 & 0 \\ 0 & 0 & 4i & -4 & 0 \\ 0 & 0 & 4 & 4i & 0 \end{array}\right]$$

$$R_2 = iR_1 + R_2 \Longrightarrow \left[\begin{array}{cccc|c} 4i & -4 & 1 & 0 & 0 \\ 0 & 0 & i & 1 & 0 \\ 0 & 0 & 4i & -4 & 0 \\ 0 & 0 & 4 & 4i & 0 \end{array}\right]$$

$$R_3 = R_3 - 4R_2 \Longrightarrow \left[\begin{array}{cccc|c} 4i & -4 & 1 & 0 & 0 \\ 0 & 0 & i & 1 & 0 \\ 0 & 0 & 0 & -8 & 0 \\ 0 & 0 & 4 & 4i & 0 \end{array}\right]$$

$$R_4 = 4iR_2 + R_4 \Longrightarrow \left[\begin{array}{cccc|c} 4i & -4 & 1 & 0 & 0 \\ 0 & 0 & i & 1 & 0 \\ 0 & 0 & 0 & -8 & 0 \\ 0 & 0 & 0 & 8i & 0 \end{array}\right]$$

$$
R_4 = iR_3 + R_4 \Longrightarrow \left[\begin{array}{cccc|c} 4i & -4 & 1 & 0 & 0 \\ 0 & 0 & i & 1 & 0 \\ 0 & 0 & 0 & -8 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]
$$

Therefore the system in Echelon form is

$$
\begin{bmatrix} 4i & -4 & 1 & 0 \\ 0 & 0 & i & 1 \\ 0 & 0 & 0 & -8 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1, v_3, v_4\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -it, v_3 = 0, v_4 = 0\}$

Hence the solution is

$$
\begin{bmatrix} \text{-I}\,t \\ t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -it \\ t \\ 0 \\ 0 \end{bmatrix}
$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$
\begin{bmatrix} \text{-I}\,t \\ t \\ 0 \\ 0 \end{bmatrix} = t \begin{bmatrix} -i \\ 1 \\ 0 \\ 0 \end{bmatrix}
$$

Let $t = 1$ the eigenvector becomes

$$
\begin{bmatrix} \text{-I}\,t \\ t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -i \\ 1 \\ 0 \\ 0 \end{bmatrix}
$$

Considering the eigenvalue $\lambda_2 = 3 + 4i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & -4 & 1 & 0 \\ 4 & 3 & 0 & 1 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 4 & 3 \end{bmatrix} - (3+4i)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4i & -4 & 1 & 0 \\ 4 & -4i & 0 & 1 \\ 0 & 0 & -4i & -4 \\ 0 & 0 & 4 & -4i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} -4i & -4 & 1 & 0 & 0 \\ 4 & -4i & 0 & 1 & 0 \\ 0 & 0 & -4i & -4 & 0 \\ 0 & 0 & 4 & -4i & 0 \end{array}\right]$$

$$R_2 = -iR_1 + R_2 \Longrightarrow \left[\begin{array}{cccc|c} -4i & -4 & 1 & 0 & 0 \\ 0 & 0 & -i & 1 & 0 \\ 0 & 0 & -4i & -4 & 0 \\ 0 & 0 & 4 & -4i & 0 \end{array}\right]$$

$$R_3 = R_3 - 4R_2 \Longrightarrow \left[\begin{array}{cccc|c} -4i & -4 & 1 & 0 & 0 \\ 0 & 0 & -i & 1 & 0 \\ 0 & 0 & 0 & -8 & 0 \\ 0 & 0 & 4 & -4i & 0 \end{array}\right]$$

$$R_4 = -4iR_2 + R_4 \Longrightarrow \left[\begin{array}{cccc|c} -4i & -4 & 1 & 0 & 0 \\ 0 & 0 & -i & 1 & 0 \\ 0 & 0 & 0 & -8 & 0 \\ 0 & 0 & 0 & -8i & 0 \end{array}\right]$$

$$R_4 = -iR_3 + R_4 \Longrightarrow \left[\begin{array}{cccc|c} -4i & -4 & 1 & 0 & 0 \\ 0 & 0 & -i & 1 & 0 \\ 0 & 0 & 0 & -8 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -4i & -4 & 1 & 0 \\ 0 & 0 & -i & 1 \\ 0 & 0 & 0 & -8 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1, v_3, v_4\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = it, v_3 = 0, v_4 = 0\}$

Hence the solution is

$$\begin{bmatrix} \mathrm{I}t \\ t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} it \\ t \\ 0 \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \mathrm{I}t \\ t \\ 0 \\ 0 \end{bmatrix} = t \begin{bmatrix} i \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \mathrm{I}t \\ t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} i \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity: algebraic m	multiplicity: geometric k	defective?	eigenvectors
$3+4i$	2	1	Yes	$\begin{bmatrix} i \\ 1 \\ 0 \\ 0 \end{bmatrix}$
$3-4i$	2	1	Yes	$\begin{bmatrix} -i \\ 1 \\ 0 \\ 0 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = c_1 \begin{bmatrix} i\,\mathrm{e}^{(3+4i)t} \\ \mathrm{e}^{(3+4i)t} \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -i\,\mathrm{e}^{(3-4i)t} \\ \mathrm{e}^{(3-4i)t} \\ 0 \\ 0 \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = \begin{bmatrix} i\left(c_1\mathrm{e}^{(3+4i)t} - c_2\mathrm{e}^{(3-4i)t}\right) \\ c_1\mathrm{e}^{(3+4i)t} + c_2\mathrm{e}^{(3-4i)t} \\ 0 \\ 0 \end{bmatrix}$$

✓ Solution by Maple

Time used: 0.453 (sec). Leaf size: 140

```
dsolve([diff(x__1(t),t)=3*x__1(t)-4*x__2(t)+1*x__3(t)+0*x__4(t),diff(x__2(t),t)=4*x__1(t)+3*
```

$$x_1(t) = \frac{e^{3t}(4\cos(4t)\,c_4t + 4\sin(4t)\,c_3t + 4c_1\cos(4t) + 4c_2\sin(4t) - \sin(4t)\,c_4)}{4}$$
$$x_2(t) = -\frac{e^{3t}(4\cos(4t)\,c_3t - 4\sin(4t)\,c_4t + 4c_2\cos(4t) - c_4\cos(4t) - 4c_1\sin(4t))}{4}$$
$$x_3(t) = e^{3t}(c_4\cos(4t) + c_3\sin(4t))$$
$$x_4(t) = -e^{3t}(\cos(4t)\,c_3 - \sin(4t)\,c_4)$$

✓ Solution by Mathematica

Time used: 0.1 (sec). Leaf size: 120

```
DSolve[{x1'[t]==3*x1[t]-4*x2[t]+1*x3[t]+0*x4[t],x2'[t]==4*x1[t]+3*x2[t]+0*x3[t]+1*x4[t],x3'[
```

$$\text{x1}(t) \to e^{3t}((c_3t + c_1)\cos(4t) - (c_4t + c_2)\sin(4t))$$
$$\text{x2}(t) \to e^{3t}((c_4t + c_2)\cos(4t) + (c_3t + c_1)\sin(4t))$$
$$\text{x3}(t) \to e^{3t}(c_3\cos(4t) - c_4\sin(4t))$$
$$\text{x4}(t) \to e^{3t}(c_4\cos(4t) + c_3\sin(4t))$$

6.34 problem problem 34

Internal problem ID [391]
Internal file name [OUTPUT/391_Sunday_June_05_2022_01_40_26_AM_22487972/index.tex]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Section 7.6, Multiple Eigenvalue Solutions. Page 451
Problem number: problem 34.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= 2x_1(t) - 8x_3(t) - 3x_4(t)\\
x_2'(t) &= -18x_1(t) - x_2(t)\\
x_3'(t) &= -9x_1(t) - 3x_2(t) - 25x_3(t) - 9x_4(t)\\
x_4'(t) &= 33x_1(t) + 10x_2(t) + 90x_3(t) + 32x_4(t)
\end{aligned}
$$

6.34.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \\ x_4'(t) \end{bmatrix} = \begin{bmatrix} 2 & 0 & -8 & -3 \\ -18 & -1 & 0 & 0 \\ -9 & -3 & -25 & -9 \\ 33 & 10 & 90 & 32 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \left[\begin{array}{c} e^{2t}\cos(3t) - 3t\,e^{2t}\sin(3t) \\ -9\cos(3t)\,t\,e^{2t} + \left(-\frac{3}{2}+\frac{9t}{2}\right)e^{2t}\sin(3t) + \frac{9\,e^{2t}\sin(3t)\left(-\frac{1}{3}+t\right)}{2} + i\left(\left(-\frac{3}{2}+\frac{9t}{2}\right)e^{2t}\cos(3t) - \frac{9\,e^{2t}\cos(3t)(-}{2}\right. \\ -3\,e^{2t}\sin(3t) \\ 3\cos(3t)\,t\,e^{2t} + 10\,e^{2t}\sin(3t) \end{array}\right.$$

$$= \left[\begin{array}{ccc} e^{2t}(\cos(3t) - 3\sin(3t)\,t) & -t\,e^{2t}\sin(3t) & ((-3-9t)\sin(3t) + \\ (-3+9t)\,e^{2t}\sin(3t) - 9\cos(3t)\,t\,e^{2t} & (1-3t)\,e^{2t}\cos(3t) + 3t\,e^{2t}\sin(3t) & (24t+10)\,e^{2t}\sin(3t) - \\ -3\,e^{2t}\sin(3t) & -e^{2t}\sin(3t) & e^{2t}(\cos(3t) - 9 \\ e^{2t}(3\cos(3t)\,t + 10\sin(3t)) & e^{2t}(\cos(3t)\,t + 3\sin(3t)) & (27+t)\,e^{2t}\sin(3t) + \end{array}\right.$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = e^{At}\vec{c}$$

$$= \left[\begin{array}{ccc} e^{2t}(\cos(3t) - 3\sin(3t)\,t) & -t\,e^{2t}\sin(3t) & ((-3-9t)\sin(3t) \\ (-3+9t)\,e^{2t}\sin(3t) - 9\cos(3t)\,t\,e^{2t} & (1-3t)\,e^{2t}\cos(3t) + 3t\,e^{2t}\sin(3t) & (24t+10)\,e^{2t}\sin(3t \\ -3\,e^{2t}\sin(3t) & -e^{2t}\sin(3t) & e^{2t}(\cos(3t) - \\ e^{2t}(3\cos(3t)\,t + 10\sin(3t)) & e^{2t}(\cos(3t)\,t + 3\sin(3t)) & (27+t)\,e^{2t}\sin(3t) \end{array}\right.$$

$$= \left[\begin{array}{c} e^{2t}(\cos(3t) - 3\sin(3t)\,t)\,c_1 - t\,e^{2t}\sin(3t)\,c_2 + ((-3-9t)\sin(3t \\ ((-3+9t)\,e^{2t}\sin(3t) - 9\cos(3t)\,t\,e^{2t})\,c_1 + ((1-3t)\,e^{2t}\cos(3t) + 3t\,e^{2t}\sin(3t))\,c_2 + ((24t+10 \\ -3\,e^{2t}\sin(3t)\,c_1 - e^{2t}\sin(3t)\,c_2 + e^{2t}(\cos(3t) - 9 \\ e^{2t}(3\cos(3t)\,t + 10\sin(3t))\,c_1 + e^{2t}(\cos(3t)\,t + 3\sin(3t))\,c_2 + ((27+t)\,e^{2t}\sin(3t) \end{array}\right.$$

$$= \left[\begin{array}{c} -3\left(\left(\left(c_1 + \frac{c_2}{3} + 3c_3 + c_4\right)t + c_3 + \frac{c_4}{3}\right)\sin(3t) - \frac{(tc_3+c_1)\cos(3t)}{3}\right)e^{2t} \\ -9\left(\left(\left(-c_1 - \frac{c_2}{3} - \frac{8c_3}{3} - c_4\right)t + \frac{c_1}{3} - \frac{10c_3}{9} - \frac{c_4}{3}\right)\sin(3t) + \cos(3t)\left(\left(c_1 + \frac{c_2}{3} + \frac{10c_3}{3} + c_4\right)t - \frac{c_2}{9}\right)\right) \\ -3\left(\left(c_1 + \frac{c_2}{3} + 3c_3 + c_4\right)\sin(3t) - \frac{c_3\cos(3t)}{3}\right)e^{2t} \\ 3\left(\left(\left(c_1 + \frac{c_2}{3} + 3c_3 + c_4\right)t + \frac{c_4}{3}\right)\cos(3t) + \frac{\sin(3t)(tc_3 + 10c_1 + 3c_2 + 27c_3 + 9c_4)}{3}\right)e^{2t} \end{array}\right.$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

6.34.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \\ x_4'(t) \end{bmatrix} = \begin{bmatrix} 2 & 0 & -8 & -3 \\ -18 & -1 & 0 & 0 \\ -9 & -3 & -25 & -9 \\ 33 & 10 & 90 & 32 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 2 & 0 & -8 & -3 \\ -18 & -1 & 0 & 0 \\ -9 & -3 & -25 & -9 \\ 33 & 10 & 90 & 32 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 2-\lambda & 0 & -8 & -3 \\ -18 & -1-\lambda & 0 & 0 \\ -9 & -3 & -25-\lambda & -9 \\ 33 & 10 & 90 & 32-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^4 - 8\lambda^3 + 42\lambda^2 - 104\lambda + 169 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2 + 3i$$
$$\lambda_2 = 2 - 3i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$2+3i$	1	complex eigenvalue
$2-3i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 2-3i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A-\lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 0 & -8 & -3 \\ -18 & -1 & 0 & 0 \\ -9 & -3 & -25 & -9 \\ 33 & 10 & 90 & 32 \end{bmatrix} - (2-3i)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3i & 0 & -8 & -3 \\ -18 & -3+3i & 0 & 0 \\ -9 & -3 & -27+3i & -9 \\ 33 & 10 & 90 & 30+3i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} 3i & 0 & -8 & -3 & 0 \\ -18 & -3+3i & 0 & 0 & 0 \\ -9 & -3 & -27+3i & -9 & 0 \\ 33 & 10 & 90 & 30+3i & 0 \end{array}\right]$$

$$R_2 = -6iR_1 + R_2 \Longrightarrow \left[\begin{array}{cccc|c} 3i & 0 & -8 & -3 & 0 \\ 0 & -3+3i & 48i & 18i & 0 \\ -9 & -3 & -27+3i & -9 & 0 \\ 33 & 10 & 90 & 30+3i & 0 \end{array}\right]$$

$$R_3 = -3iR_1 + R_3 \Longrightarrow \left[\begin{array}{cccc|c} 3i & 0 & -8 & -3 & 0 \\ 0 & -3+3i & 48i & 18i & 0 \\ 0 & -3 & -27+27i & -9+9i & 0 \\ 33 & 10 & 90 & 30+3i & 0 \end{array}\right]$$

$$R_4 = 11iR_1 + R_4 \Longrightarrow \left[\begin{array}{cccc|c} 3i & 0 & -8 & -3 & 0 \\ 0 & -3+3i & 48i & 18i & 0 \\ 0 & -3 & -27+27i & -9+9i & 0 \\ 0 & 10 & 90-88i & 30-30i & 0 \end{array}\right]$$

$$R_3 = R_3 + \left(-\frac{1}{2} - \frac{i}{2}\right) R_2 \Longrightarrow \left[\begin{array}{cccc|c} 3i & 0 & -8 & -3 & 0 \\ 0 & -3+3i & 48i & 18i & 0 \\ 0 & 0 & -3+3i & 0 & 0 \\ 0 & 10 & 90-88i & 30-30i & 0 \end{array}\right]$$

$$R_4 = R_4 + \left(\frac{5}{3} + \frac{5i}{3}\right) R_2 \Longrightarrow \left[\begin{array}{cccc|c} 3i & 0 & -8 & -3 & 0 \\ 0 & -3+3i & 48i & 18i & 0 \\ 0 & 0 & -3+3i & 0 & 0 \\ 0 & 0 & 10-8i & 0 & 0 \end{array}\right]$$

$$R_4 = R_4 + \left(3 + \frac{i}{3}\right) R_3 \Longrightarrow \left[\begin{array}{cccc|c} 3i & 0 & -8 & -3 & 0 \\ 0 & -3+3i & 48i & 18i & 0 \\ 0 & 0 & -3+3i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3i & 0 & -8 & -3 \\ 0 & -3+3i & 48i & 18i \\ 0 & 0 & -3+3i & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_4\}$ and the leading variables are $\{v_1, v_2, v_3\}$. Let $v_4 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -it, v_2 = (-3+3i)\,t, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} \text{-I}t \\ (-3+3\ \text{I})\,t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} -it \\ (-3+3i)\,t \\ 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \text{-I}t \\ (-3+3\ \text{I})\,t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} -i \\ -3+3i \\ 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \text{-I}t \\ (-3+3\ \text{I})\,t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} -i \\ -3+3i \\ 0 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 2 + 3i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 0 & -8 & -3 \\ -18 & -1 & 0 & 0 \\ -9 & -3 & -25 & -9 \\ 33 & 10 & 90 & 32 \end{bmatrix} - (2+3i) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3i & 0 & -8 & -3 \\ -18 & -3-3i & 0 & 0 \\ -9 & -3 & -27-3i & -9 \\ 33 & 10 & 90 & 30-3i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} -3i & 0 & -8 & -3 & 0 \\ -18 & -3-3i & 0 & 0 & 0 \\ -9 & -3 & -27-3i & -9 & 0 \\ 33 & 10 & 90 & 30-3i & 0 \end{array}\right]$$

$$R_2 = 6iR_1 + R_2 \Longrightarrow \left[\begin{array}{cccc|c} -3i & 0 & -8 & -3 & 0 \\ 0 & -3-3i & -48i & -18i & 0 \\ -9 & -3 & -27-3i & -9 & 0 \\ 33 & 10 & 90 & 30-3i & 0 \end{array}\right]$$

$$R_3 = 3iR_1 + R_3 \Longrightarrow \left[\begin{array}{cccc|c} -3i & 0 & -8 & -3 & 0 \\ 0 & -3-3i & -48i & -18i & 0 \\ 0 & -3 & -27-27i & -9-9i & 0 \\ 33 & 10 & 90 & 30-3i & 0 \end{array}\right]$$

$$R_4 = -11iR_1 + R_4 \Longrightarrow \left[\begin{array}{cccc|c} -3i & 0 & -8 & -3 & 0 \\ 0 & -3-3i & -48i & -18i & 0 \\ 0 & -3 & -27-27i & -9-9i & 0 \\ 0 & 10 & 90+88i & 30+30i & 0 \end{array}\right]$$

$$R_3 = R_3 + \left(-\frac{1}{2} + \frac{i}{2}\right) R_2 \Longrightarrow \left[\begin{array}{cccc|c} -3i & 0 & -8 & -3 & 0 \\ 0 & -3-3i & -48i & -18i & 0 \\ 0 & 0 & -3-3i & 0 & 0 \\ 0 & 10 & 90+88i & 30+30i & 0 \end{array}\right]$$

$$R_4 = R_4 + \left(\frac{5}{3} - \frac{5i}{3}\right) R_2 \Longrightarrow \left[\begin{array}{cccc|c} -3i & 0 & -8 & -3 & 0 \\ 0 & -3-3i & -48i & -18i & 0 \\ 0 & 0 & -3-3i & 0 & 0 \\ 0 & 0 & 10+8i & 0 & 0 \end{array}\right]$$

$$R_4 = R_4 + \left(3 - \frac{i}{3}\right) R_3 \Longrightarrow \left[\begin{array}{cccc|c} -3i & 0 & -8 & -3 & 0 \\ 0 & -3-3i & -48i & -18i & 0 \\ 0 & 0 & -3-3i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -3i & 0 & -8 & -3 \\ 0 & -3-3i & -48i & -18i \\ 0 & 0 & -3-3i & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_4\}$ and the leading variables are $\{v_1, v_2, v_3\}$. Let $v_4 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = it, v_2 = (-3-3i)\, t, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} \mathrm{I}\, t \\ (-3 - 3\ \mathrm{I})\, t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} it \\ (-3-3i)\, t \\ 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \mathrm{I}\, t \\ (-3 - 3\ \mathrm{I})\, t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} i \\ -3-3i \\ 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \mathrm{I}\, t \\ (-3 - 3\ \mathrm{I})\, t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} i \\ -3-3i \\ 0 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$2+3i$	2	1	Yes	$\begin{bmatrix} i \\ -3-3i \\ 0 \\ 1 \end{bmatrix}$
$2-3i$	2	1	Yes	$\begin{bmatrix} -i \\ -3+3i \\ 0 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = c_1 \begin{bmatrix} i\mathrm{e}^{(2+3i)t} \\ (-3-3i)\,\mathrm{e}^{(2+3i)t} \\ 0 \\ \mathrm{e}^{(2+3i)t} \end{bmatrix} + c_2 \begin{bmatrix} -i\mathrm{e}^{(2-3i)t} \\ (-3+3i)\,\mathrm{e}^{(2-3i)t} \\ 0 \\ \mathrm{e}^{(2-3i)t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = \begin{bmatrix} i\left(c_1\mathrm{e}^{(2+3i)t} - c_2\mathrm{e}^{(2-3i)t}\right) \\ (-3-3i)\,c_1\mathrm{e}^{(2+3i)t} + (-3+3i)\,c_2\mathrm{e}^{(2-3i)t} \\ 0 \\ c_1\mathrm{e}^{(2+3i)t} + c_2\mathrm{e}^{(2-3i)t} \end{bmatrix}$$

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 252

```
dsolve([diff(x__1(t),t)=2*x__1(t)+0*x__2(t)-8*x__3(t)-3*x__4(t),diff(x__2(t),t)=-18*x__1(t)-
```

$$x_1(t) = -\frac{e^{2t}(3\cos(3t)\,c_3t + 3\cos(3t)\,c_4t + 3\sin(3t)\,c_3t - 3\sin(3t)\,c_4t + 3c_1\cos(3t) + 3c_2\cos(3t) + \cos(3t)\,c_4}{18}$$

$$x_2(t) = e^{2t}(\cos(3t)\,c_4t + \sin(3t)\,c_3t + c_2\cos(3t) + c_1\sin(3t))$$

$$x_3(t) = -\frac{e^{2t}(\cos(3t)\,c_3 + \cos(3t)\,c_4 + \sin(3t)\,c_3 - \sin(3t)\,c_4)}{6}$$

$$x_4(t) = \frac{e^{2t}(3\cos(3t)\,c_3t - 3\cos(3t)\,c_4t - 3\sin(3t)\,c_3t - 3\sin(3t)\,c_4t + 3c_1\cos(3t) - 3c_2\cos(3t) + 10\cos(3t)\,c_3}{18}$$

✓ Solution by Mathematica

Time used: 0.026 (sec). Leaf size: 482

```
DSolve[{x1'[t]==2*x1[t]+0*x2[t]-8*x3[t]-3*x4[t],x2'[t]==-18*x1[t]-1*x2[t]+0*x3[t]+0*x4[t],x3
```

$$\begin{aligned}\text{x1}(t) \to \frac{1}{2}e^{(2-3i)t}\big(&c_1\big(e^{6it}(1+3it) - 3it + 1\big) + i(3c_3 + c_4)\left(-1 + e^{6it}\right)\\ &+ t\big(ic_2\left(-1 + e^{6it}\right) + c_3\big((1+9i)e^{6it} + (1-9i)\big) + 3ic_4\left(-1 + e^{6it}\right)\big)\big)\end{aligned}$$

$$\begin{aligned}\text{x2}(t) \to -\frac{1}{2}e^{(2-3i)t}\big(&c_1\big((9-9i)t + e^{6it}((9+9i)t - 3i) + 3i\big)\\ &+ c_2\big((3-3i)t + e^{6it}(-1 + (3+3i)t) - 1\big) + 10ic_3e^{6it} + (30+24i)c_3e^{6it}t\\ &+ (30-24i)c_3t + 3ic_4e^{6it} + (9+9i)c_4e^{6it}t + (9-9i)c_4t - 10ic_3 - 3ic_4\big)\end{aligned}$$

$$\begin{aligned}\text{x3}(t) \to \frac{1}{2}e^{(2-3i)t}\big(&3ic_1\left(-1 + e^{6it}\right) + ic_2\left(-1 + e^{6it}\right) + (1+9i)c_3e^{6it} + 3ic_4e^{6it} + (1-9i)c_3\\ &- 3ic_4\big)\end{aligned}$$

$$\begin{aligned}\text{x4}(t) \to \frac{1}{2}e^{(2-3i)t}\big(&c_1\left(3t + e^{6it}(3t - 10i) + 10i\right) + c_2\left(t + e^{6it}(t - 3i) + 3i\right) - 27ic_3e^{6it}\\ &+ (9-i)c_3e^{6it}t + (9+i)c_3t + (1-9i)c_4e^{6it} + 3c_4e^{6it}t + 3c_4t + 27ic_3 + (1+9i)c_4\big)\end{aligned}$$

7 Chapter 11 Power series methods. Section 11.1 Introduction and Review of power series. Page 615

7.1 problem problem 1

Internal problem ID [392]
Internal file name [`OUTPUT/392_Sunday_June_05_2022_01_40_28_AM_3647982/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Chapter 11 Power series methods. Section 11.1 Introduction and Review of power series. Page 615
Problem number: problem 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first order ode series method. Ordinary point", "first order ode series method. Taylor series method"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{y' - y = 0}$$

With the expansion point for the power series method at $x = 0$.

7.1.1 Solving as series ode

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving first order ode. Let

$$y' = f(x, y)$$

Where $f(x, y)$ is analytic at expansion point x_0. We can always shift to $x_0 = 0$ if x_0 is not zero. So from now we assume $x_0 = 0$. Assume also that $y(x_0) = y_0$. Using Taylor series

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y''(x_0) + \cdots \\
&= y_0 + xf + \frac{x^2}{2} \frac{df}{dx}\bigg|_{x_0, y_0} + \frac{x^3}{3!} \frac{d^2 f}{dx^2}\bigg|_{x_0, y_0} + \cdots \\
&= y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0}
\end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f \qquad (1)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f \qquad (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) f \qquad (3)\\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x, y)$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y) \qquad (4)\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) F_0 \qquad (5)
\end{aligned}
$$

For example, for $n = 1$ we see that

$$
\begin{aligned}
F_1 &= \frac{d}{dx}(F_0)\\
&= \frac{\partial}{\partial x} F_0 + \left(\frac{\partial F_0}{\partial y}\right) F_0\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f
\end{aligned}
$$

Which is (1). And when $n = 2$

$$
\begin{aligned}
F_2 &= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) F_0\\
&= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) f\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f
\end{aligned}
$$

Which is (2) and so on. Therefore (4,5) can be used from now on along with

$$y(x) = y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} F_n|_{x_0,y_0} \tag{6}$$

Hence

$$\begin{aligned}
F_0 &= y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} F_0 \\
&= y \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} F_1 \\
&= y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} F_2 \\
&= y \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} F_3 \\
&= y
\end{aligned}$$

And so on. Evaluating all the above at initial conditions $x(0) = 0$ and $y(0) = y(0)$ gives

$$\begin{aligned}
F_0 &= y(0) \\
F_1 &= y(0) \\
F_2 &= y(0) \\
F_3 &= y(0) \\
F_4 &= y(0)
\end{aligned}$$

Substituting all the above in (6) and simplifying gives the solution as

$$y = \left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5\right) y(0) + O(x^6)$$

Since $x = 0$ is also an ordinary point, then standard power series can also be used. Writing the ODE as

$$
\begin{aligned}
y' + q(x)y &= p(x) \\
y' - y &= 0
\end{aligned}
$$

Where

$$
\begin{aligned}
q(x) &= -1 \\
p(x) &= 0
\end{aligned}
$$

Next, the type of the expansion point $x = 0$ is determined. This point can be an ordinary point, a regular singular point (also called removable singularity), or irregular singular point (also called non-removable singularity or essential singularity). When $x = 0$ is an ordinary point, then the standard power series is used. If the point is a regular singular point, Frobenius series is used instead. Irregular singular point requires more advanced methods (asymptotic methods) and is not supported now. Hopefully this will be added in the future. $x = 0$ is called an ordinary point $q(x)$ has a Taylor series expansion around the point $x = 0$. $x = 0$ is called a regular singular point if $q(x)$ is not not analytic at $x = 0$ but $xq(x)$ has Taylor series expansion. And finally, $x = 0$ is an irregular singular point if the point is not ordinary and not regular singular. This is the most complicated case. Now the expansion point $x = 0$ is checked to see if it is an ordinary point or not. Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
y' = \sum_{n=1}^{\infty} n a_n x^{n-1}
$$

Substituting the above back into the ode gives

$$
\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}
$$

Which simplifies to

$$
\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + \sum_{n=0}^{\infty} (-a_n x^n) = 0 \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n\right) + \sum_{n=0}^{\infty} (-a_n x^n) = 0 \tag{3}$$

For $0 \leq n$, the recurrence equation is

$$(n+1) a_{n+1} - a_n = 0 \tag{4}$$

Solving for a_{n+1}, gives

$$a_{n+1} = \frac{a_n}{n+1} \tag{5}$$

For $n = 0$ the recurrence equation gives

$$a_1 - a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_1 = a_0$$

For $n = 1$ the recurrence equation gives

$$2a_2 - a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_2 = \frac{a_0}{2}$$

For $n = 2$ the recurrence equation gives

$$3a_3 - a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{a_0}{6}$$

For $n = 3$ the recurrence equation gives

$$4a_4 - a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{24}$$

For $n = 4$ the recurrence equation gives

$$5a_5 - a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_0}{120}$$

For $n = 5$ the recurrence equation gives

$$6a_6 - a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0}{720}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_0 x + \frac{1}{2} a_0 x^2 + \frac{1}{6} a_0 x^3 + \frac{1}{24} a_0 x^4 + \frac{1}{120} a_0 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5\right) a_0 + O\left(x^6\right) \tag{3}$$

Summary
The solution(s) found are the following

$$y = \left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5\right) y(0) + O\left(x^6\right) \tag{1}$$

$$y = \left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5\right) c_1 + O\left(x^6\right) \tag{2}$$

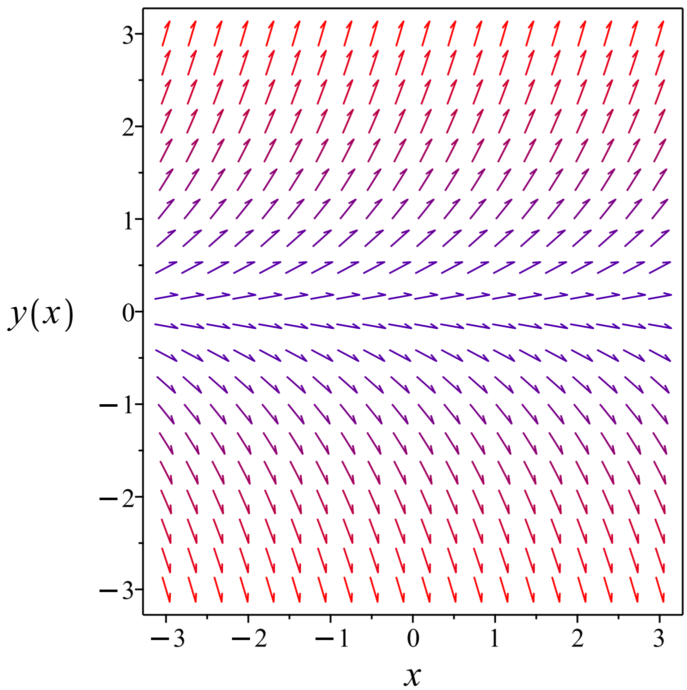


Figure 70: Slope field plot

Verification of solutions

$$y = \left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5\right) y(0) + O\left(x^6\right)$$

Verified OK.

$$y = \left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5\right) c_1 + O\left(x^6\right)$$

Verified OK.

7.1.2 Maple step by step solution

Let's solve

$y' - y = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{y}dx = \int 1dx + c_1$

- Evaluate integral

  $\ln(y) = x + c_1$

- Solve for y

  $y = e^{x+c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
Order:=6;
dsolve(diff(y(x),x)=y(x),y(x),type='series',x=0);
```

$$y(x) = \left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5\right)y(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 37

```
AsymptoticDSolveValue[y'[x]==y[x],y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(\frac{x^5}{120} + \frac{x^4}{24} + \frac{x^3}{6} + \frac{x^2}{2} + x + 1 \right)$$

7.2 problem problem 2

Internal problem ID [393]
Internal file name [`OUTPUT/393_Sunday_June_05_2022_01_40_29_AM_3185101/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Chapter 11 Power series methods. Section 11.1 Introduction and Review of power series. Page 615
Problem number: problem 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first order ode series method. Ordinary point", "first order ode series method. Taylor series method"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{y' - 4y = 0}$$

With the expansion point for the power series method at $x = 0$.

7.2.1 Solving as series ode

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving first order ode. Let

$$y' = f(x, y)$$

Where $f(x, y)$ is analytic at expansion point x_0. We can always shift to $x_0 = 0$ if x_0 is not zero. So from now we assume $x_0 = 0$. Assume also that $y(x_0) = y_0$. Using Taylor series

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y''(x_0) + \cdots \\
&= y_0 + xf + \frac{x^2}{2} \frac{df}{dx}\bigg|_{x_0,y_0} + \frac{x^3}{3!} \frac{d^2 f}{dx^2}\bigg|_{x_0,y_0} + \cdots \\
&= y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f \tag{1}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y)$ then the above can be written as

$$F_0 = f(x,y) \tag{4}$$

$$F_n = \frac{d}{dx}(F_{n-1})$$

$$= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) F_0 \tag{5}$$

For example, for $n = 1$ we see that

$$\begin{aligned} F_1 &= \frac{d}{dx}(F_0) \\ &= \frac{\partial}{\partial x} F_0 + \left(\frac{\partial F_0}{\partial y}\right) F_0 \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f \end{aligned}$$

Which is (1). And when $n = 2$

$$\begin{aligned} F_2 &= \frac{d}{dx}(F_1) \\ &= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) F_0 \\ &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) f \\ &= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f \end{aligned}$$

Which is (2) and so on. Therefore (4,5) can be used from now on along with

$$y(x) = y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} F_n|_{x_0,y_0} \tag{6}$$

Hence

$$\begin{aligned}
F_0 &= 4y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} F_0 \\
&= 16y \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} F_1 \\
&= 64y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} F_2 \\
&= 256y \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} F_3 \\
&= 1024y
\end{aligned}$$

And so on. Evaluating all the above at initial conditions $x(0) = 0$ and $y(0) = y(0)$ gives

$$\begin{aligned}
F_0 &= 4y(0) \\
F_1 &= 16y(0) \\
F_2 &= 64y(0) \\
F_3 &= 256y(0) \\
F_4 &= 1024y(0)
\end{aligned}$$

Substituting all the above in (6) and simplifying gives the solution as

$$y = \left(1 + 4x + 8x^2 + \frac{32}{3}x^3 + \frac{32}{3}x^4 + \frac{128}{15}x^5\right) y(0) + O(x^6)$$

Since $x = 0$ is also an ordinary point, then standard power series can also be used. Writing the ODE as

$$
\begin{aligned}
y' + q(x)y &= p(x) \\
y' - 4y &= 0
\end{aligned}
$$

Where

$$
\begin{aligned}
q(x) &= -4 \\
p(x) &= 0
\end{aligned}
$$

Next, the type of the expansion point $x = 0$ is determined. This point can be an ordinary point, a regular singular point (also called removable singularity), or irregular singular point (also called non-removable singularity or essential singularity). When $x = 0$ is an ordinary point, then the standard power series is used. If the point is a regular singular point, Frobenius series is used instead. Irregular singular point requires more advanced methods (asymptotic methods) and is not supported now. Hopefully this will be added in the future. $x = 0$ is called an ordinary point $q(x)$ has a Taylor series expansion around the point $x = 0$. $x = 0$ is called a regular singular point if $q(x)$ is not not analytic at $x = 0$ but $xq(x)$ has Taylor series expansion. And finally, $x = 0$ is an irregular singular point if the point is not ordinary and not regular singular. This is the most complicated case. Now the expansion point $x = 0$ is checked to see if it is an ordinary point or not. Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
y' = \sum_{n=1}^{\infty} n a_n x^{n-1}
$$

Substituting the above back into the ode gives

$$
\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - 4\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}
$$

Which simplifies to

$$
\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + \sum_{n=0}^{\infty} (-4 a_n x^n) = 0 \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n\right) + \sum_{n=0}^{\infty} (-4 a_n x^n) = 0 \tag{3}$$

For $0 \leq n$, the recurrence equation is

$$(n+1) a_{n+1} - 4a_n = 0 \tag{4}$$

Solving for a_{n+1}, gives

$$a_{n+1} = \frac{4a_n}{n+1} \tag{5}$$

For $n = 0$ the recurrence equation gives

$$a_1 - 4a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_1 = 4a_0$$

For $n = 1$ the recurrence equation gives

$$2a_2 - 4a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_2 = 8a_0$$

For $n = 2$ the recurrence equation gives

$$3a_3 - 4a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{32a_0}{3}$$

For $n = 3$ the recurrence equation gives

$$4a_4 - 4a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{32a_0}{3}$$

For $n = 4$ the recurrence equation gives

$$5a_5 - 4a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{128a_0}{15}$$

For $n = 5$ the recurrence equation gives

$$6a_6 - 4a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{256a_0}{45}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + 4a_0x + 8a_0x^2 + \frac{32}{3}a_0x^3 + \frac{32}{3}a_0x^4 + \frac{128}{15}a_0x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + 4x + 8x^2 + \frac{32}{3}x^3 + \frac{32}{3}x^4 + \frac{128}{15}x^5\right) a_0 + O(x^6) \tag{3}$$

Summary

The solution(s) found are the following

$$y = \left(1 + 4x + 8x^2 + \frac{32}{3}x^3 + \frac{32}{3}x^4 + \frac{128}{15}x^5\right) y(0) + O(x^6) \tag{1}$$

$$y = \left(1 + 4x + 8x^2 + \frac{32}{3}x^3 + \frac{32}{3}x^4 + \frac{128}{15}x^5\right) c_1 + O(x^6) \tag{2}$$

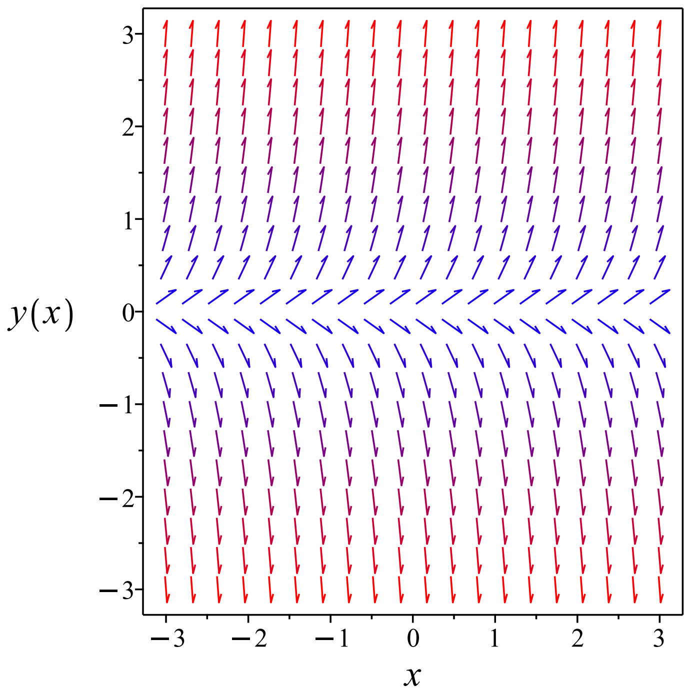


Figure 71: Slope field plot

Verification of solutions

$$y = \left(1 + 4x + 8x^2 + \frac{32}{3}x^3 + \frac{32}{3}x^4 + \frac{128}{15}x^5\right) y(0) + O(x^6)$$

Verified OK.

$$y = \left(1 + 4x + 8x^2 + \frac{32}{3}x^3 + \frac{32}{3}x^4 + \frac{128}{15}x^5\right) c_1 + O(x^6)$$

Verified OK.

7.2.2 Maple step by step solution

Let's solve

$y' - 4y = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = 4$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int 4dx + c_1$

- Evaluate integral

  $\ln(y) = 4x + c_1$

- Solve for y

  $y = e^{4x+c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 36

```
Order:=6;
dsolve(diff(y(x),x)=4*y(x),y(x),type='series',x=0);
```

$$y(x) = \left(1 + 4x + 8x^2 + \frac{32}{3}x^3 + \frac{32}{3}x^4 + \frac{128}{15}x^5\right) y(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 37

```
AsymptoticDSolveValue[y'[x]==4*y[x],y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(\frac{128x^5}{15} + \frac{32x^4}{3} + \frac{32x^3}{3} + 8x^2 + 4x + 1 \right)$$

7.3 problem problem 3

Internal problem ID [394]
Internal file name [`OUTPUT/394_Sunday_June_05_2022_01_40_30_AM_85759581/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Chapter 11 Power series methods. Section 11.1 Introduction and Review of power series. Page 615
Problem number: problem 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first order ode series method. Ordinary point", "first order ode series method. Taylor series method"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{2y' + 3y = 0}$$

With the expansion point for the power series method at $x = 0$.

7.3.1 Solving as series ode

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving first order ode. Let

$$y' = f(x, y)$$

Where $f(x,y)$ is analytic at expansion point x_0. We can always shift to $x_0 = 0$ if x_0 is not zero. So from now we assume $x_0 = 0$. Assume also that $y(x_0) = y_0$. Using Taylor series

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y''(x_0) + \cdots \\
&= y_0 + xf + \frac{x^2}{2} \frac{df}{dx}\bigg|_{x_0,y_0} + \frac{x^3}{3!} \frac{d^2 f}{dx^2}\bigg|_{x_0,y_0} + \cdots \\
&= y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f \tag{1}$$

$$\begin{aligned}\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\ &= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f \end{aligned} \tag{2}$$

$$\begin{aligned}\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\ &= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) f \end{aligned} \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x, y)$ then the above can be written as

$$F_0 = f(x, y) \tag{4}$$

$$\begin{aligned}F_n &= \frac{d}{dx}(F_{n-1}) \\ &= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) F_0 \end{aligned} \tag{5}$$

For example, for $n = 1$ we see that

$$\begin{aligned}F_1 &= \frac{d}{dx}(F_0) \\ &= \frac{\partial}{\partial x} F_0 + \left(\frac{\partial F_0}{\partial y}\right) F_0 \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f \end{aligned}$$

Which is (1). And when $n = 2$

$$\begin{aligned}F_2 &= \frac{d}{dx}(F_1) \\ &= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) F_0 \\ &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) f \\ &= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f \end{aligned}$$

Which is (2) and so on. Therefore (4,5) can be used from now on along with

$$y(x) = y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)\,!} F_n|_{x_0,y_0} \tag{6}$$

Hence

$$\begin{aligned}
F_0 &= -\frac{3y}{2} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} F_0 \\
&= \frac{9y}{4} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} F_1 \\
&= -\frac{27y}{8} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} F_2 \\
&= \frac{81y}{16} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} F_3 \\
&= -\frac{243y}{32}
\end{aligned}$$

And so on. Evaluating all the above at initial conditions $x(0) = 0$ and $y(0) = y(0)$ gives

$$\begin{aligned}
F_0 &= -\frac{3y(0)}{2} \\
F_1 &= \frac{9y(0)}{4} \\
F_2 &= -\frac{27y(0)}{8} \\
F_3 &= \frac{81y(0)}{16} \\
F_4 &= -\frac{243y(0)}{32}
\end{aligned}$$

Substituting all the above in (6) and simplifying gives the solution as

$$y=\left(1-\frac{3}{2}x+\frac{9}{8}x^{2}-\frac{9}{16}x^{3}+\frac{27}{128}x^{4}-\frac{81}{1280}x^{5}\right)y(0)+O\left(x^{6}\right)$$

Since $x=0$ is also an ordinary point, then standard power series can also be used. Writing the ODE as

$$\begin{aligned}y'+q(x)y&=p(x)\\ y'+\frac{3y}{2}&=0\end{aligned}$$

Where

$$\begin{aligned}q(x)&=\frac{3}{2}\\ p(x)&=0\end{aligned}$$

Next, the type of the expansion point $x=0$ is determined. This point can be an ordinary point, a regular singular point (also called removable singularity), or irregular singular point (also called non-removable singularity or essential singularity). When $x=0$ is an ordinary point, then the standard power series is used. If the point is a regular singular point, Frobenius series is used instead. Irregular singular point requires more advanced methods (asymptotic methods) and is not supported now. Hopefully this will be added in the future. $x=0$ is called an ordinary point $q(x)$ has a Taylor series expansion around the point $x=0$. $x=0$ is called a regular singular point if $q(x)$ is not not analytic at $x=0$ but $xq(x)$ has Taylor series expansion. And finally, $x=0$ is an irregular singular point if the point is not ordinary and not regular singular. This is the most complicated case. Now the expansion point $x=0$ is checked to see if it is an ordinary point or not. Let the solution be represented as power series of the form

$$y=\sum_{n=0}^{\infty}a_{n}x^{n}$$

Then

$$y'=\sum_{n=1}^{\infty}na_{n}x^{n-1}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=1}^{\infty}na_{n}x^{n-1}\right)+\frac{3\left(\sum_{n=0}^{\infty}a_{n}x^{n}\right)}{2}=0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + \left(\sum_{n=0}^{\infty} \frac{3 a_n x^n}{2}\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n\right) + \left(\sum_{n=0}^{\infty} \frac{3 a_n x^n}{2}\right) = 0 \tag{3}$$

For $0 \leq n$, the recurrence equation is

$$(n+1) a_{n+1} + \frac{3 a_n}{2} = 0 \tag{4}$$

Solving for a_{n+1}, gives

$$a_{n+1} = -\frac{3 a_n}{2 (n+1)} \tag{5}$$

For $n = 0$ the recurrence equation gives

$$a_1 + \frac{3 a_0}{2} = 0$$

Which after substituting the earlier terms found becomes

$$a_1 = -\frac{3 a_0}{2}$$

For $n = 1$ the recurrence equation gives

$$2 a_2 + \frac{3 a_1}{2} = 0$$

Which after substituting the earlier terms found becomes

$$a_2 = \frac{9a_0}{8}$$

For $n = 2$ the recurrence equation gives

$$3a_3 + \frac{3a_2}{2} = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{9a_0}{16}$$

For $n = 3$ the recurrence equation gives

$$4a_4 + \frac{3a_3}{2} = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{27a_0}{128}$$

For $n = 4$ the recurrence equation gives

$$5a_5 + \frac{3a_4}{2} = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{81a_0}{1280}$$

For $n = 5$ the recurrence equation gives

$$6a_6 + \frac{3a_5}{2} = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{81a_0}{5120}$$

And so on. Therefore the solution is

$$y = \sum_{n=0}^{\infty} a_n x^n$$
$$= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 - \frac{3}{2} a_0 x + \frac{9}{8} a_0 x^2 - \frac{9}{16} a_0 x^3 + \frac{27}{128} a_0 x^4 - \frac{81}{1280} a_0 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{3}{2}x + \frac{9}{8}x^2 - \frac{9}{16}x^3 + \frac{27}{128}x^4 - \frac{81}{1280}x^5\right) a_0 + O(x^6) \tag{3}$$

Summary

The solution(s) found are the following

$$y = \left(1 - \frac{3}{2}x + \frac{9}{8}x^2 - \frac{9}{16}x^3 + \frac{27}{128}x^4 - \frac{81}{1280}x^5\right) y(0) + O(x^6) \tag{1}$$
$$y = \left(1 - \frac{3}{2}x + \frac{9}{8}x^2 - \frac{9}{16}x^3 + \frac{27}{128}x^4 - \frac{81}{1280}x^5\right) c_1 + O(x^6) \tag{2}$$

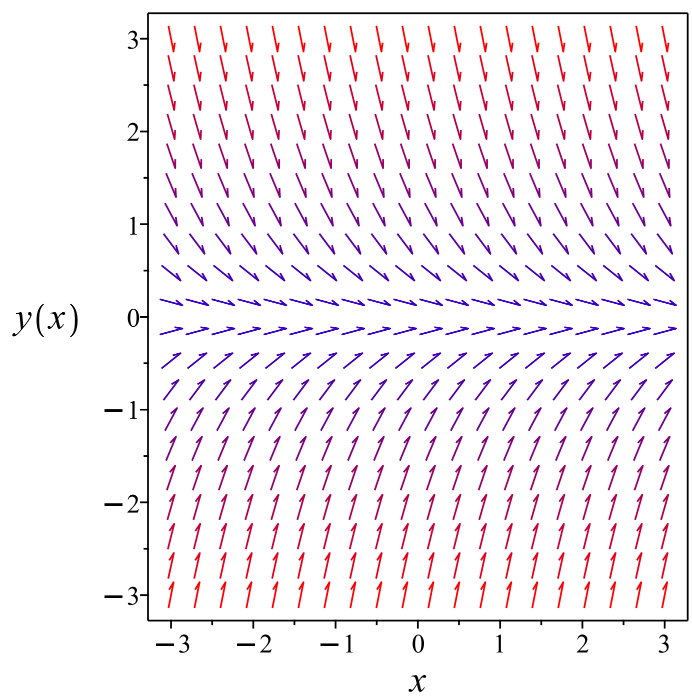


Figure 72: Slope field plot

Verification of solutions

$$y = \left(1-\frac{3}{2}x+\frac{9}{8}x^{2}-\frac{9}{16}x^{3}+\frac{27}{128}x^{4}-\frac{81}{1280}x^{5}\right)y(0)+O\left(x^{6}\right)$$

Verified OK.

$$y = \left(1-\frac{3}{2}x+\frac{9}{8}x^{2}-\frac{9}{16}x^{3}+\frac{27}{128}x^{4}-\frac{81}{1280}x^{5}\right)c_1+O\left(x^{6}\right)$$

Verified OK.

7.3.2 Maple step by step solution

Let's solve

$y'+\frac{3y}{2}=0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y}=-\frac{3}{2}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y}dx=\int -\frac{3}{2}dx+c_1$

- Evaluate integral

  $\ln(y)=-\frac{3x}{2}+c_1$

- Solve for y

  $y=\mathrm{e}^{-\frac{3x}{2}+c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 36

```
Order:=6;
dsolve(2*diff(y(x),x)+3*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{3}{2}x + \frac{9}{8}x^2 - \frac{9}{16}x^3 + \frac{27}{128}x^4 - \frac{81}{1280}x^5\right) y(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 41

```
AsymptoticDSolveValue[2*y'[x]+3*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(-\frac{81x^5}{1280} + \frac{27x^4}{128} - \frac{9x^3}{16} + \frac{9x^2}{8} - \frac{3x}{2} + 1\right)$$

7.4 problem problem 4

Internal problem ID [395]
Internal file name [`OUTPUT/395_Sunday_June_05_2022_01_40_31_AM_59354872/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Chapter 11 Power series methods. Section 11.1 Introduction and Review of power series. Page 615
Problem number: problem 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first order ode series method. Ordinary point", "first order ode series method. Taylor series method"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{2yx + y' = 0}$$

With the expansion point for the power series method at $x = 0$.

7.4.1 Solving as series ode

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving first order ode. Let

$$y' = f(x, y)$$

Where $f(x, y)$ is analytic at expansion point x_0. We can always shift to $x_0 = 0$ if x_0 is not zero. So from now we assume $x_0 = 0$. Assume also that $y(x_0) = y_0$. Using Taylor series

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y''(x_0) + \cdots \\
&= y_0 + xf + \frac{x^2}{2} \frac{df}{dx}\bigg|_{x_0, y_0} + \frac{x^3}{3!} \frac{d^2 f}{dx^2}\bigg|_{x_0, y_0} + \cdots \\
&= y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0}
\end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f && (1)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) f && (3)\\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y)$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y) && (4)\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) F_0 && (5)
\end{aligned}
$$

For example, for $n = 1$ we see that

$$
\begin{aligned}
F_1 &= \frac{d}{dx}(F_0)\\
&= \frac{\partial}{\partial x} F_0 + \left(\frac{\partial F_0}{\partial y}\right) F_0\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f
\end{aligned}
$$

Which is (1). And when $n = 2$

$$
\begin{aligned}
F_2 &= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) F_0\\
&= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) f\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f
\end{aligned}
$$

Which is (2) and so on. Therefore (4,5) can be used from now on along with

$$y(x) = y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} F_n\vert_{x_0,y_0} \tag{6}$$

Hence

$$\begin{aligned}
F_0 &= -2yx \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} F_0 \\
&= (4x^2 - 2)\, y \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} F_1 \\
&= (-8x^3 + 12x)\, y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} F_2 \\
&= 4y(4x^4 - 12x^2 + 3) \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} F_3 \\
&= -32xy\left(x^4 - 5x^2 + \frac{15}{4}\right)
\end{aligned}$$

And so on. Evaluating all the above at initial conditions $x(0) = 0$ and $y(0) = y(0)$ gives

$$\begin{aligned}
F_0 &= 0 \\
F_1 &= -2y(0) \\
F_2 &= 0 \\
F_3 &= 12y(0) \\
F_4 &= 0
\end{aligned}$$

Substituting all the above in (6) and simplifying gives the solution as

$$y = \left(1 - x^2 + \frac{1}{2}x^4\right) y(0) + O(x^6)$$

Since $x = 0$ is also an ordinary point, then standard power series can also be used. Writing the ODE as

$$
\begin{aligned}
y' + q(x)y &= p(x) \\
2yx + y' &= 0
\end{aligned}
$$

Where

$$
\begin{aligned}
q(x) &= 2x \\
p(x) &= 0
\end{aligned}
$$

Next, the type of the expansion point $x = 0$ is determined. This point can be an ordinary point, a regular singular point (also called removable singularity), or irregular singular point (also called non-removable singularity or essential singularity). When $x = 0$ is an ordinary point, then the standard power series is used. If the point is a regular singular point, Frobenius series is used instead. Irregular singular point requires more advanced methods (asymptotic methods) and is not supported now. Hopefully this will be added in the future. $x = 0$ is called an ordinary point $q(x)$ has a Taylor series expansion around the point $x = 0$. $x = 0$ is called a regular singular point if $q(x)$ is not not analytic at $x = 0$ but $xq(x)$ has Taylor series expansion. And finally, $x = 0$ is an irregular singular point if the point is not ordinary and not regular singular. This is the most complicated case. Now the expansion point $x = 0$ is checked to see if it is an ordinary point or not. Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
y' = \sum_{n=1}^{\infty} n a_n x^{n-1}
$$

Substituting the above back into the ode gives

$$
2\left(\sum_{n=0}^{\infty} a_n x^n\right) x + \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) = 0 \tag{1}
$$

Which simplifies to

$$
\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + \left(\sum_{n=0}^{\infty} 2x^{1+n} a_n\right) = 0 \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=1}^{\infty} n a_n x^{n-1} &= \sum_{n=0}^{\infty} (1+n)\, a_{1+n} x^n \\
\sum_{n=0}^{\infty} 2x^{1+n} a_n &= \sum_{n=1}^{\infty} 2a_{n-1} x^n
\end{aligned}
$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$
\left(\sum_{n=0}^{\infty} (1+n)\, a_{1+n} x^n\right) + \left(\sum_{n=1}^{\infty} 2a_{n-1} x^n\right) = 0 \tag{3}
$$

For $1 \leq n$, the recurrence equation is

$$
(1+n)\, a_{1+n} + 2a_{n-1} = 0 \tag{4}
$$

Solving for a_{1+n}, gives

$$
a_{1+n} = -\frac{2a_{n-1}}{1+n} \tag{5}
$$

For $n = 1$ the recurrence equation gives

$$
2a_2 + 2a_0 = 0
$$

Which after substituting the earlier terms found becomes

$$
a_2 = -a_0
$$

For $n = 2$ the recurrence equation gives

$$
3a_3 + 2a_1 = 0
$$

Which after substituting the earlier terms found becomes

$$
a_3 = 0
$$

For $n = 3$ the recurrence equation gives

$$4a_4 + 2a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{2}$$

For $n = 4$ the recurrence equation gives

$$5a_5 + 2a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = 0$$

For $n = 5$ the recurrence equation gives

$$6a_6 + 2a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{a_0}{6}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 - a_0 x^2 + \frac{1}{2} a_0 x^4 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - x^2 + \frac{1}{2} x^4\right) a_0 + O(x^6) \tag{3}$$

Summary

The solution(s) found are the following

$$y = \left(1 - x^2 + \frac{1}{2}x^4\right) y(0) + O(x^6) \tag{1}$$

$$y = \left(1 - x^2 + \frac{1}{2}x^4\right) c_1 + O(x^6) \tag{2}$$

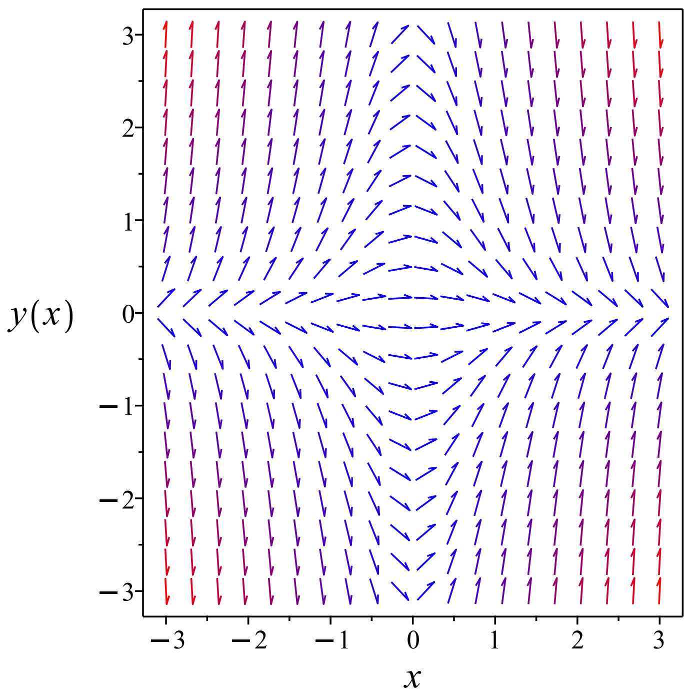


Figure 73: Slope field plot

Verification of solutions

$$y = \left(1 - x^2 + \frac{1}{2}x^4\right) y(0) + O(x^6)$$

Verified OK.

$$y = \left(1 - x^2 + \frac{1}{2}x^4\right) c_1 + O(x^6)$$

Verified OK.

7.4.2 Maple step by step solution

Let's solve

$2yx + y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = -2x$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int -2x dx + c_1$

- Evaluate integral

  $\ln(y) = -x^2 + c_1$

- Solve for y

  $y = e^{-x^2 + c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
Order:=6;
dsolve(diff(y(x),x)+2*x*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - x^2 + \frac{1}{2}x^4\right) y(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 20

```
AsymptoticDSolveValue[y'[x]+2*x*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(\frac{x^4}{2} - x^2 + 1 \right)$$

7.5 problem problem 5

Internal problem ID [396]
Internal file name [`OUTPUT/396_Sunday_June_05_2022_01_40_32_AM_23390871/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Chapter 11 Power series methods. Section 11.1 Introduction and Review of power series. Page 615
Problem number: problem 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first order ode series method. Ordinary point", "first order ode series method. Taylor series method"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y' - x^2 y = 0}$$

With the expansion point for the power series method at $x = 0$.

7.5.1 Solving as series ode

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving first order ode. Let

$$y' = f(x, y)$$

Where $f(x, y)$ is analytic at expansion point x_0. We can always shift to $x_0 = 0$ if x_0 is not zero. So from now we assume $x_0 = 0$. Assume also that $y(x_0) = y_0$. Using Taylor series

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y''(x_0) + \cdots \\ &= y_0 + xf + \frac{x^2}{2} \frac{df}{dx}\bigg|_{x_0,y_0} + \frac{x^3}{3!} \frac{d^2 f}{dx^2}\bigg|_{x_0,y_0} + \cdots \\ &= y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0} \end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f \qquad (1)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f \qquad (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) f \qquad (3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y)$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x,y) \qquad (4)\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) F_0 \qquad (5)
\end{aligned}$$

For example, for $n = 1$ we see that

$$\begin{aligned}
F_1 &= \frac{d}{dx}(F_0)\\
&= \frac{\partial}{\partial x} F_0 + \left(\frac{\partial F_0}{\partial y}\right) F_0\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f
\end{aligned}$$

Which is (1). And when $n = 2$

$$\begin{aligned}
F_2 &= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) F_0\\
&= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) f\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f
\end{aligned}$$

Which is (2) and so on. Therefore (4,5) can be used from now on along with

$$y(x) = y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} F_n|_{x_0,y_0} \tag{6}$$

Hence

$$\begin{aligned}
F_0 &= x^2 y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} F_0 \\
&= xy\left(x^3 + 2\right) \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} F_1 \\
&= y\left(x^6 + 6x^3 + 2\right) \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} F_2 \\
&= x^2 y\left(x^6 + 12x^3 + 20\right) \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} F_3 \\
&= y\left(x^9 + 20x^6 + 80x^3 + 40\right) x
\end{aligned}$$

And so on. Evaluating all the above at initial conditions $x(0) = 0$ and $y(0) = y(0)$ gives

$$\begin{aligned}
F_0 &= 0 \\
F_1 &= 0 \\
F_2 &= 2y(0) \\
F_3 &= 0 \\
F_4 &= 0
\end{aligned}$$

Substituting all the above in (6) and simplifying gives the solution as

$$y = \left(1 + \frac{x^3}{3}\right) y(0) + O\left(x^6\right)$$

Since $x = 0$ is also an ordinary point, then standard power series can also be used. Writing the ODE as

$$
\begin{aligned}
y' + q(x)y &= p(x) \\
y' - x^2 y &= 0
\end{aligned}
$$

Where

$$
\begin{aligned}
q(x) &= -x^2 \\
p(x) &= 0
\end{aligned}
$$

Next, the type of the expansion point $x = 0$ is determined. This point can be an ordinary point, a regular singular point (also called removable singularity), or irregular singular point (also called non-removable singularity or essential singularity). When $x = 0$ is an ordinary point, then the standard power series is used. If the point is a regular singular point, Frobenius series is used instead. Irregular singular point requires more advanced methods (asymptotic methods) and is not supported now. Hopefully this will be added in the future. $x = 0$ is called an ordinary point $q(x)$ has a Taylor series expansion around the point $x = 0$. $x = 0$ is called a regular singular point if $q(x)$ is not not analytic at $x = 0$ but $xq(x)$ has Taylor series expansion. And finally, $x = 0$ is an irregular singular point if the point is not ordinary and not regular singular. This is the most complicated case. Now the expansion point $x = 0$ is checked to see if it is an ordinary point or not. Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
y' = \sum_{n=1}^{\infty} n a_n x^{n-1}
$$

Substituting the above back into the ode gives

$$
\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - x^2 \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}
$$

Which simplifies to

$$
\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + \sum_{n=0}^{\infty} \left(-x^{n+2} a_n\right) = 0 \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=1}^{\infty} n a_n x^{n-1} &= \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n \\
\sum_{n=0}^{\infty} \left(-x^{n+2} a_n\right) &= \sum_{n=2}^{\infty} \left(-a_{n-2} x^n\right)
\end{aligned}
$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$
\left(\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n\right) + \sum_{n=2}^{\infty} \left(-a_{n-2} x^n\right) = 0 \tag{3}
$$

For $2 \leq n$, the recurrence equation is

$$
(n+1) a_{n+1} - a_{n-2} = 0 \tag{4}
$$

Solving for a_{n+1}, gives

$$
a_{n+1} = \frac{a_{n-2}}{n+1} \tag{5}
$$

For $n = 2$ the recurrence equation gives

$$
3a_3 - a_0 = 0
$$

Which after substituting the earlier terms found becomes

$$
a_3 = \frac{a_0}{3}
$$

For $n = 3$ the recurrence equation gives

$$
4a_4 - a_1 = 0
$$

Which after substituting the earlier terms found becomes

$$
a_4 = 0
$$

For $n = 4$ the recurrence equation gives

$$5a_5 - a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = 0$$

For $n = 5$ the recurrence equation gives

$$6a_6 - a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0}{18}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + \frac{1}{3} a_0 x^3 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + \frac{x^3}{3}\right) a_0 + O(x^6) \tag{3}$$

Summary

The solution(s) found are the following

$$y = \left(1 + \frac{x^3}{3}\right) y(0) + O(x^6) \tag{1}$$

$$y = \left(1 + \frac{x^3}{3}\right) c_1 + O(x^6) \tag{2}$$

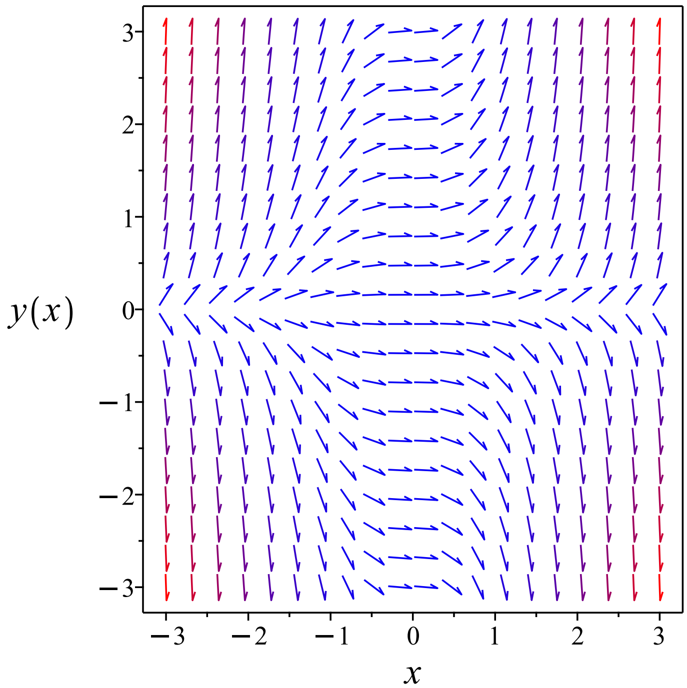


Figure 74: Slope field plot

Verification of solutions

$$y = \left(1 + \frac{x^3}{3}\right) y(0) + O(x^6)$$

Verified OK.

$$y = \left(1 + \frac{x^3}{3}\right) c_1 + O(x^6)$$

Verified OK.

7.5.2 Maple step by step solution

Let's solve

$y' - x^2 y = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = x^2$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int x^2 dx + c_1$

- Evaluate integral

  $\ln(y) = \frac{x^3}{3} + c_1$

- Solve for y

  $y = e^{\frac{x^3}{3}+c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
Order:=6;
dsolve(diff(y(x),x)=x^2*y(x),y(x),type='series',x=0);
```

$$y(x) = \left(1 + \frac{x^3}{3}\right) y(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 15

```
AsymptoticDSolveValue[y'[x]==x^2*y[x],y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(\frac{x^3}{3} + 1\right)$$

7.6 problem problem 6

Internal problem ID [397]
Internal file name [`OUTPUT/397_Sunday_June_05_2022_01_40_33_AM_28716630/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Chapter 11 Power series methods. Section 11.1 Introduction and Review of power series. Page 615
Problem number: problem 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first order ode series method. Ordinary point", "first order ode series method. Taylor series method"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{(-2+x)\,y'+y=0}$$

With the expansion point for the power series method at $x=0$.

7.6.1 Solving as series ode

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving first order ode. Let

$$y'=f(x,y)$$

Where $f(x,y)$ is analytic at expansion point x_0. We can always shift to $x_0=0$ if x_0 is not zero. So from now we assume $x_0=0$. Assume also that $y(x_0)=y_0$. Using Taylor series

$$\begin{aligned}
y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y''(x_0)+\cdots \\
&= y_0+xf+\frac{x^2}{2}\,\frac{df}{dx}\bigg|_{x_0,y_0}+\frac{x^3}{3!}\,\frac{d^2f}{dx^2}\bigg|_{x_0,y_0}+\cdots \\
&= y_0+\sum_{n=0}^{\infty}\frac{x^{n+1}}{(n+1)\,!}\,\frac{d^nf}{dx^n}\bigg|_{x_0,y_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f \tag{1}$$

$$\begin{aligned} \frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\ &= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f \end{aligned} \tag{2}$$

$$\begin{aligned} \frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\ &= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) f \\ &\vdots \end{aligned} \tag{3}$$

And so on. Hence if we name $F_0 = f(x,y)$ then the above can be written as

$$F_0 = f(x,y) \tag{4}$$

$$\begin{aligned} F_n &= \frac{d}{dx}(F_{n-1}) \\ &= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) F_0 \end{aligned} \tag{5}$$

For example, for $n = 1$ we see that

$$\begin{aligned} F_1 &= \frac{d}{dx}(F_0) \\ &= \frac{\partial}{\partial x} F_0 + \left(\frac{\partial F_0}{\partial y}\right) F_0 \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f \end{aligned}$$

Which is (1). And when $n = 2$

$$\begin{aligned} F_2 &= \frac{d}{dx}(F_1) \\ &= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) F_0 \\ &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) f \\ &= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f \end{aligned}$$

Which is (2) and so on. Therefore (4,5) can be used from now on along with

$$y(x) = y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} F_n|_{x_0,y_0} \tag{6}$$

Hence

$$\begin{aligned}
F_0 &= -\frac{y}{-2+x} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} F_0 \\
&= \frac{2y}{(-2+x)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} F_1 \\
&= -\frac{6y}{(-2+x)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} F_2 \\
&= \frac{24y}{(-2+x)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} F_3 \\
&= -\frac{120y}{(-2+x)^5}
\end{aligned}$$

And so on. Evaluating all the above at initial conditions $x(0) = 0$ and $y(0) = y(0)$ gives

$$
\begin{aligned}
F_0 &= \frac{y(0)}{2} \\
F_1 &= \frac{y(0)}{2} \\
F_2 &= \frac{3y(0)}{4} \\
F_3 &= \frac{3y(0)}{2} \\
F_4 &= \frac{15y(0)}{4}
\end{aligned}
$$

Substituting all the above in (6) and simplifying gives the solution as

$$
y = \left(1 + \frac{1}{2}x + \frac{1}{4}x^2 + \frac{1}{8}x^3 + \frac{1}{16}x^4 + \frac{1}{32}x^5\right) y(0) + O(x^6)
$$

Since $x = 0$ is also an ordinary point, then standard power series can also be used. Writing the ODE as

$$
\begin{aligned}
y' + q(x)y &= p(x) \\
y' + \frac{y}{-2 + x} &= 0
\end{aligned}
$$

Where

$$
\begin{aligned}
q(x) &= \frac{1}{-2 + x} \\
p(x) &= 0
\end{aligned}
$$

Next, the type of the expansion point $x = 0$ is determined. This point can be an ordinary point, a regular singular point (also called removable singularity), or irregular singular point (also called non-removable singularity or essential singularity). When $x = 0$ is an ordinary point, then the standard power series is used. If the point is a regular singular point, Frobenius series is used instead. Irregular singular point requires more advanced methods (asymptotic methods) and is not supported now. Hopefully this will be added in the future. $x = 0$ is called an ordinary point $q(x)$ has a Taylor series expansion around the point $x = 0$. $x = 0$ is called a regular singular point if $q(x)$ is not not analytic at $x = 0$ but $xq(x)$ has Taylor series expansion. And finally, $x = 0$ is an irregular singular point if the point is not ordinary and not regular singular. This is the most complicated case. Now the expansion point $x = 0$ is checked to see if it is an ordinary point or not. Now the ode is normalized by writing it as

$$
(-2 + x)\, y' + y = 0
$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

Substituting the above back into the ode gives

$$(-2+x)\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}$$

Which simplifies to

$$\sum_{n=1}^{\infty}\left(-2n a_n x^{n-1}\right) + \left(\sum_{n=1}^{\infty} n a_n x^n\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=1}^{\infty}\left(-2n a_n x^{n-1}\right) = \sum_{n=0}^{\infty}\left(-2(n+1) a_{n+1} x^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\sum_{n=0}^{\infty}\left(-2(n+1) a_{n+1} x^n\right) + \left(\sum_{n=1}^{\infty} n a_n x^n\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$-2a_1 + a_0 = 0$$
$$a_1 = \frac{a_0}{2}$$

For $1 \leq n$, the recurrence equation is

$$-2(n+1) a_{n+1} + n a_n + a_n = 0 \tag{4}$$

Solving for a_{n+1}, gives

$$a_{n+1} = \frac{a_n}{2} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$-4a_2 + 2a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_2 = \frac{a_0}{4}$$

For $n = 2$ the recurrence equation gives

$$-6a_3 + 3a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{a_0}{8}$$

For $n = 3$ the recurrence equation gives

$$-8a_4 + 4a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{16}$$

For $n = 4$ the recurrence equation gives

$$-10a_5 + 5a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_0}{32}$$

For $n = 5$ the recurrence equation gives

$$-12a_6 + 6a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0}{64}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + \frac{1}{2} a_0 x + \frac{1}{4} a_0 x^2 + \frac{1}{8} a_0 x^3 + \frac{1}{16} a_0 x^4 + \frac{1}{32} a_0 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + \frac{1}{2}x + \frac{1}{4}x^2 + \frac{1}{8}x^3 + \frac{1}{16}x^4 + \frac{1}{32}x^5\right) a_0 + O(x^6) \tag{3}$$

Summary

The solution(s) found are the following

$$y = \left(1 + \frac{1}{2}x + \frac{1}{4}x^2 + \frac{1}{8}x^3 + \frac{1}{16}x^4 + \frac{1}{32}x^5\right) y(0) + O(x^6) \tag{1}$$

$$y = \left(1 + \frac{1}{2}x + \frac{1}{4}x^2 + \frac{1}{8}x^3 + \frac{1}{16}x^4 + \frac{1}{32}x^5\right) c_1 + O(x^6) \tag{2}$$

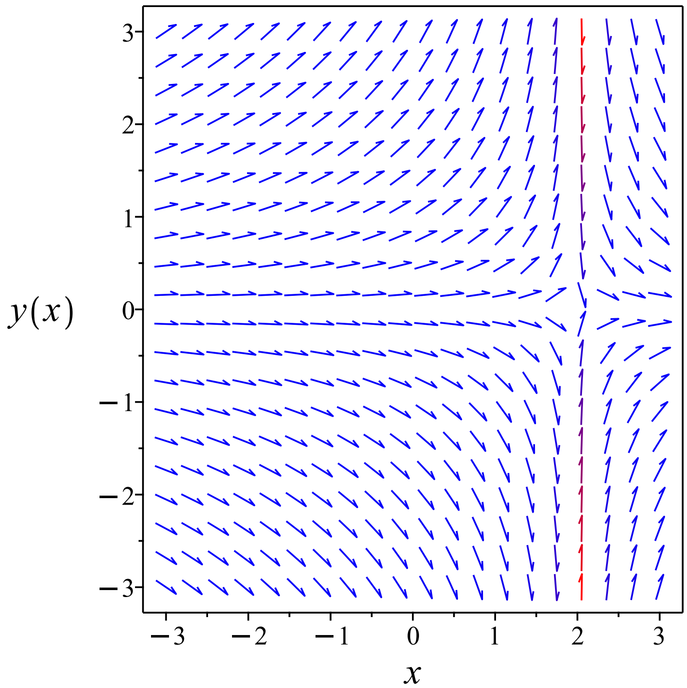


Figure 75: Slope field plot

Verification of solutions

$$y = \left(1 + \frac{1}{2}x + \frac{1}{4}x^2 + \frac{1}{8}x^3 + \frac{1}{16}x^4 + \frac{1}{32}x^5\right) y(0) + O(x^6)$$

Verified OK.

$$y = \left(1 + \frac{1}{2}x + \frac{1}{4}x^2 + \frac{1}{8}x^3 + \frac{1}{16}x^4 + \frac{1}{32}x^5\right) c_1 + O(x^6)$$

Verified OK.

7.6.2 Maple step by step solution

Let's solve

$(-2 + x)\, y' + y = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int ((-2 + x)\, y' + y)\, dx = \int 0 dx + c_1$

- Evaluate integral

  $(-2+x)\,y = c_1$

- Solve for y

  $y = \frac{c_1}{-2+x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 36

```
Order:=6;
dsolve((x-2)*diff(y(x),x)+y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1+\frac{1}{2}x+\frac{1}{4}x^2+\frac{1}{8}x^3+\frac{1}{16}x^4+\frac{1}{32}x^5\right)y(0)+O\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 41

```
AsymptoticDSolveValue[(x-2)*y'[x]+y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{x^5}{32}+\frac{x^4}{16}+\frac{x^3}{8}+\frac{x^2}{4}+\frac{x}{2}+1\right)$$

7.7 problem problem 7

Internal problem ID [398]
Internal file name [`OUTPUT/398_Sunday_June_05_2022_01_40_34_AM_60817607/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Chapter 11 Power series methods. Section 11.1 Introduction and Review of power series. Page 615
Problem number: problem 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first order ode series method. Ordinary point", "first order ode series method. Taylor series method"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{(2x-1)\,y'+2y=0}$$

With the expansion point for the power series method at $x=0$.

7.7.1 Solving as series ode

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving first order ode. Let

$$y'=f(x,y)$$

Where $f(x,y)$ is analytic at expansion point x_0. We can always shift to $x_0=0$ if x_0 is not zero. So from now we assume $x_0=0$. Assume also that $y(x_0)=y_0$. Using Taylor series

$$\begin{aligned}
y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y''(x_0)+\cdots \\
&= y_0+xf+\frac{x^2}{2}\frac{df}{dx}\bigg|_{x_0,y_0}+\frac{x^3}{3!}\frac{d^2f}{dx^2}\bigg|_{x_0,y_0}+\cdots \\
&= y_0+\sum_{n=0}^{\infty}\frac{x^{n+1}}{(n+1)\,!}\frac{d^nf}{dx^n}\bigg|_{x_0,y_0}
\end{aligned}$$

But

$$\begin{aligned}\frac{df}{dx} &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f && (1)\\ \frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\ &= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f && (2)\\ \frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\ &= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) f && (3)\\ &\vdots\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y)$ then the above can be written as

$$\begin{aligned}F_0 &= f(x,y) && (4)\\ F_n &= \frac{d}{dx}(F_{n-1})\\ &= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) F_0 && (5)\end{aligned}$$

For example, for $n = 1$ we see that

$$\begin{aligned}F_1 &= \frac{d}{dx}(F_0)\\ &= \frac{\partial}{\partial x} F_0 + \left(\frac{\partial F_0}{\partial y}\right) F_0\\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\end{aligned}$$

Which is (1). And when $n = 2$

$$\begin{aligned}F_2 &= \frac{d}{dx}(F_1)\\ &= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) F_0\\ &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) f\\ &= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f\end{aligned}$$

Which is (2) and so on. Therefore (4,5) can be used from now on along with

$$y(x) = y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)\,!} F_n|_{x_0,y_0} \tag{6}$$

Hence

$$\begin{aligned}
F_0 &= -\frac{2y}{2x-1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} F_0 \\
&= \frac{8y}{(2x-1)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} F_1 \\
&= -\frac{48y}{(2x-1)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} F_2 \\
&= \frac{384y}{(2x-1)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} F_3 \\
&= -\frac{3840y}{(2x-1)^5}
\end{aligned}$$

And so on. Evaluating all the above at initial conditions $x(0) = 0$ and $y(0) = y(0)$ gives

$$\begin{aligned}
F_0 &= 2y(0) \\
F_1 &= 8y(0) \\
F_2 &= 48y(0) \\
F_3 &= 384y(0) \\
F_4 &= 3840y(0)
\end{aligned}$$

Substituting all the above in (6) and simplifying gives the solution as

$$y = \left(32x^5 + 16x^4 + 8x^3 + 4x^2 + 2x + 1\right) y(0) + O\left(x^6\right)$$

Since $x = 0$ is also an ordinary point, then standard power series can also be used. Writing the ODE as

$$\begin{aligned} y' + q(x)y &= p(x) \\ y' + \frac{2y}{2x-1} &= 0 \end{aligned}$$

Where

$$\begin{aligned} q(x) &= \frac{2}{2x-1} \\ p(x) &= 0 \end{aligned}$$

Next, the type of the expansion point $x = 0$ is determined. This point can be an ordinary point, a regular singular point (also called removable singularity), or irregular singular point (also called non-removable singularity or essential singularity). When $x = 0$ is an ordinary point, then the standard power series is used. If the point is a regular singular point, Frobenius series is used instead. Irregular singular point requires more advanced methods (asymptotic methods) and is not supported now. Hopefully this will be added in the future. $x = 0$ is called an ordinary point $q(x)$ has a Taylor series expansion around the point $x = 0$. $x = 0$ is called a regular singular point if $q(x)$ is not not analytic at $x = 0$ but $xq(x)$ has Taylor series expansion. And finally, $x = 0$ is an irregular singular point if the point is not ordinary and not regular singular. This is the most complicated case. Now the expansion point $x = 0$ is checked to see if it is an ordinary point or not. Now the ode is normalized by writing it as

$$(2x-1)\, y' + 2y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

Substituting the above back into the ode gives

$$(2x-1)\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right)+2\left(\sum_{n=0}^{\infty} a_n x^n\right)=0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=1}^{\infty} 2n a_n x^n\right)+\sum_{n=1}^{\infty}\left(-n a_n x^{n-1}\right)+\left(\sum_{n=0}^{\infty} 2a_n x^n\right)=0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=1}^{\infty}\left(-n a_n x^{n-1}\right)=\sum_{n=0}^{\infty}\left(-(n+1)\, a_{n+1} x^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=1}^{\infty} 2n a_n x^n\right)+\sum_{n=0}^{\infty}\left(-(n+1)\, a_{n+1} x^n\right)+\left(\sum_{n=0}^{\infty} 2a_n x^n\right)=0 \tag{3}$$

$n=0$ gives

$$\begin{aligned}-a_1+2a_0&=0\\ a_1&=2a_0\end{aligned}$$

For $1\leq n$, the recurrence equation is

$$2n a_n-(n+1)\, a_{n+1}+2a_n=0 \tag{4}$$

Solving for a_{n+1}, gives

$$a_{n+1}=2a_n \tag{5}$$

For $n=1$ the recurrence equation gives

$$4a_1-2a_2=0$$

Which after substituting the earlier terms found becomes

$$a_2 = 4a_0$$

For $n = 2$ the recurrence equation gives

$$6a_2 - 3a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = 8a_0$$

For $n = 3$ the recurrence equation gives

$$8a_3 - 4a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = 16a_0$$

For $n = 4$ the recurrence equation gives

$$10a_4 - 5a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = 32a_0$$

For $n = 5$ the recurrence equation gives

$$12a_5 - 6a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = 64a_0$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = 32a_0x^5 + 16a_0x^4 + 8a_0x^3 + 4a_0x^2 + 2a_0x + a_0 + \dots$$

Collecting terms, the solution becomes

$$y = (32x^5 + 16x^4 + 8x^3 + 4x^2 + 2x + 1) a_0 + O(x^6) \tag{3}$$

Summary

The solution(s) found are the following

$$y = (32x^5 + 16x^4 + 8x^3 + 4x^2 + 2x + 1) y(0) + O(x^6) \tag{1}$$

$$y = (32x^5 + 16x^4 + 8x^3 + 4x^2 + 2x + 1) c_1 + O(x^6) \tag{2}$$

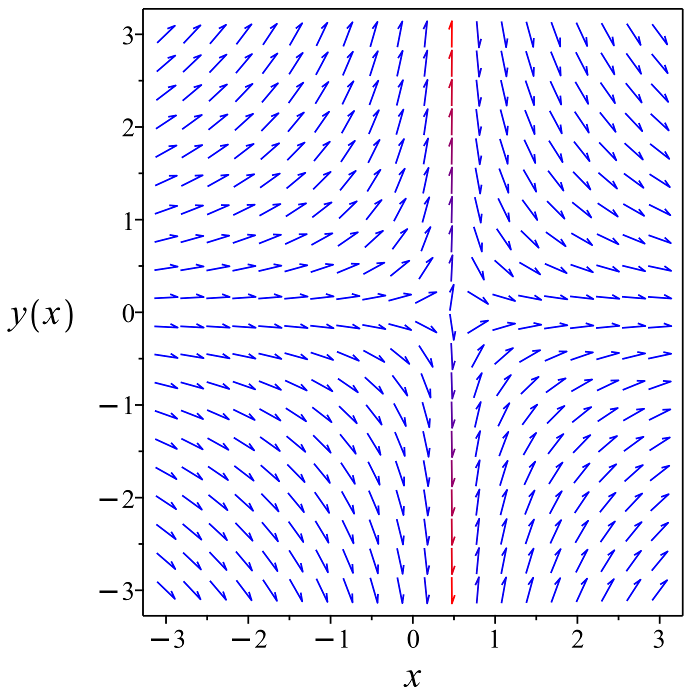


Figure 76: Slope field plot

Verification of solutions

$$y = (32x^5 + 16x^4 + 8x^3 + 4x^2 + 2x + 1) y(0) + O(x^6)$$

Verified OK.

$$y = (32x^5 + 16x^4 + 8x^3 + 4x^2 + 2x + 1) c_1 + O(x^6)$$

Verified OK.

7.7.2 Maple step by step solution

Let's solve

$(2x-1)\,y'+2y=0$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int \left((2x-1)\,y'+2y\right)dx=\int 0dx+c_1$

- Evaluate integral

  $y(2x-1)=c_1$

- Solve for y

  $y=\frac{c_1}{2x-1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 36

```
Order:=6;
dsolve((2*x-1)*diff(y(x),x)+2*y(x)=0,y(x),type='series',x=0);
```

$$y(x)=\left(32x^5+16x^4+8x^3+4x^2+2x+1\right)y(0)+O\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 31

```
AsymptoticDSolveValue[(2*x-1)*y'[x]+2*y[x]==0,y[x],{x,0,5}]
```

$$y(x)\to c_1\left(32x^5+16x^4+8x^3+4x^2+2x+1\right)$$

7.8 problem problem 8

Internal problem ID [399]
Internal file name [`OUTPUT/399_Sunday_June_05_2022_01_40_35_AM_1587067/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Chapter 11 Power series methods. Section 11.1 Introduction and Review of power series. Page 615
Problem number: problem 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first order ode series method. Ordinary point", "first order ode series method. Taylor series method"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{2(x+1)\,y'-y=0}$$

With the expansion point for the power series method at $x=0$.

7.8.1 Solving as series ode

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving first order ode. Let

$$y'=f(x,y)$$

Where $f(x,y)$ is analytic at expansion point x_0. We can always shift to $x_0=0$ if x_0 is not zero. So from now we assume $x_0=0$. Assume also that $y(x_0)=y_0$. Using Taylor series

$$\begin{aligned}
y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y''(x_0)+\cdots \\
&= y_0+xf+\frac{x^2}{2}\frac{df}{dx}\Big|_{x_0,y_0}+\frac{x^3}{3!}\frac{d^2f}{dx^2}\Big|_{x_0,y_0}+\cdots \\
&= y_0+\sum_{n=0}^{\infty}\frac{x^{n+1}}{(n+1)\,!}\frac{d^nf}{dx^n}\Big|_{x_0,y_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f \tag{1}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y)$ then the above can be written as

$$F_0 = f(x,y) \tag{4}$$

$$F_n = \frac{d}{dx}(F_{n-1})$$

$$= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) F_0 \tag{5}$$

For example, for $n = 1$ we see that

$$F_1 = \frac{d}{dx}(F_0)$$

$$= \frac{\partial}{\partial x} F_0 + \left(\frac{\partial F_0}{\partial y}\right) F_0$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f$$

Which is (1). And when $n = 2$

$$F_2 = \frac{d}{dx}(F_1)$$

$$= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) F_0$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) f$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f$$

Which is (2) and so on. Therefore (4,5) can be used from now on along with

$$y(x) = y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)\,!} F_n|_{x_0,y_0} \tag{6}$$

Hence

$$\begin{aligned}
F_0 &= \frac{y}{2+2x} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} F_0 \\
&= -\frac{y}{4\,(x+1)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} F_1 \\
&= \frac{3y}{8\,(x+1)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} F_2 \\
&= -\frac{15y}{16\,(x+1)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} F_3 \\
&= \frac{105y}{32\,(x+1)^5}
\end{aligned}$$

And so on. Evaluating all the above at initial conditions $x(0) = 0$ and $y(0) = y(0)$ gives

$$
\begin{aligned}
F_0 &= \frac{y(0)}{2} \\
F_1 &= -\frac{y(0)}{4} \\
F_2 &= \frac{3y(0)}{8} \\
F_3 &= -\frac{15y(0)}{16} \\
F_4 &= \frac{105y(0)}{32}
\end{aligned}
$$

Substituting all the above in (6) and simplifying gives the solution as

$$
y = \left(1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \frac{7}{256}x^5\right) y(0) + O(x^6)
$$

Since $x = 0$ is also an ordinary point, then standard power series can also be used. Writing the ODE as

$$
\begin{aligned}
y' + q(x)y &= p(x) \\
y' - \frac{y}{2 + 2x} &= 0
\end{aligned}
$$

Where

$$
\begin{aligned}
q(x) &= -\frac{1}{2 + 2x} \\
p(x) &= 0
\end{aligned}
$$

Next, the type of the expansion point $x = 0$ is determined. This point can be an ordinary point, a regular singular point (also called removable singularity), or irregular singular point (also called non-removable singularity or essential singularity). When $x = 0$ is an ordinary point, then the standard power series is used. If the point is a regular singular point, Frobenius series is used instead. Irregular singular point requires more advanced methods (asymptotic methods) and is not supported now. Hopefully this will be added in the future. $x = 0$ is called an ordinary point $q(x)$ has a Taylor series expansion around the point $x = 0$. $x = 0$ is called a regular singular point if $q(x)$ is not not analytic at $x = 0$ but $xq(x)$ has Taylor series expansion. And finally, $x = 0$ is an irregular singular point if the point is not ordinary and not regular singular. This is the most complicated case. Now the expansion point $x = 0$ is checked to see if it is an ordinary point or not. Now the ode is normalized by writing it as

$$
(2 + 2x)\, y' - y = 0
$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

Substituting the above back into the ode gives

$$(2+2x)\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=1}^{\infty} 2n a_n x^{n-1}\right) + \left(\sum_{n=1}^{\infty} 2n a_n x^n\right) + \sum_{n=0}^{\infty} (-a_n x^n) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=1}^{\infty} 2n a_n x^{n-1} = \sum_{n=0}^{\infty} 2(n+1) a_{n+1} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} 2(n+1) a_{n+1} x^n\right) + \left(\sum_{n=1}^{\infty} 2n a_n x^n\right) + \sum_{n=0}^{\infty} (-a_n x^n) = 0 \tag{3}$$

$n = 0$ gives

$$2a_1 - a_0 = 0$$
$$a_1 = \frac{a_0}{2}$$

For $1 \leq n$, the recurrence equation is

$$2(n+1) a_{n+1} + 2n a_n - a_n = 0 \tag{4}$$

Solving for a_{n+1}, gives

$$a_{n+1} = -\frac{a_n(2n-1)}{2(n+1)} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$4a_2 + a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_2 = -\frac{a_0}{8}$$

For $n = 2$ the recurrence equation gives

$$6a_3 + 3a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{a_0}{16}$$

For $n = 3$ the recurrence equation gives

$$8a_4 + 5a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{5a_0}{128}$$

For $n = 4$ the recurrence equation gives

$$10a_5 + 7a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{7a_0}{256}$$

For $n = 5$ the recurrence equation gives

$$12a_6 + 9a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{21a_0}{1024}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + \frac{1}{2}a_0 x - \frac{1}{8}a_0 x^2 + \frac{1}{16}a_0 x^3 - \frac{5}{128}a_0 x^4 + \frac{7}{256}a_0 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \frac{7}{256}x^5\right) a_0 + O\left(x^6\right) \tag{3}$$

Summary

The solution(s) found are the following

$$y = \left(1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \frac{7}{256}x^5\right) y(0) + O\left(x^6\right) \tag{1}$$

$$y = \left(1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \frac{7}{256}x^5\right) c_1 + O\left(x^6\right) \tag{2}$$

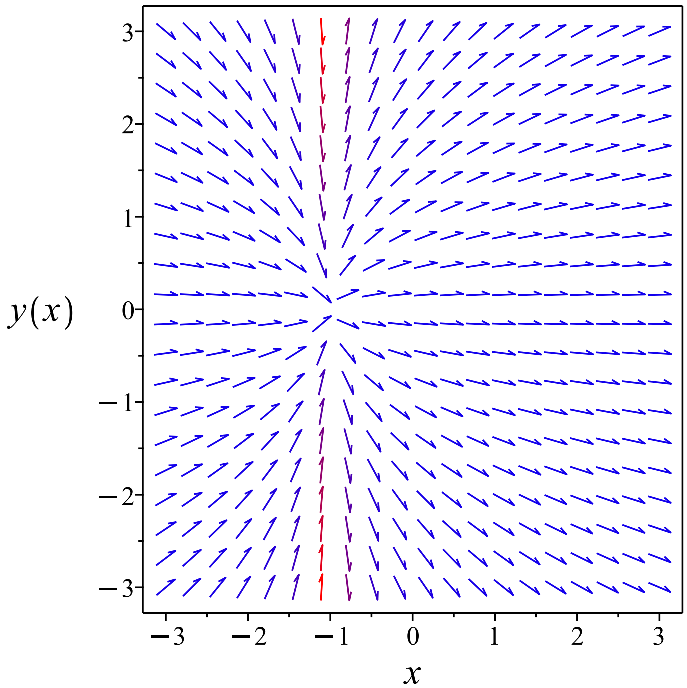


Figure 77: Slope field plot

Verification of solutions

$$y = \left(1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3-\frac{5}{128}x^4+\frac{7}{256}x^5\right)y(0)+O(x^6)$$

Verified OK.

$$y = \left(1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3-\frac{5}{128}x^4+\frac{7}{256}x^5\right)c_1+O(x^6)$$

Verified OK.

7.8.2 Maple step by step solution

Let's solve

$(2+2x)\,y'-y=0$

- Highest derivative means the order of the ODE is 1

y'

- Separate variables

$\frac{y'}{y}=\frac{1}{2+2x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y}dx = \int \frac{1}{2+2x}dx + c_1$
- Evaluate integral

  $\ln(y) = \frac{\ln(x+1)}{2} + c_1$
- Solve for y

  $y = e^{\frac{\ln(x+1)}{2}+c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 36

```
Order:=6;
dsolve(2*(x+1)*diff(y(x),x)=y(x),y(x),type='series',x=0);
```

$$y(x) = \left(1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \frac{7}{256}x^5\right)y(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 41

```
AsymptoticDSolveValue[2*(x+1)*y'[x]==y[x],y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{7x^5}{256} - \frac{5x^4}{128} + \frac{x^3}{16} - \frac{x^2}{8} + \frac{x}{2} + 1\right)$$

7.9 problem problem 9

Internal problem ID [400]
Internal file name [`OUTPUT/400_Sunday_June_05_2022_01_40_36_AM_15613461/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Chapter 11 Power series methods. Section 11.1 Introduction and Review of power series. Page 615
Problem number: problem 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first order ode series method. Ordinary point", "first order ode series method. Taylor series method"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{(x-1)\,y'+2y=0}$$

With the expansion point for the power series method at $x=0$.

7.9.1 Solving as series ode

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving first order ode. Let

$$y'=f(x,y)$$

Where $f(x,y)$ is analytic at expansion point x_0. We can always shift to $x_0=0$ if x_0 is not zero. So from now we assume $x_0=0$. Assume also that $y(x_0)=y_0$. Using Taylor series

$$\begin{aligned}
y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y''(x_0)+\cdots \\
&= y_0+xf+\frac{x^2}{2}\frac{df}{dx}\bigg|_{x_0,y_0}+\frac{x^3}{3!}\frac{d^2f}{dx^2}\bigg|_{x_0,y_0}+\cdots \\
&= y_0+\sum_{n=0}^{\infty}\frac{x^{n+1}}{(n+1)\,!}\frac{d^nf}{dx^n}\bigg|_{x_0,y_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f \tag{1}$$

$$\begin{aligned}\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\ &= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f \end{aligned}\tag{2}$$

$$\begin{aligned}\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\ &= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) f \end{aligned}\tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y)$ then the above can be written as

$$F_0 = f(x,y) \tag{4}$$

$$\begin{aligned}F_n &= \frac{d}{dx}(F_{n-1})\\ &= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) F_0 \end{aligned}\tag{5}$$

For example, for $n = 1$ we see that

$$\begin{aligned}F_1 &= \frac{d}{dx}(F_0)\\ &= \frac{\partial}{\partial x}F_0 + \left(\frac{\partial F_0}{\partial y}\right) F_0\\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\end{aligned}$$

Which is (1). And when $n = 2$

$$\begin{aligned}F_2 &= \frac{d}{dx}(F_1)\\ &= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right) F_0\\ &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) f\\ &= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f\end{aligned}$$

Which is (2) and so on. Therefore (4,5) can be used from now on along with

$$y(x) = y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} F_n|_{x_0,y_0} \tag{6}$$

Hence

$$\begin{aligned}
F_0 &= -\frac{2y}{x-1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} F_0 \\
&= \frac{6y}{(x-1)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} F_1 \\
&= -\frac{24y}{(x-1)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} F_2 \\
&= \frac{120y}{(x-1)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} F_3 \\
&= -\frac{720y}{(x-1)^5}
\end{aligned}$$

And so on. Evaluating all the above at initial conditions $x(0) = 0$ and $y(0) = y(0)$ gives

$$\begin{aligned}
F_0 &= 2y(0) \\
F_1 &= 6y(0) \\
F_2 &= 24y(0) \\
F_3 &= 120y(0) \\
F_4 &= 720y(0)
\end{aligned}$$

Substituting all the above in (6) and simplifying gives the solution as

$$y = (6x^5 + 5x^4 + 4x^3 + 3x^2 + 2x + 1)\, y(0) + O(x^6)$$

Since $x = 0$ is also an ordinary point, then standard power series can also be used. Writing the ODE as

$$\begin{aligned} y' + q(x)y &= p(x) \\ y' + \frac{2y}{x-1} &= 0 \end{aligned}$$

Where

$$\begin{aligned} q(x) &= \frac{2}{x-1} \\ p(x) &= 0 \end{aligned}$$

Next, the type of the expansion point $x = 0$ is determined. This point can be an ordinary point, a regular singular point (also called removable singularity), or irregular singular point (also called non-removable singularity or essential singularity). When $x = 0$ is an ordinary point, then the standard power series is used. If the point is a regular singular point, Frobenius series is used instead. Irregular singular point requires more advanced methods (asymptotic methods) and is not supported now. Hopefully this will be added in the future. $x = 0$ is called an ordinary point $q(x)$ has a Taylor series expansion around the point $x = 0$. $x = 0$ is called a regular singular point if $q(x)$ is not not analytic at $x = 0$ but $xq(x)$ has Taylor series expansion. And finally, $x = 0$ is an irregular singular point if the point is not ordinary and not regular singular. This is the most complicated case. Now the expansion point $x = 0$ is checked to see if it is an ordinary point or not. Now the ode is normalized by writing it as

$$(x-1)\, y' + 2y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

Substituting the above back into the ode gives

$$(x-1)\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right)+2\left(\sum_{n=0}^{\infty} a_n x^n\right)=0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=1}^{\infty} n a_n x^n\right)+\sum_{n=1}^{\infty}\left(-n a_n x^{n-1}\right)+\left(\sum_{n=0}^{\infty} 2 a_n x^n\right)=0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=1}^{\infty}\left(-n a_n x^{n-1}\right)=\sum_{n=0}^{\infty}\left(-(n+1) a_{n+1} x^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=1}^{\infty} n a_n x^n\right)+\sum_{n=0}^{\infty}\left(-(n+1) a_{n+1} x^n\right)+\left(\sum_{n=0}^{\infty} 2 a_n x^n\right)=0 \tag{3}$$

$n=0$ gives

$$\begin{aligned} -a_1+2a_0&=0 \\ a_1&=2a_0 \end{aligned}$$

For $1 \leq n$, the recurrence equation is

$$n a_n-(n+1) a_{n+1}+2 a_n=0 \tag{4}$$

Solving for a_{n+1}, gives

$$a_{n+1}=\frac{a_n(n+2)}{n+1} \tag{5}$$

For $n=1$ the recurrence equation gives

$$3a_1-2a_2=0$$

Which after substituting the earlier terms found becomes

$$a_2 = 3a_0$$

For $n = 2$ the recurrence equation gives

$$4a_2 - 3a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = 4a_0$$

For $n = 3$ the recurrence equation gives

$$5a_3 - 4a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = 5a_0$$

For $n = 4$ the recurrence equation gives

$$6a_4 - 5a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = 6a_0$$

For $n = 5$ the recurrence equation gives

$$7a_5 - 6a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = 7a_0$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = 6a_0x^5 + 5a_0x^4 + 4a_0x^3 + 3a_0x^2 + 2a_0x + a_0 + \dots$$

Collecting terms, the solution becomes

$$y = \left(6x^5 + 5x^4 + 4x^3 + 3x^2 + 2x + 1\right) a_0 + O\left(x^6\right) \tag{3}$$

Summary

The solution(s) found are the following

$$y = \left(6x^5 + 5x^4 + 4x^3 + 3x^2 + 2x + 1\right) y(0) + O\left(x^6\right) \tag{1}$$

$$y = \left(6x^5 + 5x^4 + 4x^3 + 3x^2 + 2x + 1\right) c_1 + O\left(x^6\right) \tag{2}$$

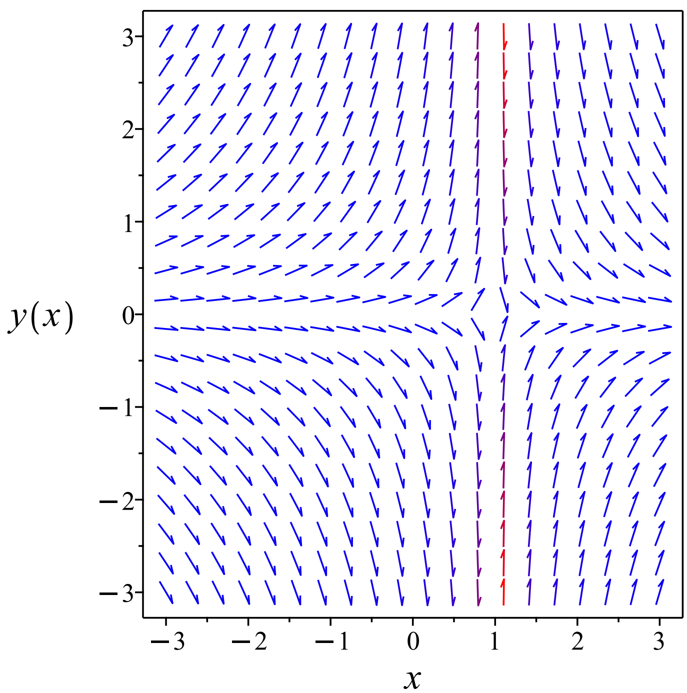


Figure 78: Slope field plot

Verification of solutions

$$y = \left(6x^5 + 5x^4 + 4x^3 + 3x^2 + 2x + 1\right) y(0) + O\left(x^6\right)$$

Verified OK.

$$y = \left(6x^5 + 5x^4 + 4x^3 + 3x^2 + 2x + 1\right) c_1 + O\left(x^6\right)$$

Verified OK.

7.9.2 Maple step by step solution

Let's solve

$(x-1)\,y'+2y=0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y}=-\frac{2}{x-1}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y}dx=\int -\frac{2}{x-1}dx+c_1$

- Evaluate integral

  $\ln(y)=-2\ln(x-1)+c_1$

- Solve for y

  $y=\frac{e^{c_1}}{(x-1)^2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 36

```
Order:=6;
dsolve((x-1)*diff(y(x),x)+2*y(x)=0,y(x),type='series',x=0);
```

$$y(x)=\left(6x^5+5x^4+4x^3+3x^2+2x+1\right)y(0)+O\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 31

```
AsymptoticDSolveValue[(x-1)*y'[x]+2*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(6x^5 + 5x^4 + 4x^3 + 3x^2 + 2x + 1\right)$$

7.10 problem problem 10

Internal problem ID [401]
Internal file name [`OUTPUT/401_Sunday_June_05_2022_01_40_36_AM_2674200/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Chapter 11 Power series methods. Section 11.1 Introduction and Review of power series. Page 615
Problem number: problem 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first order ode series method. Ordinary point", "first order ode series method. Taylor series method"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{2(x-1)\, y'-3y=0}$$

With the expansion point for the power series method at $x=0$.

7.10.1 Solving as series ode

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving first order ode. Let

$$y'=f(x,y)$$

Where $f(x,y)$ is analytic at expansion point x_0. We can always shift to $x_0=0$ if x_0 is not zero. So from now we assume $x_0=0$. Assume also that $y(x_0)=y_0$. Using Taylor series

$$\begin{aligned}
y(x) &= y(x_0)+(x-x_0)\, y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y''(x_0)+\cdots \\
&= y_0+xf+\frac{x^2}{2}\frac{df}{dx}\bigg|_{x_0,y_0}+\frac{x^3}{3!}\frac{d^2f}{dx^2}\bigg|_{x_0,y_0}+\cdots \\
&= y_0+\sum_{n=0}^{\infty}\frac{x^{n+1}}{(n+1)\,!}\frac{d^nf}{dx^n}\bigg|_{x_0,y_0}
\end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f && (1) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y)$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y) && (4) \\
F_n &= \frac{d}{dx}(F_{n-1}) \\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) F_0 && (5)
\end{aligned}
$$

For example, for $n = 1$ we see that

$$
\begin{aligned}
F_1 &= \frac{d}{dx}(F_0) \\
&= \frac{\partial}{\partial x} F_0 + \left(\frac{\partial F_0}{\partial y}\right) F_0 \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f
\end{aligned}
$$

Which is (1). And when $n = 2$

$$
\begin{aligned}
F_2 &= \frac{d}{dx}(F_1) \\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) F_0 \\
&= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) f \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f
\end{aligned}
$$

Which is (2) and so on. Therefore (4,5) can be used from now on along with

$$y(x) = y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)\,!} F_n|_{x_0,y_0} \tag{6}$$

Hence

$$\begin{aligned}
F_0 &= \frac{3y}{2\,(x-1)} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} F_0 \\
&= \frac{3y}{4\,(x-1)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} F_1 \\
&= -\frac{3y}{8\,(x-1)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} F_2 \\
&= \frac{9y}{16\,(x-1)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} F_3 \\
&= -\frac{45y}{32\,(x-1)^5}
\end{aligned}$$

And so on. Evaluating all the above at initial conditions $x(0) = 0$ and $y(0) = y(0)$ gives

$$
\begin{aligned}
F_0 &= -\frac{3y(0)}{2} \\
F_1 &= \frac{3y(0)}{4} \\
F_2 &= \frac{3y(0)}{8} \\
F_3 &= \frac{9y(0)}{16} \\
F_4 &= \frac{45y(0)}{32}
\end{aligned}
$$

Substituting all the above in (6) and simplifying gives the solution as

$$
y = \left(1 - \frac{3}{2}x + \frac{3}{8}x^2 + \frac{1}{16}x^3 + \frac{3}{128}x^4 + \frac{3}{256}x^5\right) y(0) + O(x^6)
$$

Since $x = 0$ is also an ordinary point, then standard power series can also be used. Writing the ODE as

$$
\begin{aligned}
y' + q(x)y &= p(x) \\
y' - \frac{3y}{2x - 2} &= 0
\end{aligned}
$$

Where

$$
\begin{aligned}
q(x) &= -\frac{3}{2x - 2} \\
p(x) &= 0
\end{aligned}
$$

Next, the type of the expansion point $x = 0$ is determined. This point can be an ordinary point, a regular singular point (also called removable singularity), or irregular singular point (also called non-removable singularity or essential singularity). When $x = 0$ is an ordinary point, then the standard power series is used. If the point is a regular singular point, Frobenius series is used instead. Irregular singular point requires more advanced methods (asymptotic methods) and is not supported now. Hopefully this will be added in the future. $x = 0$ is called an ordinary point $q(x)$ has a Taylor series expansion around the point $x = 0$. $x = 0$ is called a regular singular point if $q(x)$ is not not analytic at $x = 0$ but $xq(x)$ has Taylor series expansion. And finally, $x = 0$ is an irregular singular point if the point is not ordinary and not regular singular. This is the most complicated case. Now the expansion point $x = 0$ is checked to see if it is an ordinary point or not. Now the ode is normalized by writing it as

$$
(2x - 2)\, y' - 3y = 0
$$

Let the solution be represented as power series of the form

$$y=\sum_{n=0}^{\infty} a_n x^n$$

Then

$$y'=\sum_{n=1}^{\infty} n a_n x^{n-1}$$

Substituting the above back into the ode gives

$$(2x-2)\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right)-3\left(\sum_{n=0}^{\infty} a_n x^n\right)=0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=1}^{\infty} 2n a_n x^n\right)+\sum_{n=1}^{\infty}\left(-2n a_n x^{n-1}\right)+\sum_{n=0}^{\infty}\left(-3a_n x^n\right)=0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=1}^{\infty}\left(-2n a_n x^{n-1}\right)=\sum_{n=0}^{\infty}\left(-2(n+1)\, a_{n+1} x^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=1}^{\infty} 2n a_n x^n\right)+\sum_{n=0}^{\infty}\left(-2(n+1)\, a_{n+1} x^n\right)+\sum_{n=0}^{\infty}\left(-3a_n x^n\right)=0 \tag{3}$$

$n=0$ gives

$$-2a_1-3a_0=0$$
$$a_1=-\frac{3a_0}{2}$$

For $1\leq n$, the recurrence equation is

$$2n a_n-2(n+1)\, a_{n+1}-3a_n=0 \tag{4}$$

Solving for a_{n+1}, gives

$$a_{n+1} = \frac{a_n(2n-3)}{2n+2} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$-a_1 - 4a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_2 = \frac{3a_0}{8}$$

For $n = 2$ the recurrence equation gives

$$a_2 - 6a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{a_0}{16}$$

For $n = 3$ the recurrence equation gives

$$3a_3 - 8a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{3a_0}{128}$$

For $n = 4$ the recurrence equation gives

$$5a_4 - 10a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{3a_0}{256}$$

For $n = 5$ the recurrence equation gives

$$7a_5 - 12a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{7a_0}{1024}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 - \frac{3}{2} a_0 x + \frac{3}{8} a_0 x^2 + \frac{1}{16} a_0 x^3 + \frac{3}{128} a_0 x^4 + \frac{3}{256} a_0 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{3}{2}x + \frac{3}{8}x^2 + \frac{1}{16}x^3 + \frac{3}{128}x^4 + \frac{3}{256}x^5\right) a_0 + O(x^6) \tag{3}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \left(1 - \frac{3}{2}x + \frac{3}{8}x^2 + \frac{1}{16}x^3 + \frac{3}{128}x^4 + \frac{3}{256}x^5\right) y(0) + O(x^6) \tag{1}$$

$$y = \left(1 - \frac{3}{2}x + \frac{3}{8}x^2 + \frac{1}{16}x^3 + \frac{3}{128}x^4 + \frac{3}{256}x^5\right) c_1 + O(x^6) \tag{2}$$

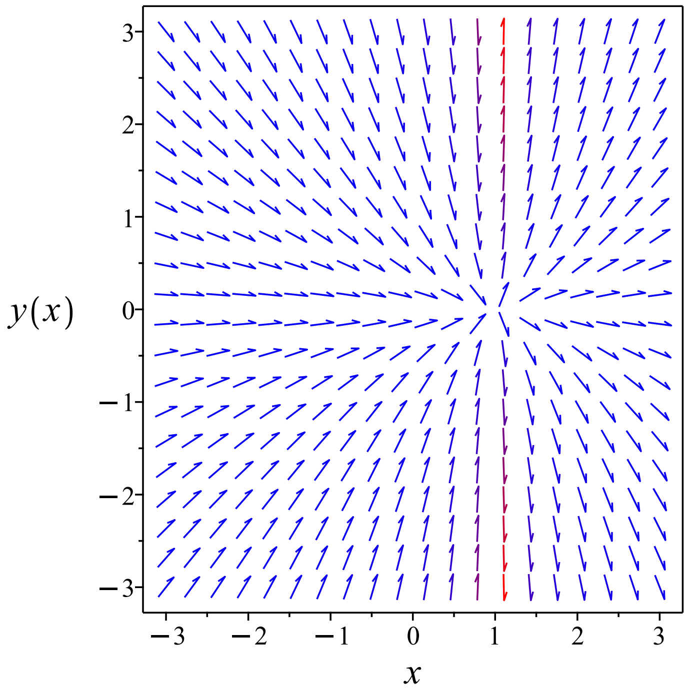


Figure 79: Slope field plot

Verification of solutions

$$y = \left(1 - \frac{3}{2}x + \frac{3}{8}x^2 + \frac{1}{16}x^3 + \frac{3}{128}x^4 + \frac{3}{256}x^5\right) y(0) + O(x^6)$$

Verified OK.

$$y = \left(1 - \frac{3}{2}x + \frac{3}{8}x^2 + \frac{1}{16}x^3 + \frac{3}{128}x^4 + \frac{3}{256}x^5\right) c_1 + O(x^6)$$

Verified OK.

7.10.2 Maple step by step solution

Let's solve

$(2x-2)\,y' - 3y = 0$

- Highest derivative means the order of the ODE is 1

y'

- Separate variables

$\frac{y'}{y} = \frac{3}{2x-2}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y}dx = \int \frac{3}{2x-2}dx + c_1$

- Evaluate integral

  $\ln(y) = \frac{3\ln(x-1)}{2} + c_1$

- Solve for y

  $y = e^{\frac{3\ln(x-1)}{2}+c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 36

```
Order:=6;
dsolve(2*(x-1)*diff(y(x),x)=3*y(x),y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{3}{2}x + \frac{3}{8}x^2 + \frac{1}{16}x^3 + \frac{3}{128}x^4 + \frac{3}{256}x^5\right)y(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 41

```
AsymptoticDSolveValue[2*(x-1)*y'[x]==3*y[x],y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{3x^5}{256} + \frac{3x^4}{128} + \frac{x^3}{16} + \frac{3x^2}{8} - \frac{3x}{2} + 1\right)$$

7.11 problem problem 11

Internal problem ID [402]
Internal file name [`OUTPUT/402_Sunday_June_05_2022_01_40_37_AM_74838344/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Chapter 11 Power series methods. Section 11.1 Introduction and Review of power series. Page 615
Problem number: problem 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (46)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (47)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{align}
F_0 &= f(x,y,y') \tag{4}\\
F_1 &= \frac{df}{dx} \nonumber\\
&= \frac{dF_0}{dx} \nonumber\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \nonumber\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{5}\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0 \nonumber\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right) \nonumber\\
&= \frac{d}{dx}(F_1) \nonumber\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y'' \nonumber\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0 \nonumber\\
&\vdots \nonumber\\
F_n &= \frac{d}{dx}(F_{n-1}) \nonumber\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y'' \nonumber\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \tag{6}
\end{align}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n\Big|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= y' \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= y
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= y(0) \\
F_1 &= y'(0) \\
F_2 &= y(0) \\
F_3 &= y'(0) \\
F_4 &= y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \frac{1}{720}x^6\right) y(0) + \left(x + \frac{1}{6}x^3 + \frac{1}{120}x^5\right) y'(0) + O(x^6)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard

power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} a_n x^n \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \sum_{n=0}^{\infty} (-a_n x^n) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n\right) + \sum_{n=0}^{\infty} (-a_n x^n) = 0 \tag{3}$$

For $0 \leq n$, the recurrence equation is

$$(n+2) a_{n+2}(n+1) - a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{a_n}{(n+2)(n+1)} \tag{5}$$

For $n = 0$ the recurrence equation gives

$$2a_2 - a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_2 = \frac{a_0}{2}$$

For $n = 1$ the recurrence equation gives

$$6a_3 - a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{a_1}{6}$$

For $n = 2$ the recurrence equation gives

$$12a_4 - a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{24}$$

For $n = 3$ the recurrence equation gives

$$20a_5 - a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_1}{120}$$

For $n = 4$ the recurrence equation gives

$$30a_6 - a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0}{720}$$

For $n = 5$ the recurrence equation gives

$$42a_7 - a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{a_1}{5040}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \frac{1}{2} a_0 x^2 + \frac{1}{6} a_1 x^3 + \frac{1}{24} a_0 x^4 + \frac{1}{120} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + \frac{1}{2}x^2 + \frac{1}{24}x^4\right) a_0 + \left(x + \frac{1}{6}x^3 + \frac{1}{120}x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 + \frac{1}{2}x^2 + \frac{1}{24}x^4\right) c_1 + \left(x + \frac{1}{6}x^3 + \frac{1}{120}x^5\right) c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = \left(1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \frac{1}{720}x^6\right) y(0) + \left(x + \frac{1}{6}x^3 + \frac{1}{120}x^5\right) y'(0) + O(x^6) \tag{1}$$

$$y = \left(1 + \frac{1}{2}x^2 + \frac{1}{24}x^4\right) c_1 + \left(x + \frac{1}{6}x^3 + \frac{1}{120}x^5\right) c_2 + O(x^6) \tag{2}$$

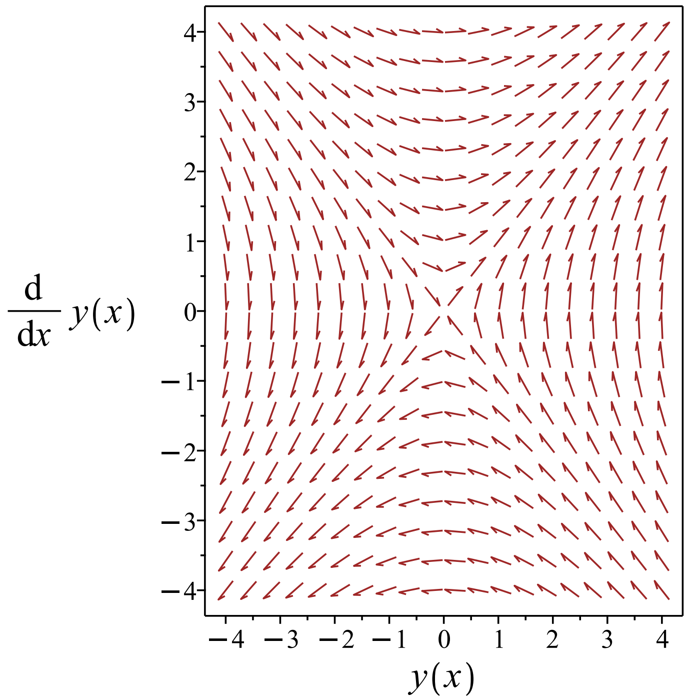


Figure 80: Slope field plot

Verification of solutions

$$y = \left(1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \frac{1}{720}x^6\right) y(0) + \left(x + \frac{1}{6}x^3 + \frac{1}{120}x^5\right) y'(0) + O(x^6)$$

Verified OK.

$$y = \left(1 + \frac{1}{2}x^2 + \frac{1}{24}x^4\right) c_1 + \left(x + \frac{1}{6}x^3 + \frac{1}{120}x^5\right) c_2 + O(x^6)$$

Verified OK.

7.11.1 Maple step by step solution

Let's solve

$y'' = y$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - y = 0$

- Characteristic polynomial of ODE

  $r^2 - 1 = 0$
- Factor the characteristic polynomial

  $(r-1)(r+1) = 0$
- Roots of the characteristic polynomial

  $r = (-1, 1)$
- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^{-x}$
- 2nd solution of the ODE

  $y_2(x) = \mathrm{e}^{x}$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$
- Substitute in solutions

  $y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
Order:=6;
dsolve(diff(y(x),x$2)=y(x),y(x),type='series',x=0);
```

$$y(x) = \left(1 + \frac{1}{2}x^2 + \frac{1}{24}x^4\right) y(0) + \left(x + \frac{1}{6}x^3 + \frac{1}{120}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 42

```
AsymptoticDSolveValue[y''[x]==y[x],y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{x^5}{120} + \frac{x^3}{6} + x\right) + c_1\left(\frac{x^4}{24} + \frac{x^2}{2} + 1\right)$$

7.12 problem problem 12

Internal problem ID [403]
Internal file name [`OUTPUT/403_Sunday_June_05_2022_01_40_38_AM_64020814/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Chapter 11 Power series methods. Section 11.1 Introduction and Review of power series. Page 615
Problem number: problem 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 4y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\ &= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0} \end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (49)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (50)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y'_0} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= 4y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= 4y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= 16y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= 16y' \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= 64y
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 4y(0) \\
F_1 &= 4y'(0) \\
F_2 &= 16y(0) \\
F_3 &= 16y'(0) \\
F_4 &= 64y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 + 2x^2 + \frac{2}{3}x^4 + \frac{4}{45}x^6\right) y(0) + \left(x + \frac{2}{3}x^3 + \frac{2}{15}x^5\right) y'(0) + O(x^6)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard

power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = 4\left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \sum_{n=0}^{\infty} (-4 a_n x^n) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n\right) + \sum_{n=0}^{\infty} (-4 a_n x^n) = 0 \tag{3}$$

For $0 \leq n$, the recurrence equation is

$$(n+2) a_{n+2}(n+1) - 4a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{4a_n}{(n+2)(n+1)} \tag{5}$$

For $n = 0$ the recurrence equation gives

$$2a_2 - 4a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_2 = 2a_0$$

For $n = 1$ the recurrence equation gives

$$6a_3 - 4a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{2a_1}{3}$$

For $n = 2$ the recurrence equation gives

$$12a_4 - 4a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{2a_0}{3}$$

For $n = 3$ the recurrence equation gives

$$20a_5 - 4a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{2a_1}{15}$$

For $n = 4$ the recurrence equation gives

$$30a_6 - 4a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{4a_0}{45}$$

For $n = 5$ the recurrence equation gives

$$42a_7 - 4a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{4a_1}{315}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + 2a_0 x^2 + \frac{2}{3} a_1 x^3 + \frac{2}{3} a_0 x^4 + \frac{2}{15} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + 2x^2 + \frac{2}{3}x^4\right) a_0 + \left(x + \frac{2}{3}x^3 + \frac{2}{15}x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 + 2x^2 + \frac{2}{3}x^4\right) c_1 + \left(x + \frac{2}{3}x^3 + \frac{2}{15}x^5\right) c_2 + O(x^6)$$

<u>Summary</u>
The solution(s) found are the following

$$y = \left(1 + 2x^2 + \frac{2}{3}x^4 + \frac{4}{45}x^6\right) y(0) + \left(x + \frac{2}{3}x^3 + \frac{2}{15}x^5\right) y'(0) + O(x^6) \tag{1}$$

$$y = \left(1 + 2x^2 + \frac{2}{3}x^4\right) c_1 + \left(x + \frac{2}{3}x^3 + \frac{2}{15}x^5\right) c_2 + O(x^6) \tag{2}$$

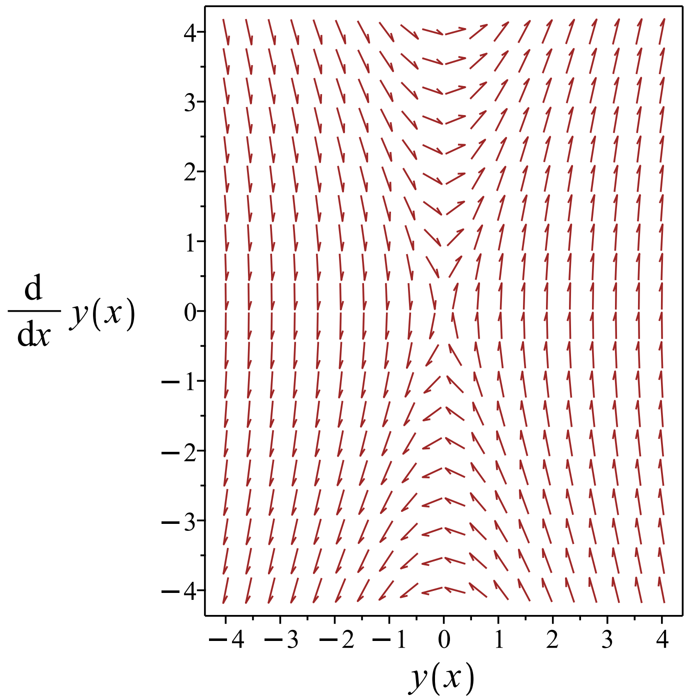


Figure 81: Slope field plot

Verification of solutions

$$y = \left(1 + 2x^2 + \frac{2}{3}x^4 + \frac{4}{45}x^6\right) y(0) + \left(x + \frac{2}{3}x^3 + \frac{2}{15}x^5\right) y'(0) + O(x^6)$$

Verified OK.

$$y = \left(1 + 2x^2 + \frac{2}{3}x^4\right) c_1 + \left(x + \frac{2}{3}x^3 + \frac{2}{15}x^5\right) c_2 + O(x^6)$$

Verified OK.

7.12.1 Maple step by step solution

Let's solve

$y'' = 4y$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - 4y = 0$

- Characteristic polynomial of ODE

  $r^2 - 4 = 0$
- Factor the characteristic polynomial

  $(r - 2)(r + 2) = 0$
- Roots of the characteristic polynomial

  $r = (-2, 2)$
- 1st solution of the ODE

  $y_1(x) = e^{-2x}$
- 2nd solution of the ODE

  $y_2(x) = e^{2x}$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$
- Substitute in solutions

  $y = c_1 e^{-2x} + c_2 e^{2x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
Order:=6;
dsolve(diff(y(x),x$2)=4*y(x),y(x),type='series',x=0);
```

$$y(x) = \left(1 + 2x^2 + \frac{2}{3}x^4\right) y(0) + \left(x + \frac{2}{3}x^3 + \frac{2}{15}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 40

```
AsymptoticDSolveValue[y''[x]==4*y[x],y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{2x^5}{15} + \frac{2x^3}{3} + x\right) + c_1\left(\frac{2x^4}{3} + 2x^2 + 1\right)$$

7.13 problem problem 13

Internal problem ID [404]
Internal file name [`OUTPUT/404_Sunday_June_05_2022_01_40_39_AM_89349105/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Chapter 11 Power series methods. Section 11.1 Introduction and Review of power series. Page 615
Problem number: problem 13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 9y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (52)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (53)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{align}
F_0 &= f(x,y,y') \tag{4}\\
F_1 &= \frac{df}{dx} \nonumber\\
&= \frac{dF_0}{dx} \nonumber\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' \nonumber\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \tag{5}\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \nonumber\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right) \nonumber\\
&= \frac{d}{dx}(F_1) \nonumber\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y'' \nonumber\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0 \nonumber\\
&\vdots \nonumber\\
F_n &= \frac{d}{dx}(F_{n-1}) \nonumber\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y'' \nonumber\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \tag{6}
\end{align}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -9y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= -9y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= 81y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= 81y' \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= -729y
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and $y'(0)=y'(0)$ gives

$$
\begin{aligned}
F_0 &= -9y(0) \\
F_1 &= -9y'(0) \\
F_2 &= 81y(0) \\
F_3 &= 81y'(0) \\
F_4 &= -729y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 - \frac{9}{2}x^2 + \frac{27}{8}x^4 - \frac{81}{80}x^6\right) y(0) + \left(x - \frac{3}{2}x^3 + \frac{27}{40}x^5\right) y'(0) + O(x^6)
$$

Since the expansion point $x=0$ is an ordinary, we can also solve this using standard

power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = -9\left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \left(\sum_{n=0}^{\infty} 9 a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n\right) + \left(\sum_{n=0}^{\infty} 9 a_n x^n\right) = 0 \tag{3}$$

For $0 \leq n$, the recurrence equation is

$$(n+2) a_{n+2}(n+1) + 9 a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{9a_n}{(n+2)(n+1)} \tag{5}$$

For $n = 0$ the recurrence equation gives

$$2a_2 + 9a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_2 = -\frac{9a_0}{2}$$

For $n = 1$ the recurrence equation gives

$$6a_3 + 9a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{3a_1}{2}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + 9a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{27a_0}{8}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + 9a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{27a_1}{40}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + 9a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{81a_0}{80}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + 9a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{81a_1}{560}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1x - \frac{9}{2}a_0x^2 - \frac{3}{2}a_1x^3 + \frac{27}{8}a_0x^4 + \frac{27}{40}a_1x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{9}{2}x^2 + \frac{27}{8}x^4\right)a_0 + \left(x - \frac{3}{2}x^3 + \frac{27}{40}x^5\right)a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{9}{2}x^2 + \frac{27}{8}x^4\right)c_1 + \left(x - \frac{3}{2}x^3 + \frac{27}{40}x^5\right)c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = \left(1 - \frac{9}{2}x^2 + \frac{27}{8}x^4 - \frac{81}{80}x^6\right)y(0) + \left(x - \frac{3}{2}x^3 + \frac{27}{40}x^5\right)y'(0) + O(x^6) \tag{1}$$

$$y = \left(1 - \frac{9}{2}x^2 + \frac{27}{8}x^4\right)c_1 + \left(x - \frac{3}{2}x^3 + \frac{27}{40}x^5\right)c_2 + O(x^6) \tag{2}$$

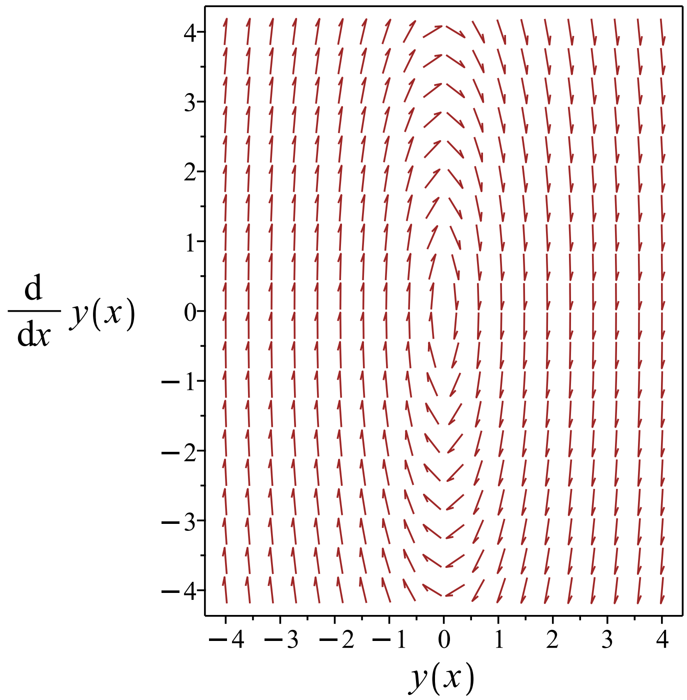


Figure 82: Slope field plot

Verification of solutions

$$y = \left(1 - \frac{9}{2}x^2 + \frac{27}{8}x^4 - \frac{81}{80}x^6\right) y(0) + \left(x - \frac{3}{2}x^3 + \frac{27}{40}x^5\right) y'(0) + O(x^6)$$

Verified OK.

$$y = \left(1 - \frac{9}{2}x^2 + \frac{27}{8}x^4\right) c_1 + \left(x - \frac{3}{2}x^3 + \frac{27}{40}x^5\right) c_2 + O(x^6)$$

Verified OK.

7.13.1 Maple step by step solution

Let's solve

$y'' = -9y$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + 9y = 0$

- Characteristic polynomial of ODE

  $r^2 + 9 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-36})}{2}$

- Roots of the characteristic polynomial

  $r = (-3\,\mathrm{I}, 3\,\mathrm{I})$

- 1st solution of the ODE

  $y_1(x) = \cos(3x)$

- 2nd solution of the ODE

  $y_2(x) = \sin(3x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \cos(3x) + c_2 \sin(3x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 34

```
Order:=6;
dsolve(diff(y(x),x$2)+9*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{9}{2}x^2 + \frac{27}{8}x^4\right) y(0) + \left(x - \frac{3}{2}x^3 + \frac{27}{40}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 42

```
AsymptoticDSolveValue[y''[x]+9*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{27x^5}{40} - \frac{3x^3}{2} + x\right) + c_1\left(\frac{27x^4}{8} - \frac{9x^2}{2} + 1\right)$$

7.14 problem problem 14

Internal problem ID [405]
Internal file name [`OUTPUT/405_Sunday_June_05_2022_01_40_40_AM_87187974/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Chapter 11 Power series methods. Section 11.1 Introduction and Review of power series. Page 615
Problem number: problem 14.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_linear_constant_coeff", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + y = x}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \left.\frac{d^n f}{dx^n}\right|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{55}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{56}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= x-y\\
F_1 &= \frac{dF_0}{dx}\\
&= \frac{\partial F_0}{\partial x}+\frac{\partial F_0}{\partial y}y'+\frac{\partial F_0}{\partial y'}F_0\\
&= 1-y'\\
F_2 &= \frac{dF_1}{dx}\\
&= \frac{\partial F_1}{\partial x}+\frac{\partial F_1}{\partial y}y'+\frac{\partial F_1}{\partial y'}F_1\\
&= -x+y\\
F_3 &= \frac{dF_2}{dx}\\
&= \frac{\partial F_2}{\partial x}+\frac{\partial F_2}{\partial y}y'+\frac{\partial F_2}{\partial y'}F_2\\
&= -1+y'\\
F_4 &= \frac{dF_3}{dx}\\
&= \frac{\partial F_3}{\partial x}+\frac{\partial F_3}{\partial y}y'+\frac{\partial F_3}{\partial y'}F_3\\
&= x-y
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and $y'(0)=y'(0)$ gives

$$
\begin{aligned}
F_0 &= -y(0)\\
F_1 &= 1-y'(0)\\
F_2 &= y(0)\\
F_3 &= y'(0)-1\\
F_4 &= -y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y=\left(1-\frac{1}{2}x^2+\frac{1}{24}x^4-\frac{1}{720}x^6\right)y(0)+\left(x-\frac{1}{6}x^3+\frac{1}{120}x^5\right)y'(0)+\frac{x^3}{6}-\frac{x^5}{120}+O\left(x^6\right)
$$

Since the expansion point $x=0$ is an ordinary, we can also solve this using standard

power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = x - \left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Expanding x as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned} x &= x + \dots \\ &= x \end{aligned}$$

Hence the ODE in Eq (1) becomes

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = x$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = x \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^{n}\right) + \left(\sum_{n=0}^{\infty} a_{n} x^{n}\right) = x \tag{3}$$

For $0 \leq n$, the recurrence equation is

$$\left((n+2)\, a_{n+2}(n+1) + a_{n}\right) x^{n} = x \tag{4}$$

For $n = 0$ the recurrence equation gives

$$\begin{aligned}(2a_{2} + a_{0})\, 1 &= 0\\ 2a_{2} + a_{0} &= 0\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_{2} = -\frac{a_{0}}{2}$$

For $n = 1$ the recurrence equation gives

$$\begin{aligned}(6a_{3} + a_{1})\, x &= x\\ 6a_{3} + a_{1} &= 1\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_{3} = \frac{1}{6} - \frac{a_{1}}{6}$$

For $n = 2$ the recurrence equation gives

$$\begin{aligned}(12a_{4} + a_{2})\, x^{2} &= 0\\ 12a_{4} + a_{2} &= 0\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_{4} = \frac{a_{0}}{24}$$

For $n = 3$ the recurrence equation gives

$$
\begin{aligned}
(20a_5 + a_3)\, x^3 &= 0 \\
20a_5 + a_3 &= 0
\end{aligned}
$$

Which after substituting the earlier terms found becomes

$$
a_5 = -\frac{1}{120} + \frac{a_1}{120}
$$

For $n = 4$ the recurrence equation gives

$$
\begin{aligned}
(30a_6 + a_4)\, x^4 &= 0 \\
30a_6 + a_4 &= 0
\end{aligned}
$$

Which after substituting the earlier terms found becomes

$$
a_6 = -\frac{a_0}{720}
$$

For $n = 5$ the recurrence equation gives

$$
\begin{aligned}
(42a_7 + a_5)\, x^5 &= 0 \\
42a_7 + a_5 &= 0
\end{aligned}
$$

Which after substituting the earlier terms found becomes

$$
a_7 = \frac{1}{5040} - \frac{a_1}{5040}
$$

And so on. Therefore the solution is

$$
\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^n \\
&= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots
\end{aligned}
$$

Substituting the values for a_n found above, the solution becomes

$$
y = a_0 + a_1 x - \frac{a_0 x^2}{2} + \left(\frac{1}{6} - \frac{a_1}{6}\right) x^3 + \frac{a_0 x^4}{24} + \left(-\frac{1}{120} + \frac{a_1}{120}\right) x^5 + \dots
$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4\right) a_0 + \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5\right) a_1 + \frac{x^3}{6} - \frac{x^5}{120} + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4\right) c_1 + \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5\right) c_2 + \frac{x^3}{6} - \frac{x^5}{120} + O(x^6)$$

Summary

The solution(s) found are the following

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6\right) y(0) + \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5\right) y'(0) + \frac{x^3}{6} - \frac{x^5}{120} + O(x^6) \tag{1}$$

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4\right) c_1 + \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5\right) c_2 + \frac{x^3}{6} - \frac{x^5}{120} + O(x^6) \tag{2}$$

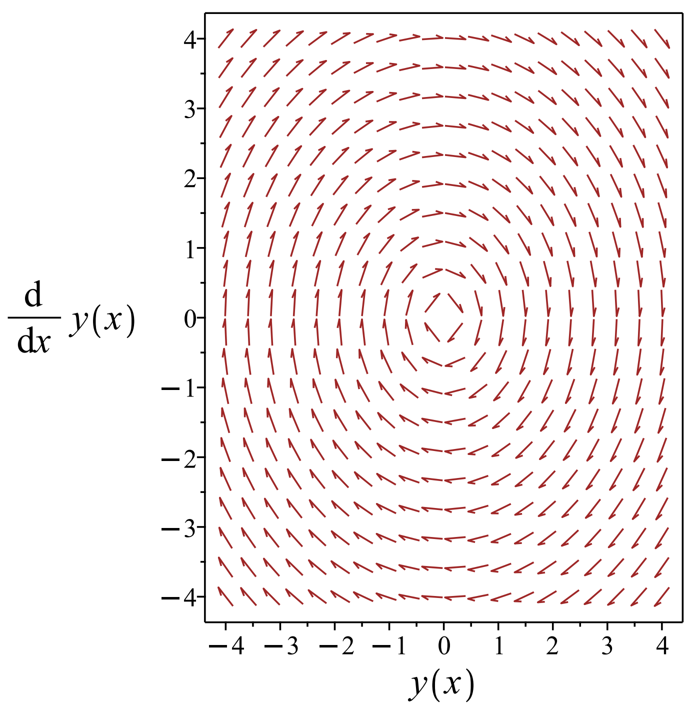


Figure 83: Slope field plot

Verification of solutions

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6\right) y(0) + \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5\right) y'(0) + \frac{x^3}{6} - \frac{x^5}{120} + O(x^6)$$

Verified OK.

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4\right) c_1 + \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5\right) c_2 + \frac{x^3}{6} - \frac{x^5}{120} + O(x^6)$$

Verified OK.

7.14.1 Maple step by step solution

Let's solve

$y'' = x - y$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + y = x$

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-I, I)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = x\right]$
- Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$
- Compute Wronskian

  $W(y_1(x), y_2(x)) = 1$
- Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\cos(x)\left(\int \sin(x) x dx\right) + \sin(x)\left(\int x\cos(x) dx\right)$
- Compute integrals

  $y_p(x) = x$

- Substitute particular solution into general solution to ODE

  $y = c_1\cos(x) + c_2\sin(x) + x$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 38

```
Order:=6;
dsolve(diff(y(x),x$2)+y(x)=x,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4\right) y(0) + \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5\right) D(y)(0) + \frac{x^3}{6} - \frac{x^5}{120} + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.043 (sec). Leaf size: 56

```
AsymptoticDSolveValue[y''[x]+y[x]==x,y[x],{x,0,5}]
```

$$y(x) \to -\frac{x^5}{120} + \frac{x^3}{6} + c_2\left(\frac{x^5}{120} - \frac{x^3}{6} + x\right) + c_1\left(\frac{x^4}{24} - \frac{x^2}{2} + 1\right)$$

7.15 problem problem 15

Internal problem ID [406]
Internal file name [OUTPUT/406_Sunday_June_05_2022_01_40_41_AM_98177842/index.tex]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Chapter 11 Power series methods. Section 11.1 Introduction and Review of power series. Page 615
Problem number: problem 15.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first order ode series method. Regular singular point"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y'x + y = 0}$$

With the expansion point for the power series method at $x = 0$.

7.15.1 Solving as series ode

Writing the ODE as

$$y' + q(x)y = p(x)$$
$$y' + \frac{y}{x} = 0$$

Where

$$q(x) = \frac{1}{x}$$
$$p(x) = 0$$

Next, the type of the expansion point $x = 0$ is determined. This point can be an ordinary point, a regular singular point (also called removable singularity), or irregular singular point (also called non-removable singularity or essential singularity). When

$x = 0$ is an ordinary point, then the standard power series is used. If the point is a regular singular point, Frobenius series is used instead. Irregular singular point requires more advanced methods (asymptotic methods) and is not supported now. Hopefully this will be added in the future. $x = 0$ is called an ordinary point $q(x)$ has a Taylor series expansion around the point $x = 0$. $x = 0$ is called a regular singular point if $q(x)$ is not not analytic at $x = 0$ but $xq(x)$ has Taylor series expansion. And finally, $x = 0$ is an irregular singular point if the point is not ordinary and not regular singular. This is the most complicated case. Now the expansion point $x = 0$ is checked to see if it is an ordinary point or not.

Since $x = 0$ is not an ordinary point, we now check to see if it is a regular singular point. $xq(x) = 1$ has a Taylor series around $x = 0$. Since $x = 0$ is regular singular point, then Frobenius power series is used. Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \frac{\sum_{n=0}^{\infty} a_n x^{n+r}}{x} = 0 \tag{1}$$

Hence the ODE in Eq (1) becomes

$$\left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \frac{\sum_{n=0}^{\infty} a_n x^{n+r}}{x} = 0 \tag{1}$$

Expanding the second term in (1) gives

$$\left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + x \cdot \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) - 1 \cdot \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} x^{n+r-1} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} x^{n+r-1} a_n\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n = 0$. From Eq (2) this gives

$$(n+r)\, a_n x^{n+r-1} + x^{n+r-1} a_n = 0$$

When $n = 0$ the above becomes

$$r a_0 x^{-1+r} + x^{-1+r} a_0 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(r+1)\, x^{-1+r} = 0$$

Since the above is true for all x then the indicial equation simplifies to

$$r + 1 = 0$$

Solving for r gives the root of the indicial equation as

$$r = -1$$

We start by finding y_h. Replacing $r = -1$ found above results in

$$\left(\sum_{n=0}^{\infty} (n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=0}^{\infty} x^{n-2} a_n\right) = 0$$

From the above we see that there is no recurrence relation since there is only one summation term. Therefore all a_n terms are zero except for a_0. Hence

$$y_h = a_0 x^r$$

Therefore the homogeneous solution is

$$y_h(x) = a_0 \left(\frac{1}{x} + O(x^6)\right)$$

At $x = 0$ the solution above becomes

$$y = c_1 \left(\frac{1}{x} + O(x^6)\right)$$

Summary

The solution(s) found are the following

$$y = c_1 \left(\frac{1}{x} + O(x^6)\right) \tag{1}$$

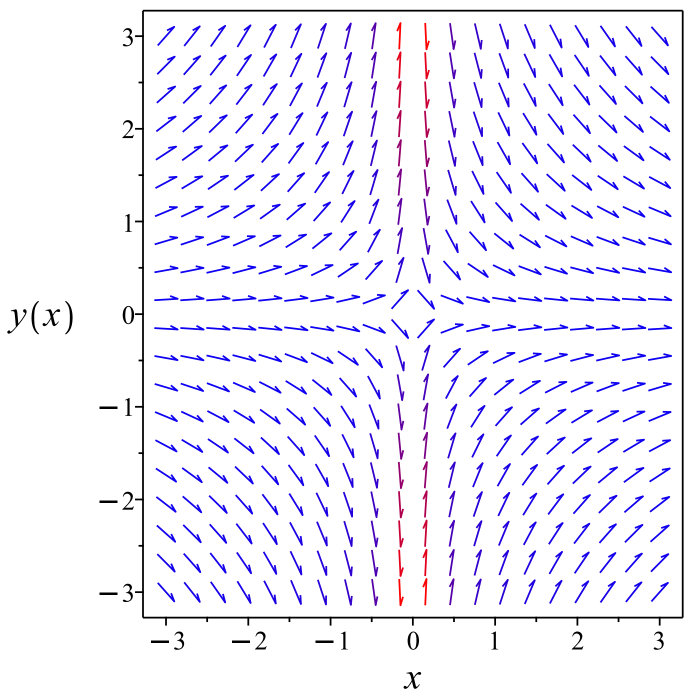


Figure 84: Slope field plot

Verification of solutions

$$y = c_1 \left(\frac{1}{x} + O(x^6)\right)$$

Verified OK.

7.15.2 Maple step by step solution

Let's solve

$y' + \frac{y}{x} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = -\frac{1}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y}dx = \int -\frac{1}{x}dx + c_1$

- Evaluate integral

  $\ln(y) = -\ln(x) + c_1$

- Solve for y

  $y = \frac{e^{c_1}}{x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
Order:=6;
dsolve(x*diff(y(x),x)+y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_1}{x} + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 9

```
AsymptoticDSolveValue[x*y'[x]+y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to \frac{c_1}{x}$$

7.16 problem problem 16

Internal problem ID [407]
Internal file name [`OUTPUT/407_Sunday_June_05_2022_01_40_42_AM_95030683/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Chapter 11 Power series methods. Section 11.1 Introduction and Review of power series. Page 615
Problem number: problem 16.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first order ode series method. Regular singular point"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{2y'x - y = 0}$$

With the expansion point for the power series method at $x = 0$.

7.16.1 Solving as series ode

Writing the ODE as

$$\begin{aligned} y' + q(x)y &= p(x) \\ y' - \frac{y}{2x} &= 0 \end{aligned}$$

Where

$$\begin{aligned} q(x) &= -\frac{1}{2x} \\ p(x) &= 0 \end{aligned}$$

Next, the type of the expansion point $x = 0$ is determined. This point can be an ordinary point, a regular singular point (also called removable singularity), or irregular singular point (also called non-removable singularity or essential singularity). When

$x = 0$ is an ordinary point, then the standard power series is used. If the point is a regular singular point, Frobenius series is used instead. Irregular singular point requires more advanced methods (asymptotic methods) and is not supported now. Hopefully this will be added in the future. $x = 0$ is called an ordinary point $q(x)$ has a Taylor series expansion around the point $x = 0$. $x = 0$ is called a regular singular point if $q(x)$ is not not analytic at $x = 0$ but $xq(x)$ has Taylor series expansion. And finally, $x = 0$ is an irregular singular point if the point is not ordinary and not regular singular. This is the most complicated case. Now the expansion point $x = 0$ is checked to see if it is an ordinary point or not.

Since $x = 0$ is not an ordinary point, we now check to see if it is a regular singular point. $xq(x) = -\frac{1}{2}$ has a Taylor series around $x = 0$. Since $x = 0$ is regular singular point, then Frobenius power series is used. Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) - \frac{\sum_{n=0}^{\infty} a_n x^{n+r}}{2x} = 0 \tag{1}$$

Hence the ODE in Eq (1) becomes

$$\left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) - \frac{\sum_{n=0}^{\infty} a_n x^{n+r}}{2x} = 0 \tag{1}$$

Expanding the second term in (1) gives

$$\left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + -\frac{1}{2} \cdot \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) + \frac{1}{x} \cdot \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-\frac{x^{n+r-1} a_n}{2}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-\frac{x^{n+r-1} a_n}{2}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2) this gives

$$(n+r) a_n x^{n+r-1} - \frac{x^{n+r-1} a_n}{2} = 0$$

When $n=0$ the above becomes

$$r a_0 x^{-1+r} - \frac{x^{-1+r} a_0}{2} = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(r - \frac{1}{2}\right) x^{-1+r} = 0$$

Since the above is true for all x then the indicial equation simplifies to

$$r - \frac{1}{2} = 0$$

Solving for r gives the root of the indicial equation as

$$r = \frac{1}{2}$$

We start by finding y_h. Replacing $r = \frac{1}{2}$ found above results in

$$\left(\sum_{n=0}^{\infty} \left(n+\frac{1}{2}\right) a_n x^{n-\frac{1}{2}}\right) + \sum_{n=0}^{\infty} \left(-\frac{x^{n-\frac{1}{2}} a_n}{2}\right) = 0$$

From the above we see that there is no recurrence relation since there is only one summation term. Therefore all a_n terms are zero except for a_0. Hence

$$y_h = a_0 x^r$$

Therefore the homogeneous solution is

$$y_h(x) = a_0\left(\sqrt{x} + O\left(x^6\right)\right)$$

At $x = 0$ the solution above becomes

$$y = c_1\left(\sqrt{x} + O\left(x^6\right)\right)$$

Summary

The solution(s) found are the following

$$y = c_1\left(\sqrt{x} + O\left(x^6\right)\right) \tag{1}$$

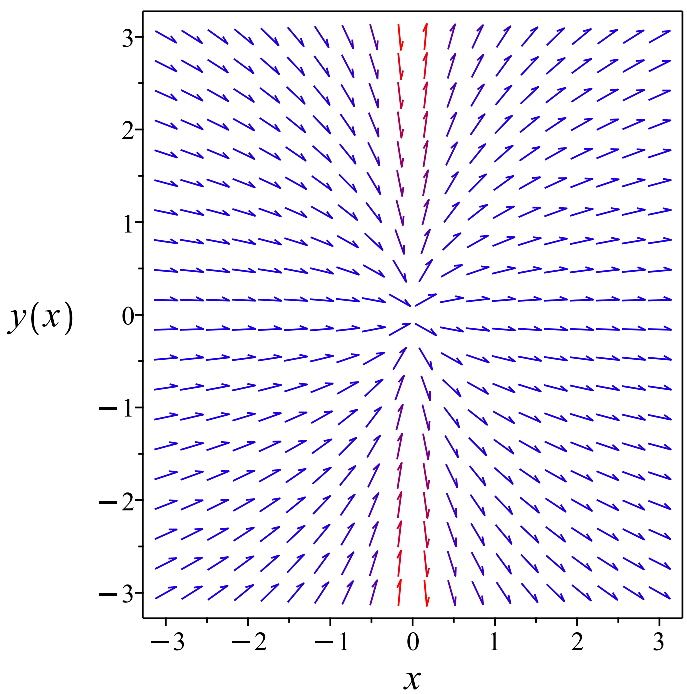


Figure 85: Slope field plot

Verification of solutions

$$y = c_1\left(\sqrt{x} + O\left(x^6\right)\right)$$

Verified OK.

7.16.2 Maple step by step solution

Let's solve

$y' - \frac{y}{2x} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = \frac{1}{2x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y}dx = \int \frac{1}{2x}dx + c_1$

- Evaluate integral

  $\ln(y) = \frac{\ln(x)}{2} + c_1$

- Solve for y

  $\left\{y = \frac{\sqrt{e^{-2c_1}x}}{e^{-2c_1}}, y = -\frac{\sqrt{e^{-2c_1}x}}{e^{-2c_1}}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 15

```
Order:=6;
dsolve(2*x*diff(y(x),x)=y(x),y(x),type='series',x=0);
```

$$y(x) = c_1\sqrt{x} + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 11

```
AsymptoticDSolveValue[2*x*y'[x]==y[x],y[x],{x,0,5}]
```

$$y(x) \to c_1\sqrt{x}$$

7.17 problem problem 17

Internal problem ID [408]
Internal file name [`OUTPUT/408_Sunday_June_05_2022_01_40_43_AM_15302160/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Chapter 11 Power series methods. Section 11.1 Introduction and Review of power series. Page 615
Problem number: problem 17.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first order ode series method. Irregular singular point"**

Maple gives the following as the ode type

`[_separable]`

Unable to solve or complete the solution.

$$\boxed{y' x^2 + y = 0}$$

With the expansion point for the power series method at $x = 0$.

7.17.1 Solving as series ode

Writing the ODE as

$$\begin{aligned} y' + q(x)y &= p(x) \\ y' + \frac{y}{x^2} &= 0 \end{aligned}$$

Where

$$\begin{aligned} q(x) &= \frac{1}{x^2} \\ p(x) &= 0 \end{aligned}$$

Next, the type of the expansion point $x = 0$ is determined. This point can be an ordinary point, a regular singular point (also called removable singularity), or irregular singular point (also called non-removable singularity or essential singularity). When $x = 0$ is an ordinary point, then the standard power series is used. If the point is a regular singular point, Frobenius series is used instead. Irregular singular point requires more advanced methods (asymptotic methods) and is not supported now. Hopefully this will be added in the future. $x = 0$ is called an ordinary point $q(x)$ has a Taylor series expansion around the point $x = 0$. $x = 0$ is called a regular singular point if $q(x)$ is not not analytic at $x = 0$ but $xq(x)$ has Taylor series expansion. And finally, $x = 0$ is an irregular singular point if the point is not ordinary and not regular singular. This is the most complicated case. Now the expansion point $x = 0$ is checked to see if it is an ordinary point or not.

Since $x = 0$ is not an ordinary point, we now check to see if it is a regular singular point.

$$xq(x) = \frac{1}{x}$$

does not have a Taylor series around $x = 0$.

Unable to solve since $x = 0$ is not regular singular point. Terminating.

Verification of solutions N/A

7.17.2 Maple step by step solution

Let's solve

$y'x^2 + y = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = -\frac{1}{x^2}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int -\frac{1}{x^2} dx + c_1$

- Evaluate integral

  $\ln(y) = \frac{1}{x} + c_1$

- Solve for y

  $y = \mathrm{e}^{\frac{c_1 x + 1}{x}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✗ Solution by Maple

```
Order:=6;
dsolve(x^2*diff(y(x),x)+y(x)=0,y(x),type='series',x=0);
```

No solution found

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 11

```
AsymptoticDSolveValue[x^2*y'[x]+y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 e^{\frac{1}{x}}$$

7.18 problem problem 18

Internal problem ID [409]
Internal file name [`OUTPUT/409_Sunday_June_05_2022_01_40_44_AM_31609085/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Chapter 11 Power series methods. Section 11.1 Introduction and Review of power series. Page 615
Problem number: problem 18.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first order ode series method. Irregular singular point"**

Maple gives the following as the ode type

`[_separable]`

Unable to solve or complete the solution.

$$\boxed{x^3y' - 2y = 0}$$

With the expansion point for the power series method at $x = 0$.

7.18.1 Solving as series ode

Writing the ODE as

$$y' + q(x)y = p(x)$$
$$y' - \frac{2y}{x^3} = 0$$

Where

$$q(x) = -\frac{2}{x^3}$$
$$p(x) = 0$$

Next, the type of the expansion point $x = 0$ is determined. This point can be an ordinary point, a regular singular point (also called removable singularity), or irregular singular point (also called non-removable singularity or essential singularity). When $x = 0$ is an ordinary point, then the standard power series is used. If the point is a regular singular point, Frobenius series is used instead. Irregular singular point requires more advanced methods (asymptotic methods) and is not supported now. Hopefully this will be added in the future. $x = 0$ is called an ordinary point $q(x)$ has a Taylor series expansion around the point $x = 0$. $x = 0$ is called a regular singular point if $q(x)$ is not not analytic at $x = 0$ but $xq(x)$ has Taylor series expansion. And finally, $x = 0$ is an irregular singular point if the point is not ordinary and not regular singular. This is the most complicated case. Now the expansion point $x = 0$ is checked to see if it is an ordinary point or not.

Since $x = 0$ is not an ordinary point, we now check to see if it is a regular singular point.

$$xq(x) = -\frac{2}{x^2}$$

does not have a Taylor series around $x = 0$.

Unable to solve since $x = 0$ is not regular singular point. Terminating.

Verification of solutions N/A

7.18.2 Maple step by step solution

Let's solve

$x^3y' - 2y = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = \frac{2}{x^3}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y}dx = \int \frac{2}{x^3}dx + c_1$

- Evaluate integral

  $\ln(y) = -\frac{1}{x^2} + c_1$

- Solve for y

  $y = e^{\frac{c_1 x^2 - 1}{x^2}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✗ Solution by Maple

```
Order:=6;
dsolve(x^3*diff(y(x),x)=2*y(x),y(x),type='series',x=0);
```

No solution found

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 13

```
AsymptoticDSolveValue[x^3*y'[x]==2*y[x],y[x],{x,0,5}]
```

$$y(x) \to c_1 e^{-\frac{1}{x^2}}$$

7.19 problem problem 19

Internal problem ID [410]
Internal file name [`OUTPUT/410_Sunday_June_05_2022_01_40_45_AM_82679318/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Chapter 11 Power series methods. Section 11.1 Introduction and Review of power series. Page 615
Problem number: problem 19.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 4y = 0}$$

With initial conditions

$$[y(0) = 0, y'(0) = 3]$$

With the expansion point for the power series method at $x = 0$.

7.19.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= 4 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + 4y = 0$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 4$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\ &= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0} \end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (60)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (61)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -4y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= -4y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= 16y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= 16y' \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= -64y
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = 0$ and $y'(0) = 3$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= -12 \\
F_2 &= 0 \\
F_3 &= 48 \\
F_4 &= 0
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = -2x^3 + 3x + \frac{2x^5}{5} + O(x^6)
$$

$$
y = -2x^3 + 3x + \frac{2x^5}{5} + O(x^6)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = -4\left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \left(\sum_{n=0}^{\infty} 4a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n\right) + \left(\sum_{n=0}^{\infty} 4a_n x^n\right) = 0 \tag{3}$$

For $0 \leq n$, the recurrence equation is

$$(n+2) a_{n+2}(n+1) + 4a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{4a_n}{(n+2)(n+1)} \tag{5}$$

For $n = 0$ the recurrence equation gives

$$2a_2 + 4a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_2 = -2a_0$$

For $n = 1$ the recurrence equation gives

$$6a_3 + 4a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{2a_1}{3}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + 4a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{2a_0}{3}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + 4a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{2a_1}{15}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + 4a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{4a_0}{45}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + 4a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{4a_1}{315}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - 2a_0 x^2 - \frac{2}{3}a_1 x^3 + \frac{2}{3}a_0 x^4 + \frac{2}{15}a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - 2x^2 + \frac{2}{3}x^4\right) a_0 + \left(x - \frac{2}{3}x^3 + \frac{2}{15}x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - 2x^2 + \frac{2}{3}x^4\right) c_1 + \left(x - \frac{2}{3}x^3 + \frac{2}{15}x^5\right) c_2 + O(x^6)$$

$$y = -2x^3 + 3x + \frac{2x^5}{5} + O(x^6)$$

Summary

The solution(s) found are the following

$$y = -2x^3 + 3x + \frac{2x^5}{5} + O(x^6) \tag{1}$$

$$y = -2x^3 + 3x + \frac{2x^5}{5} + O(x^6) \tag{2}$$

Verification of solutions

$$y = -2x^3 + 3x + \frac{2x^5}{5} + O(x^6)$$

Verified OK.

$$y = -2x^3 + 3x + \frac{2x^5}{5} + O(x^6)$$

Verified OK.

7.19.2 Maple step by step solution

Let's solve

$\left[y'' = -4y, y(0) = 0, y'\Big|_{\{x=0\}} = 3\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + 4y = 0$

- Characteristic polynomial of ODE

  $r^2 + 4 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (-2\,\mathrm{I}, 2\,\mathrm{I})$

- 1st solution of the ODE

  $y_1(x) = \cos(2x)$

- 2nd solution of the ODE

  $y_2(x) = \sin(2x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \cos(2x) + c_2 \sin(2x)$

- Check validity of solution $y = c_1 \cos(2x) + c_2 \sin(2x)$
  - Use initial condition $y(0) = 0$

    $0 = c_1$
  - Compute derivative of the solution

    $y' = -2c_1 \sin(2x) + 2c_2 \cos(2x)$
  - Use the initial condition $y'\Big|_{\{x=0\}} = 3$

    $3 = 2c_2$
  - Solve for c_1 and c_2

    $\{c_1 = 0, c_2 = \frac{3}{2}\}$
  - Substitute constant values into general solution and simplify

    $y = \frac{3\sin(2x)}{2}$
- Solution to the IVP

  $y = \frac{3\sin(2x)}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
Order:=6;
dsolve([diff(y(x),x$2)+4*y(x)=0,y(0) = 0, D(y)(0) = 3],y(x),type='series',x=0);
```

$$y(x) = 3x - 2x^3 + \frac{2}{5}x^5 + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 19

```
AsymptoticDSolveValue[{y''[x]+4*y[x]==0,{y[0]==0,y'[0]==3}},y[x],{x,0,5}]
```

$$y(x) \to \frac{2x^5}{5} - 2x^3 + 3x$$

7.20 problem problem 20

Internal problem ID [411]
Internal file name [`OUTPUT/411_Sunday_June_05_2022_01_40_47_AM_38293236/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Chapter 11 Power series methods. Section 11.1 Introduction and Review of power series. Page 615
Problem number: problem 20.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 4y = 0}$$

With initial conditions

$$[y(0) = 2, y'(0) = 0]$$

With the expansion point for the power series method at $x = 0$.

7.20.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= -4 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - 4y = 0$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = -4$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{63}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{64}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y'_0} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= 4y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= 4y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= 16y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= 16y' \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= 64y
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = 2$ and $y'(0) = 0$ gives

$$
\begin{aligned}
F_0 &= 8 \\
F_1 &= 0 \\
F_2 &= 32 \\
F_3 &= 0 \\
F_4 &= 128
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = 4x^2 + 2 + \frac{4x^4}{3} + \frac{8x^6}{45} + O(x^6)
$$

$$
y = 4x^2 + 2 + \frac{4x^4}{3} + \frac{8x^6}{45} + O(x^6)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = 4\left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \sum_{n=0}^{\infty} (-4 a_n x^n) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \sum_{n=0}^{\infty} (-4 a_n x^n) = 0 \tag{3}$$

For $0 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) - 4a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{4a_n}{(n+2)(n+1)} \tag{5}$$

For $n = 0$ the recurrence equation gives

$$2a_2 - 4a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_2 = 2a_0$$

For $n = 1$ the recurrence equation gives

$$6a_3 - 4a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{2a_1}{3}$$

For $n = 2$ the recurrence equation gives

$$12a_4 - 4a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{2a_0}{3}$$

For $n = 3$ the recurrence equation gives

$$20a_5 - 4a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{2a_1}{15}$$

For $n = 4$ the recurrence equation gives

$$30a_6 - 4a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{4a_0}{45}$$

For $n = 5$ the recurrence equation gives

$$42a_7 - 4a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{4a_1}{315}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1x + 2a_0x^2 + \frac{2}{3}a_1x^3 + \frac{2}{3}a_0x^4 + \frac{2}{15}a_1x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + 2x^2 + \frac{2}{3}x^4\right)a_0 + \left(x + \frac{2}{3}x^3 + \frac{2}{15}x^5\right)a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 + 2x^2 + \frac{2}{3}x^4\right)c_1 + \left(x + \frac{2}{3}x^3 + \frac{2}{15}x^5\right)c_2 + O(x^6)$$

$$y = 4x^2 + 2 + \frac{4x^4}{3} + O(x^6)$$

Summary

The solution(s) found are the following

$$y = 4x^2 + 2 + \frac{4x^4}{3} + \frac{8x^6}{45} + O(x^6) \tag{1}$$

$$y = 4x^2 + 2 + \frac{4x^4}{3} + O(x^6) \tag{2}$$

Verification of solutions

$$y = 4x^2 + 2 + \frac{4x^4}{3} + \frac{8x^6}{45} + O(x^6)$$

Verified OK.

$$y = 4x^2 + 2 + \frac{4x^4}{3} + O(x^6)$$

Verified OK.

7.20.2 Maple step by step solution

Let's solve

$\left[y'' = 4y, y(0) = 2, y'\Big|_{\{x=0\}} = 0\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - 4y = 0$

- Characteristic polynomial of ODE

  $r^2 - 4 = 0$

- Factor the characteristic polynomial

  $(r-2)(r+2) = 0$

- Roots of the characteristic polynomial

  $r = (-2, 2)$

- 1st solution of the ODE

  $y_1(x) = e^{-2x}$

- 2nd solution of the ODE

  $y_2(x) = e^{2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{-2x} + c_2 e^{2x}$

□ Check validity of solution $y = c_1 e^{-2x} + c_2 e^{2x}$

- Use initial condition $y(0) = 2$

  $2 = c_1 + c_2$

- Compute derivative of the solution

  $y' = -2c_1 e^{-2x} + 2c_2 e^{2x}$

- Use the initial condition $y'\big|_{\{x=0\}} = 0$

  $0 = -2c_1 + 2c_2$

- Solve for c_1 and c_2

  $\{c_1 = 1, c_2 = 1\}$

- Substitute constant values into general solution and simplify

  $y = e^{-2x} + e^{2x}$

- Solution to the IVP

  $y = e^{-2x} + e^{2x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
Order:=6;
dsolve([diff(y(x),x$2)-4*y(x)=0,y(0) = 2, D(y)(0) = 0],y(x),type='series',x=0);
```

$$y(x) = 2 + 4x^2 + \frac{4}{3}x^4 + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 17

```
AsymptoticDSolveValue[{y''[x]-4*y[x]==0,{y[0]==2,y'[0]==0}},y[x],{x,0,5}]
```

$$y(x) \to \frac{4x^4}{3} + 4x^2 + 2$$

7.21 problem problem 21

Internal problem ID [412]
Internal file name [`OUTPUT/412_Sunday_June_05_2022_01_40_48_AM_9667363/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Chapter 11 Power series methods. Section 11.1 Introduction and Review of power series. Page 615
Problem number: problem 21.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_linear_constant_coeff", "second order series method. Ordinary point", "linear_second_order_ode_solved_by_an_integrating_factor", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 2y' + y = 0}$$

With initial conditions

$$[y(0) = 0, y'(0) = 1]$$

With the expansion point for the power series method at $x = 0$.

7.21.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -2 \\ q(x) &= 1 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - 2y' + y = 0$$

The domain of $p(x) = -2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\ &= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n} \bigg|_{x_0, y_0, y'_0} \end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{66}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{67}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$\begin{aligned}
F_0 &= 2y' - y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= 3y' - 2y \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= 4y' - 3y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= 5y' - 4y \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= 6y' - 5y
\end{aligned}$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = 0$ and $y'(0) = 1$ gives

$$\begin{aligned}
F_0 &= 2 \\
F_1 &= 3 \\
F_2 &= 4 \\
F_3 &= 5 \\
F_4 &= 6
\end{aligned}$$

Substituting all the above in (7) and simplifying gives the solution as

$$y = x^2 + x + \frac{x^3}{2} + \frac{x^4}{6} + \frac{x^5}{24} + \frac{x^6}{120} + O(x^6)$$

$$y = x^2 + x + \frac{x^3}{2} + \frac{x^4}{6} + \frac{x^5}{24} + \frac{x^6}{120} + O(x^6)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = 2\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - \left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \sum_{n=1}^{\infty} \left(-2n a_n x^{n-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

$$\sum_{n=1}^{\infty} \left(-2n a_n x^{n-1}\right) = \sum_{n=0}^{\infty} \left(-2(n+1)\, a_{n+1} x^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \sum_{n=0}^{\infty} \left(-2(n+1)\, a_{n+1} x^n\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{3}$$

For $0 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) - 2(n+1)\, a_{n+1} + a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$
\begin{aligned}
a_{n+2} &= \frac{2na_{n+1} - a_n + 2a_{n+1}}{(n+2)(n+1)} \\
&= -\frac{a_n}{(n+2)(n+1)} + \frac{(2n+2)a_{n+1}}{(n+2)(n+1)}
\end{aligned} \tag{5}
$$

For $n = 0$ the recurrence equation gives

$$2a_2 - 2a_1 + a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_2 = a_1 - \frac{a_0}{2}$$

For $n = 1$ the recurrence equation gives

$$6a_3 - 4a_2 + a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{a_1}{2} - \frac{a_0}{3}$$

For $n = 2$ the recurrence equation gives

$$12a_4 - 6a_3 + a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_1}{6} - \frac{a_0}{8}$$

For $n = 3$ the recurrence equation gives

$$20a_5 - 8a_4 + a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_1}{24} - \frac{a_0}{30}$$

For $n = 4$ the recurrence equation gives

$$30a_6 - 10a_5 + a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_1}{120} - \frac{a_0}{144}$$

For $n = 5$ the recurrence equation gives

$$42a_7 - 12a_6 + a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{a_1}{720} - \frac{a_0}{840}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \left(a_1 - \frac{a_0}{2}\right) x^2 + \left(\frac{a_1}{2} - \frac{a_0}{3}\right) x^3 + \left(\frac{a_1}{6} - \frac{a_0}{8}\right) x^4 + \left(\frac{a_1}{24} - \frac{a_0}{30}\right) x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{8}x^4 - \frac{1}{30}x^5\right) a_0 + \left(x^2 + x + \frac{1}{2}x^3 + \frac{1}{6}x^4 + \frac{1}{24}x^5\right) a_1 + O(x^6) \quad (3)$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{8}x^4 - \frac{1}{30}x^5\right) c_1 + \left(x^2 + x + \frac{1}{2}x^3 + \frac{1}{6}x^4 + \frac{1}{24}x^5\right) c_2 + O(x^6)$$

$$y = x^2 + x + \frac{x^3}{2} + \frac{x^4}{6} + \frac{x^5}{24} + O(x^6)$$

Summary
The solution(s) found are the following

$$
y = x^2 + x + \frac{x^3}{2} + \frac{x^4}{6} + \frac{x^5}{24} + \frac{x^6}{120} + O(x^6) \tag{1}
$$

$$
y = x^2 + x + \frac{x^3}{2} + \frac{x^4}{6} + \frac{x^5}{24} + O(x^6) \tag{2}
$$

Verification of solutions

$$
y = x^2 + x + \frac{x^3}{2} + \frac{x^4}{6} + \frac{x^5}{24} + \frac{x^6}{120} + O(x^6)
$$

Verified OK.

$$
y = x^2 + x + \frac{x^3}{2} + \frac{x^4}{6} + \frac{x^5}{24} + O(x^6)
$$

Verified OK.

7.21.2 Maple step by step solution

Let's solve

$\left[y'' = 2y' - y, y(0) = 0, y'\Big|_{\{x=0\}} = 1\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - 2y' + y = 0$

- Characteristic polynomial of ODE

  $r^2 - 2r + 1 = 0$

- Factor the characteristic polynomial

  $(r - 1)^2 = 0$

- Root of the characteristic polynomial

  $r = 1$

- 1st solution of the ODE

  $y_1(x) = e^x$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\, e^x$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^{x} + c_2 x\, \mathrm{e}^{x}$

☐ Check validity of solution $y = c_1 \mathrm{e}^{x} + c_2 x \mathrm{e}^{x}$

  - Use initial condition $y(0) = 0$

    $0 = c_1$

  - Compute derivative of the solution

    $y' = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{x} + c_2 x\, \mathrm{e}^{x}$

  - Use the initial condition $y'\Big|_{\{x=0\}} = 1$

    $1 = c_1 + c_2$

  - Solve for c_1 and c_2

    $\{c_1 = 0, c_2 = 1\}$

  - Substitute constant values into general solution and simplify

    $y = x\, \mathrm{e}^{x}$

- Solution to the IVP

  $y = x\, \mathrm{e}^{x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
Order:=6;
dsolve([diff(y(x),x$2)-2*diff(y(x),x)+y(x)=0,y(0) = 0, D(y)(0) = 1],y(x),type='series',x=0);
```

$$y(x) = x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4 + \frac{1}{24}x^5 + \mathrm{O}\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 29

```
AsymptoticDSolveValue[{y''[x]-2*y'[x]+y[x]==0,{y[0]==0,y'[0]==1}},y[x],{x,0,5}]
```

$$y(x) \to \frac{x^5}{24} + \frac{x^4}{6} + \frac{x^3}{2} + x^2 + x$$

7.22 problem problem 22

Internal problem ID [413]
Internal file name [`OUTPUT/413_Sunday_June_05_2022_01_40_50_AM_76871572/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Chapter 11 Power series methods. Section 11.1 Introduction and Review of power series. Page 615
Problem number: problem 22.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_linear_constant_coeff", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + y' - 2y = 0}$$

With initial conditions

$$[y(0) = 1, y'(0) = -2]$$

With the expansion point for the power series method at $x = 0$.

7.22.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 1 \\ q(x) &= -2 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + y' - 2y = 0$$

The domain of $p(x) = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = -2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left. \frac{d^n f}{dx^n} \right|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (69)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (70)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{align}
F_0 &= f(x,y,y') \tag{4}\\
F_1 &= \frac{df}{dx} \nonumber\\
&= \frac{dF_0}{dx} \nonumber\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \nonumber\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{5}\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0 \nonumber\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right) \nonumber\\
&= \frac{d}{dx}(F_1) \nonumber\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y'' \nonumber\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0 \nonumber\\
&\vdots \nonumber\\
F_n &= \frac{d}{dx}(F_{n-1}) \nonumber\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y'' \nonumber\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \tag{6}
\end{align}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x=0$. Hence

$$\begin{aligned}
F_0 &= -y' + 2y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= 3y' - 2y \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= -5y' + 6y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= 11y' - 10y \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= -21y' + 22y
\end{aligned}$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=1$ and $y'(0)=-2$ gives

$$\begin{aligned}
F_0 &= 4 \\
F_1 &= -8 \\
F_2 &= 16 \\
F_3 &= -32 \\
F_4 &= 64
\end{aligned}$$

Substituting all the above in (7) and simplifying gives the solution as

$$y = 2x^2 - 2x + 1 - \frac{4x^3}{3} + \frac{2x^4}{3} - \frac{4x^5}{15} + \frac{4x^6}{45} + O(x^6)$$

$$y = 2x^2 - 2x + 1 - \frac{4x^3}{3} + \frac{2x^4}{3} - \frac{4x^5}{15} + \frac{4x^6}{45} + O(x^6)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = -\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + 2\left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + \sum_{n=0}^{\infty} (-2 a_n x^n) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$
$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)\, a_{n+1} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \left(\sum_{n=0}^{\infty} (n+1)\, a_{n+1} x^n\right) + \sum_{n=0}^{\infty} (-2 a_n x^n) = 0 \tag{3}$$

For $0 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) + (n+1)\, a_{n+1} - 2a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$
\begin{aligned}
a_{n+2} &= -\frac{n a_{n+1} - 2a_n + a_{n+1}}{(n+2)(n+1)} \\
&= \frac{2a_n}{(n+2)(n+1)} - \frac{a_{n+1}}{n+2}
\end{aligned}
\tag{5}
$$

For $n = 0$ the recurrence equation gives

$$2a_2 + a_1 - 2a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_2 = -\frac{a_1}{2} + a_0$$

For $n = 1$ the recurrence equation gives

$$6a_3 + 2a_2 - 2a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{a_1}{2} - \frac{a_0}{3}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + 3a_3 - 2a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{5a_1}{24} + \frac{a_0}{4}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + 4a_4 - 2a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{11a_1}{120} - \frac{a_0}{12}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + 5a_5 - 2a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{7a_1}{240} + \frac{11a_0}{360}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + 6a_6 - 2a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{43a_1}{5040} - \frac{a_0}{120}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \left(-\frac{a_1}{2} + a_0\right) x^2 + \left(\frac{a_1}{2} - \frac{a_0}{3}\right) x^3 + \left(-\frac{5a_1}{24} + \frac{a_0}{4}\right) x^4 + \left(\frac{11a_1}{120} - \frac{a_0}{12}\right) x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + x^2 - \frac{1}{3}x^3 + \frac{1}{4}x^4 - \frac{1}{12}x^5\right) a_0 + \left(x - \frac{1}{2}x^2 + \frac{1}{2}x^3 - \frac{5}{24}x^4 + \frac{11}{120}x^5\right) a_1 + O(x^6) \quad (3)$$

At $x = 0$ the solution above becomes

$$y = \left(1 + x^2 - \frac{1}{3}x^3 + \frac{1}{4}x^4 - \frac{1}{12}x^5\right) c_1 + \left(x - \frac{1}{2}x^2 + \frac{1}{2}x^3 - \frac{5}{24}x^4 + \frac{11}{120}x^5\right) c_2 + O(x^6)$$

$$y = 1 + 2x^2 - \frac{4x^3}{3} + \frac{2x^4}{3} - \frac{4x^5}{15} - 2x + O(x^6)$$

Summary
The solution(s) found are the following

$$y = 2x^2 - 2x + 1 - \frac{4x^3}{3} + \frac{2x^4}{3} - \frac{4x^5}{15} + \frac{4x^6}{45} + O(x^6) \tag{1}$$

$$y = 1 + 2x^2 - \frac{4x^3}{3} + \frac{2x^4}{3} - \frac{4x^5}{15} - 2x + O(x^6) \tag{2}$$

Verification of solutions

$$y = 2x^2 - 2x + 1 - \frac{4x^3}{3} + \frac{2x^4}{3} - \frac{4x^5}{15} + \frac{4x^6}{45} + O(x^6)$$

Verified OK.

$$y = 1 + 2x^2 - \frac{4x^3}{3} + \frac{2x^4}{3} - \frac{4x^5}{15} - 2x + O(x^6)$$

Verified OK.

7.22.2 Maple step by step solution

Let's solve

$\left[y'' = -y' + 2y, y(0) = 1, y'\Big|_{\{x=0\}} = -2\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + y' - 2y = 0$

- Characteristic polynomial of ODE

  $r^2 + r - 2 = 0$

- Factor the characteristic polynomial

  $(r + 2)(r - 1) = 0$

- Roots of the characteristic polynomial

  $r = (-2, 1)$

- 1st solution of the ODE

  $y_1(x) = e^{-2x}$

- 2nd solution of the ODE

  $y_2(x) = e^{x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{x}$

□ Check validity of solution $y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{x}$

  - Use initial condition $y(0) = 1$

    $1 = c_1 + c_2$

  - Compute derivative of the solution

    $y' = -2c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{x}$

  - Use the initial condition $y'\Big|_{\{x=0\}} = -2$

    $-2 = -2c_1 + c_2$

  - Solve for c_1 and c_2

    $\{c_1 = 1, c_2 = 0\}$

  - Substitute constant values into general solution and simplify

    $y = \mathrm{e}^{-2x}$

- Solution to the IVP

  $y = \mathrm{e}^{-2x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
Order:=6;
dsolve([diff(y(x),x$2)+diff(y(x),x)-2*y(x)=0,y(0) = 1, D(y)(0) = -2],y(x),type='series',x=0)
```

$$y(x) = 1 - 2x + 2x^2 - \frac{4}{3}x^3 + \frac{2}{3}x^4 - \frac{4}{15}x^5 + \mathrm{O}\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 34

```
AsymptoticDSolveValue[{y''[x]+y'[x]-2*y[x]==0,{y[0]==1,y'[0]==-2}},y[x],{x,0,5}]
```

$$y(x) \to -\frac{4x^5}{15} + \frac{2x^4}{3} - \frac{4x^3}{3} + 2x^2 - 2x + 1$$

7.23 problem problem 23

Internal problem ID [414]
Internal file name [OUTPUT/414_Sunday_June_05_2022_01_40_52_AM_2426009/index.tex]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Chapter 11 Power series methods. Section 11.1 Introduction and Review of power series. Page 615
Problem number: problem 23.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Complex roots"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + y'x^2 + y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + y'x^2 + y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = 1$$
$$q(x) = \frac{1}{x^2}$$

Table 102: Table $p(x), q(x)$ singularites.

$p(x) = 1$	
singularity	type

$q(x) = \frac{1}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2 y'' + y' x^2 + y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} \right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} \right) x^2 + \left(\sum_{n=0}^{\infty} a_n x^{n+r} \right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1) \right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r) \right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r} \right) = 0 \quad \text{(2A)}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} a_{n-1}(n+r-1)\, x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) + a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) + a_0 x^r = 0$$

Or

$$(x^r r(-1+r) + x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2 - r + 1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - r + 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{2} + \frac{i\sqrt{3}}{2}$$
$$r_2 = \frac{1}{2} - \frac{i\sqrt{3}}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(r^2 - r + 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the roots are complex conjugates, then two linearly independent solutions can be constructed using

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}+\frac{i\sqrt{3}}{2}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{2}-\frac{i\sqrt{3}}{2}}$$

$y_1(x)$ is found first. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_{n-1}(n+r-1) + a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}(n+r-1)}{n^2+2nr+r^2-n-r+1} \tag{4}$$

Which for the root $r = \frac{1}{2} + \frac{i\sqrt{3}}{2}$ becomes

$$a_n = -\frac{a_{n-1}\left(i\sqrt{3}+2n-1\right)}{2n\left(i\sqrt{3}+n\right)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2} + \frac{i\sqrt{3}}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{r}{r^2+r+1}$$

Which for the root $r = \frac{1}{2} + \frac{i\sqrt{3}}{2}$ becomes

$$a_1 = -\frac{1}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r}{r^2+r+1}$	$-\frac{1}{2}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{r(1+r)}{(r^2+r+1)(r^2+3r+3)}$$

Which for the root $r = \frac{1}{2} + \frac{i\sqrt{3}}{2}$ becomes

$$a_2 = \frac{i\sqrt{3}+3}{16+8i\sqrt{3}}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r}{r^2+r+1}$	$-\frac{1}{2}$
a_2	$\frac{r(1+r)}{(r^2+r+1)(r^2+3r+3)}$	$\frac{i\sqrt{3}+3}{16+8i\sqrt{3}}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{r(1+r)(2+r)}{(r^2+r+1)(r^2+3r+3)(r^2+5r+7)}$$

Which for the root $r = \frac{1}{2} + \frac{i\sqrt{3}}{2}$ becomes

$$a_3 = \frac{-i\sqrt{3}-5}{48i\sqrt{3}+96}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r}{r^2+r+1}$	$-\frac{1}{2}$
a_2	$\frac{r(1+r)}{(r^2+r+1)(r^2+3r+3)}$	$\frac{i\sqrt{3}+3}{16+8i\sqrt{3}}$
a_3	$-\frac{r(1+r)(2+r)}{(r^2+r+1)(r^2+3r+3)(r^2+5r+7)}$	$\frac{-i\sqrt{3}-5}{48i\sqrt{3}+96}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{r(1+r)(2+r)(3+r)}{(r^2+r+1)(r^2+3r+3)(r^2+5r+7)(r^2+7r+13)}$$

Which for the root $r = \frac{1}{2} + \frac{i\sqrt{3}}{2}$ becomes

$$a_4 = \frac{\left(i\sqrt{3}+5\right)\left(i\sqrt{3}+7\right)}{384\left(i\sqrt{3}+4\right)\left(2+i\sqrt{3}\right)}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r}{r^2+r+1}$	$-\frac{1}{2}$
a_2	$\frac{r(1+r)}{(r^2+r+1)(r^2+3r+3)}$	$\frac{i\sqrt{3}+3}{16+8i\sqrt{3}}$
a_3	$-\frac{r(1+r)(2+r)}{(r^2+r+1)(r^2+3r+3)(r^2+5r+7)}$	$\frac{-i\sqrt{3}-5}{48i\sqrt{3}+96}$
a_4	$\frac{r(1+r)(2+r)(3+r)}{(r^2+r+1)(r^2+3r+3)(r^2+5r+7)(r^2+7r+13)}$	$\frac{\left(i\sqrt{3}+5\right)\left(i\sqrt{3}+7\right)}{384\left(i\sqrt{3}+4\right)\left(2+i\sqrt{3}\right)}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{r(1+r)(2+r)(3+r)(4+r)}{(r^2+r+1)(r^2+3r+3)(r^2+5r+7)(r^2+7r+13)(r^2+9r+21)}$$

Which for the root $r = \frac{1}{2} + \frac{i\sqrt{3}}{2}$ becomes

$$a_5 = -\frac{\left(i\sqrt{3}+7\right)\left(i\sqrt{3}+9\right)}{3840\left(i\sqrt{3}+4\right)\left(2+i\sqrt{3}\right)}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r}{r^2+r+1}$	$-\frac{1}{2}$
a_2	$\frac{r(1+r)}{(r^2+r+1)(r^2+3r+3)}$	$\frac{i\sqrt{3}+3}{16+8i\sqrt{3}}$
a_3	$-\frac{r(1+r)(2+r)}{(r^2+r+1)(r^2+3r+3)(r^2+5r+7)}$	$\frac{-i\sqrt{3}-5}{48i\sqrt{3}+96}$
a_4	$\frac{r(1+r)(2+r)(3+r)}{(r^2+r+1)(r^2+3r+3)(r^2+5r+7)(r^2+7r+13)}$	$\frac{\left(i\sqrt{3}+5\right)\left(i\sqrt{3}+7\right)}{384\left(i\sqrt{3}+4\right)\left(2+i\sqrt{3}\right)}$
a_5	$-\frac{r(1+r)(2+r)(3+r)(4+r)}{(r^2+r+1)(r^2+3r+3)(r^2+5r+7)(r^2+7r+13)(r^2+9r+21)}$	$-\frac{\left(i\sqrt{3}+7\right)\left(i\sqrt{3}+9\right)}{3840\left(i\sqrt{3}+4\right)\left(2+i\sqrt{3}\right)}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}
y_1(x) &= x^{\frac{1}{2}+\frac{i\sqrt{3}}{2}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= x^{\frac{1}{2}+\frac{i\sqrt{3}}{2}}\left(1 - \frac{x}{2} + \frac{\left(i\sqrt{3}+3\right)x^2}{16+8i\sqrt{3}} + \frac{\left(-i\sqrt{3}-5\right)x^3}{48i\sqrt{3}+96} + \frac{\left(i\sqrt{3}+5\right)\left(i\sqrt{3}+7\right)x^4}{384\left(i\sqrt{3}+4\right)\left(2+i\sqrt{3}\right)}\right. \\
&\qquad \left. - \frac{\left(i\sqrt{3}+7\right)\left(i\sqrt{3}+9\right)x^5}{3840\left(i\sqrt{3}+4\right)\left(2+i\sqrt{3}\right)} + O\left(x^6\right)\right)
\end{aligned}$$

The second solution $y_2(x)$ is found by taking the complex conjugate of $y_1(x)$ which gives

$$\begin{aligned}
y_2(x) = x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}}&\left(1 - \frac{x}{2} + \frac{\left(-i\sqrt{3}+3\right)x^2}{16-8i\sqrt{3}} + \frac{\left(i\sqrt{3}-5\right)x^3}{-48i\sqrt{3}+96} + \frac{\left(-i\sqrt{3}+5\right)\left(-i\sqrt{3}+7\right)x^4}{384\left(-i\sqrt{3}+4\right)\left(2-i\sqrt{3}\right)}\right. \\
&\qquad \left. - \frac{\left(-i\sqrt{3}+7\right)\left(-i\sqrt{3}+9\right)x^5}{3840\left(-i\sqrt{3}+4\right)\left(2-i\sqrt{3}\right)} + O\left(x^6\right)\right)
\end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 y_1(x) + c_2 y_2(x)$$

$$= c_1 x^{\frac{1}{2}+\frac{i\sqrt{3}}{2}} \left(1 - \frac{x}{2} + \frac{(i\sqrt{3}+3)\, x^2}{16+8i\sqrt{3}} + \frac{(-i\sqrt{3}-5)\, x^3}{48i\sqrt{3}+96} + \frac{(i\sqrt{3}+5)\,(i\sqrt{3}+7)\, x^4}{384\,(i\sqrt{3}+4)\,(2+i\sqrt{3})} - \frac{(i\sqrt{3}+7)\,(i\sqrt{3}+9)\, x^5}{3840\,(i\sqrt{3}+4)\,(2+i\sqrt{3})} + O(x^6)\right) + c_2 x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}} \left(1 - \frac{x}{2} + \frac{(-i\sqrt{3}+3)\, x^2}{16-8i\sqrt{3}} + \frac{(i\sqrt{3}-5)\, x^3}{-48i\sqrt{3}+96} + \frac{(-i\sqrt{3}+5)\,(-i\sqrt{3}+7)\, x^4}{384\,(-i\sqrt{3}+4)\,(2-i\sqrt{3})} - \frac{(-i\sqrt{3}+7)\,(-i\sqrt{3}+9)\, x^5}{3840\,(-i\sqrt{3}+4)\,(2-i\sqrt{3})} + O(x^6)\right)$$

Hence the final solution is

$$y = y_h$$

$$= c_1 x^{\frac{1}{2}+\frac{i\sqrt{3}}{2}} \left(1 - \frac{x}{2} + \frac{(i\sqrt{3}+3)\, x^2}{16+8i\sqrt{3}} + \frac{(-i\sqrt{3}-5)\, x^3}{48i\sqrt{3}+96} + \frac{(i\sqrt{3}+5)\,(i\sqrt{3}+7)\, x^4}{384\,(i\sqrt{3}+4)\,(2+i\sqrt{3})} - \frac{(i\sqrt{3}+7)\,(i\sqrt{3}+9)\, x^5}{3840\,(i\sqrt{3}+4)\,(2+i\sqrt{3})} + O(x^6)\right) + c_2 x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}} \left(1 - \frac{x}{2} + \frac{(-i\sqrt{3}+3)\, x^2}{16-8i\sqrt{3}} + \frac{(i\sqrt{3}-5)\, x^3}{-48i\sqrt{3}+96} + \frac{(-i\sqrt{3}+5)\,(-i\sqrt{3}+7)\, x^4}{384\,(-i\sqrt{3}+4)\,(2-i\sqrt{3})} - \frac{(-i\sqrt{3}+7)\,(-i\sqrt{3}+9)\, x^5}{3840\,(-i\sqrt{3}+4)\,(2-i\sqrt{3})} + O(x^6)\right)$$

Summary

The solution(s) found are the following

$$y = c_1 x^{\frac{1}{2}+\frac{i\sqrt{3}}{2}} \left(1 - \frac{x}{2} + \frac{(i\sqrt{3}+3)\, x^2}{16+8i\sqrt{3}} + \frac{(-i\sqrt{3}-5)\, x^3}{48i\sqrt{3}+96} + \frac{(i\sqrt{3}+5)\,(i\sqrt{3}+7)\, x^4}{384\,(i\sqrt{3}+4)\,(2+i\sqrt{3})} - \frac{(i\sqrt{3}+7)\,(i\sqrt{3}+9)\, x^5}{3840\,(i\sqrt{3}+4)\,(2+i\sqrt{3})} + O(x^6)\right) + c_2 x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}} \left(1 - \frac{x}{2} + \frac{(-i\sqrt{3}+3)\, x^2}{16-8i\sqrt{3}} + \frac{(i\sqrt{3}-5)\, x^3}{-48i\sqrt{3}+96} + \frac{(-i\sqrt{3}+5)\,(-i\sqrt{3}+7)\, x^4}{384\,(-i\sqrt{3}+4)\,(2-i\sqrt{3})} - \frac{(-i\sqrt{3}+7)\,(-i\sqrt{3}+9)\, x^5}{3840\,(-i\sqrt{3}+4)\,(2-i\sqrt{3})} + O(x^6)\right) \tag{1}$$

Verification of solutions

$$y = c_1 x^{\frac{1}{2}+\frac{i\sqrt{3}}{2}} \left(1-\frac{x}{2}+\frac{(i\sqrt{3}+3)\,x^2}{16+8i\sqrt{3}}+\frac{(-i\sqrt{3}-5)\,x^3}{48i\sqrt{3}+96}+\frac{(i\sqrt{3}+5)\,(i\sqrt{3}+7)\,x^4}{384\,(i\sqrt{3}+4)\,(2+i\sqrt{3})} - \frac{(i\sqrt{3}+7)\,(i\sqrt{3}+9)\,x^5}{3840\,(i\sqrt{3}+4)\,(2+i\sqrt{3})}+O(x^6)\right) + c_2 x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}} \left(1-\frac{x}{2}+\frac{(-i\sqrt{3}+3)\,x^2}{16-8i\sqrt{3}}+\frac{(i\sqrt{3}-5)\,x^3}{-48i\sqrt{3}+96}+\frac{(-i\sqrt{3}+5)\,(-i\sqrt{3}+7)\,x^4}{384\,(-i\sqrt{3}+4)\,(2-i\sqrt{3})} - \frac{(-i\sqrt{3}+7)\,(-i\sqrt{3}+9)\,x^5}{3840\,(-i\sqrt{3}+4)\,(2-i\sqrt{3})}+O(x^6)\right)$$

Verified OK.

7.23.1 Maple step by step solution

Let's solve

$x^2 y'' + y' x^2 + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -y' - \frac{y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + y' + \frac{y}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ○ Define functions

  $\left[P_2(x) = 1, P_3(x) = \frac{1}{x^2}\right]$

  ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = 0$

  ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 1$

  ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

  $x^2 y'' + y' x^2 + y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^2 \cdot y'$ to series expansion

    $x^2 \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r+1}$

  - Shift index using $k->k-1$

    $x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1}(k-1+r)\, x^{k+r}$

  - Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k-1+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(r^2 - r + 1)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k^2 + 2kr + r^2 - k - r + 1) + a_{k-1}(k-1+r)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r^2 - r + 1 = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{\frac{1}{2} - \frac{\mathrm{I}\sqrt{3}}{2}, \frac{1}{2} + \frac{\mathrm{I}\sqrt{3}}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $(k^2 + (2r-1)\, k + r^2 - r + 1)\, a_k + a_{k-1}(k-1+r) = 0$

- Shift index using $k->k+1$

  $\left((k+1)^2 + (2r-1)\,(k+1) + r^2 - r + 1\right) a_{k+1} + a_k(k+r) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k(k+r)}{k^2+2kr+r^2+k+r+1}$

- Recursion relation for $r = \frac{1}{2} - \frac{\mathrm{I}\sqrt{3}}{2}$

$$a_{k+1} = -\frac{a_k\left(k+\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)}{k^2+2k\left(\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)+\left(\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^2+k+\frac{3}{2}-\frac{I\sqrt{3}}{2}}$$

- Solution for $r = \frac{1}{2} - \frac{I\sqrt{3}}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}-\frac{I\sqrt{3}}{2}}, a_{k+1} = -\frac{a_k\left(k+\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)}{k^2+2k\left(\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)+\left(\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^2+k+\frac{3}{2}-\frac{I\sqrt{3}}{2}}\right]$$

- Recursion relation for $r = \frac{1}{2} + \frac{I\sqrt{3}}{2}$

$$a_{k+1} = -\frac{a_k\left(k+\frac{1}{2}+\frac{I\sqrt{3}}{2}\right)}{k^2+2k\left(\frac{1}{2}+\frac{I\sqrt{3}}{2}\right)+\left(\frac{1}{2}+\frac{I\sqrt{3}}{2}\right)^2+k+\frac{3}{2}+\frac{I\sqrt{3}}{2}}$$

- Solution for $r = \frac{1}{2} + \frac{I\sqrt{3}}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}+\frac{I\sqrt{3}}{2}}, a_{k+1} = -\frac{a_k\left(k+\frac{1}{2}+\frac{I\sqrt{3}}{2}\right)}{k^2+2k\left(\frac{1}{2}+\frac{I\sqrt{3}}{2}\right)+\left(\frac{1}{2}+\frac{I\sqrt{3}}{2}\right)^2+k+\frac{3}{2}+\frac{I\sqrt{3}}{2}}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}-\frac{I\sqrt{3}}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}+\frac{I\sqrt{3}}{2}}\right), a_{k+1} = -\frac{a_k\left(k+\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)}{k^2+2k\left(\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)+\left(\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^2+k+\frac{3}{2}-\frac{I\sqrt{3}}{2}}, b_{k+1} =\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 907

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+x^2*diff(y(x),x)+y(x)=0,y(x),type='series',x=0);
```

$$
\begin{aligned}
y(x) = \sqrt{x}\,\Bigg(& c_2 x^{\frac{i\sqrt{3}}{2}} \Bigg(1 - \frac{1}{2}x + \frac{i\sqrt{3}+3}{8i\sqrt{3}+16}x^2 + \frac{-i\sqrt{3}-5}{48i\sqrt{3}+96}x^3 \\
& + \frac{1}{384}\frac{\left(i\sqrt{3}+5\right)\left(i\sqrt{3}+7\right)}{\left(i\sqrt{3}+4\right)\left(i\sqrt{3}+2\right)}x^4 - \frac{1}{3840}\frac{\left(i\sqrt{3}+7\right)\left(i\sqrt{3}+9\right)}{\left(i\sqrt{3}+4\right)\left(i\sqrt{3}+2\right)}x^5 + \mathrm{O}\left(x^6\right)\Bigg) \\
& + c_1 x^{-\frac{i\sqrt{3}}{2}} \Bigg(1 - \frac{1}{2}x + \frac{\sqrt{3}+3i}{8\sqrt{3}+16i}x^2 + \frac{-\sqrt{3}-5i}{48\sqrt{3}+96i}x^3 + \frac{3i\sqrt{3}-8}{576i\sqrt{3}-480}x^4 \\
& - \frac{1}{3840}\frac{\left(\sqrt{3}+7i\right)\left(\sqrt{3}+9i\right)}{\left(\sqrt{3}+4i\right)\left(\sqrt{3}+2i\right)}x^5 + \mathrm{O}\left(x^6\right)\Bigg)\Bigg)
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 886

```
AsymptoticDSolveValue[x^2*y''[x]+x^2*y'[x]+y[x]==0,y[x],{x,0,5}]
```

$$
\begin{aligned}
& y(x) \\
& \to \Bigg(\frac{(-1)^{2/3}\left(1-(-1)^{2/3}\right)\left(2-(-1)^{2/3}\right)\left(3-(-1)^{2/3}\right)(4}{\left(1-(-1)^{2/3}\left(1-(-1)^{2/3}\right)\right)\left(1+\left(1-(-1)^{2/3}\right)\left(2-(-1)^{2/3}\right)\right)\left(1+\left(2-(-1)^{2/3}\right)\left(3-(-1)^{2/3}\right)\right)(1+(} \\
& - \frac{(-1)^{2/3}\left(1-(-1)^{2/3}\right)\left(2-(-1)^{2/3}\right)\left(3-(-1)^{2/3}\right)x^4}{\left(1-(-1)^{2/3}\left(1-(-1)^{2/3}\right)\right)\left(1+\left(1-(-1)^{2/3}\right)\left(2-(-1)^{2/3}\right)\right)\left(1+\left(2-(-1)^{2/3}\right)\left(3-(-1)^{2/3}\right)\right)(1+(} \\
& + \frac{(-1)^{2/3}\left(1-(-1)^{2/3}\right)\left(2-(-1)^{2/3}\right)x^3}{\left(1-(-1)^{2/3}\left(1-(-1)^{2/3}\right)\right)\left(1+\left(1-(-1)^{2/3}\right)\left(2-(-1)^{2/3}\right)\right)\left(1+\left(2-(-1)^{2/3}\right)\left(3-(-1)^{2/3}\right)\right)} \\
& - \frac{(-1)^{2/3}\left(1-(-1)^{2/3}\right)x^2}{\left(1-(-1)^{2/3}\left(1-(-1)^{2/3}\right)\right)\left(1+\left(1-(-1)^{2/3}\right)\left(2-(-1)^{2/3}\right)\right)} \\
& + \frac{(-1)^{2/3}x}{1-(-1)^{2/3}\left(1-(-1)^{2/3}\right)} \\
& + 1\Bigg) c_1 x^{-(-1)^{2/3}} + \Bigg(- \frac{\sqrt[3]{-1}\left(1+\sqrt[3]{-1}\right)\left(2+\sqrt[3]{-1}\right)(3+\sqrt[3]{-}}{\left(1+\sqrt[3]{-1}\left(1+\sqrt[3]{-1}\right)\right)\left(1+\left(1+\sqrt[3]{-1}\right)\left(2+\sqrt[3]{-1}\right)\right)\left(1+\left(2+\sqrt[3]{-1}\right)(3+\sqrt[3]{-1}\right.}
\end{aligned}
$$

7.24 problem problem 26(a)

Internal problem ID [415]
Internal file name [`OUTPUT/415_Sunday_June_05_2022_01_40_54_AM_45682351/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Chapter 11 Power series methods. Section 11.1 Introduction and Review of power series. Page 615
Problem number: problem 26(a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - y^2 = 1}$$

With initial conditions

$$[y(0) = 0]$$

7.24.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= y^2 + 1 \end{aligned}$$

The y domain of $f(x, y)$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(y^2 + 1) \\ &= 2y \end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Therefore solution exists and is unique.

7.24.2 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{y^2+1} dy = x + c_1$$
$$\arctan(y) = x + c_1$$

Solving for y gives these solutions

$$y_1 = \tan(x + c_1)$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \tan(c_1)$$

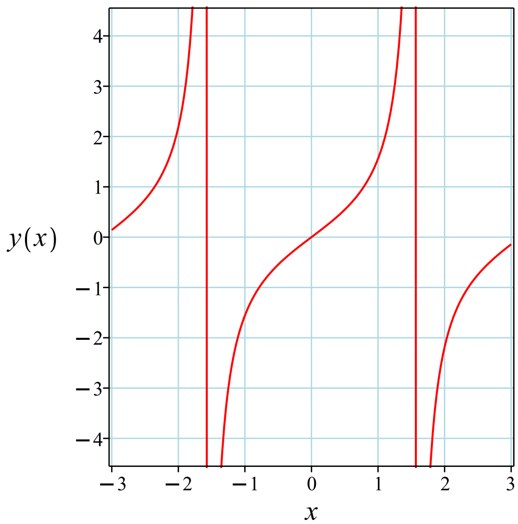
$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

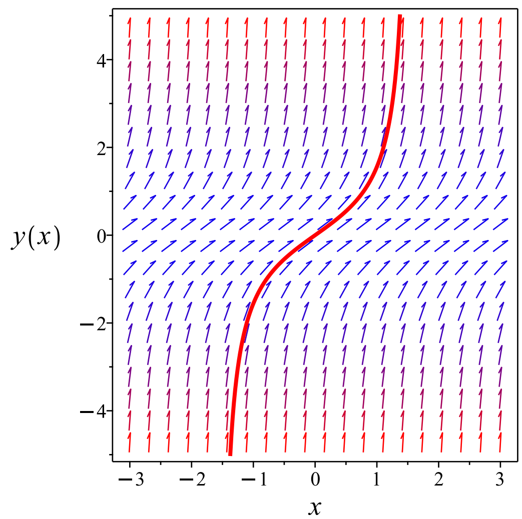
$$y = \tan(x)$$

Summary
The solution(s) found are the following

$$y = \tan(x) \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \tan(x)$$

Verified OK.

7.24.3 Maple step by step solution

Let's solve

$[y' - y^2 = 1, y(0) = 0]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{1+y^2} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{1+y^2}dx = \int 1dx + c_1$

- Evaluate integral

  $\arctan(y) = x + c_1$

- Solve for y

  $y = \tan(x + c_1)$

- Use initial condition $y(0) = 0$

  $0 = \tan(c_1)$
- Solve for c_1

  $c_1 = 0$
- Substitute $c_1 = 0$ into general solution and simplify

  $y = \tan(x)$
- Solution to the IVP

  $y = \tan(x)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple
Time used: 0.015 (sec). Leaf size: 6

```
dsolve([diff(y(x),x)=1+y(x)^2,y(0) = 0],y(x), singsol=all)
```

$$y(x) = \tan(x)$$

✓ Solution by Mathematica
Time used: 0.004 (sec). Leaf size: 7

```
DSolve[{y'[x]==1+y[x]^2,{y[0]==0}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \tan(x)$$

8 Chapter 11 Power series methods. Section 11.2 Power series solutions. Page 624

8.1 problem problem 1

Internal problem ID [416]
Internal file name [OUTPUT/416_Sunday_June_05_2022_01_40_55_AM_89930341/index.tex]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Chapter 11 Power series methods. Section 11.2 Power series solutions. Page 624
Problem number: problem 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second order series method. Ordinary point", "second_order_change_of_variable_on_y_method_1", "linear_second_order_ode_solved_by_an_integrating_factor", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{\left(x^2-1\right) y''+4y'x+2y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using

Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0, y_0, y_0'} + \frac{x^3}{3!} f'|_{x_0, y_0, y_0'} + \cdots \\ &= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y_0'} \end{aligned}$$

But

$$\begin{aligned} \frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (74) \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (75) \\ \frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\ &= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\ \frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\ &= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\ &\vdots \end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{2(y+2y'x)}{x^2-1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{18y'x^2+12yx+6y'}{(x^2-1)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{-96x^3y'-72x^2y-96y'x-24y}{(x^2-1)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{(600x^4+1200x^2+120)\,y'+480xy(x^2+1)}{(x^2-1)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{(-4320x^5-14400x^3-4320x)\,y'-3600y\left(x^4+2x^2+\frac{1}{5}\right)}{(x^2-1)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and $y'(0)=y'(0)$ gives

$$
\begin{aligned}
F_0 &= 2y(0) \\
F_1 &= 6y'(0) \\
F_2 &= 24y(0) \\
F_3 &= 120y'(0) \\
F_4 &= 720y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(x^6+x^4+x^2+1\right) y(0) + \left(x^5+x^3+x\right) y'(0) + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(x^2 - 1) y'' + 4y'x + 2y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the above back into the ode gives

$$(x^2 - 1) \left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + 4\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x + 2\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \sum_{n=2}^{\infty} \left(-n(n-1) a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} 4n a_n x^n\right) + \left(\sum_{n=0}^{\infty} 2a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} \left(-n(n-1) a_n x^{n-2}\right) = \sum_{n=0}^{\infty} \left(-(n+2) a_{n+2}(n+1) x^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\begin{aligned} &\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \sum_{n=0}^{\infty} \left(-(n+2) a_{n+2}(n+1) x^n\right) \\ &+ \left(\sum_{n=1}^{\infty} 4n a_n x^n\right) + \left(\sum_{n=0}^{\infty} 2a_n x^n\right) = 0 \end{aligned} \tag{3}$$

$n = 0$ gives

$$-2a_2 + 2a_0 = 0$$

$$a_2 = a_0$$

$n = 1$ gives

$$-6a_3 + 6a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = a_1$$

For $2 \leq n$, the recurrence equation is

$$na_n(n-1) - (n+2)\, a_{n+2}(n+1) + 4na_n + 2a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = a_n \tag{5}$$

For $n = 2$ the recurrence equation gives

$$12a_2 - 12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = a_0$$

For $n = 3$ the recurrence equation gives

$$20a_3 - 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = a_1$$

For $n = 4$ the recurrence equation gives

$$30a_4 - 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = a_0$$

For $n = 5$ the recurrence equation gives

$$42a_5 - 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = a_1$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_1 x^5 + a_0 x^4 + a_1 x^3 + a_0 x^2 + a_1 x + a_0 + \dots$$

Collecting terms, the solution becomes

$$y = \left(x^4 + x^2 + 1\right) a_0 + \left(x^5 + x^3 + x\right) a_1 + O\left(x^6\right) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(x^4 + x^2 + 1\right) c_1 + \left(x^5 + x^3 + x\right) c_2 + O\left(x^6\right)$$

Summary

The solution(s) found are the following

$$y = \left(x^6 + x^4 + x^2 + 1\right) y(0) + \left(x^5 + x^3 + x\right) y'(0) + O\left(x^6\right) \tag{1}$$

$$y = \left(x^4 + x^2 + 1\right) c_1 + \left(x^5 + x^3 + x\right) c_2 + O\left(x^6\right) \tag{2}$$

Verification of solutions

$$y = \left(x^6 + x^4 + x^2 + 1\right) y(0) + \left(x^5 + x^3 + x\right) y'(0) + O\left(x^6\right)$$

Verified OK.

$$y = \left(x^4 + x^2 + 1\right) c_1 + \left(x^5 + x^3 + x\right) c_2 + O\left(x^6\right)$$

Verified OK.

8.1.1 Maple step by step solution

Let's solve

$(x^2-1)\, y'' + 4y'x + 2y = 0$

- Highest derivative means the order of the ODE is 2

y''

- Isolate 2nd derivative

$y'' = -\frac{4xy'}{x^2-1} - \frac{2y}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' + \frac{4xy'}{x^2-1} + \frac{2y}{x^2-1} = 0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

$\left[P_2(x) = \frac{4x}{x^2-1}, P_3(x) = \frac{2}{x^2-1}\right]$

- ◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

$\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1} = 2$

- ◦ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

$\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1} = 0$

- ◦ $x=-1$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0 = -1$

- Multiply by denominators

$(x^2-1)\, y'' + 4y'x + 2y = 0$

- Change variables using $x = u-1$ so that the regular singular point is at $u=0$

$(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right) + (4u-4)\left(\frac{d}{du}y(u)\right) + 2y(u) = 0$

- Assume series solution for $y(u)$

$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

$$u^{m}\cdot \left(\frac{d}{d u}y \left(u \right)\right)=\sum_{k =0}^{\infty}a_{k} \left(k +r \right) u^{k +r -1+m}$$

- Shift index using $k->k +1-m$

$$u^{m}\cdot \left(\frac{d}{d u}y \left(u \right)\right)=\sum_{k =-1+m}^{\infty}a_{k +1-m} \left(k +1-m +r \right) u^{k +r}$$

- Convert $u^{m}\cdot \left(\frac{d^{2}}{d u^{2}}y \left(u \right)\right)$ to series expansion for $m =1..2$

$$u^{m}\cdot \left(\frac{d^{2}}{d u^{2}}y \left(u \right)\right)=\sum_{k =0}^{\infty}a_{k} \left(k +r \right) \left(k +r -1\right) u^{k +r -2+m}$$

- Shift index using $k->k +2-m$

$$u^{m}\cdot \left(\frac{d^{2}}{d u^{2}}y \left(u \right)\right)=\sum_{k =-2+m}^{\infty}a_{k +2-m} \left(k +2-m +r \right) \left(k +1-m +r \right) u^{k +r}$$

Rewrite ODE with series expansions

$$-2 a_{0} r \left(1+r \right) u^{-1+r}+\left(\sum_{k =0}^{\infty}\left(-2 a_{k +1} \left(k +r +1\right) \left(k +r +2\right)+a_{k} \left(k +r +2\right) \left(k +r +1\right)\right) u^{k +r}\right)=$$

- a_{0}cannot be 0 by assumption, giving the indicial equation

$$-2 r \left(1+r \right)=0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{-1, 0\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$\left(k +r +2\right) \left(k +r +1\right) \left(-2 a_{k +1}+a_{k}\right)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k +1}=\frac{a_{k}}{2}$$

- Recursion relation for $r =-1$

$$a_{k +1}=\frac{a_{k}}{2}$$

- Solution for $r =-1$

$$\left[y \left(u \right)=\sum_{k =0}^{\infty}a_{k} u^{k -1}, a_{k +1}=\frac{a_{k}}{2}\right]$$

- Revert the change of variables $u =x +1$

$$\left[y =\sum_{k =0}^{\infty}a_{k} \left(x +1\right)^{k -1}, a_{k +1}=\frac{a_{k}}{2}\right]$$

- Recursion relation for $r =0$

$$a_{k +1}=\frac{a_{k}}{2}$$

- Solution for $r = 0$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k}{2}\right]$$

- Revert the change of variables $u = x + 1$

  $$\left[y = \sum_{k=0}^{\infty} a_k (x+1)^k, a_{k+1} = \frac{a_k}{2}\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k (x+1)^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k (x+1)^k\right), a_{k+1} = \frac{a_k}{2}, b_{k+1} = \frac{b_k}{2}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
Order:=6;
dsolve((x^2-1)*diff(y(x),x$2)+4*x*diff(y(x),x)+2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = (x^4 + x^2 + 1)\, y(0) + (x^5 + x^3 + x)\, D(y)\,(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 26

```
AsymptoticDSolveValue[(x^2-1)*y''[x]+4*x*y'[x]+2*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2 (x^5 + x^3 + x) + c_1 (x^4 + x^2 + 1)$$

8.2 problem problem 2

Internal problem ID [417]
Internal file name [OUTPUT/417_Sunday_June_05_2022_01_40_56_AM_25828976/index.tex]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Chapter 11 Power series methods. Section 11.2 Power series solutions. Page 624
Problem number: problem 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second order series method. Ordinary point", "second_order_change_of_variable_on_y_method_1", "linear_second_order_ode_solved_by_an_integrating_factor", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{(x^2+2)\,y''+4y'x+2y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y_0'$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots\\
&= y_0+xy_0'+\frac{x^2}{2}f|_{x_0,y_0,y_0'}+\frac{x^3}{3!}f'|_{x_0,y_0,y_0'}+\cdots\\
&= y_0+xy_0'+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}\,\frac{d^nf}{dx^n}\bigg|_{x_0,y_0,y_0'}
\end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (77)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (78)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{align}
F_0 &= f(x,y,y') \tag{4}\\
F_1 &= \frac{df}{dx} \nonumber\\
&= \frac{dF_0}{dx} \nonumber\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' \nonumber\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \tag{5}\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \nonumber\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right) \nonumber\\
&= \frac{d}{dx}(F_1) \nonumber\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y'' \nonumber\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0 \nonumber\\
&\vdots \nonumber\\
F_n &= \frac{d}{dx}(F_{n-1}) \nonumber\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y'' \nonumber\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \tag{6}
\end{align}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$\begin{aligned}
F_0 &= -\frac{2(y + 2y'x)}{x^2 + 2} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{18y'x^2 + 12yx - 12y'}{(x^2 + 2)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{-96x^3y' - 72x^2y + 192y'x + 48y}{(x^2 + 2)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{(600x^4 - 2400x^2 + 480)\, y' + 480xy(x^2 - 2)}{(x^2 + 2)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{(-4320x^5 + 28800x^3 - 17280x)\, y' - 3600y\left(x^4 - 4x^2 + \frac{4}{5}\right)}{(x^2 + 2)^5}
\end{aligned}$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$\begin{aligned}
F_0 &= -y(0) \\
F_1 &= -3y'(0) \\
F_2 &= 6y(0) \\
F_3 &= 30y'(0) \\
F_4 &= -90y(0)
\end{aligned}$$

Substituting all the above in (7) and simplifying gives the solution as

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{4}x^4 - \frac{1}{8}x^6\right) y(0) + \left(x - \frac{1}{2}x^3 + \frac{1}{4}x^5\right) y'(0) + O\left(x^6\right)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(x^2 + 2) y'' + 4y'x + 2y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the above back into the ode gives

$$(x^2 + 2) \left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \right) + 4 \left(\sum_{n=1}^{\infty} n a_n x^{n-1} \right) x + 2 \left(\sum_{n=0}^{\infty} a_n x^n \right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} x^n a_n n(n-1) \right) + \left(\sum_{n=2}^{\infty} 2n(n-1) a_n x^{n-2} \right) + \left(\sum_{n=1}^{\infty} 4n a_n x^n \right) + \left(\sum_{n=0}^{\infty} 2a_n x^n \right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} 2n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} 2(n+2) a_{n+2}(n+1) x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\begin{aligned} &\left(\sum_{n=2}^{\infty} x^n a_n n(n-1) \right) + \left(\sum_{n=0}^{\infty} 2(n+2) a_{n+2}(n+1) x^n \right) \\ &+ \left(\sum_{n=1}^{\infty} 4n a_n x^n \right) + \left(\sum_{n=0}^{\infty} 2a_n x^n \right) = 0 \end{aligned} \tag{3}$$

$n = 0$ gives

$$4a_2 + 2a_0 = 0$$

$$a_2 = -\frac{a_0}{2}$$

$n = 1$ gives

$$12a_3 + 6a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{a_1}{2}$$

For $2 \leq n$, the recurrence equation is

$$na_n(n-1) + 2(n+2)\, a_{n+2}(n+1) + 4na_n + 2a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_n}{2} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$12a_2 + 24a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{4}$$

For $n = 3$ the recurrence equation gives

$$20a_3 + 40a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_1}{4}$$

For $n = 4$ the recurrence equation gives

$$30a_4 + 60a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{a_0}{8}$$

For $n = 5$ the recurrence equation gives

$$42a_5 + 84a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{a_1}{8}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{1}{2} a_0 x^2 - \frac{1}{2} a_1 x^3 + \frac{1}{4} a_0 x^4 + \frac{1}{4} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{4}x^4\right) a_0 + \left(x - \frac{1}{2}x^3 + \frac{1}{4}x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{4}x^4\right) c_1 + \left(x - \frac{1}{2}x^3 + \frac{1}{4}x^5\right) c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{4}x^4 - \frac{1}{8}x^6\right) y(0) + \left(x - \frac{1}{2}x^3 + \frac{1}{4}x^5\right) y'(0) + O(x^6) \tag{1}$$

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{4}x^4\right) c_1 + \left(x - \frac{1}{2}x^3 + \frac{1}{4}x^5\right) c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1-\frac{1}{2}x^2+\frac{1}{4}x^4-\frac{1}{8}x^6\right)y(0)+\left(x-\frac{1}{2}x^3+\frac{1}{4}x^5\right)y'(0)+O(x^6)$$

Verified OK.

$$y = \left(1-\frac{1}{2}x^2+\frac{1}{4}x^4\right)c_1+\left(x-\frac{1}{2}x^3+\frac{1}{4}x^5\right)c_2+O(x^6)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
Order:=6;
dsolve((x^2+2)*diff(y(x),x$2)+4*x*diff(y(x),x)+2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1-\frac{1}{2}x^2+\frac{1}{4}x^4\right)y(0)+\left(x-\frac{1}{2}x^3+\frac{1}{4}x^5\right)D(y)(0)+O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 68

```
AsymptoticDSolveValue[(x^2+2)*y''[x]+4*y'[x]+2*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(-\frac{x^5}{30}-\frac{x^4}{12}+\frac{x^3}{3}-\frac{x^2}{2}+1\right)+c_2\left(-\frac{x^5}{15}-\frac{x^4}{12}+\frac{x^3}{2}-x^2+x\right)$$

8.3 problem problem 3

Internal problem ID [418]
Internal file name [`OUTPUT/418_Sunday_June_05_2022_01_40_57_AM_16875638/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Chapter 11 Power series methods. Section 11.2 Power series solutions. Page 624
Problem number: problem 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{y'' + y'x + y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + xy'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\ &= y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y'_0} \end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{80}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{81}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \qquad (7)$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -y'x - y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= y'x^2 + yx - 2y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= -x^3y' - x^2y + 5y'x + 3y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \left(x^4 - 9x^2 + 8\right) y' + xy\left(x^2 - 7\right) \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \left(-x^5 + 14x^3 - 33x\right) y' - y\left(x^4 - 12x^2 + 15\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and $y'(0)=y'(0)$ gives

$$
\begin{aligned}
F_0 &= -y(0) \\
F_1 &= -2y'(0) \\
F_2 &= 3y(0) \\
F_3 &= 8y'(0) \\
F_4 &= -15y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{1}{48}x^6\right) y(0) + \left(x - \frac{1}{3}x^3 + \frac{1}{15}x^5\right) y'(0) + O\left(x^6\right)
$$

Since the expansion point $x=0$ is an ordinary, we can also solve this using standard

power series Let the solution be represented as power series of the form

$$y=\sum_{n=0}^{\infty} a_n x^n$$

Then

$$y'=\sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y''=\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}=-\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x-\left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right)+\left(\sum_{n=1}^{\infty} n\, x^n a_n\right)+\left(\sum_{n=0}^{\infty} a_n x^n\right)=0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}=\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right)+\left(\sum_{n=1}^{\infty} n\, x^n a_n\right)+\left(\sum_{n=0}^{\infty} a_n x^n\right)=0 \tag{3}$$

$n=0$ gives

$$2a_2+a_0=0$$

$$a_2=-\frac{a_0}{2}$$

For $1 \leq n$, the recurrence equation is

$$(n+2)\,a_{n+2}(n+1)+na_n+a_n=0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2}=-\frac{a_n}{n+2} \tag{5}$$

For $n=1$ the recurrence equation gives

$$6a_3+2a_1=0$$

Which after substituting the earlier terms found becomes

$$a_3=-\frac{a_1}{3}$$

For $n=2$ the recurrence equation gives

$$12a_4+3a_2=0$$

Which after substituting the earlier terms found becomes

$$a_4=\frac{a_0}{8}$$

For $n=3$ the recurrence equation gives

$$20a_5+4a_3=0$$

Which after substituting the earlier terms found becomes

$$a_5=\frac{a_1}{15}$$

For $n=4$ the recurrence equation gives

$$30a_6+5a_4=0$$

Which after substituting the earlier terms found becomes

$$a_6=-\frac{a_0}{48}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + 6a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{a_1}{105}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{1}{2} a_0 x^2 - \frac{1}{3} a_1 x^3 + \frac{1}{8} a_0 x^4 + \frac{1}{15} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{1}{2} x^2 + \frac{1}{8} x^4\right) a_0 + \left(x - \frac{1}{3} x^3 + \frac{1}{15} x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{1}{2} x^2 + \frac{1}{8} x^4\right) c_1 + \left(x - \frac{1}{3} x^3 + \frac{1}{15} x^5\right) c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = \left(1 - \frac{1}{2} x^2 + \frac{1}{8} x^4 - \frac{1}{48} x^6\right) y(0) + \left(x - \frac{1}{3} x^3 + \frac{1}{15} x^5\right) y'(0) + O(x^6) \tag{1}$$

$$y = \left(1 - \frac{1}{2} x^2 + \frac{1}{8} x^4\right) c_1 + \left(x - \frac{1}{3} x^3 + \frac{1}{15} x^5\right) c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1 - \frac{1}{2} x^2 + \frac{1}{8} x^4 - \frac{1}{48} x^6\right) y(0) + \left(x - \frac{1}{3} x^3 + \frac{1}{15} x^5\right) y'(0) + O(x^6)$$

Verified OK.

$$y = \left(1 - \frac{1}{2} x^2 + \frac{1}{8} x^4\right) c_1 + \left(x - \frac{1}{3} x^3 + \frac{1}{15} x^5\right) c_2 + O(x^6)$$

Verified OK.

8.3.1 Maple step by step solution

Let's solve

$y'' = -y'x - y$

- Highest derivative means the order of the ODE is 2

y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' + y'x + y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ◦ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert y'' to series expansion

  $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- >k+2$

  $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

Rewrite DE with series expansions

$\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_k(k+1)\right) x^k = 0$

- Each term in the series must be 0, giving the recursion relation

$(k+1)\,(a_{k+2}(k+2) + a_k) = 0$

- Recursion relation that defines the series solution to the ODE

$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k}{k+2}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
Order:=6;
dsolve(diff(y(x),x$2)+x*diff(y(x),x)+y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4\right) y(0) + \left(x - \frac{1}{3}x^3 + \frac{1}{15}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 42

```
AsymptoticDSolveValue[y''[x]+x*y'[x]+y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{x^5}{15} - \frac{x^3}{3} + x\right) + c_1\left(\frac{x^4}{8} - \frac{x^2}{2} + 1\right)$$

8.4 problem problem 4

Internal problem ID [419]
Internal file name [`OUTPUT/419_Sunday_June_05_2022_01_40_58_AM_65107965/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Chapter 11 Power series methods. Section 11.2 Power series solutions. Page 624
Problem number: problem 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{(x^2+1)\,y''+6y'x+4y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y_0'$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots \\ &= y_0+xy_0'+\frac{x^2}{2}\,f|_{x_0,y_0,y_0'}+\frac{x^3}{3!}\,f'|_{x_0,y_0,y_0'}+\cdots \\ &= y_0+xy_0'+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}\,\frac{d^nf}{dx^n}\bigg|_{x_0,y_0,y_0'} \end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{83}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{84}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{align}
F_0 &= f(x,y,y') \tag{4}\\
F_1 &= \frac{df}{dx} \\
&= \frac{dF_0}{dx} \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{5}\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0 \\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right) \\
&= \frac{d}{dx}(F_1) \\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y'' \\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0 \\
&\vdots \\
F_n &= \frac{d}{dx}(F_{n-1}) \\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y'' \\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \tag{6}
\end{align}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n\Big|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{2(3y'x+2y)}{x^2+1}\\
F_1 &= \frac{dF_0}{dx}\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
&= \frac{38y'x^2+32yx-10y'}{(x^2+1)^2}\\
F_2 &= \frac{dF_1}{dx}\\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y}y' + \frac{\partial F_1}{\partial y'}F_1\\
&= \frac{-272x^3y'-248x^2y+208y'x+72y}{(x^2+1)^3}\\
F_3 &= \frac{dF_2}{dx}\\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y}y' + \frac{\partial F_2}{\partial y'}F_2\\
&= \frac{(2200x^4-3280x^2+280)\,y'+(2080x^3-1760x)\,y}{(x^2+1)^4}\\
F_4 &= \frac{dF_3}{dx}\\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y}y' + \frac{\partial F_3}{\partial y'}F_3\\
&= \frac{(-19920x^5+48480x^3-12240x)\,y'-19200\left(x^4-\frac{33}{20}x^2+\frac{3}{20}\right)y}{(x^2+1)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and $y'(0)=y'(0)$ gives

$$
\begin{aligned}
F_0 &= -4y(0)\\
F_1 &= -10y'(0)\\
F_2 &= 72y(0)\\
F_3 &= 280y'(0)\\
F_4 &= -2880y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(-4x^6+3x^4-2x^2+1\right)y(0) + \left(x-\frac{5}{3}x^3+\frac{7}{3}x^5\right)y'(0) + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(x^2+1)\,y'' + 6y'x + 4y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}$$

Substituting the above back into the ode gives

$$(x^2+1)\left(\sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}\right) + 6\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x + 4\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \quad (1)$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \left(\sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} 6n a_n x^n\right) + \left(\sum_{n=0}^{\infty} 4a_n x^n\right) = 0 \quad (2)$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\,a_{n+2}(n+1)\,x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \left(\sum_{n=0}^{\infty} (n+2)\,a_{n+2}(n+1)\,x^n\right) + \left(\sum_{n=1}^{\infty} 6n a_n x^n\right) + \left(\sum_{n=0}^{\infty} 4a_n x^n\right) = 0 \quad (3)$$

$n = 0$ gives

$$2a_2 + 4a_0 = 0$$

$$a_2 = -2a_0$$

$n = 1$ gives

$$6a_3 + 10a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{5a_1}{3}$$

For $2 \leq n$, the recurrence equation is

$$n a_n (n-1) + (n+2) a_{n+2} (n+1) + 6 n a_n + 4 a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{(n+4) a_n}{n+2} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$18a_2 + 12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = 3a_0$$

For $n = 3$ the recurrence equation gives

$$28a_3 + 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{7a_1}{3}$$

For $n = 4$ the recurrence equation gives

$$40a_4 + 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -4a_0$$

For $n = 5$ the recurrence equation gives

$$54a_5 + 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -3a_1$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1x - 2a_0x^2 - \frac{5}{3}a_1x^3 + 3a_0x^4 + \frac{7}{3}a_1x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(3x^4 - 2x^2 + 1\right) a_0 + \left(x - \frac{5}{3}x^3 + \frac{7}{3}x^5\right) a_1 + O\left(x^6\right) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(3x^4 - 2x^2 + 1\right) c_1 + \left(x - \frac{5}{3}x^3 + \frac{7}{3}x^5\right) c_2 + O\left(x^6\right)$$

Summary

The solution(s) found are the following

$$y = \left(-4x^6 + 3x^4 - 2x^2 + 1\right) y(0) + \left(x - \frac{5}{3}x^3 + \frac{7}{3}x^5\right) y'(0) + O\left(x^6\right) \tag{1}$$

$$y = \left(3x^4 - 2x^2 + 1\right) c_1 + \left(x - \frac{5}{3}x^3 + \frac{7}{3}x^5\right) c_2 + O\left(x^6\right) \tag{2}$$

Verification of solutions

$$y = (-4x^6 + 3x^4 - 2x^2 + 1)\, y(0) + \left(x - \frac{5}{3}x^3 + \frac{7}{3}x^5\right) y'(0) + O(x^6)$$

Verified OK.

$$y = (3x^4 - 2x^2 + 1)\, c_1 + \left(x - \frac{5}{3}x^3 + \frac{7}{3}x^5\right) c_2 + O(x^6)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
Order:=6;
dsolve((x^2+1)*diff(y(x),x$2)+6*x*diff(y(x),x)+4*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = (3x^4 - 2x^2 + 1)\, y(0) + \left(x - \frac{5}{3}x^3 + \frac{7}{3}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 60

```
AsymptoticDSolveValue[(x^2+1)*y''[x]+6*y'[x]+4*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 (4x^5 - 5x^4 + 4x^3 - 2x^2 + 1) + c_2 \left(\frac{77x^5}{15} - \frac{13x^4}{2} + \frac{16x^3}{3} - 3x^2 + x\right)$$

8.5 problem problem 5

Internal problem ID [420]
Internal file name [`OUTPUT/420_Sunday_June_05_2022_01_40_58_AM_37968535/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Chapter 11 Power series methods. Section 11.2 Power series solutions. Page 624
Problem number: problem 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second__order__integrable__as__is", "second__order__ode__missing__y", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{(x^2+1)\, y''+2y'x=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y_0'$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0)+(x-x_0)\, y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots \\
&= y_0+xy_0'+\frac{x^2}{2}\, f|_{x_0,y_0,y_0'}+\frac{x^3}{3!}\, f'|_{x_0,y_0,y_0'}+\cdots \\
&= y_0+xy_0'+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}\,\frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y_0'}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{86}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{87}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x, y, y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 && (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n \big|_{x_0, y_0, y_0'} \tag{7}$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{2xy'}{x^2+1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{(6x^2-2)\,y'}{(x^2+1)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{(-24x^3+24x)\,y'}{(x^2+1)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{120\left(x^4-2x^2+\frac{1}{5}\right) y'}{(x^2+1)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{(-720x^5+2400x^3-720x)\,y'}{(x^2+1)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= -2y'(0) \\
F_2 &= 0 \\
F_3 &= 24y'(0) \\
F_4 &= 0
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = y(0) + \left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5\right) y'(0) + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(x^2 + 1)\, y'' + 2y'x = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$(x^2 + 1) \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + 2\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} 2n a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \left(\sum_{n=1}^{\infty} 2n a_n x^n\right) = 0 \tag{3}$$

$n = 1$ gives

$$6a_3 + 2a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{a_1}{3}$$

For $2 \leq n$, the recurrence equation is

$$na_n(n-1) + (n+2)\, a_{n+2}(n+1) + 2na_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{na_n}{n+2} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$6a_2 + 12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = 0$$

For $n = 3$ the recurrence equation gives

$$12a_3 + 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_1}{5}$$

For $n = 4$ the recurrence equation gives

$$20a_4 + 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = 0$$

For $n = 5$ the recurrence equation gives

$$30a_5 + 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{a_1}{7}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{1}{3} a_1 x^3 + \frac{1}{5} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = a_0 + \left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = c_1 + \left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5\right) c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = y(0) + \left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5\right) y'(0) + O(x^6) \tag{1}$$

$$y = c_1 + \left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5\right) c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$y = y(0) + \left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5\right) y'(0) + O(x^6)$$

Verified OK.

$$y = c_1 + \left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5\right) c_2 + O(x^6)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
<- LODE missing y successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
Order:=6;
dsolve((x^2+1)*diff(y(x),x$2)+2*x*diff(y(x),x)=0,y(x),type='series',x=0);
```

$$y(x) = y(0) + \left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 25

```
AsymptoticDSolveValue[(x^2-3)*y''[x]+2*x*y'[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{x^5}{45} + \frac{x^3}{9} + x\right) + c_1$$

8.6 problem problem 6

Internal problem ID [421]
Internal file name [OUTPUT/421_Sunday_June_05_2022_01_40_59_AM_27112025/index.tex]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Chapter 11 Power series methods. Section 11.2 Power series solutions. Page 624
Problem number: problem 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

`[_Gegenbauer]`

$$\boxed{(x^2-1)\, y''-6y'x+12y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y_0'$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0)+(x-x_0)\, y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots \\
&= y_0+xy_0'+\frac{x^2}{2}\, f|_{x_0,y_0,y_0'}+\frac{x^3}{3!}\, f'|_{x_0,y_0,y_0'}+\cdots \\
&= y_0+xy_0'+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}\,\frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y_0'}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (89)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (90)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= \frac{6y'x - 12y}{x^2 - 1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{18y'x^2 - 48yx + 6y'}{\left(x^2 - 1\right)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{24x^3y' - 72x^2y + 24y'x - 24y}{\left(x^2 - 1\right)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= 0 \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= 0
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 12y(0) \\
F_1 &= 6y'(0) \\
F_2 &= 24y(0) \\
F_3 &= 0 \\
F_4 &= 0
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(x^4 + 6x^2 + 1\right) y(0) + \left(x^3 + x\right) y'(0) + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(x^2 - 1) y'' - 6y'x + 12y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the above back into the ode gives

$$(x^2 - 1) \left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) - 6\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x + 12\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \sum_{n=2}^{\infty} \left(-n(n-1) a_n x^{n-2}\right) + \sum_{n=1}^{\infty} \left(-6n a_n x^n\right) + \left(\sum_{n=0}^{\infty} 12 a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} \left(-n(n-1) a_n x^{n-2}\right) = \sum_{n=0}^{\infty} \left(-(n+2) a_{n+2}(n+1) x^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\begin{aligned} &\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \sum_{n=0}^{\infty} \left(-(n+2) a_{n+2}(n+1) x^n\right) \\ &+ \sum_{n=1}^{\infty} \left(-6n a_n x^n\right) + \left(\sum_{n=0}^{\infty} 12 a_n x^n\right) = 0 \end{aligned} \tag{3}$$

$n = 0$ gives

$$-2a_2 + 12a_0 = 0$$

$$a_2 = 6a_0$$

$n = 1$ gives

$$-6a_3 + 6a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = a_1$$

For $2 \leq n$, the recurrence equation is

$$na_n(n-1) - (n+2) a_{n+2}(n+1) - 6na_n + 12a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{a_n(n^2 - 7n + 12)}{(n+2)(n+1)} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$2a_2 - 12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = a_0$$

For $n = 3$ the recurrence equation gives

$$-20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = 0$$

For $n = 4$ the recurrence equation gives

$$-30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = 0$$

For $n = 5$ the recurrence equation gives

$$2a_5 - 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = 0$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 x^4 + a_1 x^3 + 6a_0 x^2 + a_1 x + a_0 + \dots$$

Collecting terms, the solution becomes

$$y = (x^4 + 6x^2 + 1)\, a_0 + (x^3 + x)\, a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = (x^4 + 6x^2 + 1)\, c_1 + (x^3 + x)\, c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = (x^4 + 6x^2 + 1)\, y(0) + (x^3 + x)\, y'(0) + O(x^6) \tag{1}$$
$$y = (x^4 + 6x^2 + 1)\, c_1 + (x^3 + x)\, c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$y = (x^4 + 6x^2 + 1)\, y(0) + (x^3 + x)\, y'(0) + O(x^6)$$

Verified OK.

$$y = (x^4 + 6x^2 + 1)\, c_1 + (x^3 + x)\, c_2 + O(x^6)$$

Verified OK.

8.6.1 Maple step by step solution

Let's solve

$(x^2-1)\,y''-6y'x+12y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{6xy'}{x^2-1}-\frac{12y}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{6xy'}{x^2-1}+\frac{12y}{x^2-1}=0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=-\frac{6x}{x^2-1}, P_3(x)=\frac{12}{x^2-1}\right]$

  - $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

    $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1}=-3$

  - $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1}=0$

  - $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $(x^2-1)\,y''-6y'x+12y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right)+(-6u+6)\left(\frac{d}{du}y(u)\right)+12y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$-2a_0r(-4+r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2a_{k+1}(k+1+r)\,(k+r-3) + a_k(k+r-3)\,(k+r-4)\right) u^{k+r}\right)$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-2r(-4+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 4\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k+r-3)\left((-2k-2r-2)\, a_{k+1} + a_k(k+r-4)\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k(k+r-4)}{2(k+1+r)}$$

- Recursion relation for $r = 0$; series terminates at $k = 4$

$$a_{k+1} = \frac{a_k(k-4)}{2(k+1)}$$

- Apply recursion relation for $k = 0$

$$a_1 = -2a_0$$

- Apply recursion relation for $k = 1$

$$a_2 = -\frac{3a_1}{4}$$

- Express in terms of a_0

$$a_2 = \frac{3a_0}{2}$$

- Apply recursion relation for $k = 2$

$a_3 = -\frac{a_2}{3}$

- Express in terms of a_0

  $a_3 = -\frac{a_0}{2}$

- Apply recursion relation for $k = 3$

  $a_4 = -\frac{a_3}{8}$

- Express in terms of a_0

  $a_4 = \frac{a_0}{16}$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot \left(1 - 2u + \frac{3}{2}u^2 - \frac{1}{2}u^3 + \frac{1}{16}u^4\right)$

- Revert the change of variables $u = x + 1$

  $\left[y = \frac{a_0(x-1)^4}{16}\right]$

- Recursion relation for $r = 4$

  $a_{k+1} = \frac{a_k k}{2(k+5)}$

- Solution for $r = 4$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+4}, a_{k+1} = \frac{a_k k}{2(k+5)}\right]$

- Revert the change of variables $u = x + 1$

  $\left[y = \sum_{k=0}^{\infty} a_k (x+1)^{k+4}, a_{k+1} = \frac{a_k k}{2(k+5)}\right]$

- Combine solutions and rename parameters

  $\left[y = \frac{a_0(x-1)^4}{16} + \left(\sum_{k=0}^{\infty} b_k (x+1)^{k+4}\right), b_{k+1} = \frac{b_k k}{2(k+5)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 25

```
Order:=6;
dsolve((x^2-1)*diff(y(x),x$2)-6*x*diff(y(x),x)+12*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = (x^4 + 6x^2 + 1)\, y(0) + (x^3 + x)\, D(y)\,(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 25

```
AsymptoticDSolveValue[(x^2-1)*y''[x]-6*x*y'[x]+12*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2(x^3 + x) + c_1(x^4 + 6x^2 + 1)$$

8.7 problem problem 7

Internal problem ID [422]
Internal file name [`OUTPUT/422_Sunday_June_05_2022_01_41_00_AM_24441164/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Chapter 11 Power series methods. Section 11.2 Power series solutions. Page 624
Problem number: problem 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x^2+3)\,y''-7y'x+16y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots \\
&= y_0+xy'_0+\frac{x^2}{2}\,f|_{x_0,y_0,y'_0}+\frac{x^3}{3!}\,f'|_{x_0,y_0,y'_0}+\cdots \\
&= y_0+xy'_0+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}\,\frac{d^nf}{dx^n}\bigg|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (92)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (93)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{align}
F_0 &= f(x,y,y') \tag{4}\\
F_1 &= \frac{df}{dx} \nonumber\\
&= \frac{dF_0}{dx} \nonumber\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \nonumber\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{5}\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0 \nonumber\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right) \nonumber\\
&= \frac{d}{dx}(F_1) \nonumber\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y'' \nonumber\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0 \nonumber\\
&\vdots \nonumber\\
F_n &= \frac{d}{dx}(F_{n-1}) \nonumber\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y'' \nonumber\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \tag{6}
\end{align}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} F_n|_{x_0,y_0,y'_0} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$\begin{aligned}
F_0 &= \frac{7y'x - 16y}{x^2 + 3} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0 \\
&= \frac{26y'x^2 - 80yx - 27y'}{(x^2+3)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y}y' + \frac{\partial F_1}{\partial y'}F_1 \\
&= \frac{50x^3y' - 176x^2y - 165y'x + 192y}{(x^2+3)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y}y' + \frac{\partial F_2}{\partial y'}F_2 \\
&= \frac{(24x^4 - 216x^2 + 81)\, y' + (-96x^3 + 432x)\, y}{(x^2+3)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y}y' + \frac{\partial F_3}{\partial y'}F_3 \\
&= -\frac{24x\left(\left(x^4 - 9x^2 + \frac{27}{8}\right) y' + (-4x^3 + 18x)\, y\right)}{(x^2+3)^5}
\end{aligned}$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$\begin{aligned}
F_0 &= -\frac{16y(0)}{3} \\
F_1 &= -3y'(0) \\
F_2 &= \frac{64y(0)}{9} \\
F_3 &= y'(0) \\
F_4 &= 0
\end{aligned}$$

Substituting all the above in (7) and simplifying gives the solution as

$$y = \left(1 - \frac{8}{3}x^2 + \frac{8}{27}x^4\right) y(0) + \left(x - \frac{1}{2}x^3 + \frac{1}{120}x^5\right) y'(0) + O(x^6)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(x^2 + 3) y'' - 7y'x + 16y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the above back into the ode gives

$$(x^2 + 3) \left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) - 7\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x + 16\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \quad (1)$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \left(\sum_{n=2}^{\infty} 3n(n-1) a_n x^{n-2}\right) + \sum_{n=1}^{\infty} (-7n a_n x^n) + \left(\sum_{n=0}^{\infty} 16 a_n x^n\right) = 0 \quad (2)$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} 3n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} 3(n+2) a_{n+2}(n+1) x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \left(\sum_{n=0}^{\infty} 3(n+2)\, a_{n+2}(n+1)\, x^n\right) + \sum_{n=1}^{\infty} (-7n a_n x^n) + \left(\sum_{n=0}^{\infty} 16 a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$6a_2 + 16a_0 = 0$$

$$a_2 = -\frac{8a_0}{3}$$

$n = 1$ gives

$$18a_3 + 9a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{a_1}{2}$$

For $2 \leq n$, the recurrence equation is

$$n a_n(n-1) + 3(n+2)\, a_{n+2}(n+1) - 7n a_n + 16 a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_n(n^2 - 8n + 16)}{3\,(n+2)\,(n+1)} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$4a_2 + 36a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{8a_0}{27}$$

For $n = 3$ the recurrence equation gives

$$a_3 + 60a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_1}{120}$$

For $n = 4$ the recurrence equation gives

$$90a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = 0$$

For $n = 5$ the recurrence equation gives

$$a_5 + 126a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{a_1}{15120}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{8}{3} a_0 x^2 - \frac{1}{2} a_1 x^3 + \frac{8}{27} a_0 x^4 + \frac{1}{120} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{8}{3}x^2 + \frac{8}{27}x^4\right) a_0 + \left(x - \frac{1}{2}x^3 + \frac{1}{120}x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{8}{3}x^2 + \frac{8}{27}x^4\right)c_1 + \left(x - \frac{1}{2}x^3 + \frac{1}{120}x^5\right)c_2 + O(x^6)$$

Summary
The solution(s) found are the following

$$y = \left(1 - \frac{8}{3}x^2 + \frac{8}{27}x^4\right)y(0) + \left(x - \frac{1}{2}x^3 + \frac{1}{120}x^5\right)y'(0) + O(x^6) \tag{1}$$

$$y = \left(1 - \frac{8}{3}x^2 + \frac{8}{27}x^4\right)c_1 + \left(x - \frac{1}{2}x^3 + \frac{1}{120}x^5\right)c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1 - \frac{8}{3}x^2 + \frac{8}{27}x^4\right)y(0) + \left(x - \frac{1}{2}x^3 + \frac{1}{120}x^5\right)y'(0) + O(x^6)$$

Verified OK.

$$y = \left(1 - \frac{8}{3}x^2 + \frac{8}{27}x^4\right)c_1 + \left(x - \frac{1}{2}x^3 + \frac{1}{120}x^5\right)c_2 + O(x^6)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
Order:=6;
dsolve((x^2+3)*diff(y(x),x$2)-7*x*diff(y(x),x)+16*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{8}{3}x^2 + \frac{8}{27}x^4\right) y(0) + \left(x - \frac{1}{2}x^3 + \frac{1}{120}x^5\right) D(y)\,(0) + O\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 42

```
AsymptoticDSolveValue[(x^2+3)*y''[x]-7*x*y'[x]+16*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{x^5}{120} - \frac{x^3}{2} + x\right) + c_1\left(\frac{8x^4}{27} - \frac{8x^2}{3} + 1\right)$$

8.8 problem problem 8

Internal problem ID [423]
Internal file name [`OUTPUT/423_Sunday_June_05_2022_01_41_01_AM_41111291/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Chapter 11 Power series methods. Section 11.2 Power series solutions. Page 624
Problem number: problem 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries], [_2nd_order, _linear, `
   _with_symmetry_[0,F(x)]`]]
```

$$\boxed{\left(-x^{2}+2\right) y''-y'x+16y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots \\
&= y_0+xy'_0+\frac{x^2}{2}\,f|_{x_0,y_0,y'_0}+\frac{x^3}{3!}\,f'|_{x_0,y_0,y'_0}+\cdots \\
&= y_0+xy'_0+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}\,\frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (95) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (96) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$\begin{aligned}
F_0 &= -\frac{y'x - 16y}{x^2 - 2} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{18y'x^2 - 48yx - 30y'}{(x^2 - 2)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{-102x^3y' + 432x^2y + 174y'x - 384y}{(x^2 - 2)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{420(2x^4 - 4x^2 + 1)\, y' + 3360(-x^3 + x)\, y}{(x^2 - 2)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= -\frac{3780(2y'x^4 - 8yx^3 - 4y'x^2 + 8yx + y')\, x}{(x^2 - 2)^5}
\end{aligned}$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$\begin{aligned}
F_0 &= -8y(0) \\
F_1 &= -\frac{15y'(0)}{2} \\
F_2 &= 48y(0) \\
F_3 &= \frac{105y'(0)}{4} \\
F_4 &= 0
\end{aligned}$$

Substituting all the above in (7) and simplifying gives the solution as

$$y = (2x^4 - 4x^2 + 1)\, y(0) + \left(x - \frac{5}{4}x^3 + \frac{7}{32}x^5\right) y'(0) + O(x^6)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(-x^2 + 2)\, y'' - y'x + 16y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$(-x^2 + 2) \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) - \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x + 16 \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \quad (1)$$

Which simplifies to

$$\sum_{n=2}^{\infty} (-x^n a_n n(n-1)) + \left(\sum_{n=2}^{\infty} 2n(n-1)\, a_n x^{n-2}\right) + \sum_{n=1}^{\infty} (-n a_n x^n) + \left(\sum_{n=0}^{\infty} 16 a_n x^n\right) = 0 \quad (2)$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} 2n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} 2(n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\sum_{n=2}^{\infty}\left(-x^{n} a_{n} n(n-1)\right)+\left(\sum_{n=0}^{\infty} 2(n+2) a_{n+2}(n+1) x^{n}\right) + \sum_{n=1}^{\infty}\left(-n a_{n} x^{n}\right)+\left(\sum_{n=0}^{\infty} 16 a_{n} x^{n}\right)=0 \tag{3}$$

$n=0$ gives

$$4a_2+16a_0=0$$

$$a_2=-4a_0$$

$n=1$ gives

$$12a_3+15a_1=0$$

Which after substituting earlier equations, simplifies to

$$a_3=-\frac{5a_1}{4}$$

For $2\leq n$, the recurrence equation is

$$-n a_{n}(n-1)+2(n+2) a_{n+2}(n+1)-n a_{n}+16 a_{n}=0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2}=\frac{a_{n}\left(n^{2}-16\right)}{2\left(n+2\right)\left(n+1\right)} \tag{5}$$

For $n=2$ the recurrence equation gives

$$12a_2+24a_4=0$$

Which after substituting the earlier terms found becomes

$$a_4=2a_0$$

For $n = 3$ the recurrence equation gives

$$7a_3 + 40a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{7a_1}{32}$$

For $n = 4$ the recurrence equation gives

$$60a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = 0$$

For $n = 5$ the recurrence equation gives

$$-9a_5 + 84a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{3a_1}{128}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - 4a_0 x^2 - \frac{5}{4} a_1 x^3 + 2a_0 x^4 + \frac{7}{32} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(2x^4 - 4x^2 + 1\right) a_0 + \left(x - \frac{5}{4}x^3 + \frac{7}{32}x^5\right) a_1 + O\left(x^6\right) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = (2x^4 - 4x^2 + 1) c_1 + \left(x - \frac{5}{4}x^3 + \frac{7}{32}x^5\right) c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = (2x^4 - 4x^2 + 1) y(0) + \left(x - \frac{5}{4}x^3 + \frac{7}{32}x^5\right) y'(0) + O(x^6) \tag{1}$$

$$y = (2x^4 - 4x^2 + 1) c_1 + \left(x - \frac{5}{4}x^3 + \frac{7}{32}x^5\right) c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$y = (2x^4 - 4x^2 + 1) y(0) + \left(x - \frac{5}{4}x^3 + \frac{7}{32}x^5\right) y'(0) + O(x^6)$$

Verified OK.

$$y = (2x^4 - 4x^2 + 1) c_1 + \left(x - \frac{5}{4}x^3 + \frac{7}{32}x^5\right) c_2 + O(x^6)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
Order:=6;
dsolve((2-x^2)*diff(y(x),x$2)-x*diff(y(x),x)+16*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = (2x^4 - 4x^2 + 1) y(0) + \left(x - \frac{5}{4}x^3 + \frac{7}{32}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 38

```
AsymptoticDSolveValue[(2-x^2)*y''[x]-x*y'[x]+16*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{7x^5}{32} - \frac{5x^3}{4} + x\right) + c_1\left(2x^4 - 4x^2 + 1\right)$$

8.9 problem problem 9

Internal problem ID [424]
Internal file name [`OUTPUT/424_Sunday_June_05_2022_01_41_02_AM_34500257/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Chapter 11 Power series methods. Section 11.2 Power series solutions. Page 624
Problem number: problem 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

`[_Gegenbauer]`

$$\boxed{(x^2-1)\,y''+8y'x+12y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots \\
&= y_0+xy'_0+\frac{x^2}{2}\,f|_{x_0,y_0,y'_0}+\frac{x^3}{3!}\,f'|_{x_0,y_0,y'_0}+\cdots \\
&= y_0+xy'_0+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}\,\frac{d^nf}{dx^n}\bigg|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (98)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (99)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{4(2y'x+3y)}{x^2-1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{60y'x^2 + 120yx + 20y'}{(x^2-1)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= -\frac{120(4x^3y' + 9x^2y + 4y'x + 3y)}{(x^2-1)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{(4200x^4 + 8400x^2 + 840)\, y' + 10080xy(x^2+1)}{(x^2-1)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{(-40320x^5 - 134400x^3 - 40320x)\, y' - 100800y\left(x^4 + 2x^2 + \frac{1}{5}\right)}{(x^2-1)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and $y'(0)=y'(0)$ gives

$$
\begin{aligned}
F_0 &= 12y(0) \\
F_1 &= 20y'(0) \\
F_2 &= 360y(0) \\
F_3 &= 840y'(0) \\
F_4 &= 20160y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(28x^6 + 15x^4 + 6x^2 + 1\right) y(0) + \left(x + \frac{10}{3}x^3 + 7x^5\right) y'(0) + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(x^2 - 1)\, y'' + 8y'x + 12y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$(x^2 - 1)\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + 8\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x + 12\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \sum_{n=2}^{\infty} \left(-n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} 8n a_n x^n\right) + \left(\sum_{n=0}^{\infty} 12 a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} \left(-n(n-1)\, a_n x^{n-2}\right) = \sum_{n=0}^{\infty} \left(-(n+2)\, a_{n+2}(n+1)\, x^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \sum_{n=0}^{\infty} \left(-(n+2)\, a_{n+2}(n+1)\, x^n\right) + \left(\sum_{n=1}^{\infty} 8n a_n x^n\right) + \left(\sum_{n=0}^{\infty} 12 a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$-2a_2 + 12a_0 = 0$$

$$a_2 = 6a_0$$

$n = 1$ gives

$$-6a_3 + 20a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = \frac{10a_1}{3}$$

For $2 \leq n$, the recurrence equation is

$$na_n(n-1) - (n+2)\, a_{n+2}(n+1) + 8na_n + 12a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{a_n(n^2 + 7n + 12)}{(n+2)\,(n+1)} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$30a_2 - 12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = 15a_0$$

For $n = 3$ the recurrence equation gives

$$42a_3 - 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = 7a_1$$

For $n = 4$ the recurrence equation gives

$$56a_4 - 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = 28a_0$$

For $n = 5$ the recurrence equation gives

$$72a_5 - 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = 12a_1$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + 6a_0 x^2 + \frac{10}{3} a_1 x^3 + 15a_0 x^4 + 7a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = (15x^4 + 6x^2 + 1) a_0 + \left(x + \frac{10}{3}x^3 + 7x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = (15x^4 + 6x^2 + 1) c_1 + \left(x + \frac{10}{3}x^3 + 7x^5\right) c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = (28x^6 + 15x^4 + 6x^2 + 1) y(0) + \left(x + \frac{10}{3}x^3 + 7x^5\right) y'(0) + O(x^6) \tag{1}$$

$$y = (15x^4 + 6x^2 + 1) c_1 + \left(x + \frac{10}{3}x^3 + 7x^5\right) c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$y = (28x^6 + 15x^4 + 6x^2 + 1)\, y(0) + \left(x + \frac{10}{3}x^3 + 7x^5\right) y'(0) + O(x^6)$$

Verified OK.

$$y = (15x^4 + 6x^2 + 1)\, c_1 + \left(x + \frac{10}{3}x^3 + 7x^5\right) c_2 + O(x^6)$$

Verified OK.

8.9.1 Maple step by step solution

Let's solve

$(x^2 - 1)\, y'' + 8y'x + 12y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{8xy'}{x^2-1} - \frac{12y}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{8xy'}{x^2-1} + \frac{12y}{x^2-1} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{8x}{x^2-1}, P_3(x) = \frac{12}{x^2-1}\right]$

  ◦ $(x+1) \cdot P_2(x)$ is analytic at $x = -1$

    $\left((x+1) \cdot P_2(x)\right)\Big|_{x=-1} = 4$

  ◦ $(x+1)^2 \cdot P_3(x)$ is analytic at $x = -1$

    $\left((x+1)^2 \cdot P_3(x)\right)\Big|_{x=-1} = 0$

  ◦ $x = -1$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = -1$

- Multiply by denominators

$\left(x^2-1\right) y''+8y'x+12y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

$\left(u^2-2u\right)\left(\frac{d^2}{du^2}y(u)\right)+(8u-8)\left(\frac{d}{du}y(u)\right)+12y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

  ◦ Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

  ◦ Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$

  ◦ Shift index using $k->k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$

Rewrite ODE with series expansions

$-2a_0 r(3+r)\, u^{-1+r}+\left(\sum_{k=0}^{\infty} \left(-2a_{k+1}(k+1+r)\,(k+r+4)+a_k(k+r+4)\,(k+r+3)\right) u^{k+r}\right)=$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$-2r(3+r)=0$

- Values of r that satisfy the indicial equation

$r\in \{-3,0\}$

- Each term in the series must be 0, giving the recursion relation

$(k+r+4)\left((-2k-2r-2)\, a_{k+1}+a_k(k+r+3)\right)=0$

- Recursion relation that defines series solution to ODE

$a_{k+1}=\frac{a_k(k+r+3)}{2(k+1+r)}$

- Recursion relation for $r=-3$

$a_{k+1} = \frac{a_k k}{2(k-2)}$

- Series not valid for $r = -3$, division by 0 in the recursion relation at $k = 2$

  $a_{k+1} = \frac{a_k k}{2(k-2)}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k(k+3)}{2(k+1)}$

- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k(k+3)}{2(k+1)}\right]$

- Revert the change of variables $u = x + 1$

  $\left[y = \sum_{k=0}^{\infty} a_k (x+1)^k, a_{k+1} = \frac{a_k(k+3)}{2(k+1)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
Order:=6;
dsolve((x^2-1)*diff(y(x),x$2)+8*x*diff(y(x),x)+12*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = (15x^4 + 6x^2 + 1)\, y(0) + \left(x + \frac{10}{3}x^3 + 7x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 36

```
AsymptoticDSolveValue[(x^2-1)*y''[x]+8*x*y'[x]+12*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2 \left(7x^5 + \frac{10x^3}{3} + x\right) + c_1 \left(15x^4 + 6x^2 + 1\right)$$

8.10 problem problem 10

Internal problem ID [425]
Internal file name [OUTPUT/425_Sunday_June_05_2022_01_41_03_AM_10644048/index.tex]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Chapter 11 Power series methods. Section 11.2 Power series solutions. Page 624
Problem number: problem 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{3y'' + xy' - 4y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (101)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (102)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \tag{7}$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{xy'}{3} + \frac{4y}{3} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{x^2 y'}{9} - \frac{4xy}{9} + y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{(-x^3 - 15x)\, y'}{27} + \frac{4y(x^2 + 6)}{27} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{(x^4 + 18x^2 + 27)\, y'}{81} - \frac{4yx(x^2 + 9)}{81} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= -\frac{((x^4 + 18x^2 + 27)\, y' - 4yx(x^2 + 9))\, x}{243}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= \frac{4y(0)}{3} \\
F_1 &= y'(0) \\
F_2 &= \frac{8y(0)}{9} \\
F_3 &= \frac{y'(0)}{3} \\
F_4 &= 0
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$y=\left(1+\frac{2}{3}x^{2}+\frac{1}{27}x^{4}\right)y(0)+\left(x+\frac{1}{6}x^{3}+\frac{1}{360}x^{5}\right)y'(0)+O(x^{6})$$

Since the expansion point $x=0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y=\sum_{n=0}^{\infty}a_{n}x^{n}$$

Then

$$y'=\sum_{n=1}^{\infty}na_{n}x^{n-1}$$
$$y''=\sum_{n=2}^{\infty}n(n-1)\,a_{n}x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty}n(n-1)\,a_{n}x^{n-2}=-\frac{x\left(\sum_{n=1}^{\infty}na_{n}x^{n-1}\right)}{3}+\frac{4\left(\sum_{n=0}^{\infty}a_{n}x^{n}\right)}{3} \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty}3n(n-1)\,a_{n}x^{n-2}\right)+\left(\sum_{n=1}^{\infty}n\,x^{n}a_{n}\right)+\sum_{n=0}^{\infty}(-4a_{n}x^{n})=0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^{n} and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty}3n(n-1)\,a_{n}x^{n-2}=\sum_{n=0}^{\infty}3(n+2)\,a_{n+2}(n+1)\,x^{n}$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty}3(n+2)\,a_{n+2}(n+1)\,x^{n}\right)+\left(\sum_{n=1}^{\infty}n\,x^{n}a_{n}\right)+\sum_{n=0}^{\infty}(-4a_{n}x^{n})=0 \tag{3}$$

$n = 0$ gives

$$6a_2 - 4a_0 = 0$$

$$a_2 = \frac{2a_0}{3}$$

For $1 \leq n$, the recurrence equation is

$$3(n+2)\, a_{n+2}(n+1) + na_n - 4a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_n(n-4)}{3\,(n+2)\,(n+1)} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$18a_3 - 3a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{a_1}{6}$$

For $n = 2$ the recurrence equation gives

$$36a_4 - 2a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{27}$$

For $n = 3$ the recurrence equation gives

$$60a_5 - a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_1}{360}$$

For $n = 4$ the recurrence equation gives

$$90a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = 0$$

For $n = 5$ the recurrence equation gives

$$126a_7 + a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{a_1}{45360}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \frac{2}{3} a_0 x^2 + \frac{1}{6} a_1 x^3 + \frac{1}{27} a_0 x^4 + \frac{1}{360} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + \frac{2}{3}x^2 + \frac{1}{27}x^4\right) a_0 + \left(x + \frac{1}{6}x^3 + \frac{1}{360}x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 + \frac{2}{3}x^2 + \frac{1}{27}x^4\right) c_1 + \left(x + \frac{1}{6}x^3 + \frac{1}{360}x^5\right) c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = \left(1 + \frac{2}{3}x^2 + \frac{1}{27}x^4\right) y(0) + \left(x + \frac{1}{6}x^3 + \frac{1}{360}x^5\right) y'(0) + O(x^6) \tag{1}$$

$$y = \left(1 + \frac{2}{3}x^2 + \frac{1}{27}x^4\right) c_1 + \left(x + \frac{1}{6}x^3 + \frac{1}{360}x^5\right) c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1+\frac{2}{3}x^{2}+\frac{1}{27}x^{4}\right) y(0)+\left(x+\frac{1}{6}x^{3}+\frac{1}{360}x^{5}\right) y'(0)+O\left(x^{6}\right)$$

Verified OK.

$$y = \left(1+\frac{2}{3}x^{2}+\frac{1}{27}x^{4}\right) c_{1}+\left(x+\frac{1}{6}x^{3}+\frac{1}{360}x^{5}\right) c_{2}+O\left(x^{6}\right)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
Order:=6;
dsolve(3*diff(y(x),x$2)+x*diff(y(x),x)-4*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 + \frac{2}{3}x^2 + \frac{1}{27}x^4\right) y(0) + \left(x + \frac{1}{6}x^3 + \frac{1}{360}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 42

```
AsymptoticDSolveValue[3*y''[x]+x*y'[x]-4*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{x^5}{360} + \frac{x^3}{6} + x\right) + c_1\left(\frac{x^4}{27} + \frac{2x^2}{3} + 1\right)$$

8.11 problem problem 11

Internal problem ID [426]
Internal file name [`OUTPUT/426_Sunday_June_05_2022_01_41_05_AM_20911788/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Chapter 11 Power series methods. Section 11.2 Power series solutions. Page 624
Problem number: problem 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{5y'' - 2y'x + 10y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + xy'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\
&= y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{104}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{105}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= \frac{2y'x}{5} - 2y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{4y'x^2}{25} - \frac{4yx}{5} - \frac{8y'}{5} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{4(2x^3 - 35x)\, y'}{125} + \frac{4(-2x^2 + 15)\, y}{25} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{8(2x^4 - 45x^2 + 100)\, y'}{625} + \frac{8(-2x^3 + 25x)\, y}{125} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{8(4x^5 - 100x^3 + 375x)\, y'}{3125} - \frac{32\left(x^4 - 15x^2 + \frac{75}{4}\right) y}{625}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -2y(0) \\
F_1 &= -\frac{8y'(0)}{5} \\
F_2 &= \frac{12y(0)}{5} \\
F_3 &= \frac{32y'(0)}{25} \\
F_4 &= -\frac{24y(0)}{25}
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$y = \left(1 - x^2 + \frac{1}{10}x^4 - \frac{1}{750}x^6\right) y(0) + \left(x - \frac{4}{15}x^3 + \frac{4}{375}x^5\right) y'(0) + O(x^6)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \frac{2\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x}{5} - 2\left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} 5n(n-1)\, a_n x^{n-2}\right) + \sum_{n=1}^{\infty} \left(-2n\, x^n a_n\right) + \left(\sum_{n=0}^{\infty} 10 a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} 5n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} 5(n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} 5(n+2)\, a_{n+2}(n+1)\, x^n\right) + \sum_{n=1}^{\infty} \left(-2n\, x^n a_n\right) + \left(\sum_{n=0}^{\infty} 10 a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$10a_2 + 10a_0 = 0$$

$$a_2 = -a_0$$

For $1 \leq n$, the recurrence equation is

$$5(n+2)\, a_{n+2}(n+1) - 2na_n + 10a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{2a_n(n-5)}{5\,(n+2)\,(n+1)} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$30a_3 + 8a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{4a_1}{15}$$

For $n = 2$ the recurrence equation gives

$$60a_4 + 6a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{10}$$

For $n = 3$ the recurrence equation gives

$$100a_5 + 4a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{4a_1}{375}$$

For $n = 4$ the recurrence equation gives

$$150a_6 + 2a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{a_0}{750}$$

For $n = 5$ the recurrence equation gives

$$210a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = 0$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1x - a_0x^2 - \frac{4}{15}a_1x^3 + \frac{1}{10}a_0x^4 + \frac{4}{375}a_1x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - x^2 + \frac{1}{10}x^4\right) a_0 + \left(x - \frac{4}{15}x^3 + \frac{4}{375}x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - x^2 + \frac{1}{10}x^4\right) c_1 + \left(x - \frac{4}{15}x^3 + \frac{4}{375}x^5\right) c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = \left(1 - x^2 + \frac{1}{10}x^4 - \frac{1}{750}x^6\right) y(0) + \left(x - \frac{4}{15}x^3 + \frac{4}{375}x^5\right) y'(0) + O(x^6) \tag{1}$$

$$y = \left(1 - x^2 + \frac{1}{10}x^4\right) c_1 + \left(x - \frac{4}{15}x^3 + \frac{4}{375}x^5\right) c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1-x^2+\frac{1}{10}x^4-\frac{1}{750}x^6\right)y(0)+\left(x-\frac{4}{15}x^3+\frac{4}{375}x^5\right)y'(0)+O(x^6)$$

Verified OK.

$$y = \left(1-x^2+\frac{1}{10}x^4\right)c_1+\left(x-\frac{4}{15}x^3+\frac{4}{375}x^5\right)c_2+O(x^6)$$

Verified OK.

8.11.1 Maple step by step solution

Let's solve

$y'' = \frac{2y'x}{5} - 2y$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{2y'x}{5} + 2y = 0$

- Multiply by denominators

  $5y'' - 2y'x + 10y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ◦ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert y'' to series expansion

  $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k->k+2$

  $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\, x^k$

  Rewrite DE with series expansions

$$\sum_{k=0}^{\infty} \left(5a_{k+2}(k+2)\,(k+1) - 2a_k(k-5)\right) x^k = 0$$

- Each term in the series must be 0, giving the recursion relation

  $5(k^2+3k+2)\,a_{k+2} - 2a_k(k-5) = 0$

- Recursion relation that defines the series solution to the ODE

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{2a_k(k-5)}{5(k^2+3k+2)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric solution.
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
Order:=6;
dsolve(5*diff(y(x),x$2)-2*x*diff(y(x),x)+10*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - x^2 + \frac{1}{10}x^4\right) y(0) + \left(\frac{4}{375}x^5 - \frac{4}{15}x^3 + x\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 40

```
AsymptoticDSolveValue[5*y''[x]-2*x*y'[x]+10*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{4x^5}{375} - \frac{4x^3}{15} + x\right) + c_1\left(\frac{x^4}{10} - x^2 + 1\right)$$

8.12 problem problem 12

Internal problem ID [427]
Internal file name [OUTPUT/427_Sunday_June_05_2022_01_41_06_AM_54462440/index.tex]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Chapter 11 Power series methods. Section 11.2 Power series solutions. Page 624
Problem number: problem 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - y'x^2 - 3yx = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + xy'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\
&= y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (107)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (108)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= y' x^2 + 3yx \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \left(x^4 + 5x\right) y' + \left(3x^3 + 3\right) y \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \left(x^6 + 12x^3 + 8\right) y' + 3\left(x^5 + 8x^2\right) y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \left(\left(x^7 + 21x^4 + 68x\right) y' + 3y\left(x^6 + 17x^3 + 24\right)\right) x \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \left(x^{10} + 32x^7 + 224x^4 + 208x\right) y' + 3y\left(x^3 + 6\right)\left(x^6 + 22x^3 + 4\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= 3y(0) \\
F_2 &= 8y'(0) \\
F_3 &= 0 \\
F_4 &= 72y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 + \frac{1}{2}x^3 + \frac{1}{10}x^6\right) y(0) + \left(x + \frac{1}{3}x^4\right) y'(0) + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard

power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x^2 + 3\left(\sum_{n=0}^{\infty} a_n x^n\right) x \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \sum_{n=1}^{\infty} \left(-n\, x^{1+n} a_n\right) + \sum_{n=0}^{\infty} \left(-3x^{1+n} a_n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2) a_{n+2}(1+n) x^n$$
$$\sum_{n=1}^{\infty} \left(-n\, x^{1+n} a_n\right) = \sum_{n=2}^{\infty} \left(-(n-1) a_{n-1} x^n\right)$$
$$\sum_{n=0}^{\infty} \left(-3x^{1+n} a_n\right) = \sum_{n=1}^{\infty} \left(-3a_{n-1} x^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2) a_{n+2}(1+n) x^n\right) + \sum_{n=2}^{\infty} \left(-(n-1) a_{n-1} x^n\right) + \sum_{n=1}^{\infty} \left(-3a_{n-1} x^n\right) = 0 \tag{3}$$

$n = 1$ gives

$$6a_3 - 3a_0 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = \frac{a_0}{2}$$

For $2 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(1+n) - (n-1)\, a_{n-1} - 3a_{n-1} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{a_{n-1}}{1+n} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$12a_4 - 4a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_1}{3}$$

For $n = 3$ the recurrence equation gives

$$20a_5 - 5a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = 0$$

For $n = 4$ the recurrence equation gives

$$30a_6 - 6a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0}{10}$$

For $n = 5$ the recurrence equation gives

$$42a_7 - 7a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{a_1}{18}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \frac{1}{2} a_0 x^3 + \frac{1}{3} a_1 x^4 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + \frac{x^3}{2}\right) a_0 + \left(x + \frac{1}{3} x^4\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 + \frac{x^3}{2}\right) c_1 + \left(x + \frac{1}{3} x^4\right) c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = \left(1 + \frac{1}{2} x^3 + \frac{1}{10} x^6\right) y(0) + \left(x + \frac{1}{3} x^4\right) y'(0) + O(x^6) \tag{1}$$

$$y = \left(1 + \frac{x^3}{2}\right) c_1 + \left(x + \frac{1}{3} x^4\right) c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1 + \frac{1}{2} x^3 + \frac{1}{10} x^6\right) y(0) + \left(x + \frac{1}{3} x^4\right) y'(0) + O(x^6)$$

Verified OK.

$$y = \left(1 + \frac{x^3}{2}\right) c_1 + \left(x + \frac{1}{3} x^4\right) c_2 + O(x^6)$$

Verified OK.

8.12.1 Maple step by step solution

Let's solve

$y'' = y'x^2 + 3yx$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - y'x^2 - 3yx = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ∘ Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

  ∘ Shift index using $k- > k-1$

    $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$

  ∘ Convert $x^2 \cdot y'$ to series expansion

    $x^2 \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^{k+1}$

  ∘ Shift index using $k- > k-1$

    $x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1}(k-1)\, x^k$

  ∘ Convert y'' to series expansion

    $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ∘ Shift index using $k- > k+2$

    $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite ODE with series expansions

  $2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - a_{k-1}(k+2)\right) x^k\right) = 0$

- Each term must be 0

  $2a_2 = 0$
- Each term in the series must be 0, giving the recursion relation

  $(k+2)\left(k a_{k+2} - a_{k-1} + a_{k+2}\right) = 0$
- Shift index using $k- >k+1$

  $(k+3)\left((k+1)\, a_{k+3} - a_k + a_{k+3}\right) = 0$
- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{a_k}{k+2}, 2a_2 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
Order:=6;
dsolve(diff(y(x),x$2)-x^2*diff(y(x),x)-3*x*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 + \frac{x^3}{2}\right) y(0) + \left(x + \frac{1}{3}x^4\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 28

```
AsymptoticDSolveValue[y''[x]-x^2*y'[x]-3*x*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{x^4}{3}+x\right)+c_1\left(\frac{x^3}{2}+1\right)$$

8.13 problem problem 13

Internal problem ID [428]
Internal file name [OUTPUT/428_Sunday_June_05_2022_01_41_07_AM_13020367/index.tex]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Chapter 11 Power series methods. Section 11.2 Power series solutions. Page 624
Problem number: problem 13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{y'' + y'x^2 + 2yx = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + xy'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (110)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (111)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y'_0} \qquad (7)
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -y'x^2 - 2yx \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \left(x^4 - 4x\right) y' + \left(2x^3 - 2\right) y \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \left(-x^6 + 10x^3 - 6\right) y' - 2x^2 y\left(x^3 - 7\right) \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \left(\left(x^7 - 18x^4 + 50x\right) y' + 2y\left(x^6 - 15x^3 + 20\right)\right) x \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \left(-x^{10} + 28x^7 - 170x^4 + 140x\right) y' - 2y\left(x^9 - 25x^6 + 110x^3 - 20\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and $y'(0)=y'(0)$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= -2y(0) \\
F_2 &= -6y'(0) \\
F_3 &= 0 \\
F_4 &= 40y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 - \frac{1}{3}x^3 + \frac{1}{18}x^6\right) y(0) + \left(x - \frac{1}{4}x^4\right) y'(0) + O\left(x^6\right)
$$

Since the expansion point $x=0$ is an ordinary, we can also solve this using standard

power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = -\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x^2 - 2\left(\sum_{n=0}^{\infty} a_n x^n\right) x \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} n\, x^{1+n} a_n\right) + \left(\sum_{n=0}^{\infty} 2x^{1+n} a_n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n$$
$$\sum_{n=1}^{\infty} n\, x^{1+n} a_n = \sum_{n=2}^{\infty} (n-1)\, a_{n-1} x^n$$
$$\sum_{n=0}^{\infty} 2x^{1+n} a_n = \sum_{n=1}^{\infty} 2a_{n-1} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n\right) + \left(\sum_{n=2}^{\infty} (n-1)\, a_{n-1} x^n\right) + \left(\sum_{n=1}^{\infty} 2a_{n-1} x^n\right) = 0 \tag{3}$$

$n = 1$ gives

$$6a_3 + 2a_0 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{a_0}{3}$$

For $2 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(1+n) + (n-1)\, a_{n-1} + 2a_{n-1} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_{n-1}}{n+2} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + 3a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{a_1}{4}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + 4a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = 0$$

For $n = 4$ the recurrence equation gives

$$30a_6 + 5a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0}{18}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + 6a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{a_1}{28}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{1}{3} a_0 x^3 - \frac{1}{4} a_1 x^4 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{x^3}{3}\right) a_0 + \left(x - \frac{1}{4}x^4\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{x^3}{3}\right) c_1 + \left(x - \frac{1}{4}x^4\right) c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = \left(1 - \frac{1}{3}x^3 + \frac{1}{18}x^6\right) y(0) + \left(x - \frac{1}{4}x^4\right) y'(0) + O(x^6) \tag{1}$$

$$y = \left(1 - \frac{x^3}{3}\right) c_1 + \left(x - \frac{1}{4}x^4\right) c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1 - \frac{1}{3}x^3 + \frac{1}{18}x^6\right) y(0) + \left(x - \frac{1}{4}x^4\right) y'(0) + O(x^6)$$

Verified OK.

$$y = \left(1 - \frac{x^3}{3}\right) c_1 + \left(x - \frac{1}{4}x^4\right) c_2 + O(x^6)$$

Verified OK.

8.13.1 Maple step by step solution

Let's solve

$y'' = -y'x^2 - 2yx$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + y'x^2 + 2yx = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  - Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

  - Shift index using $k- > k - 1$

    $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$

  - Convert $x^2 \cdot y'$ to series expansion

    $x^2 \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^{k+1}$

  - Shift index using $k- > k - 1$

    $x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1}(k-1)\, x^k$

  - Convert y'' to series expansion

    $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - Shift index using $k- > k + 2$

    $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite ODE with series expansions

  $2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_{k-1}(k+1)\right) x^k\right) = 0$

- Each term must be 0

  $2a_2 = 0$
- Each term in the series must be 0, giving the recursion relation

  $(k+1)(a_{k+2}(k+2) + a_{k-1}) = 0$
- Shift index using $k- >k+1$

  $(k+2)(a_{k+3}(k+3) + a_k) = 0$
- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = -\frac{a_k}{k+3}, 2a_2 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
Order:=6;
dsolve(diff(y(x),x$2)+x^2*diff(y(x),x)+2*x*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{x^3}{3}\right) y(0) + \left(x - \frac{1}{4}x^4\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 28

```
AsymptoticDSolveValue[y''[x]+x^2*y'[x]+2*x*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(x - \frac{x^4}{4}\right) + c_1\left(1 - \frac{x^3}{3}\right)$$

8.14 problem problem 14

Internal problem ID [429]
Internal file name [`OUTPUT/429_Sunday_June_05_2022_01_41_08_AM_52240698/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Chapter 11 Power series methods. Section 11.2 Power series solutions. Page 624
Problem number: problem 14.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_airy", "second_order_bessel_ode", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{y'' + yx = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\ &= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0} \end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{113}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{114}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -yx \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= -y'x - y \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= x^2 y - 2y' \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= x(y'x + 4y) \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= -yx^3 + 6y'x + 4y
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= -y(0) \\
F_2 &= -2y'(0) \\
F_3 &= 0 \\
F_4 &= 4y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 - \frac{1}{6}x^3 + \frac{1}{180}x^6\right) y(0) + \left(x - \frac{1}{12}x^4\right) y'(0) + O(x^6)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard

power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = -\left(\sum_{n=0}^{\infty} a_n x^n\right) x \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=0}^{\infty} x^{1+n} a_n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n$$
$$\sum_{n=0}^{\infty} x^{1+n} a_n = \sum_{n=1}^{\infty} a_{n-1} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^n\right) = 0 \tag{3}$$

For $1 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(1+n) + a_{n-1} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_{n-1}}{(n+2)(1+n)} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 + a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{a_0}{6}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{a_1}{12}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = 0$$

For $n = 4$ the recurrence equation gives

$$30a_6 + a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0}{180}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{a_1}{504}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{1}{6} a_0 x^3 - \frac{1}{12} a_1 x^4 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{x^3}{6}\right) a_0 + \left(x - \frac{1}{12} x^4\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{x^3}{6}\right) c_1 + \left(x - \frac{1}{12} x^4\right) c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = \left(1 - \frac{1}{6} x^3 + \frac{1}{180} x^6\right) y(0) + \left(x - \frac{1}{12} x^4\right) y'(0) + O(x^6) \tag{1}$$

$$y = \left(1 - \frac{x^3}{6}\right) c_1 + \left(x - \frac{1}{12} x^4\right) c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1 - \frac{1}{6} x^3 + \frac{1}{180} x^6\right) y(0) + \left(x - \frac{1}{12} x^4\right) y'(0) + O(x^6)$$

Verified OK.

$$y = \left(1 - \frac{x^3}{6}\right) c_1 + \left(x - \frac{1}{12} x^4\right) c_2 + O(x^6)$$

Verified OK.

8.14.1 Maple step by step solution

Let's solve

$y'' = -yx$

- Highest derivative means the order of the ODE is 2

y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' + yx = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x \cdot y$ to series expansion

  $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

  ◦ Shift index using $k- >k - 1$

  $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$

  ◦ Convert y'' to series expansion

  $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- >k + 2$

  $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite ODE with series expansions

  $2a_2 + \left(\sum_{k=1}^{\infty} (a_{k+2}(k+2)\,(k+1) + a_{k-1})\, x^k\right) = 0$

- Each term must be 0

$2a_2 = 0$

- Each term in the series must be 0, giving the recursion relation

$(k^2 + 3k + 2)\, a_{k+2} + a_{k-1} = 0$

- Shift index using $k- >k + 1$

$$\left((k+1)^2+3k+5\right)a_{k+3}+a_k=0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y=\sum_{k=0}^{\infty}a_k x^k, a_{k+3}=-\frac{a_k}{k^2+5k+6}, 2a_2=0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
Order:=6;
dsolve(diff(y(x),x$2)+x*y(x)=0,y(x),type='series',x=0);
```

$$y(x)=\left(1-\frac{x^3}{6}\right)y(0)+\left(x-\frac{1}{12}x^4\right)D(y)(0)+O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 28

```
AsymptoticDSolveValue[y''[x]+x*y[x]==0,y[x],{x,0,5}]
```

$$y(x)\to c_2\left(x-\frac{x^4}{12}\right)+c_1\left(1-\frac{x^3}{6}\right)$$

8.15 problem problem 15

Internal problem ID [430]
Internal file name [`OUTPUT/430_Sunday_June_05_2022_01_41_09_AM_50501255/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Chapter 11 Power series methods. Section 11.2 Power series solutions. Page 624
Problem number: problem 15.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{y'' + x^2 y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\ &= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \left. \frac{d^n f}{dx^n} \right|_{x_0, y_0, y'_0} \end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{116}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{117}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -x^2 y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= -x(2y + y'x) \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= y x^4 - 4y'x - 2y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= y'x^4 + 8yx^3 - 6y' \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= 12x^3 y' - x^2 y\left(x^4 - 30\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and $y'(0)=y'(0)$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= 0 \\
F_2 &= -2y(0) \\
F_3 &= -6y'(0) \\
F_4 &= 0
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 - \frac{x^4}{12}\right) y(0) + \left(x - \frac{1}{20} x^5\right) y'(0) + O\left(x^6\right)
$$

Since the expansion point $x=0$ is an ordinary, we can also solve this using standard

power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$\begin{aligned} y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\ y'' &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \end{aligned}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = -x^2 \left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \left(\sum_{n=0}^{\infty} x^{n+2} a_n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\begin{aligned} \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} &= \sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n \\ \sum_{n=0}^{\infty} x^{n+2} a_n &= \sum_{n=2}^{\infty} a_{n-2} x^n \end{aligned}$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n\right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^n\right) = 0 \tag{3}$$

For $2 \leq n$, the recurrence equation is

$$(n+2) a_{n+2}(n+1) + a_{n-2} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_{n-2}}{(n+2)(n+1)} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{a_0}{12}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{a_1}{20}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = 0$$

For $n = 5$ the recurrence equation gives

$$42a_7 + a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = 0$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{1}{12} a_0 x^4 - \frac{1}{20} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{x^4}{12}\right) a_0 + \left(x - \frac{1}{20} x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{x^4}{12}\right) c_1 + \left(x - \frac{1}{20} x^5\right) c_2 + O(x^6)$$

Summary
The solution(s) found are the following

$$y = \left(1 - \frac{x^4}{12}\right) y(0) + \left(x - \frac{1}{20} x^5\right) y'(0) + O(x^6) \tag{1}$$

$$y = \left(1 - \frac{x^4}{12}\right) c_1 + \left(x - \frac{1}{20} x^5\right) c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1 - \frac{x^4}{12}\right) y(0) + \left(x - \frac{1}{20} x^5\right) y'(0) + O(x^6)$$

Verified OK.

$$y = \left(1 - \frac{x^4}{12}\right) c_1 + \left(x - \frac{1}{20} x^5\right) c_2 + O(x^6)$$

Verified OK.

8.15.1 Maple step by step solution

Let's solve

$y'' = -x^2 y$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$y'' + x^2 y = 0$$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^k$$

□ Rewrite ODE with series expansions

  - Convert $x^2 \cdot y$ to series expansion

$$x^2 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+2}$$

  - Shift index using $k- >k - 2$

$$x^2 \cdot y = \sum_{k=2}^{\infty} a_{k-2} x^k$$

  - Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

  - Shift index using $k- >k + 2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$$

  Rewrite ODE with series expansions

$$6a_3 x + 2a_2 + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_{k-2}\right) x^k\right) = 0$$

- The coefficients of each power of x must be 0

$$[2a_2 = 0, 6a_3 = 0]$$

- Solve for the dependent coefficient(s)

$$\{a_2 = 0, a_3 = 0\}$$

- Each term in the series must be 0, giving the recursion relation

$$\left(k^2 + 3k + 2\right) a_{k+2} + a_{k-2} = 0$$

- Shift index using $k- >k + 2$

$$\left((k+2)^2 + 3k + 8\right) a_{k+4} + a_k = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{a_k}{k^2+7k+12}, a_2 = 0, a_3 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
Order:=6;
dsolve(diff(y(x),x$2)+x^2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{x^4}{12}\right) y(0) + \left(x - \frac{1}{20}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 28

```
AsymptoticDSolveValue[y''[x]+x^2*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(x - \frac{x^5}{20}\right) + c_1\left(1 - \frac{x^4}{12}\right)$$

8.16 problem problem 16

Internal problem ID [431]
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Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Chapter 11 Power series methods. Section 11.2 Power series solutions. Page 624
Problem number: problem 16.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second_order_change_of_variable_on_y_method_2", "second order series method. Taylor series method", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x^2+1)\,y''+2y'x-2y=0}$$

With initial conditions

$$[y(0)=0, y'(0)=1]$$

With the expansion point for the power series method at $x=0$.

8.16.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y''+p(x)y'+q(x)y=F$$

Where here

$$\begin{aligned} p(x) &= \frac{2x}{x^2+1} \\ q(x) &= -\frac{2}{x^2+1} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + \frac{2xy'}{x^2+1} - \frac{2y}{x^2+1} = 0$$

The domain of $p(x) = \frac{2x}{x^2+1}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = -\frac{2}{x^2+1}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y_0'$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0, y_0, y_0'} + \frac{x^3}{3!} f'|_{x_0, y_0, y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y_0'}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (119) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (120) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!}F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{2(-y+y'x)}{x^2+1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0 \\
&= \frac{8(-y+y'x)\,x}{(x^2+1)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y}y' + \frac{\partial F_1}{\partial y'}F_1 \\
&= -\frac{8(-y+y'x)\,(5x^2-1)}{(x^2+1)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y}y' + \frac{\partial F_2}{\partial y'}F_2 \\
&= \frac{240(-y+y'x)\,x\left(x^2-\frac{3}{5}\right)}{(x^2+1)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y}y' + \frac{\partial F_3}{\partial y'}F_3 \\
&= -\frac{48(-y+y'x)\,(35x^4-42x^2+3)}{(x^2+1)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=0$ and $y'(0)=1$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= 0 \\
F_2 &= 0 \\
F_3 &= 0 \\
F_4 &= 0
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = x + O\left(x^6\right)
$$

$$y = x + O(x^6)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(x^2 + 1)\, y'' + 2y'x - 2y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$(x^2 + 1)\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + 2\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x - 2\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \quad (1)$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} 2n a_n x^n\right) + \sum_{n=0}^{\infty} (-2a_n x^n) = 0 \quad (2)$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \left(\sum_{n=1}^{\infty} 2n a_n x^n\right) + \sum_{n=0}^{\infty} (-2a_n x^n) = 0 \quad (3)$$

$n = 0$ gives

$$2a_2 - 2a_0 = 0$$

$$a_2 = a_0$$

For $2 \leq n$, the recurrence equation is

$$na_n(n-1) + (n+2)\,a_{n+2}(n+1) + 2na_n - 2a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{(n-1)\,a_n}{n+1} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$4a_2 + 12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{a_0}{3}$$

For $n = 3$ the recurrence equation gives

$$10a_3 + 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = 0$$

For $n = 4$ the recurrence equation gives

$$18a_4 + 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0}{5}$$

For $n = 5$ the recurrence equation gives

$$28a_5 + 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = 0$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + a_0 x^2 - \frac{1}{3} a_0 x^4 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + x^2 - \frac{1}{3}x^4\right) a_0 + a_1 x + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 + x^2 - \frac{1}{3}x^4\right) c_1 + c_2 x + O(x^6)$$

$$y = x + O(x^6)$$

Summary

The solution(s) found are the following

$$y = x + O(x^6) \tag{1}$$

$$y = x + O(x^6) \tag{2}$$

Verification of solutions

$$y = x + O(x^6)$$

Verified OK.

$$y = x + O(x^6)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 5

```
Order:=6;
dsolve([(1+x^2)*diff(y(x),x$2)+2*x*diff(y(x),x)-2*y(x)=0,y(0) = 0, D(y)(0) = 1],y(x),type='s
```

$$y(x) = x$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 4

```
AsymptoticDSolveValue[{(1+x^2)*y''[x]+2*x*y'[x]-2*y[x]==0,{y[0]==0,y'[0]==1}},y[x],{x,0,5}]
```

$$y(x) \to x$$

8.17 problem problem 17

Internal problem ID [432]
Internal file name [`OUTPUT/432_Sunday_June_05_2022_01_41_12_AM_23931826/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Chapter 11 Power series methods. Section 11.2 Power series solutions. Page 624
Problem number: problem 17.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + y'x - 2y = 0}$$

With initial conditions

$$[y(0) = 1, y'(0) = 0]$$

With the expansion point for the power series method at $x = 0$.

8.17.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= x \\ q(x) &= -2 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + y'x - 2y = 0$$

The domain of $p(x) = x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = -2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \left.\frac{d^n f}{dx^n}\right|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (122)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (123)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$\begin{aligned} F_0 &= f(x,y,y') \\ F_1 &= \frac{df}{dx} \\ &= \frac{dF_0}{dx} \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \\ &= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0 \\ F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right) \\ &= \frac{d}{dx}(F_1) \\ &= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y'' \\ &= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0 \\ &\vdots \\ F_n &= \frac{d}{dx}(F_{n-1}) \\ &= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y'' \\ &= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \end{aligned}$$

(4), (5), (6)

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n\big|_{x_0,y_0,y_0'} \tag{7}$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -y'x + 2y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= y'x^2 - 2yx + y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= -xy'\left(x^2 + 1\right) + 2x^2 y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \left(x^4 - 1\right) y' + \left(-2x^3 + 2x\right) y \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= -\left(x^2 - 3\right)\left(\left(x^2 + 1\right) y' - 2yx\right) x
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = 1$ and $y'(0) = 0$ gives

$$
\begin{aligned}
F_0 &= 2 \\
F_1 &= 0 \\
F_2 &= 0 \\
F_3 &= 0 \\
F_4 &= 0
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = x^2 + 1 + O\left(x^6\right)
$$

$$
y = x^2 + 1 + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = -\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x + 2\left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} n\, x^n a_n\right) + \sum_{n=0}^{\infty} (-2 a_n x^n) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \left(\sum_{n=1}^{\infty} n\, x^n a_n\right) + \sum_{n=0}^{\infty} (-2 a_n x^n) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 - 2a_0 = 0$$

$$a_2 = a_0$$

For $1 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) + n a_n - 2a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_n(n-2)}{(n+2)\,(n+1)} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 - a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{a_1}{6}$$

For $n = 2$ the recurrence equation gives

$$12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = 0$$

For $n = 3$ the recurrence equation gives

$$20a_5 + a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{a_1}{120}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + 2a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = 0$$

For $n = 5$ the recurrence equation gives

$$42a_7 + 3a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{a_1}{1680}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + a_0 x^2 + \frac{1}{6} a_1 x^3 - \frac{1}{120} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = (x^2 + 1) a_0 + \left(x + \frac{1}{6}x^3 - \frac{1}{120}x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = (x^2 + 1) c_1 + \left(x + \frac{1}{6}x^3 - \frac{1}{120}x^5\right) c_2 + O(x^6)$$

$$y = x^2 + 1 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = x^2 + 1 + O(x^6) \tag{1}$$
$$y = x^2 + 1 + O(x^6) \tag{2}$$

Verification of solutions

$$y = x^2 + 1 + O(x^6)$$

Verified OK.

$$y = x^2 + 1 + O(x^6)$$

Verified OK.

8.17.2 Maple step by step solution

Let's solve

$$\left[y'' = -y'x + 2y, y(0) = 1, y'\Big|_{\{x=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' + y'x - 2y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ◦ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert y'' to series expansion

  $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- > k+2$

  $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

Rewrite DE with series expansions

$\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_k(k-2)\right) x^k = 0$

- Each term in the series must be 0, giving the recursion relation

$(k^2 + 3k + 2)\, a_{k+2} + a_k(k-2) = 0$

- Recursion relation; series terminates at $k = 2$

$a_{k+2} = -\frac{a_k(k-2)}{k^2+3k+2}$

- Apply recursion relation for $k = 0$

$a_2 = a_0$

- Terminating series solution of the ODE. Use reduction of order to find the second linearly ind

$$y = A_2x^2 + A_1x + a_0$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 9

```
Order:=6;
dsolve([diff(y(x),x$2)+x*diff(y(x),x)-2*y(x)=0,y(0) = 1, D(y)(0) = 0],y(x),type='series',x=0
```

$$y(x) = x^2 + 1$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 19

```
AsymptoticDSolveValue[{y''[x]+x*y'[x]-2*y[x]==0,{y[0]==0,y'[0]==1}},y[x],{x,0,5}]
```

$$y(x) \to -\frac{x^5}{120} + \frac{x^3}{6} + x$$

8.18 problem problem 18

Internal problem ID [433]
Internal file name [`OUTPUT/433_Sunday_June_05_2022_01_41_14_AM_37236895/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Chapter 11 Power series methods. Section 11.2 Power series solutions. Page 624
Problem number: problem 18.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{y'' + (x - 1) y' + y = 0}$$

With initial conditions

$$[y(1) = 2, y'(1) = 0]$$

With the expansion point for the power series method at $x = 1$.

8.18.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= x - 1 \\ q(x) &= 1 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + (x - 1) y' + y = 0$$

The domain of $p(x) = x - 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

The ode does not have its expansion point at $x = 0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t = x - 1$$

The ode is converted to be in terms of the new independent variable t. This results in

$$\frac{d^2}{dt^2} y(t) + t\left(\frac{d}{dt} y(t)\right) + y(t) = 0$$

With its expansion point and initial conditions now at $t = 0$. With initial conditions now becoming

$$y(0) = 2$$
$$y'(0) = 0$$

The transformed ODE is now solved. Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the

case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0,y_0,y'_0}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (125) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (126) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x,y,y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 && (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}\,F_n|_{x_0,y_0,y_0'} \tag{7}$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -t\left(\frac{d}{dt}y(t)\right) - y(t) \\
F_1 &= \frac{dF_0}{dt} \\
&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_0}{\partial \frac{d}{dt}y(t)}F_0 \\
&= \left(\frac{d}{dt}y(t)\right)t^2 + y(t)\,t - 2\frac{d}{dt}y(t) \\
F_2 &= \frac{dF_1}{dt} \\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_1}{\partial \frac{d}{dt}y(t)}F_1 \\
&= -\left(\frac{d}{dt}y(t)\right)t^3 - y(t)\,t^2 + 5t\left(\frac{d}{dt}y(t)\right) + 3y(t) \\
F_3 &= \frac{dF_2}{dt} \\
&= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_2}{\partial \frac{d}{dt}y(t)}F_2 \\
&= \left(t^4 - 9t^2 + 8\right)\left(\frac{d}{dt}y(t)\right) + ty(t)\left(t^2 - 7\right) \\
F_4 &= \frac{dF_3}{dt} \\
&= \frac{\partial F_3}{\partial t} + \frac{\partial F_3}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_3}{\partial \frac{d}{dt}y(t)}F_3 \\
&= \left(-t^5 + 14t^3 - 33t\right)\left(\frac{d}{dt}y(t)\right) - y(t)\left(t^4 - 12t^2 + 15\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $t = 0$ and $y(0) = 2$ and $y'(0) = 0$ gives

$$
\begin{aligned}
F_0 &= -2 \\
F_1 &= 0 \\
F_2 &= 6 \\
F_3 &= 0 \\
F_4 &= -30
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y(t) = -t^2 + 2 + \frac{t^4}{4} - \frac{t^6}{24} + O\left(t^6\right)
$$

$$
y(t) = -t^2 + 2 + \frac{t^4}{4} - \frac{t^6}{24} + O(t^6)
$$

Since the expansion point $t = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$
y(t) = \sum_{n=0}^{\infty} a_n t^n
$$

Then

$$
\frac{d}{dt}y(t) = \sum_{n=1}^{\infty} n a_n t^{n-1}
$$

$$
\frac{d^2}{dt^2}y(t) = \sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}
$$

Substituting the above back into the ode gives

$$
\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2} = -t\left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right) - \left(\sum_{n=0}^{\infty} a_n t^n\right) \tag{1}
$$

Which simplifies to

$$
\left(\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}\right) + \left(\sum_{n=1}^{\infty} n\, t^n a_n\right) + \left(\sum_{n=0}^{\infty} a_n t^n\right) = 0 \tag{2}
$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the power and the corresponding index gives

$$
\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, t^n
$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$
\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, t^n\right) + \left(\sum_{n=1}^{\infty} n\, t^n a_n\right) + \left(\sum_{n=0}^{\infty} a_n t^n\right) = 0 \tag{3}
$$

$n = 0$ gives

$$
2a_2 + a_0 = 0
$$

$$a_2 = -\frac{a_0}{2}$$

For $1 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) + n a_n + a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_n}{n+2} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 + 2a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{a_1}{3}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + 3a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{8}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + 4a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_1}{15}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + 5a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{a_0}{48}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + 6a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{a_1}{105}$$

And so on. Therefore the solution is

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} a_n t^n \\ &= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y(t) = a_0 + a_1 t - \frac{1}{2} a_0 t^2 - \frac{1}{3} a_1 t^3 + \frac{1}{8} a_0 t^4 + \frac{1}{15} a_1 t^5 + \dots$$

Collecting terms, the solution becomes

$$y(t) = \left(1 - \frac{1}{2}t^2 + \frac{1}{8}t^4\right) a_0 + \left(t - \frac{1}{3}t^3 + \frac{1}{15}t^5\right) a_1 + O(t^6) \tag{3}$$

At $t = 0$ the solution above becomes

$$y(t) = \left(1 - \frac{1}{2}t^2 + \frac{1}{8}t^4\right) c_1 + \left(t - \frac{1}{3}t^3 + \frac{1}{15}t^5\right) c_2 + O(t^6)$$

$$y(t) = -t^2 + 2 + \frac{t^4}{4} + O(t^6)$$

Replacing t in the above with the original independent variable $x$using $t = x-1$ results in

$$y = -(x-1)^2 + 2 + \frac{(x-1)^4}{4} - \frac{(x-1)^6}{24} + O\left((x-1)^6\right)$$

Summary
The solution(s) found are the following

$$y = -(x-1)^2 + 2 + \frac{(x-1)^4}{4} - \frac{(x-1)^6}{24} + O\left((x-1)^6\right) \tag{1}$$

Verification of solutions

$$y = -(x-1)^2 + 2 + \frac{(x-1)^4}{4} - \frac{(x-1)^6}{24} + O\left((x-1)^6\right)$$

Verified OK.

8.18.2 Maple step by step solution

Let's solve

$\left[y'' + (x-1)\,y' + y = 0, y(1) = 2, y'\Big|_{\{x=1\}} = 0\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ◦ Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

    $x^m \cdot y' = \sum_{k=\max(0,1-m)}^{\infty} a_k k\, x^{k-1+m}$

  ◦ Shift index using $k- >k+1-m$

    $x^m \cdot y' = \sum_{k=\max(0,1-m)+m-1}^{\infty} a_{k+1-m}(k+1-m)\, x^k$

  ◦ Convert y'' to series expansion

    $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- >k+2$

    $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite DE with series expansions

$$\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - a_{k+1}(k+1) + a_k(k+1)\right) x^k = 0$$

- Each term in the series must be 0, giving the recursion relation

$$(k+1)\left(a_{k+2}(k+2) - a_{k+1} + a_k\right) = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{-a_{k+1}+a_k}{k+2}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
Order:=6;
dsolve([diff(y(x),x$2)+(x-1)*diff(y(x),x)+y(x)=0,y(1) = 2, D(y)(1) = 0],y(x),type='series',x
```

$$y(x) = 2 - (x-1)^2 + \frac{1}{4}(x-1)^4 + \mathrm{O}\left((x-1)^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 21

```
AsymptoticDSolveValue[{y''[x]+(x-1)*y'[x]+y[x]==0,{y[1]==2,y'[1]==0}},y[x],{x,1,5}]
```

$$y(x) \to \frac{1}{4}(x-1)^4 - (x-1)^2 + 2$$

8.19 problem problem 19

Internal problem ID [434]
Internal file name [`OUTPUT/434_Sunday_June_05_2022_01_41_16_AM_60641738/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Chapter 11 Power series methods. Section 11.2 Power series solutions. Page 624
Problem number: problem 19.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{(-x^2+2x)\,y''-6(x-1)\,y'-4y=0}$$

With initial conditions

$$[y(1)=0,y'(1)=1]$$

With the expansion point for the power series method at $x=1$.

8.19.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y''+p(x)y'+q(x)y=F$$

Where here

$$\begin{aligned} p(x) &= \frac{-6x+6}{-x^2+2x} \\ q(x) &= -\frac{4}{-x^2+2x} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + \frac{(-6x+6)\, y'}{-x^2+2x} - \frac{4y}{-x^2+2x} = 0$$

The domain of $p(x) = \frac{-6x+6}{-x^2+2x}$ is

$$\{-\infty \le x < 0, 0 < x < 2, 2 < x \le \infty\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = -\frac{4}{-x^2+2x}$ is

$$\{-\infty \le x < 0, 0 < x < 2, 2 < x \le \infty\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

The ode does not have its expansion point at $x = 0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t = x - 1$$

The ode is converted to be in terms of the new independent variable t. This results in

$$\left(-(t+1)^2 + 2t + 2\right) \left(\frac{d^2}{dt^2} y(t)\right) - 6t \left(\frac{d}{dt} y(t)\right) - 4y(t) = 0$$

With its expansion point and initial conditions now at $t = 0$. With initial conditions now becoming

$$y(0) = 0$$
$$y'(0) = 1$$

The transformed ODE is now solved. Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the

case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y_0'$. Using Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0, y_0, y_0'} + \frac{x^3}{3!} f'|_{x_0, y_0, y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left. \frac{d^n f}{dx^n} \right|_{x_0, y_0, y_0'}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (128) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (129) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{2\left(3t\left(\frac{d}{dt}y(t)\right)+2y(t)\right)}{t^2-1}\\
F_1 &= \frac{dF_0}{dt}\\
&= \frac{\partial F_0}{\partial t}+\frac{\partial F_0}{\partial y}\frac{d}{dt}y(t)+\frac{\partial F_0}{\partial \frac{d}{dt}y\left(t\right)}F_0\\
&= \frac{38\left(\frac{d}{dt}y(t)\right)t^2+32y(t)\,t+10\frac{d}{dt}y(t)}{\left(t^2-1\right)^2}\\
F_2 &= \frac{dF_1}{dt}\\
&= \frac{\partial F_1}{\partial t}+\frac{\partial F_1}{\partial y}\frac{d}{dt}y(t)+\frac{\partial F_1}{\partial \frac{d}{dt}y\left(t\right)}F_1\\
&= \frac{-272\left(\frac{d}{dt}y(t)\right)t^3-248y(t)\,t^2-208t\left(\frac{d}{dt}y(t)\right)-72y(t)}{\left(t^2-1\right)^3}\\
F_3 &= \frac{dF_2}{dt}\\
&= \frac{\partial F_2}{\partial t}+\frac{\partial F_2}{\partial y}\frac{d}{dt}y(t)+\frac{\partial F_2}{\partial \frac{d}{dt}y\left(t\right)}F_2\\
&= \frac{\left(2200t^4+3280t^2+280\right)\left(\frac{d}{dt}y(t)\right)+\left(2080t^3+1760t\right)y(t)}{\left(t^2-1\right)^4}\\
F_4 &= \frac{dF_3}{dt}\\
&= \frac{\partial F_3}{\partial t}+\frac{\partial F_3}{\partial y}\frac{d}{dt}y(t)+\frac{\partial F_3}{\partial \frac{d}{dt}y\left(t\right)}F_3\\
&= \frac{\left(-19920t^5-48480t^3-12240t\right)\left(\frac{d}{dt}y(t)\right)-19200y(t)\left(t^4+\frac{33}{20}t^2+\frac{3}{20}\right)}{\left(t^2-1\right)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $t=0$ and $y(0)=0$ and $y'(0)=1$ gives

$$
\begin{aligned}
F_0 &= 0\\
F_1 &= 10\\
F_2 &= 0\\
F_3 &= 280\\
F_4 &= 0
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$y(t) = t + \frac{5t^3}{3} + \frac{7t^5}{3} + O(t^6)$$

$$y(t) = t + \frac{5t^3}{3} + \frac{7t^5}{3} + O(t^6)$$

Since the expansion point $t = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(-t^2 + 1)\left(\frac{d^2}{dt^2}y(t)\right) - 6t\left(\frac{d}{dt}y(t)\right) - 4y(t) = 0$$

Let the solution be represented as power series of the form

$$y(t) = \sum_{n=0}^{\infty} a_n t^n$$

Then

$$\frac{d}{dt}y(t) = \sum_{n=1}^{\infty} n a_n t^{n-1}$$
$$\frac{d^2}{dt^2}y(t) = \sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}$$

Substituting the above back into the ode gives

$$(-t^2 + 1)\left(\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}\right) - 6t\left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right) - 4\left(\sum_{n=0}^{\infty} a_n t^n\right) = 0 \tag{1}$$

Which simplifies to

$$\sum_{n=2}^{\infty}(-t^n a_n n(n-1)) + \left(\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}\right) + \sum_{n=1}^{\infty}(-6n a_n t^n) + \sum_{n=0}^{\infty}(-4a_n t^n) = 0 \tag{2}$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, t^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$\sum_{n=2}^{\infty} \left(-t^n a_n n(n-1)\right) + \left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, t^n\right) + \sum_{n=1}^{\infty} \left(-6n a_n t^n\right) + \sum_{n=0}^{\infty} \left(-4a_n t^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 - 4a_0 = 0$$

$$a_2 = 2a_0$$

$n = 1$ gives

$$6a_3 - 10a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = \frac{5a_1}{3}$$

For $2 \leq n$, the recurrence equation is

$$-n a_n(n-1) + (n+2)\, a_{n+2}(n+1) - 6n a_n - 4a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{(n+4)\, a_n}{n+2} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$-18a_2 + 12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = 3a_0$$

For $n = 3$ the recurrence equation gives

$$-28a_3 + 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{7a_1}{3}$$

For $n = 4$ the recurrence equation gives

$$-40a_4 + 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = 4a_0$$

For $n = 5$ the recurrence equation gives

$$-54a_5 + 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = 3a_1$$

And so on. Therefore the solution is

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} a_n t^n \\ &= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y(t) = a_0 + a_1 t + 2a_0 t^2 + \frac{5}{3} a_1 t^3 + 3a_0 t^4 + \frac{7}{3} a_1 t^5 + \dots$$

Collecting terms, the solution becomes

$$y(t) = \left(3t^4 + 2t^2 + 1\right) a_0 + \left(t + \frac{5}{3}t^3 + \frac{7}{3}t^5\right) a_1 + O\left(t^6\right) \tag{3}$$

At $t = 0$ the solution above becomes

$$y(t) = \left(3t^4 + 2t^2 + 1\right) c_1 + \left(t + \frac{5}{3}t^3 + \frac{7}{3}t^5\right) c_2 + O\left(t^6\right)$$

$$y(t) = t + \frac{5t^3}{3} + \frac{7t^5}{3} + O(t^6)$$

Replacing t in the above with the original independent variable $x$using $t = x-1$ results in

$$y = x - 1 + \frac{5(x-1)^3}{3} + \frac{7(x-1)^5}{3} + O\left((x-1)^6\right)$$

Summary
The solution(s) found are the following

$$y = x - 1 + \frac{5(x-1)^3}{3} + \frac{7(x-1)^5}{3} + O\left((x-1)^6\right) \tag{1}$$

Verification of solutions

$$y = x - 1 + \frac{5(x-1)^3}{3} + \frac{7(x-1)^5}{3} + O\left((x-1)^6\right)$$

Verified OK.

8.19.2 Maple step by step solution

Let's solve

$\left[(-x^2+2x)\, y'' + (-6x+6)\, y' - 4y = 0, y(1) = 0, y'\Big|_{\{x=1\}} = 1\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{4y}{x(-2+x)} - \frac{6(x-1)y'}{x(-2+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{6(x-1)y'}{x(-2+x)} + \frac{4y}{x(-2+x)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{6(x-1)}{x(-2+x)}, P_3(x) = \frac{4}{x(-2+x)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

$\left(x \cdot P_2(x)\right)\Big|_{x=0} = 3$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$

- ○ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- • Multiply by denominators

  $y''x(-2+x) + (6x-6)\,y' + 4y = 0$

- • Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions

  - ○ Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  - ○ Shift index using $k- >k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  - ○ Convert $x^m \cdot y''$ to series expansion for $m = 1..2$

    $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

  - ○ Shift index using $k- >k+2-m$

    $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $-2a_0r(2+r)\,x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2a_{k+1}(k+r+1)\,(k+3+r) + a_k(k+r+4)\,(k+r+1)\right) x^{k+r}\right) =$

- • $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r(2+r) = 0$

- • Values of r that satisfy the indicial equation

$r \in \{-2, 0\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r+1)\left((-2k-2r-6)\,a_{k+1}+a_k(k+r+4)\right)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{a_k(k+r+4)}{2(k+3+r)}$

- Recursion relation for $r=-2$

  $a_{k+1}=\frac{a_k(k+2)}{2(k+1)}$

- Solution for $r=-2$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+1}=\frac{a_k(k+2)}{2(k+1)}\right]$

- Recursion relation for $r=0$

  $a_{k+1}=\frac{a_k(k+4)}{2(k+3)}$

- Solution for $r=0$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k}, a_{k+1}=\frac{a_k(k+4)}{2(k+3)}\right]$

- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-2}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k}\right), a_{k+1}=\frac{a_k(k+2)}{2(k+1)}, b_{k+1}=\frac{b_k(k+4)}{2(k+3)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
Order:=6;
dsolve([(2*x-x^2)*diff(y(x),x$2)-6*(x-1)*diff(y(x),x)-4*y(x)=0,y(1) = 0, D(y)(1) = 1],y(x),t
```

$$y(x) = (x-1) + \frac{5}{3}(x-1)^3 + \frac{7}{3}(x-1)^5 + \mathrm{O}\left((x-1)^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 24

```
AsymptoticDSolveValue[{(2*x-x^2)*y''[x]-6*(x-1)*y'[x]-4*y[x]==0,{y[1]==0,y'[1]==1}},y[x],{x,
```

$$y(x) \to \frac{7}{3}(x-1)^5 + \frac{5}{3}(x-1)^3 + x - 1$$

8.20 problem problem 20

Internal problem ID [435]
Internal file name [OUTPUT/435_Sunday_June_05_2022_01_41_18_AM_90937312/index.tex]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Chapter 11 Power series methods. Section 11.2 Power series solutions. Page 624
Problem number: problem 20.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x^2 - 6x + 10)\, y'' - 4(x - 3)\, y' + 6y = 0}$$

With initial conditions

$$[y(3) = 2, y'(3) = 0]$$

With the expansion point for the power series method at $x = 3$.

8.20.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= \frac{-4x + 12}{x^2 - 6x + 10} \\ q(x) &= \frac{6}{x^2 - 6x + 10} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + \frac{(-4x+12)\, y'}{x^2-6x+10} + \frac{6y}{x^2-6x+10} = 0$$

The domain of $p(x) = \frac{-4x+12}{x^2-6x+10}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 3$ is inside this domain. The domain of $q(x) = \frac{6}{x^2-6x+10}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 3$ is also inside this domain. Hence solution exists and is unique.

The ode does not have its expansion point at $x = 0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t = x - 3$$

The ode is converted to be in terms of the new independent variable t. This results in

$$\left((t+3)^2 - 6t - 8\right) \left(\frac{d^2}{dt^2}y(t)\right) - 4t\left(\frac{d}{dt}y(t)\right) + 6y(t) = 0$$

With its expansion point and initial conditions now at $t = 0$. With initial conditions now becoming

$$y(0) = 2$$
$$y'(0) = 0$$

The transformed ODE is now solved. Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the

case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (131) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (132) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= \frac{4t\left(\frac{d}{dt}y(t)\right) - 6y(t)}{t^2+1} \\
F_1 &= \frac{dF_0}{dt} \\
&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_0}{\partial \frac{d}{dt}y\left(t\right)} F_0 \\
&= \frac{6\left(\frac{d}{dt}y(t)\right)t^2 - 12y(t)\,t - 2\frac{d}{dt}y(t)}{\left(t^2+1\right)^2} \\
F_2 &= \frac{dF_1}{dt} \\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_1}{\partial \frac{d}{dt}y\left(t\right)} F_1 \\
&= 0 \\
F_3 &= \frac{dF_2}{dt} \\
&= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_2}{\partial \frac{d}{dt}y\left(t\right)} F_2 \\
&= 0 \\
F_4 &= \frac{dF_3}{dt} \\
&= \frac{\partial F_3}{\partial t} + \frac{\partial F_3}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_3}{\partial \frac{d}{dt}y\left(t\right)} F_3 \\
&= 0
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $t = 0$ and $y(0) = 2$ and $y'(0) = 0$ gives

$$
\begin{aligned}
F_0 &= -12 \\
F_1 &= 0 \\
F_2 &= 0 \\
F_3 &= 0 \\
F_4 &= 0
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y(t) = -6t^2 + 2 + O\left(t^6\right)
$$

$$
y(t) = -6t^2 + 2 + O\left(t^6\right)
$$

Since the expansion point $t = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$\left(\frac{d^2}{dt^2}y(t)\right)(t^2+1) - 4t\left(\frac{d}{dt}y(t)\right) + 6y(t) = 0$$

Let the solution be represented as power series of the form

$$y(t) = \sum_{n=0}^{\infty} a_n t^n$$

Then

$$\frac{d}{dt}y(t) = \sum_{n=1}^{\infty} n a_n t^{n-1}$$
$$\frac{d^2}{dt^2}y(t) = \sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}\right)(t^2+1) - 4t\left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right) + 6\left(\sum_{n=0}^{\infty} a_n t^n\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} t^n a_n n(n-1)\right) + \left(\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}\right) + \sum_{n=1}^{\infty} (-4n a_n t^n) + \left(\sum_{n=0}^{\infty} 6 a_n t^n\right) = 0 \tag{2}$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, t^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} t^n a_n n(n-1)\right) + \left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, t^n\right) + \sum_{n=1}^{\infty} (-4n a_n t^n) + \left(\sum_{n=0}^{\infty} 6 a_n t^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + 6a_0 = 0$$

$$a_2 = -3a_0$$

$n = 1$ gives

$$6a_3 + 2a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{a_1}{3}$$

For $2 \leq n$, the recurrence equation is

$$na_n(n-1) + (n+2)\,a_{n+2}(n+1) - 4na_n + 6a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_n(n^2 - 5n + 6)}{(n+2)\,(n+1)} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = 0$$

For $n = 3$ the recurrence equation gives

$$20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = 0$$

For $n = 4$ the recurrence equation gives

$$2a_4 + 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = 0$$

For $n = 5$ the recurrence equation gives

$$6a_5 + 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = 0$$

And so on. Therefore the solution is

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} a_n t^n \\ &= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y(t) = a_0 + a_1 t - 3a_0 t^2 - \frac{1}{3} a_1 t^3 + \dots$$

Collecting terms, the solution becomes

$$y(t) = \left(-3t^2 + 1\right) a_0 + \left(t - \frac{1}{3} t^3\right) a_1 + O\left(t^6\right) \tag{3}$$

At $t = 0$ the solution above becomes

$$y(t) = \left(-3t^2 + 1\right) c_1 + \left(t - \frac{1}{3} t^3\right) c_2 + O\left(t^6\right)$$

$$y(t) = -6t^2 + 2 + O\left(t^6\right)$$

Replacing t in the above with the original independent variable $x$susing $t = x - 3$ results in

$$y = -6(x-3)^2 + 2 + O\left((x-3)^6\right)$$

Summary

The solution(s) found are the following

$$y = -6(x-3)^2 + 2 + O\left((x-3)^6\right) \tag{1}$$

Verification of solutions

$$y = -6(x-3)^2 + 2 + O\left((x-3)^6\right)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
Order:=6;
dsolve([(x^2-6*x+10)*diff(y(x),x$2)-4*(x-3)*diff(y(x),x)+6*y(x)=0,y(3) = 2, D(y)(3) = 0],y(x
```

$$y(x) = -6x^2 + 36x - 52$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 12

```
AsymptoticDSolveValue[{(x^2-6*x+10)*y''[x]-4*(x-3)*y'[x]+6*y[x]==0,{y[3]==2,y'[3]==0}},y[x],
```

$$y(x) \to 2 - 6(x-3)^2$$

8.21 problem problem 21

Internal problem ID [436]
Internal file name [`OUTPUT/436_Sunday_June_05_2022_01_41_21_AM_15042728/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Chapter 11 Power series methods. Section 11.2 Power series solutions. Page 624
Problem number: problem 21.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{\left(4x^2+16x+17\right) y''-8y=0}$$

With initial conditions

$$[y(-2)=1, y'(-2)=0]$$

With the expansion point for the power series method at $x=-2$.

8.21.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y''+p(x)y'+q(x)y=F$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= -\frac{8}{4x^2+16x+17} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - \frac{8y}{4x^2 + 16x + 17} = 0$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = -2$ is inside this domain. The domain of $q(x) = -\frac{8}{4x^2+16x+17}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = -2$ is also inside this domain. Hence solution exists and is unique.

The ode does not have its expansion point at $x = 0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t = 2 + x$$

The ode is converted to be in terms of the new independent variable t. This results in

$$\left(4(t-2)^2 + 16t - 15\right)\left(\frac{d^2}{dt^2}y(t)\right) - 8y(t) = 0$$

With its expansion point and initial conditions now at $t = 0$. With initial conditions now becoming

$$y(0) = 1$$
$$y'(0) = 0$$

The transformed ODE is now solved. Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the

case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0,y_0,y'_0}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (134) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (135) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= \frac{8y(t)}{4t^2+1} \\
F_1 &= \frac{dF_0}{dt} \\
&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_0}{\partial \frac{d}{dt}y(t)}F_0 \\
&= \frac{32\left(\frac{d}{dt}y(t)\right)t^2 - 64y(t)\,t + 8\frac{d}{dt}y(t)}{\left(4t^2+1\right)^2} \\
F_2 &= \frac{dF_1}{dt} \\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_1}{\partial \frac{d}{dt}y(t)}F_1 \\
&= \frac{128\left(-4\left(\frac{d}{dt}y(t)\right)t^2 + 8y(t)\,t - \frac{d}{dt}y(t)\right)t}{\left(4t^2+1\right)^3} \\
F_3 &= \frac{dF_2}{dt} \\
&= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_2}{\partial \frac{d}{dt}y(t)}F_2 \\
&= \frac{10240\left(\left(t^2+\frac{1}{4}\right)\left(\frac{d}{dt}y(t)\right) - 2y(t)\,t\right)\left(t^2-\frac{1}{20}\right)}{\left(4t^2+1\right)^4} \\
F_4 &= \frac{dF_3}{dt} \\
&= \frac{\partial F_3}{\partial t} + \frac{\partial F_3}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_3}{\partial \frac{d}{dt}y(t)}F_3 \\
&= -\frac{245760\left(\left(t^2+\frac{1}{4}\right)\left(\frac{d}{dt}y(t)\right) - 2y(t)\,t\right)t\left(t^2-\frac{3}{20}\right)}{\left(4t^2+1\right)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $t=0$ and $y(0)=1$ and $y'(0)=0$ gives

$$
\begin{aligned}
F_0 &= 8 \\
F_1 &= 0 \\
F_2 &= 0 \\
F_3 &= 0 \\
F_4 &= 0
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y(t) = 4t^2 + 1 + O\left(t^6\right)
$$

$$y(t) = 4t^2 + 1 + O(t^6)$$

Since the expansion point $t = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$\left(\frac{d^2}{dt^2}y(t)\right)(4t^2+1) - 8y(t) = 0$$

Let the solution be represented as power series of the form

$$y(t) = \sum_{n=0}^{\infty} a_n t^n$$

Then

$$\frac{d}{dt}y(t) = \sum_{n=1}^{\infty} n a_n t^{n-1}$$
$$\frac{d^2}{dt^2}y(t) = \sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}\right)(4t^2+1) - 8\left(\sum_{n=0}^{\infty} a_n t^n\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} 4t^n a_n n(n-1)\right) + \left(\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}\right) + \sum_{n=0}^{\infty} (-8a_n t^n) = 0 \tag{2}$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, t^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} 4t^n a_n n(n-1)\right) + \left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, t^n\right) + \sum_{n=0}^{\infty} (-8a_n t^n) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 - 8a_0 = 0$$

$$a_2 = 4a_0$$

$n = 1$ gives

$$6a_3 - 8a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = \frac{4a_1}{3}$$

For $2 \leq n$, the recurrence equation is

$$4na_n(n-1) + (n+2)\,a_{n+2}(n+1) - 8a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{4(n-2)\,a_n}{n+2} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = 0$$

For $n = 3$ the recurrence equation gives

$$16a_3 + 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{16a_1}{15}$$

For $n = 4$ the recurrence equation gives

$$40a_4 + 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = 0$$

For $n = 5$ the recurrence equation gives

$$72a_5 + 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{64a_1}{35}$$

And so on. Therefore the solution is

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} a_n t^n \\ &= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y(t) = a_0 + a_1 t + 4a_0 t^2 + \frac{4}{3} a_1 t^3 - \frac{16}{15} a_1 t^5 + \dots$$

Collecting terms, the solution becomes

$$y(t) = (4t^2 + 1)\, a_0 + \left(t + \frac{4}{3} t^3 - \frac{16}{15} t^5\right) a_1 + O(t^6) \tag{3}$$

At $t = 0$ the solution above becomes

$$y(t) = (4t^2 + 1)\, c_1 + \left(t + \frac{4}{3} t^3 - \frac{16}{15} t^5\right) c_2 + O(t^6)$$

$$y(t) = 4t^2 + 1 + O(t^6)$$

Replacing t in the above with the original independent variable $x$using $t = 2 + x$ results in

$$y = 4(2 + x)^2 + 1 + O\left((2 + x)^6\right)$$

Summary
The solution(s) found are the following

$$y = 4(2 + x)^2 + 1 + O\left((2 + x)^6\right) \tag{1}$$

Verification of solutions

$$y = 4(2 + x)^2 + 1 + O\left((2 + x)^6\right)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 13

```
Order:=6;
dsolve([(4*x^2+16*x+17)*diff(y(x),x$2)=8*y(x),y(-2) = 1, D(y)(-2) = 0],y(x),type='series',x=
```

$$y(x) = 4x^2 + 16x + 17$$

✓ Solution by Mathematica
Time used: 0.001 (sec). Leaf size: 12

```
AsymptoticDSolveValue[{(4*x^2+16*x+17)*y''[x]==8*y[x],{y[-2]==1,y'[-2]==0}},y[x],{x,-2,5}]
```

$$y(x) \to 4(x + 2)^2 + 1$$

8.22 problem problem 22

Internal problem ID [437]
Internal file name [`OUTPUT/437_Sunday_June_05_2022_01_41_23_AM_37533081/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Chapter 11 Power series methods. Section 11.2 Power series solutions. Page 624
Problem number: problem 22.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second_order_change_of_variable_on_y_method_2", "second order series method. Taylor series method", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x^2+6x)\,y''+(3x+9)\,y'-3y=0}$$

With initial conditions

$$[y(-3)=1, y'(-3)=0]$$

With the expansion point for the power series method at $x=-3$.

8.22.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y''+p(x)y'+q(x)y=F$$

Where here

$$\begin{aligned} p(x) &= \frac{3x+9}{x^2+6x} \\ q(x) &= -\frac{3}{x^2+6x} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + \frac{(3x+9)\,y'}{x^2+6x} - \frac{3y}{x^2+6x} = 0$$

The domain of $p(x) = \frac{3x+9}{x^2+6x}$ is

$$\{-\infty \le x < -6, -6 < x < 0, 0 < x \le \infty\}$$

And the point $x_0 = -3$ is inside this domain. The domain of $q(x) = -\frac{3}{x^2+6x}$ is

$$\{-\infty \le x < -6, -6 < x < 0, 0 < x \le \infty\}$$

And the point $x_0 = -3$ is also inside this domain. Hence solution exists and is unique.

The ode does not have its expansion point at $x = 0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t = x + 3$$

The ode is converted to be in terms of the new independent variable t. This results in

$$\left((t-3)^2 + 6t - 18\right)\left(\frac{d^2}{dt^2}y(t)\right) + 3t\left(\frac{d}{dt}y(t)\right) - 3y(t) = 0$$

With its expansion point and initial conditions now at $t = 0$. With initial conditions now becoming

$$\begin{aligned} y(0) &= 1 \\ y'(0) &= 0 \end{aligned}$$

The transformed ODE is now solved. Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the

case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0,y_0,y'_0}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (137) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (138) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\ \vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{3\left(t\left(\frac{d}{dt}y(t)\right) - y(t)\right)}{t^2 - 9} \\
F_1 &= \frac{dF_0}{dt} \\
&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_0}{\partial \frac{d}{dt}y\left(t\right)}F_0 \\
&= \frac{15\left(t\left(\frac{d}{dt}y(t)\right) - y(t)\right)t}{\left(t^2 - 9\right)^2} \\
F_2 &= \frac{dF_1}{dt} \\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_1}{\partial \frac{d}{dt}y\left(t\right)}F_1 \\
&= -\frac{90\left(t\left(\frac{d}{dt}y(t)\right) - y(t)\right)\left(t^2 + \frac{3}{2}\right)}{\left(t^2 - 9\right)^3} \\
F_3 &= \frac{dF_2}{dt} \\
&= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_2}{\partial \frac{d}{dt}y\left(t\right)}F_2 \\
&= \frac{630\left(t^2 + \frac{9}{2}\right)\left(t\left(\frac{d}{dt}y(t)\right) - y(t)\right)t}{\left(t^2 - 9\right)^4} \\
F_4 &= \frac{dF_3}{dt} \\
&= \frac{\partial F_3}{\partial t} + \frac{\partial F_3}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_3}{\partial \frac{d}{dt}y\left(t\right)}F_3 \\
&= -\frac{5040\left(t\left(\frac{d}{dt}y(t)\right) - y(t)\right)\left(t^4 + 9t^2 + \frac{81}{16}\right)}{\left(t^2 - 9\right)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $t = 0$ and $y(0) = 1$ and $y'(0) = 0$ gives

$$
\begin{aligned}
F_0 &= -\frac{1}{3} \\
F_1 &= 0 \\
F_2 &= -\frac{5}{27} \\
F_3 &= 0 \\
F_4 &= -\frac{35}{81}
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$y(t) = 1 - \frac{t^2}{6} - \frac{5t^4}{648} - \frac{7t^6}{11664} + O(t^6)$$

$$y(t) = 1 - \frac{t^2}{6} - \frac{5t^4}{648} - \frac{7t^6}{11664} + O(t^6)$$

Since the expansion point $t = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$\left(\frac{d^2}{dt^2}y(t)\right)(t^2 - 9) + 3t\left(\frac{d}{dt}y(t)\right) - 3y(t) = 0$$

Let the solution be represented as power series of the form

$$y(t) = \sum_{n=0}^{\infty} a_n t^n$$

Then

$$\frac{d}{dt}y(t) = \sum_{n=1}^{\infty} n a_n t^{n-1}$$
$$\frac{d^2}{dt^2}y(t) = \sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}\right)(t^2 - 9) + 3t\left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right) - 3\left(\sum_{n=0}^{\infty} a_n t^n\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} t^n a_n n(n-1)\right) + \sum_{n=2}^{\infty}\left(-9n(n-1)\, a_n t^{n-2}\right) + \left(\sum_{n=1}^{\infty} 3n a_n t^n\right) + \sum_{n=0}^{\infty}(-3a_n t^n) = 0 \tag{2}$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty}\left(-9n(n-1)\, a_n t^{n-2}\right) = \sum_{n=0}^{\infty}\left(-9(n+2)\, a_{n+2}(n+1)\, t^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} t^n a_n n(n-1)\right)+\sum_{n=0}^{\infty}\left(-9(n+2)\, a_{n+2}(n+1)\, t^n\right)+\left(\sum_{n=1}^{\infty} 3n a_n t^n\right)+\sum_{n=0}^{\infty}\left(-3a_n t^n\right)=0 \tag{3}$$

$n = 0$ gives

$$-18a_2 - 3a_0 = 0$$

$$a_2 = -\frac{a_0}{6}$$

For $2 \leq n$, the recurrence equation is

$$na_n(n-1) - 9(n+2)\, a_{n+2}(n+1) + 3na_n - 3a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{a_n(n^2 + 2n - 3)}{9\,(n+2)\,(n+1)} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$5a_2 - 108a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{5a_0}{648}$$

For $n = 3$ the recurrence equation gives

$$12a_3 - 180a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = 0$$

For $n = 4$ the recurrence equation gives

$$21a_4 - 270a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{7a_0}{11664}$$

For $n = 5$ the recurrence equation gives

$$32a_5 - 378a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = 0$$

And so on. Therefore the solution is

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} a_n t^n \\ &= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y(t) = a_0 + a_1 t - \frac{1}{6} a_0 t^2 - \frac{5}{648} a_0 t^4 + \dots$$

Collecting terms, the solution becomes

$$y(t) = \left(1 - \frac{1}{6}t^2 - \frac{5}{648}t^4\right) a_0 + a_1 t + O(t^6) \tag{3}$$

At $t = 0$ the solution above becomes

$$y(t) = \left(1 - \frac{1}{6}t^2 - \frac{5}{648}t^4\right) c_1 + c_2 t + O(t^6)$$

$$y(t) = 1 - \frac{t^2}{6} - \frac{5t^4}{648} + O(t^6)$$

Replacing t in the above with the original independent variable $x$using $t = x + 3$ results in

$$y = 1 - \frac{(x+3)^2}{6} - \frac{5(x+3)^4}{648} - \frac{7(x+3)^6}{11664} + O\big((x+3)^6\big)$$

Summary

The solution(s) found are the following

$$y = 1 - \frac{(x+3)^2}{6} - \frac{5(x+3)^4}{648} - \frac{7(x+3)^6}{11664} + O\big((x+3)^6\big) \tag{1}$$

Verification of solutions

$$y = 1 - \frac{(x+3)^2}{6} - \frac{5(x+3)^4}{648} - \frac{7(x+3)^6}{11664} + O\big((x+3)^6\big)$$

Verified OK.

8.22.2 Maple step by step solution

Let's solve

$$\left[(x^2+6x)\,y'' + (3x+9)\,y' - 3y = 0, y(-3) = 1, y'\Big|_{\{x=-3\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{3y}{x(x+6)} - \frac{3(x+3)y'}{x(x+6)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{3(x+3)y'}{x(x+6)} - \frac{3y}{x(x+6)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{3(x+3)}{x(x+6)}, P_3(x) = -\frac{3}{x(x+6)}\right]$

  ◦ $(x+6)\cdot P_2(x)$ is analytic at $x=-6$

    $\left((x+6)\cdot P_2(x)\right)\Big|_{x=-6} = \frac{3}{2}$

  ◦ $(x+6)^2\cdot P_3(x)$ is analytic at $x=-6$

$$\left((x+6)^2 \cdot P_3(x)\right)\Big|_{x=-6} = 0$$

- $x = -6$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -6$

- Multiply by denominators

  $y''x(x+6) + (3x+9)\, y' - 3y = 0$

- Change variables using $x = u - 6$ so that the regular singular point is at $u = 0$

  $(u^2 - 6u)\left(\frac{d^2}{du^2}y(u)\right) + (3u - 9)\left(\frac{d}{du}y(u)\right) - 3y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$

- Shift index using $k- >k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $-3a_0 r(1+2r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-3a_{k+1}(k+1+r)(2k+3+2r) + a_k(k+r+3)(k+r-1)\right) u^{k+r}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-3r(1+2r) = 0$

- Values of r that satisfy the indicial equation

$r \in \left\{0, -\frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $-6(k+1+r)\left(k+\frac{3}{2}+r\right) a_{k+1} + a_k(k+r+3)(k+r-1) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r+3)(k+r-1)}{3(k+1+r)(2k+3+2r)}$

- Recursion relation for $r = 0$; series terminates at $k = 1$

  $a_{k+1} = \frac{a_k(k+3)(k-1)}{3(k+1)(2k+3)}$

- Apply recursion relation for $k = 0$

  $a_1 = -\frac{a_0}{3}$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot \left(1 - \frac{u}{3}\right)$

- Revert the change of variables $u = x + 6$

  $\left[y = a_0\left(-1 - \frac{x}{3}\right)\right]$

- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+1} = \frac{a_k\left(k+\frac{5}{2}\right)\left(k-\frac{3}{2}\right)}{3\left(k+\frac{1}{2}\right)(2k+2)}$

- Solution for $r = -\frac{1}{2}$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{1}{2}}, a_{k+1} = \frac{a_k\left(k+\frac{5}{2}\right)\left(k-\frac{3}{2}\right)}{3\left(k+\frac{1}{2}\right)(2k+2)}\right]$

- Revert the change of variables $u = x + 6$

  $\left[y = \sum_{k=0}^{\infty} a_k(x+6)^{k-\frac{1}{2}}, a_{k+1} = \frac{a_k\left(k+\frac{5}{2}\right)\left(k-\frac{3}{2}\right)}{3\left(k+\frac{1}{2}\right)(2k+2)}\right]$

- Combine solutions and rename parameters

  $\left[y = a_0\left(-1 - \frac{x}{3}\right) + \left(\sum_{k=0}^{\infty} b_k(x+6)^{k-\frac{1}{2}}\right), b_{k+1} = \frac{b_k\left(k+\frac{5}{2}\right)\left(k-\frac{3}{2}\right)}{3\left(k+\frac{1}{2}\right)(2k+2)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
Order:=6;
dsolve([(x^2+6*x)*diff(y(x),x$2)+(3*x+9)*diff(y(x),x)-3*y(x)=0,y(-3) = 1, D(y)(-3) = 0],y(x)
```

$$y(x) = 1 - \frac{1}{6}(x+3)^2 - \frac{5}{648}(x+3)^4 + \mathrm{O}\left((x+3)^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 23

```
AsymptoticDSolveValue[{(x^2+6*x)*y''[x]+(3*x+9)*y'[x]-3*y[x]==0,{y[-3]==1,y'[-3]==0}},y[x],{
```

$$y(x) \to -\frac{5}{648}(x+3)^4 - \frac{1}{6}(x+3)^2 + 1$$

8.23 problem problem 23

Internal problem ID [438]
Internal file name [`OUTPUT/438_Sunday_June_05_2022_01_41_25_AM_63224170/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Chapter 11 Power series methods. Section 11.2 Power series solutions. Page 624
Problem number: problem 23.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second__order__bessel_ode", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + (x+1)\, y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (140)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (141)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n\big|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -(x+1)\,y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= (-x-1)\,y' - y \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= -2y' + \left(x^2+2x+1\right) y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= ((x+1)\,y' + 4y)\,(x+1) \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= -(x+1)^3\,y + (6x+6)\,y' + 4y
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and $y'(0)=y'(0)$ gives

$$
\begin{aligned}
F_0 &= -y(0) \\
F_1 &= -y'(0) - y(0) \\
F_2 &= -2y'(0) + y(0) \\
F_3 &= y'(0) + 4y(0) \\
F_4 &= 3y(0) + 6y'(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y = {} & \left(1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{30}x^5 + \frac{1}{240}x^6\right) y(0) \\
& + \left(x - \frac{1}{6}x^3 - \frac{1}{12}x^4 + \frac{1}{120}x^5 + \frac{1}{120}x^6\right) y'(0) + O\left(x^6\right)
\end{aligned}
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = -(x+1) \left(\sum_{n=0}^{\infty} a_n x^n \right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \right) + \left(\sum_{n=0}^{\infty} x^{1+n} a_n \right) + \left(\sum_{n=0}^{\infty} a_n x^n \right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2) a_{n+2}(1+n) x^n$$

$$\sum_{n=0}^{\infty} x^{1+n} a_n = \sum_{n=1}^{\infty} a_{n-1} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2) a_{n+2}(1+n) x^n \right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^n \right) + \left(\sum_{n=0}^{\infty} a_n x^n \right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + a_0 = 0$$

$$a_2 = -\frac{a_0}{2}$$

For $1 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(1+n) + a_{n-1} + a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -\frac{a_{n-1} + a_n}{(n+2)\,(1+n)} \\ &= -\frac{a_n}{(n+2)\,(1+n)} - \frac{a_{n-1}}{(n+2)\,(1+n)} \end{aligned} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 + a_0 + a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{a_0}{6} - \frac{a_1}{6}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + a_1 + a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{a_1}{12} + \frac{a_0}{24}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + a_2 + a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_0}{30} + \frac{a_1}{120}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + a_3 + a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0}{240} + \frac{a_1}{120}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + a_4 + a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{a_1}{560} - \frac{a_0}{560}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{a_0 x^2}{2} + \left(-\frac{a_0}{6} - \frac{a_1}{6}\right) x^3 + \left(-\frac{a_1}{12} + \frac{a_0}{24}\right) x^4 + \left(\frac{a_0}{30} + \frac{a_1}{120}\right) x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{30}x^5\right) a_0 + \left(x - \frac{1}{6}x^3 - \frac{1}{12}x^4 + \frac{1}{120}x^5\right) a_1 + O(x^6) \quad (3)$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{30}x^5\right) c_1 + \left(x - \frac{1}{6}x^3 - \frac{1}{12}x^4 + \frac{1}{120}x^5\right) c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = \left(1-\frac{1}{2}x^{2}-\frac{1}{6}x^{3}+\frac{1}{24}x^{4}+\frac{1}{30}x^{5}+\frac{1}{240}x^{6}\right) y(0) + \left(x-\frac{1}{6}x^{3}-\frac{1}{12}x^{4}+\frac{1}{120}x^{5}+\frac{1}{120}x^{6}\right) y'(0)+O\left(x^{6}\right) \tag{1}$$

$$y = \left(1-\frac{1}{2}x^{2}-\frac{1}{6}x^{3}+\frac{1}{24}x^{4}+\frac{1}{30}x^{5}\right) c_{1}+\left(x-\frac{1}{6}x^{3}-\frac{1}{12}x^{4}+\frac{1}{120}x^{5}\right) c_{2}+O\left(x^{6}\right) \tag{2}$$

Verification of solutions

$$y = \left(1-\frac{1}{2}x^{2}-\frac{1}{6}x^{3}+\frac{1}{24}x^{4}+\frac{1}{30}x^{5}+\frac{1}{240}x^{6}\right) y(0) + \left(x-\frac{1}{6}x^{3}-\frac{1}{12}x^{4}+\frac{1}{120}x^{5}+\frac{1}{120}x^{6}\right) y'(0)+O\left(x^{6}\right)$$

Verified OK.

$$y = \left(1-\frac{1}{2}x^{2}-\frac{1}{6}x^{3}+\frac{1}{24}x^{4}+\frac{1}{30}x^{5}\right) c_{1}+\left(x-\frac{1}{6}x^{3}-\frac{1}{12}x^{4}+\frac{1}{120}x^{5}\right) c_{2}+O\left(x^{6}\right)$$

Verified OK.

8.23.1 Maple step by step solution

Let's solve

$y'' = -(x+1)\, y$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = (-x-1)\, y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + (x+1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

$$x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$$

- Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$$

- Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k- >k + 2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$$

Rewrite ODE with series expansions

$$2a_2 + a_0 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_k + a_{k-1}\right) x^k\right) = 0$$

- Each term must be 0

$$2a_2 + a_0 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$\left(k^2 + 3k + 2\right) a_{k+2} + a_k + a_{k-1} = 0$$

- Shift index using $k- >k + 1$

$$\left((k+1)^2 + 3k + 5\right) a_{k+3} + a_{k+1} + a_k = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = -\frac{a_{k+1} + a_k}{k^2 + 5k + 6}, 2a_2 + a_0 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 49

```
Order:=6;
dsolve(diff(y(x),x$2)+(1+x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{30}x^5\right) y(0) + \left(x - \frac{1}{6}x^3 - \frac{1}{12}x^4 + \frac{1}{120}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 63

```
AsymptoticDSolveValue[y''[x]+(1+x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{x^5}{120} - \frac{x^4}{12} - \frac{x^3}{6} + x\right) + c_1\left(\frac{x^5}{30} + \frac{x^4}{24} - \frac{x^3}{6} - \frac{x^2}{2} + 1\right)$$

8.24 problem problem 24

Internal problem ID [439]
Internal file name [`OUTPUT/439_Sunday_June_05_2022_01_41_26_AM_71221353/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Chapter 11 Power series methods. Section 11.2 Power series solutions. Page 624
Problem number: problem 24.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{\left(x^2-1\right) y''+2y'x+2yx=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots \\
&= y_0+xy'_0+\frac{x^2}{2}\,f|_{x_0,y_0,y'_0}+\frac{x^3}{3!}\,f'|_{x_0,y_0,y'_0}+\cdots \\
&= y_0+xy'_0+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}\,\frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (143)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (144)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{2x(y' + y)}{x^2 - 1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{\left(-2x^3 + 6x^2 + 2x + 2\right) y' + \left(6x^2 + 2\right) y}{\left(x^2 - 1\right)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{\left(12x^4 - 24x^3 - 8x^2 - 24x - 4\right) y' + 4xy(x^3 - 6x^2 - x - 6)}{\left(x^2 - 1\right)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{\left(4x^6 - 72x^5 + 112x^4 + 244x^2 + 72x + 24\right) y' - 32\left(x^5 - \frac{15}{4}x^4 - \frac{1}{2}x^3 - \frac{15}{2}x^2 - \frac{1}{2}x - \frac{3}{4}\right) y}{\left(x^2 - 1\right)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{\left(-48x^7 + 480x^6 - 648x^5 + 480x^4 - 2400x^3 - 864x^2 - 744x - 96\right) y' - 8y(x^7 - 30x^6 + 88x^5 - 10x^4}{\left(x^2 - 1\right)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= 2y'(0) + 2y(0) \\
F_2 &= 4y'(0) \\
F_3 &= 24y(0) + 24y'(0) \\
F_4 &= 16y(0) + 96y'(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \frac{1}{45}x^6\right) y(0) + \left(x + \frac{1}{3}x^3 + \frac{1}{6}x^4 + \frac{1}{5}x^5 + \frac{2}{15}x^6\right) y'(0) + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(x^2 - 1)\, y'' + 2y'x + 2yx = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$(x^2 - 1) \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + 2\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x + 2\left(\sum_{n=0}^{\infty} a_n x^n\right) x = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \sum_{n\,=2}^{\infty} \left(-n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} 2n a_n x^n\right) + \left(\sum_{n=0}^{\infty} 2x^{1+n} a_n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n\,=2}^{\infty} \left(-n(n-1)\, a_n x^{n-2}\right) = \sum_{n=0}^{\infty} \left(-(n+2)\, a_{n+2}(1+n)\, x^n\right)$$
$$\sum_{n\,=0}^{\infty} 2x^{1+n} a_n = \sum_{n=1}^{\infty} 2a_{n-1} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} x^{n} a_{n} n(n-1)\right)+\sum_{n=0}^{\infty}\left(-(n+2) a_{n+2}(1+n) x^{n}\right) \\ +\left(\sum_{n=1}^{\infty} 2 n a_{n} x^{n}\right)+\left(\sum_{n=1}^{\infty} 2 a_{n-1} x^{n}\right)=0 \tag{3}$$

$n=1$ gives

$$-6a_3+2a_1+2a_0=0$$

Which after substituting earlier equations, simplifies to

$$a_3=\frac{a_0}{3}+\frac{a_1}{3}$$

For $2 \leq n$, the recurrence equation is

$$n a_n(n-1)-(n+2) a_{n+2}(1+n)+2 n a_n+2 a_{n-1}=0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= \frac{n^2 a_n+n a_n+2 a_{n-1}}{(n+2)(1+n)} \\ &= \frac{(n^2+n) a_n}{(n+2)(1+n)}+\frac{2 a_{n-1}}{(n+2)(1+n)} \end{aligned} \tag{5}$$

For $n=2$ the recurrence equation gives

$$6a_2-12a_4+2a_1=0$$

Which after substituting the earlier terms found becomes

$$a_4=\frac{a_1}{6}$$

For $n=3$ the recurrence equation gives

$$12a_3-20a_5+2a_2=0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_0}{5} + \frac{a_1}{5}$$

For $n = 4$ the recurrence equation gives

$$20a_4 - 30a_6 + 2a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{2a_1}{15} + \frac{a_0}{45}$$

For $n = 5$ the recurrence equation gives

$$30a_5 - 42a_7 + 2a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{a_0}{7} + \frac{19a_1}{126}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \left(\frac{a_0}{3} + \frac{a_1}{3}\right) x^3 + \frac{a_1 x^4}{6} + \left(\frac{a_0}{5} + \frac{a_1}{5}\right) x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + \frac{1}{3}x^3 + \frac{1}{5}x^5\right) a_0 + \left(x + \frac{1}{3}x^3 + \frac{1}{6}x^4 + \frac{1}{5}x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 + \frac{1}{3}x^3 + \frac{1}{5}x^5\right) c_1 + \left(x + \frac{1}{3}x^3 + \frac{1}{6}x^4 + \frac{1}{5}x^5\right) c_2 + O(x^6)$$

Summary
The solution(s) found are the following

$$y = \left(1+\frac{1}{3}x^{3}+\frac{1}{5}x^{5}+\frac{1}{45}x^{6}\right) y(0)+\left(x+\frac{1}{3}x^{3}+\frac{1}{6}x^{4}+\frac{1}{5}x^{5}+\frac{2}{15}x^{6}\right) y'(0)+O\left(x^{6}\right) \tag{1}$$

$$y = \left(1+\frac{1}{3}x^{3}+\frac{1}{5}x^{5}\right) c_{1}+\left(x+\frac{1}{3}x^{3}+\frac{1}{6}x^{4}+\frac{1}{5}x^{5}\right) c_{2}+O\left(x^{6}\right) \tag{2}$$

Verification of solutions

$$y = \left(1+\frac{1}{3}x^{3}+\frac{1}{5}x^{5}+\frac{1}{45}x^{6}\right) y(0)+\left(x+\frac{1}{3}x^{3}+\frac{1}{6}x^{4}+\frac{1}{5}x^{5}+\frac{2}{15}x^{6}\right) y'(0)+O\left(x^{6}\right)$$

Verified OK.

$$y = \left(1+\frac{1}{3}x^{3}+\frac{1}{5}x^{5}\right) c_{1}+\left(x+\frac{1}{3}x^{3}+\frac{1}{6}x^{4}+\frac{1}{5}x^{5}\right) c_{2}+O\left(x^{6}\right)$$

Verified OK.

8.24.1 Maple step by step solution

Let's solve

$(x^2-1)\, y''+2y'x+2yx=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{2xy}{x^2-1}-\frac{2xy'}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{2xy'}{x^2-1}+\frac{2xy}{x^2-1}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{2x}{x^2-1}, P_3(x)=\frac{2x}{x^2-1}\right]$

  ◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

    $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1}=1$

  ◦ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

$$\left((x+1)^2 \cdot P_3(x)\right) \bigg|_{x=-1} = 0$$

- $x = -1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $(x^2-1)\, y'' + 2y'x + 2yx = 0$

- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

  $(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right) + (2u-2)\left(\frac{d}{du}y(u)\right) + (2u-2)\, y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

  $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

- Shift index using $k- >k-m$

  $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, u^{k+r-2+m}$

- Shift index using $k- >k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

$$-2a_0r^2u^{-1+r}+\left(-2a_1(1+r)^2+a_0(2+r)\left(-1+r\right)\right)u^r+\left(\sum_{k=1}^{\infty}\left(-2a_{k+1}(k+1+r)^2+a_k(k+r\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$-2r^2=0$$

- Values of r that satisfy the indicial equation

  $$r=0$$

- Each term must be 0

  $$-2a_1(1+r)^2+a_0(2+r)\left(-1+r\right)=0$$

- Each term in the series must be 0, giving the recursion relation

  $$-2a_{k+1}(k+1)^2+\left(k^2+k-2\right)a_k+2a_{k-1}=0$$

- Shift index using $k->k+1$

  $$-2a_{k+2}(k+2)^2+\left((k+1)^2+k-1\right)a_{k+1}+2a_k=0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2}=\frac{k^2a_{k+1}+3ka_{k+1}+2a_k}{2(k+2)^2}$$

- Recursion relation for $r=0$

  $$a_{k+2}=\frac{k^2a_{k+1}+3ka_{k+1}+2a_k}{2(k+2)^2}$$

- Solution for $r=0$

  $$\left[y(u)=\sum_{k=0}^{\infty}a_ku^k,a_{k+2}=\frac{k^2a_{k+1}+3ka_{k+1}+2a_k}{2(k+2)^2},-2a_1-2a_0=0\right]$$

- Revert the change of variables $u=x+1$

  $$\left[y=\sum_{k=0}^{\infty}a_k(x+1)^k,a_{k+2}=\frac{k^2a_{k+1}+3ka_{k+1}+2a_k}{2(k+2)^2},-2a_1-2a_0=0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
<- Heun successful: received ODE is equivalent to the  HeunC  ODE, case  a <> 0, e <> 0, c =
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 39

```
Order:=6;
dsolve((x^2-1)*diff(y(x),x$2)+2*x*diff(y(x),x)+2*x*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1+\frac{1}{3}x^3+\frac{1}{5}x^5\right)y(0)+\left(x+\frac{1}{3}x^3+\frac{1}{6}x^4+\frac{1}{5}x^5\right)D(y)(0)+O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 49

```
AsymptoticDSolveValue[(x^2+1)*y''[x]+2*x*y'[x]+2*x*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(\frac{x^5}{5} - \frac{x^3}{3} + 1 \right) + c_2 \left(\frac{x^5}{5} - \frac{x^4}{6} - \frac{x^3}{3} + x \right)$$

8.25 problem problem 25

Internal problem ID [440]
Internal file name [OUTPUT/440_Sunday_June_05_2022_01_41_27_AM_62848587/index.tex]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Chapter 11 Power series methods. Section 11.2 Power series solutions. Page 624
Problem number: problem 25.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + y'x^2 + x^2y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + xy'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\ &= y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0} \end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (146) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (147) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -y'x^2 - x^2 y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \left(\left(x^3 - x - 2\right) y' + y\left(x^3 - 2\right)\right) x \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \left(-x^6 + 2x^4 + 6x^3 - 4x - 2\right) y' - y\left(x^6 - x^4 - 6x^3 + 2\right) \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \left(x^8 - 3x^6 - 12x^5 + x^4 + 18x^3 + 20x^2 - 6\right) y' + x^2 y\left(x^6 - 2x^4 - 12x^3 + 8x + 20\right) \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= -\left(\left(x^9 - 4x^7 - 20x^6 + 3x^5 + 48x^4 + 80x^3 - 12x^2 - 80x - 40\right) y' + y\left(x^9 - 3x^7 - 20x^6 + x^5 + 30x^4 -
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= 0 \\
F_2 &= -2y'(0) - 2y(0) \\
F_3 &= -6y'(0) \\
F_4 &= 0
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 - \frac{x^4}{12}\right) y(0) + \left(x - \frac{1}{12}x^4 - \frac{1}{20}x^5\right) y'(0) + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard

power series Let the solution be represented as power series of the form

$$y=\sum_{n=0}^{\infty} a_n x^n$$

Then

$$y'=\sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y''=\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}=-\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x^2-x^2\left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right)+\left(\sum_{n=1}^{\infty} n\, x^{1+n} a_n\right)+\left(\sum_{n=0}^{\infty} x^{n+2} a_n\right)=0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}=\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n$$
$$\sum_{n=1}^{\infty} n\, x^{1+n} a_n=\sum_{n=2}^{\infty} (n-1)\, a_{n-1} x^n$$
$$\sum_{n=0}^{\infty} x^{n+2} a_n=\sum_{n=2}^{\infty} a_{n-2} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n\right)+\left(\sum_{n=2}^{\infty} (n-1)\, a_{n-1} x^n\right)+\left(\sum_{n=2}^{\infty} a_{n-2} x^n\right)=0 \tag{3}$$

For $2 \leq n$, the recurrence equation is

$$(n+2) a_{n+2}(1+n) + (n-1) a_{n-1} + a_{n-2} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -\frac{n a_{n-1} + a_{n-2} - a_{n-1}}{(n+2)(1+n)} \\ &= -\frac{a_{n-2}}{(n+2)(1+n)} - \frac{(n-1) a_{n-1}}{(n+2)(1+n)} \end{aligned} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + a_1 + a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{a_1}{12} - \frac{a_0}{12}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + 2a_2 + a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{a_1}{20}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + 3a_3 + a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = 0$$

For $n = 5$ the recurrence equation gives

$$42a_7 + 4a_4 + a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{a_1}{126} + \frac{a_0}{126}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \left(-\frac{a_1}{12} - \frac{a_0}{12}\right) x^4 - \frac{a_1 x^5}{20} + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{x^4}{12}\right) a_0 + \left(x - \frac{1}{12}x^4 - \frac{1}{20}x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{x^4}{12}\right) c_1 + \left(x - \frac{1}{12}x^4 - \frac{1}{20}x^5\right) c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = \left(1 - \frac{x^4}{12}\right) y(0) + \left(x - \frac{1}{12}x^4 - \frac{1}{20}x^5\right) y'(0) + O(x^6) \tag{1}$$

$$y = \left(1 - \frac{x^4}{12}\right) c_1 + \left(x - \frac{1}{12}x^4 - \frac{1}{20}x^5\right) c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1 - \frac{x^4}{12}\right) y(0) + \left(x - \frac{1}{12}x^4 - \frac{1}{20}x^5\right) y'(0) + O(x^6)$$

Verified OK.

$$y = \left(1 - \frac{x^4}{12}\right) c_1 + \left(x - \frac{1}{12}x^4 - \frac{1}{20}x^5\right) c_2 + O(x^6)$$

Verified OK.

8.25.1 Maple step by step solution

Let's solve

$y'' = -y'x^2 - x^2y$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + y'x^2 + x^2y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

- ◦ Convert $x^2 \cdot y$ to series expansion

  $x^2 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+2}$

- ◦ Shift index using $k->k-2$

  $x^2 \cdot y = \sum_{k=2}^{\infty} a_{k-2} x^k$

- ◦ Convert $x^2 \cdot y'$ to series expansion

  $x^2 \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^{k+1}$

- ◦ Shift index using $k->k-1$

  $x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1}(k-1)\, x^k$

- ◦ Convert y'' to series expansion

  $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

- ◦ Shift index using $k->k+2$

  $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\, x^k$

Rewrite ODE with series expansions

$6a_3x + 2a_2 + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)(k+1) + a_{k-1}(k-1) + a_{k-2}\right) x^k\right) = 0$

- The coefficients of each power of x must be 0

  $[2a_2 = 0, 6a_3 = 0]$
- Solve for the dependent coefficient(s)

  $\{a_2 = 0, a_3 = 0\}$
- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2) a_{k+2} + a_{k-1}k + a_{k-2} - a_{k-1} = 0$
- Shift index using $k- >k + 2$

  $\left((k + 2)^2 + 3k + 8\right) a_{k+4} + a_{k+1}(k + 2) + a_k - a_{k+1} = 0$
- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{k a_{k+1} + a_k + a_{k+1}}{k^2 + 7k + 12}, a_2 = 0, a_3 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
<- Heun successful: received ODE is equivalent to the  HeunT  ODE, case  c = 0 `
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 29

```
Order:=6;
dsolve(diff(y(x),x$2)+x^2*diff(y(x),x)+x^2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1-\frac{x^4}{12}\right)y(0)+\left(x-\frac{1}{12}x^4-\frac{1}{20}x^5\right)D(y)(0)+O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 35

```
AsymptoticDSolveValue[y''[x]+x^2*y'[x]+x^2*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(1-\frac{x^4}{12}\right)+c_2\left(-\frac{x^5}{20}-\frac{x^4}{12}+x\right)$$

8.26 problem problem 26

Internal problem ID [441]
Internal file name [`OUTPUT/441_Sunday_June_05_2022_01_41_28_AM_31501157/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Chapter 11 Power series methods. Section 11.2 Power series solutions. Page 624
Problem number: problem 26.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x^3+1)\,y''+yx^4=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots\\
&= y_0+xy'_0+\frac{x^2}{2}\,f|_{x_0,y_0,y'_0}+\frac{x^3}{3!}\,f'|_{x_0,y_0,y'_0}+\cdots\\
&= y_0+xy'_0+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}\,\frac{d^nf}{dx^n}\bigg|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (149)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (150)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{y x^4}{x^3+1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= -\frac{\left(\left(x^4+x\right) y' + y\left(x^3+4\right)\right) x^3}{\left(x^3+1\right)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{\left(\left(-2x^7-10x^4-8x\right) y' + y\left(x^9+x^6+6x^3-12\right)\right) x^2}{\left(x^3+1\right)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{\left(\left(x^{13}+2x^{10}+19x^7-18x^4-36x\right) y' + 4y\left(x^{12}+5x^9-2x^6+\frac{57}{2}x^3-6\right)\right) x}{\left(x^3+1\right)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{6\left(x^{16}+6x^{13}-7x^{10}+64x^7+60x^4-16x\right) y' - y\left(x^{18}-2x^{15}+7x^{12}-258x^9+932x^6-720x^3+24\right)}{\left(x^3+1\right)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= 0 \\
F_2 &= 0 \\
F_3 &= 0 \\
F_4 &= -24y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 - \frac{x^6}{30}\right) y(0) + x y'(0) + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(x^3 + 1) y'' + y x^4 = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the above back into the ode gives

$$(x^3 + 1) \left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \right) + \left(\sum_{n=0}^{\infty} a_n x^n \right) x^4 = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n\, x^{1+n} a_n (n-1) \right) + \left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \right) + \left(\sum_{n=0}^{\infty} x^{n+4} a_n \right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n\, x^{1+n} a_n (n-1) = \sum_{n=3}^{\infty} (n-1) a_{n-1} (n-2) x^n$$
$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2) a_{n+2} (1+n) x^n$$
$$\sum_{n=0}^{\infty} x^{n+4} a_n = \sum_{n=4}^{\infty} a_{n-4} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=3}^{\infty} (n-1)\, a_{n-1}(n-2)\, x^n\right) + \left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n\right) + \left(\sum_{n=4}^{\infty} a_{n-4} x^n\right) = 0 \quad (3)$$

$n = 3$ gives

$$2a_2 + 20a_5 = 0$$

Which after substituting earlier equations, simplifies to

$$20a_5 = 0$$

Or

$$a_5 = 0$$

For $4 \leq n$, the recurrence equation is

$$(n-1)\, a_{n-1}(n-2) + (n+2)\, a_{n+2}(1+n) + a_{n-4} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -\frac{n^2 a_{n-1} - 3n a_{n-1} + a_{n-4} + 2a_{n-1}}{(n+2)\,(1+n)} \\ &= -\frac{a_{n-4}}{(n+2)\,(1+n)} - \frac{(n^2 - 3n + 2)\, a_{n-1}}{(n+2)\,(1+n)} \end{aligned} \tag{5}$$

For $n = 4$ the recurrence equation gives

$$6a_3 + 30a_6 + a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{a_0}{30}$$

For $n = 5$ the recurrence equation gives

$$12a_4 + 42a_7 + a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{a_1}{42}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_1 x + a_0 + \dots$$

Collecting terms, the solution becomes

$$y = a_1 x + a_0 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = c_2 x + c_1 + O(x^6)$$

<u>Summary</u>
The solution(s) found are the following

$$y = \left(1 - \frac{x^6}{30}\right) y(0) + x y'(0) + O(x^6) \tag{1}$$

$$y = c_2 x + c_1 + O(x^6) \tag{2}$$

<u>Verification of solutions</u>

$$y = \left(1 - \frac{x^6}{30}\right) y(0) + x y'(0) + O(x^6)$$

Verified OK.

$$y = c_2 x + c_1 + O(x^6)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
<- Heun successful: received ODE is equivalent to the  HeunC  ODE, case  a <> 0, e <> 0, c =
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
Order:=6;
dsolve((1+x^3)*diff(y(x),x$2)+x^4*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = y(0) + D(y)(0)\,x + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 10

```
AsymptoticDSolveValue[(1+x^3)*y''[x]+x^4*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2 x + c_1$$

8.27 problem problem 27

Internal problem ID [442]
Internal file name [`OUTPUT/442_Sunday_June_05_2022_01_41_30_AM_13553494/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Chapter 11 Power series methods. Section 11.2 Power series solutions. Page 624
Problem number: problem 27.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + y'x + y(2x^2 + 1) = 0}$$

With initial conditions

$$[y(0) = 1, y'(0) = -1]$$

With the expansion point for the power series method at $x = 0$.

8.27.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= x \\ q(x) &= 2x^2 + 1 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + y'x + y(2x^2 + 1) = 0$$

The domain of $p(x) = x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 2x^2 + 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\ &= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0} \end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (152)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (153)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -2x^2y - y'x - y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0 \\
&= 2yx^3 - y'x^2 - 3yx - 2y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y}y' + \frac{\partial F_1}{\partial y'}F_1 \\
&= \left(3x^3 - 3x\right) y' + y\left(2x^4 + 11x^2 - 1\right) \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y}y' + \frac{\partial F_2}{\partial y'}F_2 \\
&= \left(-x^4 + 23x^2 - 4\right) y' - 6\left(x^4 - \frac{11}{6}x^2 - \frac{25}{6}\right) xy \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y}y' + \frac{\partial F_3}{\partial y'}F_3 \\
&= \left(-5x^5 - 16x^3 + 75x\right) y' + 2\left(x^6 - \frac{75}{2}x^4 + 9x^2 + \frac{29}{2}\right) y
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = 1$ and $y'(0) = -1$ gives

$$
\begin{aligned}
F_0 &= -1 \\
F_1 &= 2 \\
F_2 &= -1 \\
F_3 &= 4 \\
F_4 &= 29
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = 1 - x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{24} + \frac{x^5}{30} + \frac{29x^6}{720} + O\left(x^6\right)
$$

$$
y = 1 - x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{24} + \frac{x^5}{30} + \frac{29x^6}{720} + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = -2x^2 \left(\sum_{n=0}^{\infty} a_n x^n\right) - \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x - \left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} n\, x^n a_n\right) + \left(\sum_{n=0}^{\infty} 2x^{n+2} a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n$$
$$\sum_{n=0}^{\infty} 2x^{n+2} a_n = \sum_{n=2}^{\infty} 2a_{n-2} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n\right) + \left(\sum_{n=1}^{\infty} n\, x^n a_n\right) + \left(\sum_{n=2}^{\infty} 2a_{n-2} x^n\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + a_0 = 0$$

$$a_2 = -\frac{a_0}{2}$$

$n = 1$ gives

$$6a_3 + 2a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{a_1}{3}$$

For $2 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) + na_n + 2a_{n-2} + a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -\frac{na_n + a_n + 2a_{n-2}}{(n+2)\,(n+1)} \\ &= -\frac{a_n}{n+2} - \frac{2a_{n-2}}{(n+2)\,(n+1)} \end{aligned} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + 3a_2 + 2a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{a_0}{24}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + 4a_3 + 2a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{a_1}{30}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + 5a_4 + 2a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{29a_0}{720}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + 6a_5 + 2a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{13a_1}{630}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{1}{2} a_0 x^2 - \frac{1}{3} a_1 x^3 - \frac{1}{24} a_0 x^4 - \frac{1}{30} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{1}{2}x^2 - \frac{1}{24}x^4\right) a_0 + \left(x - \frac{1}{3}x^3 - \frac{1}{30}x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{1}{2}x^2 - \frac{1}{24}x^4\right) c_1 + \left(x - \frac{1}{3}x^3 - \frac{1}{30}x^5\right) c_2 + O(x^6)$$

$$y = 1 - \frac{x^2}{2} - \frac{x^4}{24} - x + \frac{x^3}{3} + \frac{x^5}{30} + O(x^6)$$

<u>Summary</u>
The solution(s) found are the following

$$y = 1 - x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{24} + \frac{x^5}{30} + \frac{29x^6}{720} + O(x^6) \tag{1}$$

$$y = 1 - \frac{x^2}{2} - \frac{x^4}{24} - x + \frac{x^3}{3} + \frac{x^5}{30} + O(x^6) \tag{2}$$

Verification of solutions

$$y = 1 - x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{24} + \frac{x^5}{30} + \frac{29x^6}{720} + O(x^6)$$

Verified OK.

$$y = 1 - \frac{x^2}{2} - \frac{x^4}{24} - x + \frac{x^3}{3} + \frac{x^5}{30} + O(x^6)$$

Verified OK.

8.27.2 Maple step by step solution

Let's solve

$\left[y'' = -2x^2y - y'x - y, y(0) = 1, y'\Big|_{\{x=0\}} = -1\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = (-2x^2 - 1)\, y - y'x$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + y'x + y(2x^2 + 1) = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

- ○ Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

- ○ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

- ○ Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k- >k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^{k}$$

Rewrite ODE with series expansions

$$2a_2 + a_0 + (6a_3 + 2a_1)\, x + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_k(k+1) + 2a_{k-2}\right) x^k\right) = 0$$

- The coefficients of each power of x must be 0

$$[2a_2 + a_0 = 0, 6a_3 + 2a_1 = 0]$$

- Solve for the dependent coefficient(s)

$$\left\{a_2 = -\frac{a_0}{2}, a_3 = -\frac{a_1}{3}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k^2 + 3k + 2)\, a_{k+2} + a_k k + a_k + 2a_{k-2} = 0$$

- Shift index using $k- >k+2$

$$\left((k+2)^2 + 3k + 8\right) a_{k+4} + a_{k+2}(k+2) + a_{k+2} + 2a_k = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{k a_{k+2} + 2a_k + 3a_{k+2}}{k^2 + 7k + 12}, a_2 = -\frac{a_0}{2}, a_3 = -\frac{a_1}{3}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Kummer successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
Order:=6;
dsolve([diff(y(x),x$2)+x*diff(y(x),x)+(2*x^2+1)*y(x)=0,y(0) = 1, D(y)(0) = -1],y(x),type='se
```

$$y(x) = 1 - x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{24}x^4 + \frac{1}{30}x^5 + \mathrm{O}\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 49

```
AsymptoticDSolveValue[{(x^2+1)*y''[x]+2*x*y'[x]+2*x*y[x]==0,{}},y[x],{x,0,5}]
```

$$y(x) \rightarrow c_1\left(\frac{x^5}{5} - \frac{x^3}{3} + 1\right) + c_2\left(\frac{x^5}{5} - \frac{x^4}{6} - \frac{x^3}{3} + x\right)$$

8.28 problem problem 28

Internal problem ID [443]
Internal file name [`OUTPUT/443_Sunday_June_05_2022_01_41_32_AM_81640338/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Chapter 11 Power series methods. Section 11.2 Power series solutions. Page 624
Problem number: problem 28.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second__order__bessel__ode__form__A", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + y\,\mathrm{e}^{-x} = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \left.\frac{d^n f}{dx^n}\right|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \qquad (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \qquad (155) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (156) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \qquad (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \qquad (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -y\,\mathrm{e}^{-x} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \mathrm{e}^{-x}(-y' + y) \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \mathrm{e}^{-x}\left(y\,\mathrm{e}^{-x} + 2y' - y\right) \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= (y' - 4y)\,\mathrm{e}^{-2x} + (y - 3y')\,\mathrm{e}^{-x} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= (11y - 6y')\,\mathrm{e}^{-2x} + (-y + 4y')\,\mathrm{e}^{-x} - y\,\mathrm{e}^{-3x}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -y(0) \\
F_1 &= -y'(0) + y(0) \\
F_2 &= 2y'(0) \\
F_3 &= -3y(0) - 2y'(0) \\
F_4 &= 9y(0) - 2y'(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y = &\left(1 - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{40}x^5 + \frac{1}{80}x^6\right) y(0) \\
&+ \left(x - \frac{1}{6}x^3 + \frac{1}{12}x^4 - \frac{1}{60}x^5 - \frac{1}{360}x^6\right) y'(0) + O\left(x^6\right)
\end{aligned}
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$e^x y'' + y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$e^x \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}$$

Expanding e^x as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned} e^x &= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6 + \dots \\ &= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6 \end{aligned}$$

Hence the ODE in Eq (1) becomes

$$\left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6\right)\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0$$

Expanding the first term in (1) gives

$$1\cdot\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + x\cdot\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \frac{x^2}{2}\cdot\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \frac{x^3}{6}\cdot\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \frac{x^4}{24}\cdot\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \frac{x^5}{120}\cdot\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \frac{x^6}{720}\cdot\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=2}^{\infty} \frac{n\, x^{n+4} a_n (n-1)}{720}\right)+\left(\sum_{n=2}^{\infty} \frac{n\, x^{n+3} a_n (n-1)}{120}\right) \\
&+\left(\sum_{n=2}^{\infty} \frac{n\, x^{n+2} a_n (n-1)}{24}\right)+\left(\sum_{n=2}^{\infty} \frac{n\, x^{1+n} a_n (n-1)}{6}\right)+\left(\sum_{n=2}^{\infty} \frac{n a_n x^{n} (n-1)}{2}\right) \\
&+\left(\sum_{n=2}^{\infty} n\, x^{n-1} a_n (n-1)\right)+\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right)+\left(\sum_{n=0}^{\infty} a_n x^{n}\right)=0
\end{aligned} \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=2}^{\infty} \frac{n\, x^{n+4} a_n (n-1)}{720} &= \sum_{n=6}^{\infty} \frac{(n-4)\, a_{n-4} (n-5)\, x^{n}}{720} \\
\sum_{n=2}^{\infty} \frac{n\, x^{n+3} a_n (n-1)}{120} &= \sum_{n=5}^{\infty} \frac{(n-3)\, a_{n-3} (n-4)\, x^{n}}{120} \\
\sum_{n=2}^{\infty} \frac{n\, x^{n+2} a_n (n-1)}{24} &= \sum_{n=4}^{\infty} \frac{(n-2)\, a_{n-2} (n-3)\, x^{n}}{24} \\
\sum_{n=2}^{\infty} \frac{n\, x^{1+n} a_n (n-1)}{6} &= \sum_{n=3}^{\infty} \frac{(n-1)\, a_{n-1} (n-2)\, x^{n}}{6} \\
\sum_{n=2}^{\infty} n\, x^{n-1} a_n (n-1) &= \sum_{n=1}^{\infty} (1+n)\, a_{1+n} n\, x^{n} \\
\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} &= \sum_{n=0}^{\infty} (n+2)\, a_{n+2} (1+n)\, x^{n}
\end{aligned}
$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=6}^{\infty} \frac{(n-4)\, a_{n-4}(n-5)\, x^{n}}{720}\right)+\left(\sum_{n=5}^{\infty} \frac{(n-3)\, a_{n-3}(n-4)\, x^{n}}{120}\right) + \left(\sum_{n=4}^{\infty} \frac{(n-2)\, a_{n-2}(n-3)\, x^{n}}{24}\right)+\left(\sum_{n=3}^{\infty} \frac{(n-1)\, a_{n-1}(n-2)\, x^{n}}{6}\right) + \left(\sum_{n=2}^{\infty} \frac{n a_{n} x^{n}(n-1)}{2}\right)+\left(\sum_{n=1}^{\infty} (1+n)\, a_{1+n} n\, x^{n}\right) + \left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^{n}\right)+\left(\sum_{n=0}^{\infty} a_{n} x^{n}\right) = 0 \tag{3}$$

$n=0$ gives

$$2a_2 + a_0 = 0$$

$$a_2 = -\frac{a_0}{2}$$

$n=1$ gives

$$2a_2 + 6a_3 + a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = \frac{a_0}{6} - \frac{a_1}{6}$$

$n=2$ gives

$$2a_2 + 6a_3 + 12a_4 = 0$$

Which after substituting earlier equations, simplifies to

$$-a_1 + 12a_4 = 0$$

Or

$$a_4 = \frac{a_1}{12}$$

$n=3$ gives

$$\frac{a_2}{3} + 4a_3 + 12a_4 + 20a_5 = 0$$

Which after substituting earlier equations, simplifies to

$$\frac{a_0}{2}+\frac{a_1}{3}+20a_5=0$$

Or

$$a_5=-\frac{a_0}{40}-\frac{a_1}{60}$$

$n=4$ gives

$$\frac{a_2}{12}+a_3+7a_4+20a_5+30a_6=0$$

Which after substituting earlier equations, simplifies to

$$-\frac{3a_0}{8}+\frac{a_1}{12}+30a_6=0$$

Or

$$a_6=\frac{a_0}{80}-\frac{a_1}{360}$$

$n=5$ gives

$$\frac{a_2}{60}+\frac{a_3}{4}+2a_4+11a_5+30a_6+42a_7=0$$

Which after substituting earlier equations, simplifies to

$$\frac{2a_0}{15}-\frac{17a_1}{120}+42a_7=0$$

Or

$$a_7=-\frac{a_0}{315}+\frac{17a_1}{5040}$$

For $6 \leq n$, the recurrence equation is

$$\begin{aligned}&\frac{(n-4)\,a_{n-4}(n-5)}{720}+\frac{(n-3)\,a_{n-3}(n-4)}{120}\\&+\frac{(n-2)\,a_{n-2}(n-3)}{24}+\frac{(n-1)\,a_{n-1}(n-2)}{6}+\frac{na_n(n-1)}{2}\\&+(1+n)\,a_{1+n}n+(n+2)\,a_{n+2}(1+n)+a_n=0\end{aligned} \tag{4}$$

Solving for a_{n+2}, gives

$$
a_{n+2} = -\frac{360n^2 a_n + 720n^2 a_{1+n} + n^2 a_{n-4} + 6n^2 a_{n-3} + 30n^2 a_{n-2} + 120n^2 a_{n-1} - 360n a_n + 720n a_{1+n} - 9n a_{n-4}}{720\,(n+2)\,(1+}
$$

$$
\begin{aligned}
&= -\frac{(360n^2 - 360n + 720)\, a_n}{720\,(n+2)\,(1+n)} - \frac{(720n^2 + 720n)\, a_{1+n}}{720\,(n+2)\,(1+n)} - \frac{(n^2 - 9n + 20)\, a_{n-4}}{720\,(n+2)\,(1+n)} \\
&\quad - \frac{(6n^2 - 42n + 72)\, a_{n-3}}{720\,(n+2)\,(1+n)} - \frac{(30n^2 - 150n + 180)\, a_{n-2}}{720\,(n+2)\,(1+n)} \\
&\quad - \frac{(120n^2 - 360n + 240)\, a_{n-1}}{720\,(n+2)\,(1+n)}
\end{aligned} \tag{5}
$$

And so on. Therefore the solution is

$$
\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^n \\
&= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots
\end{aligned}
$$

Substituting the values for a_n found above, the solution becomes

$$
y = a_0 + a_1 x - \frac{a_0 x^2}{2} + \left(\frac{a_0}{6} - \frac{a_1}{6}\right) x^3 + \frac{a_1 x^4}{12} + \left(-\frac{a_0}{40} - \frac{a_1}{60}\right) x^5 + \dots
$$

Collecting terms, the solution becomes

$$
y = \left(1 - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{40}x^5\right) a_0 + \left(x - \frac{1}{6}x^3 + \frac{1}{12}x^4 - \frac{1}{60}x^5\right) a_1 + O(x^6) \tag{3}
$$

At $x = 0$ the solution above becomes

$$
y = \left(1 - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{40}x^5\right) c_1 + \left(x - \frac{1}{6}x^3 + \frac{1}{12}x^4 - \frac{1}{60}x^5\right) c_2 + O(x^6)
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= \left(1 - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{40}x^5 + \frac{1}{80}x^6\right) y(0) \\
&\quad + \left(x - \frac{1}{6}x^3 + \frac{1}{12}x^4 - \frac{1}{60}x^5 - \frac{1}{360}x^6\right) y'(0) + O(x^6)
\end{aligned} \tag{1}
$$

$$
y = \left(1 - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{40}x^5\right) c_1 + \left(x - \frac{1}{6}x^3 + \frac{1}{12}x^4 - \frac{1}{60}x^5\right) c_2 + O(x^6) \tag{2}
$$

Verification of solutions

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{40}x^5 + \frac{1}{80}x^6\right) y(0) + \left(x - \frac{1}{6}x^3 + \frac{1}{12}x^4 - \frac{1}{60}x^5 - \frac{1}{360}x^6\right) y'(0) + O(x^6)$$

Verified OK.

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{40}x^5\right) c_1 + \left(x - \frac{1}{6}x^3 + \frac{1}{12}x^4 - \frac{1}{60}x^5\right) c_2 + O(x^6)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a quadrature
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
   Change of variables used:
      [x = -ln(t)]
   Linear ODE actually solved:
      u(t)+diff(u(t),t)+t*diff(diff(u(t),t),t) = 0
<- change of variables successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 44

```
Order:=6;
dsolve(diff(y(x),x$2)+exp(-x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{40}x^5\right) y(0) + \left(x - \frac{1}{6}x^3 + \frac{1}{12}x^4 - \frac{1}{60}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 56

```
AsymptoticDSolveValue[y''[x]+Exp[-x]*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(-\frac{x^5}{60} + \frac{x^4}{12} - \frac{x^3}{6} + x\right) + c_1\left(-\frac{x^5}{40} + \frac{x^3}{6} - \frac{x^2}{2} + 1\right)$$

8.29 problem problem 29

Internal problem ID [444]
Internal file name [`OUTPUT/444_Sunday_June_05_2022_01_41_33_AM_7732130/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Chapter 11 Power series methods. Section 11.2 Power series solutions. Page 624
Problem number: problem 29.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{\cos(x)\, y'' + y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\ &= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0} \end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (158)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (159)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy'_0 + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n|_{x_0,y_0,y'_0} \qquad (7)$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{y}{\cos(x)} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= -\sec(x)\left(\tan(x)\, y + y'\right) \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= -2\left(\tan(x)\, y' + (-1 + \sec(x))\left(\sec(x) + \frac{1}{2}\right) y\right) \sec(x) \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \sec(x)\left(\left(-6\sec(x)^2 + \sec(x) + 3\right) y' + \tan(x)\sec(x)^2\, y\left(\cos(x)^2 + 4\cos(x) - 6\right)\right) \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= -24\left(-\frac{\sec(x)^2 \tan(x)\left(\cos(x)^2 + \frac{3\cos(x)}{2} - 6\right) y'}{6} + y\left(\sec(x)^4 - \frac{3\sec(x)^3}{4} - \frac{19\sec(x)^2}{24} + \frac{11\sec(}{24}\right.\right.
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -y(0) \\
F_1 &= -y'(0) \\
F_2 &= 0 \\
F_3 &= -2y'(0) \\
F_4 &= y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 - \frac{1}{2}x^2 + \frac{1}{720}x^6\right) y(0) + \left(x - \frac{1}{6}x^3 - \frac{1}{60}x^5\right) y'(0) + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = -\frac{\sum_{n=0}^{\infty} a_n x^n}{\cos(x)} \tag{1}$$

Expanding $\cos(x)$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned}\cos(x) &= 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \dots \\ &= 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6\end{aligned}$$

Hence the ODE in Eq (1) becomes

$$\left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6\right)\left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0$$

Expanding the first term in (1) gives

$$\begin{aligned}&1 \cdot \left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) - \frac{x^2}{2} \cdot \left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \frac{x^4}{24} \\ &\cdot \left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) - \frac{x^6}{720} \cdot \left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0\end{aligned}$$

Which simplifies to

$$\begin{aligned}&\sum_{n=2}^{\infty}\left(-\frac{n\, x^{n+4} a_n (n-1)}{720}\right) + \left(\sum_{n=2}^{\infty} \frac{n\, x^{n+2} a_n (n-1)}{24}\right) \\ &+ \sum_{n=2}^{\infty}\left(-\frac{n a_n x^n (n-1)}{2}\right) + \left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0\end{aligned} \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} \left(-\frac{n\,x^{n+4} a_n (n-1)}{720}\right) = \sum_{n=6}^{\infty} \left(-\frac{(n-4)\, a_{n-4} (n-5)\, x^n}{720}\right)$$

$$\sum_{n=2}^{\infty} \frac{n\,x^{n+2} a_n (n-1)}{24} = \sum_{n=4}^{\infty} \frac{(n-2)\, a_{n-2} (n-3)\, x^n}{24}$$

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2} (n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\sum_{n=6}^{\infty} \left(-\frac{(n-4)\, a_{n-4} (n-5)\, x^n}{720}\right) + \left(\sum_{n=4}^{\infty} \frac{(n-2)\, a_{n-2} (n-3)\, x^n}{24}\right) + \sum_{n=2}^{\infty} \left(-\frac{n a_n x^n (n-1)}{2}\right) + \left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2} (n+1)\, x^n\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + a_0 = 0$$

$$a_2 = -\frac{a_0}{2}$$

$n = 1$ gives

$$6a_3 + a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{a_1}{6}$$

$n = 3$ gives

$$-2a_3 + 20a_5 = 0$$

Which after substituting earlier equations, simplifies to

$$\frac{a_1}{3} + 20a_5 = 0$$

Or

$$a_5 = -\frac{a_1}{60}$$

$n = 4$ gives

$$\frac{a_2}{12} - 5a_4 + 30a_6 = 0$$

Which after substituting earlier equations, simplifies to

$$-\frac{a_0}{24} + 30a_6 = 0$$

Or

$$a_6 = \frac{a_0}{720}$$

$n = 5$ gives

$$\frac{a_3}{4} - 9a_5 + 42a_7 = 0$$

Which after substituting earlier equations, simplifies to

$$\frac{13a_1}{120} + 42a_7 = 0$$

Or

$$a_7 = -\frac{13a_1}{5040}$$

For $6 \leq n$, the recurrence equation is

$$-\frac{(n-4)\, a_{n-4}(n-5)}{720} + \frac{(n-2)\, a_{n-2}(n-3)}{24} - \frac{n a_n (n-1)}{2} + (n+2)\, a_{n+2}(n+1) + a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{360n^2a_n + n^2a_{n-4} - 30n^2a_{n-2} - 360na_n - 9na_{n-4} + 150na_{n-2} - 720a_n + 20a_{n-4} - 180a_{n-2}}{720(n+2)(n+1)}$$

(5)

$$= \frac{(360n^2 - 360n - 720)a_n}{720(n+2)(n+1)} + \frac{(n^2 - 9n + 20)a_{n-4}}{720(n+2)(n+1)} + \frac{(-30n^2 + 150n - 180)a_{n-2}}{720(n+2)(n+1)}$$

And so on. Therefore the solution is

$$y = \sum_{n=0}^{\infty} a_n x^n$$
$$= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1x - \frac{1}{2}a_0x^2 - \frac{1}{6}a_1x^3 - \frac{1}{60}a_1x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{x^2}{2}\right)a_0 + \left(x - \frac{1}{6}x^3 - \frac{1}{60}x^5\right)a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{x^2}{2}\right)c_1 + \left(x - \frac{1}{6}x^3 - \frac{1}{60}x^5\right)c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{720}x^6\right)y(0) + \left(x - \frac{1}{6}x^3 - \frac{1}{60}x^5\right)y'(0) + O(x^6) \tag{1}$$

$$y = \left(1 - \frac{x^2}{2}\right)c_1 + \left(x - \frac{1}{6}x^3 - \frac{1}{60}x^5\right)c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{720}x^6\right)y(0) + \left(x - \frac{1}{6}x^3 - \frac{1}{60}x^5\right)y'(0) + O(x^6)$$

Verified OK.

$$y = \left(1 - \frac{x^2}{2}\right)c_1 + \left(x - \frac{1}{6}x^3 - \frac{1}{60}x^5\right)c_2 + O(x^6)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a quadrature
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
   trying a solution in terms of MeijerG functions
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   <- Heun successful: received ODE is equivalent to the  HeunG  ODE, case  a <> 0, e <> 0,
   Change of variables used:
      [x = arccos(t)]
   Linear ODE actually solved:
      u(t)-t^2*diff(u(t),t)+(-t^3+t)*diff(diff(u(t),t),t) = 0
<- change of variables successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 29

```
Order:=6;
dsolve(cos(x)*diff(y(x),x$2)+y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{x^2}{2}\right) y(0) + \left(x - \frac{1}{6}x^3 - \frac{1}{60}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 35

```
AsymptoticDSolveValue[Cos[x]*y''[x]+y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(1 - \frac{x^2}{2}\right) + c_2 \left(-\frac{x^5}{60} - \frac{x^3}{6} + x\right)$$

8.30 problem problem 30

Internal problem ID [445]
Internal file name [`OUTPUT/445_Sunday_June_05_2022_01_41_35_AM_58847833/index.tex`]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Chapter 11 Power series methods. Section 11.2 Power series solutions. Page 624
Problem number: problem 30.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

`[_Lienard]`

$$\boxed{xy'' + \sin(x)\, y' + yx = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + xy'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\
&= y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (161)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (162)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') &(4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f &(5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 &(6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{\sin(x)\, y' + yx}{x} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{\left(\sin(x)^2 + \sin(x) - x(x + \cos(x))\right) y' + \sin(x)\, xy}{x^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{\left(-\sin(x)^3 - 3\sin(x)^2 + (3x^2 + 3x\cos(x) - 2)\sin(x) + 2x\cos(x)\right) y' + \left(-\sin(x)^2 - 2\sin(x) + x(\right.}{x^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{\left(6\sin(x)^3 + \left(-\cos(x)^2 + 12\right)\sin(x)^2 + (-7x^2 - 14x\cos(x) + 6)\sin(x) + \left(6\cos(x)^3 + 10\cos(x)^2\right.\right.}{} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{\left(\left(\cos(x)^2 - 36\right)\sin(x)^3 + \left(10\cos(x)^2 - 60\right)\sin(x)^2 + \left(-10x\cos(x)^3 - 29x^2\cos(x)^2 + (-30x^3 + 8\right.\right.}{}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -y'(0) - y(0) \\
F_1 &= y(0) \\
F_2 &= \frac{4y'(0)}{3} \\
F_3 &= -2y(0) - \frac{7y'(0)}{3} \\
F_4 &= 4y(0) + \frac{4y'(0)}{5}
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y = &\left(1-\frac{1}{2}x^{2}+\frac{1}{6}x^{3}-\frac{1}{60}x^{5}+\frac{1}{180}x^{6}\right)y(0)\\
&+\left(x-\frac{1}{2}x^{2}+\frac{1}{18}x^{4}-\frac{7}{360}x^{5}+\frac{1}{900}x^{6}\right)y'(0)+O(x^{6})
\end{aligned}
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$
xy''+\sin(x)\,y'+yx=0
$$

Let the solution be represented as power series of the form

$$
y=\sum_{n=0}^{\infty}a_{n}x^{n}
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty}n a_{n}x^{n-1}\\
y'' &= \sum_{n=2}^{\infty}n(n-1)\,a_{n}x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\left(\sum_{n=2}^{\infty}n(n-1)\,a_{n}x^{n-2}\right)x+\sin(x)\left(\sum_{n=1}^{\infty}n a_{n}x^{n-1}\right)+\left(\sum_{n=0}^{\infty}a_{n}x^{n}\right)x=0 \tag{1}
$$

Expanding $\sin(x)$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$
\begin{aligned}
\sin(x) &= x-\frac{1}{6}x^{3}+\frac{1}{120}x^{5}-\frac{1}{5040}x^{7}+\dots\\
&= x-\frac{1}{6}x^{3}+\frac{1}{120}x^{5}-\frac{1}{5040}x^{7}
\end{aligned}
$$

Hence the ODE in Eq (1) becomes

$$
\begin{aligned}
&\left(\sum_{n=2}^{\infty}n(n-1)\,a_{n}x^{n-2}\right)x\\
&+\left(x-\frac{1}{6}x^{3}+\frac{1}{120}x^{5}-\frac{1}{5040}x^{7}\right)\left(\sum_{n=1}^{\infty}n a_{n}x^{n-1}\right)+\left(\sum_{n=0}^{\infty}a_{n}x^{n}\right)x=0
\end{aligned}
$$

Expanding the second term in (1) gives

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) x + x \cdot \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - \frac{x^3}{6} \cdot \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right)$$
$$+ \frac{x^5}{120} \cdot \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - \frac{x^7}{5040} \cdot \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) x = 0$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n\, x^{n-1} a_n (n-1)\right) + \left(\sum_{n=1}^{\infty} n a_n x^n\right) + \sum_{n=1}^{\infty} \left(-\frac{n\, x^{n+2} a_n}{6}\right)$$
$$+ \left(\sum_{n=1}^{\infty} \frac{n\, x^{n+4} a_n}{120}\right) + \sum_{n=1}^{\infty} \left(-\frac{n\, x^{n+6} a_n}{5040}\right) + \left(\sum_{n=0}^{\infty} x^{1+n} a_n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n\, x^{n-1} a_n (n-1) = \sum_{n=1}^{\infty} (1+n)\, a_{1+n} n\, x^n$$

$$\sum_{n=1}^{\infty} \left(-\frac{n\, x^{n+2} a_n}{6}\right) = \sum_{n=3}^{\infty} \left(-\frac{(n-2)\, a_{n-2} x^n}{6}\right)$$

$$\sum_{n=1}^{\infty} \frac{n\, x^{n+4} a_n}{120} = \sum_{n=5}^{\infty} \frac{(n-4)\, a_{n-4} x^n}{120}$$

$$\sum_{n=1}^{\infty} \left(-\frac{n\, x^{n+6} a_n}{5040}\right) = \sum_{n=7}^{\infty} \left(-\frac{(n-6)\, a_{n-6} x^n}{5040}\right)$$

$$\sum_{n=0}^{\infty} x^{1+n} a_n = \sum_{n=1}^{\infty} a_{n-1} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=1}^{\infty} (1+n)\, a_{1+n} n\, x^{n}\right) + \left(\sum_{n=1}^{\infty} n a_{n} x^{n}\right) + \sum_{n=3}^{\infty} \left(-\frac{(n-2)\, a_{n-2} x^{n}}{6}\right) \\ + \left(\sum_{n=5}^{\infty} \frac{(n-4)\, a_{n-4} x^{n}}{120}\right) + \sum_{n=7}^{\infty} \left(-\frac{(n-6)\, a_{n-6} x^{n}}{5040}\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^{n}\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + a_1 + a_0 = 0$$

$$a_2 = -\frac{a_0}{2} - \frac{a_1}{2}$$

$n = 1$ gives

$$6a_3 + 2a_2 + a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = \frac{a_0}{6}$$

$n = 2$ gives

$$12a_4 + 3a_3 - \frac{a_1}{6} + a_2 = 0$$

Which after substituting earlier equations, simplifies to

$$a_4 = \frac{a_1}{18}$$

$n = 3$ gives

$$20a_5 + 4a_4 - \frac{a_2}{3} + a_3 = 0$$

Which after substituting earlier equations, simplifies to

$$20a_5 + \frac{7a_1}{18} + \frac{a_0}{3} = 0$$

Or

$$a_5 = -\frac{a_0}{60} - \frac{7a_1}{360}$$

$n = 4$ gives

$$30a_6 + 5a_5 - \frac{a_3}{2} + \frac{a_1}{120} + a_4 = 0$$

Which after substituting earlier equations, simplifies to

$$a_6 = \frac{a_0}{180} + \frac{a_1}{900}$$

For $7 \leq n$, the recurrence equation is

$$(1+n)\, a_{1+n} n + n a_n - \frac{(n-2)\, a_{n-2}}{6} + \frac{(n-4)\, a_{n-4}}{120} - \frac{(n-6)\, a_{n-6}}{5040} + a_{n-1} = 0 \quad (4)$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \left(-\frac{a_0}{2} - \frac{a_1}{2}\right) x^2 + \frac{a_0 x^3}{6} + \frac{a_1 x^4}{18} + \left(-\frac{a_0}{60} - \frac{7a_1}{360}\right) x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{60}x^5\right) a_0 + \left(x - \frac{1}{2}x^2 + \frac{1}{18}x^4 - \frac{7}{360}x^5\right) a_1 + O(x^6) \quad (3)$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{60}x^5\right) c_1 + \left(x - \frac{1}{2}x^2 + \frac{1}{18}x^4 - \frac{7}{360}x^5\right) c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & \left(1-\frac{1}{2}x^{2}+\frac{1}{6}x^{3}-\frac{1}{60}x^{5}+\frac{1}{180}x^{6}\right) y(0) \\
& + \left(x-\frac{1}{2}x^{2}+\frac{1}{18}x^{4}-\frac{7}{360}x^{5}+\frac{1}{900}x^{6}\right) y'(0)+O\left(x^{6}\right)
\end{aligned} \tag{1}
$$

$$
y = \left(1-\frac{1}{2}x^{2}+\frac{1}{6}x^{3}-\frac{1}{60}x^{5}\right) c_{1}+\left(x-\frac{1}{2}x^{2}+\frac{1}{18}x^{4}-\frac{7}{360}x^{5}\right) c_{2}+O\left(x^{6}\right) \tag{2}
$$

Verification of solutions

$$
\begin{aligned}
y = {} & \left(1-\frac{1}{2}x^{2}+\frac{1}{6}x^{3}-\frac{1}{60}x^{5}+\frac{1}{180}x^{6}\right) y(0) \\
& + \left(x-\frac{1}{2}x^{2}+\frac{1}{18}x^{4}-\frac{7}{360}x^{5}+\frac{1}{900}x^{6}\right) y'(0)+O\left(x^{6}\right)
\end{aligned}
$$

Verified OK.

$$
y = \left(1-\frac{1}{2}x^{2}+\frac{1}{6}x^{3}-\frac{1}{60}x^{5}\right) c_{1}+\left(x-\frac{1}{2}x^{2}+\frac{1}{18}x^{4}-\frac{7}{360}x^{5}\right) c_{2}+O\left(x^{6}\right)
$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
      trying a symmetry of the form [xi=0, eta=F(x)]
      trying 2nd order exact linear
      trying symmetries linear in x and y(x)
      trying to convert to a linear ODE with constant coefficients
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
      trying a symmetry of the form [xi=0, eta=F(x)]
      trying 2nd order exact linear
      trying symmetries linear in x and y(x)
      trying to convert to a linear ODE with constant coefficients
      -> trying with_periodic_functions in the coefficients
         --- Trying Lie symmetry methods, 2nd order ---
         `, `-> Computing symmetries using: way = 5
<- unable to find a useful change of variables
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying differential order: 2; exact nonlinear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
   -> Trying changes of variables to rationalize or make the ODE simpler
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 44

```
Order:=6;
dsolve(x*diff(y(x),x$2)+sin(x)*diff(y(x),x)+x*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{60}x^5\right) y(0) + \left(x - \frac{1}{2}x^2 + \frac{1}{18}x^4 - \frac{7}{360}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 56

```
AsymptoticDSolveValue[x*y''[x]+Sin[x]*y'[x]+x*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(-\frac{7x^5}{360} + \frac{x^4}{18} - \frac{x^2}{2} + x\right) + c_1\left(-\frac{x^5}{60} + \frac{x^3}{6} - \frac{x^2}{2} + 1\right)$$

8.31 problem problem 33

Internal problem ID [446]
Internal file name [OUTPUT/446_Sunday_June_05_2022_01_41_38_AM_54218894/index.tex]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Chapter 11 Power series methods. Section 11.2 Power series solutions. Page 624
Problem number: problem 33.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 2y'x + 2\alpha y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + xy'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\ &= y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0,y_0,y'_0} \end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (164)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (165)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= 2y'x - 2\alpha y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \left(4x^2 - 2\alpha + 2\right) y' - 4y\alpha x \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \left(8x^3 - 8\alpha x + 12x\right) y' - 8\left(x^2 - \frac{\alpha}{2} + 1\right) \alpha y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= 4\left(3 + \alpha^2 + 2\left(-3x^2 - 2\right) \alpha + 4x^4 + 12x^2\right) y' - 16\left(x^2 - \alpha + \frac{5}{2}\right) x\alpha y \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= 32\left(\frac{3\alpha^2}{4} + \left(-2x^2 - \frac{15}{4}\right) \alpha + x^4 + 5x^2 + \frac{15}{4}\right) xy' - 32\alpha\left(2 + \frac{\alpha^2}{4} + \frac{3(-x^2 - 1)\,\alpha}{2} + x^4 + \frac{9x^2}{2}\right) y
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -2y(0)\,\alpha \\
F_1 &= -2y'(0)\,\alpha + 2y'(0) \\
F_2 &= 4y(0)\,\alpha^2 - 8y(0)\,\alpha \\
F_3 &= 4y'(0)\,\alpha^2 - 16y'(0)\,\alpha + 12y'(0) \\
F_4 &= -8y(0)\,\alpha^3 + 48y(0)\,\alpha^2 - 64y(0)\,\alpha
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y = &\left(1 - \alpha\, x^2 + \frac{1}{6} x^4 \alpha^2 - \frac{1}{3} x^4 \alpha - \frac{1}{90} x^6 \alpha^3 + \frac{1}{15} x^6 \alpha^2 - \frac{4}{45} x^6 \alpha\right) y(0) \\
&+ \left(x - \frac{1}{3} x^3 \alpha + \frac{1}{3} x^3 + \frac{1}{30} x^5 \alpha^2 - \frac{2}{15} x^5 \alpha + \frac{1}{10} x^5\right) y'(0) + O\left(x^6\right)
\end{aligned}
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = 2\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x - 2\alpha\left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \sum_{n=1}^{\infty} (-2n\, x^n a_n) + \left(\sum_{n=0}^{\infty} 2\alpha a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \sum_{n=1}^{\infty} (-2n\, x^n a_n) + \left(\sum_{n=0}^{\infty} 2\alpha a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2\alpha a_0 + 2a_2 = 0$$

$$a_2 = -\alpha a_0$$

For $1 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) - 2n a_n + 2\alpha a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{2a_n(\alpha - n)}{(n+2)\,(n+1)} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$2\alpha a_1 - 2a_1 + 6a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{1}{3}\alpha a_1 + \frac{1}{3}a_1$$

For $n = 2$ the recurrence equation gives

$$2\alpha a_2 - 4a_2 + 12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{1}{6}\alpha^2 a_0 - \frac{1}{3}\alpha a_0$$

For $n = 3$ the recurrence equation gives

$$2\alpha a_3 - 6a_3 + 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{1}{30}\alpha^2 a_1 - \frac{2}{15}\alpha a_1 + \frac{1}{10}a_1$$

For $n = 4$ the recurrence equation gives

$$2\alpha a_4 - 8a_4 + 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{1}{90}\alpha^3 a_0 + \frac{1}{15}\alpha^2 a_0 - \frac{4}{45}\alpha a_0$$

For $n = 5$ the recurrence equation gives

$$2\alpha a_5 - 10a_5 + 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{1}{630}\alpha^3 a_1 + \frac{1}{70}\alpha^2 a_1 - \frac{23}{630}\alpha a_1 + \frac{1}{42}a_1$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$\begin{aligned} y &= a_0 + a_1 x - \alpha a_0 x^2 + \left(-\frac{1}{3}\alpha a_1 + \frac{1}{3}a_1\right)x^3 \\ &+ \left(\frac{1}{6}\alpha^2 a_0 - \frac{1}{3}\alpha a_0\right)x^4 + \left(\frac{1}{30}\alpha^2 a_1 - \frac{2}{15}\alpha a_1 + \frac{1}{10}a_1\right)x^5 + \dots \end{aligned}$$

Collecting terms, the solution becomes

$$\begin{aligned} y &= \left(1 - \alpha x^2 + \left(\frac{1}{6}\alpha^2 - \frac{1}{3}\alpha\right)x^4\right)a_0 \\ &+ \left(x + \left(-\frac{\alpha}{3} + \frac{1}{3}\right)x^3 + \left(\frac{1}{30}\alpha^2 - \frac{2}{15}\alpha + \frac{1}{10}\right)x^5\right)a_1 + O(x^6) \end{aligned} \tag{3}$$

At $x = 0$ the solution above becomes

$$\begin{aligned} y &= \left(1 - \alpha x^2 + \left(\frac{1}{6}\alpha^2 - \frac{1}{3}\alpha\right)x^4\right)c_1 \\ &+ \left(x + \left(-\frac{\alpha}{3} + \frac{1}{3}\right)x^3 + \left(\frac{1}{30}\alpha^2 - \frac{2}{15}\alpha + \frac{1}{10}\right)x^5\right)c_2 + O(x^6) \end{aligned}$$

Summary

The solution(s) found are the following

$$
\begin{align*}
y &= \left(1-\alpha \,x^{2}+\frac{1}{6} x^{4} \alpha^{2}-\frac{1}{3} x^{4} \alpha -\frac{1}{90} x^{6} \alpha^{3}+\frac{1}{15} x^{6} \alpha^{2}-\frac{4}{45} x^{6} \alpha \right) y(0) \\
&+ \left(x-\frac{1}{3} x^{3} \alpha +\frac{1}{3} x^{3}+\frac{1}{30} x^{5} \alpha^{2}-\frac{2}{15} x^{5} \alpha +\frac{1}{10} x^{5}\right) y'(0)+O\left(x^{6}\right) \tag{1}
\end{align*}
$$

$$
\begin{align*}
y &= \left(1-\alpha \,x^{2}+\left(\frac{1}{6} \alpha^{2}-\frac{1}{3} \alpha \right) x^{4}\right) c_{1} \\
&+ \left(x+\left(-\frac{\alpha}{3}+\frac{1}{3}\right) x^{3}+\left(\frac{1}{30} \alpha^{2}-\frac{2}{15} \alpha +\frac{1}{10}\right) x^{5}\right) c_{2}+O\left(x^{6}\right) \tag{2}
\end{align*}
$$

Verification of solutions

$$
\begin{align*}
y &= \left(1-\alpha \,x^{2}+\frac{1}{6} x^{4} \alpha^{2}-\frac{1}{3} x^{4} \alpha -\frac{1}{90} x^{6} \alpha^{3}+\frac{1}{15} x^{6} \alpha^{2}-\frac{4}{45} x^{6} \alpha \right) y(0) \\
&+ \left(x-\frac{1}{3} x^{3} \alpha +\frac{1}{3} x^{3}+\frac{1}{30} x^{5} \alpha^{2}-\frac{2}{15} x^{5} \alpha +\frac{1}{10} x^{5}\right) y'(0)+O\left(x^{6}\right)
\end{align*}
$$

Verified OK.

$$
\begin{align*}
y &= \left(1-\alpha \,x^{2}+\left(\frac{1}{6} \alpha^{2}-\frac{1}{3} \alpha \right) x^{4}\right) c_{1} \\
&+ \left(x+\left(-\frac{\alpha}{3}+\frac{1}{3}\right) x^{3}+\left(\frac{1}{30} \alpha^{2}-\frac{2}{15} \alpha +\frac{1}{10}\right) x^{5}\right) c_{2}+O\left(x^{6}\right)
\end{align*}
$$

Verified OK.

8.31.1 Maple step by step solution

Let's solve

$y'' = 2y'x - 2\alpha y$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - 2y'x + 2\alpha y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

☐ Rewrite DE with series expansions

- Convert $x \cdot y'$ to series expansion

  $$x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$$

- Convert y'' to series expansion

  $$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k- >k+2$

  $$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$$

  Rewrite DE with series expansions

  $$\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + 2a_k(\alpha - k)\right) x^k = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$(k^2 + 3k + 2)\, a_{k+2} - 2a_k(k - \alpha) = 0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{2a_k(\alpha - k)}{k^2 + 3k + 2}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Kummer successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 63

```
Order:=6;
dsolve(diff(y(x),x$2)-2*x*diff(y(x),x)+2*alpha*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1-\alpha\,x^2+\frac{\alpha(\alpha-2)\,x^4}{6}\right)y(0) + \left(x-\frac{(\alpha-1)\,x^3}{3}+\frac{(\alpha^2-4\alpha+3)\,x^5}{30}\right)D(y)\,(0)+O\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 78

```
AsymptoticDSolveValue[y''[x]-2*x*y'[x]+2*\[Alpha]*y[x]==0,y[x],{x,0,5}]
```

$$y(x)\to c_2\left(\frac{\alpha^2x^5}{30}-\frac{2\alpha x^5}{15}+\frac{x^5}{10}-\frac{\alpha x^3}{3}+\frac{x^3}{3}+x\right)+c_1\left(\frac{\alpha^2x^4}{6}-\frac{\alpha x^4}{3}-\alpha x^2+1\right)$$

8.32 problem problem 34

Internal problem ID [447]
Internal file name [OUTPUT/447_Sunday_June_05_2022_01_41_39_AM_88645353/index.tex]

Book: Differential equations and linear algebra, 4th ed., Edwards and Penney
Section: Chapter 11 Power series methods. Section 11.2 Power series solutions. Page 624
Problem number: problem 34.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_airy", "second_order_bessel_ode", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{y'' - yx = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\ &= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y'_0} \end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (167)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (168)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x, y, y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \qquad (7)$$

To find $y(x)$ series solution around $x = 0$. Hence

$$\begin{aligned}
F_0 &= yx \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= y'x + y \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= x^2 y + 2y' \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= x(y'x + 4y) \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= yx^3 + 6y'x + 4y
\end{aligned}$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$\begin{aligned}
F_0 &= 0 \\
F_1 &= y(0) \\
F_2 &= 2y'(0) \\
F_3 &= 0 \\
F_4 &= 4y(0)
\end{aligned}$$

Substituting all the above in (7) and simplifying gives the solution as

$$y = \left(1 + \frac{1}{6}x^3 + \frac{1}{180}x^6\right) y(0) + \left(x + \frac{1}{12}x^4\right) y'(0) + O(x^6)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard

power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \left(\sum_{n=0}^{\infty} a_n x^n\right) x \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \sum_{n=0}^{\infty} \left(-x^{1+n} a_n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n$$
$$\sum_{n=0}^{\infty} \left(-x^{1+n} a_n\right) = \sum_{n=1}^{\infty} \left(-a_{n-1} x^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n\right) + \sum_{n=1}^{\infty} \left(-a_{n-1} x^n\right) = 0 \tag{3}$$

For $1 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(1+n) - a_{n-1} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{a_{n-1}}{(n+2)(1+n)} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 - a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{a_0}{6}$$

For $n = 2$ the recurrence equation gives

$$12a_4 - a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_1}{12}$$

For $n = 3$ the recurrence equation gives

$$20a_5 - a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = 0$$

For $n = 4$ the recurrence equation gives

$$30a_6 - a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0}{180}$$

For $n = 5$ the recurrence equation gives

$$42a_7 - a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{a_1}{504}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \frac{1}{6} a_0 x^3 + \frac{1}{12} a_1 x^4 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + \frac{x^3}{6}\right) a_0 + \left(x + \frac{1}{12} x^4\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 + \frac{x^3}{6}\right) c_1 + \left(x + \frac{1}{12} x^4\right) c_2 + O(x^6)$$

Summary
The solution(s) found are the following

$$y = \left(1 + \frac{1}{6} x^3 + \frac{1}{180} x^6\right) y(0) + \left(x + \frac{1}{12} x^4\right) y'(0) + O(x^6) \tag{1}$$

$$y = \left(1 + \frac{x^3}{6}\right) c_1 + \left(x + \frac{1}{12} x^4\right) c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1 + \frac{1}{6} x^3 + \frac{1}{180} x^6\right) y(0) + \left(x + \frac{1}{12} x^4\right) y'(0) + O(x^6)$$

Verified OK.

$$y = \left(1 + \frac{x^3}{6}\right) c_1 + \left(x + \frac{1}{12} x^4\right) c_2 + O(x^6)$$

Verified OK.

8.32.1 Maple step by step solution

Let's solve

$y'' = yx$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - yx = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

☐ Rewrite ODE with series expansions

  ◦ Convert $x \cdot y$ to series expansion

  $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

  ◦ Shift index using $k- >k - 1$

  $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$

  ◦ Convert y'' to series expansion

  $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- >k + 2$

  $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite ODE with series expansions

  $2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - a_{k-1}\right) x^k\right) = 0$

- Each term must be 0

  $2a_2 = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2)\, a_{k+2} - a_{k-1} = 0$

- Shift index using $k- >k + 1$

$$\left((k+1)^2+3k+5\right)a_{k+3}-a_k=0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+3}=\frac{a_k}{k^2+5k+6}, 2a_2=0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
Order:=6;
dsolve(diff(y(x),x$2)=x*y(x),y(x),type='series',x=0);
```

$$y(x)=\left(1+\frac{x^3}{6}\right)y(0)+\left(x+\frac{1}{12}x^4\right)D(y)(0)+O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 28

```
AsymptoticDSolveValue[y''[x]==x*y[x],y[x],{x,0,5}]
```

$$y(x)\to c_2\left(\frac{x^4}{12}+x\right)+c_1\left(\frac{x^3}{6}+1\right)$$