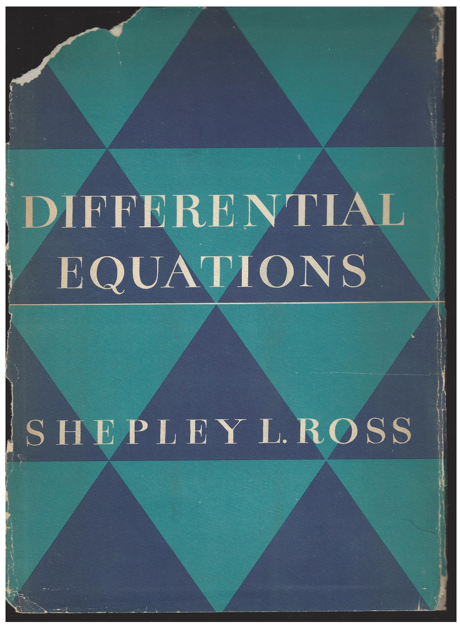
A Solution Manual For

Differential equations, Shepley L. Ross, 1964



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1.1 problem 1

Internal problem ID [2988]
Internal file name [`OUTPUT/2480_Sunday_June_05_2022_03_15_40_AM_29363400/index.tex`]

Book: Differential equations, Shepley L. Ross, 1964
Section: 2.4, page 55
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

$$\boxed{5yx + 4y^2 + (x^2 + 2yx)\, y' = -1}$$

1.1.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(x^2+2xy\right)\mathrm{d}y&=\left(-5xy-4y^2-1\right)\mathrm{d}x\\ \left(5xy+4y^2+1\right)\mathrm{d}x+\left(x^2+2xy\right)\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=5xy+4y^2+1\\ N(x,y)&=x^2+2xy\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(5xy+4y^2+1\right)\\ &=5x+8y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}\left(x^2+2xy\right)\\ &=2x+2y\end{aligned}$$

Since $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A&=\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)\\ &=\frac{1}{(x+2y)\,x}((5x+8y)-(2x+2y))\\ &=\frac{3}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int \frac{3}{x} \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{3\ln(x)} \\ &= x^3\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= x^3 \left(5xy + 4y^2 + 1\right) \\ &= \left(5xy + 4y^2 + 1\right) x^3\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= x^3 \left(x^2 + 2xy\right) \\ &= x^4 (x + 2y)\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N} \frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\left(5xy + 4y^2 + 1\right) x^3\right) + \left(x^4 (x + 2y)\right) \frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int \overline{M} \, \mathrm{d}x \\ \int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int \left(5xy + 4y^2 + 1\right) x^3 \, \mathrm{d}x\end{aligned}$$

$$\phi = x^5 y + x^4 y^2 + \frac{1}{4} x^4 + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\begin{aligned} \frac{\partial \phi}{\partial y} &= x^5 + 2x^4 y + f'(y) \\ &= x^4(x+2y) + f'(y) \end{aligned} \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^4(x+2y)$. Therefore equation (4) becomes

$$x^4(x+2y) = x^4(x+2y) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = x^5 y + x^4 y^2 + \frac{1}{4} x^4 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x^5 y + x^4 y^2 + \frac{1}{4} x^4$$

Summary

The solution(s) found are the following

$$y x^5 + y^2 x^4 + \frac{x^4}{4} = c_1 \tag{1}$$

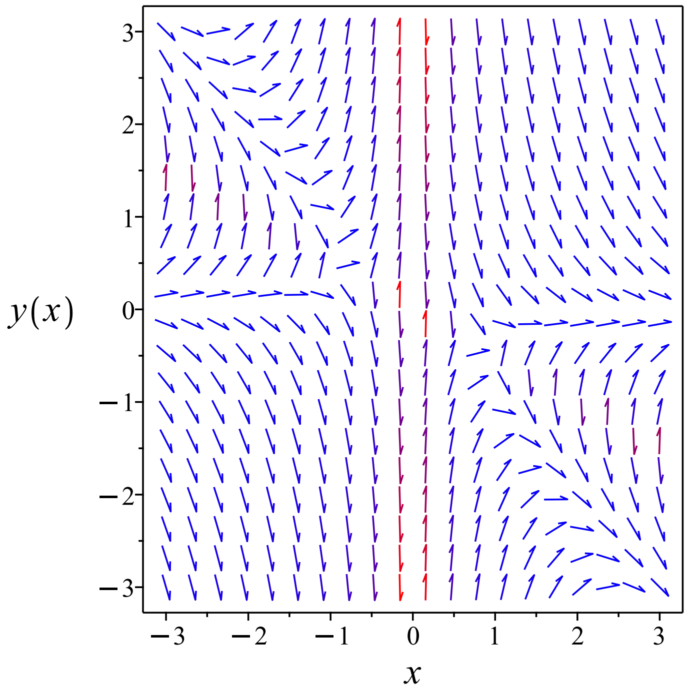


Figure 1: Slope field plot

Verification of solutions

$$
y x^{5}+y^{2} x^{4}+\frac{x^{4}}{4}=c_{1}
$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 59

```
dsolve((5*x*y(x)+4*y(x)^2+1)+(x^2+2*x*y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{-x^3 - \sqrt{x^6 - x^4 - 4c_1}}{2x^2}$$
$$y(x) = \frac{-x^3 + \sqrt{x^6 - x^4 - 4c_1}}{2x^2}$$

✓ Solution by Mathematica

Time used: 0.664 (sec). Leaf size: 84

```
DSolve[(5*x*y[x]+4*y[x]^2+1)+(x^2+2*x*y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to -\frac{x^5 + \sqrt{x^3}\sqrt{x^7 - x^5 + 4c_1x}}{2x^4}$$
$$y(x) \to -\frac{x}{2} + \frac{\sqrt{x^3}\sqrt{x^7 - x^5 + 4c_1x}}{2x^4}$$

1.2 problem 2

Internal problem ID [2989]
Internal file name [OUTPUT/2481_Sunday_June_05_2022_03_15_42_AM_22272722/index.tex]

Book: Differential equations, Shepley L. Ross, 1964
Section: 2.4, page 55
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[[_1st_order, _with_exponential_symmetries]]
```

$$\boxed{2x\tan(y) + \left(x - x^2\tan(y)\right)y' = 0}$$

1.2.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x\tan(y) - 1)\,\mathrm{d}y &= (2\tan(y))\,\mathrm{d}x\\ (-2\tan(y))\,\mathrm{d}x + (x\tan(y) - 1)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -2\tan(y)\\ N(x, y) &= x\tan(y) - 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-2\tan(y))\\ &= -2\sec(y)^2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x\tan(y) - 1)\\ &= \tan(y)\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{x\tan(y) - 1}\left(\left(-2 - 2\tan(y)^2\right) - (\tan(y))\right) \\
&= \frac{-\sin(y) - 2\sec(y)}{x\sin(y) - \cos(y)}
\end{aligned}
$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= -\frac{\cot(y)}{2}\left((\tan(y)) - \left(-2 - 2\tan(y)^2\right)\right) \\
&= -\cot(y) - \tan(y) - \frac{1}{2}
\end{aligned}
$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$
\begin{aligned}
\mu &= e^{\int B \, \mathrm{d}y} \\
&= e^{\int -\cot(y) - \tan(y) - \frac{1}{2} \, \mathrm{d}y}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-\frac{y}{2} - \ln(\sin(y)) + \ln(\cos(y))} \\
&= \frac{\cos(y)\,\mathrm{e}^{-\frac{y}{2}}}{\sin(y)}
\end{aligned}
$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{\cos(y)\,\mathrm{e}^{-\frac{y}{2}}}{\sin(y)}(-2\tan(y)) \\
&= -2\,\mathrm{e}^{-\frac{y}{2}}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{\cos(y)\,\mathrm{e}^{-\frac{y}{2}}}{\sin(y)}(x\tan(y) - 1) \\
&= (x - \cot(y))\,\mathrm{e}^{-\frac{y}{2}}
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$\left(-2\,\mathrm{e}^{-\frac{y}{2}}\right) + \left((x - \cot(y))\,\mathrm{e}^{-\frac{y}{2}}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int -2\,\mathrm{e}^{-\frac{y}{2}}\,\mathrm{d}x$$

$$\phi = -2\,\mathrm{e}^{-\frac{y}{2}}x + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{-\frac{y}{2}}x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = (x - \cot(y))\,\mathrm{e}^{-\frac{y}{2}}$. Therefore equation (4) becomes

$$(x - \cot(y))\,\mathrm{e}^{-\frac{y}{2}} = \mathrm{e}^{-\frac{y}{2}}x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\mathrm{e}^{-\frac{y}{2}}\cot(y)$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\mathrm{e}^{-\frac{y}{2}}\cot(y)\right)\mathrm{d}y$$

$$f(y) = \int_0^y -\mathrm{e}^{-\frac{\_a}{2}}\cot(\_a)\,d\_a + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -2\,\mathrm{e}^{-\frac{y}{2}}x + \int_0^y -\mathrm{e}^{-\frac{\_a}{2}} \cot(\_a)\, d\_a + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -2\,\mathrm{e}^{-\frac{y}{2}}x + \int_0^y -\mathrm{e}^{-\frac{\_a}{2}} \cot(\_a)\, d\_a$$

Summary

The solution(s) found are the following

$$-2\,\mathrm{e}^{-\frac{y}{2}}x + \int_0^y -\mathrm{e}^{-\frac{\_a}{2}} \cot(\_a)\, d\_a = c_1 \tag{1}$$

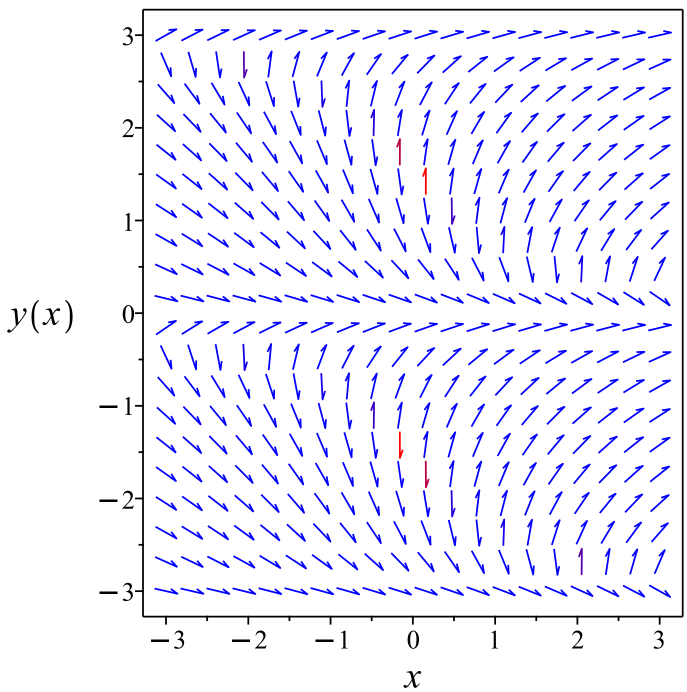


Figure 2: Slope field plot

Verification of solutions

$$-2\,\mathrm{e}^{-\frac{y}{2}}x + \int_0^y -\mathrm{e}^{-\frac{\_a}{2}} \cot(\_a)\, d\_a = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful`
```

✓ Solution by Maple

Time used: 0.093 (sec). Leaf size: 32

```
dsolve((2*x*tan(y(x)))+(x-x^2*tan(y(x)))*diff(y(x),x)=0,y(x), singsol=all)
```

$$\frac{\mathrm{e}^{\frac{y(x)}{2}}\left(\int^{y(x)}\cot(\_a)\,\mathrm{e}^{-\frac{\_a}{2}}d\_a\right)}{2}-\mathrm{e}^{\frac{y(x)}{2}}c_1+x=0$$

✓ Solution by Mathematica

Time used: 0.442 (sec). Leaf size: 78

```
DSolve[(2*x*Tan[y[x]])+(x-x^2*Tan[y[x]])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[x=\frac{1}{34}\left((8-2i)e^{2iy(x)}\,\text{Hypergeometric2F1}\left(1,1+\frac{i}{4},2+\frac{i}{4},e^{2iy(x)}\right)\right.\right.$$
$$\left.\left.-34i\,\text{Hypergeometric2F1}\left(\frac{i}{4},1,1+\frac{i}{4},e^{2iy(x)}\right)\right)+c_1e^{\frac{y(x)}{2}},y(x)\right]$$

1.3 problem 3

Internal problem ID [2990]
Internal file name [`OUTPUT/2482_Sunday_June_05_2022_03_15_45_AM_937798/index.tex`]

Book: Differential equations, Shepley L. Ross, 1964
Section: 2.4, page 55
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$\boxed{y^2(x^2+1)+y+(2yx+1)\,y'=0}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type`
```

✗ Solution by Maple

```
dsolve((y(x)^2*(x^2+1)+y(x))+(2*x*y(x)+1)*diff(y(x),x)=0,y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[(y[x]^2*(x^2+1)+y[x])+(2*x*y[x]+1)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

Not solved

1.4 problem 4

Internal problem ID [2991]
Internal file name [`OUTPUT/2483_Sunday_June_05_2022_03_15_49_AM_42950774/index.tex`]

Book: Differential equations, Shepley L. Ross, 1964
Section: 2.4, page 55
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, [_Abel, `2nd type`, `
   class B`]]
```

$$\boxed{4y^2x + 6y + (5x^2y + 8x)\, y' = 0}$$

1.4.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{2y(2xy + 3)}{x(5xy + 8)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 - \frac{2y(2xy+3)(b_3-a_2)}{x(5xy+8)} - \frac{4y^2(2xy+3)^2 a_3}{x^2(5xy+8)^2} \\ &- \left(-\frac{4y^2}{x(5xy+8)} + \frac{2y(2xy+3)}{x^2(5xy+8)} + \frac{10y^2(2xy+3)}{x(5xy+8)^2}\right)(xa_2+ya_3+a_1) \\ &- \left(-\frac{2(2xy+3)}{x(5xy+8)} - \frac{4y}{5xy+8} + \frac{10y(2xy+3)}{(5xy+8)^2}\right)(xb_2+yb_3+b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{45x^4y^2b_2 - 36x^2y^4a_3 + 20x^3y^2b_1 - 20x^2y^3a_1 + 144x^3yb_2 + 2x^2y^2a_2 + 2x^2y^2b_3 - 108x\,y^3a_3 + 64x^2yb_1 - 6}{x^2(5xy+8)^2}$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} &45x^4y^2b_2 - 36x^2y^4a_3 + 20x^3y^2b_1 - 20x^2y^3a_1 + 144x^3yb_2 + 2x^2y^2a_2 + 2x^2y^2b_3 \\ &- 108x\,y^3a_3 + 64x^2yb_1 - 60x\,y^2a_1 + 112b_2x^2 - 84y^2a_3 + 48xb_1 - 48ya_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &-36a_3v_1^2v_2^4 + 45b_2v_1^4v_2^2 - 20a_1v_1^2v_2^3 + 20b_1v_1^3v_2^2 + 2a_2v_1^2v_2^2 - 108a_3v_1v_2^3 + 144b_2v_1^3v_2 \\ &+ 2b_3v_1^2v_2^2 - 60a_1v_1v_2^2 + 64b_1v_1^2v_2 - 84a_3v_2^2 + 112b_2v_1^2 - 48a_1v_2 + 48b_1v_1 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$45b_2v_1^4v_2^2 + 20b_1v_1^3v_2^2 + 144b_2v_1^3v_2 - 36a_3v_1^2v_2^4 - 20a_1v_1^2v_2^3 + (2a_2 + 2b_3)\,v_1^2v_2^2 + 64b_1v_1^2v_2 + 112b_2v_1^2 - 108a_3v_1v_2^3 - 60a_1v_1v_2^2 + 48b_1v_1 - 84a_3v_2^2 - 48a_1v_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
-60a_1 &= 0 \\
-48a_1 &= 0 \\
-20a_1 &= 0 \\
-108a_3 &= 0 \\
-84a_3 &= 0 \\
-36a_3 &= 0 \\
20b_1 &= 0 \\
48b_1 &= 0 \\
64b_1 &= 0 \\
45b_2 &= 0 \\
112b_2 &= 0 \\
144b_2 &= 0 \\
2a_2 + 2b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0 \\
a_2 &= -b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= -x \\
\eta &= y
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x,y)\,\xi \\ &= y - \left(-\frac{2y(2xy+3)}{x\,(5xy+8)}\right)(-x) \\ &= \frac{x\,y^2+2y}{5xy+8} \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{x\,y^2+2y}{5xy+8}} dy\end{aligned}$$

Which results in

$$S = \ln(xy+2) + 4\ln(y)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{2y(2xy+3)}{x\,(5xy+8)}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{y}{xy+2} \\
S_y &= \frac{x}{xy+2} + \frac{4}{y}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = -\frac{3}{x} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = -\frac{3}{R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -3\ln(R) + c_1 \tag{4}
$$

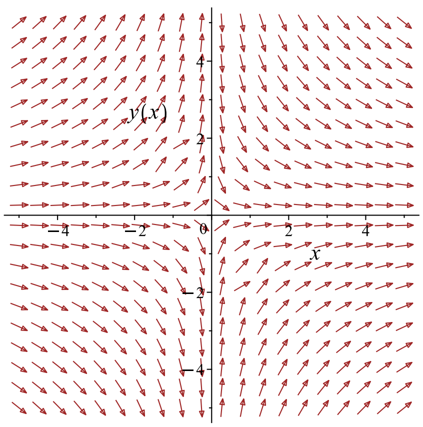
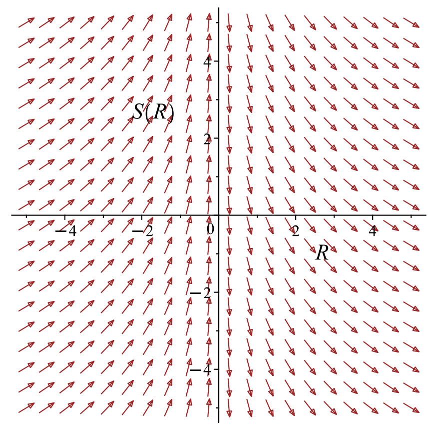
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\ln(yx+2) + 4\ln(y) = -3\ln(x) + c_1
$$

Which simplifies to

$$
\ln(yx+2) + 4\ln(y) = -3\ln(x) + c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{2y(2xy+3)}{x(5xy+8)}$ 	$R = x$ $S = \ln(xy+2) + 4\ln(y)$	$\frac{dS}{dR} = -\frac{3}{R}$ 

Summary

The solution(s) found are the following

$$\ln(yx+2) + 4\ln(y) = -3\ln(x) + c_1 \tag{1}$$

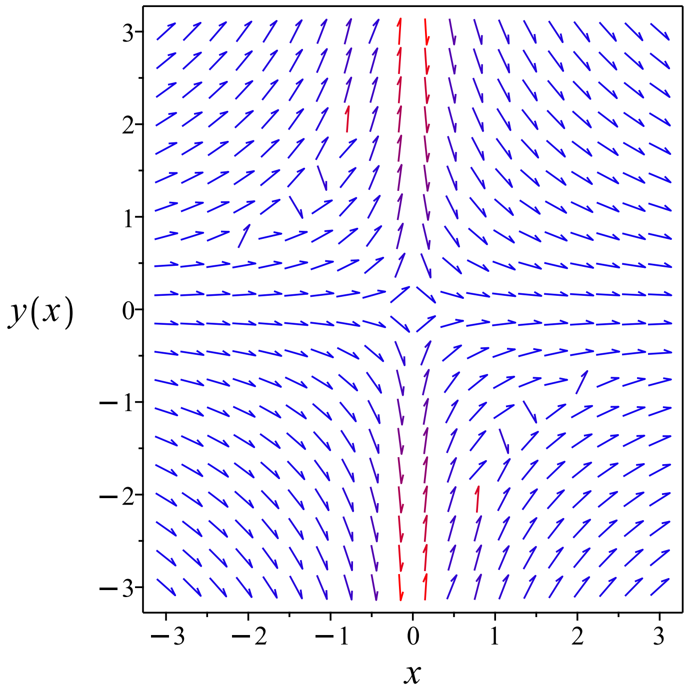


Figure 3: Slope field plot

Verification of solutions

$$\ln(yx+2)+4\ln(y) = -3\ln(x)+c_1$$

Verified OK.

1.4.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(5x^2y + 8x)\,\mathrm{d}y &= (-4x\,y^2 - 6y)\,\mathrm{d}x \\ (4x\,y^2 + 6y)\,\mathrm{d}x + (5x^2y + 8x)\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= 4x\,y^2 + 6y \\ N(x, y) &= 5x^2y + 8x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(4x\,y^2 + 6y) \\ &= 8xy + 6\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(5x^2y + 8x) \\ &= 10xy + 8\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{5x^2y + 8x}((8xy + 6) - (10xy + 8)) \\ &= \frac{-2xy - 2}{5x^2y + 8x} \end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned} B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\ &= \frac{1}{4x\,y^2 + 6y}((10xy + 8) - (8xy + 6)) \\ &= \frac{xy + 1}{2x\,y^2 + 3y} \end{aligned}$$

Since B depends on x, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$\begin{aligned} R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} \\ &= \frac{(10xy + 8) - (8xy + 6)}{x\,(4x\,y^2 + 6y) - y\,(5x^2y + 8x)} \\ &= \frac{-2xy - 2}{xy\,(xy + 2)} \end{aligned}$$

Replacing all powers of terms xy by t gives

$$R = \frac{-2t - 2}{t\,(t + 2)}$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$\begin{aligned} \mu &= e^{\int R\,\mathrm{d}t} \\ &= e^{\int \left(\frac{-2t-2}{t(t+2)}\right)\mathrm{d}t} \end{aligned}$$

The result of integrating gives

$$\mu = e^{-\ln(t(t+2))} \\ = \frac{1}{t\left(t+2\right)}$$

Now t is replaced back with xy giving

$$\mu = \frac{1}{xy\left(xy+2\right)}$$

Multiplying M and N by this integrating factor gives new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N

$$\begin{aligned}\overline{M} &= \mu M \\ &= \frac{1}{xy\left(xy+2\right)}\left(4x\,y^{2}+6y\right) \\ &= \frac{4xy+6}{x\left(xy+2\right)}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \frac{1}{xy\left(xy+2\right)}\left(5x^{2}y+8x\right) \\ &= \frac{5xy+8}{y\left(xy+2\right)}\end{aligned}$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$\begin{aligned}\overline{M}+\overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\frac{4xy+6}{x\left(xy+2\right)}\right)+\left(\frac{5xy+8}{y\left(xy+2\right)}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int \overline{M} \,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int \frac{4xy+6}{x\,(xy+2)} \,\mathrm{d}x$$
$$\phi = \ln\left(xy+2\right) + 3\ln\left(x\right) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{x}{xy+2} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{5xy+8}{y(xy+2)}$. Therefore equation (4) becomes

$$\frac{5xy+8}{y\,(xy+2)} = \frac{x}{xy+2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{4}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y) \,\mathrm{d}y = \int \left(\frac{4}{y}\right) \mathrm{d}y$$
$$f(y) = 4\ln\left(y\right) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \ln\left(xy+2\right) + 3\ln\left(x\right) + 4\ln\left(y\right) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \ln\left(xy+2\right) + 3\ln\left(x\right) + 4\ln\left(y\right)$$

Summary

The solution(s) found are the following

$$\ln(yx+2)+4\ln(y)+3\ln(x)=c_1 \tag{1}$$

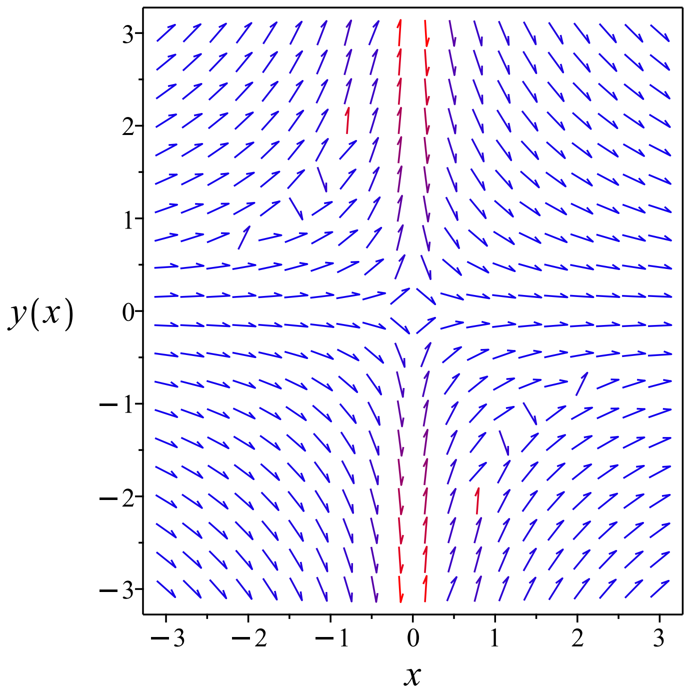


Figure 4: Slope field plot

Verification of solutions

$$\ln(yx+2)+4\ln(y)+3\ln(x)=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 23

```
dsolve((4*x*y(x)^2+6*y(x))+(5*x^2*y(x)+8*x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{\mathrm{RootOf}\left(-\ln\left(x\right)+c_1+\ln\left(2+\_Z\right)+4\ln\left(\_Z\right)\right)}{x}$$

✓ Solution by Mathematica

Time used: 1.989 (sec). Leaf size: 156

```
DSolve[(4*x*y[x]^2+6*y[x])+(5*x^2*y[x]+8*x)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to \mathrm{Root}\left[-\#1^5 - \frac{2\#1^4}{x} + \frac{e^{c_1}}{x^4}\&, 1\right]$$
$$y(x) \to \mathrm{Root}\left[-\#1^5 - \frac{2\#1^4}{x} + \frac{e^{c_1}}{x^4}\&, 2\right]$$
$$y(x) \to \mathrm{Root}\left[-\#1^5 - \frac{2\#1^4}{x} + \frac{e^{c_1}}{x^4}\&, 3\right]$$
$$y(x) \to \mathrm{Root}\left[-\#1^5 - \frac{2\#1^4}{x} + \frac{e^{c_1}}{x^4}\&, 4\right]$$
$$y(x) \to \mathrm{Root}\left[-\#1^5 - \frac{2\#1^4}{x} + \frac{e^{c_1}}{x^4}\&, 5\right]$$

1.5 problem 5

Internal problem ID [2992]
Internal file name [`OUTPUT/2484_Sunday_June_05_2022_03_15_52_AM_27302529/index.tex`]

Book: Differential equations, Shepley L. Ross, 1964
Section: 2.4, page 55
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "differentialType", "homogeneousTypeMapleC", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _exact, _rational, [_Abel, `2nd
   type`, `class A`]]
```

$$\boxed{2y + (2x + y + 1)\, y' = -5x - 1}$$

1.5.1 Solving as differentialType ode

Writing the ode as

$$y' = \frac{-5x - 2y - 1}{2x + y + 1} \tag{1}$$

Which becomes

$$(y + 1)\, dy = (-2x)\, dy + (-5x - 2y - 1)\, dx \tag{2}$$

But the RHS is complete differential because

$$(-2x)\, dy + (-5x - 2y - 1)\, dx = d\left(-\frac{5}{2}x^2 - 2xy - x\right)$$

Hence (2) becomes

$$(y+1)\, dy = d\left(-\frac{5}{2}x^2 - 2xy - x\right)$$

Integrating both sides gives gives these solutions

$$y = -2x - 1 + \sqrt{-x^2 + 2c_1 + 2x + 1} + c_1$$
$$y = -2x - 1 - \sqrt{-x^2 + 2c_1 + 2x + 1} + c_1$$

Summary

The solution(s) found are the following

$$y = -2x - 1 + \sqrt{-x^2 + 2c_1 + 2x + 1} + c_1 \tag{1}$$
$$y = -2x - 1 - \sqrt{-x^2 + 2c_1 + 2x + 1} + c_1 \tag{2}$$

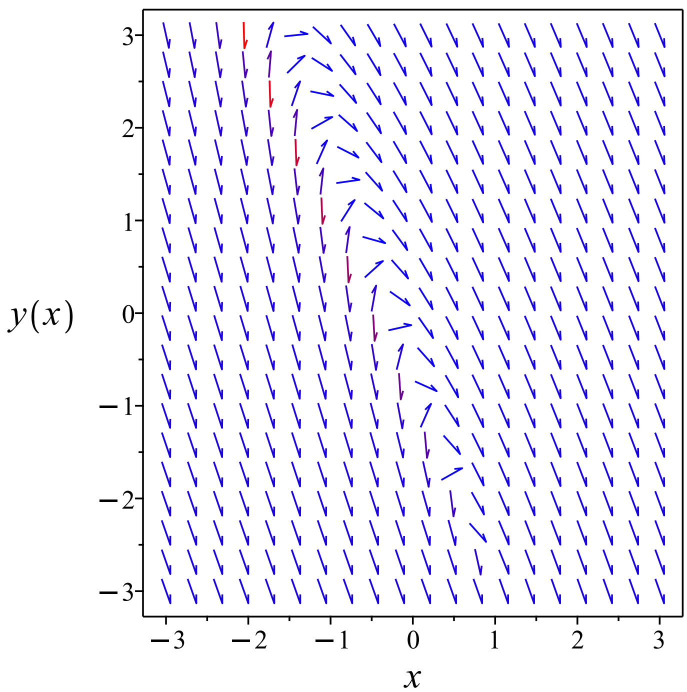


Figure 5: Slope field plot

Verification of solutions

$$y = -2x - 1 + \sqrt{-x^2 + 2c_1 + 2x + 1} + c_1$$

Verified OK.

$$y = -2x - 1 - \sqrt{-x^2 + 2c_1 + 2x + 1} + c_1$$

Verified OK.

1.5.2 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = -\frac{5X + 5x_0 + 2Y(X) + 2y_0 + 1}{2X + 2x_0 + Y(X) + y_0 + 1}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$\begin{aligned} x_0 &= 1 \\ y_0 &= -3 \end{aligned}$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = -\frac{5X + 2Y(X)}{2X + Y(X)}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X, Y) \\ &= -\frac{5X + 2Y}{2X + Y} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X, Y)$ and $N(X, Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X, Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X, Y)$$

In this case, it can be seen that both $M = -5X - 2Y$ and $N = 2X + Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}X}X + u &= \frac{-2u - 5}{u + 2} \\ \frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\frac{-2u(X) - 5}{u(X) + 2} - u(X)}{X} \end{aligned}$$

Or

$$\frac{d}{dX}u(X) - \frac{\frac{-2u(X)-5}{u(X)+2} - u(X)}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right) Xu(X) + 2\left(\frac{d}{dX}u(X)\right) X + u(X)^2 + 4u(X) + 5 = 0$$

Or

$$X(u(X) + 2) \left(\frac{d}{dX}u(X)\right) + u(X)^2 + 4u(X) + 5 = 0$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$\begin{aligned}
u' &= F(X, u) \\
&= f(X)g(u) \\
&= -\frac{u^2 + 4u + 5}{X\,(u + 2)}
\end{aligned}$$

Where $f(X) = -\frac{1}{X}$ and $g(u) = \frac{u^2+4u+5}{u+2}$. Integrating both sides gives

$$\frac{1}{\frac{u^2+4u+5}{u+2}}\,du = -\frac{1}{X}\,dX$$

$$\int \frac{1}{\frac{u^2+4u+5}{u+2}}\,du = \int -\frac{1}{X}\,dX$$

$$\frac{\ln\left(u^2 + 4u + 5\right)}{2} = -\ln\left(X\right) + c_2$$

Raising both side to exponential gives

$$\sqrt{u^2 + 4u + 5} = \mathrm{e}^{-\ln(X)+c_2}$$

Which simplifies to

$$\sqrt{u^2 + 4u + 5} = \frac{c_3}{X}$$

Which simplifies to

$$\sqrt{u\left(X\right)^2 + 4u\left(X\right) + 5} = \frac{c_3 \mathrm{e}^{c_2}}{X}$$

The solution is

$$\sqrt{u\left(X\right)^2 + 4u\left(X\right) + 5} = \frac{c_3 \mathrm{e}^{c_2}}{X}$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$\sqrt{\frac{Y(X)^{2}}{X^{2}}+\frac{4Y(X)}{X}+5} = \frac{c_3 e^{c_2}}{X}$$

Using the solution for $Y(X)$

$$\sqrt{\frac{Y(X)^{2}+4Y(X)X+5X^{2}}{X^{2}}} = \frac{c_3 e^{c_2}}{X}$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y - 3$$
$$X = x + 1$$

Then the solution in y becomes

$$\sqrt{\frac{(y+3)^{2}+4(y+3)(x-1)+5(x-1)^{2}}{(x-1)^{2}}} = \frac{c_3 e^{c_2}}{x-1}$$

Summary

The solution(s) found are the following

$$\sqrt{\frac{(y+3)^{2}+4(y+3)(x-1)+5(x-1)^{2}}{(x-1)^{2}}} = \frac{c_3 e^{c_2}}{x-1} \tag{1}$$

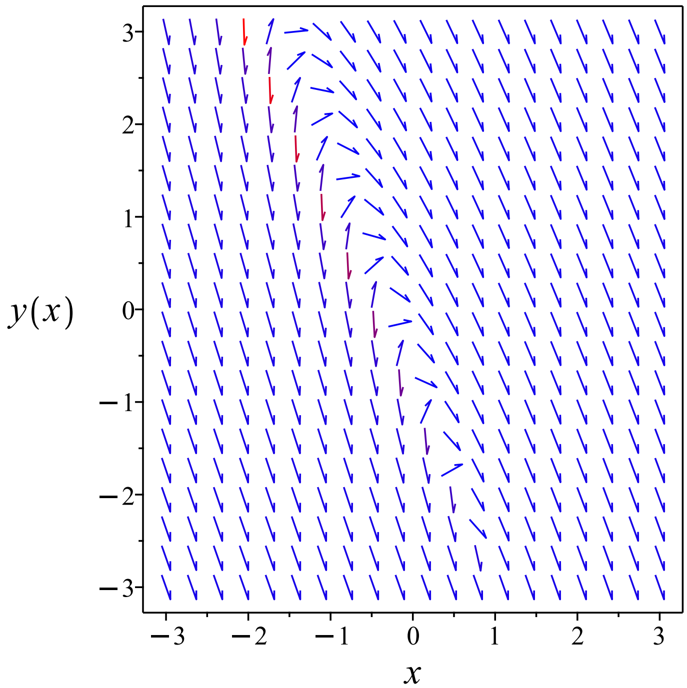


Figure 6: Slope field plot

Verification of solutions

$$\sqrt{\frac{(y+3)^2+4(y+3)(x-1)+5(x-1)^2}{(x-1)^2}}=\frac{c_3 e^{c_2}}{x-1}$$

Verified OK.

1.5.3 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y'=-\frac{5x+2y+1}{2x+y+1}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi=xa_2+ya_3+a_1 \tag{1E}$$
$$\eta=xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{(5x+2y+1)(b_3-a_2)}{2x+y+1} - \frac{(5x+2y+1)^2 a_3}{(2x+y+1)^2} - \left(-\frac{5}{2x+y+1} + \frac{10x+4y+2}{(2x+y+1)^2}\right)(xa_2+ya_3+a_1) - \left(-\frac{2}{2x+y+1} + \frac{5x+2y+1}{(2x+y+1)^2}\right)(xb_2+yb_3+b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{10x^2a_2 - 25x^2a_3 + 3x^2b_2 - 10x^2b_3 + 10xya_2 - 20xya_3 + 4xyb_2 - 10xyb_3 + 2y^2a_2 - 3y^2a_3 + y^2b_2 - 2y^2b_3}{(2x+y}$$

$= 0$

Setting the numerator to zero gives

$$10x^2a_2 - 25x^2a_3 + 3x^2b_2 - 10x^2b_3 + 10xya_2 - 20xya_3 + 4xyb_2 - 10xyb_3 + 2y^2a_2 - 3y^2a_3 + y^2b_2 - 2y^2b_3 + 10xa_2 - 10xa_3 - xb_1 + 5xb_2 - 7xb_3 + ya_1 + 3ya_2 - ya_3 + 2yb_2 - 2yb_3 + 3a_1 + a_2 - a_3 + b_1 + b_2 - b_3 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$10a_2v_1^2 + 10a_2v_1v_2 + 2a_2v_2^2 - 25a_3v_1^2 - 20a_3v_1v_2 - 3a_3v_2^2 + 3b_2v_1^2 + 4b_2v_1v_2 + b_2v_2^2 - 10b_3v_1^2 - 10b_3v_1v_2 - 2b_3v_2^2 + a_1v_2 + 10a_2v_1 + 3a_2v_2 - 10a_3v_1 - a_3v_2 - b_1v_1 + 5b_2v_1 + 2b_2v_2 - 7b_3v_1 - 2b_3v_2 + 3a_1 + a_2 - a_3 + b_1 + b_2 - b_3 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}&(10a_2 - 25a_3 + 3b_2 - 10b_3)\, v_1^2 + (10a_2 - 20a_3 + 4b_2 - 10b_3)\, v_1 v_2 \\ &+ (10a_2 - 10a_3 - b_1 + 5b_2 - 7b_3)\, v_1 + (2a_2 - 3a_3 + b_2 - 2b_3)\, v_2^2 \\ &+ (a_1 + 3a_2 - a_3 + 2b_2 - 2b_3)\, v_2 + 3a_1 + a_2 - a_3 + b_1 + b_2 - b_3 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}2a_2 - 3a_3 + b_2 - 2b_3 &= 0 \\ 10a_2 - 25a_3 + 3b_2 - 10b_3 &= 0 \\ 10a_2 - 20a_3 + 4b_2 - 10b_3 &= 0 \\ a_1 + 3a_2 - a_3 + 2b_2 - 2b_3 &= 0 \\ 10a_2 - 10a_3 - b_1 + 5b_2 - 7b_3 &= 0 \\ 3a_1 + a_2 - a_3 + b_1 + b_2 - b_3 &= 0\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1 &= -a_3 - b_3 \\ a_2 &= 4a_3 + b_3 \\ a_3 &= a_3 \\ b_1 &= 5a_3 + 3b_3 \\ b_2 &= -5a_3 \\ b_3 &= b_3\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}\xi &= x - 1 \\ \eta &= y + 3\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x, y)\, \xi \\ &= y + 3 - \left(-\frac{5x + 2y + 1}{2x + y + 1}\right)(x - 1) \\ &= \frac{5x^2 + 4xy + y^2 + 2x + 2y + 2}{2x + y + 1} \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{5x^2+4xy+y^2+2x+2y+2}{2x+y+1}} dy \end{aligned}$$

Which results in

$$S = \frac{\ln\left(5x^2 + 4xy + y^2 + 2x + 2y + 2\right)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{5x + 2y + 1}{2x + y + 1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{5x + 2y + 1}{5x^2 + (4y + 2)x + y^2 + 2y + 2} \\ S_y &= \frac{2x + y + 1}{y^2 + (4x + 2)y + 5x^2 + 2x + 2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

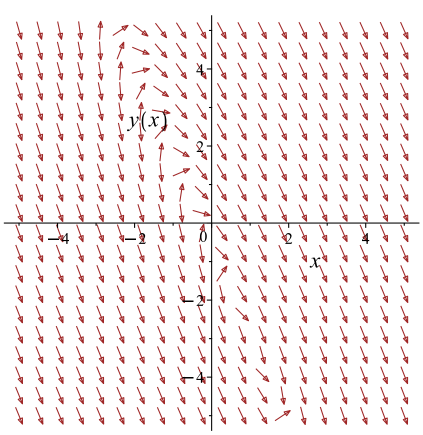
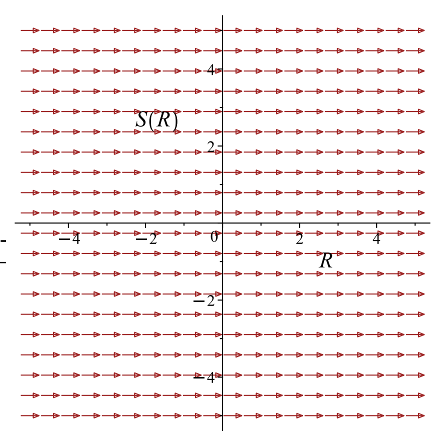
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln\left(5x^2 + (4y+2)\,x + y^2 + 2y + 2\right)}{2} = c_1$$

Which simplifies to

$$\frac{\ln\left(5x^2 + (4y+2)\,x + y^2 + 2y + 2\right)}{2} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{5x+2y+1}{2x+y+1}$ 	$R = x$ $S = \frac{\ln\left(5x^2 + (4y+2)\,x\right.}{2}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\frac{\ln \left(5x^2+(4y+2)\,x+y^2+2y+2\right)}{2} = c_1 \tag{1}$$

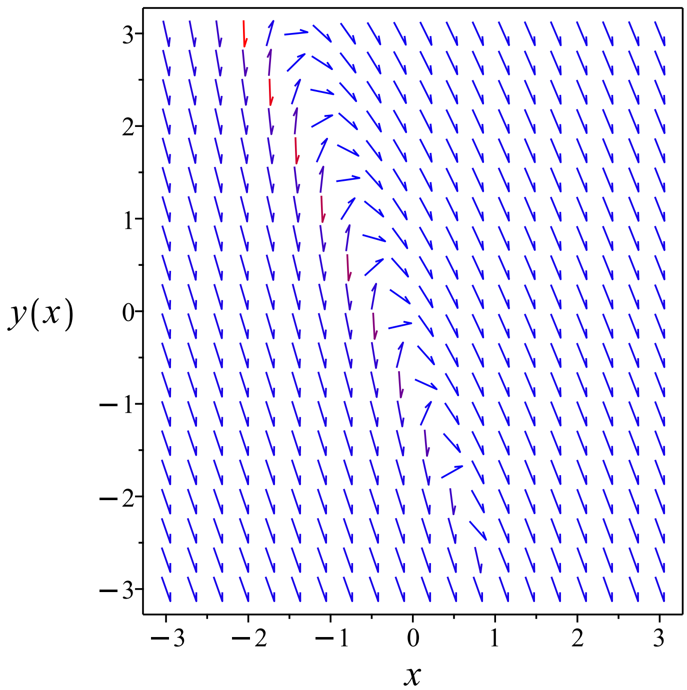


Figure 7: Slope field plot

Verification of solutions

$$\frac{\ln \left(5x^2+(4y+2)\,x+y^2+2y+2\right)}{2} = c_1$$

Verified OK.

1.5.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(2x + y + 1)\,\mathrm{d}y &= (-5x - 2y - 1)\,\mathrm{d}x \\ (5x + 2y + 1)\,\mathrm{d}x + (2x + y + 1)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= 5x + 2y + 1 \\ N(x, y) &= 2x + y + 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(5x + 2y + 1) \\ &= 2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(2x + y + 1)\\ &= 2\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int 5x + 2y + 1\,\mathrm{d}x\end{aligned}$$

$$\phi = \frac{x(5x + 4y + 2)}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 2x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 2x + y + 1$. Therefore equation (4) becomes

$$2x + y + 1 = 2x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = y + 1$$

Integrating the above w.r.t y gives

$$\begin{aligned}\int f'(y)\,\mathrm{d}y &= \int (y + 1)\,\mathrm{d}y\\ f(y) &= \frac{1}{2}y^2 + y + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x(5x+4y+2)}{2} + \frac{y^2}{2} + y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x(5x+4y+2)}{2} + \frac{y^2}{2} + y$$

Summary

The solution(s) found are the following

$$\frac{x(5x+4y+2)}{2} + \frac{y^2}{2} + y = c_1 \tag{1}$$

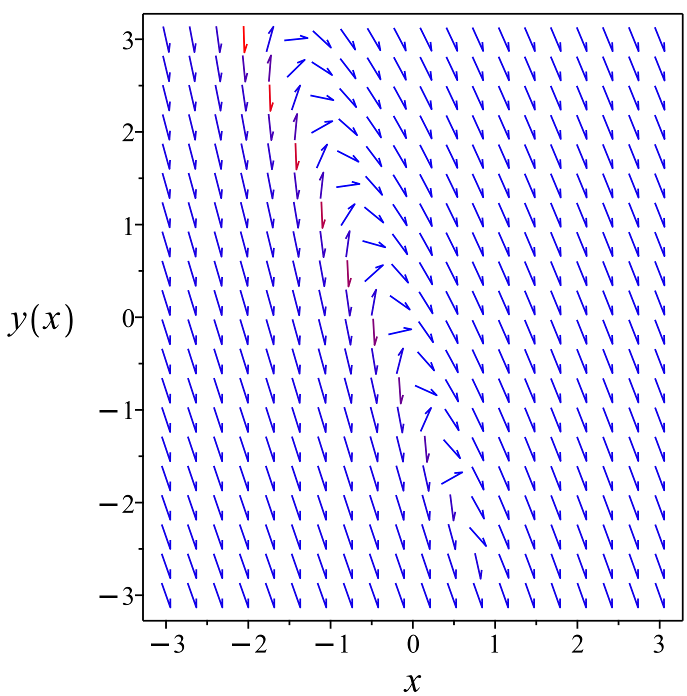


Figure 8: Slope field plot

Verification of solutions

$$\frac{x(5x+4y+2)}{2} + \frac{y^2}{2} + y = c_1$$

Verified OK.

1.5.5 Maple step by step solution

Let's solve

$2y + (2x + y + 1)\, y' = -5x - 1$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x, y) = 0$

  - Compute derivative of lhs

    $F'(x, y) + \left(\frac{\partial}{\partial y} F(x, y)\right) y' = 0$

  - Evaluate derivatives

    $2 = 2$

  - Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y)\,, N(x, y) = \frac{\partial}{\partial y} F(x, y)\right]$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int (5x + 2y + 1)\, dx + f_1(y)$

- Evaluate integral

  $F(x, y) = \frac{5x^2}{2} + 2xy + x + f_1(y)$

- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y} F(x, y)$

- Compute derivative

  $2x + y + 1 = 2x + \frac{d}{dy} f_1(y)$

- Isolate for $\frac{d}{dy} f_1(y)$

  $\frac{d}{dy} f_1(y) = y + 1$

- Solve for $f_1(y)$

  $f_1(y) = \frac{1}{2} y^2 + y$

- Substitute $f_1(y)$ into equation for $F(x, y)$

$$F(x, y) = \frac{5}{2}x^2 + 2xy + x + \frac{1}{2}y^2 + y$$

- Substitute $F(x, y)$ into the solution of the ODE

$$\frac{5}{2}x^2 + 2xy + x + \frac{1}{2}y^2 + y = c_1$$

- Solve for y

$$\left\{y = -2x - 1 - \sqrt{-x^2 + 2c_1 + 2x + 1}, y = -2x - 1 + \sqrt{-x^2 + 2c_1 + 2x + 1}\right\}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.406 (sec). Leaf size: 32

```
dsolve((5*x+2*y(x)+1)+(2*x+y(x)+1)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{-\sqrt{-(x-1)^2 c_1^2 + 1} + (-2x - 1) c_1}{c_1}$$

✓ Solution by Mathematica

Time used: 0.188 (sec). Leaf size: 53

```
DSolve[(5*x+2*y[x]+1)+(2*x+y[x]+1)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{-x^2 + 2x + 1 + c_1} - 2x - 1$$
$$y(x) \to \sqrt{-x^2 + 2x + 1 + c_1} - 2x - 1$$

1.6 problem 6

Internal problem ID [2993]
Internal file name [`OUTPUT/2485_Sunday_June_05_2022_03_15_56_AM_58613521/index.tex`]

Book: Differential equations, Shepley L. Ross, 1964
Section: 2.4, page 55
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{-y-(6x-2y-3)\,y'=-3x-1}$$

1.6.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{-3x+y-1}{-6x+2y+3}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{(-3x+y-1)(b_3-a_2)}{-6x+2y+3} - \frac{(-3x+y-1)^2 a_3}{(-6x+2y+3)^2} - \left(-\frac{3}{-6x+2y+3} + \frac{-18x+6y-6}{(-6x+2y+3)^2}\right)(xa_2+ya_3+a_1) - \left(\frac{1}{-6x+2y+3} - \frac{2(-3x+y-1)}{(-6x+2y+3)^2}\right)(xb_2+yb_3+b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{18x^2a_2 + 9x^2a_3 - 36x^2b_2 - 18x^2b_3 - 12xya_2 - 6xya_3 + 24xyb_2 + 12xyb_3 + 2y^2a_2 + y^2a_3 - 4y^2b_2 - 2y^2b}{(6x-2y}$$
$$= 0$$

Setting the numerator to zero gives

$$-18x^2a_2 - 9x^2a_3 + 36x^2b_2 + 18x^2b_3 + 12xya_2 + 6xya_3 - 24xyb_2 - 12xyb_3 - 2y^2a_2 - y^2a_3 + 4y^2b_2 + 2y^2b_3 + 18xa_2 - 6xa_3 - 41xb_2 - 3xb_3 - ya_2 + 17ya_3 + 12yb_2 - 4yb_3 + 15a_1 + 3a_2 - a_3 - 5b_1 + 9b_2 - 3b_3 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-18a_2v_1^2 + 12a_2v_1v_2 - 2a_2v_2^2 - 9a_3v_1^2 + 6a_3v_1v_2 - a_3v_2^2 + 36b_2v_1^2 - 24b_2v_1v_2 + 4b_2v_2^2 + 18b_3v_1^2 - 12b_3v_1v_2 + 2b_3v_2^2 + 18a_2v_1 - a_2v_2 - 6a_3v_1 + 17a_3v_2 - 41b_2v_1 + 12b_2v_2 - 3b_3v_1 - 4b_3v_2 + 15a_1 + 3a_2 - a_3 - 5b_1 + 9b_2 - 3b_3 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}&(-18a_2 - 9a_3 + 36b_2 + 18b_3)\, v_1^2 + (12a_2 + 6a_3 - 24b_2 - 12b_3)\, v_1 v_2 \\ &+ (18a_2 - 6a_3 - 41b_2 - 3b_3)\, v_1 + (-2a_2 - a_3 + 4b_2 + 2b_3)\, v_2^2 \\ &+ (-a_2 + 17a_3 + 12b_2 - 4b_3)\, v_2 + 15a_1 + 3a_2 - a_3 - 5b_1 + 9b_2 - 3b_3 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}-18a_2 - 9a_3 + 36b_2 + 18b_3 &= 0 \\ -2a_2 - a_3 + 4b_2 + 2b_3 &= 0 \\ -a_2 + 17a_3 + 12b_2 - 4b_3 &= 0 \\ 12a_2 + 6a_3 - 24b_2 - 12b_3 &= 0 \\ 18a_2 - 6a_3 - 41b_2 - 3b_3 &= 0 \\ 15a_1 + 3a_2 - a_3 - 5b_1 + 9b_2 - 3b_3 &= 0\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1 &= a_1 \\ a_2 &= 2b_2 \\ a_3 &= -\frac{2b_2}{3} \\ b_1 &= 3a_1 + \frac{10b_2}{3} \\ b_2 &= b_2 \\ b_3 &= -\frac{b_2}{3}\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}\xi &= 1 \\ \eta &= 3\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x,y)\,\xi \\ &= 3 - \left(\frac{-3x+y-1}{-6x+2y+3}\right)(1) \\ &= \frac{15x-5y-10}{6x-2y-3} \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{15x-5y-10}{6x-2y-3}} dy\end{aligned}$$

Which results in

$$S = \frac{2y}{5} - \frac{\ln(-3x+y+2)}{5}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{-3x+y-1}{-6x+2y+3}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{3}{15x - 5y - 10} \\
S_y &= \frac{2}{5} + \frac{1}{15x - 5y - 10}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{1}{5} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{1}{5}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \frac{R}{5} + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{2y}{5} - \frac{\ln(-3x + y + 2)}{5} = \frac{x}{5} + c_1
$$

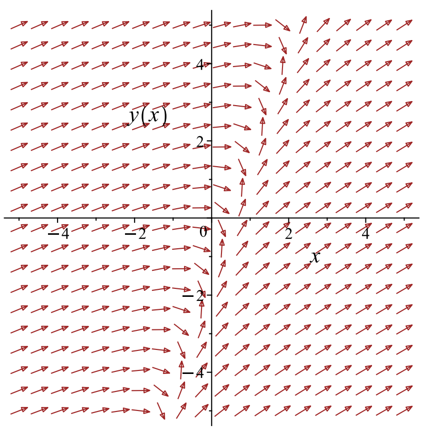
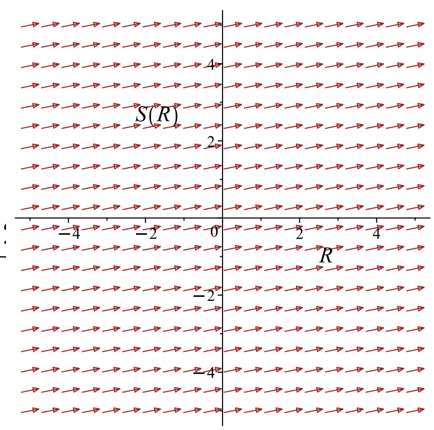
Which simplifies to

$$
\frac{2y}{5} - \frac{\ln(-3x + y + 2)}{5} = \frac{x}{5} + c_1
$$

Which gives

$$
y = -\frac{\text{LambertW}\left(-2\,\mathrm{e}^{5x-4-5c_1}\right)}{2} + 3x - 2
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{-3x+y-1}{-6x+2y+3}$ 	$R = x$ $S = \frac{2y}{5} - \frac{\ln(-3x + y + \cdots}{5}$	$\frac{dS}{dR} = \frac{1}{5}$ 

Summary

The solution(s) found are the following

$$y = -\frac{\mathrm{LambertW}\left(-2\,\mathrm{e}^{5x-4-5c_1}\right)}{2} + 3x - 2 \tag{1}$$

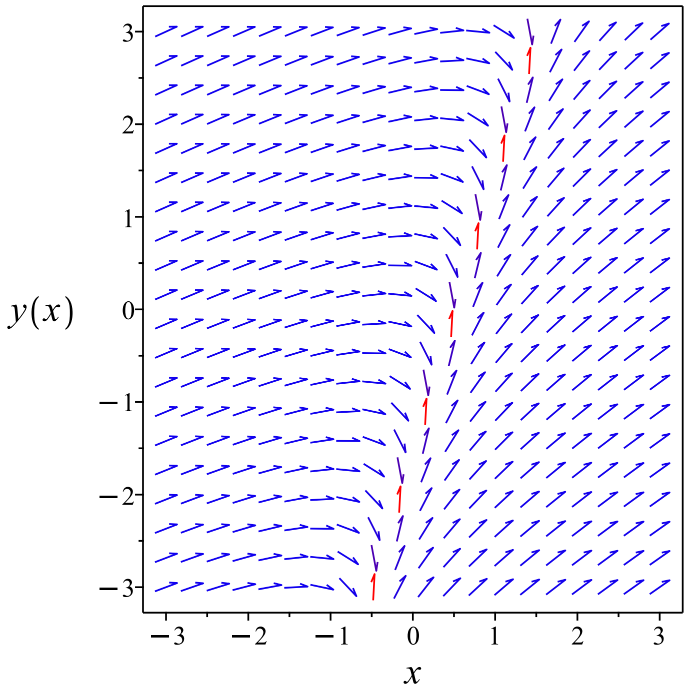


Figure 9: Slope field plot

Verification of solutions

$$y = -\frac{\mathrm{LambertW}\left(-2\,\mathrm{e}^{5x-4-5c_1}\right)}{2} + 3x - 2$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 23

```
dsolve((3*x-y(x)+1)-(6*x-2*y(x)-3)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{\text{LambertW}\left(-2\,e^{-4+5x-5c_1}\right)}{2} + 3x - 2$$

✓ Solution by Mathematica

Time used: 3.791 (sec). Leaf size: 35

```
DSolve[(3*x-y[x]+1)-(6*x-2*y[x]-3)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{2}W\left(-e^{5x-1+c_1}\right) + 3x - 2$$
$$y(x) \to 3x - 2$$

1.7 problem 7

Internal problem ID [2994]
Internal file name [`OUTPUT/2486_Sunday_June_05_2022_03_15_58_AM_51338548/index.tex`]

Book: Differential equations, Shepley L. Ross, 1964
Section: 2.4, page 55
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeMapleC", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{-2y + (2x + y - 1) y' = -x + 3}$$

1.7.1 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = \frac{-X - x_0 + 2Y(X) + 2y_0 + 3}{2X + 2x_0 + Y(X) + y_0 - 1}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = 1$$
$$y_0 = -1$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = \frac{-X + 2Y(X)}{2X + Y(X)}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X, Y) \\ &= \frac{-X + 2Y}{2X + Y} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X, Y)$$

In this case, it can be seen that both $M = -X + 2Y$ and $N = 2X + Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}X}X + u &= \frac{2u - 1}{u + 2} \\ \frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\frac{2u(X)-1}{u(X)+2} - u(X)}{X} \end{aligned}$$

Or

$$\frac{d}{dX}u(X) - \frac{\frac{2u(X)-1}{u(X)+2} - u(X)}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right) Xu(X) + 2\left(\frac{d}{dX}u(X)\right) X + u(X)^2 + 1 = 0$$

Or

$$X(u(X) + 2)\left(\frac{d}{dX}u(X)\right) + u(X)^2 + 1 = 0$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(X, u) \\ &= f(X)g(u) \\ &= -\frac{u^2 + 1}{X\,(u + 2)} \end{aligned}$$

Where $f(X) = -\frac{1}{X}$ and $g(u) = \frac{u^2+1}{u+2}$. Integrating both sides gives

$$\frac{1}{\frac{u^2+1}{u+2}}\,du = -\frac{1}{X}\,dX$$

$$\int \frac{1}{\frac{u^2+1}{u+2}}\,du = \int -\frac{1}{X}\,dX$$

$$\frac{\ln\left(u^2+1\right)}{2} + 2\arctan\left(u\right) = -\ln\left(X\right) + c_2$$

The solution is

$$\frac{\ln\left(u(X)^2+1\right)}{2} + 2\arctan\left(u(X)\right) + \ln\left(X\right) - c_2 = 0$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$\frac{\ln\left(\frac{Y(X)^2}{X^2}+1\right)}{2} + 2\arctan\left(\frac{Y(X)}{X}\right) + \ln\left(X\right) - c_2 = 0$$

Using the solution for $Y(X)$

$$\frac{\ln\left(\frac{Y(X)^2}{X^2}+1\right)}{2} + 2\arctan\left(\frac{Y(X)}{X}\right) + \ln\left(X\right) - c_2 = 0$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y - 1$$
$$X = x + 1$$

Then the solution in y becomes

$$\frac{\ln\left(\frac{(y+1)^2}{(x-1)^2}+1\right)}{2} + 2\arctan\left(\frac{y+1}{x-1}\right) + \ln\left(x-1\right) - c_2 = 0$$

Summary

The solution(s) found are the following

$$\frac{\ln\left(\frac{(y+1)^2}{(x-1)^2}+1\right)}{2}+2\arctan\left(\frac{y+1}{x-1}\right)+\ln(x-1)-c_2=0 \tag{1}$$

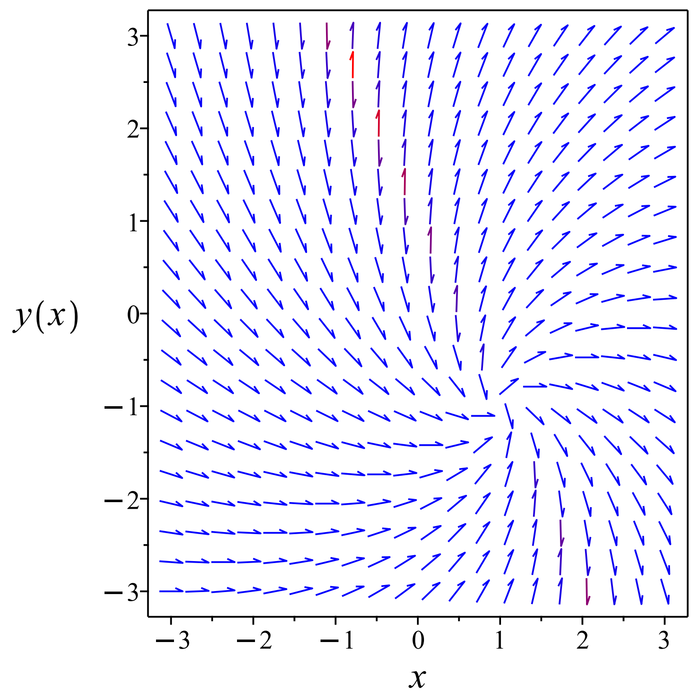


Figure 10: Slope field plot

Verification of solutions

$$\frac{\ln\left(\frac{(y+1)^2}{(x-1)^2}+1\right)}{2}+2\arctan\left(\frac{y+1}{x-1}\right)+\ln(x-1)-c_2=0$$

Verified OK.

1.7.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y'=\frac{-x+2y+3}{2x+y-1}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 + \frac{(-x+2y+3)(b_3-a_2)}{2x+y-1} - \frac{(-x+2y+3)^2 a_3}{(2x+y-1)^2} \\ &- \left(-\frac{1}{2x+y-1} - \frac{2(-x+2y+3)}{(2x+y-1)^2}\right)(xa_2+ya_3+a_1) \\ &- \left(\frac{2}{2x+y-1} - \frac{-x+2y+3}{(2x+y-1)^2}\right)(xb_2+yb_3+b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{2x^2a_2 - x^2a_3 - x^2b_2 - 2x^2b_3 + 2xya_2 + 4xya_3 + 4xyb_2 - 2xyb_3 - 2y^2a_2 + y^2a_3 + y^2b_2 + 2y^2b_3 - 2xa_2 + 6}{(2x+y-1} = 0$$

Setting the numerator to zero gives

$$\begin{aligned} &2x^2a_2 - x^2a_3 - x^2b_2 - 2x^2b_3 + 2xya_2 + 4xya_3 + 4xyb_2 - 2xyb_3 - 2y^2a_2 \\ &+ y^2a_3 + y^2b_2 + 2y^2b_3 - 2xa_2 + 6xa_3 - 5xb_1 + xb_2 + 7xb_3 + 5ya_1 \\ &- ya_2 - 7ya_3 - 2yb_2 + 6yb_3 + 5a_1 + 3a_2 - 9a_3 + 5b_1 + b_2 - 3b_3 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned}&2a_2v_1^2 + 2a_2v_1v_2 - 2a_2v_2^2 - a_3v_1^2 + 4a_3v_1v_2 + a_3v_2^2 - b_2v_1^2 + 4b_2v_1v_2 + b_2v_2^2 \\ &- 2b_3v_1^2 - 2b_3v_1v_2 + 2b_3v_2^2 + 5a_1v_2 - 2a_2v_1 - a_2v_2 + 6a_3v_1 - 7a_3v_2 - 5b_1v_1 \\ &+ b_2v_1 - 2b_2v_2 + 7b_3v_1 + 6b_3v_2 + 5a_1 + 3a_2 - 9a_3 + 5b_1 + b_2 - 3b_3 = 0\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}&(2a_2 - a_3 - b_2 - 2b_3)\, v_1^2 + (2a_2 + 4a_3 + 4b_2 - 2b_3)\, v_1v_2 \\ &+ (-2a_2 + 6a_3 - 5b_1 + b_2 + 7b_3)\, v_1 + (-2a_2 + a_3 + b_2 + 2b_3)\, v_2^2 \\ &+ (5a_1 - a_2 - 7a_3 - 2b_2 + 6b_3)\, v_2 + 5a_1 + 3a_2 - 9a_3 + 5b_1 + b_2 - 3b_3 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}-2a_2 + a_3 + b_2 + 2b_3 &= 0 \\ 2a_2 - a_3 - b_2 - 2b_3 &= 0 \\ 2a_2 + 4a_3 + 4b_2 - 2b_3 &= 0 \\ 5a_1 - a_2 - 7a_3 - 2b_2 + 6b_3 &= 0 \\ -2a_2 + 6a_3 - 5b_1 + b_2 + 7b_3 &= 0 \\ 5a_1 + 3a_2 - 9a_3 + 5b_1 + b_2 - 3b_3 &= 0\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1 &= -b_2 - b_3 \\ a_2 &= b_3 \\ a_3 &= -b_2 \\ b_1 &= -b_2 + b_3 \\ b_2 &= b_2 \\ b_3 &= b_3\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = x - 1$$
$$\eta = y + 1$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y + 1 - \left(\frac{-x+2y+3}{2x+y-1}\right)(x-1) \\
&= \frac{x^2+y^2-2x+2y+2}{2x+y-1} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{x^2+y^2-2x+2y+2}{2x+y-1}} dy
\end{aligned}$$

Which results in

$$S = \frac{\ln(x^2+y^2-2x+2y+2)}{2} + 2\arctan\left(\frac{2+2y}{2x-2}\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{-x+2y+3}{2x+y-1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{x-2y-3}{x^2+y^2-2x+2y+2} \\
S_y &= \frac{2x+y-1}{x^2+y^2-2x+2y+2}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

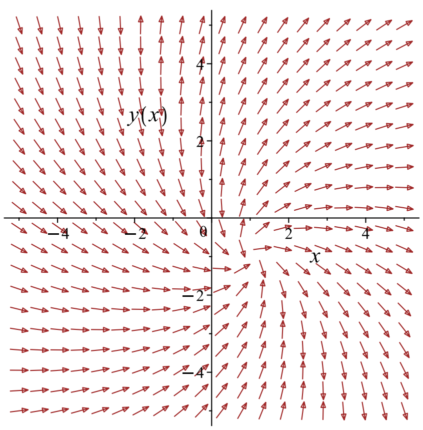
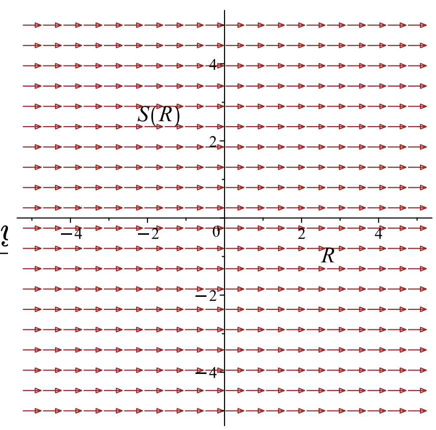
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(y^2+x^2+2y-2x+2)}{2} + 2\arctan\left(\frac{y+1}{x-1}\right) = c_1$$

Which simplifies to

$$\frac{\ln(y^2+x^2+2y-2x+2)}{2} + 2\arctan\left(\frac{y+1}{x-1}\right) = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{-x+2y+3}{2x+y-1}$ 	$R = x$ $S = \frac{\ln (x^2 + y^2 - 2x + 2y}{2}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\frac{\ln (y^2 + x^2 + 2y - 2x + 2)}{2} + 2 \arctan \left(\frac{y+1}{x-1}\right) = c_1 \tag{1}$$

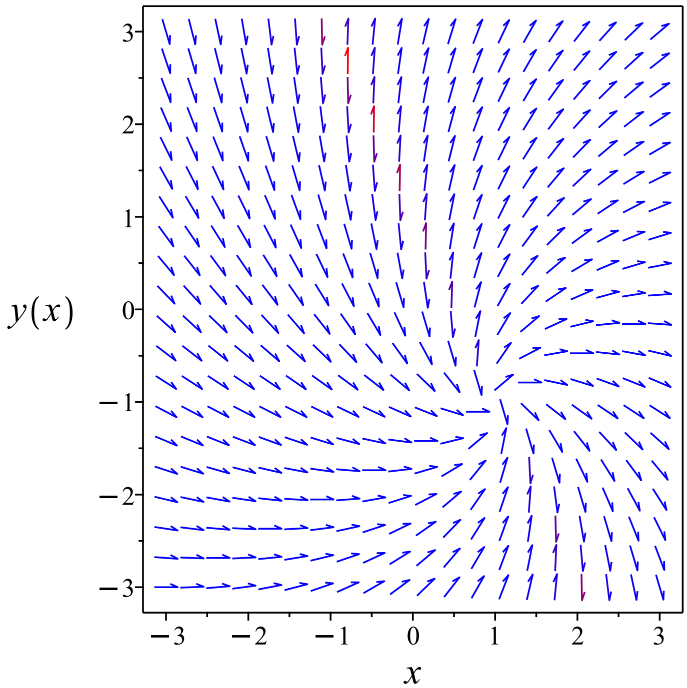


Figure 11: Slope field plot

Verification of solutions

$$\frac{\ln\left(y^2+x^2+2y-2x+2\right)}{2}+2\arctan\left(\frac{y+1}{x-1}\right)=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 31

```
dsolve((x-2*y(x)-3)+(2*x+y(x)-1)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -1 - \tan\left(\operatorname{RootOf}\left(-4\_Z + \ln\left(\sec\left(\_Z\right)^2\right) + 2\ln\left(x-1\right) + 2c_1\right)\right)(x-1)$$

✓ Solution by Mathematica

Time used: 0.061 (sec). Leaf size: 66

```
DSolve[(x-2*y[x]-3)+(2*x+y[x]-1)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[32\arctan\left(\frac{2y(x)-x+3}{y(x)+2x-1}\right)\right.$$
$$\left.+8\log\left(\frac{x^2+y(x)^2+2y(x)-2x+2}{5(x-1)^2}\right)+16\log(x-1)+5c_1=0,y(x)\right]$$

1.8 problem 8

Internal problem ID [2995]
Internal file name [`OUTPUT/2487_Sunday_June_05_2022_03_16_02_AM_89696227/index.tex`]

Book: Differential equations, Shepley L. Ross, 1964
Section: 2.4, page 55
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "differentialType", "homogeneousTypeMapleC", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _exact, _rational, [_Abel, `2nd
   type`, `class A`]]
```

$$\boxed{4y + (4x + 2y + 2)\, y' = -6x - 1}$$

With initial conditions

$$\left[y\left(\frac{1}{2}\right) = 3\right]$$

1.8.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= -\frac{6x + 4y + 1}{2\,(2x + y + 1)} \end{aligned}$$

The x domain of $f(x,y)$ when $y=3$ is

$$\{x < -2 \lor -2 < x\}$$

And the point $x_0 = \frac{1}{2}$ is inside this domain. The y domain of $f(x,y)$ when $x = \frac{1}{2}$ is

$$\{y < -2 \lor -2 < y\}$$

And the point $y_0 = 3$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{6x+4y+1}{2\left(2x+y+1\right)}\right) \\ &= -\frac{2}{2x+y+1} + \frac{6x+4y+1}{2\left(2x+y+1\right)^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y=3$ is

$$\{x < -2 \lor -2 < x\}$$

And the point $x_0 = \frac{1}{2}$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = \frac{1}{2}$ is

$$\{y < -2 \lor -2 < y\}$$

And the point $y_0 = 3$ is inside this domain. Therefore solution exists and is unique.

1.8.2 Solving as differentialType ode

Writing the ode as

$$y' = \frac{-6x-4y-1}{4x+2y+2} \tag{1}$$

Which becomes

$$(2+2y)\,dy = (-4x)\,dy + (-6x-4y-1)\,dx \tag{2}$$

But the RHS is complete differential because

$$(-4x)\,dy + (-6x-4y-1)\,dx = d\left(-3x^2-4xy-x\right)$$

Hence (2) becomes

$$(2+2y)\,dy = d\left(-3x^2-4xy-x\right)$$

Integrating both sides gives gives these solutions

$$
\begin{aligned}
y &= -2x - 1 + \sqrt{x^2 + c_1 + 3x + 1} + c_1 \\
y &= -2x - 1 - \sqrt{x^2 + c_1 + 3x + 1} + c_1
\end{aligned}
$$

Initial conditions are used to solve for c_1. Substituting $x = \frac{1}{2}$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$
3 = -2 - \frac{\sqrt{11 + 4c_1}}{2} + c_1
$$

$$
c_1 = \frac{11}{2} + 2\sqrt{2}
$$

Substituting c_1 found above in the general solution gives

$$
y = -2x + \frac{9}{2} - \frac{\sqrt{4x^2 + 26 + 8\sqrt{2} + 12x}}{2} + 2\sqrt{2}
$$

Initial conditions are used to solve for c_1. Substituting $x = \frac{1}{2}$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$
3 = -2 + \frac{\sqrt{11 + 4c_1}}{2} + c_1
$$

$$
c_1 = \frac{11}{2} - 2\sqrt{2}
$$

Substituting c_1 found above in the general solution gives

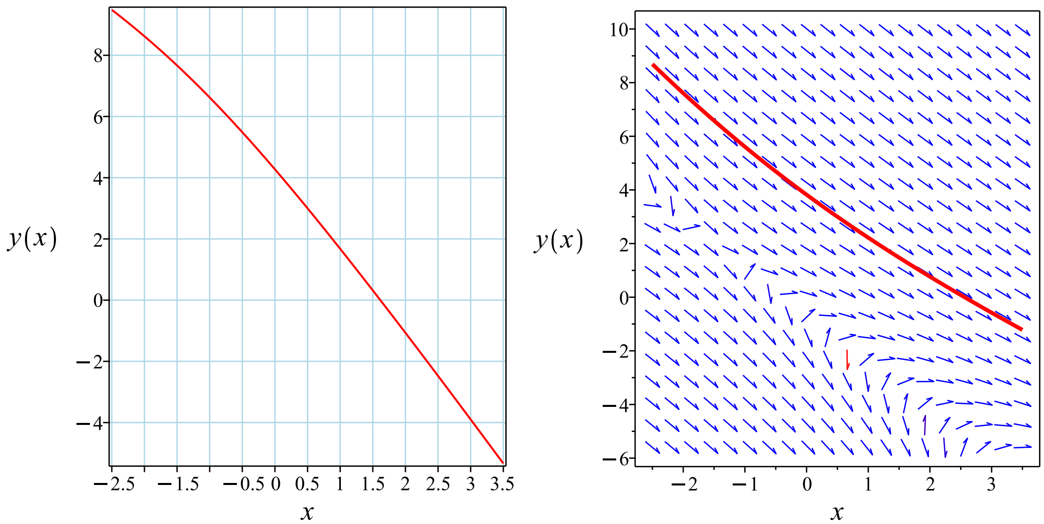
$$
y = -2x + \frac{9}{2} + \frac{\sqrt{4x^2 + 26 - 8\sqrt{2} + 12x}}{2} - 2\sqrt{2}
$$

Summary

The solution(s) found are the following

$$
y = -2x + \frac{9}{2} + \frac{\sqrt{4x^2 + 26 - 8\sqrt{2} + 12x}}{2} - 2\sqrt{2} \tag{1}
$$

$$
y = -2x + \frac{9}{2} - \frac{\sqrt{4x^2 + 26 + 8\sqrt{2} + 12x}}{2} + 2\sqrt{2} \tag{2}
$$



(a) Solution plot

(b) Slope field plot

<u>Verification of solutions</u>

$$y = -2x + \frac{9}{2} + \frac{\sqrt{4x^2 + 26 - 8\sqrt{2} + 12x}}{2} - 2\sqrt{2}$$

Verified OK.

$$y = -2x + \frac{9}{2} - \frac{\sqrt{4x^2 + 26 + 8\sqrt{2} + 12x}}{2} + 2\sqrt{2}$$

Verified OK.

1.8.3 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = -\frac{6X + 6x_0 + 4Y(X) + 4y_0 + 1}{2\,(2X + 2x_0 + Y\,(X) + y_0 + 1)}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = -\frac{3}{2}$$
$$y_0 = 2$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = -\frac{6X + 4Y(X)}{2\left(2X + Y\left(X\right)\right)}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X, Y) \\ &= -\frac{3X + 2Y}{2X + Y} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X, Y)$ and $N(X, Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X, Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X, Y)$$

In this case, it can be seen that both $M = -3X - 2Y$ and $N = 2X + Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}X}X + u &= \frac{-2u - 3}{u + 2} \\ \frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\frac{-2u(X)-3}{u(X)+2} - u(X)}{X} \end{aligned}$$

Or

$$\frac{d}{dX}u(X) - \frac{\frac{-2u(X)-3}{u(X)+2} - u(X)}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right)Xu(X) + 2\left(\frac{d}{dX}u(X)\right)X + u(X)^2 + 4u(X) + 3 = 0$$

Or

$$X(u(X) + 2)\left(\frac{d}{dX}u(X)\right) + u(X)^2 + 4u(X) + 3 = 0$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(X, u) \\ &= f(X)g(u) \\ &= -\frac{u^2 + 4u + 3}{X\left(u + 2\right)} \end{aligned}$$

Where $f(X) = -\frac{1}{X}$ and $g(u) = \frac{u^2+4u+3}{u+2}$. Integrating both sides gives

$$\frac{1}{\frac{u^2+4u+3}{u+2}}\,du = -\frac{1}{X}\,dX$$

$$\int \frac{1}{\frac{u^2+4u+3}{u+2}}\,du = \int -\frac{1}{X}\,dX$$

$$\frac{\ln(u^2+4u+3)}{2} = -\ln(X) + c_2$$

Raising both side to exponential gives

$$\sqrt{u^2+4u+3} = \mathrm{e}^{-\ln(X)+c_2}$$

Which simplifies to

$$\sqrt{u^2+4u+3} = \frac{c_3}{X}$$

Which simplifies to

$$\sqrt{u(X)^2 + 4u(X) + 3} = \frac{c_3 \mathrm{e}^{c_2}}{X}$$

The solution is

$$\sqrt{u(X)^2 + 4u(X) + 3} = \frac{c_3 \mathrm{e}^{c_2}}{X}$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$\sqrt{\frac{Y(X)^2}{X^2} + \frac{4Y(X)}{X} + 3} = \frac{c_3 \mathrm{e}^{c_2}}{X}$$

Which simplifies to

$$\sqrt{\frac{(Y(X)+3X)(Y(X)+X)}{X^2}} = \frac{c_3 \mathrm{e}^{c_2}}{X}$$

Using the solution for $Y(X)$

$$\sqrt{\frac{(Y(X)+3X)(Y(X)+X)}{X^2}} = \frac{c_3 \mathrm{e}^{c_2}}{X}$$

And replacing back terms in the above solution using

$$
\begin{aligned}
Y &= y + y_0 \\
X &= x + x_0
\end{aligned}
$$

Or

$$
\begin{aligned}
Y &= y + 2 \\
X &= x - \frac{3}{2}
\end{aligned}
$$

Then the solution in y becomes

$$
\sqrt{\frac{\left(y+\frac{5}{2}+3x\right)\left(y-\frac{1}{2}+x\right)}{\left(\frac{3}{2}+x\right)^2}} = \frac{c_3 \mathrm{e}^{c_2}}{\frac{3}{2}+x}
$$

Initial conditions are used to solve for c_2. Substituting $x = \frac{1}{2}$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$
\frac{\sqrt{21}}{2} = \frac{c_3 \mathrm{e}^{c_2}}{2}
$$

$$
c_2 = \frac{\ln\left(\frac{21}{c_3^2}\right)}{2}
$$

Substituting c_2 found above in the general solution gives

$$
\sqrt{\frac{\left(y+\frac{5}{2}+3x\right)\left(x+y-\frac{1}{2}\right)}{\left(\frac{3}{2}+x\right)^2}} = \frac{2c_3\sqrt{21}\,\sqrt{\frac{1}{c_3^2}}}{2x+3}
$$

The above simplifies to

$$
-2c_3\sqrt{21}\,\sqrt{\frac{1}{c_3^2}} + 2\sqrt{\frac{(2y+5+6x)(2x+2y-1)}{(2x+3)^2}}\,x + 3\sqrt{\frac{(2y+5+6x)(2x+2y-1)}{(2x+3)^2}} = 0
$$

Summary

The solution(s) found are the following

$$
\sqrt{\frac{(2y+5+6x)(2x+2y-1)}{(2x+3)^2}}\,(2x+3) - 2\sqrt{21}\,\operatorname{csgn}\left(\frac{1}{c_3}\right) = 0 \tag{1}
$$

Verification of solutions

$$
\sqrt{\frac{(2y+5+6x)(2x+2y-1)}{(2x+3)^2}}\,(2x+3) - 2\sqrt{21}\,\operatorname{csgn}\left(\frac{1}{c_3}\right) = 0
$$

Verified OK.

1.8.4 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{6x + 4y + 1}{2\,(2x + y + 1)} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = x a_2 + y a_3 + a_1 \tag{1E}
$$

$$
\eta = x b_2 + y b_3 + b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, b_1, b_2, b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2 - \frac{(6x + 4y + 1)\,(b_3 - a_2)}{2\,(2x + y + 1)} - \frac{(6x + 4y + 1)^2\, a_3}{4\,(2x + y + 1)^2} \\
&- \left(-\frac{3}{2x + y + 1} + \frac{6x + 4y + 1}{(2x + y + 1)^2}\right)(x a_2 + y a_3 + a_1) \\
&- \left(-\frac{2}{2x + y + 1} + \frac{6x + 4y + 1}{2\,(2x + y + 1)^2}\right)(x b_2 + y b_3 + b_1) = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
\frac{24x^2 a_2 - 36x^2 a_3 + 20x^2 b_2 - 24x^2 b_3 + 24xy a_2 - 48xy a_3 + 16xy b_2 - 24xy b_3 + 8y^2 a_2 - 20y^2 a_3 + 4y^2 b_2 - 8}{4\,(2}
$$

$$
= 0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
&24x^2 a_2 - 36x^2 a_3 + 20x^2 b_2 - 24x^2 b_3 + 24xy a_2 - 48xy a_3 + 16xy b_2 - 24xy b_3 \\
&+ 8y^2 a_2 - 20y^2 a_3 + 4y^2 b_2 - 8y^2 b_3 + 24x a_2 - 12x a_3 + 4x b_1 + 22x b_2 - 16x b_3 \\
&- 4y a_1 + 10y a_2 + 8y b_2 - 4y b_3 + 8a_1 + 2a_2 - a_3 + 6b_1 + 4b_2 - 2b_3 = 0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} & 24a_2v_1^2 + 24a_2v_1v_2 + 8a_2v_2^2 - 36a_3v_1^2 - 48a_3v_1v_2 - 20a_3v_2^2 + 20b_2v_1^2 + 16b_2v_1v_2 \\ & + 4b_2v_2^2 - 24b_3v_1^2 - 24b_3v_1v_2 - 8b_3v_2^2 - 4a_1v_2 + 24a_2v_1 + 10a_2v_2 - 12a_3v_1 + 4b_1v_1 \\ & + 22b_2v_1 + 8b_2v_2 - 16b_3v_1 - 4b_3v_2 + 8a_1 + 2a_2 - a_3 + 6b_1 + 4b_2 - 2b_3 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} & (24a_2 - 36a_3 + 20b_2 - 24b_3)\, v_1^2 + (24a_2 - 48a_3 + 16b_2 - 24b_3)\, v_1v_2 \\ & + (24a_2 - 12a_3 + 4b_1 + 22b_2 - 16b_3)\, v_1 + (8a_2 - 20a_3 + 4b_2 - 8b_3)\, v_2^2 \\ & + (-4a_1 + 10a_2 + 8b_2 - 4b_3)\, v_2 + 8a_1 + 2a_2 - a_3 + 6b_1 + 4b_2 - 2b_3 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -4a_1 + 10a_2 + 8b_2 - 4b_3 &= 0 \\ 8a_2 - 20a_3 + 4b_2 - 8b_3 &= 0 \\ 24a_2 - 48a_3 + 16b_2 - 24b_3 &= 0 \\ 24a_2 - 36a_3 + 20b_2 - 24b_3 &= 0 \\ 24a_2 - 12a_3 + 4b_1 + 22b_2 - 16b_3 &= 0 \\ 8a_1 + 2a_2 - a_3 + 6b_1 + 4b_2 - 2b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 4a_3 + \frac{3b_3}{2} \\
a_2 &= 4a_3 + b_3 \\
a_3 &= a_3 \\
b_1 &= -\frac{9a_3}{2} - 2b_3 \\
b_2 &= -3a_3 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= \frac{3}{2} + x \\
\eta &= y - 2
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - 2 - \left(-\frac{6x+4y+1}{2\,(2x+y+1)}\right)\left(\frac{3}{2} + x\right) \\
&= \frac{12x^2 + 16xy + 4y^2 + 4x + 8y - 5}{8x + 4y + 4} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{12x^2+16xy+4y^2+4x+8y-5}{8x+4y+4}} dy \end{aligned}$$

Which results in

$$S = \frac{\ln \left(12x^2 + 16xy + 4y^2 + 4x + 8y - 5\right)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{6x + 4y + 1}{2\left(2x + y + 1\right)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{3}{2y + 5 + 6x} + \frac{1}{2x + 2y - 1} \\ S_y &= \frac{1}{2y + 5 + 6x} + \frac{1}{2x + 2y - 1} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

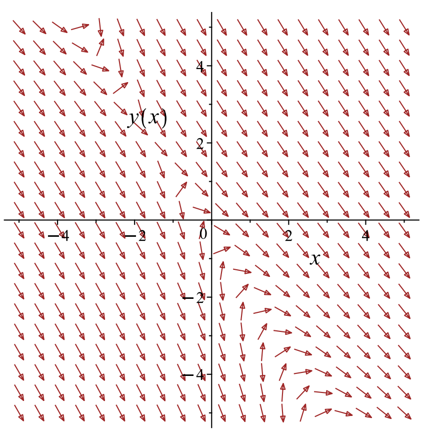
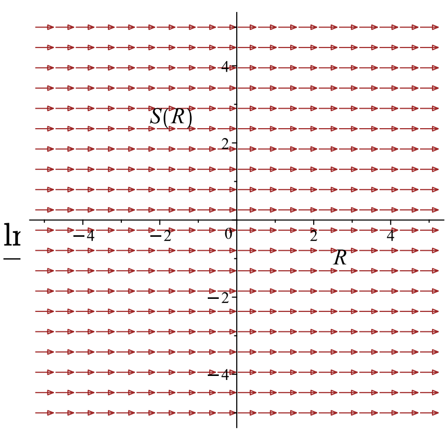
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(2y+5+6x)}{2} + \frac{\ln(2x+2y-1)}{2} = c_1$$

Which simplifies to

$$\frac{\ln(2y+5+6x)}{2} + \frac{\ln(2x+2y-1)}{2} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{6x+4y+1}{2(2x+y+1)}$ 	$R = x$ $S = \frac{\ln(2y+5+6x)}{2} + \ln$	$\frac{dS}{dR} = 0$ 

Initial conditions are used to solve for c_1. Substituting $x = \frac{1}{2}$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$\ln(2) + \frac{\ln(7)}{2} + \frac{\ln(3)}{2} = c_1$$

$$c_1 = \ln(2) + \frac{\ln(7)}{2} + \frac{\ln(3)}{2}$$

Substituting c_1 found above in the general solution gives

$$\frac{\ln(2y+5+6x)}{2}+\frac{\ln(2x+2y-1)}{2}=\ln(2)+\frac{\ln(7)}{2}+\frac{\ln(3)}{2}$$

Summary

The solution(s) found are the following

$$\frac{\ln(2y+5+6x)}{2}+\frac{\ln(2x+2y-1)}{2}=\ln(2)+\frac{\ln(7)}{2}+\frac{\ln(3)}{2} \tag{1}$$

Verification of solutions

$$\frac{\ln(2y+5+6x)}{2}+\frac{\ln(2x+2y-1)}{2}=\ln(2)+\frac{\ln(7)}{2}+\frac{\ln(3)}{2}$$

Verified OK.

1.8.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x}=M$$
$$\frac{\partial\phi}{\partial y}=N$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition

$\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(4x+2y+2)\,\mathrm{d}y &= (-6x-4y-1)\,\mathrm{d}x\\ (6x+4y+1)\,\mathrm{d}x + (4x+2y+2)\,\mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= 6x+4y+1\\ N(x,y) &= 4x+2y+2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(6x+4y+1)\\ &= 4\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(4x+2y+2)\\ &= 4\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = M \tag{1}$$

$$\frac{\partial\phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int 6x+4y+1\,\mathrm{d}x\\ \phi &= x(3x+4y+1)+f(y)\end{aligned}\tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 4x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 4x + 2y + 2$. Therefore equation (4) becomes

$$4x + 2y + 2 = 4x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 2 + 2y$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (2 + 2y)\,\mathrm{d}y$$
$$f(y) = y^2 + 2y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = x(3x + 4y + 1) + y^2 + 2y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x(3x + 4y + 1) + y^2 + 2y$$

Initial conditions are used to solve for c_1. Substituting $x = \frac{1}{2}$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{89}{4} = c_1$$

$$c_1 = \frac{89}{4}$$

Substituting c_1 found above in the general solution gives

$$x(3x + 4y + 1) + y^2 + 2y = \frac{89}{4}$$

Summary

The solution(s) found are the following

$$y^2 + (4x + 2)\, y + 3x^2 + x = \frac{89}{4} \tag{1}$$

Verification of solutions

$$y^2 + (4x + 2)\, y + 3x^2 + x = \frac{89}{4}$$

Verified OK.

1.8.6 Maple step by step solution

Let's solve

$\left[4y + (4x + 2y + 2)\, y' = -6x - 1, y\left(\frac{1}{2}\right) = 3\right]$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact
  - ◦ ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x, y) = 0$

  - ◦ Compute derivative of lhs

    $F'(x, y) + \left(\frac{\partial}{\partial y} F(x, y)\right) y' = 0$

  - ◦ Evaluate derivatives

    $4 = 4$

  - ◦ Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y)\, , N(x, y) = \frac{\partial}{\partial y} F(x, y)\right]$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int (6x + 4y + 1)\, dx + f_1(y)$

- Evaluate integral

$F(x,y) = 3x^2 + 4xy + x + f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y) = \frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $4x + 2y + 2 = 4x + \frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y) = 2 + 2y$

- Solve for $f_1(y)$

  $f_1(y) = y^2 + 2y$

- Substitute $f_1(y)$ into equation for $F(x,y)$

  $F(x,y) = 3x^2 + 4xy + y^2 + x + 2y$

- Substitute $F(x,y)$ into the solution of the ODE

  $3x^2 + 4xy + y^2 + x + 2y = c_1$

- Solve for y

  $\left\{y = -2x - 1 - \sqrt{x^2 + c_1 + 3x + 1}, y = -2x - 1 + \sqrt{x^2 + c_1 + 3x + 1}\right\}$

- Use initial condition $y\left(\frac{1}{2}\right) = 3$

  $3 = -2 - \sqrt{\frac{11}{4} + c_1}$

- Solution does not satisfy initial condition
- Use initial condition $y\left(\frac{1}{2}\right) = 3$

  $3 = -2 + \sqrt{\frac{11}{4} + c_1}$

- Solve for c_1

  $c_1 = \frac{89}{4}$

- Substitute $c_1 = \frac{89}{4}$ into general solution and simplify

  $y = -2x - 1 + \frac{\sqrt{4x^2+12x+93}}{2}$

- Solution to the IVP

  $y = -2x - 1 + \frac{\sqrt{4x^2+12x+93}}{2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.234 (sec). Leaf size: 23

```
dsolve([(6*x+4*y(x)+1)+(4*x+2*y(x)+2)*diff(y(x),x)=0,y(1/2) = 3],y(x), singsol=all)
```

$$y(x) = -2x - 1 + \frac{\sqrt{4x^2 + 12x + 93}}{2}$$

✓ Solution by Mathematica

Time used: 0.143 (sec). Leaf size: 28

```
DSolve[{(6*x+4*y[x]+1)+(4*x+2*y[x]+2)*y'[x]==0,y[1/2]==3},y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to \frac{1}{2}\left(\sqrt{4x^2 + 12x + 93} - 4x - 2\right)$$

1.9 problem 9

Internal problem ID [2996]
Internal file name [`OUTPUT/2488_Sunday_June_05_2022_03_16_06_AM_84320762/index.tex`]

Book: Differential equations, Shepley L. Ross, 1964
Section: 2.4, page 55
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeMapleC", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{-y + (x + y + 2)\, y' = -3x + 6}$$

With initial conditions

$$[y(2) = -2]$$

1.9.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= \frac{-3x + y + 6}{x + y + 2} \end{aligned}$$

The x domain of $f(x, y)$ when $y = -2$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 2$ is inside this domain. The y domain of $f(x, y)$ when $x = 2$ is

$$\{y < -4 \lor -4 < y\}$$

And the point $y_0 = -2$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{-3x + y + 6}{x + y + 2}\right) \\
&= \frac{1}{x + y + 2} - \frac{-3x + y + 6}{(x + y + 2)^2}
\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = -2$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 2$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 2$ is

$$\{y < -4 \lor -4 < y\}$$

And the point $y_0 = -2$ is inside this domain. Therefore solution exists and is unique.

1.9.2 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = \frac{-3X - 3x_0 + Y(X) + y_0 + 6}{X + x_0 + Y(X) + y_0 + 2}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$\begin{aligned}
x_0 &= 1 \\
y_0 &= -3
\end{aligned}$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = \frac{-3X + Y(X)}{X + Y(X)}$$

In canonical form, the ODE is

$$\begin{aligned}
Y' &= F(X, Y) \\
&= \frac{-3X + Y}{X + Y} \qquad (1)
\end{aligned}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X,Y)$$

In this case, it can be seen that both $M = -3X + Y$ and $N = X + Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}X}X + u &= \frac{u-3}{u+1}\\ \frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\frac{u(X)-3}{u(X)+1} - u(X)}{X}\end{aligned}$$

Or

$$\frac{d}{dX}u(X) - \frac{\frac{u(X)-3}{u(X)+1} - u(X)}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right) X u(X) + \left(\frac{d}{dX}u(X)\right) X + u(X)^2 + 3 = 0$$

Or

$$X(u(X)+1)\left(\frac{d}{dX}u(X)\right) + u(X)^2 + 3 = 0$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$\begin{aligned}u' &= F(X,u)\\ &= f(X)g(u)\\ &= -\frac{u^2+3}{X(u+1)}\end{aligned}$$

Where $f(X) = -\frac{1}{X}$ and $g(u) = \frac{u^2+3}{u+1}$. Integrating both sides gives

$$\frac{1}{\frac{u^2+3}{u+1}}\, du = -\frac{1}{X}\, dX$$

$$\int \frac{1}{\frac{u^2+3}{u+1}}\, du = \int -\frac{1}{X}\, dX$$

$$\frac{\ln(u^2+3)}{2}+\frac{\sqrt{3}\,\arctan\left(\frac{\sqrt{3}\,u}{3}\right)}{3}=-\ln(X)+c_2$$

The solution is

$$\frac{\ln\left(u(X)^2+3\right)}{2}+\frac{\sqrt{3}\,\arctan\left(\frac{\sqrt{3}\,u(X)}{3}\right)}{3}+\ln(X)-c_2=0$$

Now u in the above solution is replaced back by Y using $u=\frac{Y}{X}$ which results in the solution

$$\frac{\ln\left(\frac{Y(X)^2}{X^2}+3\right)}{2}+\frac{\sqrt{3}\,\arctan\left(\frac{\sqrt{3}\,Y(X)}{3X}\right)}{3}+\ln(X)-c_2=0$$

Using the solution for $Y(X)$

$$\frac{\ln\left(\frac{Y(X)^2}{X^2}+3\right)}{2}+\frac{\sqrt{3}\,\arctan\left(\frac{\sqrt{3}\,Y(X)}{3X}\right)}{3}+\ln(X)-c_2=0$$

And replacing back terms in the above solution using

$$Y=y+y_0$$
$$X=x+x_0$$

Or

$$Y=y-3$$
$$X=x+1$$

Then the solution in y becomes

$$\frac{\ln\left(\frac{(y+3)^2}{(x-1)^2}+3\right)}{2}+\frac{\sqrt{3}\,\arctan\left(\frac{\sqrt{3}\,(y+3)}{3x-3}\right)}{3}+\ln(x-1)-c_2=0$$

Initial conditions are used to solve for c_2. Substituting $x=2$ and $y=-2$ in the above solution gives an equation to solve for the constant of integration.

$$\ln(2)+\frac{\sqrt{3}\,\pi}{18}-c_2=0$$

$$c_2=\ln(2)+\frac{\sqrt{3}\,\pi}{18}$$

Substituting c_2 found above in the general solution gives

$$\frac{\ln\left(\frac{3x^2+y^2-6x+6y+12}{(x-1)^2}\right)}{2}+\frac{\sqrt{3}\,\arctan\left(\frac{\sqrt{3}\,(y+3)}{3x-3}\right)}{3}+\ln\left(x-1\right)-\ln\left(2\right)-\frac{\sqrt{3}\,\pi}{18}=0$$

Summary

The solution(s) found are the following

$$\frac{\ln\left(\frac{3x^2+y^2-6x+6y+12}{(x-1)^2}\right)}{2}+\frac{\sqrt{3}\,\arctan\left(\frac{\sqrt{3}\,(y+3)}{3x-3}\right)}{3}+\ln\left(x-1\right)-\ln\left(2\right)-\frac{\sqrt{3}\,\pi}{18}=0 \quad (1)$$

Verification of solutions

$$\frac{\ln\left(\frac{3x^2+y^2-6x+6y+12}{(x-1)^2}\right)}{2}+\frac{\sqrt{3}\,\arctan\left(\frac{\sqrt{3}\,(y+3)}{3x-3}\right)}{3}+\ln\left(x-1\right)-\ln\left(2\right)-\frac{\sqrt{3}\,\pi}{18}=0$$

Verified OK.

1.9.3 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned}y' &= \frac{-3x+y+6}{x+y+2}\\ y' &= \omega(x,y)\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \quad \text{(A)}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2+ya_3+a_1 \quad \text{(1E)}$$

$$\eta = xb_2+yb_3+b_1 \quad \text{(2E)}$$

Where the unknown coefficients are

$$\{a_1,a_2,a_3,b_1,b_2,b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2 + \frac{(-3x+y+6)(b_3-a_2)}{x+y+2} - \frac{(-3x+y+6)^2 a_3}{(x+y+2)^2} \\
&- \left(-\frac{3}{x+y+2} - \frac{-3x+y+6}{(x+y+2)^2}\right)(xa_2+ya_3+a_1) \\
&- \left(\frac{1}{x+y+2} - \frac{-3x+y+6}{(x+y+2)^2}\right)(xb_2+yb_3+b_1) = 0
\end{aligned}
\tag{5E}
$$

Putting the above in normal form gives

$$
\frac{3x^2a_2 - 9x^2a_3 - 3x^2b_2 - 3x^2b_3 + 6xya_2 + 6xya_3 + 2xyb_2 - 6xyb_3 - y^2a_2 + 3y^2a_3 + y^2b_2 + y^2b_3 + 12xa_2 +}{(x+y+2)}
$$

$= 0$

Setting the numerator to zero gives

$$
\begin{aligned}
&3x^2a_2 - 9x^2a_3 - 3x^2b_2 - 3x^2b_3 + 6xya_2 + 6xya_3 + 2xyb_2 - 6xyb_3 \\
&- y^2a_2 + 3y^2a_3 + y^2b_2 + y^2b_3 + 12xa_2 + 36xa_3 - 4xb_1 + 8xb_2 + 4ya_1 \\
&- 8ya_2 + 4yb_2 + 12yb_3 + 12a_1 - 12a_2 - 36a_3 + 4b_1 + 4b_2 + 12b_3 = 0
\end{aligned}
\tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&3a_2v_1^2 + 6a_2v_1v_2 - a_2v_2^2 - 9a_3v_1^2 + 6a_3v_1v_2 + 3a_3v_2^2 - 3b_2v_1^2 + 2b_2v_1v_2 + b_2v_2^2 \\
&- 3b_3v_1^2 - 6b_3v_1v_2 + b_3v_2^2 + 4a_1v_2 + 12a_2v_1 - 8a_2v_2 + 36a_3v_1 - 4b_1v_1 \\
&+ 8b_2v_1 + 4b_2v_2 + 12b_3v_2 + 12a_1 - 12a_2 - 36a_3 + 4b_1 + 4b_2 + 12b_3 = 0
\end{aligned}
\tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&(3a_2 - 9a_3 - 3b_2 - 3b_3)\, v_1^2 + (6a_2 + 6a_3 + 2b_2 - 6b_3)\, v_1 v_2 \\
&+ (12a_2 + 36a_3 - 4b_1 + 8b_2)\, v_1 + (-a_2 + 3a_3 + b_2 + b_3)\, v_2^2 \\
&+ (4a_1 - 8a_2 + 4b_2 + 12b_3)\, v_2 + 12a_1 - 12a_2 - 36a_3 + 4b_1 + 4b_2 + 12b_3 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
4a_1 - 8a_2 + 4b_2 + 12b_3 &= 0 \\
-a_2 + 3a_3 + b_2 + b_3 &= 0 \\
3a_2 - 9a_3 - 3b_2 - 3b_3 &= 0 \\
6a_2 + 6a_3 + 2b_2 - 6b_3 &= 0 \\
12a_2 + 36a_3 - 4b_1 + 8b_2 &= 0 \\
12a_1 - 12a_2 - 36a_3 + 4b_1 + 4b_2 + 12b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= -b_3 + 3a_3 \\
a_2 &= b_3 \\
a_3 &= a_3 \\
b_1 &= 3a_3 + 3b_3 \\
b_2 &= -3a_3 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x - 1 \\
\eta &= y + 3
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\, \xi \\
&= y + 3 - \left(\frac{-3x + y + 6}{x + y + 2}\right)(x - 1) \\
&= \frac{3x^2 + y^2 - 6x + 6y + 12}{x + y + 2} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{3x^2+y^2-6x+6y+12}{x+y+2}} dy \end{aligned}$$

Which results in

$$S = \frac{\ln(3x^2 + y^2 - 6x + 6y + 12)}{2} + \frac{\sqrt{3}\arctan\left(\frac{(2y+6)\sqrt{3}}{6x-6}\right)}{3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{-3x + y + 6}{x + y + 2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{3x - y - 6}{3x^2 + y^2 - 6x + 6y + 12} \\ S_y &= \frac{x + y + 2}{3x^2 + y^2 - 6x + 6y + 12} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(3x^2 + y^2 - 6x + 6y + 12)}{2} + \frac{\sqrt{3}\,\arctan\left(\frac{\sqrt{3}\,(y+3)}{3x-3}\right)}{3} = c_1$$

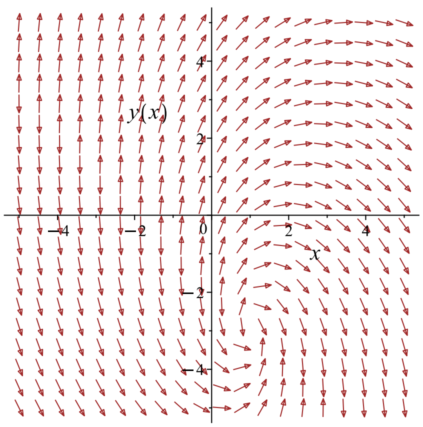
Which simplifies to

$$\frac{\ln(3x^2 + y^2 - 6x + 6y + 12)}{2} + \frac{\sqrt{3}\,\arctan\left(\frac{\sqrt{3}\,(y+3)}{3x-3}\right)}{3} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>

$\frac{dy}{dx} = \frac{-3x+y+6}{x+y+2}$



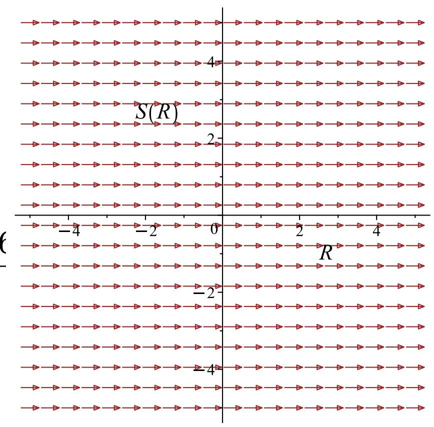
</td><td>

$R = x$

$S = \frac{\ln(3x^2 + y^2 - 6x + 6}{2}$

</td><td>

$\frac{dS}{dR} = 0$



</td></tr>
</table>

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = -2$ in the above solution gives an equation to solve for the constant of integration.

$$\ln(2) + \frac{\sqrt{3}\,\pi}{18} = c_1$$

$$c_1 = \ln(2) + \frac{\sqrt{3}\,\pi}{18}$$

Substituting c_1 found above in the general solution gives

$$\frac{\ln(3x^2 + y^2 - 6x + 6y + 12)}{2} + \frac{\sqrt{3}\arctan\left(\frac{\sqrt{3}\,(y+3)}{3x-3}\right)}{3} = \ln(2) + \frac{\sqrt{3}\,\pi}{18}$$

Summary

The solution(s) found are the following

$$\frac{\ln(3x^2 + y^2 - 6x + 6y + 12)}{2} + \frac{\sqrt{3}\arctan\left(\frac{\sqrt{3}\,(y+3)}{3x-3}\right)}{3} = \ln(2) + \frac{\sqrt{3}\,\pi}{18} \quad (1)$$

Verification of solutions

$$\frac{\ln\left(3x^2+y^2-6x+6y+12\right)}{2}+\frac{\sqrt{3}\,\arctan\left(\frac{\sqrt{3}\,(y+3)}{3x-3}\right)}{3}=\ln\left(2\right)+\frac{\sqrt{3}\,\pi}{18}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 1.968 (sec). Leaf size: 120

```
dsolve([(3*x-y(x)-6)+(x+y(x)+2)*diff(y(x),x)=0,y(2) = -2],y(x), singsol=all)
```

$$y(x)=-3-\sqrt{3}\,\tan\left(\mathrm{RootOf}\left(-3\sqrt{3}\,\ln\left(3\right)+6\sqrt{3}\,\ln\left(2\right)-3\sqrt{3}\,\ln\left(\sec\left(\_Z\right)^2\left(x-1\right)^2\right)+\pi+6\_Z\right)\right)\left(x-1\right)$$

✓ Solution by Mathematica

Time used: 0.158 (sec). Leaf size: 90

```
DSolve[{(3*x-y[x]-6)+(x+y[x]+2)*y'[x]==0,y[2]==-2},y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\frac{\arctan\left(\frac{-y(x)+3x-6}{\sqrt{3}(y(x)+x+2)}\right)}{\sqrt{3}}+\log(2)=\frac{1}{2}\log\left(\frac{3x^2+y(x)^2+6y(x)-6x+12}{(x-1)^2}\right)+\log(x-1)+\frac{1}{18}\left(\sqrt{3}\pi+18\log(2)-9\log(4)\right),y(x)\right]$$

1.10 problem 10

Internal problem ID [2997]
Internal file name [OUTPUT/2489_Sunday_June_05_2022_03_16_15_AM_52647122/index.tex]

Book: Differential equations, Shepley L. Ross, 1964
Section: 2.4, page 55
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{3y + (4x + 6y + 1)\, y' = -1 - 2x}$$

With initial conditions

$$[y(-2) = 2]$$

1.10.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= -\frac{2x + 3y + 1}{4x + 6y + 1} \end{aligned}$$

The x domain of $f(x,y)$ when $y = 2$ is

$$\left\{x < -\frac{13}{4} \lor -\frac{13}{4} < x\right\}$$

And the point $x_0 = -2$ is inside this domain. The y domain of $f(x, y)$ when $x = -2$ is

$$\left\{y < \frac{7}{6} \vee \frac{7}{6} < y\right\}$$

And the point $y_0 = 2$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{2x + 3y + 1}{4x + 6y + 1}\right) \\ &= -\frac{3}{4x + 6y + 1} + \frac{12x + 18y + 6}{(4x + 6y + 1)^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 2$ is

$$\left\{x < -\frac{13}{4} \vee -\frac{13}{4} < x\right\}$$

And the point $x_0 = -2$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = -2$ is

$$\left\{y < \frac{7}{6} \vee \frac{7}{6} < y\right\}$$

And the point $y_0 = 2$ is inside this domain. Therefore solution exists and is unique.

1.10.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned}y' &= -\frac{2x + 3y + 1}{4x + 6y + 1} \\ y' &= \omega(x, y)\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}
&b_2 - \frac{(2x+3y+1)(b_3-a_2)}{4x+6y+1} - \frac{(2x+3y+1)^2 a_3}{(4x+6y+1)^2} \\
&- \left(-\frac{2}{4x+6y+1} + \frac{8x+12y+4}{(4x+6y+1)^2}\right)(xa_2 + ya_3 + a_1) \\
&- \left(-\frac{3}{4x+6y+1} + \frac{12x+18y+6}{(4x+6y+1)^2}\right)(xb_2 + yb_3 + b_1) = 0
\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{8x^2a_2 - 4x^2a_3 + 16x^2b_2 - 8x^2b_3 + 24xya_2 - 12xya_3 + 48xyb_2 - 24xyb_3 + 18y^2a_2 - 9y^2a_3 + 36y^2b_2 - 18y}{(4x+6y+}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned}
&8x^2a_2 - 4x^2a_3 + 16x^2b_2 - 8x^2b_3 + 24xya_2 - 12xya_3 + 48xyb_2 - 24xyb_3 \\
&+ 18y^2a_2 - 9y^2a_3 + 36y^2b_2 - 18y^2b_3 + 4xa_2 - 4xa_3 + 5xb_2 - 6xb_3 \\
&+ 9ya_2 - 8ya_3 + 12yb_2 - 12yb_3 - 2a_1 + a_2 - a_3 - 3b_1 + b_2 - b_3 = 0
\end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&8a_2v_1^2 + 24a_2v_1v_2 + 18a_2v_2^2 - 4a_3v_1^2 - 12a_3v_1v_2 - 9a_3v_2^2 + 16b_2v_1^2 + 48b_2v_1v_2 \\
&+ 36b_2v_2^2 - 8b_3v_1^2 - 24b_3v_1v_2 - 18b_3v_2^2 + 4a_2v_1 + 9a_2v_2 - 4a_3v_1 - 8a_3v_2 \\
&+ 5b_2v_1 + 12b_2v_2 - 6b_3v_1 - 12b_3v_2 - 2a_1 + a_2 - a_3 - 3b_1 + b_2 - b_3 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}(8a_2 - 4a_3 + 16b_2 - 8b_3)\, v_1^2 + (24a_2 - 12a_3 + 48b_2 - 24b_3)\, v_1 v_2 \\ + (4a_2 - 4a_3 + 5b_2 - 6b_3)\, v_1 + (18a_2 - 9a_3 + 36b_2 - 18b_3)\, v_2^2 \\ + (9a_2 - 8a_3 + 12b_2 - 12b_3)\, v_2 - 2a_1 + a_2 - a_3 - 3b_1 + b_2 - b_3 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}4a_2 - 4a_3 + 5b_2 - 6b_3 &= 0 \\ 8a_2 - 4a_3 + 16b_2 - 8b_3 &= 0 \\ 9a_2 - 8a_3 + 12b_2 - 12b_3 &= 0 \\ 18a_2 - 9a_3 + 36b_2 - 18b_3 &= 0 \\ 24a_2 - 12a_3 + 48b_2 - 24b_3 &= 0 \\ -2a_1 + a_2 - a_3 - 3b_1 + b_2 - b_3 &= 0\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1 &= a_1 \\ a_2 &= -8a_1 - 12b_1 \\ a_3 &= -12a_1 - 18b_1 \\ b_1 &= b_1 \\ b_2 &= 4a_1 + 6b_1 \\ b_3 &= 6a_1 + 9b_1\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}\xi &= -12x - 18y \\ \eta &= 6x + 9y + 1\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= 6x + 9y + 1 - \left(-\frac{2x + 3y + 1}{4x + 6y + 1}\right)(-12x - 18y) \\
&= \frac{-2x - 3y + 1}{4x + 6y + 1} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-2x - 3y + 1}{4x + 6y + 1}} dy
\end{aligned}$$

Which results in

$$S = -2y - \ln(2x + 3y - 1)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{2x + 3y + 1}{4x + 6y + 1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{2}{2x+3y-1} \\ S_y &= -2-\frac{3}{2x+3y-1} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-2y - \ln(2x+3y-1) = x + c_1$$

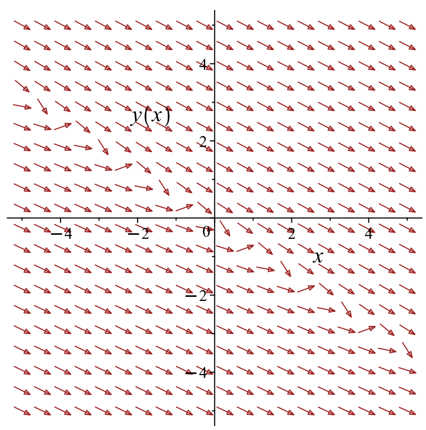
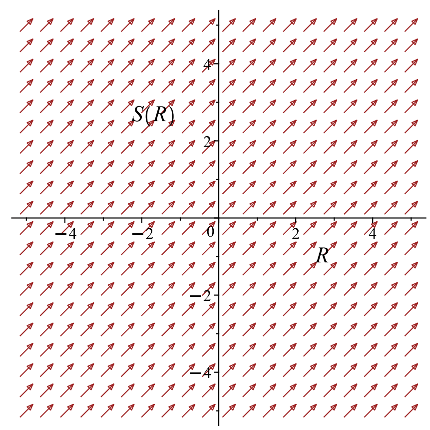
Which simplifies to

$$-2y - \ln(2x+3y-1) = x + c_1$$

Which gives

$$y = -\frac{2x}{3} + \frac{\operatorname{LambertW}\left(\frac{2\,e^{\frac{x}{3}-\frac{2}{3}-c_1}}{3}\right)}{2} + \frac{1}{3}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{2x+3y+1}{4x+6y+1}$ 	$R = x$ $S = -2y - \ln(2x + 3y -$	$\frac{dS}{dR} = 1$ 

Initial conditions are used to solve for c_1. Substituting $x = -2$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = \frac{5}{3} + \frac{\text{LambertW}\left(\frac{2\,e^{-\frac{4}{3}-c_1}}{3}\right)}{2}$$

$$c_1 = -2$$

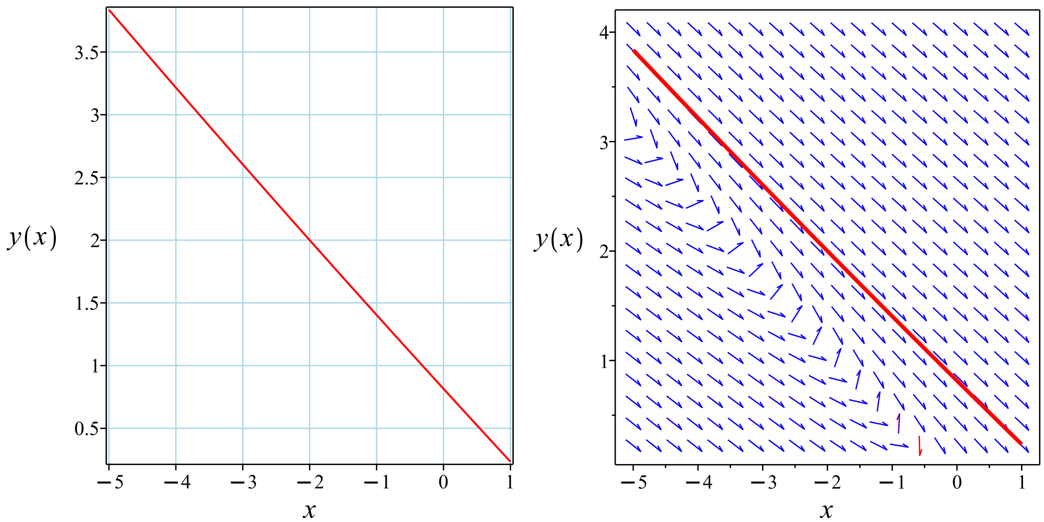
Substituting c_1 found above in the general solution gives

$$y = -\frac{2x}{3} + \frac{\text{LambertW}\left(\frac{2\,e^{\frac{x}{3}+\frac{4}{3}}}{3}\right)}{2} + \frac{1}{3}$$

Summary

The solution(s) found are the following

$$y = -\frac{2x}{3} + \frac{\text{LambertW}\left(\frac{2\,e^{\frac{x}{3}+\frac{4}{3}}}{3}\right)}{2} + \frac{1}{3} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -\frac{2x}{3} + \frac{\operatorname{LambertW}\left(\frac{2\,e^{\frac{x}{3}+\frac{4}{3}}}{3}\right)}{2} + \frac{1}{3}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
   -> Calling odsolve with the ODE`, diff(y(x), x) = -2/3, y(x)`      *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.156 (sec). Leaf size: 20

```
dsolve([(2*x+3*y(x)+1)+(4*x+6*y(x)+1)*diff(y(x),x)=0,y(-2) = 2],y(x), singsol=all)
```

$$y(x) = \frac{1}{3} - \frac{2x}{3} + \frac{\operatorname{LambertW}\left(\frac{2\,\mathrm{e}^{\frac{x}{3}+\frac{4}{3}}}{3}\right)}{2}$$

✓ Solution by Mathematica

Time used: 4.969 (sec). Leaf size: 30

```
DSolve[{(2*x+3*y[x]+1)+(4*x+6*y[x]+1)*y'[x]==0,y[-2]==2},y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to \frac{1}{6}\left(3W\left(\frac{2}{3}e^{\frac{x+4}{3}}\right) - 4x + 2\right)$$