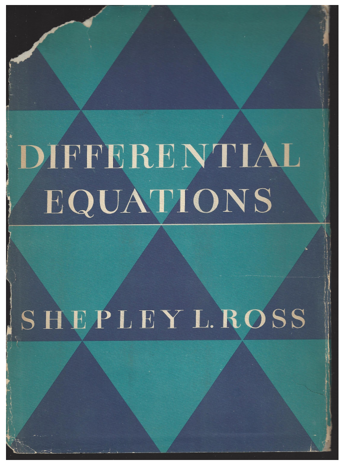
A Solution Manual For

Differential equations, Shepley L. Ross, 1964



Nasser M. Abbasi

December 13, 2025 Compiled on December 13, 2025 at 8:16am

Contents

Chapter 1

Lookup tables for all problems in current book

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Table 1.1: Lookup table for all problems in current section

ID	problem	ODE	Solved?	Maple	Mma	Sympy
4076	1	$5yx+4y^2+1+\left(x^2+2yx\right)y'=0$	✓	✓	✓	✗
4077	2	$2x\tan\left(y\right)+\left(x-x^2\tan\left(y\right)\right)y'=0$	✓	✓	✓	✗
4078	3	$y^2\left(x^2+1\right)+y+\left(2yx+1\right)y'=0$	✗	✗	✗	✗
4079	4	$4xy^2+6y+\left(5x^2y+8x\right)y'=0$	✓	✓	✓	✓
4080	5	$5x+2y+1+\left(2x+y+1\right)y'=0$	✓	✓	✓	✓
4081	6	$3x-y+1-\left(6x-2y-3\right)y'=0$	✓	✓	✓	✓
4082	7	$x-2y-3+\left(2x+y-1\right)y'=0$	✓	✓	✓	✓
4083	8	$6x+4y+1+\left(4x+2y+2\right)y'=0$ $y\left(\frac{1}{2}\right)=3$	✓	✓	✓	✓
4084	9	$3x-y-6+\left(x+y+2\right)y'=0$ $y(2)=-2$	✓	✓	✓	✓
4085	10	$2x+3y+1+\left(4x+6y+1\right)y'=0$ $y(-2)=2$	✓	✓	✓	✓

Chapter 2

Book Solved Problems

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Local contents

2.1.1 Problem 1

Local contents

Internal problem ID [4076]
Book : Differential equations, Shepley L. Ross, 1964
Section : 2.4, page 55
Problem number : 1
Date solved : `Saturday, December 06, 2025 at 04:16:17 PM`
CAS classification : `[_rational, [_Abel, ‘2nd type‘, ‘class B‘]]`

0.123 (sec)
Entering first order ode exact solver

2.1.1.1 Solved using first_order_ode_exact

$$5yx + 4y^2 + 1 + \left(x^2 + 2yx\right) y' = 0$$

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(x^2+2yx\right)\mathrm{d}y &= \left(-5yx-4y^2-1\right)\mathrm{d}x\\ \left(5yx+4y^2+1\right)\mathrm{d}x + \left(x^2+2yx\right)\mathrm{d}y &= 0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= 5yx+4y^2+1\\ N(x,y) &= x^2+2yx\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(5yx+4y^2+1\right)\\ &= 5x+8y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(x^2+2yx\right)\\ &= 2x+2y\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{x\,(x+2y)}((5x+8y)-(2x+2y))\\ &= \frac{3}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int \frac{3}{x} \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{3\ln(x)} \\ &= x^3\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= x^3\left(5yx + 4y^2 + 1\right) \\ &= \left(5yx + 4y^2 + 1\right)x^3\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= x^3\left(x^2 + 2yx\right) \\ &= x^4(x + 2y)\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\left(5yx + 4y^2 + 1\right)x^3\right) + \left(x^4(x + 2y)\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \left(5yx + 4y^2 + 1\right)x^3 \, \mathrm{d}x$$

$$\phi = y\,x^5 + x^4y^2 + \frac{1}{4}x^4 + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\begin{aligned}\frac{\partial \phi}{\partial y} &= x^5 + 2y\,x^4 + f'(y) \qquad (4)\\ &= x^4(x+2y) + f'(y)\end{aligned}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^4(x+2y)$. Therefore equation (4) becomes

$$x^4(x+2y) = x^4(x+2y) + f'(y) \qquad (5)$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = y\,x^5 + x^4y^2 + \frac{1}{4}x^4 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = y\,x^5 + x^4y^2 + \frac{1}{4}x^4$$

Solving for y gives

$$y = \frac{-x^3 - \sqrt{x^6 - x^4 + 4c_1}}{2x^2}$$

$$y = \frac{-x^3 + \sqrt{x^6 - x^4 + 4c_1}}{2x^2}$$

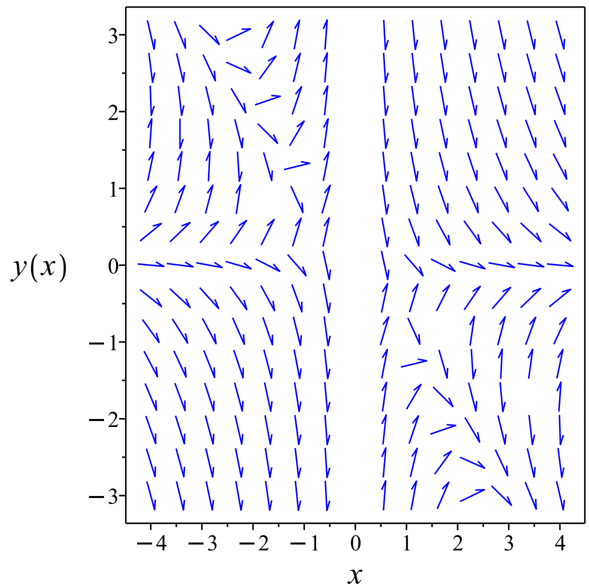


Figure 2.1: Slope field $5yx + 4y^2 + 1 + (x^2 + 2yx)\, y' = 0$

Summary of solutions found

$$y = \frac{-x^3 - \sqrt{x^6 - x^4 + 4c_1}}{2x^2}$$

$$y = \frac{-x^3 + \sqrt{x^6 - x^4 + 4c_1}}{2x^2}$$

Entering first order ode abel second kind solver

$$5yx + 4y^2 + 1 + \left(x^2 + 2yx\right) y' = 0$$

0.040 (sec)

2.1.1.2 Solved using first_order_ode_abel_second_kind_case_5

Simplifying the above gives

$$y = \frac{-x^3 + \sqrt{x^6 - x^4 + 4c_1}}{2x^2}$$

$$y = \frac{-x^3 - \sqrt{x^6 - x^4 + 4c_1}}{2x^2}$$

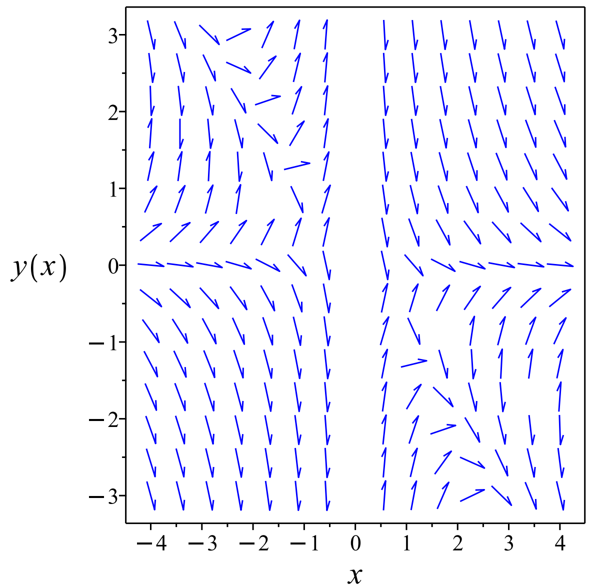


Figure 2.2: Slope field $5yx + 4y^2 + 1 + (x^2 + 2yx)\, y' = 0$

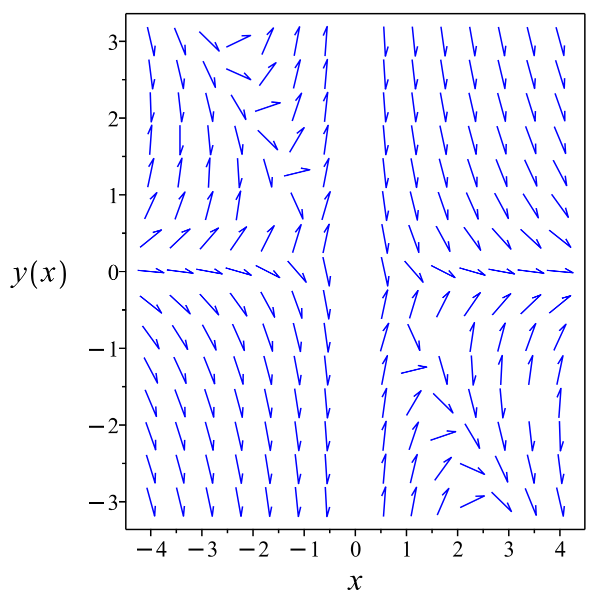


Figure 2.3: Slope field $5yx + 4y^2 + 1 + (x^2 + 2yx)\, y' = 0$

Summary of solutions found

$$y = \frac{-x^3 - \sqrt{x^6 - x^4 + 4c_1}}{2x^2}$$

$$y = \frac{-x^3 + \sqrt{x^6 - x^4 + 4c_1}}{2x^2}$$

2.1.1.3 ✓ Maple. Time used: 0.002 (sec). Leaf size: 59

```
ode:=5*y(x)*x+4*y(x)^2+1+(x^2+2*y(x)*x)*diff(y(x),x) = 0;
dsolve(ode,y(x), singsol=all);
```

$$y = \frac{-x^3 - \sqrt{x^6 - x^4 - 4c_1}}{2x^2}$$
$$y = \frac{-x^3 + \sqrt{x^6 - x^4 - 4c_1}}{2x^2}$$

Maple trace

```
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful
```

Maple step by step

Let's solve
$5xy(x) + 4y(x)^2 + 1 + (x^2 + 2xy(x))\left(\frac{d}{dx}y(x)\right) = 0$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Solve for the highest derivative
  $\frac{d}{dx}y(x) = \frac{-5xy(x) - 4y(x)^2 - 1}{x^2 + 2xy(x)}$

2.1.1.4 ✓ Mathematica. Time used: 0.455 (sec). Leaf size: 84

```
ode=(5*x*y[x]+4*y[x]^2+1)+(x^2+2*x*y[x])*D[y[x],x]==0;
ic={};
DSolve[{ode,ic},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\frac{x^5 + \sqrt{x^3}\sqrt{x^7 - x^5 + 4c_1 x}}{2x^4}$$
$$y(x) \to -\frac{x}{2} + \frac{\sqrt{x^3}\sqrt{x^7 - x^5 + 4c_1 x}}{2x^4}$$

2.1.1.5 ✗ Sympy

```
from sympy import *
x = symbols("x")
y = Function("y")
ode = Eq(5*x*y(x) + (x**2 + 2*x*y(x))*Derivative(y(x), x) + 4*y(x)**2 +
    1,0)
ics = {}
dsolve(ode,func=y(x),ics=ics)
```

```
Timed Out
```

2.1.2 Problem 2

Local contents

Internal problem ID [4077]
Book : Differential equations, Shepley L. Ross, 1964
Section : 2.4, page 55
Problem number : 2
Date solved : Saturday, December 06, 2025 at 04:16:19 PM
CAS classification : [[_1st_order, _with_exponential_symmetries]]

0.301 (sec) Entering first order ode exact solver

2.1.2.1 Solved using first_order_ode_exact

$$2x \tan(y) + \left(x - x^2 \tan(y)\right) y' = 0$$

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x\tan(y)-1)\,\mathrm{d}y &= (2\tan(y))\,\mathrm{d}x\\ (-2\tan(y))\,\mathrm{d}x + (x\tan(y)-1)\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -2\tan(y)\\ N(x,y) &= x\tan(y)-1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-2\tan(y))\\ &= -2\sec(y)^2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x\tan(y)-1)\\ &= \tan(y)\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{x\tan(y)-1}\left(\left(-2-2\tan(y)^2\right)-(\tan(y))\right)\\ &= \frac{-\sin(y)-2\sec(y)}{x\sin(y)-\cos(y)}\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned} B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\ &= -\frac{\cot(y)}{2}\left((\tan(y)) - \left(-2 - 2\tan(y)^2\right)\right) \\ &= -\frac{1}{2} - 2\csc(2y) \end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned} \mu &= e^{\int B \, \mathrm{d}y} \\ &= e^{\int -\frac{1}{2} - 2\csc(2y) \, \mathrm{d}y} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-\frac{y}{2} + \ln(\csc(2y) + \cot(2y))} \\ &= (\csc(2y) + \cot(2y))\, \mathrm{e}^{-\frac{y}{2}} \end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= (\csc(2y) + \cot(2y))\, \mathrm{e}^{-\frac{y}{2}}(-2\tan(y)) \\ &= -2\,\mathrm{e}^{-\frac{y}{2}} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= (\csc(2y) + \cot(2y))\, \mathrm{e}^{-\frac{y}{2}}(x\tan(y) - 1) \\ &= \mathrm{e}^{-\frac{y}{2}}(x - \cot(y)) \end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(-2\,\mathrm{e}^{-\frac{y}{2}}\right) + \left(\mathrm{e}^{-\frac{y}{2}}(x - \cot(y))\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int -2\,\mathrm{e}^{-\frac{y}{2}}\,\mathrm{d}x$$

$$\phi = -2\,\mathrm{e}^{-\frac{y}{2}}x + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{-\frac{y}{2}}x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{-\frac{y}{2}}(x - \cot(y))$. Therefore equation (4) becomes

$$\mathrm{e}^{-\frac{y}{2}}(x - \cot(y)) = \mathrm{e}^{-\frac{y}{2}}x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\mathrm{e}^{-\frac{y}{2}}\cot(y)$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\mathrm{e}^{-\frac{y}{2}}\cot(y)\right)\mathrm{d}y$$

$$f(y) = \int^{y} -\mathrm{e}^{-\frac{\tau}{2}}\cot(\tau)\,d\tau + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -2\,\mathrm{e}^{-\frac{y}{2}}x + \int^{y} -\mathrm{e}^{-\frac{\tau}{2}}\cot(\tau)\,d\tau + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = -2\,\mathrm{e}^{-\frac{y}{2}}x + \int^{y} -\mathrm{e}^{-\frac{\tau}{2}}\cot(\tau)\,d\tau$$

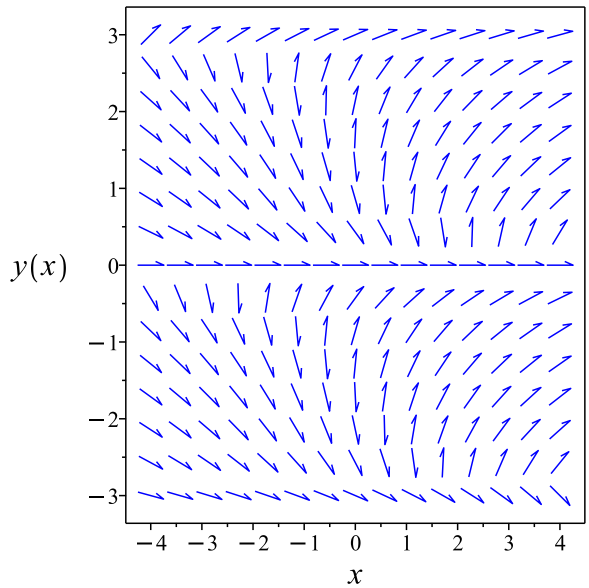


Figure 2.4: Slope field $2x\tan(y)+(x-x^2\tan(y))\,y'=0$

Summary of solutions found

$$-2\,\mathrm{e}^{-\frac{y}{2}}x+\int^{y}-\mathrm{e}^{-\frac{\tau}{2}}\cot(\tau)\,d\tau=c_1$$

2.1.2.2 ✓ Maple. Time used: 0.014 (sec). Leaf size: 28

```
ode:=2*x*tan(y(x))+(x-x^2*tan(y(x)))*diff(y(x),x) = 0;
dsolve(ode,y(x), singsol=all);
```

$$x-\left(-\frac{\int^{y}\mathrm{e}^{-\frac{\_a}{2}}\cot(\_a)\,d\_a}{2}+c_1\right)\mathrm{e}^{\frac{y}{2}}=0$$

Maple trace

```
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful
```

Maple step by step

Let's solve

$2x\tan(y(x)) + (x - x^2\tan(y(x)))\left(\frac{d}{dx}y(x)\right) = 0$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Solve for the highest derivative

  $\frac{d}{dx}y(x) = -\frac{2x\tan(y(x))}{x - x^2\tan(y(x))}$

2.1.2.3 ✓ Mathematica. Time used: 0.253 (sec). Leaf size: 78

```
ode=(2*x*Tan[y[x]])+(x-x^2*Tan[y[x]])*D[y[x],x]==0;
ic={};
DSolve[{ode,ic},y[x],x,IncludeSingularSolutions->True]
```

$$\text{Solve}\left[x = \frac{1}{34}\left((8-2i)e^{2iy(x)}\,\text{Hypergeometric2F1}\left(1, 1+\frac{i}{4}, 2+\frac{i}{4}, e^{2iy(x)}\right) - 34i\,\text{Hypergeometric2F1}\left(\frac{i}{4}, 1, 1+\frac{i}{4}, e^{2iy(x)}\right)\right) + c_1 e^{\frac{y(x)}{2}}, y(x)\right]$$

2.1.2.4 ✗ Sympy

```
from sympy import *
x = symbols("x")
y = Function("y")
ode = Eq(2*x*tan(y(x)) + (-x**2*tan(y(x)) + x)*Derivative(y(x), x),0)
ics = {}
dsolve(ode,func=y(x),ics=ics)
```

```
TypeError : NoneType object is not subscriptable
```

2.1.3 Problem 3

Local contents

Internal problem ID [4078]
Book : Differential equations, Shepley L. Ross, 1964
Section : 2.4, page 55
Problem number : 3
Date solved : `Saturday, December 06, 2025 at 04:16:21 PM`
CAS classification : `[_rational, [_Abel, '2nd type', 'class B']]`

$$y^2(x^2+1)+y+(2yx+1)y'=0$$

Unknown ode type.

2.1.3.1 ✗ Maple

```
ode:=y(x)^2*(x^2+1)+y(x)+(2*y(x)*x+1)*diff(y(x),x) = 0;
dsolve(ode,y(x), singsol=all);
```

No solution found

Maple trace

```
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
```

```
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
   -> Computing symmetries using: way = 3
   -> Computing symmetries using: way = 4
   -> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type
```

Maple step by step

Let's solve
$y(x)^2 (x^2+1)+y(x)+(2xy(x)+1)\left(\frac{d}{dx}y(x)\right)=0$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Solve for the highest derivative
  $\frac{d}{dx}y(x)=-\frac{y(x)^2 (x^2+1)+y(x)}{2xy(x)+1}$

2.1.3.2 ✗ Mathematica

```
ode=(y[x]^2*(x^2+1)+y[x])+(2*x*y[x]+1)*D[y[x],x]==0;
ic={};
DSolve[{ode,ic},y[x],x,IncludeSingularSolutions->True]
```

Not solved

2.1.3.3 ✗ Sympy

```
from sympy import *
x = symbols("x")
y = Function("y")
ode = Eq((x**2 + 1)*y(x)**2 + (2*x*y(x) + 1)*Derivative(y(x), x) + y(x),0)
ics = {}
dsolve(ode,func=y(x),ics=ics)
```

```
Timed Out
```

2.1.4 Problem 4

Local contents

Internal problem ID [4079]
Book : Differential equations, Shepley L. Ross, 1964
Section : 2.4, page 55
Problem number : 4
Date solved : `Saturday, December 06, 2025 at 04:16:29 PM`
CAS classification :
`[[_homogeneous, ‘class G‘], _rational, [_Abel, ‘2nd type‘, ‘class B‘]]`

0.174 (sec) Entering first order ode exact solver

2.1.4.1 Solved using first_order_ode_exact

$$4xy^2 + 6y + \left(5yx^2 + 8x\right) y' = 0$$

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(5y\,x^2+8x\right)\mathrm{d}y &= \left(-4y^2x-6y\right)\mathrm{d}x\\ \left(4y^2x+6y\right)\mathrm{d}x+\left(5y\,x^2+8x\right)\mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= 4y^2x+6y\\ N(x,y) &= 5y\,x^2+8x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(4y^2x+6y\right)\\ &= 8yx+6\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(5y\,x^2+8x\right)\\ &= 10yx+8\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)\\ &= \frac{1}{5y\,x^2+8x}\left((8yx+6)-(10yx+8)\right)\\ &= \frac{-2yx-2}{5y\,x^2+8x}\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned} B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\ &= \frac{1}{4y^2x + 6y}((10yx + 8) - (8yx + 6)) \\ &= \frac{yx + 1}{2y^2x + 3y} \end{aligned}$$

Since B depends on x, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$\begin{aligned} R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} \\ &= \frac{(10yx + 8) - (8yx + 6)}{x(4y^2x + 6y) - y(5y\,x^2 + 8x)} \\ &= \frac{-2yx - 2}{yx(yx + 2)} \end{aligned}$$

Replacing all powers of terms xy by t gives

$$R = \frac{-2t - 2}{t(t + 2)}$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$\begin{aligned} \mu &= e^{\int R \, \mathrm{d}t} \\ &= e^{\int \left(\frac{-2t-2}{t(t+2)}\right) \mathrm{d}t} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-\ln(t(t+2))} \\ &= \frac{1}{t(t + 2)} \end{aligned}$$

Now t is replaced back with xy giving

$$\mu = \frac{1}{yx(yx + 2)}$$

Multiplying M and N by this integrating factor gives new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{yx\left(yx+2\right)}\left(4y^{2}x+6y\right)\\ &= \frac{4yx+6}{x\left(yx+2\right)}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \frac{1}{yx\left(yx+2\right)}\left(5y\,x^{2}+8x\right)\\ &= \frac{5yx+8}{y\left(yx+2\right)}\end{aligned}$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$\begin{aligned}\overline{M}+\overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \left(\frac{4yx+6}{x\left(yx+2\right)}\right)+\left(\frac{5yx+8}{y\left(yx+2\right)}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int\overline{M}\,\mathrm{d}x$$

$$\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int\frac{4yx+6}{x\left(yx+2\right)}\,\mathrm{d}x$$

$$\phi = 3\ln\left(x\right)+\ln\left(yx+2\right)+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial\phi}{\partial y} = \frac{x}{yx+2}+f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{5yx+8}{y(yx+2)}$. Therefore equation (4) becomes

$$\frac{5yx+8}{y\,(yx+2)} = \frac{x}{yx+2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{4}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{4}{y}\right) \mathrm{d}y$$
$$f(y) = 4\ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = 3\ln(x) + \ln(yx+2) + 4\ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = 3\ln(x) + \ln(yx+2) + 4\ln(y)$$

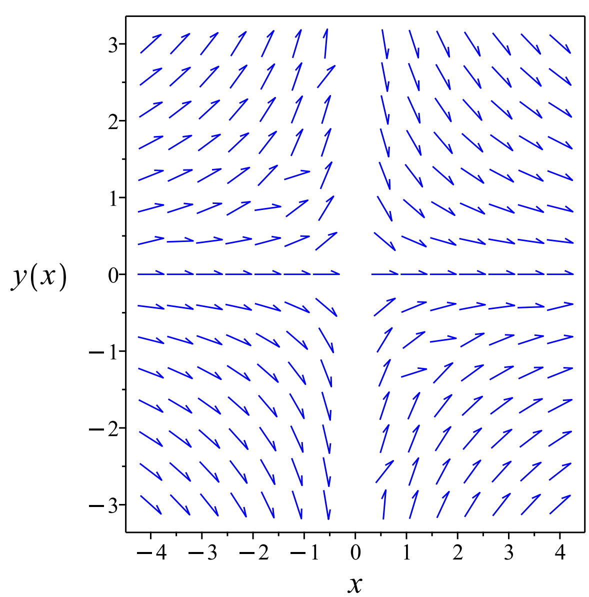


Figure 2.5: Slope field $4xy^2 + 6y + (5yx^2 + 8x)\,y' = 0$

Summary of solutions found

$$3\ln(x)+\ln(yx+2)+4\ln(y)=c_1$$

0.152 (sec) Entering first order ode isobaric solver

2.1.4.2 Solved using first_order_ode_isobaric

$$4xy^2+6y+\left(5yx^2+8x\right)y'=0$$

Solving for y' gives

$$y'=-\frac{2y(2yx+3)}{x\left(5yx+8\right)} \tag{1}$$

Each of the above ode's is now solved An ode $y'=f(x,y)$ is isobaric if

$$f(tx,t^my)=t^{m-1}f(x,y) \tag{1}$$

Where here

$$f(x,y)=-\frac{2y(2yx+3)}{x\left(5yx+8\right)} \tag{2}$$

m is the order of isobaric. Substituting (2) into (1) and solving for m gives

$$m=-1$$

Since the ode is isobaric of order $m=-1$, then the substitution

$$\begin{aligned} y&=ux^m\\ &=\frac{u}{x}\end{aligned}$$

Converts the ODE to a separable in $u(x)$. Performing this substitution gives

$$-\frac{u(x)}{x^2}+\frac{u'(x)}{x}=-\frac{2u(x)\left(2u(x)+3\right)}{x^2\left(5u\left(x\right)+8\right)}$$

The ode

$$u'(x)=\frac{u(x)\left(u(x)+2\right)}{x\left(5u\left(x\right)+8\right)} \tag{2.1}$$

is separable as it can be written as

$$\begin{aligned} u'(x)&=\frac{u(x)\left(u(x)+2\right)}{x\left(5u\left(x\right)+8\right)}\\ &=f(x)g(u)\end{aligned}$$

Where

$$
\begin{aligned}
f(x) &= \frac{1}{x} \\
g(u) &= \frac{u(u+2)}{5u+8}
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\int \frac{1}{g(u)}\,du &= \int f(x)\,dx \\
\int \frac{5u+8}{u\,(u+2)}\,du &= \int \frac{1}{x}\,dx
\end{aligned}
$$

$$
4\ln\left(u(x)\right) + \ln\left(u(x)+2\right) = \ln\left(x\right) + c_1
$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or

$$
\frac{u(u+2)}{5u+8} = 0
$$

for $u(x)$ gives

$$
\begin{aligned}
u(x) &= -2 \\
u(x) &= 0
\end{aligned}
$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$
\begin{aligned}
4\ln\left(u(x)\right) + \ln\left(u(x)+2\right) &= \ln\left(x\right) + c_1 \\
u(x) &= -2 \\
u(x) &= 0
\end{aligned}
$$

Converting $4\ln\left(u(x)\right) + \ln\left(u(x)+2\right) = \ln\left(x\right) + c_1$ back to y gives

$$
4\ln\left(yx\right) + \ln\left(yx+2\right) = \ln\left(x\right) + c_1
$$

Converting $u(x) = -2$ back to y gives

$$
yx = -2
$$

Converting $u(x) = 0$ back to y gives

$$yx = 0$$

Solving for y gives

$$4\ln(yx) + \ln(yx + 2) = \ln(x) + c_1$$
$$y = 0$$
$$y = -\frac{2}{x}$$

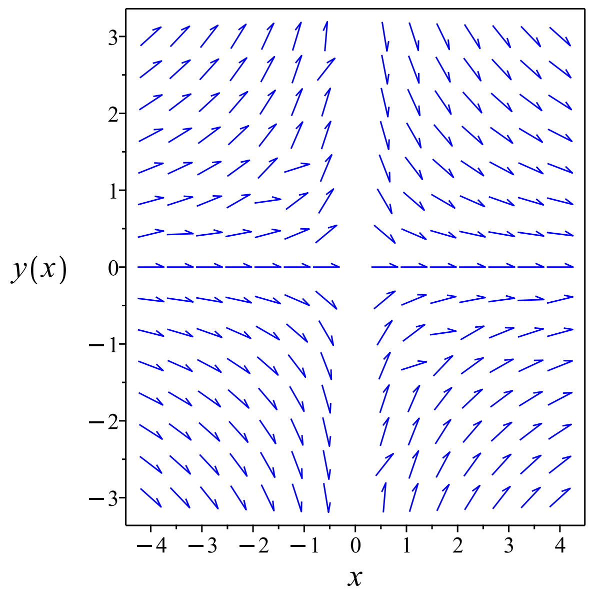


Figure 2.6: Slope field $4xy^2 + 6y + (5yx^2 + 8x)\,y' = 0$

Summary of solutions found

$$4\ln(yx) + \ln(yx + 2) = \ln(x) + c_1$$
$$y = 0$$
$$y = -\frac{2}{x}$$

0.073 (sec) Entering first order ode homog type G solver

2.1.4.3 Solved using first_order_ode_homog_type_G

$$4xy^2 + 6y + (5yx^2 + 8x)\, y' = 0$$

Multiplying the right side of the ode, which is $-\frac{2y(2yx+3)}{x(5yx+8)}$ by $\frac{x}{y}$ gives

$$\begin{aligned}
y' &= \left(\frac{x}{y}\right) - \frac{2y(2yx+3)}{x\,(5yx+8)} \\
&= -\frac{2(2yx+3)}{5yx+8} \\
&= F(x,y)
\end{aligned}$$

Since $F(x,y)$ has y, then let

$$\begin{aligned}
f_x &= x\left(\frac{\partial}{\partial x}F(x,y)\right) \\
&= -\frac{2yx}{(5yx+8)^2} \\
f_y &= y\left(\frac{\partial}{\partial y}F(x,y)\right) \\
&= -\frac{2yx}{(5yx+8)^2} \\
\alpha &= \frac{f_x}{f_y} \\
&= 1
\end{aligned}$$

Since α is independent of x, y then this is Homogeneous type G.

Let

$$\begin{aligned}
y &= \frac{z}{x^\alpha} \\
&= \frac{z}{x}
\end{aligned}$$

Substituting the above back into $F(x,y)$ gives

$$F(z) = -\frac{2(2z+3)}{5z+8}$$

We see that $F(z)$ does not depend on x nor on y. If this was not the case, then this method will not work.

Therefore, the implicit solution is given by

$$\ln(x) - c_1 - \int^{yx^\alpha} \frac{1}{z\,(\alpha + F(z))}\, dz = 0$$

Which gives

$$\ln(x) - c_1 + \int^{yx} \frac{1}{z\left(-1 + \frac{4z+6}{5z+8}\right)} dz = 0$$

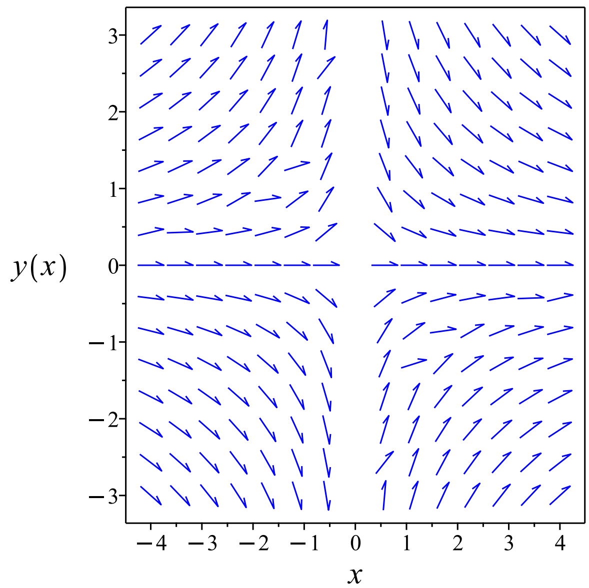


Figure 2.7: Slope field $4xy^2 + 6y + (5yx^2 + 8x)\, y' = 0$

Summary of solutions found

$$\ln(x) - c_1 + \int^{yx} \frac{1}{z\left(-1 + \frac{4z+6}{5z+8}\right)} dz = 0$$

Entering first order ode abel second kind solver

$$4xy^2 + 6y + \left(5yx^2 + 8x\right) y' = 0$$

0.179 (sec)

2.1.4.4 Solved using first_order_ode_abel_second_kind_case_3

Applying transformation

$$y = \frac{1}{u(x)} - g$$

Results in the new ode which is Abel first kind

$$\frac{4u(x)^2}{x} + \frac{6u(x)}{x} + (5xu(x) + 8x)\left(-\frac{u(x)}{x^2} + \frac{u'(x)}{x}\right) = 0$$

Entering first order ode separable solver

Which is now solved The ode

$$u'(x) = \frac{u(x)\,(u(x) + 2)}{x\,(5u(x) + 8)} \tag{2.2}$$

is separable as it can be written as

$$\begin{aligned}u'(x) &= \frac{u(x)\,(u(x)+2)}{x\,(5u\,(x)+8)}\\ &= f(x)g(u)\end{aligned}$$

Where

$$\begin{aligned}f(x) &= \frac{1}{x}\\ g(u) &= \frac{u(u+2)}{5u+8}\end{aligned}$$

Integrating gives

$$\begin{aligned}\int \frac{1}{g(u)}\,du &= \int f(x)\,dx\\ \int \frac{5u+8}{u\,(u+2)}\,du &= \int \frac{1}{x}\,dx\end{aligned}$$

$$4\ln\left(u(x)\right)+\ln\left(u(x)+2\right)=\ln\left(x\right)+c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u)=0$ or

$$\frac{u(u+2)}{5u+8}=0$$

for $u(x)$ gives

$$\begin{aligned}u(x) &= -2\\ u(x) &= 0\end{aligned}$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$4\ln\left(u(x)\right)+\ln\left(u(x)+2\right)=\ln\left(x\right)+c_1$$

$$u(x)=-2$$

$$u(x)=0$$

Substituting $u(x) = \frac{1}{y+\frac{8}{5x}}$ in the above solution gives

$$4\ln(yx) + \ln(yx+2) = \ln(x) + c_1$$

Now we transform the solution $u(x) = -2$ to y using $u(x) = \frac{1}{y+\frac{8}{5x}}$ which gives

$$\begin{aligned} y &= \frac{1}{u(x)} - \frac{8}{5x} \\ &= -\frac{1}{2} - \frac{8}{5x} \end{aligned}$$

The solution

$$y = -\frac{1}{2} - \frac{8}{5x}$$

was found not to satisfy the ode or the IC. Hence it is removed.

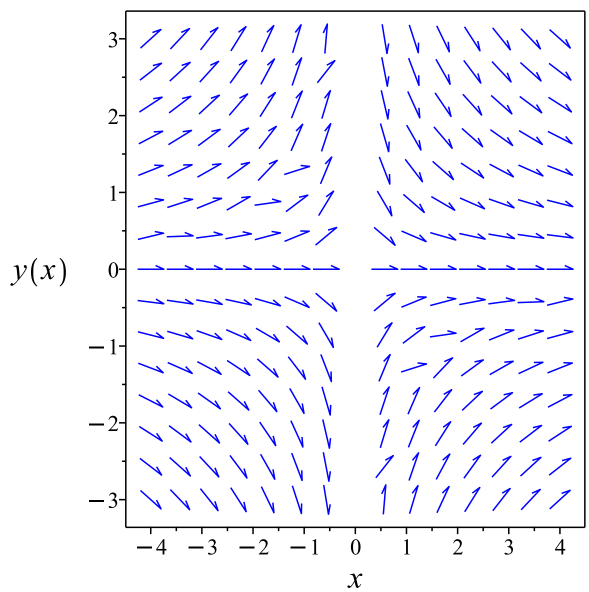


Figure 2.8: Slope field $4xy^2 + 6y + (5yx^2 + 8x)\,y' = 0$

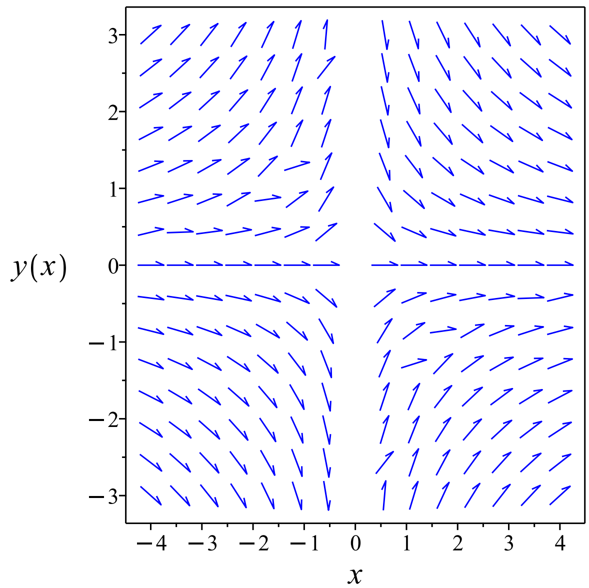


Figure 2.9: Slope field $4xy^2 + 6y + (5yx^2 + 8x)\,y' = 0$

Summary of solutions found

$$4\ln(yx) + \ln(yx + 2) = \ln(x) + c_1$$

1.111 (sec) Entering first order ode LIE solver

2.1.4.5 Solved using first_order_ode_LIE

$$4xy^2 + 6y + \left(5yx^2 + 8x\right)y' = 0$$

Writing the ode as

$$\begin{aligned} y' &= -\frac{2y(2yx + 3)}{x\,(5yx + 8)} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2-\frac{2y(2yx+3)\left(b_3-a_2\right)}{x\left(5yx+8\right)}-\frac{4y^2(2yx+3)^2 a_3}{x^2\left(5yx+8\right)^2}\\
&-\left(-\frac{4y^2}{x\left(5yx+8\right)}+\frac{2y(2yx+3)}{x^2\left(5yx+8\right)}+\frac{10y^2(2yx+3)}{x\left(5yx+8\right)^2}\right)\left(xa_2+ya_3+a_1\right)\\
&-\left(-\frac{2(2yx+3)}{x\left(5yx+8\right)}-\frac{4y}{5yx+8}+\frac{10y(2yx+3)}{\left(5yx+8\right)^2}\right)\left(xb_2+yb_3+b_1\right)=0
\end{aligned}
\tag{5E}
$$

Putting the above in normal form gives

$$
\frac{45x^4y^2b_2-36x^2y^4a_3+20x^3y^2b_1-20x^2y^3a_1+144x^3yb_2+2x^2y^2a_2+2x^2y^2b_3-108x\,y^3a_3+64x^2yb_1}{\left(5yx+8\right)^2x^2}
$$

$$=0$$

Setting the numerator to zero gives

$$
\begin{aligned}
&45x^4y^2b_2-36x^2y^4a_3+20x^3y^2b_1-20x^2y^3a_1+144x^3yb_2+2x^2y^2a_2+2x^2y^2b_3\\
&-108x\,y^3a_3+64x^2yb_1-60x\,y^2a_1+112b_2x^2-84y^2a_3+48xb_1-48ya_1=0
\end{aligned}
\tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x,y\}$ in them.

$$\{x,y\}$$

The following substitution is now made to be able to collect on all terms with $\{x,y\}$ in them

$$\{x=v_1,y=v_2\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&-36a_3v_1^2v_2^4+45b_2v_1^4v_2^2-20a_1v_1^2v_2^3+20b_1v_1^3v_2^2+2a_2v_1^2v_2^2\\
&-108a_3v_1v_2^3+144b_2v_1^3v_2+2b_3v_1^2v_2^2-60a_1v_1v_2^2\\
&+64b_1v_1^2v_2-84a_3v_2^2+112b_2v_1^2-48a_1v_2+48b_1v_1=0
\end{aligned}
\tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1,v_2\}$$

Equation (7E) now becomes

$$
\begin{aligned}
45b_2v_1^4v_2^2 + 20b_1v_1^3v_2^2 + 144b_2v_1^3v_2 - 36a_3v_1^2v_2^4 - 20a_1v_1^2v_2^3 + (2a_2 + 2b_3)\, v_1^2v_2^2 \\
+ 64b_1v_1^2v_2 + 112b_2v_1^2 - 108a_3v_1v_2^3 - 60a_1v_1v_2^2 + 48b_1v_1 - 84a_3v_2^2 - 48a_1v_2 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-60a_1 &= 0 \\
-48a_1 &= 0 \\
-20a_1 &= 0 \\
-108a_3 &= 0 \\
-84a_3 &= 0 \\
-36a_3 &= 0 \\
20b_1 &= 0 \\
48b_1 &= 0 \\
64b_1 &= 0 \\
45b_2 &= 0 \\
112b_2 &= 0 \\
144b_2 &= 0 \\
2a_2 + 2b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= -b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= -x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(-\frac{2y(2yx+3)}{x\,(5yx+8)}\right)(-x) \\
&= \frac{y(yx+2)}{5yx+8} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{y(yx+2)}{5yx+8}} dy
\end{aligned}
$$

Which results in

$$
S = 4\ln(y) + \ln(yx+2)
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$
\omega(x, y) = -\frac{2y(2yx+3)}{x\,(5yx+8)}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{y}{yx+2} \\
S_y &= \frac{4}{y} + \frac{x}{yx+2}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{3}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{3}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

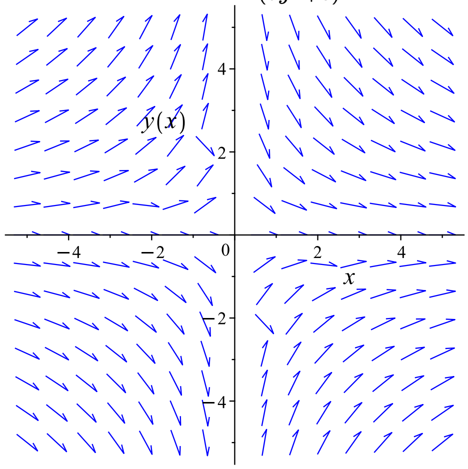
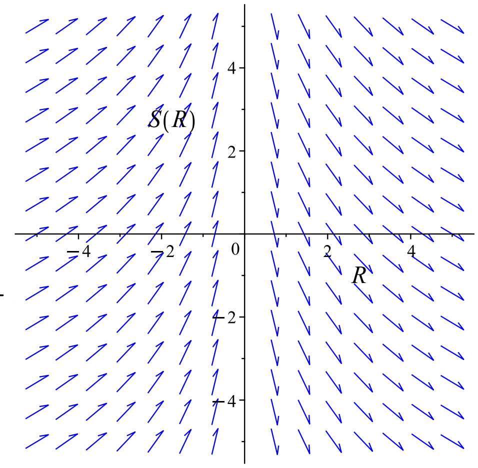
Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$
\begin{aligned}
\int dS &= \int -\frac{3}{R}\, dR \\
S(R) &= -3\ln(R) + c_2
\end{aligned}
$$

To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$4\ln(y) + \ln(yx+2) = -3\ln(x) + c_2$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{2y(2yx+3)}{x(5yx+8)}$ 	$R = x$ $S = 4\ln(y) + \ln(yx +$	$\frac{dS}{dR} = -\frac{3}{R}$ 

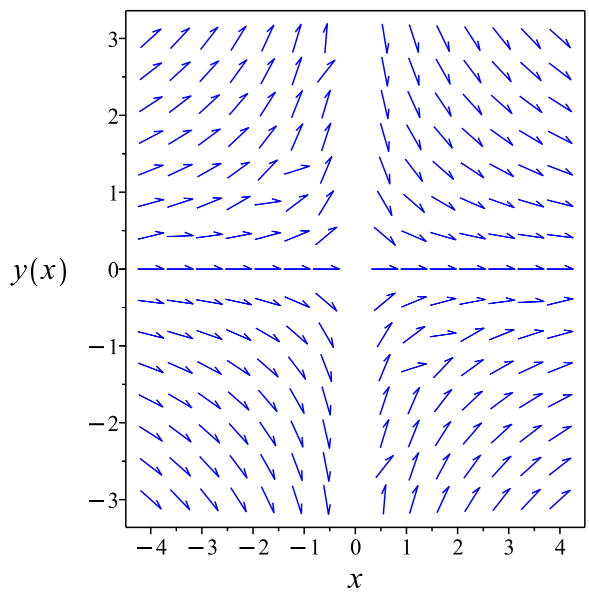


Figure 2.10: Slope field $4xy^2 + 6y + (5yx^2 + 8x)\,y' = 0$

Summary of solutions found

$$4\ln(y) + \ln(yx + 2) = -3\ln(x) + c_2$$

2.1.4.6 ✓ Maple. Time used: 0.015 (sec). Leaf size: 23

```
ode:=4*x*y(x)^2+6*y(x)+(5*x^2*y(x)+8*x)*diff(y(x),x) = 0;
dsolve(ode,y(x), singsol=all);
```

$$y = \frac{\mathrm{RootOf}\left(-\ln(x) + c_1 + 4\ln(\_Z) + \ln(2 + \_Z)\right)}{x}$$

Maple trace

```
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful
```

Maple step by step

Let's solve
$4xy(x)^2 + 6y(x) + (5x^2y(x) + 8x)\left(\frac{d}{dx}y(x)\right) = 0$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Solve for the highest derivative
  $\frac{d}{dx}y(x) = \frac{-4xy(x)^2 - 6y(x)}{5x^2y(x) + 8x}$

2.1.4.7 ✓ Mathematica. Time used: 1.996 (sec). Leaf size: 156

```
ode=(4*x*y[x]^2+6*y[x])+(5*x^2*y[x]+8*x)*D[y[x],x]==0;
ic={};
DSolve[{ode,ic},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \text{Root}\left[-\#1^5 - \frac{2\#1^4}{x} + \frac{e^{c_1}}{x^4}\&, 1\right]$$
$$y(x) \to \text{Root}\left[-\#1^5 - \frac{2\#1^4}{x} + \frac{e^{c_1}}{x^4}\&, 2\right]$$
$$y(x) \to \text{Root}\left[-\#1^5 - \frac{2\#1^4}{x} + \frac{e^{c_1}}{x^4}\&, 3\right]$$
$$y(x) \to \text{Root}\left[-\#1^5 - \frac{2\#1^4}{x} + \frac{e^{c_1}}{x^4}\&, 4\right]$$
$$y(x) \to \text{Root}\left[-\#1^5 - \frac{2\#1^4}{x} + \frac{e^{c_1}}{x^4}\&, 5\right]$$

2.1.4.8 ✓ Sympy. Time used: 0.343 (sec). Leaf size: 20

```
from sympy import *
x = symbols("x")
y = Function("y")
ode = Eq(4*x*y(x)**2 + (5*x**2*y(x) + 8*x)*Derivative(y(x), x) + 6*y(x),0)
ics = {}
dsolve(ode,func=y(x),ics=ics)
```

$$-\log(x) + 4\log(xy(x)) + \log(xy(x) + 2) = C_1$$

2.1.5 Problem 5

Local contents

Internal problem ID [4080]
Book : Differential equations, Shepley L. Ross, 1964
Section : 2.4, page 55
Problem number : 5
Date solved : `Saturday, December 06, 2025 at 04:16:32 PM`
CAS classification :
`[[_homogeneous, 'class C'], _exact, _rational, [_Abel, '2nd type', 'class A']]`

0.079 (sec) Entering first order ode exact solver

2.1.5.1 Solved using first_order_ode_exact

$$5x + 2y + 1 + (2x + y + 1)\, y' = 0$$

To solve an ode of the form

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(2x+y+1)\,\mathrm{d}y&=(-5x-2y-1)\,\mathrm{d}x\\ (5x+2y+1)\,\mathrm{d}x+(2x+y+1)\,\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=5x+2y+1\\ N(x,y)&=2x+y+1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}(5x+2y+1)\\ &=2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}(2x+y+1)\\ &=2\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x}=M \tag{1}$$

$$\frac{\partial\phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x = \int M \,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x = \int 5x + 2y + 1 \,\mathrm{d}x$$

$$\phi = \frac{x(5x + 4y + 2)}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 2x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 2x + y + 1$. Therefore equation (4) becomes

$$2x + y + 1 = 2x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = y + 1$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (y + 1)\,\mathrm{d}y$$

$$f(y) = \frac{1}{2}y^2 + y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x(5x + 4y + 2)}{2} + \frac{y^2}{2} + y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = \frac{x(5x + 4y + 2)}{2} + \frac{y^2}{2} + y$$

Simplifying the above gives

$$2yx + \frac{5x^2}{2} + x + \frac{y^2}{2} + y = c_1$$

Solving for y gives

$$y = -2x - 1 - \sqrt{-x^2 + 2c_1 + 2x + 1}$$

$$y = -2x - 1 + \sqrt{-x^2 + 2c_1 + 2x + 1}$$

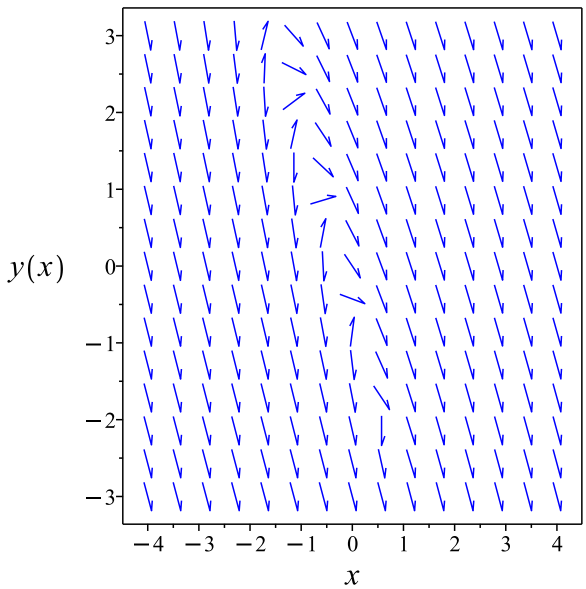


Figure 2.11: Slope field $5x + 2y + 1 + (2x + y + 1) y' = 0$

Summary of solutions found

$$y = -2x - 1 - \sqrt{-x^2 + 2c_1 + 2x + 1}$$

$$y = -2x - 1 + \sqrt{-x^2 + 2c_1 + 2x + 1}$$

0.177 (sec) Entering first order ode dAlembert solver

2.1.5.2 Solved using first_order_ode_dAlembert

$$5x + 2y + 1 + (2x + y + 1) y' = 0$$

Let $p = y'$ the ode becomes

$$5x + 2y + 1 + (2x + y + 1) p = 0$$

Solving for y from the above results in

$$y = -\frac{(2p+5)\,x}{2+p} - \frac{p+1}{2+p} \tag{1}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved.

Taking derivative of (*) w.r.t. x gives

$$\begin{aligned} p &= f + (xf' + g')\frac{dp}{dx} \\ p - f &= (xf' + g')\frac{dp}{dx} \end{aligned} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$\begin{aligned} f &= \frac{-2p-5}{2+p} \\ g &= \frac{-p-1}{2+p} \end{aligned}$$

Hence (2) becomes

$$p - \frac{-2p-5}{2+p} = \left(-\frac{2x}{2+p} + \frac{2xp}{(2+p)^2} + \frac{5x}{(2+p)^2} - \frac{1}{2+p} + \frac{p}{(2+p)^2} + \frac{1}{(2+p)^2}\right) p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$p - \frac{-2p-5}{2+p} = 0$$

Solving the above for p results in

$$\begin{aligned} p_1 &= -2 + i \\ p_2 &= -2 - i \end{aligned}$$

Substituting these in (1A) and keeping singular solution that verifies the ode gives

$$\begin{aligned} y &= -1 - i + (-2 + i)\,x \\ y &= -1 + i + (-2 - i)\,x \end{aligned}$$

The general solution is found when $\frac{dp}{dx} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{p(x) - \frac{-2p(x)-5}{2+p(x)}}{-\frac{2x}{2+p(x)} + \frac{2xp(x)}{(2+p(x))^2} + \frac{5x}{(2+p(x))^2} - \frac{1}{2+p(x)} + \frac{p(x)}{(2+p(x))^2} + \frac{1}{(2+p(x))^2}} \tag{3}$$

This ODE is now solved for $p(x)$. No inversion is needed.

The ode

$$p'(x) = \frac{(2+p(x))\left(p(x)^2 + 4p(x) + 5\right)}{-1+x} \tag{2.3}$$

is separable as it can be written as

$$\begin{aligned} p'(x) &= \frac{(2+p(x))\left(p(x)^2 + 4p(x) + 5\right)}{-1+x} \\ &= f(x)g(p) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= \frac{1}{-1+x} \\ g(p) &= (2+p)\left(p^2 + 4p + 5\right) \end{aligned}$$

Integrating gives

$$\int \frac{1}{g(p)}\,dp = \int f(x)\,dx$$

$$\int \frac{1}{(2+p)\,(p^2+4p+5)}\,dp = \int \frac{1}{-1+x}\,dx$$

$$\ln\left(\frac{2+p(x)}{\sqrt{p(x)^2 + 4p(x) + 5}}\right) = \ln(-1+x) + c_1$$

Taking the exponential of both sides the solution becomes

$$\frac{2+p(x)}{\sqrt{p(x)^2 + 4p(x) + 5}} = c_1(-1+x)$$

We now need to find the singular solutions, these are found by finding for what values $g(p)$ is zero, since we had to divide by this above. Solving $g(p) = 0$ or

$$(2+p)\left(p^2 + 4p + 5\right) = 0$$

for $p(x)$ gives

$$
\begin{aligned}
p(x) &= -2 \\
p(x) &= -2 - i
\end{aligned}
$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$
\frac{2 + p(x)}{\sqrt{p\left(x\right)^{2} + 4p\left(x\right) + 5}} = c_1(-1 + x)
$$

$$
p(x) = -2
$$

$$
p(x) = -2 - i
$$

Substituing the above solution for p in (2A) gives

$$
\begin{aligned}
y &= \frac{x\left(-2c_1\sqrt{-\frac{1}{c_1^2x^2-2c_1^2x+c_1^2-1}}\, x + 2c_1\sqrt{-\frac{1}{c_1^2x^2-2c_1^2x+c_1^2-1}} - 1\right)}{c_1\sqrt{-\frac{1}{c_1^2x^2-2c_1^2x+c_1^2-1}}\, x - c_1\sqrt{-\frac{1}{c_1^2x^2-2c_1^2x+c_1^2-1}}} \\
&\quad + \frac{-c_1\sqrt{-\frac{1}{c_1^2x^2-2c_1^2x+c_1^2-1}}\, x + c_1\sqrt{-\frac{1}{c_1^2x^2-2c_1^2x+c_1^2-1}} + 1}{c_1\sqrt{-\frac{1}{c_1^2x^2-2c_1^2x+c_1^2-1}}\, x - c_1\sqrt{-\frac{1}{c_1^2x^2-2c_1^2x+c_1^2-1}}} \\
y &= \left(-\frac{2}{5} - \frac{i}{5}\right)(5x + 1 - 3i)
\end{aligned}
$$

Simplifying the above gives

$$
\begin{aligned}
y &= -1 - i + (-2 + i)\, x \\
y &= -1 + i + (-2 - i)\, x \\
y &= -\frac{1 + \sqrt{-\frac{1}{-1+c_1^2(-1+x)^2}}\,(1 + 2x)\, c_1}{\sqrt{-\frac{1}{-1+c_1^2(-1+x)^2}}\, c_1} \\
y &= -1 + i + (-2 - i)\, x
\end{aligned}
$$

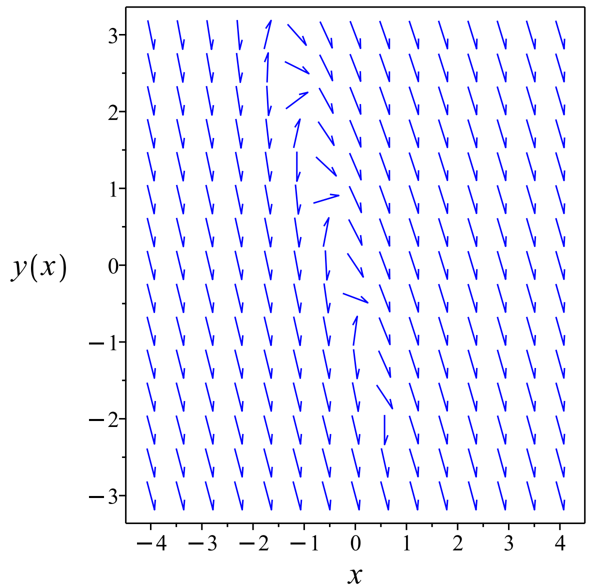


Figure 2.12: Slope field $5x + 2y + 1 + (2x + y + 1) y' = 0$

Summary of solutions found

$$y = -\frac{1 + \sqrt{-\frac{1}{-1+c_1^2(-1+x)^2}}\,(1 + 2x)\,c_1}{\sqrt{-\frac{1}{-1+c_1^2(-1+x)^2}}\,c_1}$$

$$y = -1 - i + (-2 + i)\,x$$

$$y = -1 + i + (-2 - i)\,x$$

0.509 (sec) Entering first order ode homog type maple C solver

2.1.5.3 Solved using first_order_ode_homog_type_maple_C

$$5x + 2y + 1 + (2x + y + 1) y' = 0$$

Let $Y = y - y_0$ and $X = x - x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = -\frac{5X + 5x_0 + 2Y(X) + 2y_0 + 1}{2X + 2x_0 + Y(X) + y_0 + 1}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = 1$$
$$y_0 = -3$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = -\frac{5X + 2Y(X)}{2X + Y(X)}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X, Y) \\ &= -\frac{5X + 2Y}{2X + Y} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X, Y)$ and $N(X, Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X, Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X, Y)$$

In this case, it can be seen that both $M = -5X - 2Y$ and $N = 2X + Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}X}X + u &= \frac{-2u - 5}{u + 2} \\ \frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\frac{-2u(X)-5}{u(X)+2} - u(X)}{X} \end{aligned}$$

Or

$$\frac{d}{dX}u(X) - \frac{\frac{-2u(X)-5}{u(X)+2} - u(X)}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right)Xu(X) + 2\left(\frac{d}{dX}u(X)\right)X + u(X)^2 + 4u(X) + 5 = 0$$

Or

$$X(u(X) + 2)\left(\frac{d}{dX}u(X)\right) + u(X)^2 + 4u(X) + 5 = 0$$

Which is now solved as separable in $u(X)$.

The ode

$$\frac{d}{dX}u(X) = -\frac{u(X)^2 + 4u(X) + 5}{X\,(u\,(X) + 2)} \tag{2.4}$$

is separable as it can be written as

$$
\begin{aligned}
\frac{d}{dX}u(X) &= -\frac{u(X)^2 + 4u(X) + 5}{X\,(u\,(X) + 2)} \\
&= f(X)g(u)
\end{aligned}
$$

Where

$$
\begin{aligned}
f(X) &= -\frac{1}{X} \\
g(u) &= \frac{u^2 + 4u + 5}{u + 2}
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\int \frac{1}{g(u)}\,du &= \int f(X)\,dX \\
\int \frac{u + 2}{u^2 + 4u + 5}\,du &= \int -\frac{1}{X}\,dX
\end{aligned}
$$

$$
\frac{\ln\left(u(X)^2 + 4u(X) + 5\right)}{2} = \ln\left(\frac{1}{X}\right) + c_1
$$

Taking the exponential of both sides the solution becomes

$$
\sqrt{u\,(X)^2 + 4u\,(X) + 5} = \frac{c_1}{X}
$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or

$$
\frac{u^2 + 4u + 5}{u + 2} = 0
$$

for $u(X)$ gives

$$
\begin{aligned}
u(X) &= -2 - i \\
u(X) &= -2 + i
\end{aligned}
$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\sqrt{u(X)^2 + 4u(X) + 5} = \frac{c_1}{X}$$
$$u(X) = -2 - i$$
$$u(X) = -2 + i$$

Converting $\sqrt{u(X)^2 + 4u(X) + 5} = \frac{c_1}{X}$ back to $Y(X)$ gives

$$\sqrt{\frac{Y(X)^2 + 4Y(X)X + 5X^2}{X^2}} = \frac{c_1}{X}$$

Converting $u(X) = -2 - i$ back to $Y(X)$ gives

$$Y(X) = (-2 - i)X$$

Converting $u(X) = -2 + i$ back to $Y(X)$ gives

$$Y(X) = (-2 + i)X$$

Using the solution for $Y(X)$

$$\sqrt{\frac{Y(X)^2 + 4Y(X)X + 5X^2}{X^2}} = \frac{c_1}{X} \tag{A}$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x_0 + x$$

Or

$$Y = y - 3$$
$$X = 1 + x$$

Then the solution in y becomes using EQ (A)

$$\sqrt{\frac{(y+3)^2 + 4(y+3)(-1+x) + 5(-1+x)^2}{(-1+x)^2}} = \frac{c_1}{-1+x}$$

Using the solution for $Y(X)$

$$Y(X) = (-2 - i)X \tag{A}$$

And replacing back terms in the above solution using

$$
\begin{aligned}
Y &= y + y_0 \\
X &= x_0 + x
\end{aligned}
$$

Or

$$
\begin{aligned}
Y &= y - 3 \\
X &= 1 + x
\end{aligned}
$$

Then the solution in y becomes using EQ (A)

$$
y + 3 = (-2 - i)(-1 + x)
$$

Using the solution for $Y(X)$

$$
Y(X) = (-2 + i)\,X \tag{A}
$$

And replacing back terms in the above solution using

$$
\begin{aligned}
Y &= y + y_0 \\
X &= x_0 + x
\end{aligned}
$$

Or

$$
\begin{aligned}
Y &= y - 3 \\
X &= 1 + x
\end{aligned}
$$

Then the solution in y becomes using EQ (A)

$$
y + 3 = (-2 + i)(-1 + x)
$$

Solving for y gives

$$
\begin{aligned}
y &= -2x - 1 - \sqrt{c_1^2 - x^2 + 2x - 1} \\
y &= -2x - 1 + \sqrt{c_1^2 - x^2 + 2x - 1} \\
y &= -ix - 2x + i - 1 \\
y &= ix - 2x - i - 1
\end{aligned}
$$

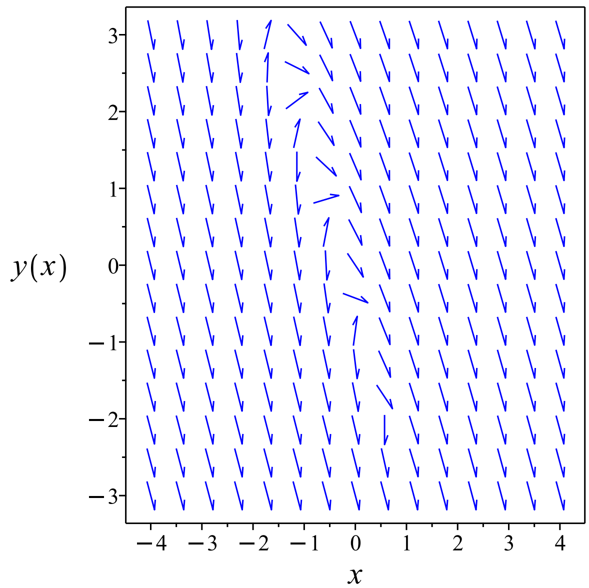


Figure 2.13: Slope field $5x + 2y + 1 + (2x + y + 1)\, y' = 0$

Summary of solutions found

$$y = -2x - 1 - \sqrt{c_1^2 - x^2 + 2x - 1}$$

$$y = -2x - 1 + \sqrt{c_1^2 - x^2 + 2x - 1}$$

$$y = -ix - 2x + i - 1$$

$$y = ix - 2x - i - 1$$

Entering first order ode abel second kind solver

$$5x + 2y + 1 + (2x + y + 1)\, y' = 0$$

0.029 (sec)

2.1.5.4 Solved using first_order_ode_abel_second_kind_case_5

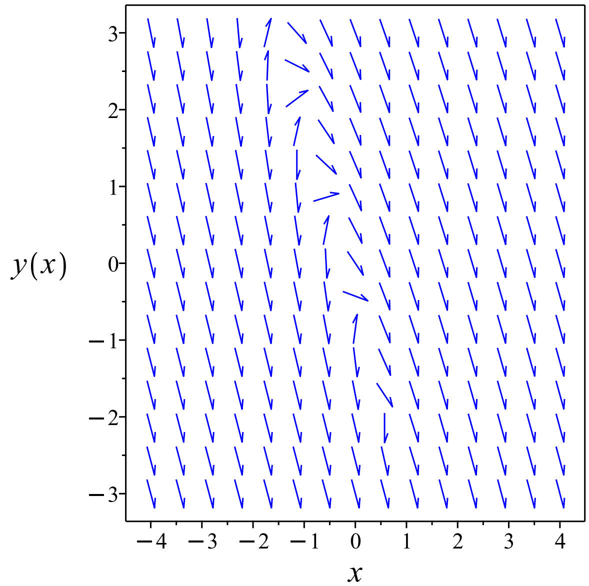


Figure 2.14: Slope field $5x + 2y + 1 + (2x + y + 1)\, y' = 0$

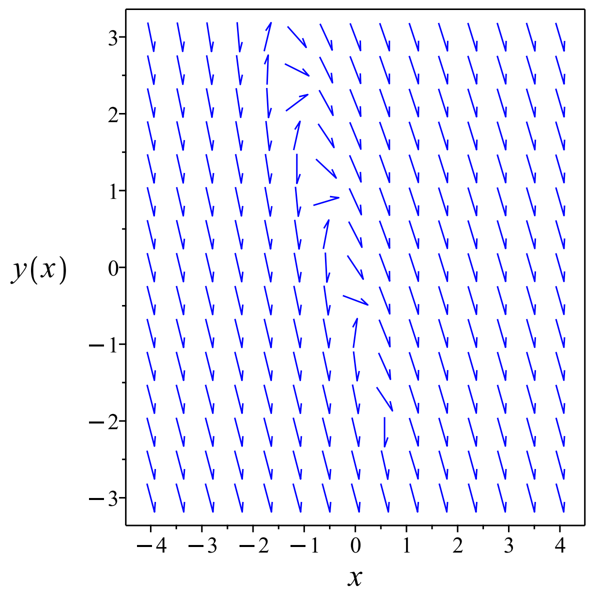


Figure 2.15: Slope field $5x + 2y + 1 + (2x + y + 1)\, y' = 0$

Summary of solutions found

$$y = -1 - 2x - \sqrt{-x^2 + c_1 + 2x}$$

$$y = -1 - 2x + \sqrt{-x^2 + c_1 + 2x}$$

0.779 (sec) Entering first order ode LIE solver

2.1.5.5 Solved using first_order_ode_LIE

$$5x+2y+1+(2x+y+1)\,y'=0$$

Writing the ode as

$$y'=-\frac{5x+2y+1}{2x+y+1}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi=xa_2+ya_3+a_1 \tag{1E}$$
$$\eta=xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1,a_2,a_3,b_1,b_2,b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}&b_2-\frac{(5x+2y+1)(b_3-a_2)}{2x+y+1}-\frac{(5x+2y+1)^2a_3}{(2x+y+1)^2}\\&-\left(-\frac{5}{2x+y+1}+\frac{10x+4y+2}{(2x+y+1)^2}\right)(xa_2+ya_3+a_1)\\&-\left(-\frac{2}{2x+y+1}+\frac{5x+2y+1}{(2x+y+1)^2}\right)(xb_2+yb_3+b_1)=0\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{10x^2a_2-25x^2a_3+3x^2b_2-10x^2b_3+10xya_2-20xya_3+4xyb_2-10xyb_3+2y^2a_2-3y^2a_3+y^2b_2-2}{(2x}$$
$$=0$$

Setting the numerator to zero gives

$$\begin{aligned}&10x^2a_2-25x^2a_3+3x^2b_2-10x^2b_3+10xya_2-20xya_3+4xyb_2-10xyb_3\\&+2y^2a_2-3y^2a_3+y^2b_2-2y^2b_3+10xa_2-10xa_3-xb_1+5xb_2-7xb_3\\&+ya_1+3ya_2-ya_3+2yb_2-2yb_3+3a_1+a_2-a_3+b_1+b_2-b_3=0\end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &10a_2v_1^2 + 10a_2v_1v_2 + 2a_2v_2^2 - 25a_3v_1^2 - 20a_3v_1v_2 - 3a_3v_2^2 + 3b_2v_1^2 + 4b_2v_1v_2 \\ &+ b_2v_2^2 - 10b_3v_1^2 - 10b_3v_1v_2 - 2b_3v_2^2 + a_1v_2 + 10a_2v_1 + 3a_2v_2 - 10a_3v_1 - a_3v_2 \\ &- b_1v_1 + 5b_2v_1 + 2b_2v_2 - 7b_3v_1 - 2b_3v_2 + 3a_1 + a_2 - a_3 + b_1 + b_2 - b_3 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} &(10a_2 - 25a_3 + 3b_2 - 10b_3)\, v_1^2 + (10a_2 - 20a_3 + 4b_2 - 10b_3)\, v_1v_2 \\ &+ (10a_2 - 10a_3 - b_1 + 5b_2 - 7b_3)\, v_1 + (2a_2 - 3a_3 + b_2 - 2b_3)\, v_2^2 \\ &+ (a_1 + 3a_2 - a_3 + 2b_2 - 2b_3)\, v_2 + 3a_1 + a_2 - a_3 + b_1 + b_2 - b_3 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} 2a_2 - 3a_3 + b_2 - 2b_3 &= 0 \\ 10a_2 - 25a_3 + 3b_2 - 10b_3 &= 0 \\ 10a_2 - 20a_3 + 4b_2 - 10b_3 &= 0 \\ a_1 + 3a_2 - a_3 + 2b_2 - 2b_3 &= 0 \\ 10a_2 - 10a_3 - b_1 + 5b_2 - 7b_3 &= 0 \\ 3a_1 + a_2 - a_3 + b_1 + b_2 - b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= -a_3 - b_3 \\ a_2 &= 4a_3 + b_3 \\ a_3 &= a_3 \\ b_1 &= 5a_3 + 3b_3 \\ b_2 &= -5a_3 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = -1 + x$$
$$\eta = y + 3$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y + 3 - \left(-\frac{5x + 2y + 1}{2x + y + 1}\right)(-1 + x) \\
&= y + 3 + \frac{(5x + 2y + 1)\,(-1 + x)}{2x + y + 1} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{y + 3 + \frac{(5x+2y+1)(-1+x)}{2x+y+1}} dy
\end{aligned}$$

Which results in

$$S = \frac{\ln (5x^2 + 4yx + y^2 + 2x + 2y + 2)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{5x+2y+1}{2x+y+1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{5x+2y+1}{5x^2+(4y+2)\,x+y^2+2y+2} \\
S_y &= \frac{2x+y+1}{y^2+(4x+2)\,y+5x^2+2x+2}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

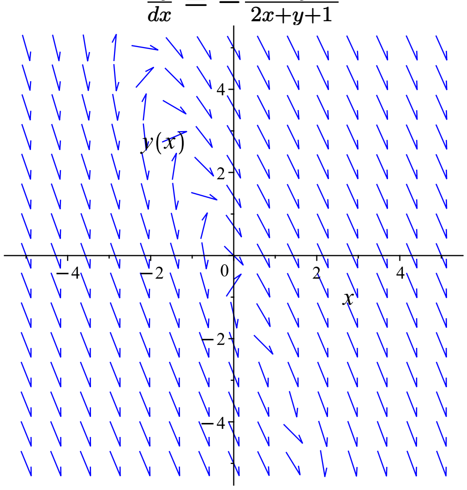
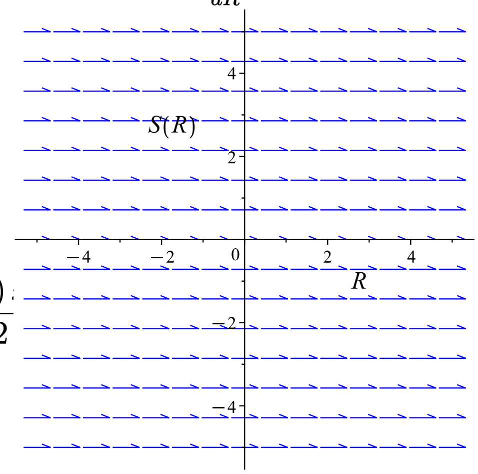
Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\begin{aligned}
\int dS &= \int 0\,dR + c_2 \\
S(R) &= c_2
\end{aligned}$$

To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$\frac{\ln\left(5x^2+(4y+2)\,x+y^2+2y+2\right)}{2} = c_2$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{5x+2y+1}{2x+y+1}$ 	$R = x$ $S = \frac{\ln(5x^2 + (4y+2)\cdots}{2}$	$\frac{dS}{dR} = 0$ 

Simplifying the above gives

$$\frac{\ln(y^2 + (4x+2)y + 5x^2 + 2x + 2)}{2} = c_2$$

Solving for y gives

$$y = -2x - 1 - \sqrt{-x^2 + e^{2c_2} + 2x - 1}$$

$$y = -2x - 1 + \sqrt{-x^2 + e^{2c_2} + 2x - 1}$$

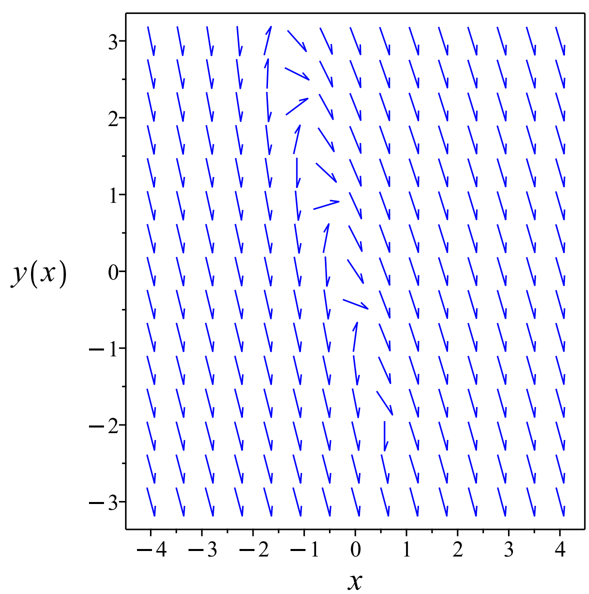


Figure 2.16: Slope field $5x + 2y + 1 + (2x + y + 1)y' = 0$

Summary of solutions found

$$y = -2x - 1 - \sqrt{-x^2 + e^{2c_2} + 2x - 1}$$

$$y = -2x - 1 + \sqrt{-x^2 + e^{2c_2} + 2x - 1}$$

2.1.5.6 ✓ Maple. Time used: 0.086 (sec). Leaf size: 32

```
ode:=5*x+2*y(x)+1+(2*x+y(x)+1)*diff(y(x),x) = 0;
dsolve(ode,y(x), singsol=all);
```

$$y = \frac{-\sqrt{-\left(x-1\right)^2 c_1^2 + 1} + \left(-2x-1\right) c_1}{c_1}$$

Maple trace

```
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful
```

Maple step by step

Let's solve
$5x + 2y(x) + 1 + (2x + y(x) + 1)\left(\frac{d}{dx}y(x)\right) = 0$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$

□ Check if ODE is exact

  ◦ ODE is exact if the lhs is the total derivative of a C^2 function
    $\frac{d}{dx}G(x, y(x)) = 0$

  ◦ Compute derivative of lhs
    $\frac{\partial}{\partial x}G(x, y) + \left(\frac{\partial}{\partial y}G(x, y)\right)\left(\frac{d}{dx}y(x)\right) = 0$

- Evaluate derivatives
  $2 = 2$
- Condition met, ODE is exact

- Exact ODE implies solution will be of this form
  $\left[G(x,y) = C1, M(x,y) = \frac{\partial}{\partial x}G(x,y), N(x,y) = \frac{\partial}{\partial y}G(x,y)\right]$
- Solve for $G(x,y)$ by integrating $M(x,y)$ with respect to x
  $G(x,y) = \int (5x+2y+1)\, dx + \_F1(y)$
- Evaluate integral
  $G(x,y) = \frac{5x^2}{2} + 2yx + x + \_F1(y)$
- Take derivative of $G(x,y)$ with respect to y
  $N(x,y) = \frac{\partial}{\partial y}G(x,y)$
- Compute derivative
  $2x + y + 1 = 2x + \frac{d}{dy}\_F1(y)$
- Isolate for $\frac{d}{dy}\_F1(y)$
  $\frac{d}{dy}\_F1(y) = y + 1$
- Solve for $\_F1(y)$
  $\_F1(y) = \frac{1}{2}y^2 + y$
- Substitute $\_F1(y)$ into equation for $G(x,y)$
  $G(x,y) = \frac{5}{2}x^2 + 2yx + x + \frac{1}{2}y^2 + y$
- Substitute $G(x,y)$ into the solution of the ODE
  $\frac{5}{2}x^2 + 2yx + x + \frac{1}{2}y^2 + y = C1$
- Solve for $y(x)$
  $\left\{y(x) = -2x - 1 - \sqrt{-x^2 + 2C1 + 2x + 1}, y(x) = -2x - 1 + \sqrt{-x^2 + 2C1 + 2x + 1}\right\}$

2.1.5.7 ✓ Mathematica. Time used: 0.096 (sec). Leaf size: 53

```
ode=(5*x+2*y[x]+1)+(2*x+y[x]+1)*D[y[x],x]==0;
ic={};
DSolve[{ode,ic},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\sqrt{-x^2 + 2x + 1 + c_1} - 2x - 1$$
$$y(x) \to \sqrt{-x^2 + 2x + 1 + c_1} - 2x - 1$$

2.1.5.8 ✓ Sympy. Time used: 1.720 (sec). Leaf size: 39

```
from sympy import *
x = symbols("x")
y = Function("y")
ode = Eq(5*x + (2*x + y(x) + 1)*Derivative(y(x), x) + 2*y(x) + 1,0)
ics = {}
dsolve(ode,func=y(x),ics=ics)
```

$$\left[y(x) = -2x - \sqrt{C_1 - x^2 + 2x} - 1,\ y(x) = -2x + \sqrt{C_1 - x^2 + 2x} - 1\right]$$

2.1.6 Problem 6

Local contents

Internal problem ID [4081]
Book : Differential equations, Shepley L. Ross, 1964
Section : 2.4, page 55
Problem number : 6
Date solved : `Saturday, December 06, 2025 at 04:16:35 PM`
CAS classification :
`[[_homogeneous, ‘class C‘], _rational, [_Abel, ‘2nd type‘, ‘class A‘]]`

0.143 (sec) Entering first order ode dAlembert solver

2.1.6.1 Solved using first_order_ode_dAlembert

$$3x - y + 1 - (6x - 2y - 3)\, y' = 0$$

Let $p = y'$ the ode becomes

$$3x - y + 1 - (6x - 2y - 3)\, p = 0$$

Solving for y from the above results in

$$y = \frac{(6p - 3)\, x}{-1 + 2p} + \frac{-3p - 1}{-1 + 2p} \tag{1}$$

This has the form

$$y = x f(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved.

Taking derivative of (*) w.r.t. x gives

$$p = f + (x f' + g') \frac{dp}{dx}$$

$$p - f = (x f' + g') \frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$f = 3$$
$$g = \frac{-3p - 1}{-1 + 2p}$$

Hence (2) becomes

$$p - 3 = \left(-\frac{3}{-1 + 2p} + \frac{6p}{(-1 + 2p)^2} + \frac{2}{(-1 + 2p)^2}\right) p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$p - 3 = 0$$

Solving the above for p results in

$$p_1 = 3$$

Substituting these in (1A) and keeping singular solution that verifies the ode gives

$$y = -2 + 3x$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{p(x) - 3}{-\frac{3}{-1+2p(x)} + \frac{6p(x)}{(-1+2p(x))^2} + \frac{2}{(-1+2p(x))^2}} \tag{3}$$

This ODE is now solved for $p(x)$. No inversion is needed.

Integrating gives

$$\int \frac{5}{(p - 3)(-1 + 2p)^2} dp = dx$$
$$\frac{\ln(p - 3)}{5} + \frac{1}{-1 + 2p} - \frac{\ln(-1 + 2p)}{5} = x + c_1$$

Singular solutions are found by solving

$$\frac{(p - 3)(-1 + 2p)^2}{5} = 0$$

for $p(x)$. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$p(x) = 3$$
$$p(x) = \frac{1}{2}$$

Substituing the above solution for p in (2A) gives

$$y = 3x + \frac{-3\,\mathrm{e}^{\mathrm{RootOf}\left(2\ln\left(2\,\mathrm{e}^{\_Z}+5\right)\mathrm{e}^{\_Z}+10c_1\,\mathrm{e}^{\_Z}-2\_Z\,\mathrm{e}^{\_Z}+10\,\mathrm{e}^{\_Z}x+5\ln\left(2\,\mathrm{e}^{\_Z}+5\right)+25c_1-5\_Z+25x-5\right)}-10}{2\,\mathrm{e}^{\mathrm{RootOf}\left(2\ln\left(2\,\mathrm{e}^{\_Z}+5\right)\mathrm{e}^{\_Z}+10c_1\,\mathrm{e}^{\_Z}-2\_Z\,\mathrm{e}^{\_Z}+10\,\mathrm{e}^{\_Z}x+5\ln\left(2\,\mathrm{e}^{\_Z}+5\right)+25c_1-5\_Z+25x-5\right)}+5}$$

$$y = -2 + 3x$$

Simplifying the above gives

$$y = -2 + 3x$$

$$y = \frac{(6x-3)\,\mathrm{e}^{\mathrm{RootOf}\left(2\ln\left(2\,\mathrm{e}^{\_Z}+5\right)\mathrm{e}^{\_Z}+10c_1\,\mathrm{e}^{\_Z}-2\_Z\,\mathrm{e}^{\_Z}+10\,\mathrm{e}^{\_Z}x+5\ln\left(2\,\mathrm{e}^{\_Z}+5\right)+25c_1-5\_Z+25x-5\right)}+15x-10}{2\,\mathrm{e}^{\mathrm{RootOf}\left(2\ln\left(2\,\mathrm{e}^{\_Z}+5\right)\mathrm{e}^{\_Z}+10c_1\,\mathrm{e}^{\_Z}-2\_Z\,\mathrm{e}^{\_Z}+10\,\mathrm{e}^{\_Z}x+5\ln\left(2\,\mathrm{e}^{\_Z}+5\right)+25c_1-5\_Z+25x-5\right)}+5}$$

$$y = -2 + 3x$$

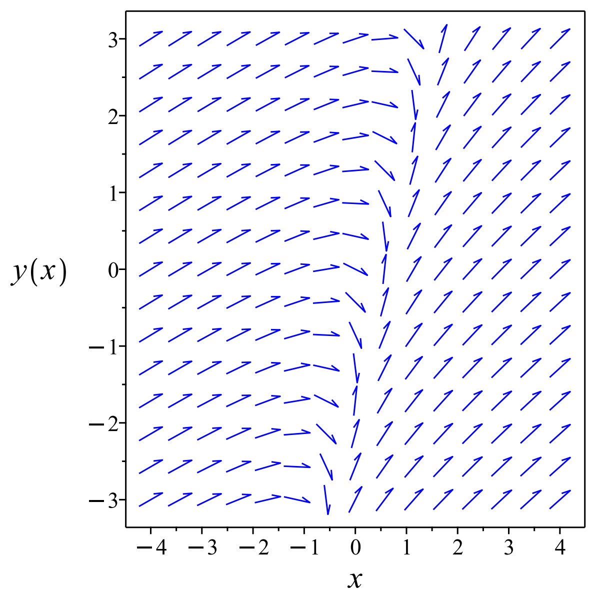


Figure 2.17: Slope field $3x - y + 1 - (6x - 2y - 3)\,y' = 0$

Summary of solutions found

$$y = \frac{(6x-3)\,\mathrm{e}^{\mathrm{RootOf}\left(2\ln\left(2\,\mathrm{e}^{\_Z}+5\right)\mathrm{e}^{\_Z}+10c_1\,\mathrm{e}^{\_Z}-2\_Z\,\mathrm{e}^{\_Z}+10\,\mathrm{e}^{\_Z}x+5\ln\left(2\,\mathrm{e}^{\_Z}+5\right)+25c_1-5\_Z+25x-5\right)}+15x-10}{2\,\mathrm{e}^{\mathrm{RootOf}\left(2\ln\left(2\,\mathrm{e}^{\_Z}+5\right)\mathrm{e}^{\_Z}+10c_1\,\mathrm{e}^{\_Z}-2\_Z\,\mathrm{e}^{\_Z}+10\,\mathrm{e}^{\_Z}x+5\ln\left(2\,\mathrm{e}^{\_Z}+5\right)+25c_1-5\_Z+25x-5\right)}+5}$$

$$y = -2 + 3x$$

Entering first order ode abel second kind solver

$$3x - y + 1 - (6x - 2y - 3)\,y' = 0$$

Applying transformation

$$y = \frac{1}{u(x)} - g$$

Results in the new ode which is Abel first kind

$$3x - xu(x) + 1 - (6x - 2xu(x) - 3)(u(x) + xu'(x)) = 0$$

Which is now solved Unknown ode type.

2.1.6.2 Solved using first_order_ode_abel_second_kind_solved_by_converting_to_first_kind

0.289 (sec)

This is Abel second kind ODE, it has the form

$$(y(x) + g)\, y'(x) = f_0(x) + f_1(x)y(x) + f_2(x)y(x)^2 + f_3(x)y(x)^3$$

Comparing the above to given ODE which is

$$3x - y(x) + 1 - (6x - 2y(x) - 3)\, y'(x) = 0 \tag{1}$$

Shows that

$$\begin{aligned} g &= -3x + \frac{3}{2} \\ f_0 &= -\frac{3x}{2} - \frac{1}{2} \\ f_1 &= \frac{1}{2} \\ f_2 &= 0 \\ f_3 &= 0 \end{aligned}$$

Applying transformation

$$y(x) = \frac{1}{u(x)} - g$$

Results in the new ode which is Abel first kind

$$u'(x) = \frac{5u(x)^3}{4} + \frac{5u(x)^2}{2}$$

Which is now solved.

Entering first order ode autonomous solver

Integrating gives

$$\int \frac{1}{\frac{5}{4}u^3 + \frac{5}{2}u^2} du = dx$$

$$-\frac{2}{5u} - \frac{\ln(u)}{5} + \frac{\ln(u+2)}{5} = x + c_1$$

Singular solutions are found by solving

$$\frac{5}{4}u^3 + \frac{5}{2}u^2 = 0$$

for $u(x)$. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$u(x) = -2$$
$$u(x) = 0$$

Substituting $u(x) = \frac{1}{-3x+y(x)+\frac{3}{2}}$ in the above solution gives

$$\frac{6x}{5} - \frac{2y(x)}{5} - \frac{3}{5} - \frac{\ln\left(\frac{1}{-3x+y(x)+\frac{3}{2}}\right)}{5} + \frac{\ln\left(\frac{1}{-3x+y(x)+\frac{3}{2}} + 2\right)}{5} = x + c_1$$

Now we transform the solution $u(x) = -2$ to $y(x)$ using $u(x) = \frac{1}{-3x+y(x)+\frac{3}{2}}$ which gives

$$y(x) = -2 + 3x$$

Simplifying the above gives

$$\frac{6x}{5} - \frac{2y(x)}{5} - \frac{3}{5} + \frac{\ln(2)}{5} - \frac{\ln\left(\frac{1}{-6x+2y(x)+3}\right)}{5} + \frac{\ln\left(\frac{y(x)-3x+2}{-6x+2y(x)+3}\right)}{5} = x + c_1$$
$$y(x) = -2 + 3x$$

Solving for $y(x)$ gives

$$y(x) = -2 + 3x$$
$$y(x) = 3x - 2 - \frac{\operatorname{LambertW}\left(-e^{5c_1+5x-1}\right)}{2}$$

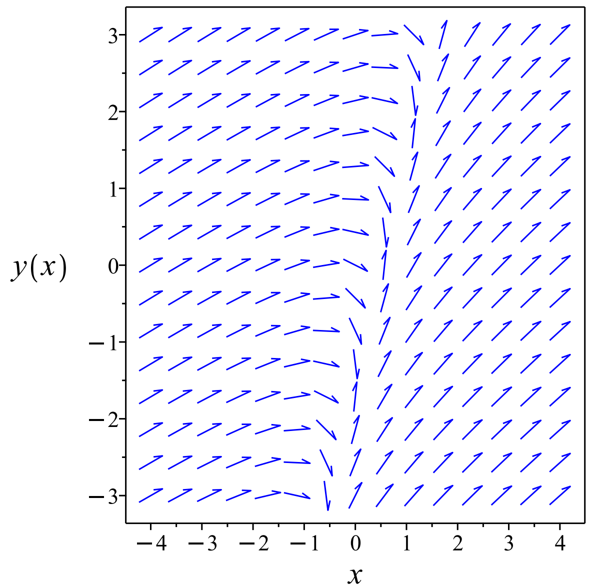


Figure 2.18: Slope field $3x - y(x) + 1 - (6x - 2y(x) - 3)\, y'(x) = 0$

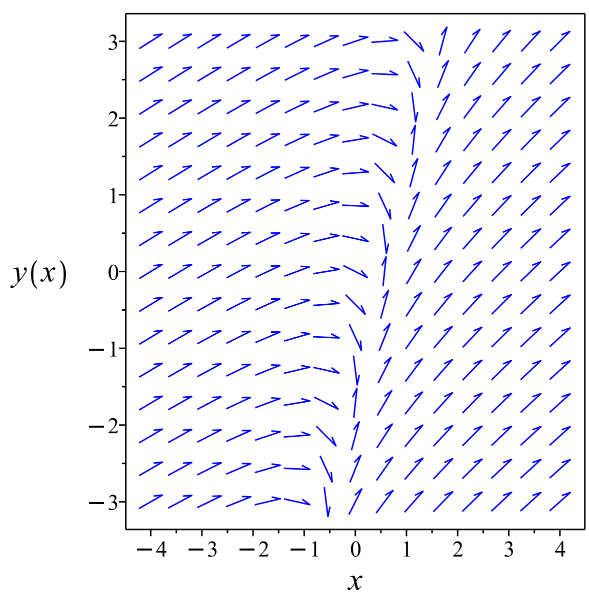


Figure 2.19: Slope field $3x - y(x) + 1 - (6x - 2y(x) - 3)\, y'(x) = 0$

Summary of solutions found

$$y(x) = -2 + 3x$$

$$y(x) = 3x - 2 - \frac{\text{LambertW}\left(-\mathrm{e}^{5c_1+5x-1}\right)}{2}$$

0.191 (sec) Entering first order ode LIE solver

2.1.6.3 Solved using first_order_ode_LIE

$$3x - y(x) + 1 - (6x - 2y(x) - 3)\, y'(x) = 0$$

Writing the ode as

$$y'(x) = \frac{-3x + y - 1}{-6x + 2y + 3}$$
$$y'(x) = \omega(x, y(x))$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega\left(\eta_{y(x)} - \xi_x\right) - \omega^2 \xi_{y(x)} - \omega_x \xi - \omega_{y(x)} \eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Using these anstaz

$$\xi = 1 \tag{1E}$$

$$\eta = \frac{Ax + By}{Cx} \tag{2E}$$

Where the unknown coefficients are

$$\{A, B, C\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\frac{A}{Cx} - \frac{Ax + By}{C\, x^2} + \frac{(-3x + y - 1)\, B}{(-6x + 2y + 3)\, Cx} + \frac{3}{-6x + 2y + 3} - \frac{6(-3x + y - 1)}{(-6x + 2y + 3)^2} - \frac{\left(\frac{1}{-6x+2y+3} - \frac{2(-3x+y-1)}{(-6x+2y+3)^2}\right)(Ax + By)}{Cx} = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{-18B\, x^3 + 48B\, x^2 y - 26Bx\, y^2 + 4B\, y^3 + 5A\, x^2 + 3B\, x^2 - 32Bxy + 12B\, y^2 - 15C\, x^2 + 3Bx + 9B}{C\, x^2 \left(6x - 2y - 3\right)^2} = 0$$

Setting the numerator to zero gives

$$18B\, x^3 - 48B\, x^2 y + 26Bx\, y^2 - 4B\, y^3 - 5A\, x^2 - 3B\, x^2 + 32Bxy - 12B\, y^2 + 15C\, x^2 - 3Bx - 9By = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} 18Bv_1^3 - 48Bv_1^2v_2 + 26Bv_1v_2^2 - 4Bv_2^3 - 5Av_1^2 - 3Bv_1^2 \\ + 32Bv_1v_2 - 12Bv_2^2 + 15Cv_1^2 - 3Bv_1 - 9Bv_2 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} 18Bv_1^3 - 48Bv_1^2v_2 + (-5A - 3B + 15C)\, v_1^2 + 26Bv_1v_2^2 \\ + 32Bv_1v_2 - 3Bv_1 - 4Bv_2^3 - 12Bv_2^2 - 9Bv_2 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -48B &= 0 \\ -12B &= 0 \\ -9B &= 0 \\ -4B &= 0 \\ -3B &= 0 \\ 18B &= 0 \\ 26B &= 0 \\ 32B &= 0 \\ -5A - 3B + 15C &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} A &= 3C \\ B &= 0 \\ C &= C \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = 1$$
$$\eta = 3$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Therefore

$$\begin{aligned}\frac{dy}{dx} &= \frac{\eta}{\xi} \\ &= \frac{3}{1} \\ &= 3\end{aligned}$$

This is easily solved to give

$$y(x) = 3x + c_1$$

Where now the coordinate R is taken as the constant of integration. Hence

$$R = -3x + y$$

And S is found from

$$\begin{aligned}dS &= \frac{dx}{\xi} \\ &= \frac{dx}{1}\end{aligned}$$

Integrating gives

$$\begin{aligned}S &= \int \frac{dx}{T} \\ &= x\end{aligned}$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{-3x+y-1}{-6x+2y+3}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= -3 \\ R_y &= 1 \\ S_x &= 1 \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{-6x+2y+3}{15x-5y-10} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{2R+3}{-5R-10}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

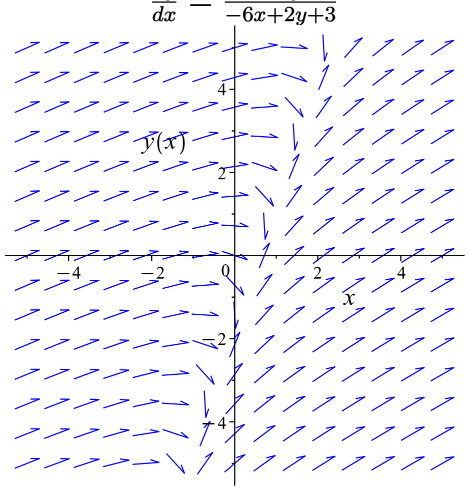
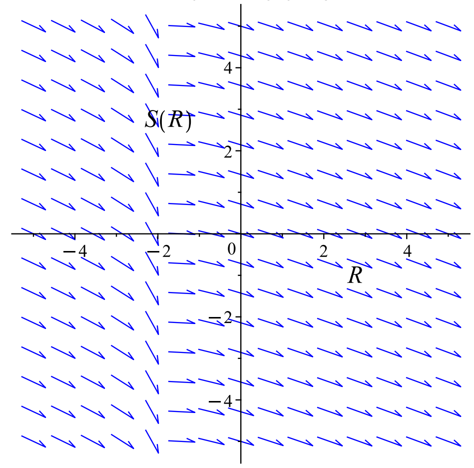
Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\begin{aligned} \int dS &= \int -\frac{2R+3}{5(R+2)}\, dR \\ S(R) &= -\frac{2R}{5} + \frac{\ln(R+2)}{5} + c_2 \end{aligned}$$

To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$x = \frac{6x}{5} - \frac{2y(x)}{5} + \frac{\ln(y(x)-3x+2)}{5} + c_2$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{-3x+y-1}{-6x+2y+3}$ 	$R = -3x + y$ $S = x$	$\frac{dS}{dR} = \frac{2R+3}{-5R-10}$ 

Solving for $y(x)$ gives

$$y(x) = -\frac{\text{LambertW}\left(-2\,e^{5x-4-5c_2}\right)}{2} + 3x - 2$$

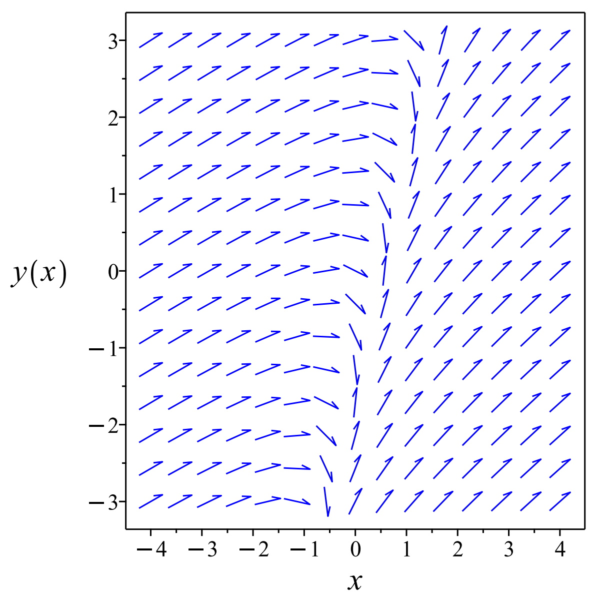


Figure 2.20: Slope field $3x - y(x) + 1 - (6x - 2y(x) - 3)\, y'(x) = 0$

Summary of solutions found

$$y(x) = -\frac{\text{LambertW}\left(-2\,e^{5x-4-5c_2}\right)}{2} + 3x - 2$$

2.1.6.4 ✓ Maple. Time used: 0.012 (sec). Leaf size: 23

```
ode:=3*x-y(x)+1-(6*x-2*y(x)-3)*diff(y(x),x) = 0;
dsolve(ode,y(x), singsol=all);
```

$$y = -\frac{\mathrm{LambertW}\left(-2\,\mathrm{e}^{5x-4-5c_1}\right)}{2} + 3x - 2$$

Maple trace

```
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
   -> Calling odsolve with the ODE, diff(y(x),x) = 3, y(x)
      *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
<- 1st order, canonical coordinates successful
<- homogeneous successful
```

Maple step by step

Let's solve
$3x - y(x) + 1 - (6x - 2y(x) - 3)\left(\frac{d}{dx}y(x)\right) = 0$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Solve for the highest derivative
  $\frac{d}{dx}y(x) = -\frac{-3x+y(x)-1}{6x-2y(x)-3}$

2.1.6.5 ✓ Mathematica. Time used: 2.097 (sec). Leaf size: 35

```
ode=(3*x-y[x]+1)-(6*x-2*y[x]-3)*D[y[x],x]==0;
ic={};
DSolve[{ode,ic},y[x],x,IncludeSingularSolutions->True]
```

$$
\begin{aligned}
y(x) &\to -\frac{1}{2}W\left(-e^{5x-1+c_1}\right) + 3x - 2 \\
y(x) &\to 3x - 2
\end{aligned}
$$

2.1.6.6 ✓ Sympy. Time used: 0.903 (sec). Leaf size: 19

```
from sympy import *
x = symbols("x")
y = Function("y")
ode = Eq(3*x - (6*x - 2*y(x) - 3)*Derivative(y(x), x) - y(x) + 1,0)
ics = {}
dsolve(ode,func=y(x),ics=ics)
```

$$
y(x) = 3x - \frac{W(C_1e^{5x-4})}{2} - 2
$$

2.1.7 Problem 7

Local contents

Internal problem ID [4082]
Book : Differential equations, Shepley L. Ross, 1964
Section : 2.4, page 55
Problem number : 7
Date solved : `Saturday, December 06, 2025 at 04:16:40 PM`
CAS classification :
`[[_homogeneous, 'class C'], _rational, [_Abel, '2nd type', 'class A']]`

0.254 (sec) Entering first order ode dAlembert solver

2.1.7.1 Solved using first_order_ode_dAlembert

$$x - 2y - 3 + (2x + y - 1)\, y' = 0$$

Let $p = y'$ the ode becomes

$$x - 2y - 3 + (2x + y - 1)\, p = 0$$

Solving for y from the above results in

$$y = -\frac{(2p + 1)\, x}{-2 + p} - \frac{-p - 3}{-2 + p} \tag{1}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved.

Taking derivative of (*) w.r.t. x gives

$$\begin{aligned} p &= f + (xf' + g')\frac{dp}{dx} \\ p - f &= (xf' + g')\frac{dp}{dx} \end{aligned} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$\begin{aligned} f &= \frac{-2p - 1}{-2 + p} \\ g &= \frac{p + 3}{-2 + p} \end{aligned}$$

Hence (2) becomes

$$p - \frac{-2p - 1}{-2 + p} = \left(-\frac{2x}{-2 + p} + \frac{2xp}{(-2 + p)^2} + \frac{x}{(-2 + p)^2} + \frac{1}{-2 + p} - \frac{p}{(-2 + p)^2} - \frac{3}{(-2 + p)^2}\right) p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$p - \frac{-2p - 1}{-2 + p} = 0$$

Solving the above for p results in

$$\begin{aligned} p_1 &= i \\ p_2 &= -i \end{aligned}$$

Substituting these in (1A) and keeping singular solution that verifies the ode gives

$$\begin{aligned} y &= i(-1 + i + x) \\ y &= -ix + i - 1 \end{aligned}$$

The general solution is found when $\frac{dp}{dx} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{p(x) - \frac{-2p(x)-1}{-2+p(x)}}{-\frac{2x}{-2+p(x)} + \frac{2xp(x)}{(-2+p(x))^2} + \frac{x}{(-2+p(x))^2} + \frac{1}{-2+p(x)} - \frac{p(x)}{(-2+p(x))^2} - \frac{3}{(-2+p(x))^2}} \tag{3}$$

This ODE is now solved for $p(x)$. No inversion is needed.

The ode

$$p'(x) = \frac{(-2 + p(x))\left(p(x)^2 + 1\right)}{5x - 5} \tag{2.5}$$

is separable as it can be written as

$$
\begin{aligned}
p'(x) &= \frac{(-2+p(x))\left(p(x)^2+1\right)}{5x-5} \\
&= f(x)g(p)
\end{aligned}
$$

Where

$$
\begin{aligned}
f(x) &= \frac{1}{5x-5} \\
g(p) &= (-2+p)\left(p^2+1\right)
\end{aligned}
$$

Integrating gives

$$
\int \frac{1}{g(p)}\, dp = \int f(x)\, dx
$$

$$
\int \frac{1}{(-2+p)\left(p^2+1\right)}\, dp = \int \frac{1}{5x-5}\, dx
$$

$$
\ln\left(\frac{(-2+p(x))^{1/5}}{\left(p(x)^2+1\right)^{1/10}}\right) - \frac{2\arctan\left(p(x)\right)}{5} = \ln\left((-1+x)^{1/5}\right) + c_1
$$

We now need to find the singular solutions, these are found by finding for what values $g(p)$ is zero, since we had to divide by this above. Solving $g(p) = 0$ or

$$
(-2+p)\left(p^2+1\right) = 0
$$

for $p(x)$ gives

$$
\begin{aligned}
p(x) &= 2 \\
p(x) &= -i
\end{aligned}
$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$
\ln\left(\frac{(-2+p(x))^{1/5}}{\left(p(x)^2+1\right)^{1/10}}\right) - \frac{2\arctan\left(p(x)\right)}{5} = \ln\left((-1+x)^{1/5}\right) + c_1
$$

$$
p(x) = 2
$$

$$
p(x) = -i
$$

Substituing the above solution for p in (2A) gives

$$y = -i(x - 1 - i)$$

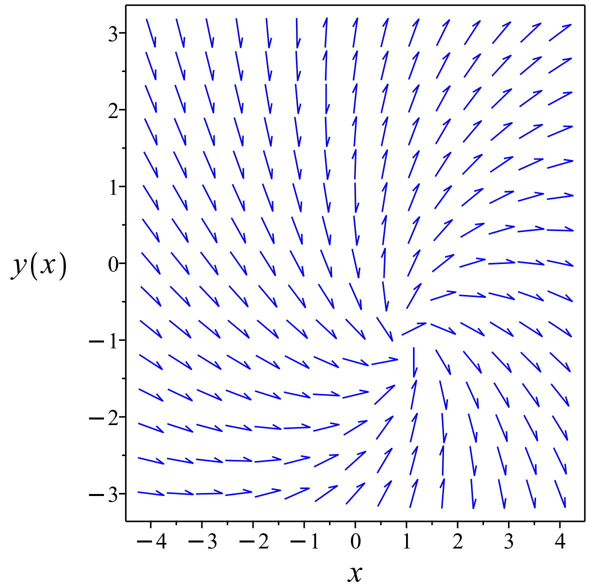


Figure 2.21: Slope field $x - 2y - 3 + (2x + y - 1)\, y' = 0$

Summary of solutions found

$$y = i(-1 + i + x)$$

$$y = -ix + i - 1$$

0.413 (sec) Entering first order ode homog type maple C solver

2.1.7.2 Solved using first_order_ode_homog_type_maple_C

$$x - 2y - 3 + (2x + y - 1)\, y' = 0$$

Let $Y = y - y_0$ and $X = x - x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = \frac{-X - x_0 + 2Y(X) + 2y_0 + 3}{2X + 2x_0 + Y(X) + y_0 - 1}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = 1$$

$$y_0 = -1$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = \frac{-X + 2Y(X)}{2X + Y(X)}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X, Y) \\ &= \frac{-X + 2Y}{2X + Y} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X, Y)$ and $N(X, Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X, Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X, Y)$$

In this case, it can be seen that both $M = -X + 2Y$ and $N = 2X + Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{dY}{dX} = \frac{du}{dX}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned} \frac{du}{dX}X + u &= \frac{2u - 1}{u + 2} \\ \frac{du}{dX} &= \frac{\frac{2u(X)-1}{u(X)+2} - u(X)}{X} \end{aligned}$$

Or

$$\frac{d}{dX}u(X) - \frac{\frac{2u(X)-1}{u(X)+2} - u(X)}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right) Xu(X) + u(X)^2 + 2\left(\frac{d}{dX}u(X)\right) X + 1 = 0$$

Or

$$X(u(X) + 2)\left(\frac{d}{dX}u(X)\right) + u(X)^2 + 1 = 0$$

Which is now solved as separable in $u(X)$.

The ode

$$\frac{d}{dX}u(X) = -\frac{u(X)^2 + 1}{X(u(X) + 2)} \tag{2.6}$$

is separable as it can be written as

$$
\begin{aligned}
\frac{d}{dX}u(X) &= -\frac{u(X)^2+1}{X\left(u\left(X\right)+2\right)} \\
&= f(X)g(u)
\end{aligned}
$$

Where

$$
\begin{aligned}
f(X) &= -\frac{1}{X} \\
g(u) &= \frac{u^2+1}{u+2}
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\int \frac{1}{g(u)}\, du &= \int f(X)\, dX \\
\int \frac{u+2}{u^2+1}\, du &= \int -\frac{1}{X}\, dX
\end{aligned}
$$

$$
\frac{\ln\left(u(X)^2+1\right)}{2} + 2\arctan\left(u(X)\right) = \ln\left(\frac{1}{X}\right) + c_1
$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u)=0$ or

$$
\frac{u^2+1}{u+2} = 0
$$

for $u(X)$ gives

$$
\begin{aligned}
u(X) &= -i \\
u(X) &= i
\end{aligned}
$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$
\frac{\ln\left(u(X)^2+1\right)}{2} + 2\arctan\left(u(X)\right) = \ln\left(\frac{1}{X}\right) + c_1
$$

$$
u(X) = -i
$$

$$
u(X) = i
$$

Converting $\frac{\ln\left(u(X)^2+1\right)}{2}+2\arctan\left(u(X)\right)=\ln\left(\frac{1}{X}\right)+c_1$ back to $Y(X)$ gives

$$\frac{\ln\left(\frac{Y(X)^2+X^2}{X^2}\right)}{2}+2\arctan\left(\frac{Y(X)}{X}\right)=\ln\left(\frac{1}{X}\right)+c_1$$

Converting $u(X)=-i$ back to $Y(X)$ gives

$$Y(X)=-iX$$

Converting $u(X)=i$ back to $Y(X)$ gives

$$Y(X)=iX$$

Using the solution for $Y(X)$

$$\frac{\ln\left(\frac{Y(X)^2+X^2}{X^2}\right)}{2}+2\arctan\left(\frac{Y(X)}{X}\right)=\ln\left(\frac{1}{X}\right)+c_1 \tag{A}$$

And replacing back terms in the above solution using

$$Y=y+y_0$$
$$X=x_0+x$$

Or

$$Y=y-1$$
$$X=1+x$$

Then the solution in y becomes using EQ (A)

$$\frac{\ln\left(\frac{(y+1)^2+(-1+x)^2}{(-1+x)^2}\right)}{2}+2\arctan\left(\frac{y+1}{-1+x}\right)=\ln\left(\frac{1}{-1+x}\right)+c_1$$

Using the solution for $Y(X)$

$$Y(X)=-iX \tag{A}$$

And replacing back terms in the above solution using

$$Y=y+y_0$$
$$X=x_0+x$$

Or

$$Y=y-1$$
$$X=1+x$$

Then the solution in y becomes using EQ (A)

$$y + 1 = -i(-1 + x)$$

Using the solution for $Y(X)$

$$Y(X) = iX \tag{A}$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x_0 + x$$

Or

$$Y = y - 1$$
$$X = 1 + x$$

Then the solution in y becomes using EQ (A)

$$y + 1 = i(-1 + x)$$

Simplifying the above gives

$$\frac{\ln\left(\frac{(y+1)^2+(-1+x)^2}{(-1+x)^2}\right)}{2} + 2\arctan\left(\frac{y+1}{-1+x}\right) = \ln\left(\frac{1}{-1+x}\right) + c_1$$
$$y + 1 = -ix + i$$
$$y + 1 = i(-1 + x)$$

Solving for y gives

$$\frac{\ln\left(\frac{(y+1)^2+(-1+x)^2}{(-1+x)^2}\right)}{2} + 2\arctan\left(\frac{y+1}{-1+x}\right) = \ln\left(\frac{1}{-1+x}\right) + c_1$$
$$y = ix - i - 1$$
$$y = -ix + i - 1$$

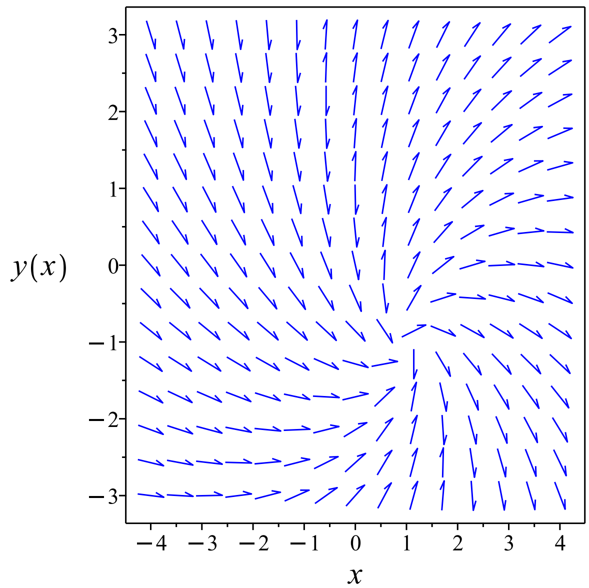


Figure 2.22: Slope field $x - 2y - 3 + (2x + y - 1)\, y' = 0$

Summary of solutions found

$$\frac{\ln\left(\frac{(y+1)^2+(-1+x)^2}{(-1+x)^2}\right)}{2} + 2\arctan\left(\frac{y+1}{-1+x}\right) = \ln\left(\frac{1}{-1+x}\right) + c_1$$

$$y = ix - i - 1$$

$$y = -ix + i - 1$$

Entering first order ode abel second kind solver

$$x - 2y - 3 + (2x + y - 1)\, y' = 0$$

Applying transformation

$$y = \frac{1}{u(x)} - g$$

Results in the new ode which is Abel first kind

$$x - 2xu(x) - 3 + (2x + xu(x) - 1)\,(u(x) + xu'(x)) = 0$$

Which is now solved Unknown ode type.

2.1.7.3 Solved using first_order_ode_abel_second_kind_solved_by_converting_to_first_kind

4.800 (sec)

This is Abel second kind ODE, it has the form

$$(y(x)+g)\,y'(x) = f_0(x)+f_1(x)y(x)+f_2(x)y(x)^2+f_3(x)y(x)^3$$

Comparing the above to given ODE which is

$$x-2y(x)-3+(2x+y(x)-1)\,y'(x) = 0 \tag{1}$$

Shows that

$$\begin{aligned} g &= 2x-1 \\ f_0 &= 3-x \\ f_1 &= 2 \\ f_2 &= 0 \\ f_3 &= 0 \end{aligned}$$

Applying transformation

$$y(x) = \frac{1}{u(x)}-g$$

Results in the new ode which is Abel first kind

$$u'(x) = (5x-5)\,u(x)^3-4u(x)^2$$

Which is now solved.

Entering first order ode abel first kind solver

This is Abel first kind ODE, it has the form

$$u'(x) = f_0(x)+f_1(x)u(x)+f_2(x)u(x)^2+f_3(x)u(x)^3$$

Comparing the above to given ODE which is

$$u'(x) = (5x-5)\,u(x)^3-4u(x)^2 \tag{1}$$

Therefore

$$\begin{aligned} f_0 &= 0 \\ f_1 &= 0 \\ f_2 &= -4 \\ f_3 &= 5x-5 \end{aligned}$$

Hence

$$
\begin{aligned}
f_0' &= 0 \\
f_3' &= 5
\end{aligned}
$$

Since $f_2(x) = -4$ is not zero, then the followingtransformation is used to remove f_2. Let $u(x) = u(x) - \frac{f_2}{3f_3}$ or

$$
\begin{aligned}
u(x) &= u(x) - \left(\frac{-4}{15x - 15}\right) \\
&= u(x) + \frac{4}{15x - 15}
\end{aligned}
$$

The above transformation applied to (1) gives a new ODE as

$$
u'(x) = \frac{(3375x^3 - 10125x^2 + 10125x - 3375)\, u(x)^3}{675\,(-1+x)^2} + \frac{(-720x + 720)\, u(x)}{675\,(-1+x)^2} + \frac{52}{675\,(-1+x)^2} \tag{2}
$$

The above ODE (2) can now be solved.

Entering first order ode LIE solver

Writing the ode as

$$
\begin{aligned}
u'(x) &= \frac{3375u^3x^3 - 10125u^3x^2 + 10125u^3x - 3375u^3 - 720ux + 720u + 52}{675\,(-1+x)^2} \\
u'(x) &= \omega(x, u(x))
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega\left(\eta_{u(x)} - \xi_x\right) - \omega^2 \xi_{u(x)} - \omega_x \xi - \omega_{u(x)} \eta = 0 \tag{A}
$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = ua_3 + xa_2 + a_1 \tag{1E}
$$

$$
\eta = ub_3 + xb_2 + b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, b_1, b_2, b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2 + \frac{(3375u^3x^3 - 10125u^3x^2 + 10125u^3x - 3375u^3 - 720ux + 720u + 52)(b_3 - a_2)}{675(-1+x)^2} \\
&- \frac{(3375u^3x^3 - 10125u^3x^2 + 10125u^3x - 3375u^3 - 720ux + 720u + 52)^2 a_3}{455625(-1+x)^4} \\
&- \left(\frac{10125u^3x^2 - 20250u^3x + 10125u^3 - 720u}{675(-1+x)^2}\right. \\
&\left. - \frac{2(3375u^3x^3 - 10125u^3x^2 + 10125u^3x - 3375u^3 - 720ux + 720u + 52)}{675(-1+x)^3}\right)(ua_3 \\
&+ xa_2 + a_1) \\
&- \frac{(10125u^2x^3 - 30375u^2x^2 + 30375u^2x - 10125u^2 - 720x + 720)(ub_3 + xb_2 + b_1)}{675(-1+x)^2} = 0
\end{aligned}
\tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{11390625u^6x^6a_3 - 68343750u^6x^5a_3 + 170859375u^6x^4a_3 - 227812500u^6x^3a_3 + 170859375u^6x^2a_3 - \cdots}{\cdots}
$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned}
&-11390625u^6x^6a_3 + 68343750u^6x^5a_3 \\
&- 170859375u^6x^4a_3 + 227812500u^6x^3a_3 \\
&- 170859375u^6x^2a_3 + 2581875u^4x^4a_3 - 4556250u^3x^5a_2 \\
&- 4556250u^3x^5b_3 - 6834375u^2x^6b_2 + 68343750u^6xa_3 \\
&- 10327500u^4x^3a_3 - 2278125u^3x^4a_1 + 20503125u^3x^4a_2 \\
&+ 22781250u^3x^4b_3 - 6834375u^2x^5b_1 + 34171875u^2x^5b_2 \\
&- 11390625u^6a_3 + 15491250u^4x^2a_3 + 9112500u^3x^3a_1 \\
&- 36450000u^3x^3a_2 - 351000u^3x^3a_3 - 45562500u^3x^3b_3 \\
&+ 34171875u^2x^4b_1 - 68343750u^2x^4b_2 - 10327500u^4xa_3 \\
&- 13668750u^3x^2a_1 + 31893750u^3x^2a_2 + 1053000u^3x^2a_3 \\
&+ 45562500u^3x^2b_3 - 68343750u^2x^3b_1 + 68343750u^2x^3b_2 \\
&+ 2581875u^4a_3 + 9112500u^3xa_1 - 13668750u^3xa_2 \\
&- 1053000u^3xa_3 - 22781250u^3xb_3 - 1004400u^2x^2a_3 \\
&+ 68343750u^2x^2b_1 - 34171875u^2x^2b_2 + 941625x^4b_2 \\
&- 2278125u^3a_1 + 2278125u^3a_2 + 351000u^3a_3 \\
&+ 4556250u^3b_3 + 2008800u^2xa_3 - 34171875u^2xb_1 \\
&+ 6834375u^2xb_2 - 486000u\,x^2a_1 - 486000u\,x^2a_2 \\
&+ 486000x^3b_1 - 3280500x^3b_2 - 1004400u^2a_3 \\
&+ 6834375u^2b_1 + 972000uxa_1 + 972000uxa_2 \\
&+ 145080uxa_3 + 35100x^2a_2 - 1458000x^2b_1 \\
&+ 4191750x^2b_2 + 35100x^2b_3 - 486000ua_1 \\
&- 486000ua_2 - 145080ua_3 + 70200xa_1 + 1458000xb_1 \\
&- 2308500xb_2 - 70200xb_3 - 70200a_1 - 35100a_2 \\
&- 2704a_3 - 486000b_1 + 455625b_2 + 35100b_3 = 0
\end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{u, x\}$ in them.

$$\{u, x\}$$

The following substitution is now made to be able to collect on all terms with $\{u, x\}$ in them

$$\{u = v_1, x = v_2\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&-11390625a_3v_1^6v_2^6 + 68343750a_3v_1^6v_2^5 \\
&- 170859375a_3v_1^6v_2^4 + 227812500a_3v_1^6v_2^3 \\
&- 4556250a_2v_1^3v_2^5 - 170859375a_3v_1^6v_2^2 + 2581875a_3v_1^4v_2^4 \\
&- 6834375b_2v_1^2v_2^6 - 4556250b_3v_1^3v_2^5 - 2278125a_1v_1^3v_2^4 \\
&+ 20503125a_2v_1^3v_2^4 + 68343750a_3v_1^6v_2 - 10327500a_3v_1^4v_2^3 \\
&- 6834375b_1v_1^2v_2^5 + 34171875b_2v_1^2v_2^5 + 22781250b_3v_1^3v_2^4 \\
&+ 9112500a_1v_1^3v_2^3 - 36450000a_2v_1^3v_2^3 - 11390625a_3v_1^6 \\
&+ 15491250a_3v_1^4v_2^2 - 351000a_3v_1^3v_2^3 + 34171875b_1v_1^2v_2^4 \\
&- 68343750b_2v_1^2v_2^4 - 45562500b_3v_1^3v_2^3 - 13668750a_1v_1^3v_2^2 \\
&+ 31893750a_2v_1^3v_2^2 - 10327500a_3v_1^4v_2 + 1053000a_3v_1^3v_2^2 \\
&- 68343750b_1v_1^2v_2^3 + 68343750b_2v_1^2v_2^3 + 45562500b_3v_1^3v_2^2 \\
&+ 9112500a_1v_1^3v_2 - 13668750a_2v_1^3v_2 + 2581875a_3v_1^4 \\
&- 1053000a_3v_1^3v_2 - 1004400a_3v_1^2v_2^2 + 68343750b_1v_1^2v_2^2 \\
&- 34171875b_2v_1^2v_2^2 + 941625b_2v_2^4 - 22781250b_3v_1^3v_2 \\
&- 2278125a_1v_1^3 - 486000a_1v_1v_2^2 + 2278125a_2v_1^3 \\
&- 486000a_2v_1v_2^2 + 351000a_3v_1^3 + 2008800a_3v_1^2v_2 \\
&- 34171875b_1v_1^2v_2 + 486000b_1v_2^3 + 6834375b_2v_1^2v_2 \\
&- 3280500b_2v_2^3 + 4556250b_3v_1^3 + 972000a_1v_1v_2 \\
&+ 972000a_2v_1v_2 + 35100a_2v_2^2 - 1004400a_3v_1^2 \\
&+ 145080a_3v_1v_2 + 6834375b_1v_1^2 - 1458000b_1v_2^2 \\
&+ 4191750b_2v_2^2 + 35100b_3v_2^2 - 486000a_1v_1 + 70200a_1v_2 \\
&- 486000a_2v_1 - 145080a_3v_1 + 1458000b_1v_2 \\
&- 2308500b_2v_2 - 70200b_3v_2 - 70200a_1 - 35100a_2 \\
&- 2704a_3 - 486000b_1 + 455625b_2 + 35100b_3 = 0
\end{aligned}
\tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&-11390625a_3v_1^6 + 2581875a_3v_1^4 + 941625b_2v_2^4 - 70200a_1 \\
&- 35100a_2 - 2704a_3 - 486000b_1 + 455625b_2 \\
&+ 35100b_3 + (-1004400a_3 + 6834375b_1)\, v_1^2 \\
&+ (-486000a_1 - 486000a_2 - 145080a_3)\, v_1 \\
&+ (486000b_1 - 3280500b_2)\, v_2^3 \\
&+ (35100a_2 - 1458000b_1 + 4191750b_2 + 35100b_3)\, v_2^2 \\
&+ (70200a_1 + 1458000b_1 - 2308500b_2 - 70200b_3)\, v_2 \\
&+ (-2278125a_1 + 2278125a_2 + 351000a_3 + 4556250b_3)\, v_1^3 \\
&- 10327500a_3v_1^4v_2^3 + 15491250a_3v_1^4v_2^2 \\
&- 10327500a_3v_1^4v_2 - 11390625a_3v_1^6v_2^6 + 68343750a_3v_1^6v_2^5 \\
&- 170859375a_3v_1^6v_2^4 + 227812500a_3v_1^6v_2^3 \\
&- 170859375a_3v_1^6v_2^2 + 2581875a_3v_1^4v_2^4 - 6834375b_2v_1^2v_2^6 \\
&+ 68343750a_3v_1^6v_2 + (-4556250a_2 - 4556250b_3)\, v_1^3v_2^5 \\
&+ (-2278125a_1 + 20503125a_2 + 22781250b_3)\, v_1^3v_2^4 \\
&+ (9112500a_1 - 36450000a_2 - 351000a_3 \\
&- 45562500b_3)\, v_1^3v_2^3 + (-13668750a_1 + 31893750a_2 \\
&+ 1053000a_3 + 45562500b_3)\, v_1^3v_2^2 + (9112500a_1 \\
&- 13668750a_2 - 1053000a_3 - 22781250b_3)\, v_1^3v_2 \\
&+ (-6834375b_1 + 34171875b_2)\, v_1^2v_2^5 \\
&+ (34171875b_1 - 68343750b_2)\, v_1^2v_2^4 \\
&+ (-68343750b_1 + 68343750b_2)\, v_1^2v_2^3 \\
&+ (-1004400a_3 + 68343750b_1 - 34171875b_2)\, v_1^2v_2^2 \\
&+ (2008800a_3 - 34171875b_1 + 6834375b_2)\, v_1^2v_2 \\
&+ (-486000a_1 - 486000a_2)\, v_1v_2^2 \\
&+ (972000a_1 + 972000a_2 + 145080a_3)\, v_1v_2 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-170859375a_3 &= 0\\
-11390625a_3 &= 0\\
-10327500a_3 &= 0\\
2581875a_3 &= 0\\
15491250a_3 &= 0\\
68343750a_3 &= 0\\
227812500a_3 &= 0\\
-6834375b_2 &= 0\\
941625b_2 &= 0\\
-486000a_1 - 486000a_2 &= 0\\
-4556250a_2 - 4556250b_3 &= 0\\
-1004400a_3 + 6834375b_1 &= 0\\
-68343750b_1 + 68343750b_2 &= 0\\
-6834375b_1 + 34171875b_2 &= 0\\
486000b_1 - 3280500b_2 &= 0\\
34171875b_1 - 68343750b_2 &= 0\\
-2278125a_1 + 20503125a_2 + 22781250b_3 &= 0\\
-486000a_1 - 486000a_2 - 145080a_3 &= 0\\
972000a_1 + 972000a_2 + 145080a_3 &= 0\\
-1004400a_3 + 68343750b_1 - 34171875b_2 &= 0\\
2008800a_3 - 34171875b_1 + 6834375b_2 &= 0\\
-13668750a_1 + 31893750a_2 + 1053000a_3 + 45562500b_3 &= 0\\
-2278125a_1 + 2278125a_2 + 351000a_3 + 4556250b_3 &= 0\\
70200a_1 + 1458000b_1 - 2308500b_2 - 70200b_3 &= 0\\
9112500a_1 - 36450000a_2 - 351000a_3 - 45562500b_3 &= 0\\
9112500a_1 - 13668750a_2 - 1053000a_3 - 22781250b_3 &= 0\\
35100a_2 - 1458000b_1 + 4191750b_2 + 35100b_3 &= 0\\
-70200a_1 - 35100a_2 - 2704a_3 - 486000b_1 + 455625b_2 + 35100b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= b_3 \\
a_2 &= -b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= 1 - x \\
\eta &= u
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,u)\,\xi \\
&= u - \left(\frac{3375u^3x^3 - 10125u^3x^2 + 10125u^3x - 3375u^3 - 720ux + 720u + 52}{675\,(-1+x)^2}\right)(1-x) \\
&= \frac{52 + 3375(-1+x)^3\,u^3 + (45 - 45x)\,u}{-675 + 675x} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,u) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{du}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial u}\right) S(x,u) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{52+3375(-1+x)^3u^3+(45-45x)u}{-675+675x}} dy
\end{aligned}
$$

Which results in

$$S = (-675 + 675x)\left(\frac{(-15x+15)\ln\left(225u^2x^2 - 450u^2x + 225u^2 - 60ux + 60u + 13\right)}{20250x^2 - 40500x + 20250} + \frac{2\left(8 - \frac{(-15x}{2(225}\right.}{}\right.$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,u)S_u}{R_x + \omega(x,u)R_u} \tag{2}$$

Where in the above R_x, R_u, S_x, S_u are all partial derivatives and $\omega(x,u)$ is the right hand side of the original ode given by

$$\omega(x,u) = \frac{3375u^3x^3 - 10125u^3x^2 + 10125u^3x - 3375u^3 - 720ux + 720u + 52}{675\left(-1+x\right)^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_u &= 0 \\
S_x &= \frac{u}{5\left(\frac{4}{15} + (-1+x)\,u\right)\left(\frac{13}{225} + (-1+x)^2 u^2 + \frac{4(1-x)u}{15}\right)} \\
S_u &= \frac{-1+x}{5\left(\frac{4}{15} + (-1+x)\,u\right)\left(\frac{13}{225} + (-1+x)^2 u^2 + \frac{4(1-x)u}{15}\right)}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{-1+x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, u in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{-1+R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\begin{aligned}
\int dS &= \int \frac{1}{-1+R}\, dR \\
S(R) &= \ln\left(-1+R\right) + c_2
\end{aligned}$$

To complete the solution, we just need to transform the above back to x, u coordinates. This results in

$$2\arctan\left(-\frac{2}{3}+5(-1+x)\,u(x)\right)-\frac{\ln\left(13+225(-1+x)^{2}\,u(x)^{2}+(-60x+60)\,u(x)\right)}{2}+\ln\left(4+15(-\right.$$

Substituting $u = u(x) - \frac{4}{3(5x-5)}$ in the above solution gives

$$-2\arctan\left(\frac{2}{3}-5(-1+x)\left(u(x)-\frac{4}{3\,(5x-5)}\right)\right)-\frac{\ln\left(13+225(-1+x)^{2}\left(u(x)-\frac{4}{3(5x-5)}\right)^{2}+(-\right.}{2}$$

Simplifying the above gives

$$2\arctan\left(-2+(5x-5)\,u(x)\right)+\frac{\ln(5)}{2}-\frac{\ln\left(1+5(-1+x)^{2}\,u(x)^{2}+(-4x+4)\,u(x)\right)}{2}+\ln\left(u(x)\,(-1+x)\right)=\ln(-1+x)+c_{2}$$

Substituting $u(x) = \frac{1}{2x+y(x)-1}$ in the above solution gives

$$2\arctan\left(-2+\frac{5x-5}{2x+y(x)-1}\right)+\frac{\ln(5)}{2}-\frac{\ln\left(1+\frac{5(-1+x)^{2}}{(2x+y(x)-1)^{2}}+\frac{-4x+4}{2x+y(x)-1}\right)}{2}+\ln\left(\frac{-1+x}{2x+y(x)-1}\right)=\ln(-1+x)+c_{2}$$

Simplifying the above gives

$$2\arctan\left(\frac{x-2y(x)-3}{2x+y(x)-1}\right)+\frac{\ln(5)}{2}-\frac{\ln\left(\frac{y(x)^{2}+x^{2}+2y(x)-2x+2}{(2x+y(x)-1)^{2}}\right)}{2}+\ln\left(\frac{-1+x}{2x+y(x)-1}\right)=\ln(-1+x)+c_{2}$$

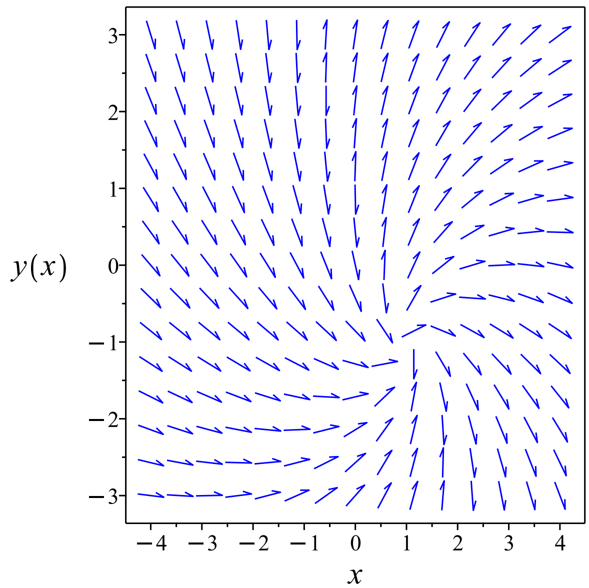


Figure 2.23: Slope field $x - 2y(x) - 3 + (2x + y(x) - 1)\, y'(x) = 0$

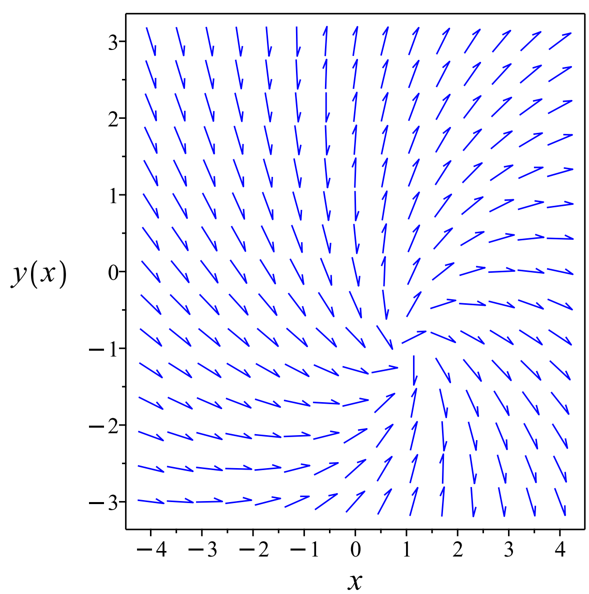


Figure 2.24: Slope field $x - 2y(x) - 3 + (2x + y(x) - 1)\, y'(x) = 0$

Summary of solutions found

$$2\arctan\left(\frac{x - 2y(x) - 3}{2x + y(x) - 1}\right) + \frac{\ln(5)}{2} - \frac{\ln\left(\frac{y(x)^2 + x^2 + 2y(x) - 2x + 2}{(2x + y(x) - 1)^2}\right)}{2} + \ln\left(\frac{-1 + x}{2x + y(x) - 1}\right) = \ln(-1 + x) + c_2$$

0.880 (sec) Entering first order ode LIE solver

2.1.7.4 Solved using first_order_ode_LIE

$$x-2y(x)-3+(2x+y(x)-1)\,y'(x)=0$$

Writing the ode as

$$y'(x)=\frac{-x+2y+3}{2x+y-1}$$
$$y'(x)=\omega(x,y(x))$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega\left(\eta_{y(x)}-\xi_x\right)-\omega^2\xi_{y(x)}-\omega_x\xi-\omega_{y(x)}\eta=0 \tag{A}$$

To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi=xa_2+ya_3+a_1 \tag{1E}$$
$$\eta=xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1,a_2,a_3,b_1,b_2,b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}b_2&+\frac{(-x+2y+3)\,(b_3-a_2)}{2x+y-1}-\frac{(-x+2y+3)^2\,a_3}{(2x+y-1)^2}\\&-\left(-\frac{1}{2x+y-1}-\frac{2(-x+2y+3)}{(2x+y-1)^2}\right)(xa_2+ya_3+a_1)\\&-\left(\frac{2}{2x+y-1}-\frac{-x+2y+3}{(2x+y-1)^2}\right)(xb_2+yb_3+b_1)=0\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{2x^2a_2-x^2a_3-x^2b_2-2x^2b_3+2xya_2+4xya_3+4xyb_2-2xyb_3-2y^2a_2+y^2a_3+y^2b_2+2y^2b_3-2xa}{(2x+}$$
$$=0$$

Setting the numerator to zero gives

$$\begin{aligned}&2x^2a_2-x^2a_3-x^2b_2-2x^2b_3+2xya_2+4xya_3+4xyb_2-2xyb_3-2y^2a_2\\&+y^2a_3+y^2b_2+2y^2b_3-2xa_2+6xa_3-5xb_1+xb_2+7xb_3+5ya_1\\&-ya_2-7ya_3-2yb_2+6yb_3+5a_1+3a_2-9a_3+5b_1+b_2-3b_3=0\end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &2a_2v_1^2 + 2a_2v_1v_2 - 2a_2v_2^2 - a_3v_1^2 + 4a_3v_1v_2 + a_3v_2^2 - b_2v_1^2 + 4b_2v_1v_2 + b_2v_2^2 \\ &- 2b_3v_1^2 - 2b_3v_1v_2 + 2b_3v_2^2 + 5a_1v_2 - 2a_2v_1 - a_2v_2 + 6a_3v_1 - 7a_3v_2 - 5b_1v_1 \\ &+ b_2v_1 - 2b_2v_2 + 7b_3v_1 + 6b_3v_2 + 5a_1 + 3a_2 - 9a_3 + 5b_1 + b_2 - 3b_3 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} &(2a_2 - a_3 - b_2 - 2b_3)\, v_1^2 + (2a_2 + 4a_3 + 4b_2 - 2b_3)\, v_1v_2 \\ &+ (-2a_2 + 6a_3 - 5b_1 + b_2 + 7b_3)\, v_1 + (-2a_2 + a_3 + b_2 + 2b_3)\, v_2^2 \\ &+ (5a_1 - a_2 - 7a_3 - 2b_2 + 6b_3)\, v_2 + 5a_1 + 3a_2 - 9a_3 + 5b_1 + b_2 - 3b_3 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -2a_2 + a_3 + b_2 + 2b_3 &= 0 \\ 2a_2 - a_3 - b_2 - 2b_3 &= 0 \\ 2a_2 + 4a_3 + 4b_2 - 2b_3 &= 0 \\ 5a_1 - a_2 - 7a_3 - 2b_2 + 6b_3 &= 0 \\ -2a_2 + 6a_3 - 5b_1 + b_2 + 7b_3 &= 0 \\ 5a_1 + 3a_2 - 9a_3 + 5b_1 + b_2 - 3b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= -b_2 - b_3 \\ a_2 &= b_3 \\ a_3 &= -b_2 \\ b_1 &= -b_2 + b_3 \\ b_2 &= b_2 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = -1 + x$$
$$\eta = y + 1$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x,y)\,\xi \\ &= y + 1 - \left(\frac{-x+2y+3}{2x+y-1}\right)(-1+x) \\ &= \frac{x^2+y^2-2x+2y+2}{2x+y-1} \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{x^2+y^2-2x+2y+2}{2x+y-1}} dy\end{aligned}$$

Which results in

$$S = \frac{\ln(x^2+y^2-2x+2y+2)}{2} + 2\arctan\left(\frac{2y+2}{2x-2}\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{-x+2y+3}{2x+y-1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{x-2y-3}{x^2+y^2-2x+2y+2} \\
S_y &= \frac{2x+y-1}{x^2+y^2-2x+2y+2}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

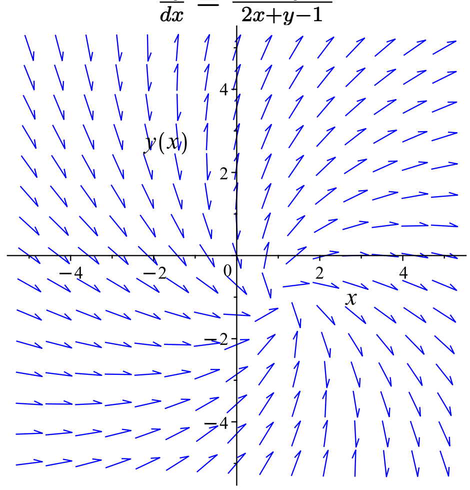
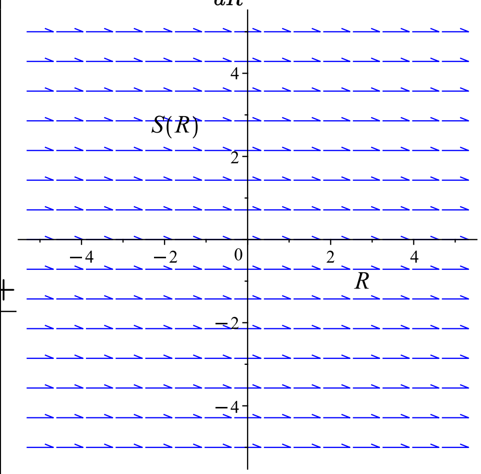
Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\begin{aligned}
\int dS &= \int 0\, dR + c_2 \\
S(R) &= c_2
\end{aligned}$$

To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$\frac{\ln\left(y(x)^2 + x^2 + 2y(x) - 2x + 2\right)}{2} + 2\arctan\left(\frac{y(x)+1}{-1+x}\right) = c_2$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{-x+2y+3}{2x+y-1}$ 	$R = x$ $S = \frac{\ln(x^2 + y^2 - 2x +}{2}$	$\frac{dS}{dR} = 0$ 

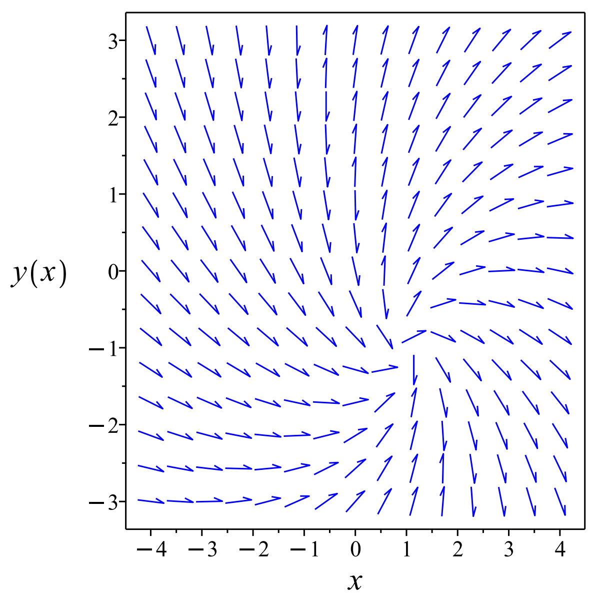


Figure 2.25: Slope field $x - 2y(x) - 3 + (2x + y(x) - 1)\, y'(x) = 0$

Summary of solutions found

$$\frac{\ln\left(y(x)^2 + x^2 + 2y(x) - 2x + 2\right)}{2} + 2\arctan\left(\frac{y(x) + 1}{-1 + x}\right) = c_2$$

2.1.7.5 ✓ Maple. Time used: 0.017 (sec). Leaf size: 31

```
ode:=x-2*y(x)-3+(2*x+y(x)-1)*diff(y(x),x) = 0;
dsolve(ode,y(x), singsol=all);
```

$$y = -1 - \tan\left(\operatorname{RootOf}\left(-4\_Z + \ln\left(\sec\left(\_Z\right)^2\right) + 2\ln\left(x-1\right) + 2c_1\right)\right)\left(x-1\right)$$

Maple trace

```
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful
```

Maple step by step

Let's solve
$x - 2y(x) - 3 + (2x + y(x) - 1)\left(\frac{d}{dx}y(x)\right) = 0$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Solve for the highest derivative
  $\frac{d}{dx}y(x) = \frac{-x+2y(x)+3}{2x+y(x)-1}$

2.1.7.6 ✓ Mathematica. Time used: 0.035 (sec). Leaf size: 66

```
ode=(x-2*y[x]-3)+(2*x+y[x]-1)*D[y[x],x]==0;
ic={};
DSolve[{ode,ic},y[x],x,IncludeSingularSolutions->True]
```

$$\text{Solve}\left[32\arctan\left(\frac{2y(x)-x+3}{y(x)+2x-1}\right) + 8\log\left(\frac{x^2+y(x)^2+2y(x)-2x+2}{5(x-1)^2}\right) + 16\log(x-1) + 5c_1 = 0, y(x)\right]$$

2.1.7.7 ✓ Sympy. Time used: 2.882 (sec). Leaf size: 36

```
from sympy import *
x = symbols("x")
y = Function("y")
ode = Eq(x + (2*x + y(x) - 1)*Derivative(y(x), x) - 2*y(x) - 3,0)
ics = {}
dsolve(ode,func=y(x),ics=ics)
```

$$\log(x-1) = C_1 - \log\left(\sqrt{1 + \frac{(y(x)+1)^2}{(x-1)^2}}\right) - 2\,\mathrm{atan}\left(\frac{y(x)+1}{x-1}\right)$$

2.1.8 Problem 8

Local contents

Internal problem ID [4083]
Book : Differential equations, Shepley L. Ross, 1964
Section : 2.4, page 55
Problem number : 8
Date solved : `Saturday, December 06, 2025 at 04:16:51 PM`
CAS classification :
`[[_homogeneous, ‘class C‘], _exact, _rational, [_Abel, ‘2nd type‘, ‘class A‘]]`

2.1.8.1 Existence and uniqueness analysis

$$6x + 4y + 1 + (4x + 2y + 2)\, y' = 0$$
$$y\left(\frac{1}{2}\right) = 3$$

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= -\frac{6x + 4y + 1}{2\,(2x + y + 1)} \end{aligned}$$

The x domain of $f(x,y)$ when $y = 3$ is

$$\{x < -2 \lor -2 < x\}$$

And the point $x_0 = \frac{1}{2}$ is inside this domain. The y domain of $f(x,y)$ when $x = \frac{1}{2}$ is

$$\{y < -2 \lor -2 < y\}$$

And the point $y_0 = 3$ is inside this domain. Now we will look at the continuity of

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{6x+4y+1}{2\,(2x+y+1)}\right)$$
$$= -\frac{2}{2x+y+1} + \frac{6x+4y+1}{2\,(2x+y+1)^2}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 3$ is

$$\{x < -2 \lor -2 < x\}$$

And the point $x_0 = \frac{1}{2}$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = \frac{1}{2}$ is

$$\{y < -2 \lor -2 < y\}$$

And the point $y_0 = 3$ is inside this domain. Therefore solution exists and is unique.

0.087 (sec) Entering first order ode exact solver

2.1.8.2 Solved using first_order_ode_exact

$$6x+4y+1+(4x+2y+2)\,y' = 0$$
$$y\left(\frac{1}{2}\right) = 3$$

To solve an ode of the form

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(4x+2y+2)\,\mathrm{d}y &= (-6x-4y-1)\,\mathrm{d}x\\ (6x+4y+1)\,\mathrm{d}x + (4x+2y+2)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= 6x+4y+1\\ N(x,y) &= 4x+2y+2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(6x+4y+1)\\ &= 4\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(4x+2y+2)\\ &= 4\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int M\,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int 6x + 4y + 1\,\mathrm{d}x$$

$$\phi = x(3x + 4y + 1) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 4x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 4x + 2y + 2$. Therefore equation (4) becomes

$$4x + 2y + 2 = 4x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 2y + 2$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (2y + 2)\,\mathrm{d}y$$

$$f(y) = y^2 + 2y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = x(3x + 4y + 1) + y^2 + 2y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = x(3x + 4y + 1) + y^2 + 2y$$

Simplifying the above gives

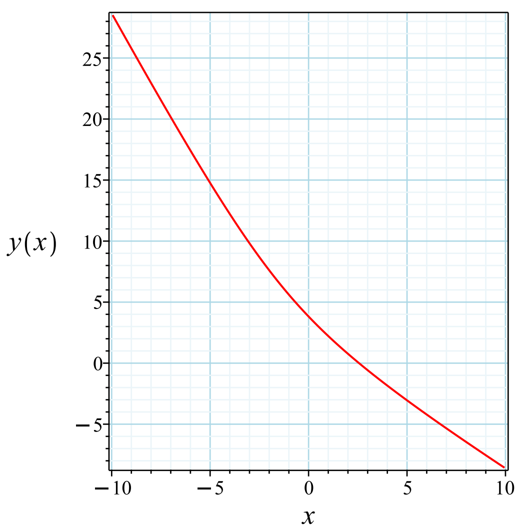
$$y^2 + (4x + 2)\,y + 3x^2 + x = c_1$$

Solving for initial conditions the solution is

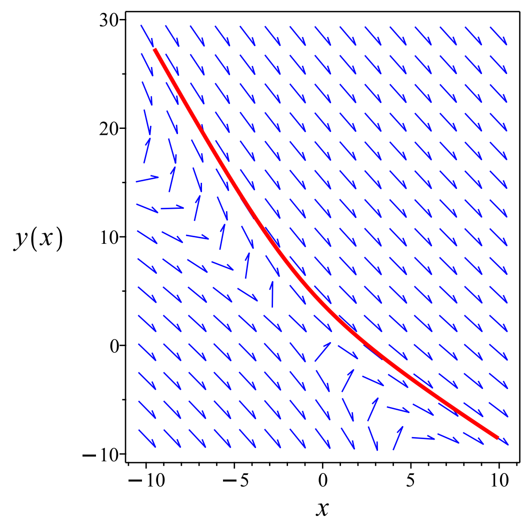
$$y^2+(4x+2)\,y+3x^2+x=\frac{89}{4}$$

Solving for y gives

$$y=-2x-1+\frac{\sqrt{4x^2+12x+93}}{2}$$



(a) Solution plot



(b) Slope field $6x+4y+1+(4x+2y+2)\,y'=0$

Summary of solutions found

$$y=-2x-1+\frac{\sqrt{4x^2+12x+93}}{2}$$

0.408 (sec) Entering first order ode dAlembert solver

2.1.8.3 Solved using first_order_ode_dAlembert

$$6x+4y+1+(4x+2y+2)\,y'=0$$
$$y\left(\frac{1}{2}\right)=3$$

Let $p=y'$ the ode becomes

$$6x+4y+1+(4x+2y+2)\,p=0$$

Solving for y from the above results in

$$y = -\frac{(4p+6)\,x}{2\,(2+p)} - \frac{2p+1}{2\,(2+p)} \tag{1}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved.

Taking derivative of (*) w.r.t. x gives

$$\begin{aligned} p &= f + (xf' + g')\frac{dp}{dx} \\ p - f &= (xf' + g')\frac{dp}{dx} \end{aligned} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$\begin{aligned} f &= \frac{-2p-3}{2+p} \\ g &= \frac{-2p-1}{4+2p} \end{aligned}$$

Hence (2) becomes

$$p - \frac{-2p-3}{2+p} = \left(-\frac{2x}{2+p} + \frac{2xp}{(2+p)^2} + \frac{3x}{(2+p)^2} - \frac{2}{4+2p} + \frac{4p}{(4+2p)^2} + \frac{2}{(4+2p)^2}\right)p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$p - \frac{-2p-3}{2+p} = 0$$

No valid singular solutions found.

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{p(x) - \frac{-2p(x)-3}{2+p(x)}}{-\frac{2x}{2+p(x)} + \frac{2xp(x)}{(2+p(x))^2} + \frac{3x}{(2+p(x))^2} - \frac{2}{4+2p(x)} + \frac{4p(x)}{(4+2p(x))^2} + \frac{2}{(4+2p(x))^2}} \tag{3}$$

This ODE is now solved for $p(x)$. No inversion is needed.

The ode

$$p'(x) = -\frac{2(2+p(x))\,(p(x)+3)\,(p(x)+1)}{2x+3} \tag{2.7}$$

is separable as it can be written as

$$\begin{aligned} p'(x) &= -\frac{2(2+p(x))\,(p(x)+3)\,(p(x)+1)}{2x+3} \\ &= f(x)g(p) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= -\frac{2}{2x+3} \\ g(p) &= (p+3)\,(2+p)\,(p+1) \end{aligned}$$

Integrating gives

$$\int \frac{1}{g(p)}\,dp = \int f(x)\,dx$$

$$\int \frac{1}{(p+3)\,(2+p)\,(p+1)}\,dp = \int -\frac{2}{2x+3}\,dx$$

$$-\ln(2+p(x)) + \frac{\ln(p(x)+1)}{2} + \frac{\ln(p(x)+3)}{2} = \ln\left(\frac{1}{2x+3}\right) + c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(p)$ is zero, since we had to divide by this above. Solving $g(p) = 0$ or

$$(p+3)\,(2+p)\,(p+1) = 0$$

for $p(x)$ gives

$$\begin{aligned} p(x) &= -3 \\ p(x) &= -2 \end{aligned}$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$-\ln(2+p(x)) + \frac{\ln(p(x)+1)}{2} + \frac{\ln(p(x)+3)}{2} = \ln\left(\frac{1}{2x+3}\right) + c_1$$

$$p(x) = -3$$

$$p(x) = -2$$

Substituing the above solution for p in (2A) gives

$$y = \frac{x\left(\frac{2\,e^{2c_1}-8x^2+2\sqrt{-4\,e^{2c_1}x^2+16x^4-12\,e^{2c_1}x+96x^3-9\,e^{2c_1}+216x^2+216x+81}-24x-18}{e^{2c_1}-4x^2-12x-9}-1\right)}{1-\frac{e^{2c_1}-4x^2+\sqrt{-4\,e^{2c_1}x^2+16x^4-12\,e^{2c_1}x+96x^3-9\,e^{2c_1}+216x^2+216x+81}-12x-9}{e^{2c_1}-4x^2-12x-9}} + \frac{\frac{2\,e^{2c_1}-8x^2+2\sqrt{-4\,e^{2c_1}x^2+16x^4-12\,e^{2c_1}x+96x^3-9\,e^{2c_1}+216x^2+216x+81}-24x-18}{e^{2c_1}-4x^2-12x-9}+1}{2-\frac{2\left(e^{2c_1}-4x^2+\sqrt{-4\,e^{2c_1}x^2+16x^4-12\,e^{2c_1}x+96x^3-9\,e^{2c_1}+216x^2+216x+81}-12x-9\right)}{e^{2c_1}-4x^2-12x-9}}$$

$$y = -3x - \frac{5}{2}$$

Simplifying the above gives

$$y = \frac{(-1-2x)\sqrt{-\left(2x+3\right)^2 e^{2c_1}+16\left(x+\frac{3}{2}\right)^4}+4\left(x+\frac{3}{2}\right)\left(-\frac{e^{2c_1}}{4}+\left(x+\frac{3}{2}\right)^2\right)}{\sqrt{-\left(2x+3\right)^2 e^{2c_1}+16\left(x+\frac{3}{2}\right)^4}}$$

$$y = -3x - \frac{5}{2}$$

Solving for initial conditions the solution is

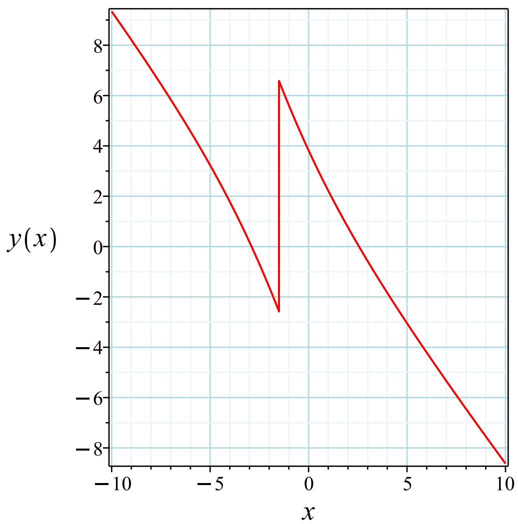
$$y = \frac{(-4x-2)\sqrt{\left(4x^2+12x+93\right)\left(2x+3\right)^2}+8x^3+36x^2+222x+279}{2\sqrt{\left(4x^2+12x+93\right)\left(2x+3\right)^2}}$$

$$y = -3x - \frac{5}{2}$$

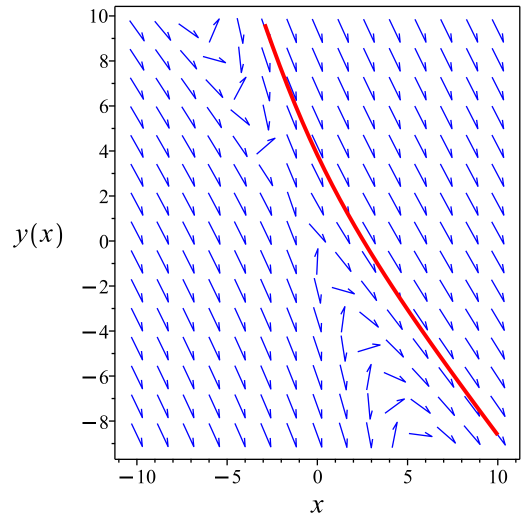
The solution

$$y = -3x - \frac{5}{2}$$

was found not to satisfy the ode or the IC. Hence it is removed.



(a) Solution plot



(b) Slope field $6x+4y+1+(4x+2y+2)\,y'=0$

Summary of solutions found

$$y=\frac{(-4x-2)\sqrt{(4x^{2}+12x+93)\,(2x+3)^{2}}+8x^{3}+36x^{2}+222x+279}{2\sqrt{(4x^{2}+12x+93)\,(2x+3)^{2}}}$$

0.546 (sec) Entering first order ode homog type maple C solver

2.1.8.4 Solved using first_order_ode_homog_type_maple_C

$$6x+4y+1+(4x+2y+2)\,y'=0$$
$$y\left(\frac{1}{2}\right)=3$$

Let $Y=y-y_0$ and $X=x-x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X)=-\frac{6X+6x_0+4Y(X)+4y_0+1}{2\,(2X+2x_0+Y\,(X)+y_0+1)}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0=-\frac{3}{2}$$
$$y_0=2$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = -\frac{6X + 4Y(X)}{2\left(2X + Y\left(X\right)\right)}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X, Y) \\ &= -\frac{3X + 2Y}{2X + Y} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X, Y)$ and $N(X, Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X, Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X, Y)$$

In this case, it can be seen that both $M = -3X - 2Y$ and $N = 2X + Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}X}X + u &= \frac{-2u - 3}{u + 2} \\ \frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\frac{-2u(X)-3}{u(X)+2} - u(X)}{X} \end{aligned}$$

Or

$$\frac{d}{dX}u(X) - \frac{\frac{-2u(X)-3}{u(X)+2} - u(X)}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right)Xu(X) + u(X)^2 + 2\left(\frac{d}{dX}u(X)\right)X + 4u(X) + 3 = 0$$

Or

$$X(u(X) + 2)\left(\frac{d}{dX}u(X)\right) + u(X)^2 + 4u(X) + 3 = 0$$

Which is now solved as separable in $u(X)$.

The ode

$$\frac{d}{dX}u(X) = -\frac{(u(X) + 3)\left(u(X) + 1\right)}{X\left(u\left(X\right) + 2\right)} \tag{2.8}$$

is separable as it can be written as

$$\begin{aligned}\frac{d}{dX}u(X) &= -\frac{(u(X)+3)\,(u(X)+1)}{X\,(u\,(X)+2)}\\ &= f(X)g(u)\end{aligned}$$

Where

$$\begin{aligned}f(X) &= -\frac{1}{X}\\ g(u) &= \frac{(u+3)\,(u+1)}{u+2}\end{aligned}$$

Integrating gives

$$\int \frac{1}{g(u)}\,du = \int f(X)\,dX$$

$$\int \frac{u+2}{(u+3)\,(u+1)}\,du = \int -\frac{1}{X}\,dX$$

$$\frac{\ln\left((u(X)+3)\,(u(X)+1)\right)}{2} = \ln\left(\frac{1}{X}\right) + c_1$$

Taking the exponential of both sides the solution becomes

$$\sqrt{(u\,(X)+3)\,(u\,(X)+1)} = \frac{c_1}{X}$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or

$$\frac{(u+3)\,(u+1)}{u+2} = 0$$

for $u(X)$ gives

$$\begin{aligned}u(X) &= -3\\ u(X) &= -1\end{aligned}$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\sqrt{(u\,(X)+3)\,(u\,(X)+1)} = \frac{c_1}{X}$$

$$u(X) = -3$$

$$u(X) = -1$$

Converting $\sqrt{\left(u\left(X\right)+3\right)\left(u\left(X\right)+1\right)}=\frac{c_1}{X}$ back to $Y(X)$ gives

$$\sqrt{\frac{\left(Y\left(X\right)+3X\right)\left(Y\left(X\right)+X\right)}{X^2}}=\frac{c_1}{X}$$

Converting $u(X)=-3$ back to $Y(X)$ gives

$$Y(X)=-3X$$

Converting $u(X)=-1$ back to $Y(X)$ gives

$$Y(X)=-X$$

Using the solution for $Y(X)$

$$\sqrt{\frac{\left(Y\left(X\right)+3X\right)\left(Y\left(X\right)+X\right)}{X^2}}=\frac{c_1}{X} \tag{A}$$

And replacing back terms in the above solution using

$$Y=y+y_0$$
$$X=x_0+x$$

Or

$$Y=y+2$$
$$X=x-\frac{3}{2}$$

Then the solution in y becomes using EQ (A)

$$\sqrt{\frac{\left(y+3x+\frac{5}{2}\right)\left(y+x-\frac{1}{2}\right)}{\left(x+\frac{3}{2}\right)^2}}=\frac{c_1}{x+\frac{3}{2}}$$

Using the solution for $Y(X)$

$$Y(X)=-3X \tag{A}$$

And replacing back terms in the above solution using

$$Y=y+y_0$$
$$X=x_0+x$$

Or

$$Y=y+2$$
$$X=x-\frac{3}{2}$$

Then the solution in y becomes using EQ (A)

$$y - 2 = -3x - \frac{9}{2}$$

Using the solution for $Y(X)$

$$Y(X) = -X \tag{A}$$

And replacing back terms in the above solution using

$$\begin{aligned} Y &= y + y_0 \\ X &= x_0 + x \end{aligned}$$

Or

$$\begin{aligned} Y &= y + 2 \\ X &= x - \frac{3}{2} \end{aligned}$$

Then the solution in y becomes using EQ (A)

$$y - 2 = -x - \frac{3}{2}$$

Simplifying the above gives

$$\begin{aligned} \sqrt{\frac{(2y + 6x + 5)(2y + 2x - 1)}{(2x + 3)^2}} &= \frac{2c_1}{2x + 3} \\ y - 2 &= -3x - \frac{9}{2} \\ y - 2 &= -x - \frac{3}{2} \end{aligned}$$

Solving for initial conditions the solution is

$$\begin{aligned} \sqrt{\frac{(2y + 6x + 5)(2y + 2x - 1)}{(2x + 3)^2}} &= \frac{2\sqrt{21}}{2x + 3} \\ y - 2 &= -3x - \frac{9}{2} \\ y - 2 &= -x - \frac{3}{2} \end{aligned}$$

Solving for y gives

$$y-2=-3x-\frac{9}{2}$$

$$y-2=-x-\frac{3}{2}$$

$$y=-2x-1+\frac{\sqrt{4x^2+12x+93}}{2}$$

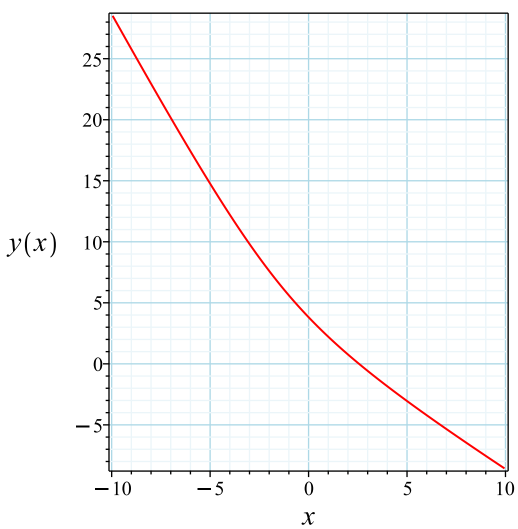
The solution

$$y-2=-3x-\frac{9}{2}$$

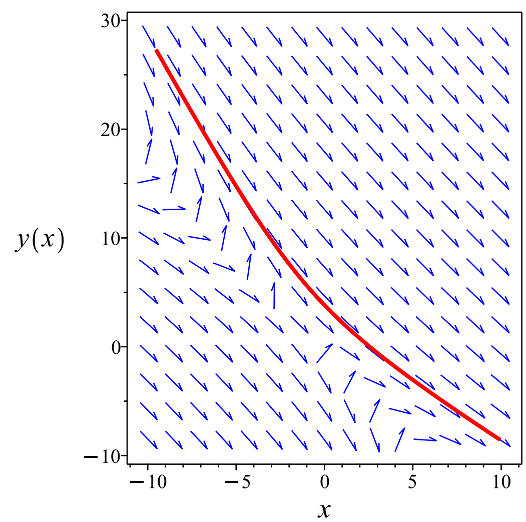
was found not to satisfy the ode or the IC. Hence it is removed. The solution

$$y-2=-x-\frac{3}{2}$$

was found not to satisfy the ode or the IC. Hence it is removed.



(a) Solution plot



(b) Slope field $6x+4y+1+(4x+2y+2)\,y'=0$

<u>Summary of solutions found</u>

$$y=-2x-1+\frac{\sqrt{4x^2+12x+93}}{2}$$

Entering first order ode abel second kind solver

$$6x+4y+1+(4x+2y+2)\,y'=0$$

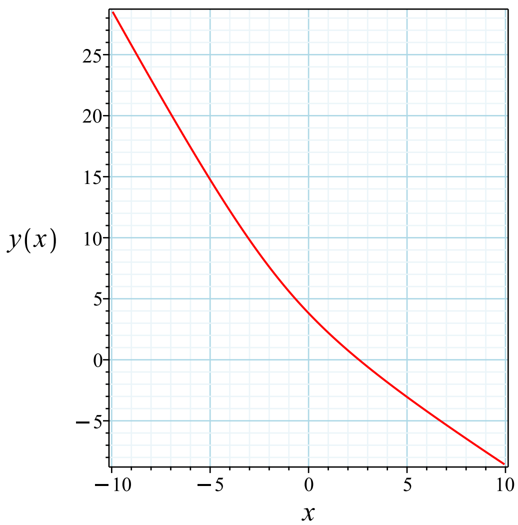
$$y\left(\frac{1}{2}\right)=3$$

0.076 (sec)

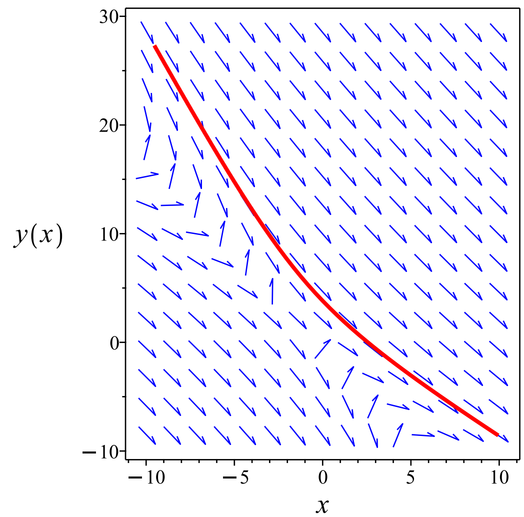
2.1.8.5 Solved using first_order_ode_abel_second_kind_case_5

Solving for initial conditions the solution is

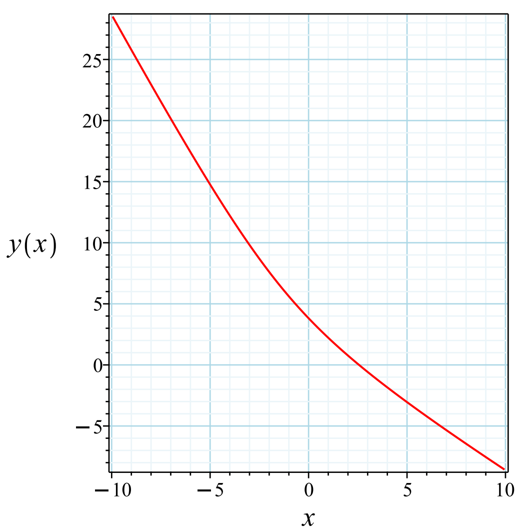
$$y = -2x - 1 + \frac{\sqrt{4x^2 + 12x + 93}}{2}$$



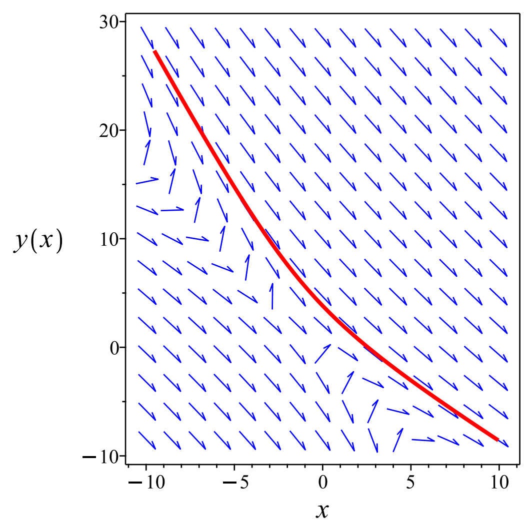
(a) Solution plot



(b) Slope field $6x+4y+1+(4x+2y+2)\,y' = 0$



(a) Solution plot



(b) Slope field $6x+4y+1+(4x+2y+2)\,y' = 0$

Summary of solutions found

$$y = -2x - 1 + \frac{\sqrt{4x^2 + 12x + 93}}{2}$$

0.939 (sec) Entering first order ode LIE solver

2.1.8.6 Solved using first_order_ode_LIE

$$6x + 4y + 1 + (4x + 2y + 2)\, y' = 0$$
$$y\left(\frac{1}{2}\right) = 3$$

Writing the ode as

$$y' = -\frac{6x + 4y + 1}{2\,(2x + y + 1)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$
$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{(6x + 4y + 1)(b_3 - a_2)}{2\,(2x + y + 1)} - \frac{(6x + 4y + 1)^2 a_3}{4\,(2x + y + 1)^2} - \left(-\frac{3}{2x + y + 1} + \frac{6x + 4y + 1}{(2x + y + 1)^2}\right)(x a_2 + y a_3 + a_1) - \left(-\frac{2}{2x + y + 1} + \frac{6x + 4y + 1}{2\,(2x + y + 1)^2}\right)(x b_2 + y b_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{24x^2 a_2 - 36x^2 a_3 + 20x^2 b_2 - 24x^2 b_3 + 24xy a_2 - 48xy a_3 + 16xy b_2 - 24xy b_3 + 8y^2 a_2 - 20y^2 a_3 + 4y^2 b_2}{}$$

$= 0$

Setting the numerator to zero gives

$$\begin{aligned}24x^2a_2 - 36x^2a_3 + 20x^2b_2 - 24x^2b_3 + 24xya_2 - 48xya_3 + 16xyb_2 - 24xyb_3 \\ + 8y^2a_2 - 20y^2a_3 + 4y^2b_2 - 8y^2b_3 + 24xa_2 - 12xa_3 + 4xb_1 + 22xb_2 - 16xb_3 \\ - 4ya_1 + 10ya_2 + 8yb_2 - 4yb_3 + 8a_1 + 2a_2 - a_3 + 6b_1 + 4b_2 - 2b_3 = 0\end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned}24a_2v_1^2 + 24a_2v_1v_2 + 8a_2v_2^2 - 36a_3v_1^2 - 48a_3v_1v_2 - 20a_3v_2^2 + 20b_2v_1^2 + 16b_2v_1v_2 \\ + 4b_2v_2^2 - 24b_3v_1^2 - 24b_3v_1v_2 - 8b_3v_2^2 - 4a_1v_2 + 24a_2v_1 + 10a_2v_2 - 12a_3v_1 \\ + 4b_1v_1 + 22b_2v_1 + 8b_2v_2 - 16b_3v_1 - 4b_3v_2 + 8a_1 + 2a_2 - a_3 + 6b_1 + 4b_2 - 2b_3 = 0\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}(24a_2 - 36a_3 + 20b_2 - 24b_3)\, v_1^2 + (24a_2 - 48a_3 + 16b_2 - 24b_3)\, v_1v_2 \\ + (24a_2 - 12a_3 + 4b_1 + 22b_2 - 16b_3)\, v_1 + (8a_2 - 20a_3 + 4b_2 - 8b_3)\, v_2^2 \\ + (-4a_1 + 10a_2 + 8b_2 - 4b_3)\, v_2 + 8a_1 + 2a_2 - a_3 + 6b_1 + 4b_2 - 2b_3 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}-4a_1 + 10a_2 + 8b_2 - 4b_3 &= 0 \\ 8a_2 - 20a_3 + 4b_2 - 8b_3 &= 0 \\ 24a_2 - 48a_3 + 16b_2 - 24b_3 &= 0 \\ 24a_2 - 36a_3 + 20b_2 - 24b_3 &= 0 \\ 24a_2 - 12a_3 + 4b_1 + 22b_2 - 16b_3 &= 0 \\ 8a_1 + 2a_2 - a_3 + 6b_1 + 4b_2 - 2b_3 &= 0\end{aligned}$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 4a_3 + \frac{3b_3}{2} \\
a_2 &= 4a_3 + b_3 \\
a_3 &= a_3 \\
b_1 &= -\frac{9a_3}{2} - 2b_3 \\
b_2 &= -3a_3 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x + \frac{3}{2} \\
\eta &= y - 2
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - 2 - \left(-\frac{6x+4y+1}{2\,(2x+y+1)}\right)\left(x+\frac{3}{2}\right) \\
&= \frac{3\left(x+\frac{y}{3}+\frac{5}{6}\right)\left(x+y-\frac{1}{2}\right)}{2x+y+1} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{3\left(x+\frac{y}{3}+\frac{5}{6}\right)\left(x+y-\frac{1}{2}\right)}{2x+y+1}} dy
\end{aligned}
$$

Which results in

$$S = \frac{\ln\left(\left(6x+2y+5\right)\left(2x+2y-1\right)\right)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{6x+4y+1}{2\left(2x+y+1\right)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{3}{6x+2y+5} + \frac{1}{2x+2y-1} \\
S_y &= \frac{1}{6x+2y+5} + \frac{1}{2x+2y-1}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

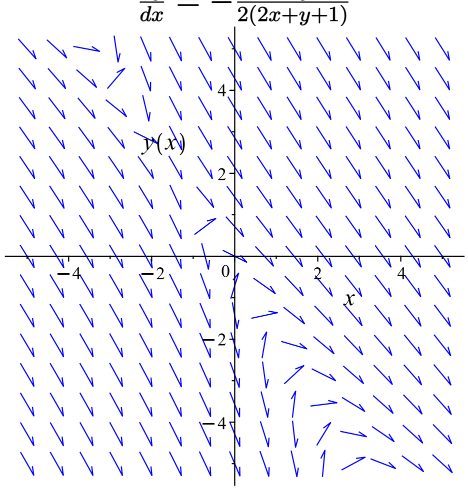
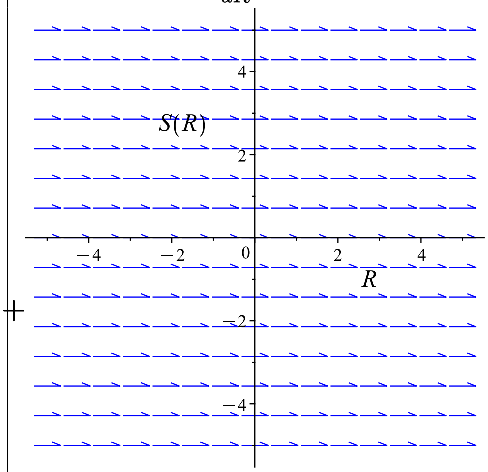
Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\begin{aligned}
\int dS &= \int 0\, dR + c_2 \\
S(R) &= c_2
\end{aligned}$$

To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$\frac{\ln\left(2y+6x+5\right)}{2} + \frac{\ln\left(2y+2x-1\right)}{2} = c_2$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

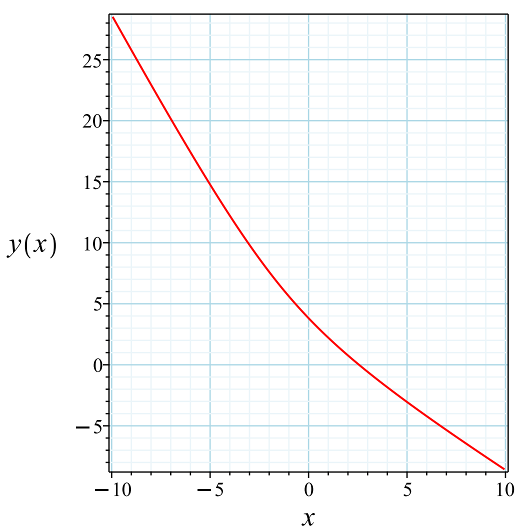
Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{6x+4y+1}{2(2x+y+1)}$ 	$R = x$ $S = \frac{\ln(6x+2y+5)}{2} + \frac{\ln(2x+2y-1)}{2}$	$\frac{dS}{dR} = 0$ 

Solving for initial conditions the solution is

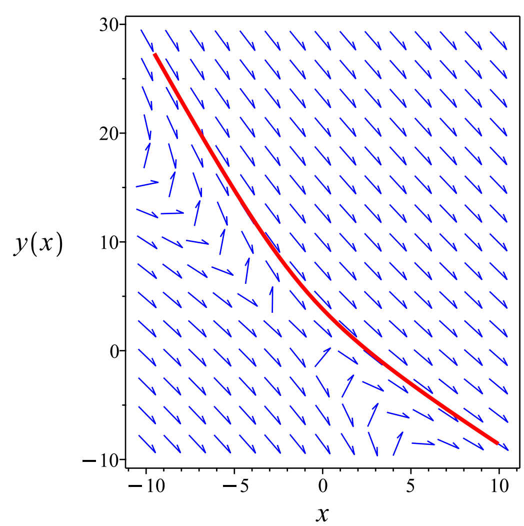
$$\frac{\ln(2y+6x+5)}{2} + \frac{\ln(2y+2x-1)}{2} = \ln(2) + \frac{\ln(7)}{2} + \frac{\ln(3)}{2}$$

Solving for y gives

$$y = -2x - 1 + \frac{\sqrt{4x^2+12x+93}}{2}$$



(a) Solution plot



(b) Slope field $6x+4y+1+(4x+2y+2)\,y' = 0$

Summary of solutions found

$$y = -2x - 1 + \frac{\sqrt{4x^2 + 12x + 93}}{2}$$

2.1.8.7 ✓ Maple. Time used: 0.125 (sec). Leaf size: 23

```
ode:=6*x+4*y(x)+1+(4*x+2*y(x)+2)*diff(y(x),x) = 0;
ic:=[y(1/2) = 3];
dsolve([ode,op(ic)],y(x), singsol=all);
```

$$y = -2x - 1 + \frac{\sqrt{4x^2 + 12x + 93}}{2}$$

Maple trace

```
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful
```

Maple step by step

Let's solve
$\left[6x + 4y(x) + 1 + (4x + 2y(x) + 2)\left(\frac{d}{dx}y(x)\right) = 0, y\left(\frac{1}{2}\right) = 3\right]$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- □ Check if ODE is exact
  - ODE is exact if the lhs is the total derivative of a C^2 function
    $\frac{d}{dx}G(x, y(x)) = 0$
  - Compute derivative of lhs
    $\frac{\partial}{\partial x}G(x, y) + \left(\frac{\partial}{\partial y}G(x, y)\right)\left(\frac{d}{dx}y(x)\right) = 0$

  - Evaluate derivatives
    $4 = 4$
  - Condition met, ODE is exact
- Exact ODE implies solution will be of this form
  $\left[G(x,y) = C1, M(x,y) = \frac{\partial}{\partial x}G(x,y), N(x,y) = \frac{\partial}{\partial y}G(x,y)\right]$
- Solve for $G(x,y)$ by integrating $M(x,y)$ with respect to x
  $G(x,y) = \int (6x + 4y + 1)\,dx + \_F1(y)$
- Evaluate integral
  $G(x,y) = 3x^2 + 4yx + x + \_F1(y)$
- Take derivative of $G(x,y)$ with respect to y
  $N(x,y) = \frac{\partial}{\partial y}G(x,y)$
- Compute derivative
  $4x + 2y + 2 = 4x + \frac{d}{dy}\_F1(y)$
- Isolate for $\frac{d}{dy}\_F1(y)$
  $\frac{d}{dy}\_F1(y) = 2y + 2$
- Solve for $\_F1(y)$
  $\_F1(y) = y^2 + 2y$
- Substitute $\_F1(y)$ into equation for $G(x,y)$
  $G(x,y) = 3x^2 + 4yx + y^2 + x + 2y$
- Substitute $G(x,y)$ into the solution of the ODE
  $3x^2 + 4yx + y^2 + x + 2y = C1$
- Solve for $y(x)$
  $\left\{y(x) = -2x - 1 - \sqrt{x^2 + C1 + 3x + 1}, y(x) = -2x - 1 + \sqrt{x^2 + C1 + 3x + 1}\right\}$
- Use initial condition $y\left(\frac{1}{2}\right) = 3$
  $3 = -2 - \sqrt{C1 + \frac{11}{4}}$
- Solve for $\_C1$
  No solution
- Solution does not satisfy initial condition
- Use initial condition $y\left(\frac{1}{2}\right) = 3$
  $3 = -2 + \sqrt{C1 + \frac{11}{4}}$
- Solve for $\_C1$
  $C1 = \frac{89}{4}$
- Substitute $\_C1 = \frac{89}{4}$ into general solution and simplify
  $y(x) = -2x - 1 + \frac{\sqrt{4x^2+12x+93}}{2}$
- Solution to the IVP
  $y(x) = -2x - 1 + \frac{\sqrt{4x^2+12x+93}}{2}$

2.1.8.8 ✓ Mathematica. Time used: 0.085 (sec). Leaf size: 28

```
ode=(6*x+4*y[x]+1)+(4*x+2*y[x]+2)*D[y[x],x]==0;
ic=y[1/2]==3;
DSolve[{ode,ic},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2}\left(\sqrt{4x^2+12x+93}-4x-2\right)$$

2.1.8.9 ✓ Sympy. Time used: 2.098 (sec). Leaf size: 22

```
from sympy import *
x = symbols("x")
y = Function("y")
ode = Eq(6*x + (4*x + 2*y(x) + 2)*Derivative(y(x), x) + 4*y(x) + 1,0)
ics = {y(1/2): 3}
dsolve(ode,func=y(x),ics=ics)
```

$$y(x) = -2x + \frac{\sqrt{4x^2+12x+93}}{2} - 1$$

2.1.9 Problem 9

Local contents

Internal problem ID [4084]
Book : Differential equations, Shepley L. Ross, 1964
Section : 2.4, page 55
Problem number : 9
Date solved : `Saturday, December 06, 2025 at 04:16:56 PM`
CAS classification :
`[[_homogeneous, 'class C'], _rational, [_Abel, '2nd type', 'class A']]`

2.1.9.1 Existence and uniqueness analysis

$$3x - y(x) - 6 + (x + y(x) + 2)\, y'(x) = 0$$
$$y(2) = -2$$

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y'(x) &= f(x, y(x)) \\ &= \frac{-3x + y + 6}{x + y + 2} \end{aligned}$$

The x domain of $f(x, y(x))$ when $y = -2$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 2$ is inside this domain. The y domain of $f(x, y(x))$ when $x = 2$ is

$$\{y < -4 \lor -4 < y\}$$

And the point $y_0 = -2$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{-3x+y+6}{x+y+2}\right)\\ &= \frac{1}{x+y+2} - \frac{-3x+y+6}{(x+y+2)^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = -2$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 2$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 2$ is

$$\{y < -4 \lor -4 < y\}$$

And the point $y_0 = -2$ is inside this domain. Therefore solution exists and is unique.

0.475 (sec) Entering first order ode homog type maple C solver

2.1.9.2 Solved using first_order_ode_homog_type_maple_C

$$3x - y - 6 + (x+y+2)\,y' = 0$$
$$y(2) = -2$$

Let $Y = y - y_0$ and $X = x - x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = \frac{-3X - 3x_0 + Y(X) + y_0 + 6}{X + x_0 + Y(X) + y_0 + 2}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$\begin{aligned}x_0 &= 1\\ y_0 &= -3\end{aligned}$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = \frac{-3X + Y(X)}{X + Y(X)}$$

In canonical form, the ODE is

$$\begin{aligned}Y' &= F(X,Y)\\ &= \frac{-3X+Y}{X+Y} \qquad (1)\end{aligned}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X,Y)$$

In this case, it can be seen that both $M = -3X + Y$ and $N = X + Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}X}X + u &= \frac{u-3}{u+1}\\ \frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\frac{u(X)-3}{u(X)+1} - u(X)}{X}\end{aligned}$$

Or

$$\frac{d}{dX}u(X) - \frac{\frac{u(X)-3}{u(X)+1} - u(X)}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right)Xu(X) + u(X)^2 + \left(\frac{d}{dX}u(X)\right)X + 3 = 0$$

Or

$$X(u(X)+1)\left(\frac{d}{dX}u(X)\right) + u(X)^2 + 3 = 0$$

Which is now solved as separable in $u(X)$.

The ode

$$\frac{d}{dX}u(X) = -\frac{u(X)^2 + 3}{X\left(u\left(X\right)+1\right)} \tag{2.9}$$

is separable as it can be written as

$$\begin{aligned}\frac{d}{dX}u(X) &= -\frac{u(X)^2 + 3}{X\left(u\left(X\right)+1\right)}\\ &= f(X)g(u)\end{aligned}$$

Where

$$\begin{aligned}f(X) &= -\frac{1}{X}\\ g(u) &= \frac{u^2+3}{u+1}\end{aligned}$$

Integrating gives

$$\int \frac{1}{g(u)}\, du = \int f(X)\, dX$$
$$\int \frac{u+1}{u^2+3}\, du = \int -\frac{1}{X}\, dX$$

$$\frac{\ln\left(u(X)^2+3\right)}{2} + \frac{\sqrt{3}\,\arctan\left(\frac{u(X)\sqrt{3}}{3}\right)}{3} = \ln\left(\frac{1}{X}\right) + c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or

$$\frac{u^2+3}{u+1} = 0$$

for $u(X)$ gives

$$u(X) = -i\sqrt{3}$$
$$u(X) = i\sqrt{3}$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\frac{\ln\left(u(X)^2+3\right)}{2} + \frac{\sqrt{3}\,\arctan\left(\frac{u(X)\sqrt{3}}{3}\right)}{3} = \ln\left(\frac{1}{X}\right) + c_1$$
$$u(X) = -i\sqrt{3}$$
$$u(X) = i\sqrt{3}$$

Converting $\frac{\ln\left(u(X)^2+3\right)}{2} + \frac{\sqrt{3}\,\arctan\left(\frac{u(X)\sqrt{3}}{3}\right)}{3} = \ln\left(\frac{1}{X}\right) + c_1$ back to $Y(X)$ gives

$$\frac{\sqrt{3}\left(\sqrt{3}\,\ln\left(\frac{Y(X)^2+3X^2}{X^2}\right) + 2\arctan\left(\frac{Y(X)\sqrt{3}}{3X}\right)\right)}{6} = \ln\left(\frac{1}{X}\right) + c_1$$

Converting $u(X) = -i\sqrt{3}$ back to $Y(X)$ gives

$$Y(X) = -i\sqrt{3}\,X$$

Converting $u(X) = i\sqrt{3}$ back to $Y(X)$ gives

$$Y(X) = i\sqrt{3}\,X$$

Using the solution for $Y(X)$

$$\frac{\sqrt{3}\left(\sqrt{3}\,\ln\left(\frac{Y(X)^2+3X^2}{X^2}\right)+2\arctan\left(\frac{Y(X)\sqrt{3}}{3X}\right)\right)}{6} = \ln\left(\frac{1}{X}\right)+c_1 \tag{A}$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x_0 + x$$

Or

$$Y = y - 3$$
$$X = 1 + x$$

Then the solution in y becomes using EQ (A)

$$\frac{\sqrt{3}\left(\sqrt{3}\,\ln\left(\frac{(y+3)^2+3(-1+x)^2}{(-1+x)^2}\right)+2\arctan\left(\frac{(y+3)\sqrt{3}}{-3+3x}\right)\right)}{6} = \ln\left(\frac{1}{-1+x}\right)+c_1$$

Using the solution for $Y(X)$

$$Y(X) = -i\sqrt{3}\,X \tag{A}$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x_0 + x$$

Or

$$Y = y - 3$$
$$X = 1 + x$$

Then the solution in y becomes using EQ (A)

$$y + 3 = -i\sqrt{3}\,(-1+x)$$

Using the solution for $Y(X)$

$$Y(X) = i\sqrt{3}\,X \tag{A}$$

And replacing back terms in the above solution using

$$
\begin{aligned}
Y &= y + y_0 \\
X &= x_0 + x
\end{aligned}
$$

Or

$$
\begin{aligned}
Y &= y - 3 \\
X &= 1 + x
\end{aligned}
$$

Then the solution in y becomes using EQ (A)

$$
y + 3 = i\sqrt{3}\,(-1 + x)
$$

Simplifying the above gives

$$
\begin{aligned}
\frac{\sqrt{3}\left(\sqrt{3}\,\ln\left(\frac{(y+3)^2+3(-1+x)^2}{(-1+x)^2}\right) + 2\arctan\left(\frac{(y+3)\sqrt{3}}{-3+3x}\right)\right)}{6} &= \ln\left(\frac{1}{-1+x}\right) + c_1 \\
y + 3 &= -i\sqrt{3}\,(-1 + x) \\
y + 3 &= i\sqrt{3}\,(-1 + x)
\end{aligned}
$$

Solving for initial conditions the solution is

$$
\begin{aligned}
\frac{\sqrt{3}\left(\sqrt{3}\,\ln\left(\frac{(y+3)^2+3(-1+x)^2}{(-1+x)^2}\right) + 2\arctan\left(\frac{(y+3)\sqrt{3}}{-3+3x}\right)\right)}{6} &= \ln\left(\frac{1}{-1+x}\right) + \frac{\sqrt{3}\,\pi}{18} + \ln(2) \\
y + 3 &= -i\sqrt{3}\,(-1 + x) \\
y + 3 &= i\sqrt{3}\,(-1 + x)
\end{aligned}
$$

The solution

$$
y + 3 = -i\sqrt{3}\,(-1 + x)
$$

was found not to satisfy the ode or the IC. Hence it is removed. The solution

$$
y + 3 = i\sqrt{3}\,(-1 + x)
$$

was found not to satisfy the ode or the IC. Hence it is removed.

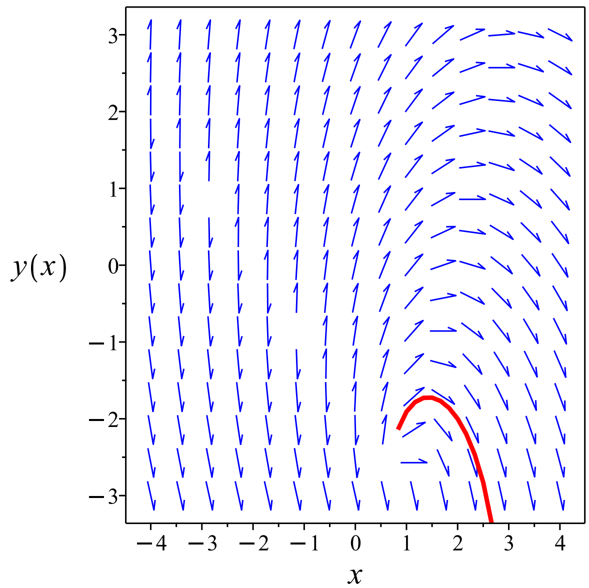


Figure 2.32: Slope field $3x - y - 6 + (x + y + 2)\, y' = 0$

Summary of solutions found

$$\frac{\sqrt{3}\left(\sqrt{3}\,\ln\left(\frac{(y+3)^2+3(-1+x)^2}{(-1+x)^2}\right)+2\arctan\left(\frac{(y+3)\sqrt{3}}{-3+3x}\right)\right)}{6} = \ln\left(\frac{1}{-1+x}\right) + \frac{\sqrt{3}\,\pi}{18} + \ln(2)$$

Entering first order ode abel second kind solver

$$3x - y - 6 + (x + y + 2)\, y' = 0$$
$$y(2) = -2$$

Applying transformation

$$y = \frac{1}{u(x)} - g$$

Results in the new ode which is Abel first kind

$$3x - xu(x) - 6 + (x + xu(x) + 2)\,(u(x) + xu'(x)) = 0$$

Which is now solved Unknown ode type.

2.1.9.3 Solved using first_order_ode_abel_second_kind_solved_by_converting_to_first_kind

7.655 (sec)

This is Abel second kind ODE, it has the form

$$(y(x)+g)\,y'(x) = f_0(x)+f_1(x)y(x)+f_2(x)y(x)^2+f_3(x)y(x)^3$$

Comparing the above to given ODE which is

$$3x - y(x) - 6 + (x + y(x) + 2)\,y'(x) = 0 \tag{1}$$

Shows that

$$\begin{aligned} g &= 2+x \\ f_0 &= -3x+6 \\ f_1 &= 1 \\ f_2 &= 0 \\ f_3 &= 0 \end{aligned}$$

Applying transformation

$$y(x) = \frac{1}{u(x)} - g$$

Results in the new ode which is Abel first kind

$$u'(x) = (4x-4)\,u(x)^3 - 2u(x)^2$$

Which is now solved.

Entering first order ode abel first kind solver

This is Abel first kind ODE, it has the form

$$u'(x) = f_0(x)+f_1(x)u(x)+f_2(x)u(x)^2+f_3(x)u(x)^3$$

Comparing the above to given ODE which is

$$u'(x) = (4x-4)\,u(x)^3 - 2u(x)^2 \tag{1}$$

Therefore

$$\begin{aligned} f_0 &= 0 \\ f_1 &= 0 \\ f_2 &= -2 \\ f_3 &= 4x-4 \end{aligned}$$

Hence

$$\begin{aligned} f_0' &= 0 \\ f_3' &= 4 \end{aligned}$$

Since $f_2(x) = -2$ is not zero, then the followingtransformation is used to remove f_2. Let $u(x) = u(x) - \frac{f_2}{3f_3}$ or

$$\begin{aligned} u(x) &= u(x) - \left(\frac{-2}{12x - 12}\right) \\ &= u(x) + \frac{1}{-6 + 6x} \end{aligned}$$

The above transformation applied to (1) gives a new ODE as

$$u'(x) = \frac{(216x^3 - 648x^2 + 648x - 216)\, u(x)^3}{54\,(-1 + x)^2} + \frac{(-18x + 18)\, u(x)}{54\,(-1 + x)^2} + \frac{7}{54\,(-1 + x)^2} \tag{2}$$

The above ODE (2) can now be solved.

Entering first order ode LIE solver

Writing the ode as

$$\begin{aligned} u'(x) &= \frac{216u^3x^3 - 648u^3x^2 + 648u^3x - 216u^3 - 18ux + 18u + 7}{54\,(-1 + x)^2} \\ u'(x) &= \omega(x, u(x)) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega\left(\eta_{u(x)} - \xi_x\right) - \omega^2 \xi_{u(x)} - \omega_x \xi - \omega_{u(x)} \eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = ua_3 + xa_2 + a_1 \tag{1E}$$

$$\eta = ub_3 + xb_2 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
b_2 &+ \frac{(216u^3x^3 - 648u^3x^2 + 648u^3x - 216u^3 - 18ux + 18u + 7)(b_3 - a_2)}{54(-1+x)^2} \\
&- \frac{(216u^3x^3 - 648u^3x^2 + 648u^3x - 216u^3 - 18ux + 18u + 7)^2 a_3}{2916(-1+x)^4} \\
&- \left(\frac{648u^3x^2 - 1296u^3x + 648u^3 - 18u}{54(-1+x)^2}\right. \\
&\left.- \frac{216u^3x^3 - 648u^3x^2 + 648u^3x - 216u^3 - 18ux + 18u + 7}{27(-1+x)^3}\right)(ua_3 + xa_2 + a_1) \\
&- \frac{(648u^2x^3 - 1944u^2x^2 + 1944u^2x - 648u^2 - 18x + 18)(ub_3 + xb_2 + b_1)}{54(-1+x)^2} = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{46656u^6x^6a_3 - 279936u^6x^5a_3 + 699840u^6x^4a_3 - 933120u^6x^3a_3 + 699840u^6x^2a_3 + 3888u^4x^4a_3 + 23}{}
$$

$$= 0$$

Setting the numerator to zero gives

$$
\begin{aligned}
&-46656u^6x^6a_3 + 279936u^6x^5a_3 - 699840u^6x^4a_3 \\
&+ 933120u^6x^3a_3 - 699840u^6x^2a_3 - 3888u^4x^4a_3 \\
&- 23328u^3x^5a_2 - 23328u^3x^5b_3 - 34992u^2x^6b_2 \\
&+ 279936u^6xa_3 + 15552u^4x^3a_3 - 11664u^3x^4a_1 \\
&+ 104976u^3x^4a_2 + 116640u^3x^4b_3 - 34992u^2x^5b_1 \\
&+ 174960u^2x^5b_2 - 46656u^6a_3 - 23328u^4x^2a_3 \\
&+ 46656u^3x^3a_1 - 186624u^3x^3a_2 - 3024u^3x^3a_3 \\
&- 233280u^3x^3b_3 + 174960u^2x^4b_1 - 349920u^2x^4b_2 \\
&+ 15552u^4xa_3 - 69984u^3x^2a_1 + 163296u^3x^2a_2 + 9072u^3x^2a_3 \\
&+ 233280u^3x^2b_3 - 349920u^2x^3b_1 + 349920u^2x^3b_2 - 3888u^4a_3 \\
&+ 46656u^3xa_1 - 69984u^3xa_2 - 9072u^3xa_3 - 116640u^3xb_3 \\
&- 1296u^2x^2a_3 + 349920u^2x^2b_1 - 174960u^2x^2b_2 + 3888x^4b_2 \\
&- 11664u^3a_1 + 11664u^3a_2 + 3024u^3a_3 + 23328u^3b_3 \\
&+ 2592u^2xa_3 - 174960u^2xb_1 + 34992u^2xb_2 - 972u\,x^2a_1 \\
&- 972u\,x^2a_2 + 972x^3b_1 - 14580x^3b_2 - 1296u^2a_3 \\
&+ 34992u^2b_1 + 1944uxa_1 + 1944uxa_2 + 1008uxa_3 + 378x^2a_2 \\
&- 2916x^2b_1 + 20412x^2b_2 + 378x^2b_3 - 972ua_1 - 972ua_2 \\
&- 1008ua_3 + 756xa_1 + 2916xb_1 - 12636xb_2 - 756xb_3 \\
&- 756a_1 - 378a_2 - 49a_3 - 972b_1 + 2916b_2 + 378b_3 = 0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{u, x\}$ in them.

$$\{u, x\}$$

The following substitution is now made to be able to collect on all terms with $\{u, x\}$ in them

$$\{u = v_1, x = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} & -46656a_3v_1^6v_2^6 + 279936a_3v_1^6v_2^5 - 699840a_3v_1^6v_2^4 \\ & + 933120a_3v_1^6v_2^3 - 23328a_2v_1^3v_2^5 - 699840a_3v_1^6v_2^2 \\ & - 3888a_3v_1^4v_2^4 - 34992b_2v_1^2v_2^6 - 23328b_3v_1^3v_2^5 \\ & - 11664a_1v_1^3v_2^4 + 104976a_2v_1^3v_2^4 + 279936a_3v_1^6v_2 \\ & + 15552a_3v_1^4v_2^3 - 34992b_1v_1^2v_2^5 + 174960b_2v_1^2v_2^5 \\ & + 116640b_3v_1^3v_2^4 + 46656a_1v_1^3v_2^3 - 186624a_2v_1^3v_2^3 \\ & - 46656a_3v_1^6 - 23328a_3v_1^4v_2^2 - 3024a_3v_1^3v_2^3 + 174960b_1v_1^2v_2^4 \\ & - 349920b_2v_1^2v_2^4 - 233280b_3v_1^3v_2^3 - 69984a_1v_1^3v_2^2 \\ & + 163296a_2v_1^3v_2^2 + 15552a_3v_1^4v_2 + 9072a_3v_1^3v_2^2 \\ & - 349920b_1v_1^2v_2^3 + 349920b_2v_1^2v_2^3 + 233280b_3v_1^3v_2^2 \\ & + 46656a_1v_1^3v_2 - 69984a_2v_1^3v_2 - 3888a_3v_1^4 - 9072a_3v_1^3v_2 \\ & - 1296a_3v_1^2v_2^2 + 349920b_1v_1^2v_2^2 - 174960b_2v_1^2v_2^2 \\ & + 3888b_2v_2^4 - 116640b_3v_1^3v_2 - 11664a_1v_1^3 - 972a_1v_1v_2^2 \\ & + 11664a_2v_1^3 - 972a_2v_1v_2^2 + 3024a_3v_1^3 + 2592a_3v_1^2v_2 \\ & - 174960b_1v_1^2v_2 + 972b_1v_2^3 + 34992b_2v_1^2v_2 - 14580b_2v_2^3 \\ & + 23328b_3v_1^3 + 1944a_1v_1v_2 + 1944a_2v_1v_2 + 378a_2v_2^2 \\ & - 1296a_3v_1^2 + 1008a_3v_1v_2 + 34992b_1v_1^2 - 2916b_1v_2^2 \\ & + 20412b_2v_2^2 + 378b_3v_2^2 - 972a_1v_1 + 756a_1v_2 - 972a_2v_1 \\ & - 1008a_3v_1 + 2916b_1v_2 - 12636b_2v_2 - 756b_3v_2 - 756a_1 \\ & - 378a_2 - 49a_3 - 972b_1 + 2916b_2 + 378b_3 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&15552a_3v_1^4v_2+(-11664a_1+11664a_2+3024a_3+23328b_3)\,v_1^3\\
&+(-1296a_3+34992b_1)\,v_1^2\\
&+(-972a_1-972a_2-1008a_3)\,v_1+(972b_1-14580b_2)\,v_2^3\\
&+(378a_2-2916b_1+20412b_2+378b_3)\,v_2^2\\
&+(756a_1+2916b_1-12636b_2-756b_3)\,v_2-46656a_3v_1^6v_2^6\\
&+279936a_3v_1^6v_2^5-699840a_3v_1^6v_2^4+933120a_3v_1^6v_2^3\\
&-699840a_3v_1^6v_2^2-3888a_3v_1^4v_2^4-34992b_2v_1^2v_2^6+279936a_3v_1^6v_2\\
&+15552a_3v_1^4v_2^3-23328a_3v_1^4v_2^2-756a_1-378a_2-49a_3\\
&-972b_1+2916b_2+378b_3+(-23328a_2-23328b_3)\,v_1^3v_2^5\\
&+(-11664a_1+104976a_2+116640b_3)\,v_1^3v_2^4\\
&+(46656a_1-186624a_2-3024a_3-233280b_3)\,v_1^3v_2^3\\
&+(-69984a_1+163296a_2+9072a_3+233280b_3)\,v_1^3v_2^2\\
&+(46656a_1-69984a_2-9072a_3-116640b_3)\,v_1^3v_2\\
&+(-34992b_1+174960b_2)\,v_1^2v_2^5+(174960b_1-349920b_2)\,v_1^2v_2^4\\
&+(-349920b_1+349920b_2)\,v_1^2v_2^3\\
&+(-1296a_3+349920b_1-174960b_2)\,v_1^2v_2^2\\
&+(2592a_3-174960b_1+34992b_2)\,v_1^2v_2\\
&+(-972a_1-972a_2)\,v_1v_2^2+(1944a_1+1944a_2+1008a_3)\,v_1v_2\\
&-46656a_3v_1^6-3888a_3v_1^4+3888b_2v_2^4=0
\end{aligned}
\tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-699840a_3 &= 0\\
-46656a_3 &= 0\\
-23328a_3 &= 0\\
-3888a_3 &= 0\\
15552a_3 &= 0\\
279936a_3 &= 0\\
933120a_3 &= 0\\
-34992b_2 &= 0\\
3888b_2 &= 0\\
-972a_1 - 972a_2 &= 0\\
-23328a_2 - 23328b_3 &= 0\\
-1296a_3 + 34992b_1 &= 0\\
-349920b_1 + 349920b_2 &= 0\\
-34992b_1 + 174960b_2 &= 0\\
972b_1 - 14580b_2 &= 0\\
174960b_1 - 349920b_2 &= 0\\
-11664a_1 + 104976a_2 + 116640b_3 &= 0\\
-972a_1 - 972a_2 - 1008a_3 &= 0\\
1944a_1 + 1944a_2 + 1008a_3 &= 0\\
-1296a_3 + 349920b_1 - 174960b_2 &= 0\\
2592a_3 - 174960b_1 + 34992b_2 &= 0\\
-69984a_1 + 163296a_2 + 9072a_3 + 233280b_3 &= 0\\
-11664a_1 + 11664a_2 + 3024a_3 + 23328b_3 &= 0\\
756a_1 + 2916b_1 - 12636b_2 - 756b_3 &= 0\\
46656a_1 - 186624a_2 - 3024a_3 - 233280b_3 &= 0\\
46656a_1 - 69984a_2 - 9072a_3 - 116640b_3 &= 0\\
378a_2 - 2916b_1 + 20412b_2 + 378b_3 &= 0\\
-756a_1 - 378a_2 - 49a_3 - 972b_1 + 2916b_2 + 378b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= b_3 \\
a_2 &= -b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= 1 - x \\
\eta &= u
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,u)\,\xi \\
&= u - \left(\frac{216u^3x^3 - 648u^3x^2 + 648u^3x - 216u^3 - 18ux + 18u + 7}{54\left(-1+x\right)^2}\right)(1-x) \\
&= \frac{7 + 216(-1+x)^3\,u^3 + (-36+36x)\,u}{-54+54x} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,u) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{du}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial u}\right) S(x,u) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta}dy \\
&= \int \frac{1}{\frac{7+216(-1+x)^3u^3+(-36+36x)u}{-54+54x}}dy
\end{aligned}$$

Which results in

$$S = (-54 + 54x) \left(\frac{(6 - 6x) \ln (36u^2x^2 - 72u^2x + 36u^2 - 6ux + 6u + 7)}{648x^2 - 1296x + 648} + \frac{\left(2 - \frac{(6-6x)^2}{2(36x^2-72x+36)}\right) \sqrt{3}\ \mathrm{a}}{-2} \right.$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,u)S_u}{R_x + \omega(x,u)R_u} \tag{2}$$

Where in the above R_x, R_u, S_x, S_u are all partial derivatives and $\omega(x,u)$ is the right hand side of the original ode given by

$$\omega(x,u) = \frac{216u^3x^3 - 648u^3x^2 + 648u^3x - 216u^3 - 18ux + 18u + 7}{54\left(-1 + x\right)^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_u &= 0 \\
S_x &= \frac{u}{4\left(\frac{7}{36} + (-1+x)^2 u^2 + \frac{(1-x)u}{6}\right)\left(\frac{1}{6} + (-1+x)u\right)} \\
S_u &= \frac{-1+x}{4\left(\frac{7}{36} + (-1+x)^2 u^2 + \frac{(1-x)u}{6}\right)\left(\frac{1}{6} + (-1+x)u\right)}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{-1+x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, u in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{-1+R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\begin{aligned}
\int dS &= \int \frac{1}{-1+R}\, dR \\
S(R) &= \ln(-1+R) + c_2
\end{aligned}$$

To complete the solution, we just need to transform the above back to x, u coordinates. This results in

$$\frac{\sqrt{3}\arctan\left(\frac{(-1+12(-1+x)u(x))\sqrt{3}}{9}\right)}{3}-\frac{\ln\left(7+36(-1+x)^2u(x)^2+(6-6x)u(x)\right)}{2}+\ln\left(u(x)\left(-6+6x\right)\right.$$

Simplifying the above gives

$$\frac{\sqrt{3}\arctan\left(\frac{(-1+12(-1+x)u(x))\sqrt{3}}{9}\right)}{3}-\frac{\ln\left(7+36(-1+x)^2u(x)^2+(6-6x)u(x)\right)}{2}+\ln\left(1+6(-1+x)u(x)\right)=\ln\left(-1+x\right)+c_2$$

Substituting $u=u(x)-\frac{2}{3(4x-4)}$ in the above solution gives

$$\frac{\sqrt{3}\arctan\left(\frac{\left(-1+12(-1+x)\left(u(x)-\frac{2}{3(4x-4)}\right)\right)\sqrt{3}}{9}\right)}{3}-\frac{\ln\left(7+36(-1+x)^2\left(u(x)-\frac{2}{3(4x-4)}\right)^2+(6-6x)\left(u(x\right.\right.}{2}$$

Simplifying the above gives

$$\frac{\sqrt{3}\arctan\left(\frac{(-1+(4x-4)u(x))\sqrt{3}}{3}\right)}{3}-\frac{\ln\left(1+4(-1+x)^2u(x)^2+(-2x+2)u(x)\right)}{2}+\ln\left(2\right)+\ln\left(u(x)\left(-1+x\right)\right)=\ln\left(-1+x\right)+c_2$$

Substituting $u(x)=\frac{1}{x+y(x)+2}$ in the above solution gives

$$\frac{\sqrt{3}\arctan\left(\frac{\left(-1+\frac{4x-4}{x+y(x)+2}\right)\sqrt{3}}{3}\right)}{3}-\frac{\ln\left(1+\frac{4(-1+x)^2}{(x+y(x)+2)^2}+\frac{-2x+2}{x+y(x)+2}\right)}{2}+\ln\left(2\right)+\ln\left(\frac{-1+x}{x+y\left(x\right)+2}\right)=\ln\left(-1+x\right)+c_2$$

Simplifying the above gives

$$-\frac{\sqrt{3}\arctan\left(\frac{(-3x+y(x)+6)\sqrt{3}}{6+3y(x)+3x}\right)}{3}-\frac{\ln\left(\frac{y(x)^2+3x^2+6y(x)-6x+12}{(x+y(x)+2)^2}\right)}{2}+\ln\left(2\right)+\ln\left(\frac{-1+x}{x+y\left(x\right)+2}\right)=\ln\left(-1+x\right)+c_2$$

Solving for initial conditions the solution is

$$-\frac{\sqrt{3}\arctan\left(\frac{(-3x+y(x)+6)\sqrt{3}}{6+3y(x)+3x}\right)}{3}-\frac{\ln\left(\frac{y(x)^2+3x^2+6y(x)-6x+12}{(x+y(x)+2)^2}\right)}{2}+\ln\left(2\right)+\ln\left(\frac{-1+x}{x+y\left(x\right)+2}\right)=\ln\left(-1+x\right)+\frac{\sqrt{3}\,\pi}{18}$$

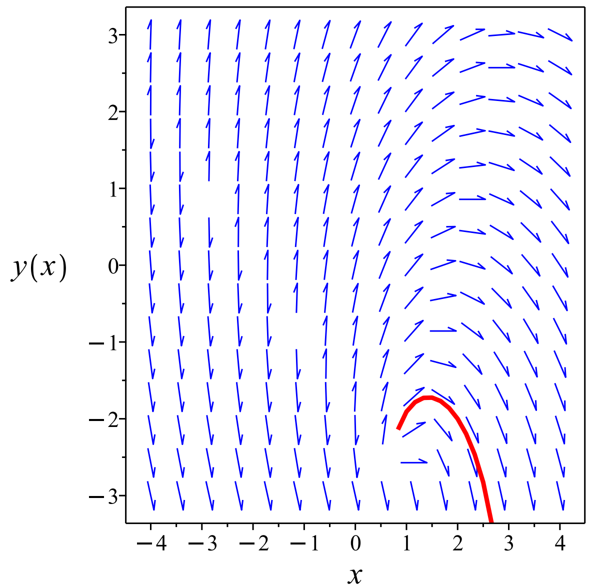


Figure 2.33: Slope field $3x - y(x) - 6 + (x + y(x) + 2)\,y'(x) = 0$

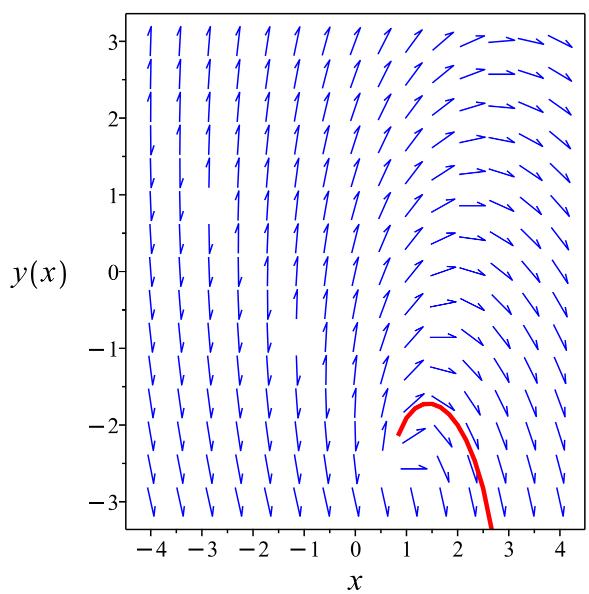


Figure 2.34: Slope field $3x - y(x) - 6 + (x + y(x) + 2)\,y'(x) = 0$

Summary of solutions found

$$
-\frac{\sqrt{3}\,\arctan\left(\frac{(-3x+y(x)+6)\sqrt{3}}{6+3y(x)+3x}\right)}{3} - \frac{\ln\left(\frac{y(x)^2+3x^2+6y(x)-6x+12}{(x+y(x)+2)^2}\right)}{2} + \ln(2) + \ln\left(\frac{x-1}{x+y(x)+2}\right) = \ln(x-1) + \frac{\sqrt{3}\,\pi}{18}
$$

1.086 (sec) Entering first order ode LIE solver

2.1.9.4 Solved using first_order_ode_LIE

$$3x - y(x) - 6 + (x + y(x) + 2)\, y'(x) = 0$$
$$y(2) = -2$$

Writing the ode as

$$y'(x) = \frac{-3x + y + 6}{x + y + 2}$$
$$y'(x) = \omega(x, y(x))$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega\left(\eta_{y(x)} - \xi_x\right) - \omega^2 \xi_{y(x)} - \omega_x \xi - \omega_{y(x)} \eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$
$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 + \frac{(-3x + y + 6)(b_3 - a_2)}{x + y + 2} - \frac{(-3x + y + 6)^2 a_3}{(x + y + 2)^2} \\ &- \left(-\frac{3}{x + y + 2} - \frac{-3x + y + 6}{(x + y + 2)^2}\right)(x a_2 + y a_3 + a_1) \\ &- \left(\frac{1}{x + y + 2} - \frac{-3x + y + 6}{(x + y + 2)^2}\right)(x b_2 + y b_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{3x^2 a_2 - 9x^2 a_3 - 3x^2 b_2 - 3x^2 b_3 + 6xy a_2 + 6xy a_3 + 2xy b_2 - 6xy b_3 - y^2 a_2 + 3y^2 a_3 + y^2 b_2 + y^2 b_3 + 12}{(x + y}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} &3x^2a_2 - 9x^2a_3 - 3x^2b_2 - 3x^2b_3 + 6xya_2 + 6xya_3 + 2xyb_2 - 6xyb_3 \\ &- y^2a_2 + 3y^2a_3 + y^2b_2 + y^2b_3 + 12xa_2 + 36xa_3 - 4xb_1 + 8xb_2 + 4ya_1 \\ &- 8ya_2 + 4yb_2 + 12yb_3 + 12a_1 - 12a_2 - 36a_3 + 4b_1 + 4b_2 + 12b_3 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &3a_2v_1^2 + 6a_2v_1v_2 - a_2v_2^2 - 9a_3v_1^2 + 6a_3v_1v_2 + 3a_3v_2^2 - 3b_2v_1^2 + 2b_2v_1v_2 \\ &+ b_2v_2^2 - 3b_3v_1^2 - 6b_3v_1v_2 + b_3v_2^2 + 4a_1v_2 + 12a_2v_1 - 8a_2v_2 + 36a_3v_1 - 4b_1v_1 \\ &+ 8b_2v_1 + 4b_2v_2 + 12b_3v_2 + 12a_1 - 12a_2 - 36a_3 + 4b_1 + 4b_2 + 12b_3 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} &(3a_2 - 9a_3 - 3b_2 - 3b_3)\, v_1^2 + (6a_2 + 6a_3 + 2b_2 - 6b_3)\, v_1v_2 \\ &+ (12a_2 + 36a_3 - 4b_1 + 8b_2)\, v_1 + (-a_2 + 3a_3 + b_2 + b_3)\, v_2^2 \\ &+ (4a_1 - 8a_2 + 4b_2 + 12b_3)\, v_2 + 12a_1 - 12a_2 - 36a_3 + 4b_1 + 4b_2 + 12b_3 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} 4a_1 - 8a_2 + 4b_2 + 12b_3 &= 0 \\ -a_2 + 3a_3 + b_2 + b_3 &= 0 \\ 3a_2 - 9a_3 - 3b_2 - 3b_3 &= 0 \\ 6a_2 + 6a_3 + 2b_2 - 6b_3 &= 0 \\ 12a_2 + 36a_3 - 4b_1 + 8b_2 &= 0 \\ 12a_1 - 12a_2 - 36a_3 + 4b_1 + 4b_2 + 12b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= -b_3 + 3a_3 \\
a_2 &= b_3 \\
a_3 &= a_3 \\
b_1 &= 3a_3 + 3b_3 \\
b_2 &= -3a_3 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x - 1 \\
\eta &= y + 3
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y + 3 - \left(\frac{-3x + y + 6}{x + y + 2}\right)(x - 1) \\
&= \frac{3x^2 + y^2 - 6x + 6y + 12}{x + y + 2} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{3x^2 + y^2 - 6x + 6y + 12}{x + y + 2}} dy
\end{aligned}
$$

Which results in

$$S = \frac{\ln(3x^2 + y^2 - 6x + 6y + 12)}{2} + \frac{\sqrt{3}\,\arctan\left(\frac{(2y+6)\sqrt{3}}{-6+6x}\right)}{3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{-3x + y + 6}{x + y + 2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{3x - y - 6}{3x^2 + y^2 - 6x + 6y + 12} \\ S_y &= \frac{x + y + 2}{3x^2 + y^2 - 6x + 6y + 12} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

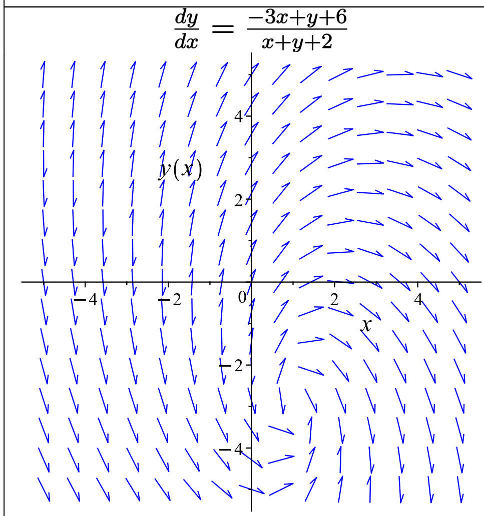
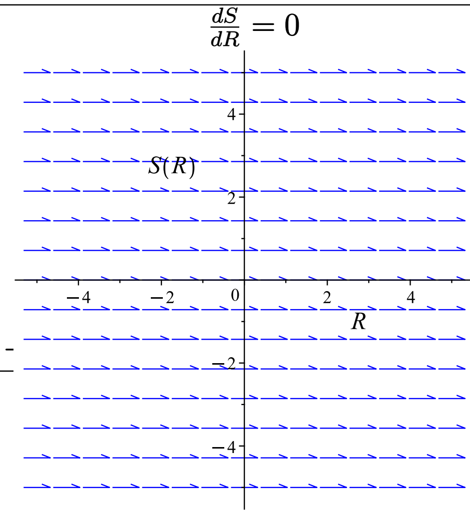
Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\begin{aligned} \int dS &= \int 0\,dR + c_2 \\ S(R) &= c_2 \end{aligned}$$

To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$\frac{\ln\left(y(x)^2+3x^2+6y(x)-6x+12\right)}{2}+\frac{\sqrt{3}\,\arctan\left(\frac{(y(x)+3)\sqrt{3}}{-3+3x}\right)}{3}=c_2$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
	$R = x$ $S = \frac{\ln\left(3x^2+y^2-6x\right.}{2}$	

Solving for initial conditions the solution is

$$\frac{\ln\left(y(x)^2+3x^2+6y(x)-6x+12\right)}{2}+\frac{\sqrt{3}\,\arctan\left(\frac{(y(x)+3)\sqrt{3}}{-3+3x}\right)}{3}=\frac{\sqrt{3}\,\pi}{18}+\ln(2)$$

Solving for $y(x)$ gives

$$\begin{aligned}y(x)=\tan\Big(\mathrm{RootOf}\Big(&6\sqrt{3}\,\ln(2)\\&-3\sqrt{3}\,\ln\left(3x^2\tan(\_Z)^2-6x\tan(\_Z)^2+3x^2+3\tan(\_Z)^2-6x+3\right)+\pi\\&-6\_Z\Big)\Big)\sqrt{3}\,x-\tan\Big(\mathrm{RootOf}\Big(6\sqrt{3}\,\ln(2)\\&-3\sqrt{3}\,\ln\left(3x^2\tan(\_Z)^2-6x\tan(\_Z)^2+3x^2+3\tan(\_Z)^2-6x+3\right)+\pi\\&-6\_Z\Big)\Big)\sqrt{3}-3\end{aligned}$$

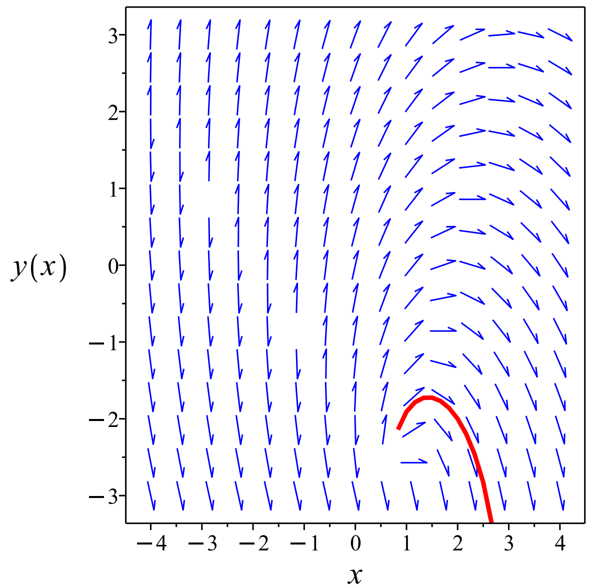


Figure 2.35: Slope field $3x - y(x) - 6 + (x + y(x) + 2)\, y'(x) = 0$

Summary of solutions found

$$y(x) = \tan\left(\mathrm{RootOf}\left(6\sqrt{3}\,\ln(2) - 3\sqrt{3}\,\ln\left(3x^2\tan(\_Z)^2 - 6x\tan(\_Z)^2 + 3x^2 + 3\tan(\_Z)^2 - 6x + 3\right) + \pi - 6\_Z\right)\right)\sqrt{3}\,x - \tan\left(\mathrm{RootOf}\left(6\sqrt{3}\,\ln(2) - 3\sqrt{3}\,\ln\left(3x^2\tan(\_Z)^2 - 6x\tan(\_Z)^2 + 3x^2 + 3\tan(\_Z)^2 - 6x + 3\right) + \pi - 6\_Z\right)\right)\sqrt{3} - 3$$

2.1.9.5 ✓ Maple. `Time used: 1.166 (sec). Leaf size: 51`

```
ode:=3*x-y(x)-6+(x+y(x)+2)*diff(y(x),x) = 0;
ic:=[y(2) = -2];
dsolve([ode,op(ic)],y(x), singsol=all);
```

$$y = -3 - \tan\left(\mathrm{RootOf}\left(-3\sqrt{3}\,\ln\left((x-1)^2\sec(\_Z)^2\right) - 3\sqrt{3}\,\ln(3) + 6\sqrt{3}\,\ln(2) + \pi + 6\_Z\right)\right)\sqrt{3}\,(x-1)$$

Maple trace

```
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
```

```
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful
```

2.1.9.6 ✓ Mathematica. Time used: 0.085 (sec). Leaf size: 90

```
ode=(3*x-y[x]-6)+(x+y[x]+2)*D[y[x],x]==0;
ic=y[2]==-2;
DSolve[{ode,ic},y[x],x,IncludeSingularSolutions->True]
```

$$\text{Solve}\left[\frac{\arctan\left(\frac{-y(x)+3x-6}{\sqrt{3}(y(x)+x+2)}\right)}{\sqrt{3}}+\log(2)=\frac{1}{2}\log\left(\frac{3x^2+y(x)^2+6y(x)-6x+12}{(x-1)^2}\right)+\log(x-1)+\frac{1}{18}\left(\sqrt{3}\pi+18\log(2)-9\log(4)\right),y(x)\right]$$

2.1.9.7 ✓ Sympy. Time used: 3.961 (sec). Leaf size: 58

```
from sympy import *
x = symbols("x")
y = Function("y")
ode = Eq(3*x + (x + y(x) + 2)*Derivative(y(x), x) - y(x) - 6,0)
ics = {y(2): -2}
dsolve(ode,func=y(x),ics=ics)
```

$$\log(x-1)=-\log\left(\sqrt{3+\frac{(y(x)+3)^2}{(x-1)^2}}\right)-\frac{\sqrt{3}\,\text{atan}\left(\frac{\sqrt{3}(y(x)+3)}{3(x-1)}\right)}{3}+\frac{\sqrt{3}\pi}{18}+\log(2)$$

2.1.10 Problem 10

Local contents

Internal problem ID [4085]
Book : Differential equations, Shepley L. Ross, 1964
Section : 2.4, page 55
Problem number : 10
Date solved : `Saturday, December 06, 2025 at 04:17:19 PM`
CAS classification :
`[[_homogeneous, ‘class C‘], _rational, [_Abel, ‘2nd type‘, ‘class A‘]]`

2.1.10.1 Existence and uniqueness analysis

$$2x + 3y(x) + 1 + (4x + 6y(x) + 1)\, y'(x) = 0$$
$$y(-2) = 2$$

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y'(x) &= f(x, y(x)) \\ &= -\frac{2x + 3y + 1}{4x + 6y + 1} \end{aligned}$$

The x domain of $f(x, y(x))$ when $y = 2$ is

$$\left\{x < -\frac{13}{4} \vee -\frac{13}{4} < x\right\}$$

And the point $x_0 = -2$ is inside this domain. The y domain of $f(x, y(x))$ when $x = -2$ is

$$\left\{y < \frac{7}{6} \vee \frac{7}{6} < y\right\}$$

And the point $y_0 = 2$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{2x+3y+1}{4x+6y+1}\right)\\ &= -\frac{3}{4x+6y+1} + \frac{12x+18y+6}{(4x+6y+1)^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 2$ is

$$\left\{x < -\frac{13}{4} \lor -\frac{13}{4} < x\right\}$$

And the point $x_0 = -2$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = -2$ is

$$\left\{y < \frac{7}{6} \lor \frac{7}{6} < y\right\}$$

And the point $y_0 = 2$ is inside this domain. Therefore solution exists and is unique.

0.168 (sec) Entering first order ode dAlembert solver

2.1.10.2 Solved using first_order_ode_dAlembert

$$\begin{aligned}2x+3y+1+(4x+6y+1)\,y' &= 0\\ y(-2) &= 2\end{aligned}$$

Let $p = y'$ the ode becomes

$$2x+3y+1+(4x+6y+1)\,p = 0$$

Solving for y from the above results in

$$y = -\frac{(4p+2)\,x}{3\,(1+2p)} - \frac{p+1}{3\,(1+2p)} \tag{1}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved.

Taking derivative of (*) w.r.t. x gives

$$\begin{aligned}p &= f + (xf' + g')\frac{dp}{dx}\\ p - f &= (xf' + g')\frac{dp}{dx}\end{aligned} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$
\begin{aligned}
f &= -\frac{2}{3} \\
g &= \frac{-p - 1}{3 + 6p}
\end{aligned}
$$

Hence (2) becomes

$$
p + \frac{2}{3} = \left(-\frac{1}{3 + 6p} + \frac{6p}{(3 + 6p)^2} + \frac{6}{(3 + 6p)^2}\right) p'(x) \tag{2A}
$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$
p + \frac{2}{3} = 0
$$

No valid singular solutions found.

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$
p'(x) = \frac{p(x) + \frac{2}{3}}{-\frac{1}{3+6p(x)} + \frac{6p(x)}{(3+6p(x))^2} + \frac{6}{(3+6p(x))^2}} \tag{3}
$$

This ODE is now solved for $p(x)$. No inversion is needed.

Integrating gives

$$
\begin{aligned}
\int \frac{1}{(3p + 2)(1 + 2p)^2} dp &= dx \\
3 \ln (3p + 2) - \frac{1}{1 + 2p} - 3 \ln (1 + 2p) &= x + c_1
\end{aligned}
$$

Singular solutions are found by solving

$$
(3p + 2)(1 + 2p)^2 = 0
$$

for $p(x)$. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$
\begin{aligned}
p(x) &= -\frac{2}{3} \\
p(x) &= -\frac{1}{2}
\end{aligned}
$$

Substituing the above solution for p in (2A) gives

$$y=-\frac{2x}{3}+\frac{-\frac{e^{\operatorname{RootOf}\left(6\ln\left(\frac{2\,e^{\_Z}}{3}-\frac{1}{3}\right)e^{\_Z}+2c_1\,e^{\_Z}-6\_Z\,e^{\_Z}+2\,e^{\_Z}x-3\ln\left(\frac{2\,e^{\_Z}}{3}-\frac{1}{3}\right)-c_1+3\_Z-x+3\right)}}{3}-\frac{1}{3}}{2\,e^{\operatorname{RootOf}\left(6\ln\left(\frac{2\,e^{\_Z}}{3}-\frac{1}{3}\right)e^{\_Z}+2c_1\,e^{\_Z}-6\_Z\,e^{\_Z}+2\,e^{\_Z}x-3\ln\left(\frac{2\,e^{\_Z}}{3}-\frac{1}{3}\right)-c_1+3\_Z-x+3\right)}-1}$$

$$y=-\frac{2x}{3}+\frac{1}{3}$$

Simplifying the above gives

$$y=\frac{(-4x-1)\,e^{\operatorname{RootOf}\left(-6\,e^{\_Z}\ln(3)+6\,e^{\_Z}\ln\left(2\,e^{\_Z}-1\right)+2c_1\,e^{\_Z}-6\_Z\,e^{\_Z}+2\,e^{\_Z}x+3\ln(3)-3\ln\left(2\,e^{\_Z}-1\right)-c_1+3\_Z-x+3\right)}+2x}{6\,e^{\operatorname{RootOf}\left(-6\,e^{\_Z}\ln(3)+6\,e^{\_Z}\ln\left(2\,e^{\_Z}-1\right)+2c_1\,e^{\_Z}-6\_Z\,e^{\_Z}+2\,e^{\_Z}x+3\ln(3)-3\ln\left(2\,e^{\_Z}-1\right)-c_1+3\_Z-x+3\right)}-3}$$

$$y=-\frac{2x}{3}+\frac{1}{3}$$

Solving for initial conditions the solution is

$$y=\frac{(-4x-1)\,e^{\operatorname{RootOf}\left(6i\pi\,e^{\_Z}-6\,e^{\_Z}\ln\left(2\,e^{\_Z}-1\right)+6\,e^{\_Z}\ln(3)-3i\pi+6\_Z\,e^{\_Z}-2\,e^{\_Z}x+3\ln\left(2\,e^{\_Z}-1\right)-3\ln(3)-14\,e^{\_Z}-3\_Z+x+4\right)}}{6\,e^{\operatorname{RootOf}\left(6i\pi\,e^{\_Z}-6\,e^{\_Z}\ln\left(2\,e^{\_Z}-1\right)+6\,e^{\_Z}\ln(3)-3i\pi+6\_Z\,e^{\_Z}-2\,e^{\_Z}x+3\ln\left(2\,e^{\_Z}-1\right)-3\ln(3)-14\,e^{\_Z}-3\_Z+x+4\right)}-}$$

$$y=-\frac{2x}{3}+\frac{1}{3}$$

The solution

$$y=-\frac{2x}{3}+\frac{1}{3}$$

was found not to satisfy the ode or the IC. Hence it is removed.

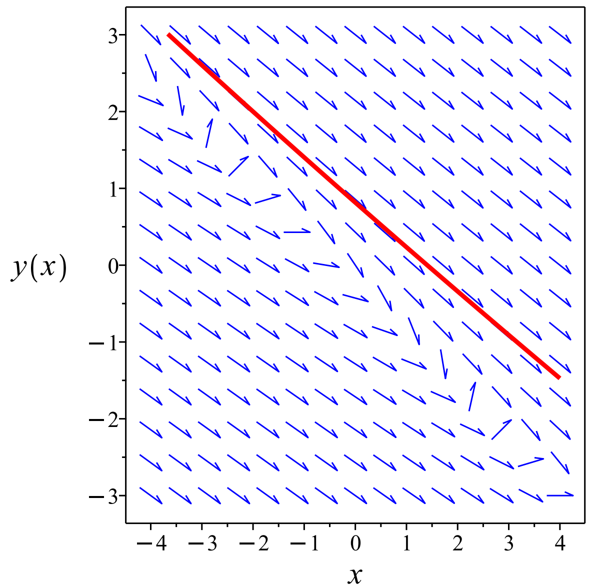


Figure 2.36: Slope field $2x + 3y + 1 + (4x + 6y + 1)\,y' = 0$

Summary of solutions found

$$y = \frac{(-4x-1)\,\mathrm{e}^{\mathrm{RootOf}(6i\pi\,\mathrm{e}^{\_Z}-6\,\mathrm{e}^{\_Z}\ln(2\,\mathrm{e}^{\_Z}-1)+6\,\mathrm{e}^{\_Z}\ln(3)-3i\pi+6\_Z\,\mathrm{e}^{\_Z}-2\,\mathrm{e}^{\_Z}x+3\ln(2\,\mathrm{e}^{\_Z}-1)-3\ln(3)-14\,\mathrm{e}^{\_Z}-3\_Z+x+4)}}{6\,\mathrm{e}^{\mathrm{RootOf}(6i\pi\,\mathrm{e}^{\_Z}-6\,\mathrm{e}^{\_Z}\ln(2\,\mathrm{e}^{\_Z}-1)+6\,\mathrm{e}^{\_Z}\ln(3)-3i\pi+6\_Z\,\mathrm{e}^{\_Z}-2\,\mathrm{e}^{\_Z}x+3\ln(2\,\mathrm{e}^{\_Z}-1)-3\ln(3)-14\,\mathrm{e}^{\_Z}-3\_Z+x+4)} -}$$

Entering first order ode abel second kind solver

$$2x + 3y + 1 + (4x + 6y + 1)\,y' = 0$$
$$y(-2) = 2$$

Applying transformation

$$y = \frac{1}{u(x)} - g$$

Results in the new ode which is Abel first kind

$$2x + 3xu(x) + 1 + (4x + 6xu(x) + 1)\,(u(x) + xu'(x)) = 0$$

Which is now solved Unknown ode type.

0.430 (sec)

2.1.10.3 Solved using first_order_ode_abel_second_kind_solved_by_converting_to_first_kind

This is Abel second kind ODE, it has the form

$$(y(x)+g)\,y'(x) = f_0(x)+f_1(x)y(x)+f_2(x)y(x)^2+f_3(x)y(x)^3$$

Comparing the above to given ODE which is

$$2x+3y(x)+1+(4x+6y(x)+1)\,y'(x) = 0 \tag{1}$$

Shows that

$$\begin{aligned} g &= \frac{2x}{3}+\frac{1}{6} \\ f_0 &= -\frac{1}{6}-\frac{x}{3} \\ f_1 &= -\frac{1}{2} \\ f_2 &= 0 \\ f_3 &= 0 \end{aligned}$$

Applying transformation

$$y(x) = \frac{1}{u(x)}-g$$

Results in the new ode which is Abel first kind

$$u'(x) = \frac{u(x)^3}{12}-\frac{u(x)^2}{6}$$

Which is now solved.

Entering first order ode autonomous solver

Integrating gives

$$\begin{gathered} \int \frac{1}{\frac{1}{12}u^3-\frac{1}{6}u^2}du = dx \\ \frac{6}{u}-3\ln(u)+3\ln(u-2) = x+c_1 \end{gathered}$$

Singular solutions are found by solving

$$\frac{1}{12}u^3-\frac{1}{6}u^2 = 0$$

for $u(x)$. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$\begin{aligned} u(x) &= 0 \\ u(x) &= 2 \end{aligned}$$

Substituting $u(x) = \frac{1}{\frac{2x}{3}+y(x)+\frac{1}{6}}$ in the above solution gives

$$4x + 6y(x) + 1 - 3\ln\left(\frac{1}{\frac{2x}{3} + y(x) + \frac{1}{6}}\right) + 3\ln\left(\frac{1}{\frac{2x}{3} + y(x) + \frac{1}{6}} - 2\right) = x + c_1$$

Now we transform the solution $u(x) = 2$ to $y(x)$ using $u(x) = \frac{1}{\frac{2x}{3}+y(x)+\frac{1}{6}}$ which gives

$$y(x) = -\frac{2x}{3} + \frac{1}{3}$$

Simplifying the above gives

$$4x + 6y(x) + 1 + 3\ln(2) - 3\ln(3) - 3\ln\left(\frac{1}{4x + 6y(x) + 1}\right) + 3\ln\left(\frac{1 - 2x - 3y(x)}{4x + 6y(x) + 1}\right) = x + c_1$$

$$y(x) = -\frac{2x}{3} + \frac{1}{3}$$

Solving for initial conditions the solution is

$$4x + 6y(x) + 1 + 3\ln(2) - 3\ln(3) - 3\ln\left(\frac{1}{4x + 6y(x) + 1}\right) + 3\ln\left(\frac{1 - 2x - 3y(x)}{4x + 6y(x) + 1}\right) = x + 7 + 3\ln(2) - 3\ln(3) + 3i\pi$$

$$y(x) = -\frac{2x}{3} + \frac{1}{3}$$

Solving for $y(x)$ gives

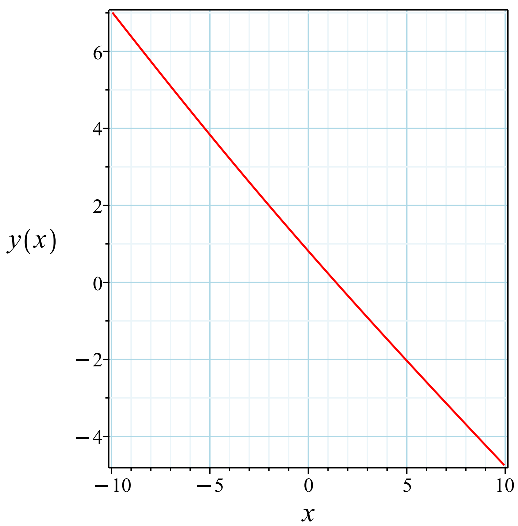
$$y(x) = -\frac{2x}{3} + \frac{1}{3}$$

$$y(x) = -\frac{2x}{3} + \frac{1}{3} + \frac{\operatorname{LambertW}\left(\frac{2\,e^{\frac{x}{3}+\frac{4}{3}}}{3}\right)}{2}$$

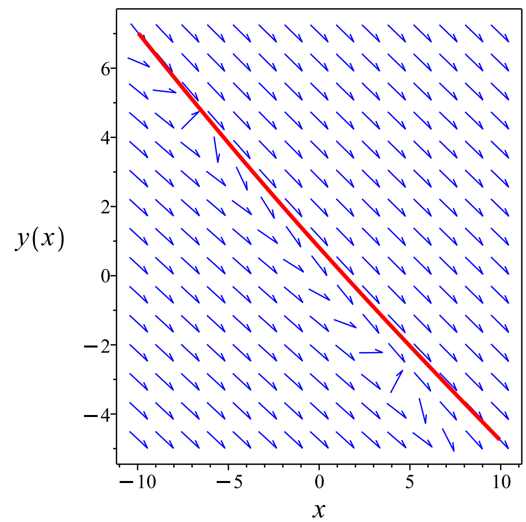
The solution

$$y(x) = -\frac{2x}{3} + \frac{1}{3}$$

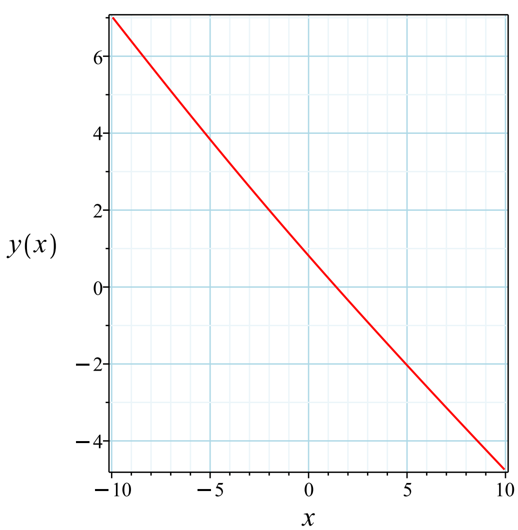
was found not to satisfy the ode or the IC. Hence it is removed.



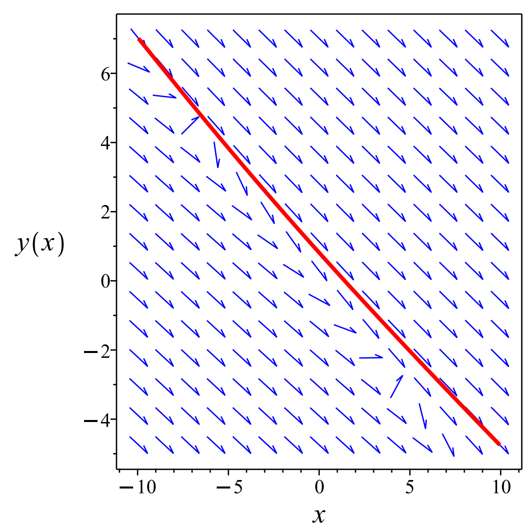
(a) Solution plot



(b) Slope field $2x + 3y(x) + 1 + (4x + 6y(x) + 1)\, y'(x) = 0$



(a) Solution plot



(b) Slope field $2x + 3y(x) + 1 + (4x + 6y(x) + 1)\, y'(x) = 0$

Summary of solutions found

$$y(x) = -\frac{2x}{3} + \frac{1}{3} + \frac{\operatorname{LambertW}\left(\frac{2\,e^{\frac{x}{3}+\frac{4}{3}}}{3}\right)}{2}$$

0.210 (sec) Entering first order ode LIE solver

2.1.10.4 Solved using first_order_ode_LIE

$$2x + 3y(x) + 1 + (4x + 6y(x) + 1)\, y'(x) = 0$$
$$y(-2) = 2$$

Writing the ode as

$$y'(x) = -\frac{2x + 3y + 1}{4x + 6y + 1}$$
$$y'(x) = \omega(x, y(x))$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega\left(\eta_{y(x)} - \xi_x\right) - \omega^2 \xi_{y(x)} - \omega_x \xi - \omega_{y(x)} \eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Using these anstaz

$$\xi = 1 \tag{1E}$$
$$\eta = \frac{Ax + By}{Cx} \tag{2E}$$

Where the unknown coefficients are

$$\{A, B, C\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\frac{A}{Cx} - \frac{Ax + By}{C\,x^2} - \frac{(2x + 3y + 1)\,B}{(4x + 6y + 1)\,Cx} + \frac{2}{4x + 6y + 1} - \frac{4(2x + 3y + 1)}{(4x + 6y + 1)^2} - \frac{\left(-\frac{3}{4x+6y+1} + \frac{12x+18y+6}{(4x+6y+1)^2}\right)(Ax + By)}{Cx} = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{8B\,x^3 + 40B\,x^2 y + 66Bx\,y^2 + 36B\,y^3 + 3A\,x^2 + 6B\,x^2 + 20Bxy + 12B\,y^2 + 2C\,x^2 + Bx + By}{C\,x^2\,(4x + 6y + 1)^2} = 0$$

Setting the numerator to zero gives

$$-8B\,x^3 - 40B\,x^2 y - 66Bx\,y^2 - 36B\,y^3 - 3A\,x^2 - 6B\,x^2 - 20Bxy - 12B\,y^2 - 2C\,x^2 - Bx - By = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &-8Bv_1^3 - 40Bv_1^2v_2 - 66Bv_1v_2^2 - 36Bv_2^3 - 3Av_1^2 - 6Bv_1^2 \\ &- 20Bv_1v_2 - 12Bv_2^2 - 2Cv_1^2 - Bv_1 - Bv_2 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} &-8Bv_1^3 - 40Bv_1^2v_2 + (-3A - 6B - 2C)\, v_1^2 - 66Bv_1v_2^2 \\ &- 20Bv_1v_2 - Bv_1 - 36Bv_2^3 - 12Bv_2^2 - Bv_2 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -66B &= 0 \\ -40B &= 0 \\ -36B &= 0 \\ -20B &= 0 \\ -12B &= 0 \\ -8B &= 0 \\ -B &= 0 \\ -3A - 6B - 2C &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} A &= A \\ B &= 0 \\ C &= -\frac{3A}{2} \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = 1$$
$$\eta = -\frac{2}{3}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Therefore

$$\begin{aligned} \frac{dy}{dx} &= \frac{\eta}{\xi} \\ &= \frac{-\frac{2}{3}}{1} \\ &= -\frac{2}{3} \end{aligned}$$

This is easily solved to give

$$y(x) = -\frac{2x}{3} + c_1$$

Where now the coordinate R is taken as the constant of integration. Hence

$$R = \frac{2x}{3} + y$$

And S is found from

$$\begin{aligned} dS &= \frac{dx}{\xi} \\ &= \frac{dx}{1} \end{aligned}$$

Integrating gives

$$\begin{aligned} S &= \int \frac{dx}{T} \\ &= x \end{aligned}$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{2x + 3y + 1}{4x + 6y + 1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= \frac{2}{3} \\ R_y &= 1 \\ S_x &= 1 \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{12x + 18y + 3}{2x + 3y - 1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{18R + 3}{3R - 1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

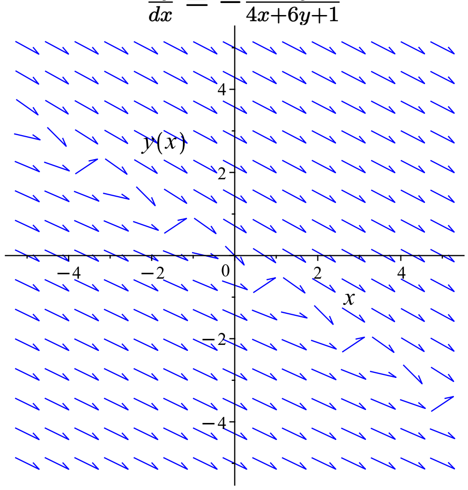
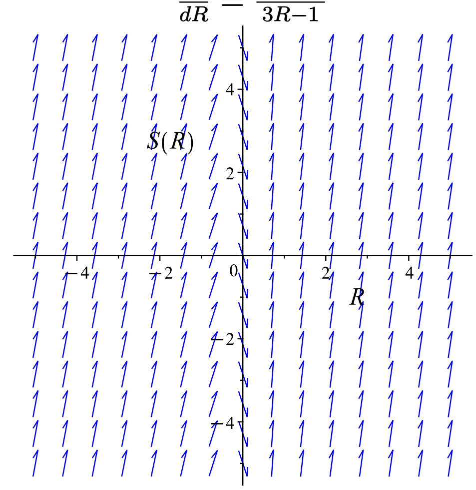
Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\begin{aligned} \int dS &= \int \frac{18R + 3}{3R - 1}\, dR \\ S(R) &= 6R + 3\ln(3R - 1) + c_2 \end{aligned}$$

To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$x = 4x + 6y(x) + 3\ln(-1 + 2x + 3y(x)) + c_2$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

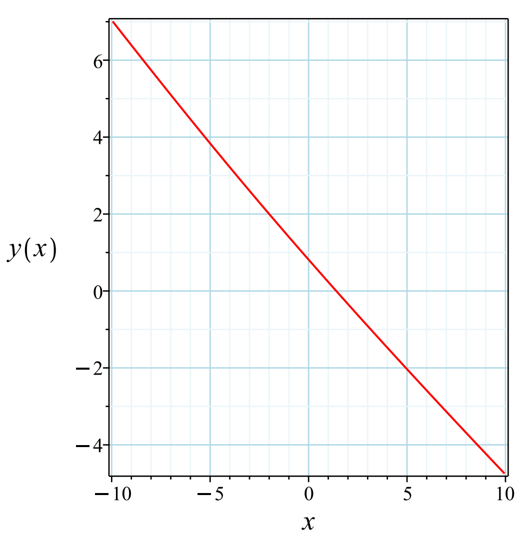
Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{2x+3y+1}{4x+6y+1}$ 	$R = \frac{2x}{3} + y$ $S = x$	$\frac{dS}{dR} = \frac{18R+3}{3R-1}$ 

Solving for initial conditions the solution is

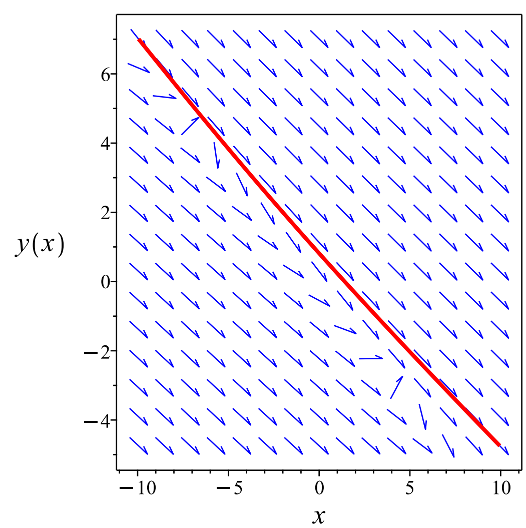
$$x = 4x + 6y(x) + 3\ln(-1 + 2x + 3y(x)) - 6$$

Solving for $y(x)$ gives

$$y(x) = -\frac{2x}{3} + \frac{1}{3} + \frac{\operatorname{LambertW}\left(\frac{2\,e^{\frac{x}{3}+\frac{4}{3}}}{3}\right)}{2}$$



(a) Solution plot



(b) Slope field $2x + 3y(x) + 1 + (4x + 6y(x) + 1)\,y'(x) = 0$

Summary of solutions found

$$y(x) = -\frac{2x}{3} + \frac{1}{3} + \frac{\text{LambertW}\left(\frac{2\,e^{\frac{x}{3}+\frac{4}{3}}}{3}\right)}{2}$$

2.1.10.5 ✓ Maple. Time used: 0.109 (sec). Leaf size: 20

```
ode:=2*x+3*y(x)+1+(4*x+6*y(x)+1)*diff(y(x),x) = 0;
ic:=[y(-2) = 2];
dsolve([ode,op(ic)],y(x), singsol=all);
```

$$y = \frac{1}{3} - \frac{2x}{3} + \frac{\text{LambertW}\left(\frac{2\,e^{\frac{4}{3}+\frac{x}{3}}}{3}\right)}{2}$$

Maple trace

```
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
   -> Calling odsolve with the ODE, diff(y(x),x) = -2/3, y(x)
      *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
<- 1st order, canonical coordinates successful
<- homogeneous successful
```

2.1.10.6 ✓ Mathematica. Time used: 2.368 (sec). Leaf size: 30

```
ode=(2*x+3*y[x]+1)+(4*x+6*y[x]+1)*D[y[x],x]==0;
ic=y[-2]==2;
DSolve[{ode,ic},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{6}\left(3W\left(\frac{2}{3}e^{\frac{x+4}{3}}\right) - 4x + 2\right)$$

2.1.10.7 ✓ Sympy. Time used: 3.114 (sec). Leaf size: 37

```
from sympy import *
x = symbols("x")
y = Function("y")
ode = Eq(2*x + (4*x + 6*y(x) + 1)*Derivative(y(x), x) + 3*y(x) + 1,0)
ics = {y(-2): 2}
dsolve(ode,func=y(x),ics=ics)
```

$$y(x) = -\frac{2x}{3} + \frac{W\left(\frac{\sqrt[3]{-e^{x}}\left(1-\sqrt{3}i\right)e^{\frac{4}{3}}}{3}\right)}{2} + \frac{1}{3}$$