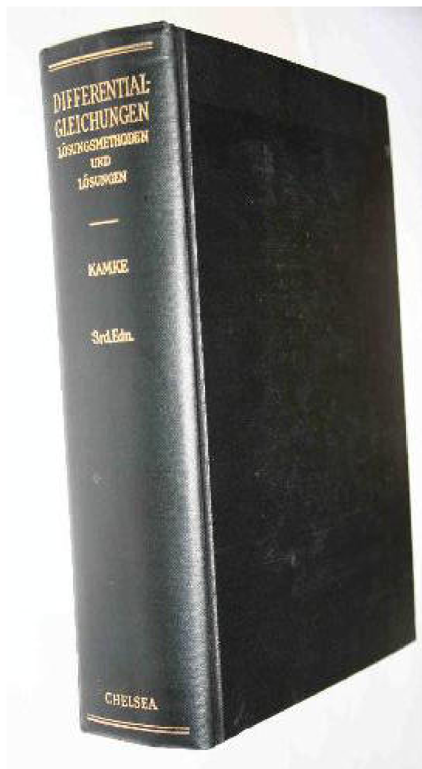
A Solution Manual For

Differential Gleichungen, Kamke, 3rd ed, Abel ODEs



Nasser M. Abbasi

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Contents

1 Abel ODE's with constant invariant

1.1 problem problem 38

Internal problem ID [4675]
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Book: Differential Gleichungen, Kamke, 3rd ed, Abel ODEs
Section: Abel ODE's with constant invariant
Problem number: problem 38.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"abelFirstKind", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, _Abel]
```

$$\boxed{-ay^3 + y' = \frac{b}{x^{\frac{3}{2}}}}$$

1.1.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{a\,y^3 x^{\frac{3}{2}} + b}{x^{\frac{3}{2}}}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 + \frac{\left(a\,y^3 x^{\frac{3}{2}} + b\right)(b_3 - a_2)}{x^{\frac{3}{2}}} - \frac{\left(a\,y^3 x^{\frac{3}{2}} + b\right)^2 a_3}{x^3} \\ & - \left(\frac{3a\,y^3}{2x} - \frac{3\left(a\,y^3 x^{\frac{3}{2}} + b\right)}{2x^{\frac{5}{2}}}\right)(xa_2 + ya_3 + a_1) - 3a\,y^2(xb_2 + yb_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{2x^{\frac{11}{2}}a^2y^6a_3 + 4x^4ab\,y^3a_3 + 6x^{\frac{13}{2}}a\,y^2b_2 + 2x^{\frac{11}{2}}a\,y^3a_2 + 4x^{\frac{11}{2}}a\,y^3b_3 + 6x^{\frac{11}{2}}a\,y^2b_1 - 2b_2x^{\frac{11}{2}} + 2x^{\frac{5}{2}}b^2a_3 - b\,x}{2x^{\frac{11}{2}}}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} & -2x^{\frac{11}{2}}a^2y^6a_3 - 6x^{\frac{13}{2}}a\,y^2b_2 - 2x^{\frac{11}{2}}a\,y^3a_2 - 4x^{\frac{11}{2}}a\,y^3b_3 - 6x^{\frac{11}{2}}a\,y^2b_1 + 2b_2x^{\frac{11}{2}} \\ & - 4x^4ab\,y^3a_3 - 2x^{\frac{5}{2}}b^2a_3 + b\,x^4a_2 + 2x^4bb_3 + 3b\,x^3ya_3 + 3b\,x^3a_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, x^{\frac{5}{2}}, x^{\frac{11}{2}}, x^{\frac{13}{2}}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, x^{\frac{5}{2}} = v_3, x^{\frac{11}{2}} = v_4, x^{\frac{13}{2}} = v_5\right\}$$

The above PDE (6E) now becomes

$$\begin{aligned} & -2v_4a^2v_2^6a_3 - 4v_1^4abv_2^3a_3 - 2v_4av_2^3a_2 - 4v_4av_2^3b_3 + bv_1^4a_2 + 3bv_1^3v_2a_3 \\ & + 2v_1^4bb_3 - 6v_4av_2^2b_1 - 6v_5av_2^2b_2 + 3bv_1^3a_1 - 2v_3b^2a_3 + 2b_2v_4 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4, v_5\}$$

Equation (7E) now becomes

$$-4v_1^4abv_2^3a_3 + (ba_2 + 2bb_3)\,v_1^4 + 3bv_1^3v_2a_3 + 3bv_1^3a_1 - 2v_4a^2v_2^6a_3 + (-2aa_2 - 4ab_3)\,v_2^3v_4 - 6v_4av_2^2b_1 - 6v_5av_2^2b_2 - 2v_3b^2a_3 + 2b_2v_4 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} 2b_2 &= 0 \\ -6ab_1 &= 0 \\ -6ab_2 &= 0 \\ -2a^2a_3 &= 0 \\ 3ba_1 &= 0 \\ 3ba_3 &= 0 \\ -2b^2a_3 &= 0 \\ -4aba_3 &= 0 \\ -2aa_2 - 4ab_3 &= 0 \\ ba_2 + 2bb_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= -2b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= -2x \\ \eta &= y \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Therefore

$$\begin{aligned}\frac{dy}{dx} &= \frac{\eta}{\xi} \\ &= \frac{y}{-2x} \\ &= -\frac{y}{2x}\end{aligned}$$

This is easily solved to give

$$y = \frac{c_1}{\sqrt{x}}$$

Where now the coordinate R is taken as the constant of integration. Hence

$$R = y\sqrt{x}$$

And S is found from

$$\begin{aligned}dS &= \frac{dx}{\xi} \\ &= \frac{dx}{-2x}\end{aligned}$$

Integrating gives

$$\begin{aligned}S &= \int \frac{dx}{T} \\ &= -\frac{\ln(x)}{2}\end{aligned}$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{a\,y^3 x^{\frac{3}{2}} + b}{x^{\frac{3}{2}}}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= \frac{y}{2\sqrt{x}} \\
R_y &= \sqrt{x} \\
S_x &= -\frac{1}{2x} \\
S_y &= 0
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{\sqrt{x}}{-2x^2 a\,y^3 - 2\sqrt{x}\,b - xy} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{2R^3 a + R + 2b}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \int -\frac{1}{2R^3 a + R + 2b} dR + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{\ln(x)}{2} = \int^{y\sqrt{x}} -\frac{1}{2\_a^3 a + \_a + 2b} d\_a + c_1$$

Which simplifies to

$$-\frac{\ln(x)}{2} = \int^{y\sqrt{x}} -\frac{1}{2\_a^3 a + \_a + 2b} d\_a + c_1$$

Summary
The solution(s) found are the following

$$-\frac{\ln(x)}{2} = \int^{y\sqrt{x}} -\frac{1}{2\_a^3a + \_a + 2b} d\_a + c_1 \tag{1}$$

Verification of solutions

$$-\frac{\ln(x)}{2} = \int^{y\sqrt{x}} -\frac{1}{2\_a^3a + \_a + 2b} d\_a + c_1$$

Verified OK.

1.1.2 Solving as abelFirstKind ode

This is Abel first kind ODE, it has the form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2 + f_3(x)y^3$$

Comparing the above to given ODE which is

$$y' = ay^3 + \frac{b}{x^{\frac{3}{2}}} \tag{1}$$

Therefore

$$\begin{aligned}
f_0(x) &= \frac{b}{x^{\frac{3}{2}}} \\
f_1(x) &= 0 \\
f_2(x) &= 0 \\
f_3(x) &= a
\end{aligned}$$

Since $f_2(x) = 0$ then we check the Abel invariant to see if it depends on x or not. The Abel invariant is given by

$$-\frac{f_1^3}{f_0^2 f_3}$$

Which when evaluating gives

$$-\frac{1}{8b^2a}$$

Since the Abel invariant does not depend on x then this ode can be solved directly.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
dsolve(-a*y(x)^3-b/(x^(3/2))+diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{\operatorname{RootOf}\left(-\ln(x) + c_1 + 2\left(\int^{\_Z} \frac{1}{2a\_a^3 + \_a + 2b} d\_a\right)\right)}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.332 (sec). Leaf size: 320

```
DSolve[-a*y[x]^3-b/(x^(3/2))+y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\operatorname{Solve}\left[\frac{2}{3}ab^2\operatorname{RootSum}\left[8\#1^9ab^2 + 24\#1^6ab^2 + 24\#1^3ab^2 + \#1^3 + 8ab^2\&, \frac{4\#1^6\log\left(y(x)\sqrt[3]{\frac{ax^{3/2}}{b}} - \#1\right) + 2\#1^4\sqrt[3]{-\frac{1}{ab^2}}\log\left(y(x)\sqrt[3]{\frac{ax^{3/2}}{b}} - \#1\right) + 8\#1^3\log\left(y(x)\sqrt[3]{\frac{ax}{}}\right.}{24\#1^8}\right.\right.$$

$$\left. + c_1, y(x)\right]$$

1.2 problem problem 41

Internal problem ID [4676]
Internal file name [`OUTPUT/4169_Sunday_June_05_2022_12_32_31_PM_62580225/index.tex`]

Book: Differential Gleichungen, Kamke, 3rd ed, Abel ODEs
Section: Abel ODE's with constant invariant
Problem number: problem 41.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"abelFirstKind", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _Abel]
```

$$\boxed{axy^3 + by^2 + y' = 0}$$

1.2.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= -xa\,y^3 - b\,y^2 \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \left(-xa\,y^3 - b\,y^2\right)(b_3 - a_2) - \left(-xa\,y^3 - b\,y^2\right)^2 a_3 \\ + a\,y^3(xa_2 + ya_3 + a_1) - \left(-3a\,y^2 x - 2by\right)(xb_2 + yb_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-a^2x^2y^6a_3 - 2abx\,y^5a_3 - b^2y^4a_3 + 3a\,x^2y^2b_2 + 2ax\,y^3a_2 + 2ax\,y^3b_3 \\ + a\,y^4a_3 + 3ax\,y^2b_1 + a\,y^3a_1 + 2bxyb_2 + b\,y^2a_2 + b\,y^2b_3 + 2byb_1 + b_2 = 0$$

Setting the numerator to zero gives

$$-a^2x^2y^6a_3 - 2abx\,y^5a_3 - b^2y^4a_3 + 3a\,x^2y^2b_2 + 2ax\,y^3a_2 + 2ax\,y^3b_3 \\ + a\,y^4a_3 + 3ax\,y^2b_1 + a\,y^3a_1 + 2bxyb_2 + b\,y^2a_2 + b\,y^2b_3 + 2byb_1 + b_2 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-a^2a_3v_1^2v_2^6 - 2aba_3v_1v_2^5 - b^2a_3v_2^4 + 2aa_2v_1v_2^3 + aa_3v_2^4 + 3ab_2v_1^2v_2^2 + 2ab_3v_1v_2^3 \\ + aa_1v_2^3 + 3ab_1v_1v_2^2 + ba_2v_2^2 + 2bb_2v_1v_2 + bb_3v_2^2 + 2bb_1v_2 + b_2 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&-a^2 a_3 v_1^2 v_2^6 + 3ab_2 v_1^2 v_2^2 - 2aba_3 v_1 v_2^5 + (2aa_2 + 2ab_3) v_1 v_2^3 + 3ab_1 v_1 v_2^2 \\
&+ 2bb_2 v_1 v_2 + \left(-b^2 a_3 + aa_3\right) v_2^4 + aa_1 v_2^3 + (ba_2 + bb_3) v_2^2 + 2bb_1 v_2 + b_2 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
b_2 &= 0 \\
aa_1 &= 0 \\
3ab_1 &= 0 \\
3ab_2 &= 0 \\
-a^2 a_3 &= 0 \\
2bb_1 &= 0 \\
2bb_2 &= 0 \\
-2aba_3 &= 0 \\
-b^2 a_3 + aa_3 &= 0 \\
2aa_2 + 2ab_3 &= 0 \\
ba_2 + bb_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= -b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= -x \\
\eta &= y
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Therefore

$$\begin{aligned}
\frac{dy}{dx} &= \frac{\eta}{\xi} \\
&= \frac{y}{-x} \\
&= -\frac{y}{x}
\end{aligned}$$

This is easily solved to give

$$y = \frac{c_1}{x}$$

Where now the coordinate R is taken as the constant of integration. Hence

$$R = xy$$

And S is found from

$$\begin{aligned}
dS &= \frac{dx}{\xi} \\
&= \frac{dx}{-x}
\end{aligned}$$

Integrating gives

$$\begin{aligned}
S &= \int \frac{dx}{T} \\
&= -\ln(x)
\end{aligned}$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -xa\, y^3 - b\, y^2$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= y \\
R_y &= x \\
S_x &= -\frac{1}{x} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{1}{xy\left(x^2 a\, y^2 + bxy - 1\right)} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{1}{R\left(a\, R^2 + bR - 1\right)}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \frac{\ln\left(a\, R^2 + bR - 1\right)}{2} - \frac{b\, \operatorname{arctanh}\left(\frac{2Ra+b}{\sqrt{b^2+4a}}\right)}{\sqrt{b^2+4a}} - \ln(R) + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
-\ln(x) = \frac{\ln\left(y^2 a\, x^2 + bxy - 1\right)}{2} - \frac{b\, \operatorname{arctanh}\left(\frac{2yax+b}{\sqrt{b^2+4a}}\right)}{\sqrt{b^2+4a}} - \ln(xy) + c_1
$$

Which simplifies to

$$
-\ln(x) = \frac{\ln\left(y^2 a\, x^2 + bxy - 1\right)}{2} - \frac{b\, \operatorname{arctanh}\left(\frac{2yax+b}{\sqrt{b^2+4a}}\right)}{\sqrt{b^2+4a}} - \ln(xy) + c_1
$$

Summary

The solution(s) found are the following

$$
-\ln(x) = \frac{\ln\left(y^2 a\, x^2 + bxy - 1\right)}{2} - \frac{b\, \operatorname{arctanh}\left(\frac{2yax+b}{\sqrt{b^2+4a}}\right)}{\sqrt{b^2+4a}} - \ln(xy) + c_1 \tag{1}
$$

Verification of solutions

$$-\ln(x) = \frac{\ln(y^2 a\, x^2 + bxy - 1)}{2} - \frac{b \operatorname{arctanh}\left(\frac{2yax+b}{\sqrt{b^2+4a}}\right)}{\sqrt{b^2+4a}} - \ln(xy) + c_1$$

Verified OK.

1.2.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$
\begin{aligned}
\mathrm{d}y &= \left(-xa\,y^{3}-b\,y^{2}\right)\mathrm{d}x \\
\left(xa\,y^{3}+b\,y^{2}\right)\mathrm{d}x+\mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= xa\,y^{3}+b\,y^{2} \\
N(x,y) &= 1
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(xa\,y^{3}+b\,y^{2}\right) \\
&= 3a\,y^{2}x+2by
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right) \\
&= 1\left(\left(3a\,y^{2}x+2by\right)-(0)\right) \\
&= 3a\,y^{2}x+2by
\end{aligned}
$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right) \\
&= \frac{1}{y^{2}\left(axy+b\right)}\left((0)-\left(3a\,y^{2}x+2by\right)\right) \\
&= \frac{-3axy-2b}{y\left(axy+b\right)}
\end{aligned}
$$

Since B depends on x, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$\begin{aligned} R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} \\ &= \frac{(0) - (3a\,y^2x + 2by)}{x\,(xa\,y^3 + b\,y^2) - y\,(1)} \\ &= \frac{-3axy - 2b}{x^2a\,y^2 + bxy - 1} \end{aligned}$$

Replacing all powers of terms xy by t gives

$$R = \frac{-3at - 2b}{a\,t^2 + bt - 1}$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$\begin{aligned} \mu &= e^{\int R \,\mathrm{d}t} \\ &= e^{\int \left(\frac{-3at-2b}{a\,t^2+bt-1}\right) \mathrm{d}t} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-\frac{3\ln\left(a\,t^2+bt-1\right)}{2} + \frac{b\,\operatorname{arctanh}\left(\frac{2at+b}{\sqrt{b^2+4a}}\right)}{\sqrt{b^2+4a}}} \\ &= \frac{\mathrm{e}^{\frac{b\,\operatorname{arctanh}\left(\frac{2at+b}{\sqrt{b^2+4a}}\right)}{\sqrt{b^2+4a}}}}{(a\,t^2 + bt - 1)^{\frac{3}{2}}} \end{aligned}$$

Now t is replaced back with xy giving

$$\mu = \frac{\mathrm{e}^{\frac{b\,\operatorname{arctanh}\left(\frac{2axy+b}{\sqrt{b^2+4a}}\right)}{\sqrt{b^2+4a}}}}{(x^2a\,y^2 + bxy - 1)^{\frac{3}{2}}}$$

Multiplying M and N by this integrating factor gives new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{\mathrm{e}^{\frac{b\,\operatorname{arctanh}\left(\frac{2axy+b}{\sqrt{b^2+4a}}\right)}{\sqrt{b^2+4a}}}}{\left(x^2 a\, y^2 + bxy - 1\right)^{\frac{3}{2}}} \left(xa\, y^3 + b\, y^2\right) \\
&= \frac{y^2 (axy+b)\, \mathrm{e}^{\frac{b\,\operatorname{arctanh}\left(\frac{2axy+b}{\sqrt{b^2+4a}}\right)}{\sqrt{b^2+4a}}}}{\left(x^2 a\, y^2 + bxy - 1\right)^{\frac{3}{2}}}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{\mathrm{e}^{\frac{b\,\operatorname{arctanh}\left(\frac{2axy+b}{\sqrt{b^2+4a}}\right)}{\sqrt{b^2+4a}}}}{\left(x^2 a\, y^2 + bxy - 1\right)^{\frac{3}{2}}} (1) \\
&= \frac{\mathrm{e}^{\frac{b\,\operatorname{arctanh}\left(\frac{2axy+b}{\sqrt{b^2+4a}}\right)}{\sqrt{b^2+4a}}}}{\left(x^2 a\, y^2 + bxy - 1\right)^{\frac{3}{2}}}
\end{aligned}
$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N} \frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{y^2 (axy+b)\, \mathrm{e}^{\frac{b\,\operatorname{arctanh}\left(\frac{2axy+b}{\sqrt{b^2+4a}}\right)}{\sqrt{b^2+4a}}}}{\left(x^2 a\, y^2 + bxy - 1\right)^{\frac{3}{2}}}\right) + \left(\frac{\mathrm{e}^{\frac{b\,\operatorname{arctanh}\left(\frac{2axy+b}{\sqrt{b^2+4a}}\right)}{\sqrt{b^2+4a}}}}{\left(x^2 a\, y^2 + bxy - 1\right)^{\frac{3}{2}}}\right) \frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x
$$

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \frac{y^2(axy+b)\,\mathrm{e}^{\frac{b\,\operatorname{arctanh}\left(\frac{2axy+b}{\sqrt{b^2+4a}}\right)}{\sqrt{b^2+4a}}}}{\left(x^2a\,y^2+bxy-1\right)^{\frac{3}{2}}} \mathrm{d}x$$

$$\phi = -\frac{y\,\mathrm{e}^{\frac{b\,\operatorname{arctanh}\left(\frac{2axy+b}{\sqrt{b^2+4a}}\right)}{\sqrt{b^2+4a}}}}{\sqrt{x^2a\,y^2+bxy-1}} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\begin{aligned}
\frac{\partial \phi}{\partial y} &= \frac{y\,\mathrm{e}^{\frac{b\,\operatorname{arctanh}\left(\frac{2axy+b}{\sqrt{b^2+4a}}\right)}{\sqrt{b^2+4a}}}\left(2a\,x^2y+bx\right)}{2\left(x^2a\,y^2+bxy-1\right)^{\frac{3}{2}}} - \frac{\mathrm{e}^{\frac{b\,\operatorname{arctanh}\left(\frac{2axy+b}{\sqrt{b^2+4a}}\right)}{\sqrt{b^2+4a}}}}{\sqrt{x^2a\,y^2+bxy-1}} \\
&\quad - \frac{2ybax\,\mathrm{e}^{\frac{b\,\operatorname{arctanh}\left(\frac{2axy+b}{\sqrt{b^2+4a}}\right)}{\sqrt{b^2+4a}}}}{\sqrt{x^2a\,y^2+bxy-1}\,\left(b^2+4a\right)\left(-\frac{(2axy+b)^2}{b^2+4a}+1\right)} + f'(y) \\
&= \frac{\mathrm{e}^{\frac{b\,\operatorname{arctanh}\left(\frac{2axy+b}{\sqrt{b^2+4a}}\right)}{\sqrt{b^2+4a}}}}{\left(x^2a\,y^2+bxy-1\right)^{\frac{3}{2}}} + f'(y)
\end{aligned} \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{\mathrm{e}^{\frac{b\,\operatorname{arctanh}\left(\frac{2axy+b}{\sqrt{b^2+4a}}\right)}{\sqrt{b^2+4a}}}}{\left(x^2a\,y^2+bxy-1\right)^{\frac{3}{2}}}$. Therefore equation (4) becomes

$$\frac{\mathrm{e}^{\frac{b\,\operatorname{arctanh}\left(\frac{2axy+b}{\sqrt{b^2+4a}}\right)}{\sqrt{b^2+4a}}}}{\left(x^2a\,y^2+bxy-1\right)^{\frac{3}{2}}} = \frac{\mathrm{e}^{\frac{b\,\operatorname{arctanh}\left(\frac{2axy+b}{\sqrt{b^2+4a}}\right)}{\sqrt{b^2+4a}}}}{\left(x^2a\,y^2+bxy-1\right)^{\frac{3}{2}}} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{y\,\mathrm{e}^{\frac{b\,\operatorname{arctanh}\left(\frac{2axy+b}{\sqrt{b^2+4a}}\right)}{\sqrt{b^2+4a}}}}{\sqrt{x^2a\,y^2+bxy-1}} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{y\,\mathrm{e}^{\frac{b\,\operatorname{arctanh}\left(\frac{2axy+b}{\sqrt{b^2+4a}}\right)}{\sqrt{b^2+4a}}}}{\sqrt{x^2a\,y^2+bxy-1}}$$

Summary

The solution(s) found are the following

$$-\frac{y\,\mathrm{e}^{\frac{b\,\operatorname{arctanh}\left(\frac{2yax+b}{\sqrt{b^2+4a}}\right)}{\sqrt{b^2+4a}}}}{\sqrt{y^2a\,x^2+bxy-1}} = c_1 \tag{1}$$

Verification of solutions

$$-\frac{y\,\mathrm{e}^{\frac{b\,\operatorname{arctanh}\left(\frac{2yax+b}{\sqrt{b^2+4a}}\right)}{\sqrt{b^2+4a}}}}{\sqrt{y^2a\,x^2+bxy-1}} = c_1$$

Verified OK.

1.2.3 Solving as abelFirstKind ode

This is Abel first kind ODE, it has the form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2 + f_3(x)y^3$$

Comparing the above to given ODE which is

$$y' = -axy^3 - by^2 \tag{1}$$

Therefore

$$\begin{aligned} f_0(x) &= 0 \\ f_1(x) &= 0 \\ f_2(x) &= -b \\ f_3(x) &= -ax \end{aligned}$$

Since $f_2(x) = -b$ is not zero, then the first step is to apply the following transformation to remove f_2. Let $y = u(x) - \frac{f_2}{3f_3}$ or

$$\begin{aligned} y &= u(x) - \left(\frac{-b}{-3ax}\right) \\ &= u(x) - \frac{b}{3ax} \end{aligned}$$

The above transformation applied to (1) gives a new ODE as

$$u'(x) = -axu(x)^3 + \frac{u(x)\,b^2}{3ax} - \frac{2b^3}{27a^2x^2} - \frac{b}{3a\,x^2} \tag{2}$$

The above ODE (2) can now be solved as separable.

Writing the ode as

$$\begin{aligned} u'(x) &= -\frac{27a^3x^3u^3 - 9u\,b^2ax + 2b^3 + 9ab}{27a^2x^2} \\ u'(x) &= \omega(x,u) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_u - \xi_x) - \omega^2\xi_u - \omega_x\xi - \omega_u\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = ua_3 + xa_2 + a_1 \tag{1E}$$

$$\eta = ub_3 + xb_2 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 - \frac{(27a^3x^3u^3 - 9u\,b^2ax + 2b^3 + 9ab)\,(b_3 - a_2)}{27a^2x^2} \\ &- \frac{(27a^3x^3u^3 - 9u\,b^2ax + 2b^3 + 9ab)^2\,a_3}{729a^4x^4} - \left(-\frac{81x^2a^3u^3 - 9a\,b^2u}{27a^2x^2}\right. \\ &\left. + \frac{2a^3x^3u^3 - \frac{2}{3}u\,b^2ax + \frac{4}{27}b^3 + \frac{2}{3}ab}{a^2x^3}\right)(ua_3 + xa_2 + a_1) \\ &+ \frac{(81a^3u^2x^3 - 9a\,b^2x)\,(ub_3 + xb_2 + b_1)}{27a^2x^2} = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{729a^6u^6x^6a_3 - 486a^4b^2u^4x^4a_3 - 729a^5u^4x^4a_3 - 1458a^5u^3x^5a_2 - 1458a^5u^3x^5b_3 - 2187a^5u^2x^6b_2 - 729a^5}{}$$

$$= 0$$

Setting the numerator to zero gives

$$
\begin{aligned}
&-729a^6u^6x^6a_3 + 486a^4b^2u^4x^4a_3 + 729a^5u^4x^4a_3 + 1458a^5u^3x^5a_2 \\
&+ 1458a^5u^3x^5b_3 + 2187a^5u^2x^6b_2 + 729a^5u^3x^4a_1 + 2187a^5u^2x^5b_1 \\
&- 108a^3b^3u^3x^3a_3 - 486a^4b\,u^3x^3a_3 - 81a^2b^4u^2x^2a_3 + 243a^3b^2u^2x^2a_3 \\
&- 243a^3b^2x^4b_2 + 729b_2a^4x^4 + 243a^3b^2u\,x^2a_1 - 243a^3b^2x^3b_1 + 36a\,b^5uxa_3 \\
&+ 54a^2b^3uxa_3 - 54a^2b^3x^2a_2 - 54a^2b^3x^2b_3 - 486a^3buxa_3 - 243a^3b\,x^2a_2 \\
&- 243a^3b\,x^2b_3 - 108a^2b^3xa_1 - 4b^6a_3 - 486a^3bxa_1 - 36a\,b^4a_3 - 81a^2b^2a_3 = 0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{u, x\}$ in them.

$$
\{u, x\}
$$

The following substitution is now made to be able to collect on all terms with $\{u, x\}$ in them

$$
\{u = v_1, x = v_2\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&-729a^6a_3v_1^6v_2^6 + 486a^4b^2a_3v_1^4v_2^4 + 1458a^5a_2v_1^3v_2^5 + 729a^5a_3v_1^4v_2^4 \\
&+ 2187a^5b_2v_1^2v_2^6 + 1458a^5b_3v_1^3v_2^5 + 729a^5a_1v_1^3v_2^4 + 2187a^5b_1v_1^2v_2^5 \\
&- 108a^3b^3a_3v_1^3v_2^3 - 486a^4ba_3v_1^3v_2^3 - 81a^2b^4a_3v_1^2v_2^2 \\
&+ 243a^3b^2a_3v_1^2v_2^2 - 243a^3b^2b_2v_2^4 + 729a^4b_2v_2^4 + 243a^3b^2a_1v_1v_2^2 \\
&- 243a^3b^2b_1v_2^3 + 36a\,b^5a_3v_1v_2 - 54a^2b^3a_2v_2^2 + 54a^2b^3a_3v_1v_2 \\
&- 54a^2b^3b_3v_2^2 - 243a^3ba_2v_2^2 - 486a^3ba_3v_1v_2 - 243a^3bb_3v_2^2 \\
&- 108a^2b^3a_1v_2 - 4b^6a_3 - 486a^3ba_1v_2 - 36a\,b^4a_3 - 81a^2b^2a_3 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&-729a^6a_3v_1^6v_2^6 + \left(486a^4b^2a_3 + 729a^5a_3\right)v_1^4v_2^4 \\
&+ \left(1458a^5a_2 + 1458a^5b_3\right)v_1^3v_2^5 + 729a^5a_1v_1^3v_2^4 \\
&+ \left(-108a^3b^3a_3 - 486a^4ba_3\right)v_1^3v_2^3 + 2187a^5b_2v_1^2v_2^6 + 2187a^5b_1v_1^2v_2^5 \\
&+ \left(-81a^2b^4a_3 + 243a^3b^2a_3\right)v_1^2v_2^2 + 243a^3b^2a_1v_1v_2^2 \\
&+ \left(36a\,b^5a_3 + 54a^2b^3a_3 - 486a^3ba_3\right)v_1v_2 + \left(-243a^3b^2b_2 + 729a^4b_2\right)v_2^4 \\
&- 243a^3b^2b_1v_2^3 + \left(-54a^2b^3a_2 - 54a^2b^3b_3 - 243a^3ba_2 - 243a^3bb_3\right)v_2^2 \\
&+ \left(-108a^2b^3a_1 - 486a^3ba_1\right)v_2 - 4b^6a_3 - 36a\,b^4a_3 - 81a^2b^2a_3 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
729a^5a_1 &= 0 \\
2187a^5b_1 &= 0 \\
2187a^5b_2 &= 0 \\
-729a^6a_3 &= 0 \\
243a^3b^2a_1 &= 0 \\
-243a^3b^2b_1 &= 0 \\
-108a^2b^3a_1 - 486a^3ba_1 &= 0 \\
-81a^2b^4a_3 + 243a^3b^2a_3 &= 0 \\
-108a^3b^3a_3 - 486a^4ba_3 &= 0 \\
486a^4b^2a_3 + 729a^5a_3 &= 0 \\
-243a^3b^2b_2 + 729a^4b_2 &= 0 \\
1458a^5a_2 + 1458a^5b_3 &= 0 \\
-4b^6a_3 - 36a\,b^4a_3 - 81a^2b^2a_3 &= 0 \\
36a\,b^5a_3 + 54a^2b^3a_3 - 486a^3ba_3 &= 0 \\
-54a^2b^3a_2 - 54a^2b^3b_3 - 243a^3ba_2 - 243a^3bb_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= -b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= -x \\
\eta &= u
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,u)\,\xi \\
&= u - \left(-\frac{27a^3x^3u^3 - 9u\,b^2ax + 2b^3 + 9ab}{27a^2x^2}\right)(-x) \\
&= \frac{-27a^3x^3u^3 + 9u\,b^2ax + 27u\,a^2x - 2b^3 - 9ab}{27a^2x} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,u) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{du}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial u}\right) S(x,u) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-27a^3x^3u^3 + 9u\,b^2ax + 27u\,a^2x - 2b^3 - 9ab}{27a^2x}} dy
\end{aligned}
$$

Which results in

$$
S = \ln(3uxa - b) - \frac{\ln(9a^2x^2u^2 + 3abux - 2b^2 - 9a)}{2} + \frac{axb \operatorname{arctanh}\left(\frac{18u\,a^2x^2 + 3abx}{9\sqrt{a^2b^2x^2 + 4a^3x^2}}\right)}{\sqrt{a^2b^2x^2 + 4a^3x^2}}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,u)S_u}{R_x + \omega(x,u)R_u} \tag{2}
$$

Where in the above R_x, R_u, S_x, S_u are all partial derivatives and $\omega(x,u)$ is the right hand side of the original ode given by

$$
\omega(x,u) = -\frac{27a^3x^3u^3 - 9u\,b^2ax + 2b^3 + 9ab}{27a^2x^2}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_u &= 0 \\
S_x &= -\frac{27a^2u}{27a^3x^3u^3 - 27u\,a^2x - 9b\,(bux - 1)\,a + 2b^3} \\
S_u &= -\frac{27a^2x}{27a^3x^3u^3 - 27u\,a^2x - 9b\,(bux - 1)\,a + 2b^3}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{1}{x} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, u in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{1}{R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \ln(R) + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, u coordinates. This results in

$$
\frac{2\ln(3u(x)\,ax - b)\sqrt{b^2+4a} - \ln\left(9a^2x^2u(x)^2 + (3bu(x)\,x - 9)\,a - 2b^2\right)\sqrt{b^2+4a} + 2b\,\operatorname{arctanh}\left(\frac{6u(x)ax+b}{3\sqrt{b^2+4a}}\right.}{2\sqrt{b^2+4a}}
$$

Which simplifies to

$$
\frac{2\ln(3u(x)\,ax - b)\sqrt{b^2+4a} - \ln\left(9a^2x^2u(x)^2 + (3bu(x)\,x - 9)\,a - 2b^2\right)\sqrt{b^2+4a} + 2b\,\operatorname{arctanh}\left(\frac{6u(x)ax+b}{3\sqrt{b^2+4a}}\right.}{2\sqrt{b^2+4a}}
$$

Substituting $u = y - \frac{b}{3ax}$ in the above solution gives

$$
\frac{2\ln\left(3\left(y - \frac{b}{3ax}\right)ax - b\right)\sqrt{b^2+4a} - \ln\left(9a^2x^2\left(y - \frac{b}{3ax}\right)^2 + \left(3b\left(y - \frac{b}{3ax}\right)x - 9\right)a - 2b^2\right)\sqrt{b^2+4a} + 2b\ \mathrm{ar}}{2\sqrt{b^2+4a}}
$$

Summary

The solution(s) found are the following

$$\frac{2\ln\left(3\left(y-\frac{b}{3ax}\right)ax-b\right)\sqrt{b^2+4a}-\ln\left(9a^2x^2\left(y-\frac{b}{3ax}\right)^2+\left(3b\left(y-\frac{b}{3ax}\right)x-9\right)a-2b^2\right)\sqrt{b^2+4a}+2b\,\mathrm{ar}}{2\sqrt{b^2+4a}} \tag{1}$$

$= \ln(x) + c_1$

Verification of solutions

$$\frac{2\ln\left(3\left(y-\frac{b}{3ax}\right)ax-b\right)\sqrt{b^2+4a}-\ln\left(9a^2x^2\left(y-\frac{b}{3ax}\right)^2+\left(3b\left(y-\frac{b}{3ax}\right)x-9\right)a-2b^2\right)\sqrt{b^2+4a}+2b\,\mathrm{ar}}{2\sqrt{b^2+4a}}$$

$= \ln(x) + c_1$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 103

```
dsolve(a*x*y(x)^3+b*y(x)^2+diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{e^{\operatorname{RootOf}\left(2\sqrt{b^2+4a}\,b\,\operatorname{arctanh}\left(\frac{2a\,e^{\_Z}+b}{\sqrt{b^2+4a}}\right)-\ln\left(x^2\left(a\,e^{2\_Z}+b\,e^{\_Z}-1\right)\right)b^2+2c_1b^2+2\_Z b^2-4\ln\left(x^2\left(a\,e^{2\_Z}+b\,e^{\_Z}-1\right)\right)a+8c_1a+8\_Z a\right)}}{x}$$

✓ Solution by Mathematica

Time used: 0.192 (sec). Leaf size: 103

```
DSolve[a*x*y[x]^3+b*y[x]^2+y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[-\frac{b^2\left(\frac{2\arctan\left(\frac{-2axy(x)-b}{b\sqrt{-\frac{4a}{b^2}-1}}\right)}{\sqrt{-\frac{4a}{b^2}-1}}-\log\left(\frac{a(-x)y(x)(-axy(x)-b)-a}{a^2x^2y(x)^2}\right)\right)}{2a}=-\frac{b^2\log(x)}{a}+c_1,y(x)\right]$$

1.3 problem problem 46

Internal problem ID [4677]
Internal file name [OUTPUT/4170_Sunday_June_05_2022_12_32_42_PM_72807646/index.tex]

Book: Differential Gleichungen, Kamke, 3rd ed, Abel ODEs
Section: Abel ODE's with constant invariant
Problem number: problem 46.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"abelFirstKind"**

Maple gives the following as the ode type

`[_Abel]`

$$\boxed{y' - x^a y^3 + 3y^2 - x^{-a} y = x^{-2a} - a\, x^{-a-1}}$$

1.3.1 Solving as abelFirstKind ode

This is Abel first kind ODE, it has the form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2 + f_3(x)y^3$$

Comparing the above to given ODE which is

$$y' = x^a y^3 - 3y^2 + x^{-a} y + x^{-2a} - a\, x^{-a-1} \tag{1}$$

Therefore

$$\begin{aligned}
f_0(x) &= x^{-2a} - \frac{x^{-a} a}{x} \\
f_1(x) &= x^{-a} \\
f_2(x) &= -3 \\
f_3(x) &= x^a
\end{aligned}$$

Since $f_2(x) = -3$ is not zero, then the first step is to apply the following transformation to remove f_2. Let $y = u(x) - \frac{f_2}{3 f_3}$ or

$$\begin{aligned}
y &= u(x) - \left(\frac{-3}{3x^a}\right) \\
&= u(x) + x^{-a}
\end{aligned}$$

The above transformation applied to (1) gives a new ODE as

$$u'(x) = x^a u(x)^3 - 2x^{-a} u(x) \tag{2}$$

The above ODE (2) can now be solved as separable.

Writing the ode as

$$\begin{aligned} u'(x) &= u\left(x^{2a} u^2 - 2\right) x^{-a} \\ u'(x) &= \omega(x, u) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_u - \xi_x) - \omega^2 \xi_u - \omega_x \xi - \omega_u \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 1: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x,u) = 0$$
$$\eta(x,u) = u^3 e^{-\frac{4x\,x^{-a}}{a-1}} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,u) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{du}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial u}\right) S(x,u) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{u^3 e^{-\frac{4x\,x^{-a}}{a-1}}} dy \end{aligned}$$

Which results in

$$S = -\frac{e^{\frac{4x^{-a+1}}{a-1}}}{2u^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,u) S_u}{R_x + \omega(x,u) R_u} \tag{2}$$

Where in the above R_x, R_u, S_x, S_u are all partial derivatives and $\omega(x,u)$ is the right hand side of the original ode given by

$$\omega(x,u) = u\left(x^{2a}u^2 - 2\right) x^{-a}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_u &= 0 \\
S_x &= \frac{2x^{-a} e^{\frac{4x^{-a+1}}{a-1}}}{u^2} \\
S_u &= \frac{e^{\frac{4x^{-a+1}}{a-1}}}{u^3}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = x^a e^{\frac{4x^{-a+1}}{a-1}} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, u in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = R^a e^{\frac{4R^{-a+1}}{a-1}}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$
S(R) = \frac{2^{-\frac{2a}{-a+1}-\frac{2}{-a+1}} \left(\frac{1}{-a+1}\right)^{\frac{a+1}{a-1}} \left(-\frac{2^{-3+\frac{2a}{-a+1}+\frac{2}{-a+1}+\frac{2}{a-1}} (a-1) R^{-\frac{a^2}{-a+1}+\frac{1}{-a+1}+a-1} \left(\frac{1}{-a+1}\right)^{-\frac{a+1}{a-1}} \left(-\frac{4R^{-a+1}a^2}{-a+1}+\frac{8R^{-a+1}a}{-a+1}-\right.}{(a+1}\right.}{} \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, u coordinates. This results in

$$
-\frac{e^{\frac{4x^{-a+1}}{a-1}}}{2u(x)^2} = \frac{2^{-\frac{2a}{-a+1}-\frac{2}{-a+1}} \left(\frac{1}{-a+1}\right)^{\frac{a+1}{a-1}} \left(-\frac{2^{-3+\frac{2a}{-a+1}+\frac{2}{-a+1}+\frac{2}{a-1}} (a-1) x^{-\frac{a^2}{-a+1}+\frac{1}{-a+1}+a-1} \left(\frac{1}{-a+1}\right)^{-\frac{a+1}{a-1}} \left(-\frac{4x^{-a+1}a^2}{-a+1}+\frac{8x^{-a+1}}{-a+1}\right.}{}\right.}{}
$$

Which simplifies to

$$
\frac{-32 e^{\frac{i\pi+2x^{-a+1}}{a-1}} u(x)^2 (a-1)^{\frac{-2+a}{a-1}} \left(\left(x-\frac{x^{2a-1}}{4}\right) 2^{\frac{-3a+5}{a-1}} + \frac{x^{2a-1} 4^{\frac{1}{a-1}}}{32}\right) \mathrm{WhittakerM}\left(-\frac{1}{a-1}, \frac{a-3}{2a-2}, -\frac{4x^{-a+1}}{a-1}\right) -}{2(a+1)(a-3)u(x)^2}
$$

Substituting $u = y + x^{-a}$ in the above solution gives

$$-\frac{32\,\mathrm{e}^{\frac{i\pi+2x^{-a+1}}{a-1}} (y+x^{-a})^2 (a-1)^{\frac{-2+a}{a-1}} \left(\left(x-\frac{x^{2a-1}}{4}\right) 2^{\frac{-3a+5}{a-1}} + \frac{x^{2a-1} 4^{\frac{1}{a-1}}}{32}\right) \mathrm{WhittakerM}\left(-\frac{1}{a-1}, \frac{a-3}{2a-2}, -\frac{4x^{-a+}}{a-1}\right.}{2(a+1)(a-3)(y-}$$

Summary

The solution(s) found are the following

$$-\frac{32\,\mathrm{e}^{\frac{i\pi+2x^{-a+1}}{a-1}} (y+x^{-a})^2 (a-1)^{\frac{-2+a}{a-1}} \left(\left(x-\frac{x^{2a-1}}{4}\right) 2^{\frac{-3a+5}{a-1}} + \frac{x^{2a-1} 4^{\frac{1}{a-1}}}{32}\right) \mathrm{WhittakerM}\left(-\frac{1}{a-1}, \frac{a-3}{2a-2}, -\frac{4x^{-a+}}{a-1}\right.}{2(a+1)(a-3)(y-} \tag{1}$$

$= 0$

Verification of solutions

$$-\frac{32\,\mathrm{e}^{\frac{i\pi+2x^{-a+1}}{a-1}} (y+x^{-a})^2 (a-1)^{\frac{-2+a}{a-1}} \left(\left(x-\frac{x^{2a-1}}{4}\right) 2^{\frac{-3a+5}{a-1}} + \frac{x^{2a-1} 4^{\frac{1}{a-1}}}{32}\right) \mathrm{WhittakerM}\left(-\frac{1}{a-1}, \frac{a-3}{2a-2}, -\frac{4x^{-a+}}{a-1}\right.}{2(a+1)(a-3)(y-}$$

$= 0$

Warning, solution could not be verified

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 2084

```
dsolve(diff(y(x),x)-x^a*y(x)^3+3*y(x)^2-x^(-a)*y(x)-x^(-2*a)+a*x^(-a-1) = 0,y(x), singsol=al
```

Expression too large to display
Expression too large to display

✓ Solution by Mathematica

Time used: 13.424 (sec). Leaf size: 231

```
DSolve[y'[x]-x^a*y[x]^3+3*y[x]^2-x^(-a)*y[x]-x^(-2*a)+a*x^(-a-1) == 0,y[x],x,IncludeSingular
```

$$y(x) \to x^{-a} - \frac{e^{\frac{2x^{1-a}}{a-1}}}{\sqrt{-\frac{2^{\frac{3a+1}{a-1}} x^{a+1} \left(\frac{x^{1-a}}{1-a}\right)^{\frac{a+1}{a-1}} \Gamma\left(\frac{a+1}{1-a}, -\frac{4x^{1-a}}{a-1}\right)}{a-1} + c_1}}$$
$$y(x) \to x^{-a} + \frac{e^{\frac{2x^{1-a}}{a-1}}}{\sqrt{-\frac{2^{\frac{3a+1}{a-1}} x^{a+1} \left(\frac{x^{1-a}}{1-a}\right)^{\frac{a+1}{a-1}} \Gamma\left(\frac{a+1}{1-a}, -\frac{4x^{1-a}}{a-1}\right)}{a-1} + c_1}}$$
$$y(x) \to x^{-a}$$

1.4 problem problem 51

Internal problem ID [4678]
Internal file name [OUTPUT/4171_Sunday_June_05_2022_12_35_19_PM_64352952/index.tex]

Book: Differential Gleichungen, Kamke, 3rd ed, Abel ODEs
Section: Abel ODE's with constant invariant
Problem number: problem 51.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"abelFirstKind"**

Maple gives the following as the ode type

```
[_Abel]
```

Unable to solve or complete the solution.

$$y' - (y - f(x))(y - g(x))\left(y - \frac{f(x)a + bg(x)}{a + b}\right)h(x) - \frac{f'(x)(y - g(x))}{f(x) - g(x)} - \frac{g'(x)(y - f(x))}{g(x) - f(x)} = 0$$

1.4.1 Solving as abelFirstKind ode

This is Abel first kind ODE, it has the form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2 + f_3(x)y^3$$

Comparing the above to given ODE which is

$$y' = -\frac{(-f(x)h(x)a - f(x)h(x)b + g(x)h(x)a + g(x)h(x)b)y^3}{(a+b)(f(x) - g(x))} - \frac{(2f(x)^2 h(x)a + f(x)^2 h(x)b - f(x)}{} \tag{1}$$

Therefore

$$
\begin{aligned}
f_0(x) &= -\frac{f(x)^3\,g(x)\,h(x)\,a}{(a+b)\,(f(x)-g(x))} + \frac{f(x)^2\,g(x)^2\,h(x)\,a}{(a+b)\,(f(x)-g(x))} - \frac{f(x)^2\,g(x)^2\,h(x)\,b}{(a+b)\,(f(x)-g(x))} + \frac{f(x)\,g(x)^3\,h(x)\,b}{(a+b)\,(f(x)-g(x}\\
f_1(x) &= \frac{af'(x)}{(a+b)\,(f(x)-g(x))} + \frac{bf'(x)}{(a+b)\,(f(x)-g(x))} - \frac{ag'(x)}{(a+b)\,(f(x)-g(x))} - \frac{g'(x)\,b}{(a+b)\,(f(x)-g(x)}\\
f_2(x) &= -\frac{2f(x)^2\,h(x)\,a}{(a+b)\,(f(x)-g(x))} - \frac{f(x)^2\,h(x)\,b}{(a+b)\,(f(x)-g(x))} + \frac{g(x)^2\,h(x)\,a}{(a+b)\,(f(x)-g(x))} + \frac{2g(x)^2\,h(x)\,b}{(a+b)\,(f(x)-g(}\\
f_3(x) &= \frac{af(x)\,h(x)}{(a+b)\,(f(x)-g(x))} + \frac{f(x)\,h(x)\,b}{(a+b)\,(f(x)-g(x))} - \frac{g(x)\,h(x)\,a}{(a+b)\,(f(x)-g(x))} - \frac{g(x)\,h(x)\,b}{(a+b)\,(f(x)-g(x)}
\end{aligned}
$$

Since $f_2(x) = -\frac{2f(x)^2h(x)a}{(a+b)(f(x)-g(x))} - \frac{f(x)^2h(x)b}{(a+b)(f(x)-g(x))} + \frac{g(x)^2h(x)a}{(a+b)(f(x)-g(x))} + \frac{2g(x)^2h(x)b}{(a+b)(f(x)-g(x))} + \frac{f(x)g(x)h(x)a}{(a+b)(f(x)-g(x))} - \frac{f(x)g(x)h(x)b}{(a+b)(f(x)-g(x))}$ is not zero, then the first step is to apply the following transformation to remove f_2. Let $y = u(x) - \frac{f_2}{3f_3}$ or

$$
\begin{aligned}
y &= u(x) - \left(\frac{-\frac{2f(x)^2h(x)a}{(a+b)(f(x)-g(x))} - \frac{f(x)^2h(x)b}{(a+b)(f(x)-g(x))} + \frac{g(x)^2h(x)a}{(a+b)(f(x)-g(x))} + \frac{2g(x)^2h(x)b}{(a+b)(f(x)-g(x))} + \frac{f(x)g(x)h(x)a}{(a+b)(f(x)-g(x))} - \frac{f(x)g}{(a+b)(f}}{\frac{3af(x)h(x)}{(a+b)(f(x)-g(x))} + \frac{3f(x)h(x)b}{(a+b)(f(x)-g(x))} - \frac{3g(x)h(x)a}{(a+b)(f(x)-g(x))} - \frac{3g(x)h(x)b}{(a+b)(f(x)-g(x))}} \right.\\
&= \frac{f(x)\,(2a+b) + (a+2b)\,g(x) + 3(a+b)\,u(x)}{3a+3b}
\end{aligned}
$$

The above transformation applied to (1) gives a new ODE as

$$
\text{Expression too large to display} \tag{2}
$$

This is Abel first kind ODE, it has the form

$$
u'(x) = f_0(x) + f_1(x)u(x) + f_2(x)u(x)^2 + f_3(x)u(x)^3
$$

Comparing the above to given ODE which is

$$
u'(x) = \text{Expression too large to display} \tag{1}
$$

Therefore

$$
\begin{aligned}
f_0(x) &= -\frac{f(x)^4\,h(x)\,a\,b^2}{9\,(a+b)^3\,(f(x)-g(x))} + \frac{f(x)^4\,h(x)\,a^2b}{9\,(a+b)^3\,(f(x)-g(x))} + \frac{2f(x)^4\,h(x)\,a^3}{27\,(a+b)^3\,(f(x)-g(x))} - \frac{2f(x}{27\,(a+b)^3}\\
f_1(x) &= \text{Expression too large to display}\\
f_2(x) &= 0\\
f_3(x) &= \frac{a^3f(x)\,h(x)}{(a+b)^3\,(f(x)-g(x))} + \frac{h(x)\,f(x)\,b^3}{(a+b)^3\,(f(x)-g(x))} - \frac{a^3g(x)\,h(x)}{(a+b)^3\,(f(x)-g(x))} - \frac{g(x)\,h(x)\,b^3}{(a+b)^3\,(f(x)-g}
\end{aligned}
$$

Since $f_2(x) = 0$ then we check the Abel invariant to see if it depends on x or not. The Abel invariant is given by

$$-\frac{f_1^3}{f_0^2 f_3}$$

Which when evaluating gives

Expression too large to display

Since the Abel invariant depends on x then unable to solve this ode at this time.

Unable to complete the solution now.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.156 (sec). Leaf size: 648

```
dsolve(diff(y(x),x)-(y(x)-f(x))*(y(x)-g(x))*(y(x)-(a*f(x)+b*g(x))/(a+b))*h(x)-diff(f(x),x)*(
```

$$y(x) = \frac{(f(x)-g(x))(a+2b)\,e^{\mathrm{RootOf}\left(-\_Z a^4 - b^4 \ln\left(\frac{9a^3+18a^2b+18a\,b^2+9b^3+a\,e^{\_Z}+2b\,e^{\_Z}}{2a+b}\right)+\ln\left(\frac{9a^2b+9a\,b^2+9b^3+a\,e^{\_Z}+2b\,e^{\_Z}}{a-b}\right)a^4+\ln\left(\right.\right.}}{}$$

✓ Solution by Mathematica

Time used: 1.124 (sec). Leaf size: 355

```
DSolve[y'[x]-(y[x]-f[x])*(y[x]-g[x])*(y[x]-(a*f[x]+b*g[x])/(a+b))*h[x]-f'[x]*(y[x]-g[x])/(f[
```

$$\text{Solve}\left[-\frac{1}{3}(a\right.$$

$$-b)^{2/3}(2a+b)^{2/3}(a+2b)^{2/3}\text{RootSum}\left[\#1^3(a-b)^{2/3}(2a+b)^{2/3}(a+2b)^{2/3}-3\#1a^2-3\#1ab-3\#1b^2+(a-b)^{2/\right.$$

1.5 problem problem 146

Internal problem ID [4679]
Internal file name [`OUTPUT/4172_Sunday_June_05_2022_12_36_12_PM_19241691/index.tex`]

Book: Differential Gleichungen, Kamke, 3rd ed, Abel ODEs
Section: Abel ODE's with constant invariant
Problem number: problem 146.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"abelFirstKind"**

Maple gives the following as the ode type

`[_rational, _Abel]`

Unable to solve or complete the solution.

$$\boxed{x^2y' + y^3x + y^2a = 0}$$

1.5.1 Solving as abelFirstKind ode

This is Abel first kind ODE, it has the form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2 + f_3(x)y^3$$

Comparing the above to given ODE which is

$$y' = -\frac{y^3}{x} - \frac{ay^2}{x^2} \tag{1}$$

Therefore

$$\begin{aligned}
f_0(x) &= 0 \\
f_1(x) &= 0 \\
f_2(x) &= -\frac{a}{x^2} \\
f_3(x) &= -\frac{1}{x}
\end{aligned}$$

Since $f_2(x) = -\frac{a}{x^2}$ is not zero, then the first step is to apply the following transformation to remove f_2. Let $y = u(x) - \frac{f_2}{3f_3}$ or

$$
\begin{aligned}
y &= u(x) - \left(\frac{-\frac{a}{x^2}}{-\frac{3}{x}}\right) \\
&= u(x) - \frac{a}{3x}
\end{aligned}
$$

The above transformation applied to (1) gives a new ODE as

$$
u'(x) = -\frac{u(x)^3}{x} + \frac{u(x)\, a^2}{3x^3} - \frac{2a^3}{27x^4} - \frac{a}{3x^2} \tag{2}
$$

This is Abel first kind ODE, it has the form

$$
u'(x) = f_0(x) + f_1(x)u(x) + f_2(x)u(x)^2 + f_3(x)u(x)^3
$$

Comparing the above to given ODE which is

$$
u'(x) = -\frac{u(x)^3}{x} + \frac{u(x)\, a^2}{3x^3} - \frac{2a^3 + 9a\, x^2}{27x^4} \tag{1}
$$

Therefore

$$
\begin{aligned}
f_0(x) &= -\frac{2a^3}{27x^4} - \frac{a}{3x^2} \\
f_1(x) &= \frac{a^2}{3x^3} \\
f_2(x) &= 0 \\
f_3(x) &= -\frac{1}{x}
\end{aligned}
$$

Since $f_2(x) = 0$ then we check the Abel invariant to see if it depends on x or not. The Abel invariant is given by

$$
-\frac{f_1^3}{f_0^2 f_3}
$$

Which when evaluating gives

$$
-\frac{\left(\frac{\frac{8a^3}{27x^5} + \frac{2a}{3x^3}}{x} + \frac{-\frac{2a^3}{27x^4} - \frac{a}{3x^2}}{x^2} - \frac{\left(-\frac{2a^3}{27x^4} - \frac{a}{3x^2}\right) a^2}{x^4}\right)^3 x^4}{27\left(-\frac{2a^3}{27x^4} - \frac{a}{3x^2}\right)^5}
$$

Since the Abel invariant depends on x then unable to solve this ode at this time.

Unable to complete the solution now.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 65

```
dsolve(x^2*diff(y(x),x)+x*y(x)^3+a*y(x)^2 = 0,y(x), singsol=all)
```

$$c_1 + \mathrm{e}^{-\frac{((a+x)y(x)+x)((a-x)y(x)+x)}{2y(x)^2x^2}} x + \frac{\operatorname{erf}\left(\frac{\sqrt{2}\,(ay(x)+x)}{2y(x)x}\right)\sqrt{2}\,\sqrt{\pi}\,a\,\mathrm{e}^{\frac{1}{2}}}{2} = 0$$

✓ Solution by Mathematica

Time used: 0.61 (sec). Leaf size: 78

```
DSolve[x^2*y'[x]+x*y[x]^3+a*y[x]^2 == 0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[-\frac{ia}{x} = \frac{2e^{\frac{1}{2}\left(-\frac{ia}{x}-\frac{i}{y(x)}\right)^2}}{\sqrt{2\pi}\text{erfi}\left(\frac{-\frac{ia}{x}-\frac{i}{y(x)}}{\sqrt{2}}\right)+2c_1}, y(x)\right]$$

1.6 problem problem 169

Internal problem ID [4680]
Internal file name [`OUTPUT/4173_Sunday_June_05_2022_12_36_26_PM_78426666/index.tex`]

Book: Differential Gleichungen, Kamke, 3rd ed, Abel ODEs
Section: Abel ODE's with constant invariant
Problem number: problem 169.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"abelFirstKind"**

Maple gives the following as the ode type

```
[_rational, _Abel]
```

Unable to solve or complete the solution.

$$\boxed{(ax+b)^2 y' + (ax+b) y^3 + cy^2 = 0}$$

1.6.1 Solving as abelFirstKind ode

This is Abel first kind ODE, it has the form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2 + f_3(x)y^3$$

Comparing the above to given ODE which is

$$y' = -\frac{y^3}{ax+b} - \frac{cy^2}{(ax+b)^2} \tag{1}$$

Therefore

$$\begin{aligned}
f_0(x) &= 0 \\
f_1(x) &= 0 \\
f_2(x) &= -\frac{c}{(ax+b)^2} \\
f_3(x) &= -\frac{1}{ax+b}
\end{aligned}$$

Since $f_2(x) = -\frac{c}{(ax+b)^2}$ is not zero, then the first step is to apply the following transformation to remove f_2. Let $y = u(x) - \frac{f_2}{3f_3}$ or

$$\begin{aligned} y &= u(x) - \left(\frac{-\frac{c}{(ax+b)^2}}{-\frac{3}{ax+b}}\right) \\ &= u(x) - \frac{c}{3ax+3b} \end{aligned}$$

The above transformation applied to (1) gives a new ODE as

$$u'(x) = -\frac{u(x)^3 a^3 x^3}{(ax+b)^4} - \frac{3u(x)^3 a^2 b x^2}{(ax+b)^4} - \frac{3u(x)^3 a b^2 x}{(ax+b)^4} - \frac{u(x)^3 b^3}{(ax+b)^4} - \frac{a^3 c x^2}{3(ax+b)^4} + \frac{u(x) a c^2 x}{3(ax+b)^4} - \frac{2a^2 bcx}{3(ax+b} \tag{2}$$

This is Abel first kind ODE, it has the form

$$u'(x) = f_0(x) + f_1(x)u(x) + f_2(x)u(x)^2 + f_3(x)u(x)^3$$

Comparing the above to given ODE which is

$$u'(x) = -\frac{(27a^3x^3 + 81a^2bx^2 + 81ab^2x + 27b^3)u(x)^3}{27(ax+b)^4} - \frac{(-9ac^2x - 9bc^2)u(x)}{27(ax+b)^4} - \frac{9a^3cx^2 + 18a^2bcx + 9a}{27(ax+b)^4} \tag{1}$$

Therefore

$$\begin{aligned} f_0(x) &= -\frac{a^3 c x^2}{3(ax+b)^4} - \frac{2a^2 bcx}{3(ax+b)^4} - \frac{a b^2 c}{3(ax+b)^4} - \frac{2c^3}{27(ax+b)^4} \\ f_1(x) &= \frac{ax c^2}{3(ax+b)^4} + \frac{b c^2}{3(ax+b)^4} \\ f_2(x) &= 0 \\ f_3(x) &= -\frac{a^3x^3}{(ax+b)^4} - \frac{3a^2 b x^2}{(ax+b)^4} - \frac{3a b^2 x}{(ax+b)^4} - \frac{b^3}{(ax+b)^4} \end{aligned}$$

Since $f_2(x) = 0$ then we check the Abel invariant to see if it depends on x or not. The Abel invariant is given by

$$-\frac{f_1^3}{f_0^2 f_3}$$

Which when evaluating gives

$$-\frac{\left(-\left(\frac{4a^4cx^2}{3(ax+b)^5} - \frac{2x a^3 c}{3(ax+b)^4} + \frac{8a^3bcx}{3(ax+b)^5} - \frac{2a^2bc}{3(ax+b)^4} + \frac{4a^2b^2c}{3(ax+b)^5} + \frac{8c^3a}{27(ax+b)^5}\right)\left(-\frac{a^3x^3}{(ax+b)^4} - \frac{3a^2bx^2}{(ax+b)^4} - \frac{3ab^2x}{(ax+b)^4} - \frac{b^3}{(ax+b)}\right.}{}$$

Since the Abel invariant depends on x then unable to solve this ode at this time.

Unable to complete the solution now.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 126

```
dsolve((a*x+b)^2*diff(y(x),x)+(a*x+b)*y(x)^3+c*y(x)^2 = 0,y(x), singsol=all)
```

$$\frac{\left(\sqrt{a}\, b+a^{\frac{3}{2}} x\right) \mathrm{e}^{-\frac{\left((a x+b+c) y(x)+a(a x+b)\right)\left((-a x-b+c) y(x)+a(a x+b)\right)}{2 y(x)^{2}(a x+b)^{2} a}}+\frac{c \sqrt{2}\, \sqrt{\pi}\, \mathrm{e}^{\frac{1}{2 a}} \operatorname{erf}\left(\frac{(c y(x)+a(a x+b)) \sqrt{2}}{2 \sqrt{a}\, y(x)(a x+b)}\right)}{2}+c_{1} a^{\frac{3}{2}}}{a^{\frac{3}{2}}}=0$$

✓ Solution by Mathematica

Time used: 1.43 (sec). Leaf size: 149

```
DSolve[(a*x+b)^2*y'[x]+(a*x+b)*y[x]^3+c*y[x]^2 == 0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[-\frac{c}{\sqrt{-a(ax+b)^2}} = \frac{2\exp\left(\frac{1}{2}\left(-\frac{c}{\sqrt{-a(ax+b)^2}} - \frac{\left(-a(ax+b)^2\right)^{3/2}}{ay(x)(ax+b)^3}\right)^2\right)}{\sqrt{2\pi}\,\text{erfi}\left(\frac{-\frac{c}{\sqrt{-a(ax+b)^2}} - \frac{\left(-a(ax+b)^2\right)^{3/2}}{ay(x)(ax+b)^3}}{\sqrt{2}}\right) + 2c_1}, y(x)\right]$$